

NON-GAUSSIAN DISTRIBUTION OF ELECTRON BEAMS DUE TO INCOHERENT STOCHASTIC PROCESSES

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A theory is presented on the equilibrium particle distribution in electron rings, when each electron is being affected by, in addition to the synchrotron radiation effects, incoherent stochastic processes, such as a collision with atoms in the residual gas.

KEY WORDS: Tail distribution, electron beams, stochastic processes

1 INTRODUCTION

The distribution function of an electron bunch, transverse or longitudinal, is often assumed to be Gaussian. One calculates or measures the rms beam sizes and fits the whole distribution to a Gaussian having these rms beam sizes. We call it the Gaussian approximation. Actually, however, there always exists some deviation. The Gaussian approximation is sometimes erroneous. At present, however, it seems that no general way exists to evaluate the deviation from a Gaussian.

The deviation is important from two points of view; The central part affects the luminosity of colliders, circular¹ or linear,² and the brightness of synchrotron light sources.³ The tail distribution affects the lifetime of the beam, when it is too large. Even when the tail distribution is not too large and has little to do with the lifetime, it can still give rise to the background in the detectors in the collision experiment. When the distribution is Gaussian in the central part but has a long tail, the single Gaussian approximation will give pessimistic predictions of the luminosity or the brightness and optimistic predictions of the lifetime and the background.

There are several major sources of deviation from a Gaussian distribution. The simplest is the incoherent (single particle) non-dynamic process: such as the synchrotron radiation diffusion process and the scattering by the atoms in the residual gas. The aim of this paper is to establish how to calculate the distribution function of the electron bunch in this case. Hence, consideration of coherent effects and incoherent dynamic effects is postponed. Since the present aim is to establish the method, we will restrict ourselves to a simple example: a ring being treated using the smooth approximation and solely taking

into account the effect of elastic scattering on the vertical distribution. It is, however, straightforward to extend our theory to more general cases.

In the next section, a basic formalism is given. This method is applied to the betatron phase space of an electron ring in section 3. The last section is devoted to discussions and the conclusion.

2 STOCHASTIC PROCESS

In this section, we give some basic discussions of stochastic processes. For simplicity, we will discuss a one dimensional distribution, although we extend this to the two dimensional betatron phase space in the next section.

Let us first consider a discrete stochastic process: in the beginning, one has a value $W = 0$ and at each later step i , a goddess chooses

1. whether she wants to throw a die (the probability is $n_i \ll 1$),
2. if she decides to do so, a number w_i is added to W , according to the die, where the probability of w obeys a density $f(w)$, which is normalized to unity:

$$\int_{-\infty}^{\infty} dw f(w) = 1. \quad (1)$$

We want to find the probability distribution of W , $\rho(W)$, which is also normalized to unity. To do this, we introduce the characteristic function of ρ :

$$\tilde{\rho}(k) = \int_{-\infty}^{\infty} dW e^{ikW} \rho(W) = \langle e^{ikW} \rangle. \quad (2)$$

The characteristic function is the Fourier transform of the probability density. Note that

$$W = \sum \Delta_i, \quad (3)$$

where Δ_i is the gain of W at the i -th step: Δ_i is either 0 with a probability $1 - n_i$ or w_i with a probability n_i . The characteristic function of Δ_i :

$$\langle \exp(ik \Delta_i) \rangle, \quad (4)$$

can be evaluated as

$$\langle e^{ik \Delta_i} \rangle = (1 - n_i) + n_i \langle e^{ikw} \rangle_1 = \exp[n_i (\langle e^{ikw} \rangle_1 - 1)], \quad (5)$$

where the subscript 1 indicates that the expectation value is evaluated by the one-throw probability f :

$$\langle e^{ikw} \rangle_1 = \tilde{f}(k) = \int dw e^{ikw} f(w) \quad (6)$$

and use is made of the fact that $n_i \ll 1$.

Since each Δ_i is independent, we obtain

$$\bar{\rho}(k) = \prod_i \langle e^{ik\Delta_i} \rangle_1 = \exp \left[\sum n_i (\tilde{f}(k) - 1) \right]. \quad (7)$$

Imagine an electron passing through a space and losing energy due to collisions with gas atoms. The volume density of the gas atoms is Q and the differential cross-section of the energy loss by the collision is $d\sigma$. Then, the average number of the collision for the length ds is

$$N(s)ds = Q ds \int \frac{d\sigma}{d\epsilon} d\epsilon. \quad (8)$$

Once the collision occurs, the probability that the electron loses the energy ϵ is given by the one-collision probability function

$$f(\epsilon)d\epsilon = \frac{1}{\sigma_{tot}} \frac{d\sigma}{d\epsilon} d\epsilon, \quad (9)$$

which is normalized to unity.

Applying the consideration above, we can obtain the probability distribution of the energy of an electron, $E(s)$, in terms of its characteristic function:

$$\langle \exp\{ikE(s)\} \rangle = \langle \exp\{ikE(0)\} \rangle \exp \left[\int_0^s ds N(s) \{ \tilde{f}(k) - 1 \} \right]. \quad (10)$$

We can deduce the Campbell-Rice law⁴ from the last expression.

3 TRANSVERSE DISTRIBUTION

Here we extend the discussion to betatron motions.

3.1 Betatron Motions

It is convenient to introduce normalized coordinates as

$$X = x_\beta / \sigma_0, \quad P = x'_\beta / \sigma'_0, \quad (11)$$

where x_β and x'_β are the betatron variables and σ_0 and $\sigma'_0 = \sigma_0/\beta$ are nominal (*nominal* means without the collision process but with synchrotron radiation effects) betatron rms beam sizes. Here β is the betatron function. We will use the smooth approximation so that σ_0 and σ'_0 will be treated as constants. Without the collision with atoms, the equilibrium beam distribution is a Gaussian:

$$\psi_0(X, P) = (2\pi)^{-1} \exp \left[-(X^2 + P^2)/2 \right].$$

The deterministic (i.e. synchrotron radiation damping included) part of the betatron oscillation is

$$\begin{pmatrix} X \\ P \end{pmatrix}_{t_1} = G(t_1 - t_0) \begin{pmatrix} X \\ P \end{pmatrix}_{t_0}, \quad (12)$$

where

$$G(t) = \exp(-dt) \begin{pmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{pmatrix}. \quad (13)$$

Here ω is the betatron frequency and d is the betatron damping rate.

Imagine that a collision occurs and a transverse kick θ is applied at $t = t_1$:

$$\delta \begin{pmatrix} X \\ P \end{pmatrix} = \begin{pmatrix} 0 \\ \theta(t_1)/\sigma'_0 \end{pmatrix}. \quad (14)$$

Its effect on $\mathbf{X} \equiv (X, P)$ at t ($t > t_1$) can be expressed as

$$\delta \begin{pmatrix} X \\ P \end{pmatrix}_t = \begin{pmatrix} G_{12}(t - t_1) \\ G_{22}(t - t_1) \end{pmatrix} \theta(t_1)/\sigma'_0. \quad (15)$$

Since θ is a stochastic variable, we have a stochastic equation

$$\mathbf{X}(t) = \mathbf{X}_0(t) + \mathbf{X}_f(t), \quad (16)$$

where

$$\mathbf{X}_f(t) = \int_0^t dt_1 \mathbf{G}(t - t_1) \theta(t_1)/\sigma'_0. \quad (17)$$

Here \mathbf{X}_0 is \mathbf{X} in the absence of the collision process and

$$\mathbf{G}(t) = (G_{12}(t), G_{22}(t))^t. \quad (18)$$

Note that \mathbf{X}_0 includes the effects of the synchrotron radiation. The latter effect can be treated in a parallel fashion. Such a treatment is not necessary, however: we are interested in an equilibrium state here so that \mathbf{X}_0 can be considered simply as a Gaussian random variable (see section 4). Since the above expression is essentially a sum of many independent stochastic variables, Eq. (10) applies and we arrive at the expression

$$\tilde{\psi}(\mathbf{K}, t) = \langle e^{i\mathbf{K} \cdot \mathbf{X}} \rangle = \langle e^{i\mathbf{K} \cdot (\mathbf{X}_0 + \mathbf{X}_f)} \rangle = \tilde{\psi}_0(\mathbf{K}) \tilde{\psi}_f(\mathbf{K}, t), \quad (19)$$

$$\tilde{\psi}_0(\mathbf{K}) = \exp \left[-\frac{K^2}{2} \right], \quad (20)$$

$$\tilde{\psi}_f(\mathbf{K}, t) = \exp \left[N \int_0^t dt_1 \{ \tilde{f}(\mathbf{K} \cdot \mathbf{G}(t - t_1)/\sigma'_0) - 1 \} \right], \quad (21)$$

where N is the number of the kicks (collisions) per unit time [see Eq. (8)], and

$$\tilde{f}(u) = \int d\theta e^{i\theta u} f(\theta). \quad (22)$$

Here $f(\theta)$ is the one-kick probability density of θ , [see Eq. (6)].

Let us evaluate the equilibrium distribution $\tilde{\psi}(\mathbf{K}, t = \infty)$. We recast Eq. (21) in the form

$$\tilde{\psi}_f(\mathbf{K}) = \exp \left[N \int_0^\infty dt \{ \tilde{f}(Ke^{-dt} \sin(\varphi + \omega t)/\sigma'_0) - 1 \} \right], \quad (23)$$

where

$$\varphi = \tan^{-1} K_2/K_1.$$

In general, $\tilde{f}(u) = \langle \exp(iu\theta) \rangle_1$ is complex. Here, we introduce an approximation that the number of betatron oscillations during one damping time is very large. This is usually the case. Then we can replace the betatron oscillation by an average over each betatron period. Let us introduce

$$\hat{f}(u) = \int_0^\infty dt \int_0^{2\pi} \frac{d\phi}{2\pi} [\tilde{f}(ue^{-t} \sin \phi) - 1] = \frac{2}{\pi} \int_0^1 dx \frac{\Re[\tilde{f}(ux)] - 1}{x} \cos^{-1} x. \quad (24)$$

Note that $\hat{f}(u)$ is a real function, regardless of the original probability density $f(\theta)$. In fact, only the real (cosine) part of $\tilde{f}(u)$ remains after the averaging over φ . Thus, under this approximation, we can always replace Eq. (22) by

$$\tilde{f}(u) = \int d\theta f(\theta) \cos u\theta. \quad (25)$$

In summary, Eq. (23) can be written as

$$\tilde{\psi}_f(\mathbf{K}) = \exp \left[\frac{N}{d} \hat{f}(K/\sigma'_0) \right]. \quad (26)$$

Thanks to this averaging, $\tilde{\psi}(\mathbf{K})$ can be expressed as a function of $K = |\mathbf{K}|$. This implies that $\psi(\mathbf{X})$ has the rotational invariance in the phase space so that it is a function only of the action.

Now, our problem is reduced to performing the inverse Fourier transformation. We have

$$\psi(\mathbf{X}) = \int \frac{d\mathbf{K}}{(2\pi)^2} e^{-i\mathbf{K} \cdot \mathbf{X}} \tilde{\psi}_0(\mathbf{K}) \tilde{\psi}_f(\mathbf{K}). \quad (27)$$

From the rotational symmetry of $\psi(\mathbf{X})$, we introduce

$$\psi(I) \equiv \int d\phi d\mathbf{X} \psi(\mathbf{X}) \delta(X - \sqrt{2I} \sin \phi) \delta(P - \sqrt{2I} \cos \phi). \quad (28)$$

Here $I = |\mathbf{X}|^2/2$ is the action normalized by the nominal emittance. The normalization of $\psi(I)$ is

$$\int_0^\infty dI \psi(I) = 1.$$

We now arrive at

$$\psi(I) = \int_0^\infty dK K J_0(K\sqrt{2I}) \exp \left[-\frac{1}{2} K^2 + \frac{N}{d} \hat{f}(K/\sigma'_0) \right]. \quad (29)$$

The projection of ψ to the X axis,

$$\rho(X) = \int dP \psi(\mathbf{X}),$$

is another useful quantity and is given by

$$\rho(X) = \frac{1}{\pi} \int dK \cos(KX) \exp \left[-\frac{1}{2} K^2 + \frac{N}{d} \hat{f}(K/\sigma'_0) \right]. \quad (30)$$

3.2 Elastic Scattering

The formalism thus obtained is quite general. It seems useful, however, to substitute an explicit and realistic form to f .

The cross-section of the elastic scattering with an atom is⁵

$$\frac{d\sigma}{d\Omega} = \left(\frac{2Zr_e}{\gamma} \right)^2 \frac{1}{(\theta^2 + \theta_{min}^2)^2},$$

where Ω is the solid angle, θ the scattering angle, Z the atomic number, r_e the classical electron radius, γ the relativistic Lorentz factor and θ_{min} is determined by the uncertainty principle as

$$\theta_{min} = Z^{1/3} \alpha / \gamma.$$

Since we are interested in the vertical distribution of the flat beam, we integrate it over the horizontal scattering angle θ_x , where

$$\theta^2 = \theta_x^2 + \theta_y^2.$$

Here we ignore the maximum scattering angle due to the finite size of the nucleus and also ignore the aperture limitation of the betatron phase space. We thus get

$$d\sigma/d\theta = 4\pi r_e^2 Z^2 \gamma^{-2} (\theta^2 + \theta_{min}^2)^{-3/2}. \quad (31)$$

Here and hereafter we denote

$$\theta \equiv \theta_y.$$

Thus,

$$f(\theta) = \theta_{min}^2 (\theta^2 + \theta_{min}^2)^{-3/2}, \quad (\theta \geq 0), \quad (32)$$

and

$$\sigma_{tot} = 4\pi r_e^2 Z^2 \gamma^{-2} \theta_{min}^{-2}. \quad (33)$$

For the elastic scattering, we assume that there is only one type of molecule so that

$$N = Q \sigma_{tot} c. \quad (34)$$

Here c is the light velocity and Q is the number of gas molecules in a unit volume and is given by

$$Q = 2.65 \times 10^{20} n P_a,$$

where n is the number of atoms in each gas molecule and P_a is the partial pressure of the gas in pascals.

Using Eq. (32), we have

$$\tilde{f}(u) = u K_1(u) \equiv C(u). \quad (35)$$

Here K_1 is the modified Bessel function of the first order. We can express $\hat{f}(K/\sigma'_0)$ in Eq. (29) as

$$\hat{f}(K/\sigma'_0) = \hat{C} \left(\frac{\theta_{min}}{\sigma'_0} K \right), \quad (36)$$

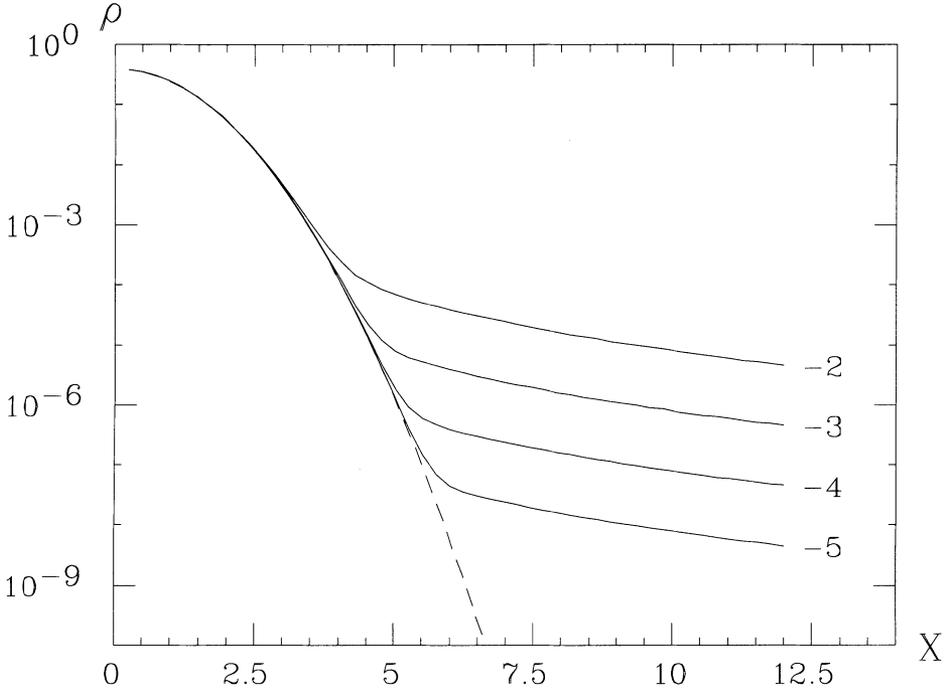


FIGURE 1: The distribution function $\rho(X)$: Dashed line is that for $N_d = 0$, and solid lines are those for $N_d = 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}$. The exponents are indicated in the figure. The value of Θ is 2.5.

$$\hat{C}(K) = \frac{2}{\pi} \int_0^1 dx \frac{C(Kx) - 1}{x} \cos^{-1} x. \quad (37)$$

It is interesting that $\psi(X)$ is characterized only by the number of collisions during one damping time

$$N_d \equiv N/d,$$

and the ratio

$$\Theta \equiv \theta_{min}/\sigma'_0.$$

3.3 Example

Let us see some numerical examples. As model parameters of the ring, according to a design of a damping ring for a linear collider,⁶ we use

$$\epsilon = 10^{-11}(\text{rad} \cdot \text{m}), \quad \beta = 3 \text{ m}, \quad d^{-1} = 5 \times 10^{-3}(\text{s}),$$

and

$$Z = \sqrt{50}, \quad \text{and} \quad n = 2,$$

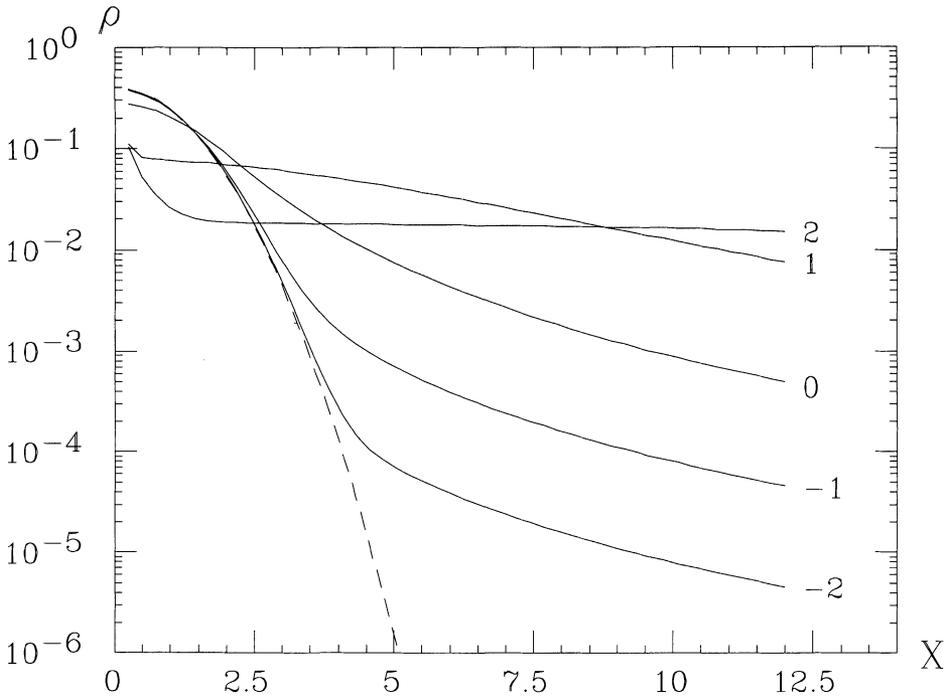


FIGURE 2: The distribution function $\rho(X)$ for $N_d = 10^{-2}, 10^{-1}, 10^0, 10^1, 10^2$. The exponents are indicated in the figure. The value of Θ is 2.5.

which represents the CO molecule. Usually, the CO molecule dominates the elastic scattering by the residual gas.⁷ With these numbers, we have

$$\Theta = 2.55,$$

and $P_a = 10^{-6}$ Pa (the typical value of the partial pressure of CO gas) corresponds to

$$N_d = 0.040.$$

The distribution function $\rho(X)$, Eq. (30), is shown in Fig. 1. It can be seen that the tail part increases as N_d becomes large, while the central part is little affected within the present range of N_d .

It is also interesting to see the case with very high pressure. The Fig. 2 is similar to Fig. 1 but corresponds to higher pressure. When N_d becomes large, the tail increases gradually and the central part becomes affected eventually.

3.4 Tail Distribution

In a storage ring, the tail distribution of the electron bunch is important from lifetime and background points of view. We will evaluate it under a reasonable assumption that

$$N_d \ll 1.$$

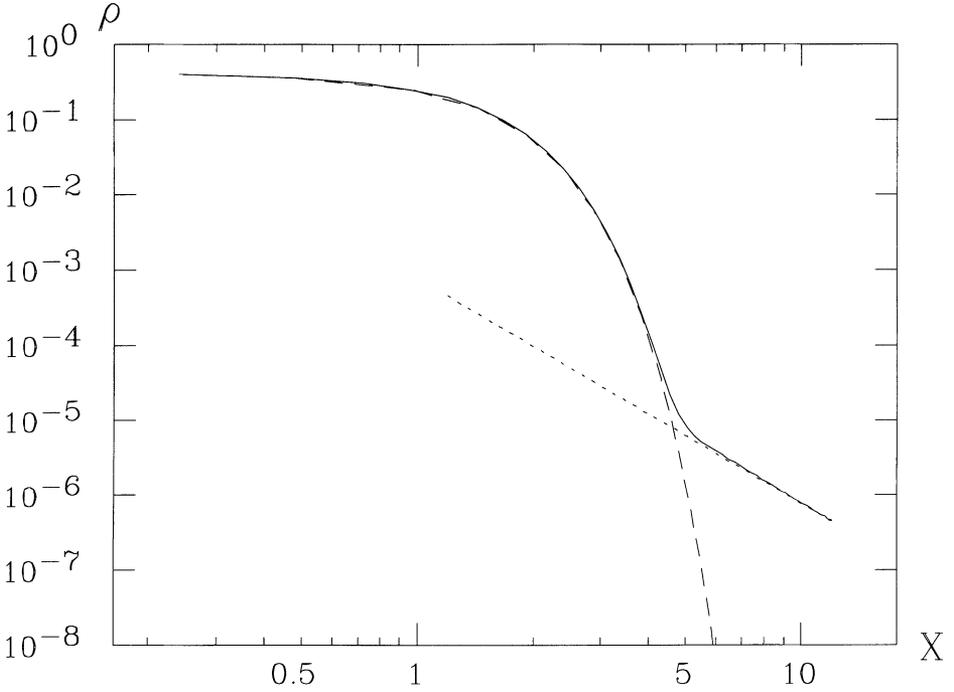


FIGURE 3: Log-log plot of $\rho(X)$ for $N_d = 10^{-3}$ and $\Theta = 2.5$. The dashed line is that for $N_d = 0$ and the dotted line is the asymptotic form given by Eq. (42).

In this limit, we can approximate $\psi(\mathbf{X})$ as

$$\psi(\mathbf{X}) = \psi_0(\mathbf{X}) + \psi_1(\mathbf{X}),$$

$$\psi_1(\mathbf{X}) = N_d \int \frac{d\mathbf{K}}{(2\pi)^2} e^{-i\mathbf{K}\cdot\mathbf{X}} e^{-K^2/2} \hat{C}(\Theta K).$$

The second term ψ_1 dominates the asymptotic distribution. When we perform the integration over K before those of t and ϕ , it can be readily evaluated as

$$\psi_1(\mathbf{X}) = N_d \int_0^\infty dt \int_0^\infty d\theta f(\theta) \left\{ \delta\left(\mathbf{X} - \frac{\theta}{\sigma_0'} e^{-t} \mathbf{g}(\phi)\right) - \delta(\mathbf{X}) \right\},$$

where

$$\mathbf{g}(\phi) = (\sin \phi, \cos \phi)^t.$$

Here, since we are interested in the tail distribution, the factor $\exp(-K^2/2)$ in ψ_1 was set at unity. (Otherwise, we would have obtained two sharply narrow Gaussian distributions.)

The integration over ϕ gives

$$\psi(I) = N_d \int_0^\infty dt \int_0^\infty d\theta f(\theta) \left[\delta\left(I - \frac{\theta^2}{2\sigma_0'^2} e^{-2t}\right) - \delta(I) \right]. \quad (38)$$

The second delta function in the above expression can be omitted, because it does not contribute to the tail. By changing variables and performing the integrations, we arrive at

$$\psi_1(I) = N_d \frac{1}{2I} F(\sigma'_0 \sqrt{2I}), \quad (39)$$

where

$$F(u) \equiv \int_u^\infty dx f(x) = 1 - \frac{u}{\sqrt{u^2 + \theta_{min}^2}}. \quad (40)$$

When $I \rightarrow \infty$, we have thus

$$\psi(I) \simeq \frac{N_d \Theta^2}{8I^2}. \quad (41)$$

Since

$$\rho(X) = \int \frac{dP}{2\pi} \psi(I),$$

we get

$$\rho(X) \simeq \frac{N_d \Theta^2}{8X^3}, \quad (X \rightarrow \infty). \quad (42)$$

From their derivations, Eqs. (41) and (42) hold where

$$\sqrt{2I} \ll \text{Max}(1, \Theta), \quad \text{and} \quad X \ll \text{Max}(1, \Theta).$$

A comparison is made between $\rho(X)$ and its asymptotic form, Eq. (42), in Fig. 3. The agreement is satisfactory. Thus it can be said that the asymptotic form of $\psi(I)$ is dominated by that of the single scattering probability function $f(\theta)$, as long as $N_d \ll 1$. (Actually, the distribution can well be approximated by the sum of the nominal one and the asymptotic form, except for the very central part.) It is worth noting that the asymptotic form of ψ (and ρ) does not depend on θ_{min} , because $N_d \propto \theta_{min}^{-2}$ and $\Theta^2 \propto \theta_{min}^2$.

3.5 Central Distribution

We have calculated the distribution function. In some other general cases, we cannot calculate \tilde{f} explicitly. In such cases, the numerical calculations of ψ and ρ would be more difficult. In such cases, however, it seems convenient to introduce quantities⁸

$$D_1 = \frac{\int \rho(X)^2 dX}{\int \rho_0(X)^2 dX} = \frac{2}{\sqrt{\pi}} \int_0^\infty dK \exp[-K^2 + 2N_d \hat{f}], \quad (43)$$

and

$$D_2 = \frac{\int \psi(\mathbf{X})^2 d\mathbf{X}}{\int \psi_0(\mathbf{X})^2 d\mathbf{X}} = 2 \int_0^\infty dK K \exp[-K^2 + 2N_d \hat{f}], \quad (44)$$

where ρ_0 and ψ_0 are those for $N_d = 0$. These represent how distributions concentrate to the central part. In particular, D_1 is proportional to the luminosity so that it is called the *luminosity reduction factor*. It can be said generally that

$$D_1 \leq 1, \quad D_2 \leq 1,$$

where the equalities hold if and only if $N_d = 0$. That is, the luminosity is always reduced by the incoherent stochastic processes. The reasoning is simple: from Eq. (25), \hat{f} is always less than unity so that \hat{f} is always negative.

The D_1 is shown as a function of N_d in Fig. 4. At $N_d \simeq 1$, D_1 starts decreasing rapidly. This reflects the fact that the central part becomes affected. This is consistent with the fact shown in Fig. 2.

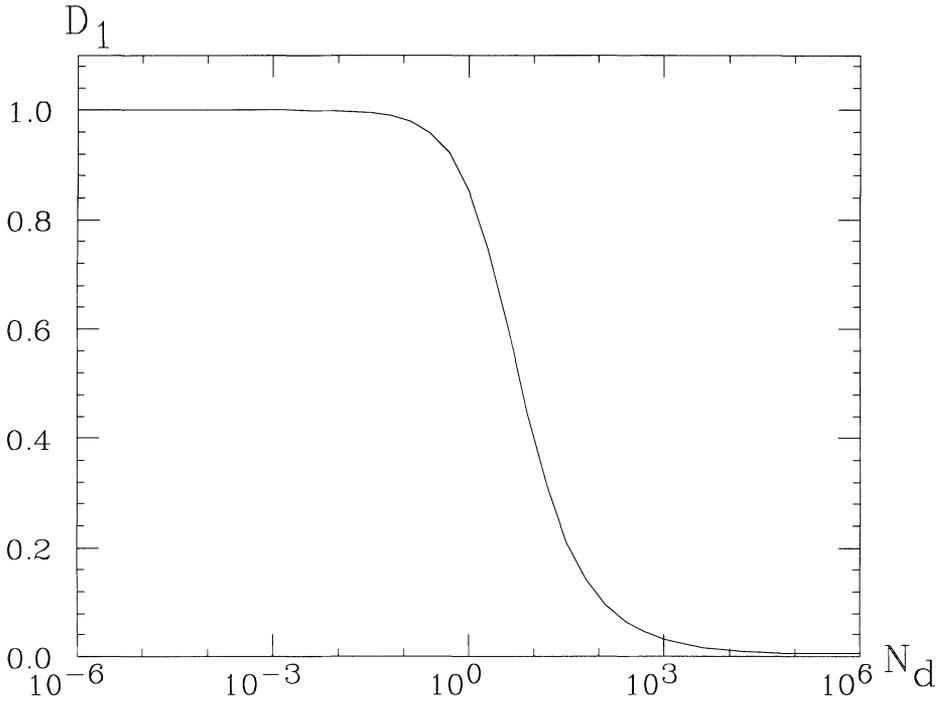


FIGURE 4: The luminosity reduction factor D_1 as a function of N_d .

4 DISCUSSION

We have treated the synchrotron radiation in a very simplified fashion. This effect can of course be treated in a similar way. When, in general, several independent processes are to be included, we can simply extend Eq. (29) as

$$\exp \left[-\frac{1}{2}K^2 + \frac{N}{d}\hat{f} \right] \longrightarrow \exp \left[\frac{1}{d} \sum_i N_i \hat{f}_i \right]. \quad (45)$$

The synchrotron radiation diffusion process can be considered as one of these processes. In the synchrotron radiation case, the \hat{f} can be obtained from

$$\int_0^{\infty} \cos(Ku) f_{SR}(u) du = \frac{\pi}{2K\sqrt{1+K^2}} \left[(\sqrt{1+K^2} + K)^{5/3} - (\sqrt{1+K^2} - K)^{5/3} \right],$$

which is to be compared with Eq. (35).

The main difference from the elastic scattering case is that $\langle u^n \rangle_1$ exists for any $n \geq 0$, while it diverges in the latter case. The saddle point analysis shows that ψ is Gaussian when N is large. This is another way of demonstrating the central limit theorem. Since N is quite large in electron rings, we do not need this way of the treatment when discussing the equilibrium state. On the other hand, in the elastic scattering case, ψ does not approach a Gaussian distribution when $N \rightarrow \infty$. The second cumulant, $\langle u^2 \rangle_1$, should be finite in order that the central limit theorem may apply.

We have established a method to calculate the distribution function in an equilibrium when the system is affected by incoherent stochastic processes. While we have treated the elastic scattering with CO molecules only, the formalism is quite general and can be extended in a straightforward way to other processes.

After the present work was completed, it came to the authors' notice that T. Raubenheimer⁹ did calculations on the same problem.

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