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HIGHER ORDER STRATONOVICH EXPANSION IN WEAK-STRONG BEAM-BEAM INTERACTION

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Higher Order Stratonovich Expansion in Weak-Strong Beam-Beam Interaction

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Abstract A general prescription is proposed for the study of coherent phenomena in electron storage rings due to a localized nonlinear force. The prescription is based on expanding the distribution function into a series using generalized Hermite polynomials in two dimensional phase space (Stratonovich expansion). When the series is truncated at the lowest order, it gives the Gaussian approximation. The prescription is applied to weak-strong beam-beam interactions in e^+e^- colliding storage rings. The approximation is introduced by the truncation but no assumption is made for the strength of the beam-beam force. This is not the perturbative approach in the traditional sense and is applicable to quite strong beam-beam force. Up to 20-th order (quasi-) moments can be included now. This is achieved by expanding the distribution function around that of the Gaussian approximation. (Gaussian gauge).

1 Introduction

In this paper, we report a part of results on extending the model proposed in Ref.[1], where a solvable model was proposed for the strong-strong and weak-strong beam-beam interaction based on the Gaussian approximation of the distribution functions. It illustrated some of the characteristic features of the problem qualitatively well. In particular, we could discuss the case of very strong beam-beam force. The agreement between multiparticle tracking results does not become worse for quite large force. Quantitatively, however, there were some disagreements. This seems to come from the lack of degrees of freedom of the model, since we represented the distribution functions by only three moments.

More recently, one possible extension was proposed[2], which is based on Stratonovich expansion[3] and truncating it at a finite order. In the next-to-lowest order approximation, the model presents improved quantitative agreement with the multiparticle tracking. There, however, we could not proceed to higher orders.

Here, we propose a variation of it using a 'Gaussian gauge'. It allows higher order expansion and we can obtain much better agreement between the multiparticle tracking results. Other extension of Ref.[1] will be reviewed in Appendix A.

2 General Stratonovich Expansion

As canonical variables in the 2-dimensional phase-space, we use

$$X^1 = \frac{x}{\sqrt{\beta\varepsilon}}, \quad X^2 = \frac{\alpha x + \beta x'}{\sqrt{\beta\varepsilon}},$$

where x and x' are transverse coordinate and its slope, α and β Twiss parameters and ε the nominal emittance (i.e. without beam-beam effect).

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General Stratonovich expansion[3], is an expansion of two dimensional distribution function $\psi(X^1, X^2)$ around the two dimensional Gaussian distribution,

$$G(\vec{X}; g) = \frac{1}{2\pi\sqrt{\det g}} \exp -\phi,$$

$$\phi = \tfrac{1}{2} g_{\alpha\beta} X^\alpha X^\beta,$$

where $g_{\alpha\beta}$ is the inverse of $g^{\alpha\beta}$ and $\det g = g^{11}g^{22} - (g^{12})^2$. Here $g^{\alpha\beta}$ is any symmetric positive definite matrix. Here and in what follows, we employ Einstein's summation convention; when the same symbol appears in both upper and lower indices simultaneously, a summation with respect to the symbol from 1 to 2 is implied. The $g^{\alpha\beta}$ will be called metric because of the similarity to Riemanian geometry. Note that A^α is different from A_α, which is $g_{\alpha\beta}A^\beta$.

We start from the following lemma:

Lemma 1 *Any distribution function $\psi(\vec{X})$, which is symmetric in phase-space, $\psi(-\vec{X}) = \psi(\vec{X})$, which is normalized to unity, and which falls exponentially at infinity, can be expanded as*

$$\psi(\vec{X}) = G(\vec{X}; g) P(\vec{X}; g, Q), \tag{1}$$

$$P(\vec{X}; g, Q) = 1 + \sum_{\substack{n=2 \\ \text{even}}} \frac{1}{n!} Q^{\alpha_1\alpha_2\cdots\alpha_n} H_{\alpha_1\alpha_2\cdots\alpha_n}(\vec{X}; g). \tag{2}$$

Here the sum extends over all even numbers from 2 to infinity.

Here, H is the generalized Hermite polynomial,

$$H_{\alpha_1\alpha_2\cdots\alpha_n}(\vec{X}; g) = e^{\phi} \prod_{i=1}^{n} (-\partial_{\alpha_i}) e^{-\phi}.$$

The Q's are called **quasi-moments:**

Lemma 2 *For given ψ, the quasi-moments are obtained as*

$$Q^{\alpha_1\alpha_2\cdots\alpha_n} = < H^{\alpha_1\alpha_2\cdots\alpha_n}(\vec{X}; g) >,$$

where $<\ \ >$ is the expectation value with respect to ψ and

$$H^{\alpha_1\alpha_2\cdots\alpha_n} = g^{\alpha_1\beta_1} g^{\alpha_2\beta_2} \cdots g^{\alpha_n\beta_n} H_{\beta_1\beta_2\cdots\beta_n}.$$

Let us introduce the moments

$$M^{\alpha_1\alpha_2\cdots\alpha_n} = < X^{\alpha_1} X^{\alpha_2} \cdots X^{\alpha_n} > . \tag{3}$$

Using Eq.(1), we have

Lemma 3 *A moment M of even order can be expanded in terms of the quasi-moments of even order as*

$$\begin{aligned}
& M^{\alpha_1\alpha_2\cdots\alpha_n} \\
= \ & Q^{\alpha_1\alpha_2\cdots\alpha_n} \\
+ \ & \sum_{apc} g^{\alpha_1\alpha_2} Q^{\alpha_3\cdots\alpha_n} \\
+ \ & \sum_{apc} g^{\alpha_1\alpha_2} g^{\alpha_3\alpha_4} Q^{\alpha_5\cdots\alpha_n} \\
+ \ & \cdots\cdots \\
+ \ & \sum_{apc} g^{\alpha_1\alpha_2} g^{\alpha_3\alpha_4} \cdots g^{\alpha_{n-3}\alpha_{n-2}} Q^{\alpha_{n-1}\alpha_n} \\
+ \ & \sum_{apc} g^{\alpha_1\alpha_2} g^{\alpha_3\alpha_4} \cdots g^{\alpha_{n-1}\alpha_n}.
\end{aligned} \tag{4}$$

Here

$$\sum_{apc}$$

means a sum over all **possible combination** of indices but restricted not to reproduce the same expression. For example,

$$\sum_{apc} g_{\alpha\beta}g_{\gamma\delta} = g_{\alpha\beta}g_{\gamma\delta} + g_{\alpha\gamma}g_{\beta\delta} + g_{\alpha\delta}g_{\beta\gamma}.$$

In the same manner, we have

Lemma 4 *A quasi-moment Q of even order can be expanded in terms of the moments of even order as*

$$\begin{aligned}
& Q^{\alpha_1\alpha_2\cdots\alpha_n} \\
= \; & M^{\alpha_1\alpha_2\cdots\alpha_n} \\
- \; & \sum_{apc} g^{\alpha_1\alpha_2} M^{\alpha_3\cdots\alpha_n} \\
+ \; & \sum_{apc} g^{\alpha_1\alpha_2} g^{\alpha_3\alpha_4} M^{\alpha_5\cdots\alpha_n} \\
- \; & \cdots\cdots \\
+ \; & \sum_{apc}(-g^{\alpha_1\alpha_2})(-g^{\alpha_3\alpha_4})\cdots(-g^{\alpha_{n-3}\alpha_{n-2}})M^{\alpha_{n-1}\alpha_n} \\
+ \; & \sum_{apc}(-g^{\alpha_1\alpha_2})(-g^{\alpha_3\alpha_4})\cdots(-g^{\alpha_{n-1}\alpha_n}). \qquad (5)
\end{aligned}$$

In the above, Q and M of only even order are considered. Inclusion of the odd order M's and Q's are straightforward, but irrelevant for the present problem.

3 Gauge Degree of Freedom

In the above, we did not specify $g^{\alpha\beta}$. Of course, $M^{\alpha\beta}$, for example, has definite meaning, which can be expressed as

$$M^{\alpha\beta} = g^{\alpha\beta} + Q^{\alpha\beta}. \qquad (6)$$

Unless we specify $g^{\alpha\beta}$, Q's are not defined.

We can freely specify g. This freedom is called Gauge freedom. It is related to how to express ψ, but has nothing to do with physical contents of ψ, **provided we do not truncate the expansion.** Once we specify the metric g, the gauge is fixed.

There are some characteristic gauges:

proper gauge We use real second moments for g,

$$g^{\alpha\beta} = M^{\alpha\beta},$$

so that, in this gauge, $Q^{\alpha\beta} = 0$ by definition. This was employed in Ref.[2]. The most important point of this gauge is that all Q's vanish if and only if ψ is a Gaussian. Another important property of it is that the convergence of the expansion is expected to be the fastest, so that we can truncate it at lower orders.

nominal gauge We use the nominal value of the second-order moment

$$g^{\alpha\beta} = \delta^{\alpha\beta}.$$

In this gauge, the generalized Hermite polynomial's are reduced to the products of the usual Hermite polynomials. This gauge is the simplest of all, but the convergence is the worst.

Gaussian gauge The Gaussian approximation[1] was shown to be rather good, i.e. it reproduces all of the qualitative features of the system. We can use the results of this approximation. That is

$$g^{\alpha\beta} = g_G^{\alpha\beta},$$

where g_G is the Gaussian approximation result so that it depends on all the parameters that characterizes the problem. (Beam-beam parameters, tunes and so on.)

The first merit of this gauge is that the changes of Q's around the ring can be expressed as linear transformation. (See next section). The theory is thus much simplified in this gauge so that we can include higher order terms rather easily. The convergence is worse, of course, than in the proper gauge.

Relation between different gauges It is easy to relate different gauges, if we use the fact that the moment M is gauge independent. Once they are related to moments, using Eqs.(4) and (5), the relation becomes clear.

4 Weak-Strong Model

In this paper, we study the weak-strong case only, because the dynamics is particularly simple,

Gaussian Approximation

Since we will use the Gaussian approximation results as a basis of the expansion, it is convenient here to review it[1].

In this approximation, we put all Q's zero so that g is the second moment. We track the changes of g around a ring. A ring is composed of one interaction point (IP) and an arc.

In the arc, the beam undergoes the betatron oscillation with the radiation effect. This is represented by the following mapping

$$g_{new} = U[\Lambda g_{old}\Lambda^t + (1-\Lambda^2)]U^t, \qquad (7)$$

where

$$g = \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix},$$

$$U = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix}, \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & \lambda^2 \end{pmatrix}, \qquad (8)$$

where $\mu = 2\pi\nu$ is the nominal phase advance for the arc.

$$\lambda = \exp(-1/T_e), \quad T_e = \frac{\text{transverse damping time}}{\text{flight time during the arc}} .$$

The prescription of radiation given above is called asymmetric prescription and was shown not the best way[4]. We use it, however, from a historical reason: The author does not have the multiparticle tracking results in the better prescription. The better way is called symmetric prescription[4], which is recapitulated in Appendix B.

The beam-beam kick, at the IP, is represented by the following:

$$g_{new}^{\alpha\beta} = <(X+F)^\alpha (X+F)^\beta>_G . \qquad (9)$$

Here, $\vec{F} = (0, F)$ is the beam-beam kick

$$F(X) = 8\pi\eta \frac{1}{X}[\exp(-X^2/2) - 1],$$

where η is the nominal beam-beam parameter,

$$\eta = \frac{N_e r_e}{4\pi\gamma\epsilon}.$$

Here N_* and γ are the number of particles of the strong beam and the relativistic Lorentz factor of the weak beam, respectively, and r_e the classical electron radius. (In general, we use η for the nominal beam-beam parameter and ξ for the real beam-beam parameter, which is defined in terms of the actual beam size instead of the nominal one). The $< \ >_G$ stands for the average with respect to the Gaussian part $G(\vec{X};g)$: to evaluate the r.h.s. of Eqs.(9), we use $G(\vec{X};g_{old})$.

It is observed that the system always falls into a period one fixed point in (g^{11}, g^{12}, g^{22}) space. This can be expressed as a complicated functions:

$$g^{\alpha\beta} = g^{\alpha\beta}(\nu, T_\epsilon, \eta).$$

Higher Order Quasi-Moments

The betatron oscillation with the radiation effect is represented by the following mapping:

$$Q_{new}^{\alpha_1\alpha_2\cdots\alpha_n} = (U\Lambda)^{\alpha_1}{}_{\beta_1}(U\Lambda)^{\alpha_2}{}_{\beta_2}\cdots(U\Lambda)^{\alpha_n}{}_{\beta_n}Q_{old}^{\beta_1\beta_2\cdots\beta_n}, \qquad (10)$$

The beam-beam effect, at the IP, is the following:

$$Q_{new}^{\alpha_1\alpha_2\cdots\alpha_n} = < H^{\alpha_1\alpha_2\cdots\alpha_n}(\vec{X}+\vec{F};g_{new}) > . \qquad (11)$$

Here, g_{new} is defined by Eq.(9). To evaluate this average, we should truncate the expansion. Let us truncate it at $2N_{max}$-th order.

Let us denote

$$Q[N,n] \equiv Q^{\overbrace{11\cdots1}^{2N-n}\overbrace{22\cdots2}^{n}},$$

for the sake of brevity. We also abbreviate the generalized Hermite polynomials as

$$H[N,n] = H_{\underbrace{11\cdots1}_{2N-n}\underbrace{22\cdots2}_{n}}.$$

Then we can define vectors

$$(\vec{Q})^k = Q[N,n], \quad (\vec{H})_k = H[N,n],$$

where

$$k = k(N,n) = N^2 + n. \ (1 \le N \le N_{max}, \ 0 \le n \le 2N)$$

That is,

$$\vec{Q} \equiv \begin{pmatrix} Q^{11} \\ Q^{12} \\ Q^{22} \\ Q^{1111} \\ \vdots \end{pmatrix}, \quad \vec{H} \equiv \begin{pmatrix} H_{11} \\ H_{12} \\ H_{22} \\ H_{1111} \\ \vdots \end{pmatrix}.$$

These vectors are $k_{max} \equiv N_{max}^2 + 2N_{max}$ dimensional.

Now, we can show that, under the beam-beam kick, $\vec{Q}$ changes as

$$\vec{Q}_{new} = A(g,g')\vec{Q}_{old} + \vec{B}(g,g'),$$

where g (g') is the second order moments in the Gaussian approximation just before (after) the beam-beam kick and A is a $k_{max} \times k_{max}$ matrix. Since we have an explicit expressions of g and g', A and $\vec{B}$ are also expressed explicitly.

Also under the betatron oscillation and radiation damping, Q are transformed linearly as

$$\vec{Q}_{new} = O(\mu,\lambda)\vec{Q}_{old}.$$

Now we have a complete set of mappings for Q. If we observe Q at the Poincaré section just before the beam-beam kick, it changes every turn as

$$\vec{Q}_{(n+1)-\text{th turn}} = O(A\vec{Q}_{n-\text{th turn}} + \vec{B}).$$

If the ψ is to fall into a fixed distribution after many turns, (although it is not assured), we expect also that $\vec{Q}$ will eventually be a single fixed point in the k_{max} dimensional vector space.

This fixed point is expressed as

$$\vec{Q} = \frac{1}{1-OA}O\vec{B}. \qquad (12)$$

As is well known[5], the mapping converges into the fixed point if and only if no eigenvalue of OA has the absolute value larger than 1. It is expected that there is some domain in the parameter space (ν, λ, η) where the above condition does not apply. We will discuss this case later.

5 Fixed Point

A Fortran program called SBS (Strong-weak Beam-beam interaction in Stratonovich expansion) solves Eq.(12). Due to the memory size restriction, $N_{max} \le 10$, for the time being.

Second Moment

One of the characteristic phenomena in weak-strong interaction is the relatively sudden increase of the (r.m.s.) beam size, when one increases η. The Gaussian approximation[1] could illustrate it only qualitatively. In this approximation, beam-size is always overestimated. This fact was understood as the unphysical increase of the entropy in Ref.[1]: if a distribution ψ was a Gaussian before the beam-beam kick, it is no longer be so after the kick. In approximating it by a Gaussian, we lose the information contained in ψ. This is almost equivalent to the case where we add an additional diffusion to the system. Thus, we can expect that the discrepancy between Gaussian approximation and the multiparticle tracking result will be reduced by introducing higher order (quasi-) moments.

η dependence In Table 1, we show the numerical results of SBS on M^{11}, which is obtained from the solution of Eq.(12) and the relation Eq.(6). We compare results with different N_{max}. Since it can be shown that, in this gauge, when $N_{max} = 1$, $\vec{B}$ vanishes and consequently $\vec{Q} = 0$, we do not show this case in the Table. In some cases, the large eigenvalue occurs so that the result of SBS is not reliable (indicated by !). In some other cases, we cannot use SBS because of the overflow in the numerical process. The latter is less difficult a problem.

The former is a fundamental problem. It seems, however, that such a thing occurs only in restricted region of η. See section 6.

We also show the multiparticle tracking results[6] in the Table. As N_{max} becomes larger, the agreement is better, except for the case of large eigenvalues.

Tune dependence A multiparticle tracking shows us that the equilibrium distribution is quite sensitive to the tune[6]. In Table 2, we show M^{11} for some parameters together with the multiparticle tracking results. In some tunes, the multiparticle tracking gives much smaller values. In this case, higher order quasi-moments seem to be required. In some other cases, the model suffers from the large eigenvalues.

N_{max}	$\eta = 0.03$	$\eta = 0.05$	$\eta = 0.06$
(Gauss)	1.08	2.06	3.60
2	0.92	1.50	2.52
3	0.89	1.14	1.81
4	0.88	0.97	1.40
6	0.88	0.87	1.01
8	0.89	!	0.87
9	0.89	!	0.81
(MPT)	~0.9	~0.8	~ 0.8
(Proper)	~0.9	BR	BR
N_{max}	$\eta = 0.1$	$\eta = 0.3$	$\eta = 0.5$
(Gauss)	12.5	83.3	179.7
2	9.89	75.1	165.8
3	8.00	69.7	157.6
4	6.69	66.2	152.5
6	5.03	61.1	144.9
8	4.05	56.6	137.4
9	3.71	*	*
(MPT)	1.95	36.0	59.5
(Proper)	9.0	72.0	

Table 1: Second moment M^{11}. Parameters are $T_s = 142.8$ and $\nu = 0.15$. (Gauss) and (MPT) stand for the Gaussian approximation and the multiparticle tracking results, respectively. The * means that the calculation fails due to overflow. The ! means that the mapping has too large eigenvalue(s). (Proper) stands for the proper gauge results obtained in Ref.[2]. BR stands for the border region. (See section 6).

Excess Along with the second moment, the excess

$$[(M^{1111} - 3(M^{11})^2]/(M^{11})^2$$

is an interesting and important quantity which expresses the deviation from a Gaussian. We will generalize it. As stated before, the Gaussian gauge can be related to the proper Gauge through the moments, Eqs.(4) and (5). Let us denote the quasi-moment in the proper gauge as Q_{prop}. The Q_{prop} is the natural extension of the excess. (See section 2). We define the normalized-proper-quasi-moment $D[N]$ as

$$D[N] = \frac{Q[N,0]_{prop}}{(M^{11})^N}.$$

As easily seen, $D[2]$ is the excess.

We show the numerical and multiparticle tracking results for $D[2]$ also in Tab.2. (Note that the multiparticle tracking results are less accurate than those of M^{11}, because of the possible fluctuation due to the finite number of the test particles.) These two types of results seem to agree with each other roughly. In particular, the sign corresponds well.

We show $D[N]$'s for some values of η in Table 3 (A).

Distribution function ψ

It is interesting to see to what extent our truncated Stratonovich expansion can reproduce ψ.

As discussed in Ref.[2], the positive definiteness of ψ is not assured when the expansion is truncated. This is the worst point of our model. In Ref.[2], however, it was shown to be harmless for the usual range of parameters, ($\eta \leq 0.1$, say). With larger values of η, it becomes more remarkable. Sometimes, $\psi(0)$ can become negative.

ν	M^{11} SBS	M^{11} MPT	$D[2]$ SBS	$D[2]$ MPT
0.1	50.4	7.58	0.345	0.973
0.1125	52.0	38.3	-0.33	-1.08
0.125	57.3	8.31	-0.526	-0.55
0.1375	52.0	31.4	-0.414	0.517
0.15	56.6	36.2	-1.104	-1.42
0.1625	!	96.6	!	-0.813
0.175	48.4	6.91	-0.96	-1.14
0.1875	50.5	16.8	-0.491	-0.148
0.2	52.1	12.7	-0.82	-1.55
0.2125	!	53.3	!	3.88
0.225	55.3	26.2	-0.79	-1.69
0.2375	!	49.4	!	-1.89

Table 2: Second moment M^{11} and normalized quasi-moment $D[2]$. Parameters are $T_s = 142.8$, $\eta = 0.3$ and $N_{max} = 8$. (SBS) and (MPT) stand for the numerical and the multiparticle tracking results, respectively.

In the multiparticle tracking, on the other hand, it is frequently be observed that even the origin of the phase space has no particle[6]. It is natural to expect that when ψ becomes negative in our model, it implies the absence of the particle there.

Since we truncated the expansion, there are at least two possibilities for the reconstruction of ψ. One is to use (g, Q), Gaussian gauge, and the other is the proper gauge, $(M^{\alpha\beta}, Q_{prop})$:

$$\psi(\vec{X})_G = G(\vec{X}; g)P(\vec{X}; g, Q), \quad (13)$$

$$\psi(\vec{X})_{prop} = G(\vec{X}; M^{\alpha\beta})P(\vec{X}; M_2, Q_{prop}). \quad (14)$$

These two need not be the same nor similar.

An example is shown in Fig.1. Here, in (B), the regions where ψ is larger than some positive value are plotted. The proper gauge seems to be better in agreement with the multiparticle tracking results. It does not seem easy, however, to predict the right form of ψ by this model.

6 Discussion

The Difference from the Proper Gauge

One merit of using the Gaussian gauge is that we can introduce quasi-moments of much higher order. There are two main reasons:

1. In the proper gauge, we should solve (or track) a set of complicated nonlinear mappings of g and Q, while in the Gaussian gauge, the nonlinear mapping appears only in g. Once the fixed point of this nonlinear mapping is obtained, (this can be done analytically), the mapping for Q is represented by a matrix with known elements.

2. It is hopeless to solve the mapping equations for (g, Q) in the proper gauge. All one can do is, therefore, to track the mapping. In tracking (g, Q), the positive definiteness of g is easily lost in the intermediate stage and this will lead an overflow of the calculation. In the Gaussian gauge, however, there is no need to track Q, since the equilibrium solution is obtained explicitly.

Since we can use much higher order moments in the Gaussian gauge, the agreement with the multiparticle tracking is much

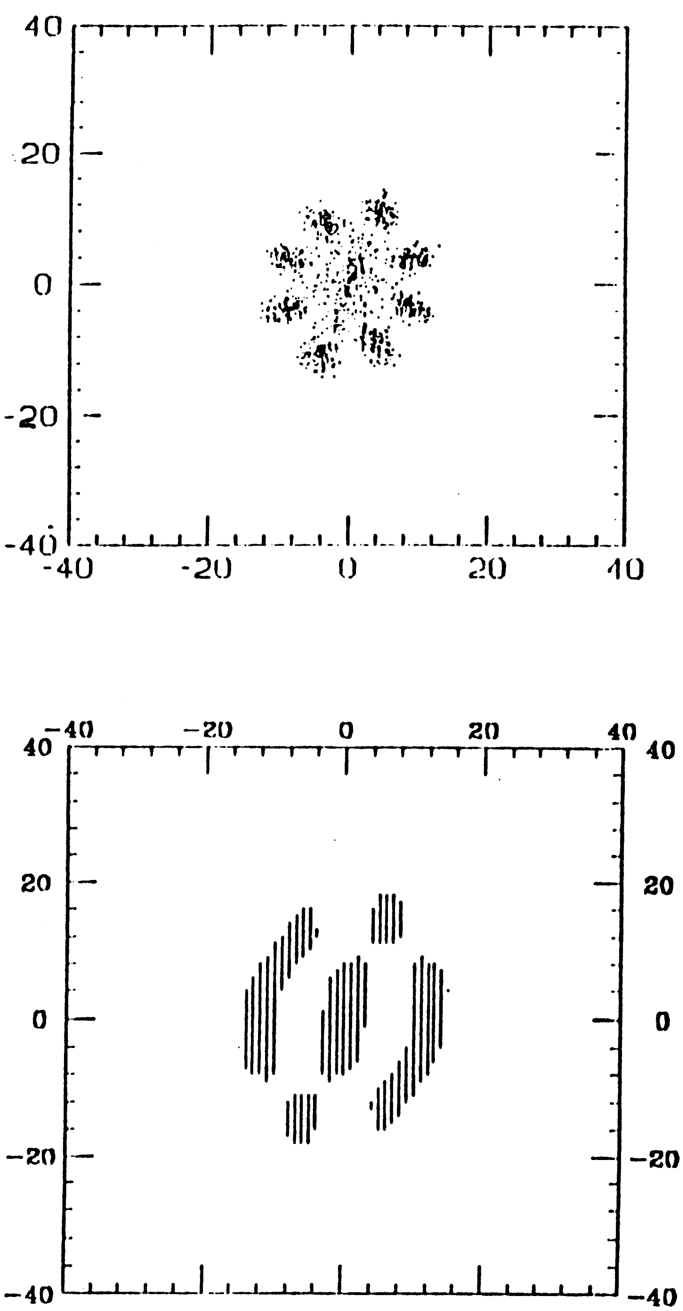

Figure 1: The equilibrium distribution. (A),The phase space obtained in a multiparticle tracking . (B), The same obtained by the present model with $N_{max} = 8$. Horizontal and vertical axes are X and P, respectively. Parameters: $T_\epsilon = 142.8$, $\nu = 0.1125$ and $\eta = 0.3$.

	η	0.03	0.1	0.3	0.5
A	M^{11}	0.886	4.05	56.6	137.4
	$D[2]$	0.10	-0.52	-1.10	-0.69
	$D[3]$	0.18	3.94	5.04	2.75
	$D[4]$	-1.30	-2.89×10^1	-1.67×10^1	1.13
	$D[5]$	1.52×10^1	-1.47×10^2	-8.70×10^2	-4.67×10^2
	$D[6]$	-1.87×10^2	1.50×10^4	3.95×10^4	1.99×10^4
	$D[7]$	2.78×10^3	-5.09×10^5	-1.35×10^6	-7.44×10^5
	$D[8]$	-3.37×10^4	1.19×10^7	4.21×10^7	2.69×10^7
B	M^{11}		1.95	36.0	59.5
	$D[2]$	~ 0.9	3.31	-1.42	-1.51
C	M^{11}	0.962	4.27	57.3	138.2
	$D[2]$	0.05	-0.47	0.23	-0.42
	$D[3]$	0.15	3.38	-3.68	0.90
	$D[4]$	-1.56	-2.80×10^1	6.94×10^1	-1.14
	$D[5]$	1.60×10^1	-2.10×10^1	-1.52×10^3	5.81×10^2
	$D[6]$	-1.90×10^2	1.13×10^4	6.38×10^4	-1.94×10^4
	$D[7]$	2.84×10^3	-4.63×10^5	-2.63×10^6	5.60×10^5
	$D[8]$	-3.61×10^4	1.42×10^7	1.00×10^8	-1.57×10^7

Table 3: Normalized quasi-moments, $D[N]$. Parameters are $T_\epsilon = 142.8$ and $\nu = 0.15$,$N_{max} = 8$. (A) Asymmetric radiation, (B) Multiparticle tracking results with asymmetric radiation, (C) Symmetric radiation (see Appendix A).

better. It is a matter of course that, for a fixed value of N_{max}, the proper gauge gives better agreement with the multiparticle tracking. In Table 1, we compared the results of the proper gauge obtained in Ref.[2] with $N_{max} = 2$. In this example, the difference between two gauges is quite small for this N_{max}. We thus conclude that the Gaussian gauge is better than the proper gauge.

Why we are interested in so large value of η

We considered $\eta \leq 0.5$. In the actual accelerators, we can not reach this value of η, usually. There are some reasons to consider such a large value:

1. We consider the weak-strong case, which is more academic than practical. Here, we are interested in the agreement between the theory and the multiparticle tracking but not interested in that between the theory and the experiment. In this case, it is helpful to consider an extreme case, since it shows the essential points of the problem more clearly.

2. In one-dimensional theory, such as that considered here, we cannot expect a good agreement with the experiments. Usually, such a theory gives larger value of η. To be more realistic, we should consider synchrotron oscillation, for example. Presumably, these effects lowers the beam-beam limit. If a model agrees well with the multiparticle tracking for larger value of η, we can expect the better agreement when the synchrotron oscillation will be included. The opposite is also true. In the one-dimensional theory, thus, we need the agreement even for the large value of η.

From the results in Table 1, we can conclude that, to obtain good agreement between the multiparticle tracking, we need higher and higher order moments when η becomes larger and larger. It seems reasonable.

Large Eigenvalue

At some set of parameters, the large eigenvalue appears, which means that the mapping of $\vec{Q}$ does not fall into the period one fixed point, Eq.(12).

It might be thought that it comes from the fact that we use the Gaussian gauge, where we expand ψ around the Gaussian approximation result. As shown in Ref.[2], however, the similar thing exists even in the proper gauge: the mapping for (g, Q) does not converge to a fixed point in some region which is called 'border region'.

It seems more fundamental.

The most characteristic point of the weak-strong case is the rapid increase of the beam size at some η. The region of this rapid increase seems to be identical with that where the large eigenvalue occurs. This seems to be related to the heart of the beam-beam interaction. The fact that ψ falls into a period one fixed point in the multiparticle tracking seems to imply that this anomaly is related to the truncation. Thus, to understand the most characteristic points of the beam-beam interaction, we should not truncate it and should use ψ itself. This is the place where the infiniteness of the degrees of freedom of ψ manifests itself.

Apart from this case, the present model gives reasonable approximation and the numerical agreement seems to be improved more and more when higher and higher order quasi moments are introduced.

More detailed and extended work will be published elsewhere.

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A Review and Planning of the Extension of Gaussian Model

It seems convenient here to review what has been done and what is considered in the future as extensions of the work Ref.[1].

Second Moment Only This is done in Ref.[1].

First Moment Only We assume that the beams are Gaussian and the second moments are given and fixed. Then we discuss the (coherent) dipole mode problem. This is done in Ref.[7].

First and Second Moment We consider first and second moments as dynamical variables within the Gaussian approximation. This will be done in Ref.[8].

Second and Higher Even Order Moments This was done in Ref.[2] in the proper gauge. In the Gaussian gauge, only strong-weak case was considered in the present paper. The strong-strong case will be considered in the future.

All Moments In the future.

Inclusion of Synchrotron Motion Under investigation.

B Better Treatment of Radiation

In Ref.[4], it is shown that the radiation should be symmetric, that is, instead of Λ in Eq.(8), we should use

$$\Lambda_{sym} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

Accordingly, the expressions of the equilibrium value of g should be modified a little[8].

In Table 3, we compared two ways of radiation treatment in terms of M^{11} and $D[N]$. There seems no remarkable difference with respect to M^{11} in the present problem. Also in the multiparticle tracking , there does not seem to be so much difference in the beam sizes. (In case of linear force insertion, there is a great difference[4]).

As for $D[N]$, the difference is more noticeable. In simulation, thus, the symmetric radiation is recommended even though it is more time consuming and even though it gives similar results as the asymmetric prescription for $M^{\alpha\beta}$.

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