Higgs branch localization of three-dimensional supersymmetric gauge theories

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Abstract

We study $\mathcal{N}=2$ supersymmetric gauge theories on a squashed three-sphere and $S^1 \times S^2$. The supersymmetric localization enables us to compute various BPS quantum quantities exactly. In the procedure, the path integrals of them usually reduce to certain finite-dimensional integrals of matrix models characterized only by the constant value of the vector multiplet scalar field. We call this procedure "the Coulomb branch localization". In particular, recently it has been shown by evaluating the matrix models that the partition functions on a three-ellipsoid and $S^1 \times S^2$ in some class of theories factorize into a product of the three-dimensional vortex and anti-vortex partition functions as well as the other factors. However, the origin of this structure has been mysterious yet. We give it a natural interpretation using "the Higgs branch localization", in which the saddle point is characterized by the value of the chiral multiplet scalar field. We also find that a large class of $\mathcal{N}=2$ theories has the same factorization structure.

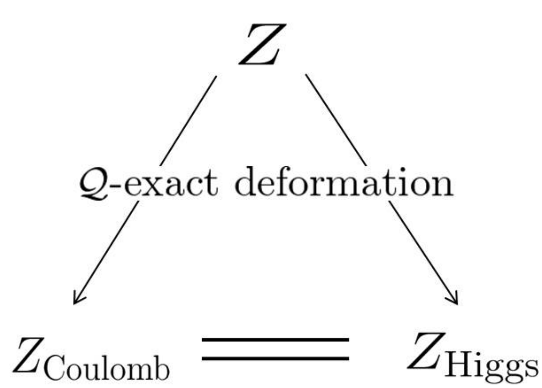

Figure 1: Sketch of Coulomb vs Higgs branch localizations

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Chapter 1

Introduction

Superstring theory is the most powerful candidate for a quantum gravity theory, that can describe physics at the Planck scale $\sim 10^{-35}m$. In addition to the quantization of gravity, this theory can have potential for explaining the Standard Model gauge group $SU(3) \times SU(2) \times U(1)$, the hierarchy of the quark masses, masses of the neutrino, the dark matter, etc., whose origins remain mysterious to this day.

There are five different types of perturbative superstring theory with ten spacetime dimensions: type IIA, type IIB, type I, $SO(32)$ heterotic and $E_8 \times E_8$ heterotic string theories. Although once these theories were thought of as independent ones, the discoveries of various dualities have revealed that the five theories are related with each other. In particular one of the most interesting discoveries is that of M-theory [1]. While it had already been known that the eleven-dimensional supergravity theory, which is the highest-dimensional supergravity theory, was related to the type IIA supergravity theory, it had not been revealed the relation with the superstring theory until then. Witten noted that there was a certain eleven-dimensional quantum theory, so-called M-theory, which gave type IIA superstring theory if one of the dimensions is compactified, and the low energy effective theory corresponded to the eleven-dimensional supergravity theory. Since the circle radius is related with the string coupling, M-theory can also be thought as type IIA superstring theory at strong coupling. Although the detail of this theory is still mysterious, even the analysis of eleven-dimensional supergvity theory has helped much understanding of the superstring theory.

After that, D-branes were found [2]. They are dynamical extended objects on which open strings can end, and are sources of the RR gauge fields. The analysis of D-branes have given us many clues of non-perturbative information in the superstring theory, and for example Strominger and Vafa succeeded to derive a Black Hole entropy using a D-brane system [3], that is to say the entropy which follows the area law could be explained in terms of the microscopic states in the string theory in the same way as the statistical mechanics.

Furthermore Maldacena has conjectured that a $d+1$-dimensional gravity theory can be encoded by just a d-dimensional gauge theory from the D-branes picture, so-called the AdS/CFT correspondence [4]. The most well-known example is a relation between a type IIB superstring theory on $\mathrm{AdS}_5 \times S^5$ and a four-dimensional $\mathcal{N}=4$ super-Yang Mills theory, which is also a conformal field theory. When we consider N-coincident D3-branes, the geometry is the $\mathrm{AdS}_5 \times S^5$ in the near horizon limit, and on the other hand a gauge theory on the branes is the four-dimensional $\mathcal{N}=4$ SYM theory. Comparing both pictures we can find that there are some coupling regions such that two theories are expected to be equivalent. In addition to this example, various other relations are expected. Although this idea has not been proven exactly, much evidence has been found up to date.

This duality strongly motivates us to study various supersymmetric gauge theories in order to understand the superstring theory. It is, however, difficult to even test it in general because this correspondence is a strong/weak duality. The difficulty is due to a technical reason that it is impossible to exactly perform path integral calculations. So we usually use an approximation method, perturbative expansions, which become ill-defined at strong coupling. Fortunately if a theory has special symmetries including some fermionic symmetries such as the BRST symmetry and the supersymmetry, the path integral over infinite field configurations can reduce to an integral or a summation over a limited configuration. Furthermore using such symmetries, the path integral can result in just a problem of calculating one-loop, where the one-loop calculation becomes exact. Such a calculation technique is called “the Localization”.

First of all, it was noted that partition function and some observables can be calculated exactly in the cohomological field theory in which the action is BRST-exact. Also even if a theory is not cohomological, by performing the topological twist it is possible to make the theory a cohomological one. For example $\mathcal{N}=2$ SYM theory on $\mathbb{R}^4$ can be regarded as a cohomological one by identifying a part of the Lorentz group as the $SU(2)$ R-symmetry [5]. Thanks to this idea, the Seiberg-Witten theory, which explains the structure of the moduli space [6, 7], has turned out to allow a certain geometrical interpretation, so-called the Donaldson invariant [8]. However, although the full moduli space was revealed by the penetrating insight of Seiberg and Witten, direct calculations for the instanton contribution were difficult at that time. Nekrasov noted that introducing the omega-background, which gives a special IR cutoff and preserves the supersymmetry, the equivariant localization theorem can be applied to the calculation of the instanton moduli space [9]. Furthermore it was found that the instanton partition function he derived was related to the prepotential [10].

After that, without introducing the omega-background, Pestun noted that the localization technique could be applied to a supersymmetric theory on a compact space [11]. We inevitably encounter divergences in considering the quantum theory, which are both ultraviolet and infrared ones. We don't have to care about the former as long as we consider renormalizable theories. While, as long as we consider a theory with only massive particles, we don't have IR divergences because of the decay of propagation. However when we consider a massless theory such as gauge theories we have to care about it. In particular when we consider the supersymmetric theories, one needs a supersymmetric regularization. Pestun has solved this problem by placing a theory to a compact space, and applied the localization technique instead of introducing the omega-background. Since then this technique has been applied to various dimensional supersymmetric theories on various manifolds. Also the field of the rigid supersymmetry on curved manifold has developed rapidly [12, 13, 14, 15, 16, 17].

One of the most important developments is to give an exact proof to a conjecture [11] that the expectation value of the half BPS Wilson loop in four-dimensional $\mathcal{N} = 4$ SYM theory can be reduced to simply a finite-dimensional integral of the Gaussian matrix model, which gives important evidence of the AdS/CFT correspondence [18, 19]. Another is to succeed in calculating the free energy of the N coincident M2-branes from the field theory side via AdS/CFT correspondence [20, 21]. The corresponding field theory can be regarded as a $U(N)_k \times U(N)_{-k}$ Chern-Simons matter theory, so-called ABJM theory [22]. More precisely, M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$ (the M2 branes in the near horizon limit) is thought to be dual to the ABJM theory. This exact calculation has revealed the free energy proportional to $N^{3/2}$ in a large N limit, which had been mysterious. The exact results were also applied to test various nontrivial dualities (Seiberg duality, Mirror symmetry, etc.) in various theories [23, 24] (there are a lot of other references), and give conjectures for AGT(-like) relations (which are nontrivial ones through the M5-brane pictures) [25, 26, 27] and an F-theorem (which is a three-dimensional counterpart of the c-theorem in two dimensions) [28, 29, 30], etc. as new discoveries. These many exact results have given a lot of developments not only for string theory and supersymmetric field theory but just for quantum field theory.

In particular the partition functions and expectation values of the BPS observables have reduced to certain finite-dimensional integrals of matrix models using the localization in most cases. Then the matrix models can be characterized by the constant Cartan value of scalar field in each vector multiplet. We call this procedure "the Coulomb branch localization" in this paper. Recently on a three-ellipsoid and $S^1 \times S^2$ it has been shown by evaluating the matrix models that the partition functions in a class of $\mathcal{N} = 2$ theories factorize into a product of the vortex and anti-vortex partition functions as well as the other factor [31, 32, 33]. These vortex and anti-vortex partition functions are not usual ones but one-dimensional lift up of usual ones, which are expressed on $S^1 \times \mathbb{R}^2_\varepsilon$ with omega background ε, so-called K-theoretic (anti-)vortex partition function. While these partition functions have such factorization structures, it has been mysterious why the vortex structure appears. Furthermore, since it is difficult to show whether an $\mathcal{N} = 2$ theory with any matter representations has the vortex structure by evaluating the corresponding matrix model that we obtain by using the Coulomb branch localization, we have a question that what kind of $\mathcal{N} = 2$ theories have the factorization properties.

Incidentally, the exact results have been also obtained in $\mathcal{N} = (2, 2)$ theories on S^2 [34, 35]. In this situation they have taken a different approach in addition to the Coulomb branch localization. They have shown that the partition functions factorize directly in terms of vortex and anti-vortex partition functions on $\mathbb{R}^2_\varepsilon$ by adding a different supersymmetric exact term, which causes a change to a different BPS configuration. Then their vortices appear on the north and south poles, and the partition functions are characterized by the discrete constant value of scalar field in the chiral multiplet. We call it "the Higgs branch localization" in contrast to the Coulomb branch one.

We extend this idea to the above three-dimensional cases, and answer the above questions [36]: Why do the vortices appear in the partition functions on the three-ellipsoid and $S^1 \times S^2$? What kind of $\mathcal{N} = 2$ theories have the factorization properties ? We show that these factorization structures can be derived by using the three-dimensional Higgs branch localization, and give a natural interpretation to the factorization structure in terms of contributions coming from the north and south poles on the (base) S^2. We also find that a large class of $\mathcal{N} = 2$ theories has the same factorization structure. More precisely speaking, we show that $U(N)$ theories with any matter representations have such structures only if the parity anomaly cancellation conditions are satisfied.

The organization of this paper is as follows. In chapter 2 we give some introduction and background of the supersymmetric gauge theories that we consider in this paper. We also discuss vortex solutions and the moduli space. In chapter 3 we give an explanation of the localization technique, and review the recent development of the rigid supersymmetry on a curved space. In chapter 4 we compute the partition functions on the three-ellipsoid and $S^1 \times S^2$ using the Coulomb branch localization. In chapter 5 we show by evaluating the some partition functions obtained in the last chapter that they factorize into a product of three-dimensional vortex and anti-vortex partition functions as well as other factors. In chapter 6 we introduce an idea of the Higgs branch localization, and give a natural interpretation of the factorization. In chapter 7 we summarize the discussion in this paper.

Chapter 2

Supersymmetric gauge theories

2.1 3d $\mathcal{N} = 2$ supersymmetric gauge theory

Before considering 3d supersummetric theory first we consider 4d $\mathcal{N} = 1$ supersymmetric theory along with [37]. The superalgebra is given by

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2\sigma^N_{\alpha\dot{\beta}} P_N, \tag{2.1.1}$$

where $\alpha, \dot{\beta}$ and N denote $SL(2, \mathbb{C})$ and 4d Lorentz group indicies respectively. We can obtain the 3d $\mathcal{N} = 2$ algebra by dimensionally reducing this along the second direction in the following (cf. appendix in [38, 39]),

$$\{Q_\alpha, \bar{Q}_\beta\} = 2\gamma^\mu_{\alpha\beta} P_\mu + 2i\epsilon_{\alpha\beta} Z, \tag{2.1.2}$$

where α, β and μ denote $SL(2, \mathbb{R})$ and 3d Lorentz group indicies. Z is a central charge, which is $Z = P_2$. We have also an R-symmetry generator R which rotates the supercharges, and is associated with the following algebras,

$$[R, Q_\alpha] = -Q_\alpha, \quad [R, \bar{Q}_\alpha] = \bar{Q}_\alpha. \tag{2.1.3}$$

Note that Q and $\bar{Q}$ are a lowering and a raising operators for the R-charge respectively. In the same way as 4d $\mathcal{N} = 1$ case there are chiral and vector multiplets as irreducible representations,

$$\text{chiral multiplet:} \quad (\phi, \psi, F), \tag{2.1.4}$$

$$\text{vector multiplet:} \quad (A_\mu, \sigma, \lambda, D). \tag{2.1.5}$$

Note that σ is real scalar field which arises from the reduced component of the 4d gauge field. We can also introduce the superspace $(x^\mu, \theta^\alpha, \bar{\theta}^\alpha)$ and define supercovariant derivatives,

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\gamma^\mu_{\alpha\beta}\bar{\theta}^\beta\partial_\mu, \quad \bar{D}_\alpha = -\frac{\partial}{\partial\bar{\theta}^\alpha} - i\theta^\beta\gamma^\mu_{\beta\alpha}\partial_\mu, \tag{2.1.6}$$

from which we can construct a chiral superfield Φ and a vector superfield V such that respectively,

$$\bar{D}^\alpha\Phi = 0, \quad V = V^\dagger. \tag{2.1.7}$$

Also we can construct a real linear multiplet Σ in the analogy of the 4d field strength superfield $W_\alpha = -\frac{1}{2}\bar{D}\bar{D}D_\alpha V$, which includes a real scalar field as the lowest component, and also includes a gauge field strength (cf. [40]),

$$\Sigma \ := \ -\frac{i}{2}\epsilon^{\alpha\beta}\bar{D}_\alpha(e^{-V}D_\beta\, e^V), \tag{2.1.8}$$

which satisfies $D^2\Sigma = \bar{D}^2\Sigma = 0$. Using these contents we can construct a supersymmetric Lagrangians in the following way (alternatively we can also obtain the above ones by dimensionally reducing the 4d $\mathcal{N} = 1$ Lagrangian),

$$\begin{aligned}\mathcal{L}_{\text{vec}} &= \int d^2\theta d^2\bar{\theta}\, \text{Tr}\Big(-\frac{1}{g^2_{\text{YM}}}\Sigma^2\Big) \\ &= \frac{1}{g^2_{\text{YM}}}\text{Tr}\Big[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}D_\mu\sigma D^\mu\sigma - \frac{1}{2}D^2 - \frac{i}{2}\bar{\lambda}\gamma^\mu D_\mu\lambda - \frac{i}{2}\bar{\lambda}[\sigma,\lambda]\ \Big],\end{aligned} \tag{2.1.9}$$

$$\begin{aligned}\mathcal{L}_{\text{chi}} &= \int d^2\theta d^2\bar{\theta}\,\Big(\bar{\Phi}\, e^V\Phi\Big) + \int d^2\theta\, W(\Phi) + \int d^2\bar{\theta}\,\bar{W}(\bar{\Phi}) \\ &= -D_\mu\bar{\phi}D^\mu\phi - \bar{\phi}\sigma^2\phi - i\bar{\phi}D\phi - \bar{F}F + i\bar{\psi}\gamma^\mu D_\mu\psi - i\bar{\psi}\sigma\psi + i\bar{\psi}\lambda\phi - i\bar{\phi}\bar{\lambda}\psi \\ &\quad - \left(F\frac{\partial W}{\partial\phi} - \frac{1}{2}\psi\psi\frac{\partial^2 W}{\partial\phi^2} + \bar{F}\frac{\partial\bar{W}}{\partial\bar{\phi}} - \frac{1}{2}\bar{\psi}\bar{\psi}\frac{\partial^2\bar{W}}{\partial\bar{\phi}^2}\right),\end{aligned} \tag{2.1.10}$$

where the bars denote complex conjugate, and $W(\Phi)$ ($\bar{W}(\bar{\Phi})$) is the superpotential. These Lagrangians are invariant under the following supersymmetry transformations:
For the vector multiplet,

$$\begin{aligned}\delta A_\mu &= \frac{i}{2}(\bar{\epsilon}\gamma_\mu\lambda - \bar{\lambda}\gamma_\mu\epsilon), \\ \delta\sigma &= \frac{1}{2}(\bar{\epsilon}\lambda - \bar{\lambda}\epsilon), \\ \delta\lambda &= -\frac{1}{2}\gamma^{\mu\nu}\epsilon F_{\mu\nu} + i\gamma^\mu\epsilon D_\mu\sigma - D\epsilon, \\ \delta D &= -\frac{i}{2}\bar{\epsilon}\gamma^\mu D_\mu\lambda - \frac{i}{2}D_\mu\bar{\lambda}\gamma^\mu\epsilon + \frac{i}{2}[\bar{\epsilon}\lambda,\sigma] + \frac{i}{2}[\bar{\lambda}\epsilon,\sigma],\end{aligned} \tag{2.1.11}$$

and for the chiral multiplet,

$$\begin{aligned}\delta\phi &= \bar{\epsilon}\psi,\\ \delta\psi &= i\gamma^\mu \epsilon D_\mu \phi + i\epsilon\sigma\phi + \bar{\epsilon}F,\\ \delta F &= \epsilon(i\gamma^\mu D_\mu \psi - i\sigma\psi - i\lambda\phi),\end{aligned} \tag{2.1.12}$$

where $D_\mu = \partial_\mu + iA_\mu$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$ and $\gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$. Note that our notation is a little different from [37] in the following way, $\lambda, \bar{\lambda} \to \frac{1}{\sqrt{2}}\lambda$, $\frac{1}{\sqrt{2}}\bar{\lambda}$, $\epsilon, \bar{\epsilon} \to \frac{1}{\sqrt{2}}\bar{\epsilon}$, $\frac{1}{\sqrt{2}}\epsilon$, $D \to -iD$, and $(\phi, \psi, F) \leftrightarrow (\bar{\phi}, \bar{\psi}, \bar{F})$ where we redefine the usual 4d chiral field as a 3d anti-chiral one and vice versa, unlike [37].

Euclidean supersymmetry

In fact when we compute the path integral in the localization technique, we have to consider a Euclidean supersymmetric theory. Because we want to consider a supersymmetric theory on a compact manifold later. Here we note just a difference from the above Minkowski notation (see the appendix in [13]). We can obtain the Euclidean supersymmetric theory reducing the time direction instead of the 2nd space direction as (2.1.2). Then three-dimansional gammma matrices are just Pauli matrices $\{\gamma_\mu\}_{\mu=1,2,3}$. We also have to note that the Lorentz group changes, $SO(1,3) \cong SL(2,\mathbb{C}) \to SO(4) \cong SU(2) \times SU(2)$ in four dimensions. So if we take a pair of a spinor and the complex conjugate spinor (undotted and dotted spinors) in Minkowski space, they are independent each other in the Wich rotated Euclidean space. If we perform the Wick rotation in the above discussion of the superfield, the corresponding complex conjugate fields are independent ones since the above θ and $\bar{\theta}$ become independent. For the vector multiplet, the condition $V = V^\dagger$ does not impose the constraint that the bosonic fields must be real. Although this causes a difficulty of choosing the contours in path integral, we take a reality condition in most cases in order for the path integral to be convergent:

$$\begin{aligned} A_\mu = A_\mu^\dagger, \quad \bar{\lambda} = \lambda^\dagger, \quad D = D^\dagger,\\ \bar{\phi} = \phi^\dagger, \quad \bar{\psi} = \psi^\dagger, \quad \bar{F} = F^\dagger.\end{aligned} \tag{2.1.13}$$

We summarize the supersymmetric Lagrangians in Euclidean space:

$$\mathcal{L}_{\text{vec}} = \frac{1}{g_{\text{YM}}^2}\text{Tr}\Big[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\sigma D^\mu\sigma + \frac{1}{2}D^2 + \frac{i}{2}\bar{\lambda}\gamma^\mu D_\mu\lambda + \frac{i}{2}\bar{\lambda}[\sigma,\lambda]\,\Big], \tag{2.1.14}$$

$$\mathcal{L}_{\text{chi}} \;=\; D_\mu\bar{\phi}D^\mu\phi + \bar{\phi}\sigma^2\phi + i\bar{\phi}D\phi + \bar{F}F - i\bar{\psi}\gamma^\mu D_\mu\psi + i\bar{\psi}\sigma\psi + i\bar{\psi}\lambda\phi - i\bar{\phi}\bar{\lambda}\psi$$

$$+\left(F\frac{\partial W}{\partial\phi}-\frac{1}{2}\psi\psi\frac{\partial^2 W}{\partial\phi^2}+\bar{F}\frac{\partial\bar{W}}{\partial\bar{\phi}}-\frac{1}{2}\bar{\psi}\bar{\psi}\frac{\partial^2 \bar{W}}{\partial\bar{\phi}^2}\right), \tag{2.1.15}$$

which are invariant under the supersymmetry transformations (2.1.11) and (2.1.12), respectively. We also emphasize that the fields and bar-fields are each independent.

Furthermore we can construct the supersymmetric Chern-Simons (CS), and Fayet-Iliopoulos (FI) terms (if the gauge group includes a $U(1)$ factor like $U(N)$) along [41, 42, 39],

$$\mathcal{L}_{\rm CS}=\int d^2\theta d^2\bar{\theta}\,\mathrm{Tr}\Big(-\frac{\kappa}{4\pi}V\Sigma\Big)=\frac{i\kappa}{4\pi}\,\mathrm{Tr}\Big[\varepsilon^{\mu\nu\rho}\Big(A_\mu\partial_\nu A_\rho+\frac{2i}{3}A_\mu A_\nu A_\rho\Big)-\bar{\lambda}\lambda+2D\sigma\Big], \tag{2.1.16}$$

$$\mathcal{L}_{\rm FI}=\int d^2\theta d^2\bar{\theta}\,\Big(-\frac{\zeta}{\pi}V\Big)=-\frac{i\zeta}{2\pi}D. \tag{2.1.17}$$

In particular, since the gauge coupling has a mass dimension in three dimensions, there is non-trivial IR dynamics in even Abelian theory. We can also give a real mass to the chiral multiplet since we can include a non-dynamical background gauge field for the flavor symmetry gauge group. In the same way, the FI term are obtained by giving a non-dynamical background gauge filed to the CS term. The effect of the real mass, the FI parameter and the CS level causes a lot of phases of the supersymmetric vacua unlike 4d $\mathcal{N}=1$ which is constrained strongly by the holomorphy [40, 43, 39].

2.2 Level shift and parity anomaly

Although we don't have to care about the gauge anomaly in the three-dimensional theory, there is a parity anomaly. The Chern-Simons level has to be an integer in order to preserve the gauge symmetry. However this level is affected from the quantum correction, so-called level shift. Even if there is no CS term classically, it is possible to be generated by the quantum correction. In fact this correction arises from integrating out charged fermions (considering Feynman diagram with two and three photons as external lines) and it has known [44, 45] (See also [43]) that this correction is given by

$$\kappa_{\rm eff}=\kappa_0+\delta\kappa=\kappa_0+\frac{1}{2}\mathrm{sgn}(m)C_2(R), \tag{2.2.1}$$

where κ_0 is the bare CS level, m is fermion mass and $C_2(R)$ is the second Casimir for representation R. For example the second Casimir is 1 for the fundamental representation.

Also this is one-loop exact [44, 45]. This effective CS-level, which consists of the bare and one-loop contribution, must be an integer otherwise the gauge symmetry would break.

For example let us consider one-flavor $(\Phi,\ \tilde{\Phi})$ in the representation $(R, \bar{R})$ of a gauge group [46]:

$$\int d^2\theta\, d^2\bar{\theta} \left[\bar{\Phi} e^{V+m_1\theta\bar{\theta}}\Phi + \tilde{\Phi} e^{-V-m_2\theta\bar{\theta}}\bar{\tilde{\Phi}}\right]. \tag{2.2.2}$$

Then integrating out their fermions, the contribution to the CS-level is from (2.2.1)

$$\kappa = \frac{1}{2}\Big[\mathrm{sgn}(m_1) - \mathrm{sgn}(m_2)\Big] C_2(R). \tag{2.2.3}$$

We can find that if $\mathrm{sgn}(m_1 m_2) > 0$, the induced CS term would cancel. Then it is convenient to define a vector mass $m^{(v)}$ and an axial mass $m^{(a)}$,

$$m^{(v)} = \frac{1}{2}(m_1 + m_2), \quad m^{(a)} = \frac{1}{2}(m_1 - m_2). \tag{2.2.4}$$

In this definition, there is the effective CS term only if $m^{(a)}$ is nonzero.

2.3 Supersymmetric vacua

First the Hamiltonian in 3d $\mathcal{N} = 2$ theory is given by from (2.1.2),

$$H = P^0 = \frac{1}{4}\sum_{\alpha=1}^{2}\{Q_\alpha, \bar{Q_\alpha}\} + iZ. \tag{2.3.1}$$

Taking the vacuum expectation value,

$$\langle 0|H|0\rangle = \frac{1}{2}\sum_{\alpha=1}^{2} ||Q_\alpha|0\rangle||^2, \tag{2.3.2}$$

where we used $\langle 0|Z|0\rangle = 0$. Namely the condition for supersymmetric vacua can be understood as that of vanishing ground state energy. We have only to examine whether the potential term vanishes except for special cases.

The moduli space of the 3d $\mathcal{N} = 2$ supersymmetric vacua is generally classified as the Coulomb branch, Higgs branch and mixed branch (and the topological vacua). The Coulomb branch is characterized only by the VEVs of scalar fields in the vector multiplets, in which the gauge symmetry is broken spontaneously to its maximal torus. On the other hand, the Higgs branch is characterized only by the VEVs of scalar fields in the chiral multiplets, in which the gauge symmetry is broken partially or completely, and the mixed branch is expressed by both values.

For an example, let us consider $U(1)$ gauge theory with one flavor, which consists of a chiral multiplet Q with charge 1 and an anti-chiral multiplet $\tilde{Q}$ with charge -1 [40, 43]. For simplicity we ignore the superpotential, CS and FI terms. The scalar potential is

$$V_{\rm cl} \;=\; \frac{1}{2e^2} D^2 + \sigma^2(|\phi|^2 + |\tilde{\phi}|^2) + D(|\phi|^2 - |\tilde{\phi}|^2) + |F|^2 + |\tilde{F}|^2. \qquad (2.3.3)$$

Then integrating out the auxiliary fields, we can obtain

$$V_{\rm cl} = -\frac{e^2}{8}(|\phi|^2 - |\tilde{\phi}|^2)^2 + \sigma^2(|\phi|^2 + |\tilde{\phi}|^2). \qquad (2.3.4)$$

This potential have to vanish to preserve the supersymmetry. The conditions are,

$$\langle \phi \rangle = \langle \tilde{\phi} \rangle \neq 0 \qquad \Rightarrow \qquad \langle \sigma \rangle = 0, \qquad (2.3.5)$$

$$\langle \sigma \rangle \neq 0 \qquad \Rightarrow \qquad \langle \phi \rangle = \langle \tilde{\phi} \rangle = 0, \qquad (2.3.6)$$

where $\langle \cdot \rangle$ expresses VEV. This relation shows that the Coulomb branch and the Higgs branch cannot mix classically, that is they can intersect at a point. Although this analysis is just at classical, note that three-dimensional gauge theory is super-renormalizable, the non-renormalization theorem works and there is no monopole-instanton in $U(1)$ gauge theory. Therefore the above potential does not change except a renormalization factor if this is affected from the quantum correction. That is to say the above relations preserve not only at the classical level.

There is also a dual photon a in the three-dimensional gauge theory:

$$F = dA \quad \Rightarrow \quad *F = \frac{1}{2\pi}\, da. \qquad (2.3.7)$$

This have a periodicity $a \sim a + 2\pi m$, $(m \in \mathbb{Z})$ from the Dirac quantization condition, and is associated with a topological $U(1)$ symmetry. That is to say, the Coulomb branch is characterized by the VEV of σ and a. For example when we consider the Coulomb branch, we can regard the classical moduli space as a cylinder because of the periodicity of a.

Returning to the above example, the fact that the Coulomb branch and Higgs branch cannot mix and intersect at a point, where $\langle \sigma \rangle = \langle a \rangle = \langle \phi \rangle = \langle \tilde{\phi} \rangle = 0$, leads the form of quantum moduli space, as depicted in Fig.2.1. Since this theory becomes classical when the absolute value of σ is large, the origin where three branches meet corresponds to an IR fixed point. Also we can consider another theory which flows to the same IR fixed point, so-called XYZ model [40, 43]. The model consists of three chiral multiplets including the each scalar field X, Y and Z, which interact with a superpotential $W = XYZ$. This relation between the SQED with 1-flavor and the XYZ model is the simplest example of the three-dimensional $\mathcal{N} = 2$ mirror symmetry.

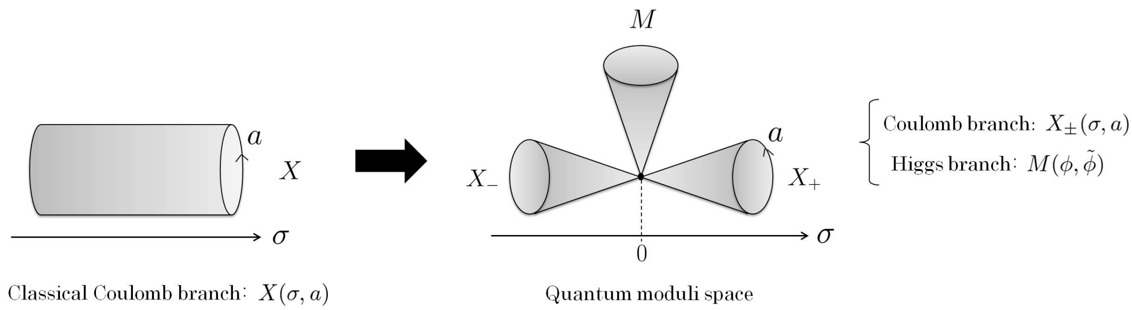


Figure 2.1: Moduli space of SQED with 1-flavor

When we consider non-Abelian theories, the situation changes significantly due to the monopole instantons. However, applying the holomorphy like 4d $\mathcal{N} = 1$ case, we can obtain an effective superpotential which includes non-perturbative contributions. Although we will not review it here, structures of the quantum moduli spaces have been analyzed well [40, 43, 39].

2.4 Dynamics of vortices

In this section, first we consider half BPS vortex solutions in a (1+2)-dimensional $\mathcal{N} = 4$ $U(N)$ Yang-Mills-Higgs theory according to [47] (see also [48]), and next consider half BPS vortices in a $\mathcal{N} = 2$ theory.

Vortex solutions and brane construction in 3d $\mathcal{N} = 4$ theories

BPS vortex solution

Let us consider a (2+1)-dimensional $\mathcal{N} = 4$ $U(N)$ supersymmetric theory with N_f fundamental hypermultiplets ($N \leq N_f$). An $\mathcal{N} = 4$ vector multiplet consists of a pair of a $\mathcal{N} = 2$ vector and an adjoint $\mathcal{N} = 2$ chiral multiplets, and a $\mathcal{N} = 4$ hypermultiplet consists of a pair of a $\mathcal{N} = 2$ chiral and a $\mathcal{N} = 2$ anti-chiral multiplets. These bosonic parts are (A_μ, σ^r) $(r = 1, 2, 3)$ and $(\phi, \tilde{\phi})$. Note that the scalar fields of the vector multiplet are triplet for an $SU(2)$ R-symmetry. For example we can understand this fact by considering a dimensional reduction of a 6d $\mathcal{N} = 1$ vector multiplet to three dimensions.

The bsonic part of the Lagrangian is

$$\begin{aligned}\mathcal{L}_{\text{bos}} &= -\text{Tr}\Big[\,\frac{1}{4g_{\text{YM}}^2}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2g_{\text{YM}}^2}D_\mu\sigma^r D^\mu\sigma^r+\frac{1}{2g_{\text{YM}}^2}[\sigma^r,\sigma^s]^2\,\Big]\\ &\quad -D_\mu\phi^\dagger D^\mu\phi-D_\mu\tilde{\phi}D^\mu\tilde{\phi}^\dagger-g_{\text{YM}}^2|\phi\tilde{\phi}|^2\\ &\quad -(\tilde{\phi}\tilde{\phi}+\phi\phi^\dagger)\sigma^r\sigma^r-\text{Tr}\Big[\,\frac{g_{\text{YM}}^2}{2}(\phi\phi^\dagger-\tilde{\phi}^\dagger\tilde{\phi}-\frac{\zeta}{2\pi}\cdot\mathbb{1}_N)^2\,\Big],\end{aligned}\tag{2.4.1}$$

where we omit the gauge and flavor indices for simplicity, and add the FI term, which has an FI parameter $\zeta>0$. The last term is often called the D-term in this context. Note that if there is no the FI term, the vacuum would be only trivial. The FI term induces the following vacuum condition,

$$\phi\phi^\dagger=\frac{\zeta}{2\pi}\cdot\mathbb{1}_N,\quad \tilde{\phi}=0,\quad \sigma^r=0.\tag{2.4.2}$$

Furthermore, up to Weyl permutations, we can choose this vacuum as

$$\begin{aligned}&\phi_{ai}=\sqrt{\frac{\zeta}{2\pi}}\delta_{ai},\quad \phi_{ai'}=0,\\ &(a=1,\cdots,N,\quad i=1,\cdots,N,\quad i'=N+1,\cdots,N_f).\end{aligned}\tag{2.4.3}$$

This vacuum breaks the symmetries in the following way,

$$U(N)_G\times SU(N_f)_F\quad\rightarrow\quad S\big[U(N)_{\text{diag}}\times U(N_f-N)_F\big].\tag{2.4.4}$$

This is called the color-flavor locked phase. Such spontaneous gauge symmetry breaking induces vortex solutions.

Next let us consider the vortex solutions. We set fields to zero except the gauge field A_μ and fundamental scalar field ϕ, that in fact they are independent of the vortex solutions, and restrict the theory to be time independent. Then the Hamiltonian is

$$\begin{aligned}\int d^2x\,\mathcal{H} &= \int d^2x\Big\{\,\text{Tr}\Big[\,\frac{1}{2g_{\text{YM}}^2}F_{12}^2+\frac{g_{\text{YM}}^2}{2}(\phi\phi^\dagger-\frac{\zeta}{2\pi}\cdot\mathbb{1}_N)^2\,\Big]+|D_1\phi|^2+|D_2\phi|^2\,\Big\}\\ &= \int d^2x\Big\{\,\text{Tr}\Big[\,\frac{1}{2g_{\text{YM}}^2}\big\{F_{12}\mp g_{\text{YM}}^2(\phi\phi^\dagger-\frac{\zeta}{2\pi}\cdot\mathbb{1}_N)\big\}^2\,\Big]\\ &\qquad\qquad +|D_1\phi\pm iD_2\phi|^2\mp\frac{\zeta}{2\pi}\,\text{Tr}\,F_{12}\,\Big\}\\ &\geq \zeta\,|k|,\end{aligned}\tag{2.4.5}$$

where in the last inequality, we used the following relation,

$$\text{Tr}\int d^2x\,F_{12}=2\pi k,\quad k\in\mathbb{Z},\tag{2.4.6}$$

where the integer k is a vortex number, and the above inequality is saturated if and only if

$$F_{12} = \pm g_{\mathrm{YM}}^2 (\phi\phi^\dagger - \zeta \cdot \mathbb{1}_N), \quad D_1\phi \pm iD_2\phi = 0. \tag{2.4.7}$$

These equations are called the vortex equations (Bogomolny equations). It is known that such solution is a half BPS solution. For example, in this case, a mass of the gauge boson equals a mass of the scalar field ϕ due to the coefficient in front of the D-term (c.f. [49]). This fact induces the fact that any BPS vortices have no forces between them.

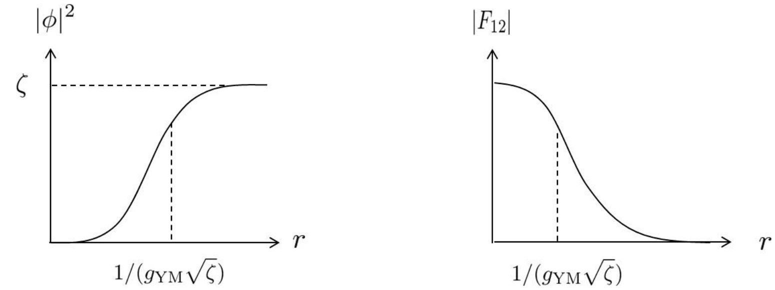


Figure 2.2: Sketch of the vortex profile

For the vortex solution, the energy density is localized at neighborhood of the vortex core, outside of which all fields approach to the vacuum asymptotically. Since the value of the flux is the largest at the core of the vortex, we expect that ϕ vanishes at the core. Then, since the flux is $|F_{12}| \sim g_{\mathrm{YM}}^2\zeta$, we estimate an order of the characteristic size of the vortex as $1/(g_{\mathrm{YM}}\sqrt{\zeta})$.

The vortex solution with vortex number k has $2kN_f$ bosonic collective collective coordinates by the index theorem [47]. Their coordinates are characterized by the position, and orientational moduli (internal degrees of freedom) of the vortex.

Brane construction

Next let us consider the moduli space using a D-branes system.[1] According to [51], we can obtain the $\mathcal{N} = 4$ $U(N)$ Yang-Mills-Higgs theory using N D3-branes stretched between two NS5-branes as the table below and the left picture of the Fig.2.3. In order to give the hypermultiplets, we connect N semi-infinite D3-branes to the right-hand NS5-brane. In the box, $\circ$ denotes a stretched direction.

[1] Although we consider the vortex moduli space using a D-brane system here, it also turns out that they are constructed in a purely field theoretic manner [50].

	0	1	2	3	4	5	6	7	8	9
D3	◦	◦	◦				◦			
D1	◦									◦
NS5	◦	◦	◦	◦	◦	◦				

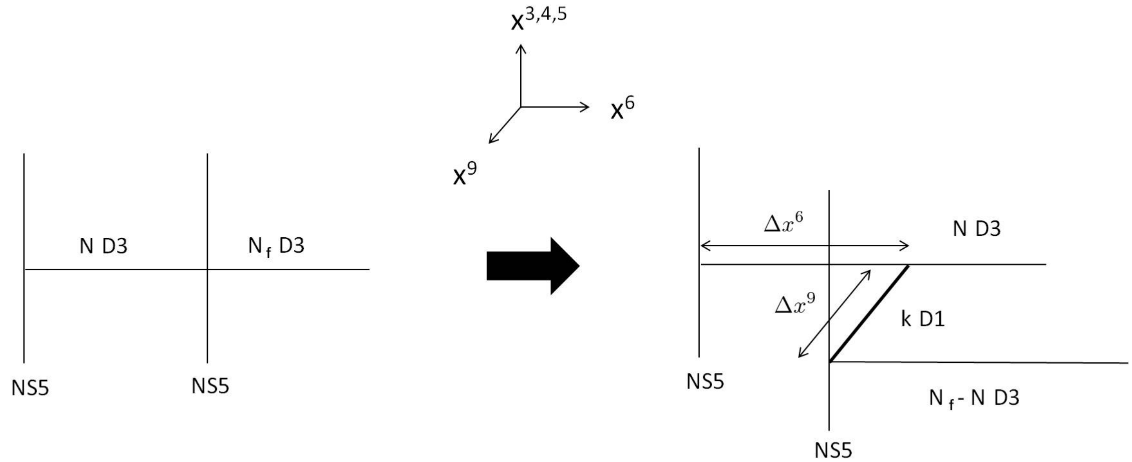


Figure 2.3: Brane construction of the 3d $\mathcal{N}=4$ SQCD and vortex solution

Furthermore we move one NS5-brane to x^9 direction, which induces a nonzero FI parameter. Since the D3-branes cannot tilt into the x^9 direction to preserve the supersymmetry, only N of the N_f D3-branes can end on the N D3-branes according to the S-rule [51]. Finally, the configuration are drawn in the right picture of the Fig.2.3 (without D1-branes). The gauge coupling and FI parameter in the system are given by the following relations,

$$\frac{1}{g_{\text{YM}}^2} \sim \frac{\Delta x^6}{g_s}, \qquad \zeta \sim \frac{\Delta x^9}{g_s l_s^2}, \tag{2.4.8}$$

where g_s is the string coupling and $l_s = \sqrt{\alpha'}$ is string length scale.

Here how is the vortex configuration? In fact the k vortex solution is realized as the k D1-branes stretched along the x^9 direction between the right-hand NS5 and N D3-branes in the right picture of the Fig.2.3. The D1-branes are identified as unique BPS-branes with the correct mass of the vortex (2.4.5) in this situation.

Let us read off the low energy effective theory on the D1-branes. First this configuration breaks 1/2 of the supersymmetry, so it would be a one-dimensional quantum mechanics with $\mathcal{N}=(2,2)$ type supersymmetry. Since one end of the D1-branes ends on the NS5-brane, the fluctuations of the (x^6, x^7, x^8, x^9) directions are fixed. Then when we consider massless modes on the D1-branes, the fluctuations of the (x^0) is a one-dimensional

$U(k)$ gauge field A_t, and the ones of the (x^3, x^4, x^5) are three adjoint scalar fields σ^r $(r = 1, 2, 3)$, which combine into a $U(k)$ vector multiplet. The ones of the (x^1, x^2) are also adjoint scalar fields, which correspond to adjoint chiral multiplet. We denote the complex scalar field as Z. Massless modes from open strings which end on the D1 and N D3-branes become N fundamental chiral multiplets [52, 53]. We denote the complex scalar fields as ϕ_i $(i = 1, \cdots, N)$. Also massless modes from open string which end on the D1 and $(N_f - N)$ D3-branes are $(N_f - N)$ anti-fundamental chiral multiplets. We denote the complex scalar field as $\tilde{\phi}_{i'}$ $(i' = N+1, \cdots, N_f)$. Summarizing them, the theory, which describes the vortex, consists of the following set of supermultiplets in one dimension:

- $U(k)$ vector multiplet: (A_t, σ^r), $\quad r = 1, 2, 3$,
- an adjoint chiral multiplet: Z ,
- N fundamental chiral multiplets: ϕ_i, $\quad i = 1, \cdots, N$,
- $N_f - N$ anti-fundamental chiral multiplets: $\tilde{\phi}_{i'}$, $\quad i' = N+1, \cdots, N_f$,

The bosonic part of the Lagrangian is

$$\begin{aligned}\mathcal{L}_{\text{vortex}}\big|_{\text{bos}} = &-\text{Tr}\Big[\ \frac{1}{2e^2} D_t\sigma^r D_t\sigma^r + D_t Z^\dagger D_t Z + \big|[Z, \sigma^r]\big|^2 + \frac{1}{2e^2}[\sigma^r, \sigma^s]^2\ \Big] \\ &- D_t\phi^\dagger D_t\phi - D_t\tilde{\phi}^\dagger D_t\tilde{\phi} - \phi_i^\dagger\phi_i\sigma^r\sigma^r - \text{Tr}\Big[\ \frac{e^2}{2}([Z, Z^\dagger] + \phi\phi^\dagger - \tilde{\phi}^\dagger\tilde{\phi} - r\cdot\mathbb{1}_k)^2\ \Big],\end{aligned} \tag{2.4.9}$$

where we omitted the flavor indices. Then the gauge coupling and FI parameter of this theory are also determined by

$$\frac{1}{e^2} \sim \frac{l_s^2 \Delta x^9}{g_s}, \quad r \sim \frac{\Delta x^6}{g_s}. \tag{2.4.10}$$

Note that the FI parameter r is related with the gauge coupling g_{YM} in the 3d $\mathcal{N} = 4$ theory: $r \sim 1/g_{\text{YM}}^2$.

The global symmetry of this theory is

$$SU(2)_R \times U(1)_F \times S\big[U(N) \times U(N_f - N)\big], \tag{2.4.11}$$

where $SU(2)_R$ is an R-symmetry which rotates the three scalar fields σ^r of the vector multiplet, $U(1)_F$ is a flavor symmetry which rotates the phase of Z, and $S[U(N) \times U(N_f - N)]$ are flavor symmetries of ϕ_i and $\tilde{\phi}_{i'}$ respectively.

Then the condition for the vacuum is

$$[Z, Z^\dagger] + \phi\phi^\dagger - \tilde{\phi}^\dagger\tilde{\phi} - r\cdot\mathbb{1}_k = 0, \tag{2.4.12}$$

so the moduli space is

$$\mathcal{M}_{k,(N,N_f)} = \Big\{(\phi, \tilde{\phi}, Z)\Big| [Z, Z^\dagger] + \phi\phi^\dagger - \tilde{\phi}^\dagger\tilde{\phi} = r \cdot \mathbb{1}_k\Big\}/U(k). \tag{2.4.13}$$

The degrees of freedom for this moduli space are

$$\dim_{\mathbb{R}}(\mathcal{M}_{k,(N,N_f)}) = 2\big\{kN + k(N_f - N) + k^2\big\} - k^2 - k^2 = 2kN_f, \tag{2.4.14}$$

which equal that of the vortex moduli space which is obtained by the index theorem.[47]

Vortices in 3d $\mathcal{N} = 2$ theories

Let us consider $\mathcal{N} = 2$ $U(N)$ theory with N_f fundamental and $\tilde{N}_f$ anti-fundamental chiral multiplets with the FI term ($\zeta > 0$) [43]. Then the bosonic part of the Lagrangian is

$$\begin{aligned}\mathcal{L}_{\text{bos}} \;\; = \;\; & -\frac{1}{g_{\text{YM}}^2}\text{Tr}\Big(\,\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\sigma D^\mu\sigma\,\Big) - D_\mu\phi^\dagger D^\mu\phi - D_\mu\tilde{\phi}^\beta D^\mu\tilde{\phi}^\dagger \\ & -\phi^\dagger\sigma^2\phi + \tilde{\phi}\sigma^2\tilde{\phi}^\dagger - \text{Tr}\Big[\,\frac{g_{\text{YM}}^2}{2}\Big(\phi^\dagger\phi - \tilde{\phi}\tilde{\phi}^\dagger - \frac{\zeta}{2\pi}\Big)^2\,\Big],\end{aligned} \tag{2.4.15}$$

where we omit the flavor indices. Here we have a special vacuum with $\phi \neq 0$ and $\tilde{\phi} = 0$ [2]. Up to Weyl permutations, we can choose the vacuum as

$$\phi_{ai} = \sqrt{\frac{\zeta}{2\pi}}\delta_{ai}, \qquad \sigma = \tilde{\phi} = \phi_{i'} = 0, \tag{2.4.16}$$

$$(a = 1, \cdots, N, \quad i = 1, \cdots, N, \quad i' = N+1, \cdots, N_f). \tag{2.4.17}$$

In this vacuum, we can show that there also exist half BPS vortex solutions in the same way as we did above.

As we have seen in the brane construction in $\mathcal{N} = 4$ theory, we also expect that the vortex moduli space is characterized by a certain one-dimensional supersymmetric theory. In particular, the authors in [54] have analyzed a half BPS vortex in a supersymmetric theories with four supercharges, and then found that the vortex solutions preserve chiral $\mathcal{N} = (0, 2)$ type supersymmetries, rather than $\mathcal{N} = (1, 1)$.

Then what is a vortex moduli space for the 3d $\mathcal{N} = 2$ $U(N)$ theory with N_f fundamental and $\tilde{N}_f$ anti-fundamental chiral multiplets? Using an analogy of the brane construction in 3d $\mathcal{N} = 4$, we expect that the moduli space is described by the following set of supermultiplets:[3]

[2] If we consider a massive theory, $\tilde{\phi}$ vanishes for $\zeta > 0$ and generic masses in the vacuum.

[3] We summarize our notations in appendix D.

- $U(k)$ vector multiplet: (A_t, φ),
- an adjoint chiral multiplet: B,
- N fundamental chiral multiplets: $I^i, \quad i = 1, \cdots N$,
- $N_f - N$ anti-fundamental chiral multiplets: $J^j, \quad j = N+1, \cdots, N_f$,
- $\tilde{N}_f$ fundamental Fermi multiplets: $F^p, \quad p = 1, \cdots, \tilde{N}_f$,

where we have displayed only the bosonic fields of the multiplets, respectively. Note that the contributions of the anti-fundamental matters in three dimensions are characterized by the Fermi multiplets. In fact, we find that the moduli space of this theory is

$$\mathcal{M}^k_{N,N_f} = \left\{ (B, I, J) \middle| [B, B^\dagger] + I\bar{I} - \bar{J}J = r \cdot \mathbb{1}_k \right\} / U(k), \tag{2.4.18}$$

where r is an FI parameter of this one-dimensional theory. We also find that the degrees of freedom for the moduli space match those of the 3d $\mathcal{N} = 2$ vortex, and the global symmetries are also the same on both side. For example, in two dimensions, it turns out that a gauged matrix model obtained as a dimensional reduction of the above contents can describe a vortex moduli space in 2d $\mathcal{N} = (2,2)$ $U(N)$ theory with N_f fundamental and $\tilde{N}_f$ anti-fundamental chiral multiplets [34].

Chapter 3

Localization and supersymmetry on a curved space

3.1 Localization

Supersymmetric localization principle

First let us see that the path integral is reduced to that only over the BPS sector when we consider any supersymmetric observable, along [55] (See also [56]). We consider some expectation value $\langle\mathcal{O}\rangle$ on the field space $\mathcal{F}$, and suppose that the theory and the operator $\mathcal{O}$ have a certain symmetry G. Furthermore we assume that G acts freely on $\mathcal{F}$, i.e. there is no fixed point on $\mathcal{F}$. Then, we have a fibration $\mathcal{F} \to \mathcal{F}/G$. Integrating over the fiber, we obtain

$$\big\langle\mathcal{O}\big\rangle \;=\; \int_{\mathcal{F}} \mathcal{O}\, e^{-S} = \mathrm{vol}(G)\cdot \int_{\mathcal{F}/G} \mathcal{O}\, e^{-S}. \tag{3.1.1}$$

Next we suppose that G is a fermionic symmetry. Then the corresponding volume for the fermionic variable θ is

$$\int d\theta\, \cdot 1 = 0. \tag{3.1.2}$$

That is to say that the contribution vanishes. However, if we consider the case of supersymmetry $\mathcal{Q}$, it cannot act freely. Note that the fixed point set of $\mathcal{Q}$ is described by

$$\mathcal{F}_{\mathrm{BPS}} = \{[X] \in \mathcal{F} \;\big|\; \forall\mathcal{Q}(\text{bosons}) = 0,\ \forall\mathcal{Q}(\text{fermions}) = 0\}. \tag{3.1.3}$$

Then with this notation, $\mathcal{Q}$ acts freely on $\mathcal{F}\setminus\mathcal{F}_{\mathrm{BPS}}$. So for this quotient space we find that the contribution vanishes. Therefore the path integral reduces to that only over the BPS sector when we consider any supersymmetric observables.

Deformation of the path integral

In the above we have seen that a supersymmetric path integral can reduce to just an integral over the BPS configurations. Furthermore we can constrain the configuration of the supersymmetric path integral. First let's consider a partiton funciton of a supersymmetric theory,

$$Z = \int \mathcal{D}\Phi \, e^{-S[\Phi]}, \tag{3.1.4}$$

where Φ is a set of fields and we assume that $\mathcal{Q}S = 0$ and $\mathcal{Q}(\mathcal{D}\Phi) = 0$ for supercharge $\mathcal{Q}$. Here we perform the following deformation as for some parameter t and a certain function $V[\Phi]$,

$$Z(t) = \int \mathcal{D}\Phi \, e^{-\left(S[\Phi]+t\mathcal{Q}V[\Phi]\right)}, \tag{3.1.5}$$

where we assume that $t \geq 0$ and $\mathcal{Q}V[\Phi] \geq 0$ for positive semi-difiniteness, and moreover $\mathcal{Q}^2 V = 0$. We note that $Z(t)$ reproduces original partition function in the case $t = 0$. Then we readily find that $Z(t)$ is independent of t since

$$\frac{d}{dt} Z(t) \;=\; \int \mathcal{D}\Phi \, \mathcal{Q}V \, e^{-\left(S[\Phi]+t\mathcal{Q}V[\Phi]\right)} = \int \mathcal{D}\Phi \, \mathcal{Q}\Big(V \, e^{-\left(S[\Phi]+t\mathcal{Q}V[\Phi]\right)}\Big) = 0, \tag{3.1.6}$$

where we used the above assumptions, and ignored the boundary contributions. From the above,

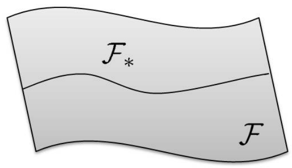
$$Z = Z(0) = Z(t) = Z(\infty) = \lim_{t\to\infty} \int \mathcal{D}\Phi \, e^{-\left(S[\Phi]+t\mathcal{Q}V[\Phi]\right)}, \tag{3.1.7}$$

so the path integral can result in just a problem of calculating one-loop around the configurations such that $\mathcal{Q}V = 0$. In the same way we can apply the same argument for any $\mathcal{Q}$ invariant observables. From this discussion, if we take $V = \sum_{\text{all fermions}} (\mathcal{Q}\psi_i)^\dagger \psi_i$, the path integral becomes a problem of calculating the one-loop around the BPS configurations.

In conclusion, we find that for any supersymmetric observable such that $\mathcal{Q}\mathcal{O}[\Phi] = 0$,

$$\begin{aligned} \big\langle \mathcal{O} \big\rangle \;&=\; \lim_{t\to\infty} \int_{\mathcal{F}_*} \mathcal{D}\Phi_* \, \mathcal{O}[\Phi_*] \, e^{-\left(S[\Phi_*]+t\mathcal{Q}V[\Phi_*]\right)}, \quad \text{for } \mathcal{F}_* = \{\Phi_* \in \mathcal{F} \mid \mathcal{F}_{\text{BPS}} \cap \mathcal{Q}V[\Phi_*] = 0\}, \\ &=\; \int_{\mathcal{F}_*} \mathcal{D}\Phi_* \, \mathcal{O}[\Phi_*] \, e^{-S[\Phi_*]} \, \frac{1}{\text{Sdet}\left[\frac{\delta^2 S[\Phi_*]}{\delta \Phi_*^2}\right]}. \end{aligned} \tag{3.1.8}$$

That is to say the infinite dimensional integral can reduce to just the integral over $\mathcal{F}_*$, and the result can be exact if the second fluctuations around the classical fields are evaluated. Note that we need a special off-shell symmetry to satisfy one of the above conditions $\mathcal{Q}^2 V = 0$.



$$\mathcal{F}_* = \left\{ \Phi \in \mathcal{F} \;\middle|\; \mathcal{F}_{\text{BPS}} \cap \mathcal{Q}V[\Phi] = 0 \right\}$$

Figure 3.1: Localized configration in whole field space

Example: Poincaré-Hopf theorem

As an application let us consider the Poincaré-Hopf theorem along [57]. Let M be a $2n$-dimensional Riemannian manifold with metric $g_{\mu\nu}$, vielbein $e^a{}_\mu$, and let V be a vector field on M. We can consider the following supercoordinates on the tangent bundle,

$$(x^\mu, \psi^\mu), \quad (B_\mu, \bar{\psi}_\mu), \quad \mu = 1, 2, \cdots, 2n, \tag{3.1.9}$$

where x^μ is a coordinate on the base tangent space, and ψ^μ is the fiber coordinate associated with the following fermionic symmetry,

$$\begin{aligned} \delta x^\mu &= \psi^\mu, \qquad & \delta \bar{\psi}_\mu &= B_\mu, \\ \delta \psi^\mu &= 0, \qquad & \delta B_\mu &= 0, \end{aligned} \tag{3.1.10}$$

where $(B_\mu, \bar{\psi}_\mu)$ is a just pair of auxiliary variables. We can verify that $\delta^2 = 0$ immediately. In this setup let us consider a partition function.

$$Z(t) \;=\; \frac{1}{(2\pi)^{2n}} \int d^{2n}x\, d^{2n}\psi\, d^{2n}\bar{\psi}\, d^{2n}B\, e^{-S(t)}, \tag{3.1.11}$$

where

$$S(t) = \delta \left[\frac{1}{2} \bar{\psi}_\mu \Big(B^\mu + 2itV^\mu + g^{\mu\tau} \Gamma^\sigma_{\tau\nu} \bar{\psi}_\sigma \psi^\nu \Big) \right]. \tag{3.1.12}$$

From the above discussion $S(t)$ is independent of the parameter t since it is δ-exact. Integrating B_μ out, we have

$$Z(t) \;=\; \frac{\sqrt{g}}{(2\pi)^n} \int dx d\psi d\bar{\psi}\, \exp\left[- \Big(\frac{t^2}{2} V_\mu V^\mu - it(\nabla_\mu V^\nu) \bar{\psi}_\nu \psi^\mu - \frac{1}{4} R^{\mu\nu}{}_{\sigma\tau} \bar{\psi}_\mu \bar{\psi}_\nu \psi^\sigma \psi^\tau \Big) \right]. \tag{3.1.13}$$

Since $Z(t)$ is independent of t, we first consider the case of $t = 0$,

$$Z(0) \;=\; \frac{1}{(2\pi)^n} \int \sqrt{g}\, dx\, d\psi\, d\bar{\psi}\, \exp\left[\frac{1}{4} R^{\mu\nu}{}_{\sigma\tau} \bar{\psi}_\mu \bar{\psi}_\nu \psi^\sigma \psi^\tau \right]$$

$$
\begin{aligned}
&= \frac{1}{(2\pi)^n} \int \sqrt{g}\, dx\, \mathrm{Pf}(R) \\
&= \int_M e(M) = \chi(M),
\end{aligned}
\tag{3.1.14}
$$

where $e(X)$ and $\chi(X)$ are the Euler class and Euler characteristic, respectively. In the last line we used the Gauss-Bonnet theorem. Next we evaluate the case of $t = \infty$. We assume that V has isolated and simple zeros p_k, $V(p_k) = 0$. Since the contribution for $V = 0$ is dominant, we expand V^μ around the zero, for $\xi^\mu = x^\mu - p_k^\mu$,

$$
V^\mu(x) = \partial_\nu V^\mu(p_k)\xi^\nu + \frac{1}{2}\partial_\nu\partial_\rho V^\mu(p_k)\xi^\nu\xi^\rho + \cdots . \tag{3.1.15}
$$

Also we rescale as

$$
\xi \to t^{-1}\xi, \quad \psi \to t^{-1/2}\psi, \quad \chi \to t^{-1/2}\chi. \tag{3.1.16}
$$

Note that the measure is invariant for this rescaling. In the limit $t \to \infty$, we find

$$
\begin{aligned}
Z(\infty) &= \sum_{p_k} \frac{1}{(2\pi)^n} \int_M \sqrt{g}\, d\xi\, d\psi\, d\bar{\psi}\, \exp\Big[-\frac{1}{2} g_{\mu\nu}\partial_\rho V^\nu(p_k)\cdot\partial_\sigma V^\mu(p_k)\xi^\rho\xi^\sigma + i\partial_\mu V^\nu(p_k)\,\bar{\psi}_\nu\psi^\mu\Big] \\
&= \sum_{p_k} \frac{\det(\partial_\mu V^\nu(p_k))}{\sqrt{\det(\partial_\mu V^\nu(p_k))^2}}.
\end{aligned}
\tag{3.1.17}
$$

Thus, from (3.1.14) and (3.1.17), we obtain the Poincaré-Hopf theorem,

$$
\chi(M) = \sum_{p_k} \frac{\det(\partial_\mu V^\nu(p_k))}{|\det(\partial_\mu V^\nu(p_k))|}. \tag{3.1.18}
$$

For example let's consider the two-sphere case. Here we set $V_i = -y_i\frac{\partial}{\partial x_i} + x_i\frac{\partial}{\partial y_i}$ ($i = N, S$) where N and S denote the north and south patches respectively. Then V has two isolated and simple zeros at the north($x_N = y_N = 0$) and south($x_S = y_S = 0$) poles. Then,

$$
\det(\partial_\mu V^\nu) = \det\begin{pmatrix} \partial_x V^x & \partial_x V^y \\ \partial_y V^x & \partial_y V^y \end{pmatrix} = 1. \tag{3.1.19}
$$

Therefore we can obtain the well-known result,

$$
\chi(S^2) = 2. \tag{3.1.20}
$$

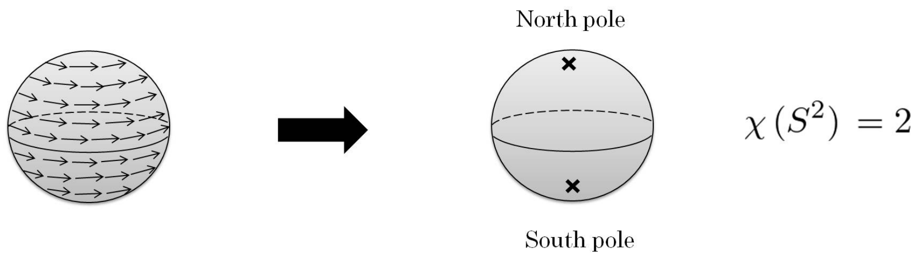


Figure 3.2: Euler characteristic on S^2

3.2 Rigid supersymmetry on a curved space

In the above discussion there is a problem, that is the IR-divergence caused by infinity of the space. Although we have to consider a theory with IR-cutoff which preserves symmetries, it is more convenient to consider a theory on a compact space, which provides the IR-cutoff automatically. In this section we consider the supersymmetric theory on a curved space. Note that we consider Euclidean theories in the following.

Construction of supersymmetry on a curved space

Let us present the outline of the construction of the supersymmetric theory on a curved space along [12] (see also [56]). We should add to the well-known flat-space SUSY Lagrangian some appropriate corrections corresponding to the curved space $\mathcal{M}$ we consider. Given the Lagrangian and supersymmetry transformation on the flat space, and the characteristic scale of $\mathcal{M}$ as $\mathcal{L}^{(0)}_{\mathcal{M}} = \mathcal{L}_{\mathbb{R}^d}$, $\delta^{(0)} = \delta_{\mathbb{R}^d}$ and r, then we would obtain a Lagragian and a supersymmetric transformation on $\mathcal{M}$ in as follows,

$$\mathcal{L}_{\mathcal{M}} = \mathcal{L}^{(0)}_{\mathcal{M}} + \delta\mathcal{L}_{\mathcal{M}} = \sum_{n=0}^{\infty} \frac{1}{r^n}\mathcal{L}^{(n)}_{\mathcal{M}}, \tag{3.2.1}$$

$$\delta_{\mathcal{M}} = \sum_{n=0}^{\infty} \frac{1}{r^n}\delta^{(n)}, \tag{3.2.2}$$

where we have to determine the each correction term order by order to preserve the supersymmetry and close the algebra. At first sight they seem to be an infinite summation, but since r has an inverse mass dimension, we do not need to consider irrelevant operators at UV similarly to the renormalization group argument. However, since this idea depends on the space, we instead consider the idea of [12]. This idea provides us with the systematic construction of a supersymmetric theory on a curved space, and as a result we can find that the above corrections in the Lagrangian terminate at second order, and those in the supersymmetry transformation terminate at first order, which are consistent with the above idea.

Let us give the outline. First we consider an appropriate supergravty theory, and take a rigid limit, i.e. taking the Newton constant G_N to zero (Planck mass M_P to infinity), while keeping the metric on $\mathcal{M}$ at the same time. Then the gravity decouples from the theory, and we can obtain a supersymmetric theory on the curved space $\mathcal{M}$, where the metric and the other auxiliary fields in the gravity multiplet become just backgrounds. Although we do not have to consider their equations of motions, we should require the conditions that the background is also supersymmetric,

$$\Psi_\mu^\alpha = 0, \quad \delta\Psi_\mu^\alpha = 0, \tag{3.2.3}$$

where Ψ is gravitino and $\delta\Psi$ implies the transformation in the supergravity theory. The conditions correspond to Killing spinor equations, where the spinors respect the supersymmetries on $\mathcal{M}$.

Minimal coupling with supergravity multiplet

If we know the corresponding supergravity theory, we should apply the above discussion. However, even if we do not know such a supergravity theory, we can still construct such a supersymmetric theory on a curved space. First recall the prescription of coupling a theory with a gauge field.

We should replace the ordinary derivative with a gauge covariant derivative: $\partial_\mu \to D_\mu = \partial_\mu - iA_\mu$. The minimally coupled Lagrangian is written as

$$\mathcal{L}(\Phi, D\Phi) = \mathcal{L}(\Phi, \partial\Phi) - j^\mu A_\mu + \mathcal{O}(A^2), \tag{3.2.4}$$

where j^μ is a conserved current for the original global symmetry, and also written by

$$j^\mu = -\frac{\partial \mathcal{L}}{\partial A_\mu}\bigg|_{A=0}. \tag{3.2.5}$$

If we want to obtain a dynamical gauge theory, we should add the Yang-Mills kinetic term. We can extend this idea to spacetime symmetries. The associated current with the Poincaré symmetry is the energy-momentum tensor $T^{\mu\nu}$. The coupled theory is obtained by replacing the flat metric $\eta_{\mu\nu}$ and the ordinary derivative with a curved metric $g_{\mu\nu} = \eta_{\mu\nu} - 2h_{\mu\nu}$ and the general covariant derivative ∇_μ in the same way. The minimally coupled Lagrangian is

$$\mathcal{L} = \mathcal{L}^{(0)} - T^{\mu\nu}h_{\mu\nu} + \mathcal{O}(h^2). \tag{3.2.6}$$

Furthermore since we consider the supersymmetric theory, we have a supercurrent $S_{\mu\alpha}$ associated with the supercharge Q_α as well as the conjugate supercharge. Then the conjugate fermionic gauge field is the gravitino $\Psi_{\mu\alpha}$. The minimally coupled Lagrangian is

$$\mathcal{L} = \mathcal{L}^{(0)} - \frac{1}{2} S^\mu_\alpha \Psi^\alpha_\mu + \text{c.c} + \mathcal{O}(S^2). \tag{3.2.7}$$

Since the supersymmetry algebra generates translations on the flat space, we expect that the supercurrent and the energy-momentum tensor belong to the same multiplet, and in the same way the conjugate fields, the graviton and the gravitini, belong to the same conjugate multiplet. In fact this idea is true, and we should consider a current multiplet with the conjugate supergravity multiplet.

3d $\mathcal{N} = 2$ theory with $U(1)_R$ on a curved space

According to [15] (see also [56]), we would like to consider a 3d $\mathcal{N} = 2$ with $U(1)_R$ theory on a curved space. First when we consider a 3d $\mathcal{N} = 2$ theory with $U(1)_R$, the set of conserved currents (the R-current $j^{(R)}_\mu$, the supercurrents $S_{\mu\alpha}$, $\bar{S}_{\mu\alpha}$, the energy momentum tensor $T_{\mu\nu}$, Z-currents $j^{(Z)}_\mu$ and a string currents $i\varepsilon_{\mu\nu\rho}\partial^\rho J^{(Z)}$) form a multiplet $\mathcal{R}_\mu$, so-called R-multiplet. The R-multiplet is characterized in the superfield formalism by

$$\bar{D}^\beta \mathcal{R}_{\alpha\beta} = -4i\bar{D}_\alpha \mathcal{J}^{(Z)}, \quad D^\beta \mathcal{R}_{\alpha\beta} = 4iD_\alpha \mathcal{J}^{(Z)}, \tag{3.2.8}$$

where $\mathcal{R}_{\alpha\beta} := -2\gamma^\mu_{\alpha\beta}\mathcal{R}_\mu$, and $\mathcal{J}^{(Z)}$ is a real linear multiplet such that $D^2\mathcal{J}^{(Z)} = \bar{D}^2\mathcal{J}^{(Z)} = 0$. They are in components,

$$\begin{aligned} \mathcal{R}_\mu = &\quad j^{(R)}_\mu - i\theta S_\mu - i\bar{\theta}\bar{S}_\mu - (\theta\gamma^\nu\bar{\theta})(2T_{\mu\nu} + i\varepsilon_{\mu\nu\rho}\partial^\rho J^{(Z)}) \\ &\quad -i\theta\bar{\theta}(2j^{(Z)}_\mu + i\varepsilon_{\mu\nu\rho}\partial^\nu j^{(R)\rho}) + \cdots , \end{aligned} \tag{3.2.9}$$

$$\mathcal{J}^{(Z)} = \quad J^{(Z)} - \frac{1}{2}\theta\gamma^\mu S_\mu + \frac{1}{2}\bar{\theta}\gamma^\mu\bar{S}_\mu + i\theta\bar{\theta}T^\mu_\mu - (\theta\gamma^\mu\bar{\theta})j^{(Z)}_\mu + \cdots, \tag{3.2.10}$$

where $\cdots$ means some terms which are written in terms of the lower components.

Then at the linear order, the R-multiplet couples to a conjugate supergravity multiplet $\mathcal{H}_\mu = (h_{\mu\nu}, \Psi_{\mu\alpha}, \bar{\Psi}_{\mu\alpha}, A_\mu, C_\mu, B_{\mu\nu})$, where the components are a graviton, two gravitini, two gauge one-forms and a gauge two-form, in the following way,

$$\delta\mathcal{L}\big|_{\text{linear}} = 2\int d^2\theta d^2\bar{\theta}\, \mathcal{R}_\mu \mathcal{H}^\mu, \tag{3.2.11}$$

which is a supersymmetric invariant from the superfield formalism, and $\mathcal{H}_\mu$ is in the Wess-Zumino gauge,

$$\mathcal{H}_\mu = \frac{1}{2}(\theta\gamma^\nu\bar{\theta})(h_{\mu\nu} + B_{\mu\nu}) - \frac{i}{2}\theta\bar{\theta}C_\mu - \frac{i}{2}\theta^2\bar{\theta}\bar{\Psi}_\mu + \frac{i}{2}\bar{\theta}^2\theta\Psi_\mu + \frac{1}{2}\theta^2\bar{\theta}^2(A_\mu - V_\mu). \tag{3.2.12}$$

Then the linearized supergraivty-matter coupling (3.2.11) is

$$\delta\mathcal{L}\big|_{\text{linear}} = -T^{\mu\nu}h_{\mu\nu} - \frac{1}{2}S^{\mu}\Psi_{\mu} + \frac{1}{2}\bar{S}^{\mu}\bar{\Psi}_{\mu} + j^{(R)\mu}\Big(A_{\mu} - \frac{3}{2}V_{\mu}\Big) + j^{(Z)\mu}C_{\mu} + J^{(Z)}H, \tag{3.2.13}$$

where we use dualized expressions, $V^{\mu} := -i\varepsilon^{\mu\nu\rho}\partial_{\mu}C_{\rho}$ and $H := \frac{i}{2}\varepsilon^{\mu\nu\rho}\partial_{\mu}B_{\nu\rho}$. If we choose the Wess-Zumino gauge, there are residual gauge transformations for the conjugate supergravity multiplet,

$$\begin{aligned} \delta h_{\mu\nu} &= \partial_{\mu}\Lambda^{(h)}_{\nu} + \partial_{\nu}\Lambda^{(h)}_{\mu}, &\qquad \delta B_{\mu\nu} &= \partial_{\mu}\Lambda^{(B)}_{\nu} + \partial_{\nu}\Lambda^{(B)}_{\mu}, \\ \delta C_{\mu} &= \partial_{\mu}\Lambda^{(C)}_{\mu}, & \delta A_{\mu} &= \partial_{\mu}\Lambda^{(A)}_{\mu}, \\ \delta\Psi_{\mu\alpha} &= \partial_{\mu}\varepsilon_{\alpha}, & \delta\bar{\Psi}_{\mu\alpha} &= \partial_{\mu}\bar{\varepsilon}_{\alpha}. \end{aligned} \tag{3.2.14}$$

Similarly, the supersymmetry transformations for the gravitini are for constant spinors ϵ and $\bar{\epsilon}$,

$$\delta_{\epsilon}\Psi_{\mu} = -i\varepsilon^{\nu\rho\lambda}\partial_{\nu}h_{\rho\mu}\gamma_{\lambda}\epsilon - 2i(A_{\mu} - V_{\mu})\epsilon + H\gamma_{\mu}\epsilon + \varepsilon_{\mu\nu\rho}V^{\nu}\gamma^{\rho}\epsilon + \partial_{\mu}(\cdots), \tag{3.2.15}$$

$$\delta_{\bar{\epsilon}}\bar{\Psi}_{\mu} = -i\varepsilon^{\nu\rho\lambda}\partial_{\nu}h_{\rho\mu}\gamma_{\lambda}\bar{\epsilon} + 2i(A_{\mu} - V_{\mu})\bar{\epsilon} + H\gamma_{\mu}\bar{\epsilon} - \varepsilon_{\mu\nu\rho}V^{\nu}\gamma^{\rho}\bar{\epsilon} + \partial_{\mu}(\cdots). \tag{3.2.16}$$

Note that we can absorb the above total derivative terms using the residual gauge transformations (3.2.14).

In the above discussion we have considered the supersymmetric theory on the flat space coupled to a linearized supergravity multiplet. Next let us consider the one on a curved space. To do so, we promote the constant spinors ϵ and $\bar{\epsilon}$ to local parameters. As usual as the Noether's theorem, the variation is

$$\delta_{\epsilon,\bar{\epsilon}}\mathcal{L}_0 = S^{\mu}\partial_{\mu}\epsilon - \bar{S}^{\mu}\partial_{\mu}\bar{\epsilon}, \tag{3.2.17}$$

where $\mathcal{L}_0$ is the above flat space Lagrangiran. We note that comparing this variation with (3.2.10), the gravitini gauge transformations (3.2.14) can absorb this variation, if we set $\varepsilon = 2\epsilon$ and $\bar{\varepsilon} = 2\bar{\epsilon}$. Therefore we can obtain the following local supersymmetry transformations for the gravitini,

$$\delta_{\epsilon}\Psi_{\mu} = 2\underbrace{\Big(\partial_{\mu} - \frac{i}{2}\varepsilon^{\nu\rho\lambda}\partial_{\nu}h_{\rho\mu}\gamma_{\lambda}\Big)\epsilon}_{=\nabla_{\mu}\epsilon} - 2i(A_{\mu} - V_{\mu})\epsilon + H\gamma_{\mu}\epsilon + \varepsilon_{\mu\nu\rho}V^{\nu}\gamma^{\rho}\epsilon, \tag{3.2.18}$$

$$\delta_{\bar{\epsilon}}\bar{\Psi}_{\mu} = 2\underbrace{\Big(\partial_{\mu} - \frac{i}{2}\varepsilon^{\nu\rho\lambda}\partial_{\nu}h_{\rho\mu}\gamma_{\lambda}\Big)\bar{\epsilon}}_{=\nabla_{\mu}\bar{\epsilon}} + 2i(A_{\mu} - V_{\mu})\bar{\epsilon} + H\gamma_{\mu}\bar{\epsilon} - \varepsilon_{\mu\nu\rho}V^{\nu}\gamma^{\rho}\bar{\epsilon}, \tag{3.2.19}$$

where the above terms in big parentheses are linearized covariant derivatives $\nabla_{\mu} := \partial_{\mu} + \frac{1}{4}\gamma^{ab}\omega^{ab}_{\mu}$. Therefore the Killing spinor equations we want to derive are

$$(\nabla_{\mu} - iA_{\mu})\epsilon = -\frac{H}{2}\gamma_{\mu}\epsilon - iV_{\mu}\epsilon - \frac{1}{2}\varepsilon_{\mu\nu\rho}V^{\nu}\gamma^{\rho}\epsilon, \tag{3.2.20}$$

$$(\nabla_\mu + iA_\mu)\bar{\epsilon} = -\frac{H}{2}\gamma_\mu\bar{\epsilon} + iV_\mu\bar{\epsilon} + \frac{1}{2}\varepsilon_{\mu\nu\rho}V^\nu\gamma^\rho\bar{\epsilon}, \tag{3.2.21}$$

where ϵ and $\bar{\epsilon}$ carry the R-charge $+1$ and -1 respectively, which induce the sign in front of each A_μ. Given a manifold, identifying the background fields $(g_{\mu\nu}, A_\mu, H, V^\mu)$, we can obtain the Killing spinor equations.

Next let us consider some examples, S^3, S^3_b and $\mathbb{R}\times S^2$.

3.3 Supersymmetries on S^3, S^3_b and $\mathbb{R}\times S^2$

In this section, we summarize the Killing spinors on the three-sphere, three-ellipsoid and $\mathbb{R}\times S^2$ that we will use later[1].

Three-sphere S^3

A three-sphere with a radius R is defined as a pair of complex coordinates $(u,v)\in\mathbb{C}^2$ s.t.

$$u\bar{u} + v\bar{v} = R^2. \tag{3.3.1}$$

The isometry is $SO(4) \cong SU(2)_L \times SU(2)_R$. Using the torus fibration coordinates $(\vartheta, \varphi_1, \varphi_2)$[2],

$$u = R\sin\vartheta\, e^{i\varphi_1}, \quad v = R\cos\vartheta\, e^{i\varphi_2}, \tag{3.3.2}$$

where $0 \le \vartheta \le \pi/2$, $0 \le \varphi_1 < 2\pi$, $0 \le \varphi_2 < 2\pi$, then the metric is

$$ds^2 = R^2(d\vartheta^2 + \cos^2\vartheta\, d\varphi_1^2 + \sin^2\vartheta\, d\varphi_2^2). \tag{3.3.3}$$

The Killing spinor equation on S^3 is given by [20]

$$D_\mu\epsilon = \frac{i}{2R}\gamma_\mu\epsilon, \quad D_\mu\bar{\epsilon} = \frac{i}{2R}\gamma_\mu\bar{\epsilon}. \tag{3.3.4}$$

This corresponds to $A_\mu = V_\mu = 0$, and $H = -\frac{i}{R}$ in (3.2.20) and (3.2.21). We choose the following frame,

$$e^1 = R\cos\vartheta\, d\varphi_2, \quad e^2 = -R\sin\vartheta\, d\varphi_1, \quad e^3 = Rd\vartheta. \tag{3.3.5}$$

Then the Levi-Civita spin connection is

$$\omega^{12} = 0, \quad \omega^{31} = \sin\vartheta\, d\varphi_2, \quad \omega^{23} = -\cos\vartheta\, d\varphi_1. \tag{3.3.6}$$

[1] We summarize supersymmeties on S^3, S^3_b and $\mathbb{R}\times S^2$ in appendix A.3

[2] See appendix A.2 for details.

We choose [20, 58, 59][3],

$$\epsilon = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{\frac{i}{2}(\varphi_1+\varphi_2+\theta)} \\ e^{\frac{i}{2}(\varphi_1+\varphi_2-\theta)} \end{pmatrix}, \quad \bar{\epsilon} = \frac{1}{\sqrt{2}} \begin{pmatrix} -e^{-\frac{i}{2}(\varphi_1+\varphi_2-\theta)} \\ e^{-\frac{i}{2}(\varphi_1+\varphi_2+\theta)} \end{pmatrix}, \tag{3.3.7}$$

as a solution to (3.3.4). Note that there are four independent solutions on S^3 corresponding to the following Killing spinor equations [59]:

$$D_\mu \psi_{(st)} = -\frac{ist}{2R}\gamma_\mu \psi_{(st)}, \quad s, t = \pm 1, \tag{3.3.8}$$

where these solutions are given by

$$\psi_{(st)} = \begin{pmatrix} e^{\frac{i}{2}(-s\varphi_1+t\varphi_2-st\vartheta)} \\ -se^{\frac{i}{2}(-s\varphi_1+t\varphi_2+st\vartheta)} \end{pmatrix}. \tag{3.3.9}$$

We have chosen $\epsilon = \psi_{-+}$ and $\bar{\epsilon} = -\psi_{(+-)}$ above.

Three-ellipsoid S_b^3

The three-ellipsoid S_b^3 is defined by the following hypersurface:

$$u\bar{u} + v\bar{v} = 1, \tag{3.3.10}$$

where $(u, v) \in \mathbb{C}^2$ with the metric

$$ds^2 = l^2 du d\bar{u} + \tilde{l}^2 dv d\bar{v}. \tag{3.3.11}$$

Note that this geometry preserves the $U(1) \times U(1)$ isometry of the original three-sphere isometry $SU(2) \times SU(2)$ [4]. Using the torus fibration coordinates $(\vartheta, \varphi_1, \varphi_2)$, the metric is

$$ds^2 = R^2 \left(f(\vartheta)^2 d\vartheta^2 + b^2 \sin^2\vartheta d\varphi_1^2 + b^{-2}\cos^2\vartheta d\varphi_2^2 \right), \tag{3.3.12}$$

where

$$R = \sqrt{l\tilde{l}}, \quad b = \sqrt{\tilde{l}/l}, \quad f(\vartheta) = \sqrt{b^{-2}\sin^2\vartheta + b^2\cos^2\vartheta}. \tag{3.3.13}$$

Note that if we choose $b = 1$, they reduce to the above three-sphere case. We choose the following orthogonal frames,

$$e^1 = Rb^{-1}\cos\vartheta d\varphi_2, \quad e^2 = -Rb\sin\vartheta d\varphi_1, \quad e^3 = Rf(\vartheta)d\vartheta, \tag{3.3.14}$$

[3] Our notation is slightly different from [59]: $(\vartheta, \varphi_1, \varphi_2)_{\text{here}} = (\theta, -\chi, \varphi)_{\text{there}}$, $(\epsilon, \bar{\epsilon})_{\text{here}} = (-\bar{\epsilon}, \epsilon)_{\text{there}}$.

[4] We can also consider other deformations which preserve supersymmetries [59, 60, 61, 15, 62, 63, 64, 16]

Then the Levi-Civita spin connection is

$$\omega^{12} = 0, \quad \omega^{31} = \frac{b^{-1}}{f(\vartheta)} \sin\vartheta \, d\varphi_2, \quad \omega^{23} = -\frac{b}{f(\vartheta)} \cos\vartheta \, d\varphi_1. \tag{3.3.15}$$

Then, the Killing spinor equations are [59]

$$D_\mu \epsilon = \frac{i}{2Rf(\vartheta)} \gamma_\mu \epsilon, \quad D_\mu \bar{\epsilon} = \frac{i}{2Rf(\vartheta)} \gamma_\mu \bar{\epsilon}, \tag{3.3.16}$$

where the covariant derivative is defined by

$$D := d + \frac{1}{4}\gamma^{ab}\omega^{ab} - i\mathcal{R}V, \quad V = \frac{1}{2}\Big(1 - \frac{b}{f(\vartheta)}\Big) d\varphi_1 + \frac{1}{2}\Big(1 - \frac{b^{-1}}{f(\vartheta)}\Big) d\varphi_2, \tag{3.3.17}$$

where $\mathcal{R}$ is a R-symmetry generator. This corresponds to $A_\mu, V_\mu \neq 0$, and $H = -\frac{i}{Rf(\vartheta)}$ in (3.2.20) and (3.2.21). The solutions are given by the same Killing spinors (3.3.7) (in fact the authors of [59] constructed (3.3.7) in order to apply the three-sphere ones to the ellipsoid ones directly). In the same as the three-sphere, in fact there are four independent solutions corresponding to the following Killing spinor equations:

$$D_\mu \psi_{(st)} = -\frac{ist}{2Rf(\vartheta)} \gamma_\mu \psi_{(st)} + iV_\mu^{(st)} \psi_{(st)}, \quad s, t = \pm 1, \tag{3.3.18}$$

where the covariant derivative is the ordinary one, and

$$V^{(st)} = -\frac{s}{2}\Big(1 - \frac{b}{f(\vartheta)}\Big) d\varphi_1 + \frac{t}{2}\Big(1 - \frac{b^{-1}}{f(\vartheta)}\Big) d\varphi_2. \tag{3.3.19}$$

As we mentioned, the solutions are given by (3.3.9), and we have just chosen $\epsilon = \psi_{(-+)}$ and $\bar{\epsilon} = -\psi_{(+-)}$.

$\mathbb{R} \times S^2$

We consider the geometry $\mathbb{R} \times S^2$. The metric is

$$ds^2 = d\tau^2 + R^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{3.3.20}$$

where $\tau \in \mathbb{R}$ has a periodicity $\tau \sim \tau + \beta R$, and $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$. We choose the following vielbein,

$$e^1 = d\tau, \quad e^2 = R d\theta, \quad e^3 = R\sin\theta d\varphi. \tag{3.3.21}$$

The Killing spinor equations are [65]

$$D_\mu \epsilon = -\frac{1}{2R}\gamma_\mu \gamma_1 \epsilon, \quad D_\mu \bar{\epsilon} = \frac{1}{2R}\gamma_\mu \gamma_1 \bar{\epsilon}. \tag{3.3.22}$$

This corresponds to the case of $A_\mu, V_\mu \neq 0$, and $H = 0$ in (3.2.20) and (3.2.21). These solutions are [65, 66]

$$\epsilon = \frac{1}{\sqrt{2}}\, e^{-\frac{\tau}{2R}} \begin{pmatrix} -e^{\frac{i}{2}(\theta-\varphi)} \\ e^{\frac{i}{2}(-\theta-\varphi)} \end{pmatrix}, \quad \bar{\epsilon} = \frac{1}{\sqrt{2}}\, e^{\frac{\tau}{2R}} \begin{pmatrix} e^{\frac{i}{2}(-\theta+\varphi)} \\ e^{\frac{i}{2}(\theta+\varphi)} \end{pmatrix}. \tag{3.3.23}$$

In the same way as the three-sphere, there are also four independent solutions in this geometry [65, 15].

There are other various geometries which can preserve some supersymmetries, also not only three dimensions [12, 13, 14, 15, 16, 17].

Chapter 4

Coulomb branch localization

In this chapter we compute the partition function of the $\mathcal{N} = 2$ gauge theory on the three-ellipsoid and $S^1 \times S^2$ using the Coulomb branch localization.

4.1 Partition function on the three-ellipsoid

First in this section, we consider a partition function on the three-ellipsoid along [59, 66], which is a one-parameter deformation of the round three-sphere. This partition function gives us more information than that on a three-sphere [20, 28, 58].

4.1.1 Supersymmetric multiplet

So far the supersymmetry transformation with a parameters $\delta = \delta_\epsilon + \delta_{\bar{\epsilon}}$ are

$$\delta = \delta_\epsilon + \delta_{\bar{\epsilon}} = \epsilon^\alpha Q_\alpha + \bar{\epsilon}^\alpha \bar{Q}_\alpha, \tag{4.1.1}$$

where $\epsilon, \bar{\epsilon}$ and $Q, \bar{Q}$ are Grassmann-odd. That is to say, δ is a "bosonic" operator. Here, using Grassmann-even spinors $\epsilon, \bar{\epsilon}$, we introduce a "fermionic" operator $\mathcal{Q}$ as

$$\mathcal{Q} = i(\epsilon^\alpha Q_\alpha + \bar{\epsilon}^\alpha \bar{Q}_\alpha), \tag{4.1.2}$$

where the coefficient is our convention, and note that $\mathcal{Q}$ is Grassmann-odd. We use this operator $\mathcal{Q}$ instead of δ below.

In this section, we choose (Grassmann-even) Killing spinors ϵ, $\bar{\epsilon}$ as (3.3.7).

Vector multiplet

Let us start with the $\mathcal{N}=2$ vector multiplet on S_b^3, which we introduced in last chapter. The action of $\mathcal{N}=2$ super Yang-Mills theory is given by [59][1]

$$\begin{aligned} S_{\text{YM}} &= \frac{1}{g_{\text{YM}}^2}\int d^3x\sqrt{g}\,\text{Tr}\Big[\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}D_\mu\sigma D^\mu\sigma+\frac{1}{2}\Big(D+\frac{\sigma}{Rf(\vartheta)}\Big)^2 \\ &\qquad +\frac{i}{2}\bar{\lambda}\gamma^\mu D_\mu\lambda+\frac{i}{2}\bar{\lambda}[\sigma,\lambda]-\frac{1}{4Rf(\vartheta)}\bar{\lambda}\lambda\Big]. \end{aligned} \tag{4.1.3}$$

This action is invariant under the following SUSY transformation:[2]

$$\begin{aligned} \mathcal{Q}A_\mu &= -\frac{1}{2}\bar{\lambda}\gamma_\mu\epsilon-\frac{1}{2}\bar{\epsilon}\gamma_\mu\lambda, \\ \mathcal{Q}\sigma &= \frac{i}{2}\bar{\lambda}\epsilon+\frac{i}{2}\bar{\epsilon}\lambda, \\ \mathcal{Q}\lambda &= \left(\frac{1}{2}\epsilon_{\mu\nu\rho}F^{\nu\rho}-D_\mu\sigma\right)\gamma^\mu\epsilon-iD\epsilon-\frac{i}{Rf(\vartheta)}\sigma\epsilon, \\ \mathcal{Q}\bar{\lambda} &= \left(\frac{1}{2}\epsilon_{\mu\nu\rho}F^{\nu\rho}+D_\mu\sigma\right)\gamma^\mu\bar{\epsilon}+iD\bar{\epsilon}+\frac{i}{Rf(\vartheta)}\sigma\bar{\epsilon}, \\ \mathcal{Q}D &= \frac{1}{2}\bar{\epsilon}\gamma^\mu D_\mu\lambda-\frac{1}{2}D_\mu\bar{\lambda}\gamma^\mu\epsilon-\frac{1}{2}[\bar{\epsilon}\lambda,\sigma]+\frac{1}{2}[\bar{\lambda}\epsilon,\sigma]-\frac{1}{4Rf(\vartheta)}(\bar{\epsilon}\lambda+\bar{\lambda}\epsilon), \end{aligned} \tag{4.1.4}$$

where we use the torus fibration coordinates (3.3.12) and the Killing spinors on S_b^3 (3.3.7), and note the definition of the covariant derivative (3.3.17). One can show that Q^2 generates

$$\mathcal{Q}^2 = i\mathcal{L}_v+i\sigma-v^\mu A_\mu+\frac{1}{2R}(b+b^{-1})\mathcal{R}, \tag{4.1.5}$$

where

$$v=(\bar{\epsilon}\gamma^{\hat{\mu}}\epsilon)e_{\hat{\mu}}=R^{-1}\left(b^{-1}\frac{\partial}{\partial\varphi_1}+b\frac{\partial}{\partial\varphi_2}\right), \tag{4.1.6}$$

$\mathcal{L}_v$ is a Lie-derivative along the vector field v, and $\mathcal{R}$ is the R-symmetry generator.

On this space, we can also consider supersymmetric CS term and FI term as

$$S_{\text{CS}}=\frac{i\kappa}{4\pi}\int d^3x\sqrt{g}\text{Tr}\left[\epsilon^{\mu\nu\rho}\left(A_\mu\partial_\nu A_\rho+\frac{2i}{3}A_\mu A_\nu A_\rho\right)-\bar{\lambda}\lambda+2D\sigma\right], \tag{4.1.7}$$

[1] Compared with the Lagrangian on the flat space (2.1.14), we note that it has corrections proportional to $1/R$ and $1/R^2$ as we mentioned in the last chapter.

[2] Since we use (4.1.2), note that it is slightly different from the supersymmetry transformation in appendix A.3.

$$S_{\mathrm{FI}} = -\frac{i\zeta}{2\pi R}\int d^3x\sqrt{g}\left(D - \frac{1}{Rf(\vartheta)}\sigma\right), \tag{4.1.8}$$

respectively. Note that they are $\mathcal{Q}$-closed but not $\mathcal{Q}$-exact.

Chiral multiplet

Let us consider the matter sector. The action is given by [59]

$$\begin{aligned} S_{\mathrm{mat}} \;=\; & \int d^3x\sqrt{g}\Big(D_\mu\bar{\phi}D^\mu\phi + \bar{\phi}\sigma^2\phi + \frac{i(2\Delta-1)}{Rf(\vartheta)}\bar{\phi}\sigma\phi + \frac{\Delta(2-\Delta)}{(Rf(\vartheta))^2}\bar{\phi}\phi + i\bar{\phi}D\phi + \bar{F}F \\ & - i\bar{\psi}\gamma^\mu D_\mu\psi + i\bar{\psi}\sigma\psi - \frac{2\Delta-1}{2Rf(\vartheta)}\bar{\psi}\psi + i\bar{\psi}\lambda\phi - i\bar{\phi}\bar{\lambda}\psi\Big), \end{aligned} \tag{4.1.9}$$

which is invariant under the SUSY transformation

$$\begin{aligned} \mathcal{Q}\phi &= i\bar{\epsilon}\psi, \\ \mathcal{Q}\bar{\phi} &= i\epsilon\bar{\psi}, \\ \mathcal{Q}\psi &= -\gamma^\mu\epsilon D_\mu\phi - \epsilon\sigma\phi - \frac{i\Delta}{Rf(\vartheta)}\epsilon\phi + i\bar{\epsilon}F, \\ \mathcal{Q}\bar{\psi} &= -\gamma^\mu\bar{\epsilon}D_\mu\bar{\phi} - \bar{\phi}\sigma\bar{\epsilon} - \frac{i\Delta}{Rf(\vartheta)}\bar{\phi}\bar{\epsilon} + i\bar{F}\epsilon, \\ \mathcal{Q}F &= \epsilon(-\gamma^\mu D_\mu\psi + \sigma\psi + \lambda\phi) + \frac{i}{2Rf(\vartheta)}(2\Delta-1)\epsilon\psi, \\ \mathcal{Q}\bar{F} &= \bar{\epsilon}(-\gamma^\mu D_\mu\bar{\psi} + \bar{\psi}\sigma - \bar{\phi}\bar{\lambda}) + \frac{i}{2Rf(\vartheta)}(2\Delta-1)\bar{\epsilon}\bar{\psi}. \end{aligned} \tag{4.1.10}$$

We have assigned R-charges: $(-\Delta,\, \Delta,\, 1-\Delta,\, \Delta-1,\, 2-\Delta,\, \Delta-2)$ to $(\phi, \bar{\phi}, \psi, \bar{\psi}, F, \bar{F})$, respectively.

4.1.2 Localized configurations

Let us consider the deformation terms in the localization.

Vector multiplet

First for the vector multiplet we note that the SYM Lagrangian becomes $\mathcal{Q}$-exact up to total derivatives:

$$\mathcal{L}_{\mathrm{YM}} = \mathcal{Q}V_{\mathrm{vec}}, \quad \text{with} \quad V_{\mathrm{vec}} = \frac{1}{4}\mathrm{Tr}\Big[(\mathcal{Q}\lambda)^\dagger\lambda + (\mathcal{Q}\bar{\lambda})^\dagger\bar{\lambda}\Big], \tag{4.1.11}$$

if we take the integral contour of D to be real (2.1.13). That is to say we can choose the SYM action as the exact deformation term. It also implies that the partition function is independent of the gauge coupling. From the (4.1.3), we can immediately find the following localized configurations,

$$F_{\mu\nu} = 0, \quad D_\mu \sigma = 0, \quad D = -\frac{1}{Rf(\vartheta)}\sigma, \tag{4.1.12}$$

and all the fermions vanish. In particular for the three-ellipsoid, this condition is

$$A_\mu = 0, \quad \sigma = \text{const.}\ , \quad D = -\frac{1}{Rf(\vartheta)}\sigma. \tag{4.1.13}$$

Chiral multiplet

Next we consider the matter sector. We choose

$$\mathcal{L}_\psi = \mathcal{Q}V_{\text{chi}}, \quad \text{with} \;\; V_{\text{chi}} = \frac{1}{2}\Big[(\mathcal{Q}\psi)^\dagger\psi + (\mathcal{Q}\bar{\psi})^\dagger\bar{\psi}\Big]. \tag{4.1.14}$$

In fact, completing the square leads to

$$\begin{aligned} \mathcal{L}_\psi|_{\text{bos.}} &= |\sin\vartheta D_1\phi + \cos\vartheta D_2\phi + iD_3\phi|^2 + |\sigma\phi|^2 + |F|^2 \\ &\quad + \Big|\cos\vartheta D_1\phi - \sin\vartheta D_2\phi + \frac{i\Delta}{Rf(\vartheta)}\phi\Big|^2, \end{aligned} \tag{4.1.15}$$

if we take the reality condition (2.1.13). From this, we can read off the localized configurations,

$$\begin{aligned} &\sin\vartheta D_1\phi + \cos\vartheta D_2\phi + iD_3\phi = 0, \quad \sigma\phi = 0, \quad F = 0, \\ &\cos\vartheta D_1\phi - \sin\vartheta D_2\phi + \frac{i\Delta}{Rf(\vartheta)}\phi = 0. \end{aligned} \tag{4.1.16}$$

Using (4.1.13), we can replace the covariant derivative with an ordinary derivative, $D_\mu \to \partial_\mu$. Since ϕ is periodic, it is expanded as

$$\phi(\varphi_1, \varphi_2, \vartheta) = \sum_{m,n\in\mathbb{Z}} \tilde{\phi}_{m,n}(\vartheta)e^{im\varphi_1 + in\varphi_2}. \tag{4.1.17}$$

Then, one of the above equations is

$$\Big(nb + mb^{-1} + \frac{\Delta}{f(\vartheta)}\Big)\phi_{m,n}(\vartheta) = 0. \tag{4.1.18}$$

If the R-charge Δ is generic, ϕ must vanish[3] . Thus this BPS configurations are

$$\phi = \bar{\phi} = F = \bar{F} = 0. \tag{4.1.19}$$

Note that $\mathcal{L}_\psi \neq \mathcal{L}_{\rm mat}$ but also $\mathcal{L}_{\rm mat} = 0$ on the localized configurations.

So far, although we do not consider the superpotential terms explicitly, we can also construct them on a curved space. However, since the terms are proportional to the auxiliary fields F, $\bar{F}$ and fermions as that on the flat space, we note that they do not affect this localization calculation from (4.1.19).

4.1.3 Gauge fixing

In order to compute the one-loop determinant, we have to perform the gauge fixing. We introduce a BRST transformation

$$Q_B A_\mu = D_\mu c, \quad Q_B c = -\frac{i}{2}[c, c], \quad Q_B \bar{c} = B, \quad Q_B B = 0, \tag{4.1.20}$$

where c, $\bar{c}$ are ghosts and B is the Nakanishi-Lautrap field. Then we find the gauge fixing term as

$$\mathcal{L}_{\rm gh} = Q_B V_{\rm gh} = Q_B {\rm Tr}\Big[\ \bar{c}\Big(G(\tilde{A}) + \frac{\xi}{2}B\Big)\ \Big], \tag{4.1.21}$$

where $G(\Phi)$ is the gauge fixing function and $\tilde{\Phi}$ stands for the fluctuation from the localized configuration $\Phi^{(0)}$ given by

$$\tilde{\Phi} = \Phi - \Phi^{(0)}. \tag{4.1.22}$$

We define SUSY transformations for the ghosts and Nakanishi-Lautrap field as

$$\mathcal{Q}c = \tilde{\sigma} + iv^\mu \tilde{A}_\mu, \quad \mathcal{Q}\bar{c} = 0, \quad \mathcal{Q}B = iv^\mu D_\mu^{(0)}\bar{c} + i[\sigma^{(0)}, \bar{c}]. \tag{4.1.23}$$

Then $\hat{\mathcal{Q}} = \mathcal{Q} + Q_B$ generates

$$\hat{\mathcal{Q}}^2 = i\mathcal{L}_v + i\sigma^{(0)} - v^\mu A_\mu^{(0)} + \frac{1}{2R}(b + b^{-1})\mathcal{R}. \tag{4.1.24}$$

We note that this is the same form as (4.1.5).

In the previous section, we have omitted the gauge fixing for simplicity: precisely speaking, we have to choose the deformation term as

$$\mathcal{Q}V \quad \to \quad \hat{\mathcal{Q}}\hat{V}, \quad \text{where} \quad \hat{V} = V + V_{\rm gh}. \tag{4.1.25}$$

In fact, this deformation does not change the localization procedure. This is because $Q_B V$ generates just gauge transformations, and hence does not give any changes. Also $\mathcal{Q}V_{\rm gh}$ generates $\bar{c}\,\mathcal{Q}(G(\tilde{A}) + \xi/2B)$ since $\mathcal{Q}\bar{c} = 0$. We can absorb the term proportional to $\bar{c}$ into the definition of c [20].

[3] We also impose the smoothness condition on the solution. In particular, if we take $\Delta = 0$, the only solution is $\phi = $ const. See appendix B.1.1 for details.

4.1.4 One-loop determinant

We compute the one-loop determinant around the saddle point by the index theorem [66]. Although we can calculate it directly as in [59], First we introduce the bosonic and fermionic coordinates (X_0, X_1) as,

$$X_0 = (X_0^{\rm vec}; X_0^{\rm chi}) = (\tilde{A}_\mu \; ; \phi, \bar{\phi}), \quad X_1 = (X_1^{\rm vec}; X_1^{\rm chi}) = (\Lambda, c, \bar{c} \; ; \epsilon\psi, \bar{\epsilon}\bar{\psi}), \quad (4.1.26)$$

where

$$\Lambda = \bar{\epsilon}\lambda + \epsilon\bar{\lambda}, \quad (4.1.27)$$

and the remaining fields are expressed by $\hat{\mathcal{Q}}X_0$ and $\hat{\mathcal{Q}}X_1$. We set the quadratic fluctuation of $\hat{V}$ in $\hat{\mathcal{Q}}\hat{V}$ to

$$\hat{V}|_{\rm quad} = (\hat{\mathcal{Q}}\Psi)^\dagger \Psi|_{\rm quad} = (\hat{\mathcal{Q}}X_0, X_1) \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} X_0 \\ \hat{\mathcal{Q}}X_1 \end{pmatrix}, \quad (4.1.28)$$

where the notation is just symbolical, and Ψ is a fermion field. Note that $\hat{\mathcal{Q}}$ is a fermionic operator. Then we can write the quadratic fluctuation part of $\hat{\mathcal{Q}}\hat{V}$ as

$$\begin{aligned} \hat{\mathcal{Q}}\hat{V}|_{\rm quad} &= -(X_0, \hat{\mathcal{Q}}X_1) \begin{pmatrix} \hat{\mathcal{Q}}^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} X_0 \\ \hat{\mathcal{Q}}X_1 \end{pmatrix} \\ &\quad -(\hat{\mathcal{Q}}X_0, X_1) \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \hat{\mathcal{Q}}^2 \end{pmatrix} \begin{pmatrix} \hat{\mathcal{Q}}X_0 \\ X_1 \end{pmatrix}. \end{aligned} \quad (4.1.29)$$

Comparing $\hat{\mathcal{Q}}\hat{V}|_{\rm quad} = X_{\rm bos}K_{\rm bos}X_{\rm bos} + X_{\rm ferm}K_{\rm ferm}X_{\rm ferm}$, we have

$$K_{\rm bos} = -\begin{pmatrix} \hat{\mathcal{Q}}^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix}, \quad K_{\rm ferm} = -\begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \hat{\mathcal{Q}}^2 \end{pmatrix}. \quad (4.1.30)$$

Then we can write the one-loop determinant as

$$Z_{\text{1-loop}} = \left(\frac{\det K_{\rm ferm}}{\det K_{\rm bos}}\right)^{1/2} = \left(\frac{\det_{X_1} \hat{\mathcal{Q}}^2}{\det_{X_0} \hat{\mathcal{Q}}^2}\right)^{1/2}. \quad (4.1.31)$$

Since D_{10} commutes with $\hat{\mathcal{Q}}^2$, we focus on the operator D_{10}. We find that there is a cancellation between the map ${\rm Im}D_{10}$ and ${\rm Im}D_{10}^\dagger$. Therefore the one-loop determinant is simply given by

$$Z_{\text{1-loop}} = \left(\frac{\det \hat{\mathcal{Q}}^2|_{{\rm CoKer}D_{10}}}{\det \hat{\mathcal{Q}}^2|_{{\rm Ker}D_{10}}}\right)^{1/2}. \quad (4.1.32)$$

Incidentally the equivariant index for the operator D_{10} is defined by

$$\mathrm{ind}D_{10} = \mathrm{Tr}_{\mathrm{Ker}D_{10}} e^{\hat{\mathcal{Q}}^2} - \mathrm{Tr}_{\mathrm{CoKer}D_{10}} e^{\hat{\mathcal{Q}}^2}. \tag{4.1.33}$$

Although this operator is infinite-dimensional, the $\mathrm{Ker}D_{10}$ and $\mathrm{CoKer}D_{10}$ are decomposed into direct sums of the eigenspaces of $\hat{\mathcal{Q}}^2$ with the weight ω_j formally,

$$\mathrm{Ker}D_{10} = \bigoplus_j V_{\omega_j}, \quad \mathrm{CoKer}D_{10} = \bigoplus_j V'_{\omega_j}. \tag{4.1.34}$$

Then the one-loop determinant and the index are respectively,

$$Z_{\text{1-loop}} = \prod_j \omega_j^{\frac{1}{2}(\mathrm{dim}V_{\omega_j} - \mathrm{dim}V'_{\omega_j})}, \quad \mathrm{ind}D_{10} = \sum_j \left(\mathrm{dim}V'_{\omega_j} - \mathrm{dim}V_{\omega_j}\right) e^{\omega_j}. \tag{4.1.35}$$

Therefore we know the equivariant index $\mathrm{ind}D_{10}$ and the one-loop determinant are related by the rule

$$\mathrm{ind}D_{10} = \sum_j c_j e^{w_j} \quad \leftrightarrow \quad Z_{\text{1-loop}} = \prod_j w_j^{-\frac{c_j}{2}}. \tag{4.1.36}$$

Note that although these operators are infinite dimensional, the index is well-defined when D_{10} is at least transversally elliptic [67]. In fact we note that the D_{10} in our cases, which we treat in this paper, is transversally elliptic. Thus our problem is reduced to computing the equivariant index.

In order to obtain the index, we would like to use the index theorem[4] as in [11, 68, 35, 69]. Roughly speaking, the index theorem states that if there are fixed points on a space with respect to a group action, the index is a sum of indices computed at each fixed point. However, we note that there is no fixed point on S^3_b with respect to the action generated by $\hat{\mathcal{Q}}^2$ (4.1.24). The authors of [66] have resolved this problem as follows. First we rewrite the vector field v in terms of the Hopf fibration coordinates (θ, φ, ψ)[5] as

$$v = \bar{\epsilon}\gamma^\mu \epsilon\, \partial_\mu = \frac{1}{R}\Big(b^{-1}\partial_{\varphi_1} + b\,\partial_{\varphi_2}\Big) = \frac{1}{R}\Big((b+b^{-1})\partial_\psi + (b-b^{-1})\partial_\varphi\Big). \tag{4.1.37}$$

Here we have two $U(1)$ actions generated by ∂_ψ and ∂_φ. In particular, ∂_ψ rotates the Hopf fiber and acts on S^3_b freely. In fact we can show that each D_{10} in vector and chiral multiplets is transversally elliptic with respect to these actions. It is known that when part of the group action is free, a transversally elliptic operator can be reduced to that on the quotient space [67]. Namely, D_{10} is reduced to that on the base S^2 in our case.

[4]We summarize the statement in appendix C.1.

[5]The relation with torus fibration coordinates $(\vartheta, \varphi_1, \varphi_2)$ is given as $\vartheta = \theta/2$, $\varphi_1 = 1/2(\psi - \varphi)$ and $\varphi_2 = 1/2(\psi + \varphi)$. See appendix A.2 for details.

Then the index theorem says that we have only to compute the contributions from fixed points of the ∂_φ-action, which are the north pole ($\theta = 0$) and south pole ($\theta = \pi$). As a result, we can find the indices are (see appendix C.2 for details)

$$\begin{aligned} \mathrm{ind}(D_{10}^{\mathrm{vec}}) &= -\sum_{n\in\mathbb{Z}} e^{inb} \sum_{\alpha} e^{\alpha(\hat{\sigma})} - \sum_{n\in\mathbb{Z}} e^{inb^{-1}} \sum_{\alpha} e^{\alpha(\hat{\sigma})}, \qquad (4.1.38) \\ \mathrm{ind}(D_{10}^{\mathrm{chi}}) &= 2\Big(\exp\Big[\prod_{\omega\in R} \prod_{m=0}^{\infty} \prod_{n=0}^{\infty} i\Big\{ mb + nb^{-1} + \frac{Q}{2} - i\omega(\hat{\sigma}) - \frac{Q}{2}(1-\Delta) \Big\} \Big] \\ &\quad - \exp\Big[\prod_{\omega\in R} \prod_{m=0}^{\infty} \prod_{n=0}^{\infty} -i\Big\{ mb + nb^{-1} + \frac{Q}{2} + i\omega(\hat{\sigma}) + \frac{Q}{2}(1-\Delta) \Big\} \Big] \Big), \qquad (4.1.39) \end{aligned}$$

where $Q \equiv b + b^{-1}$, $\hat{\sigma} \equiv R\sigma$, and ω and α denote the weights in representation R and the roots in the gauge group, respectively. Thus applying the rule (4.1.36), we find to the one-loop determinants up to overall factors,

$$Z_{\mathrm{vec}}^{(\text{1-loop})} = \prod_{\alpha>0} \sinh(\pi b\alpha(\hat{\sigma})) \sinh(\pi b^{-1}\alpha(\hat{\sigma})), \qquad (4.1.40)$$

$$Z_{\mathrm{chi}}^{(\text{1-loop})} = \prod_{w\in R} s_b\left(\frac{iQ}{2}(1-\Delta) - w(\hat{\sigma}) \right), \qquad (4.1.41)$$

where $s_b(x)$ is the double sine function defined by

$$s_b(x) = \prod_{m,n=0}^{\infty} \frac{mb + nb^{-1} + Q/2 - ix}{mb + nb^{-1} + Q/2 + ix}. \qquad (4.1.42)$$

Result

The classical part is for the saddle point $\hat{\sigma} = R\sigma$, which is constant value in Cartan subalgebra of the gauge group,

$$Z_{\mathrm{cl}}[\hat{\sigma}] = e^{-(S_{\mathrm{CS}}[\hat{\sigma}] + S_{\mathrm{FI}}[\hat{\sigma}])} = e^{i\pi\kappa \mathrm{Tr}(\hat{\sigma})^2 - 2\pi i\zeta \mathrm{Tr}(\hat{\sigma})}, \qquad (4.1.43)$$

where we used $\int_{S_b^3} d^3x\sqrt{g}\, f^{-1} = 2\pi^2 R^3$. The one-loop parts are (4.1.40) and (4.1.41). Therefore the result is

$$Z = \frac{1}{|\mathcal{W}|} \int \Big(\prod_{a=1}^{\mathrm{rank}G} d\hat{\sigma}_a \Big) Z_{\mathrm{cl}}[\hat{\sigma}] \cdot Z_{\mathrm{chi}}^{(\text{1-loop})}[\hat{\sigma}] \cdot Z_{\mathrm{vec}}^{(\text{1-loop})}[\hat{\sigma}], \qquad (4.1.44)$$

where $|\mathcal{W}|$ is the order of the Weyl group. Note that the Vandermonde determinant, which comes from restricting the integration variable to the Cartan subalgebra, cancels against the one-loop determinant for gauge fixing ghosts.

4.2 Partition function on $S^1 \times S^2$

4.2.1 Superconformal index

In this section we consider the partition function on $S^1 \times S^2$ along [70, 65, 66][6]. First we consider $\mathbb{R} \times S^2$ (3.3.20), and compactify the $\mathbb{R}$ to S^1. From (3.3.23), however, we notice that we cannot impose the periodic boundary condition to the Killing spinors, i.e.

$$\epsilon(\tau + \beta) = e^{-\frac{\beta}{2}}\epsilon(\tau), \quad \bar{\epsilon}(\tau + \beta) = e^{\frac{\beta}{2}}\bar{\epsilon}(\tau). \tag{4.2.1}$$

Furthermore recall that assigning the quantum numbers to ϵ and $\bar{\epsilon}$ as

$$\mathcal{R}(\epsilon) = \epsilon, \qquad j_3(\epsilon) = -\frac{1}{2}\epsilon, \quad F_i(\epsilon) = 0, \tag{4.2.2}$$

$$\mathcal{R}(\bar{\epsilon}) = -\bar{\epsilon}, \quad j_3(\bar{\epsilon}) = \frac{1}{2}\epsilon, \quad F_i(\bar{\epsilon}) = 0, \tag{4.2.3}$$

where $\mathcal{R}$ is an R-symmetry generator, j_3 is a generator for $SU(2)$ isometry on S^2 and F_i are the Cartan generators of the flavor symmetries, then we can rewrite the above twisted boundary conditions (4.2.1) as

$$\epsilon(\tau + \beta) = e^{\beta_1(-\mathcal{R}-j_3)+\beta_2 j_3 - i\sum_i \gamma_i F_i}\,\epsilon(\tau), \tag{4.2.4}$$

$$\bar{\epsilon}(\tau + \beta) = e^{\beta_1(-\mathcal{R}-j_3)+\beta_2 j_3 - i\sum_i \gamma_i F_i}\,\bar{\epsilon}(\tau), \tag{4.2.5}$$

where $\beta_1 + \beta_2 = \beta$. We impose the same boundary condition to all fields,

$$(\text{fields})_{\tau+\beta} = e^{\beta_1(-\mathcal{R}-j_3)+\beta_2 j_3 - i\sum_i \gamma_i F_i}\,(\text{fields})_{\tau}. \tag{4.2.6}$$

In order to make this clearer, we redefine all the fields in the following way,

$$(\text{fields})_{\text{new}} := e^{-\frac{\tau}{\beta}\left\{\beta_1(-\mathcal{R}-j_3)+\beta_2 j_3 - i\sum_i \gamma_i F_i\right\}}\,(\text{fields})_{\text{original}}, \tag{4.2.7}$$

then the new fields are periodic in $\tau \sim \tau + \beta$. For example we consider the new Killing spinors,

$$\begin{aligned} \epsilon_{\text{new}} &= e^{-\frac{\tau}{\beta}\left\{\beta_1(-\mathcal{R}-j_3)+\beta_2 j_3 - i\sum_i \gamma_i F_i\right\}}\,\epsilon_{\text{original}} \\ &= e^{\frac{\tau}{2}}\,\epsilon_{\text{original}} = \frac{1}{\sqrt{2}}\begin{pmatrix} -e^{\frac{i}{2}(\theta-\phi)} \\ e^{\frac{i}{2}(-\theta-\phi)} \end{pmatrix}, \end{aligned} \tag{4.2.8}$$

[6] Here we take the S^2 radius $R = 1$ for simplicity. We can recover the dependence on R by a dimensional analysis.

$$\bar{\epsilon}_{\rm new} \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} e^{\frac{i}{2}(-\theta+\phi)} \\ e^{\frac{i}{2}(\theta+\phi)} \end{pmatrix}, \tag{4.2.9}$$

so they are τ-independent.

This redefinition is also equivalent to the following replacement,

$$\partial_\tau \quad\to\quad \partial_\tau + \frac{1}{\beta}\Big\{\beta_1(-\mathcal{R}-j_3)+\beta_2 j_3 - i\sum_i \gamma_i F_i\Big\}. \tag{4.2.10}$$

Therefore we find that the partition function on $S^1\times S^2$ is expressed as a path integral with a periodic S^1-direction which is redefined in (4.2.10). This quantity is known as the superconformal index, and is also expressed as

$$\mathcal{I} = \mathrm{Tr}\,(-1)^{\mathrm{F}} e^{-\beta_1(\mathcal{H}-\mathcal{R}-j_3)} e^{-\beta_2(\mathcal{H}+j_3)} e^{i\sum_i \gamma_i F_i}, \tag{4.2.11}$$

where F is the fermion number operator, $\mathcal{H}=-\partial_\tau$ is the Hamiltonian. Since $\mathcal{H}-\mathcal{R}-j_3 = i\{\delta_\epsilon,\delta_{\bar\epsilon}\}$, this quantity counts only δ_ϵ and $\delta_{\bar\epsilon}$-invariant BPS states. Because if there is a state $|\varphi\rangle$ s.t. $\{\delta_\epsilon,\delta_{\bar\epsilon}\}|\varphi\rangle \neq 0$, the corresponding superpartner always exists. Furthermore since $(\mathcal{H}+j_3)$ and F_i commute with $\{\delta_\epsilon,\delta_{\bar\epsilon}\}$, we can define the superconformal index more generally as (4.2.11). From this discussion, we find that the superconformal index is independent of β_1, β_2 and γ_i.

4.2.2 Localization

Vector multiplet

The Lagrangian on $S^1\times S^2$, which we defined in (3.3.20), is given by [65][7]

$$S_{\rm YM} = \frac{1}{g_{\rm YM}^2}\int d^3x\sqrt{g}\,\mathrm{Tr}\Big[\,\frac{1}{2}V_\mu V^\mu + \frac{1}{2}D^2 + \frac{i}{2}\bar\lambda\gamma^\mu D_\mu\lambda + \frac{i}{2}\bar\lambda[\sigma,\lambda] + \frac{i}{4}\bar\lambda\gamma_1\lambda\,\Big], \tag{4.2.12}$$

where $V_a = V_\mu\, e^\mu_a$ is defined by

$$V_a = \frac{1}{2}\epsilon_{abc}F^{bc} - D_a\sigma + \delta_{a1}\sigma. \tag{4.2.13}$$

We choose a $\mathcal{Q}$-exact term in the same manner of (4.1.11):

$$\frac{1}{4}\mathcal{Q}\,\mathrm{Tr}\left[(\mathcal{Q}\lambda)^\dagger\lambda + (\mathcal{Q}\bar\lambda)^\dagger\bar\lambda\right]\Big|_{\rm bos} = \frac{1}{2}\mathrm{Tr}\Big[\,(F_{23}+\sigma)^2 + F_{31}F^{31} + F_{12}F^{12} + (D_\mu\sigma)^2 + D^2\,\Big],$$

[7] See appendix A.3 for the supersymmetry transformations on $S^1\times S^2$. Also note that we use the fermionic supercharge $\mathcal{Q}$ as we did in the ellipsoid case (4.1.2).

$$(4.2.14)$$

Note that the whole exact term is equivalent to the SYM Lagrangian up to total derivatives. The superconformal index is independent of the gauge coupling as we expected. From this, we can find the following saddle points,

$$F_{23}+\sigma=0, \quad F_{31}=F_{12}=0, \quad D_\mu\sigma=0, \quad D=0. \tag{4.2.15}$$

Using the quantization condition for the flux on S^2, the configurations for the gauge and scalar fields are[8]

$$F_{23}=\frac{m}{2}, \quad \sigma=-\frac{m}{2}, \quad \text{with} \;\; m=\frac{1}{2\pi}\int_{S^2}F, \tag{4.2.16}$$

where m is a magnetic charge (GNO charge) which takes values in the Cartan subalgebra, and the root and weight are integer values, $\alpha(m)$, $\rho(m)\in\mathbb{Z}$. In conclusion, the localized configurations are

$$A_\tau=-\frac{a}{\beta}, \quad A_\theta=0, \quad A_\varphi^{\pm}=\frac{m}{2}(\pm 1-\cos\theta), \quad \sigma=-\frac{m}{2}, \quad D=0, \tag{4.2.17}$$

where a is a holonomy around the S^1, and $A_\varphi^{\pm}$ denote the sections on the patches including the north$(+)$ and south$(-)$ poles, respectively.

Also we add the CS term (4.1.7) and FI term. The FI term on $S^1\times S^2$ is given by

$$S_{\rm FI}=-\frac{i\zeta}{2\pi}\int d^3x\sqrt{g}\,(D-A_1)\,. \tag{4.2.18}$$

Chiral multiplet

The exact term for the chiral multiplet is

$$\begin{aligned}
\mathcal{L}_\psi|_{\rm bos} &= \frac{1}{2}\mathcal{Q}\,{\rm Tr}\left[(\mathcal{Q}\psi)^\dagger\psi+(\mathcal{Q}\bar{\psi})^\dagger\bar{\psi}\right]\Big|_{\rm bos}\\
= \;& |D_1\phi|^2+\frac{1}{2}\Big|\sin\frac{\theta}{2}(D_-\phi+F)+\cos\frac{\theta}{2}(\sigma+\Delta)\phi\Big|^2\\
&+\frac{1}{2}\Big|\sin\frac{\theta}{2}(D_-\phi-F)+\cos\frac{\theta}{2}(\sigma+\Delta)\phi\Big|^2+\frac{1}{2}\Big|\cos\frac{\theta}{2}(D_+\phi+F)+\sin\frac{\theta}{2}(\sigma-\Delta)\phi\Big|^2\\
&+\frac{1}{2}\Big|\cos\frac{\theta}{2}(D_+\phi-F)+\sin\frac{\theta}{2}(\sigma-\Delta)\phi\Big|^2,
\end{aligned} \tag{4.2.19}$$

where $D_\pm=D_2\mp iD_3$, and we take the reality condition (2.1.13). We read off the localized configurations,

$$D_1\phi=0, \quad F=0,$$

[8] See appendix B.2.1 for details.

$$\sin\frac{\theta}{2}D_{-}\phi+\cos\frac{\theta}{2}(\sigma+\Delta)\phi=0,\quad \cos\frac{\theta}{2}D_{+}\phi+\sin\frac{\theta}{2}(\sigma-\Delta)\phi=0. \tag{4.2.20}$$

For the generic R-charge Δ, the saddle point configurations are[9]

$$\phi=\bar{\phi}=F=\bar{F}=0. \tag{4.2.21}$$

Although we do not express the Lagrangian for the chiral multiplet on $S^1\times S^2$ explicitly, we find that it does not any contributions to the final result using the above $\mathcal{Q}$-exact term.

Localization

Using the above Killing spinors (4.2.8), (4.2.9), the square generates

$$\mathcal{Q}^2=i\mathcal{L}_v+i(iv^\mu A_\mu+\sigma\bar{\epsilon}\epsilon)+i\mathcal{R}+\frac{i}{\beta}\Big\{\beta_1(-\mathcal{R}-j_3)+\beta_2 j_3-i\sum_i\gamma_i F_i\Big\}, \tag{4.2.22}$$

where

$$v=(\bar{\epsilon}\gamma^\mu\epsilon)\partial_\mu=\partial_\tau-i\partial_\varphi,\quad \bar{\epsilon}\epsilon=-\cos\theta. \tag{4.2.23}$$

We can understand the contribution in the last parentheses from the discussion in the last section. Substituting the localized configurations for this,

$$\mathcal{Q}^2=i\mathcal{L}_{\partial_\tau}+\frac{a}{\beta}+i\,\frac{\beta_2}{\beta}\,(2j_3+\mathcal{R})+\frac{1}{\beta}\sum_i\gamma_i F_i, \tag{4.2.24}$$

where we have used the relation $j_3=-i\partial_\varphi\pm\frac{\rho(m)}{2}$, which is an expression of the eigenvalue on the monopole background [71].

For the gauge fixing, we have only to perform the same prescription as we did in section 4.1.3. Finally we have to consider the 1-loop determinants around the localized configurations (4.2.17) and (4.2.21). In the same way as the ellipsoid case, we apply the index theorem. In fact, we find that the $D_{10}^{\rm vec}$ and $D_{10}^{\rm chi}$ are transversally elliptic with respect to the vector field ∂_τ, so they reduces to those on S^2. In conclusion, from the each index, the one-loop determinants are obtained (see appendix C.3 for details.): For the vector multiplet,

$$Z_{\rm vec}^{(\text{1-loop})} = \prod_{\alpha>0}\left[2\sinh\left(\frac{i}{2}\alpha(a)+\frac{1}{2}\alpha(m)\beta_2\right)\right]\left[2\sinh\left(\frac{i}{2}\alpha(a)-\frac{1}{2}\alpha(m)\beta_2\right)\right]$$

[9] See appendix B.2.2 for details.

$$= \prod_{\alpha\in\,\mathrm{adj}} x^{-\frac{|\alpha(m)|}{2}}\left(1-e^{-i\alpha(a)}x^{|\alpha(m)|}\right), \tag{4.2.25}$$

where we define $x = e^{-\beta_2}$. For the chiral multiplet,

$$Z_{\text{chi}}^{\text{(1-loop)}} = \prod_{\rho\in R}\left(x^{(1-\Delta)}e^{i\rho(a)}\prod_i \xi_i^{F_i}\right)^{-\frac{\rho(m)}{2}} \frac{(x^{-\rho(m)+2-\Delta}e^{i\rho(a)}(\prod_i \xi_i^{F_i});x^2)_\infty}{(x^{-\rho(m)+\Delta}e^{-i\rho(a)}(\prod_i \xi_i^{-F_i});x^2)_\infty}, \tag{4.2.26}$$

where $\xi_i = e^{i\gamma_i}$ (flavor fugacity), and $(a;q)_\infty$ is the q-Pochhammer symbol defined by

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k). \tag{4.2.27}$$

Note that this result is regularized form [70, 65, 66].

Result

The classical contribution is

$$Z_{\text{cl}}[a,m] = e^{-(S_{\text{CS}}[a,m]+S_{\text{FI}}[a,m])} = e^{i\pi\kappa\text{Tr}(am)-2\pi i\zeta\text{Tr}(a)}, \tag{4.2.28}$$

where we used $\int_{S^1\times S^2} d^3x\sqrt{g} = 4\pi^2\beta$.

Recall that the gauge group is associated with the topological $U(1)$ symmetry. In addition to the above contribution, we can consider a BF term between the topological symmetry and the background gauge field, $\int A_{\text{BG}}\wedge dA+\cdots$ [72]. This contribution to the index is

$$e^{i\text{Tr}(an)}\omega^{\text{Tr}(m)}, \tag{4.2.29}$$

where we have used (4.2.17), and n is a flux for the topological symmetry and take discrete values, and ω is a chemical potential for the topological symmetry. In fact, this first factor corresponds to the above FI term.

Finally we introduce a gauge fugacity $z_j = e^{ia_j}$ (so the contour is counterclockwise). Then the superconformal index is

$$\begin{aligned}\mathcal{I} &= \text{Tr}\,(-1)^{\text{F}}e^{-\beta_1(\mathcal{H}-\mathcal{R}-j_3)}e^{-\beta_2(\mathcal{H}+j_3)}e^{i\sum_i\gamma_iF_i}\\ &= \frac{1}{|\mathcal{W}|}\sum_{m\in\mathbb{Z}^{\text{rank}G}}\oint\Big(\prod_{j=1}^{\text{rank}G}\frac{dz_j}{2\pi i z_j}\Big)Z_{\text{cl}}\cdot Z_{\text{vec}}^{\text{(1-loop)}}\cdot Z_{\text{chi}}^{\text{(1-loop)}},\end{aligned} \tag{4.2.30}$$

where $|\mathcal{W}|$ implies the order of the Weyl group.

Chapter 5

Factorization

In this chapter we show that for a certain class of 3d $\mathcal{N} = 2$ theories, these partition functions factorize into products of 3d vortex and anti-vortex partition functions as well as the other factors following [31, 33]. Then we summarize some questions for motivating us to consider the Higgs branch localization.

5.1 Ellipsoid partition function

Abelian theory

$U(1)$ theory with $2N_f$ fundamental multiplets

Let us consider a $U(1)$ theory with $2N_f$-fundamental chiral multiplets with real masses on S_b^3. Here we take even number of fundamental chiral multiplets since otherwise we have the parity anomaly. For simplicity we set the R-charge to zero, and do not include the CS term. Then the partition function reads from (4.1.44),[1]

$$Z = \int_{-\infty}^{\infty} dx\, e^{-2\pi i\zeta x} \prod_{j=1}^{2N_f} s_b\Big(-x - m_j + \frac{iQ}{2}\Big), \tag{5.1.1}$$

where we set $\hat{\sigma} = x$, $R = 1$, $\zeta > 0$, and include real masses m_j. We would like to perform this integral exactly. Since the double sine function is

$$s_b(x) = \prod_{m,n=0}^{\infty} \frac{mb + nb^{-1} + Q/2 - ix}{mb + nb^{-1} + Q/2 + ix}, \tag{5.1.2}$$

[1]Note that our notation is different from [31, 33] (e.g. the weights have opposite signs.).

so (5.1.1) has the following simple poles,

$$x = -m_j - i(mb + nb^{-1}), \quad m, n \in \mathbb{Z}_{\geq 0}, \quad i = 1, 2, \cdots, 2N_f. \tag{5.1.3}$$

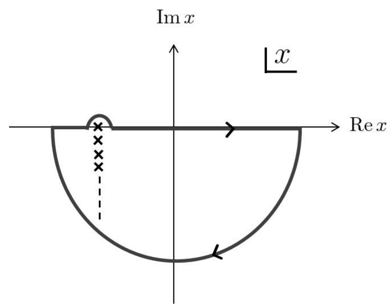


Figure 5.1: The path of the contour integral

We should choose the contour as shown in Fig.5.1 to make the integral converge at infinity. We also use the following formulas,

$$\frac{s_b(x + \frac{iQ}{2} + imb + inb^{-1})}{s_b(x + \frac{iQ}{2})} = \frac{(-1)^{mn}}{\prod_{k=1}^{m} 2i \sinh \pi b(x + ikb) \prod_{l=1}^{n} 2i \sinh \pi b(x + ilb^{-1})}, \tag{5.1.4}$$

$$\frac{s_b(x - \frac{iQ}{2} + imb + inb^{-1})}{s_b(x - \frac{iQ}{2})} = \frac{(-1)^{mn}}{\prod_{k=1}^{m} 2i \sinh \pi b(x - iQ + ikb) \prod_{l=1}^{n} 2i \sinh \pi b(x - iQ + ilb^{-1})}. \tag{5.1.5}$$

Therefore, the partition function is

$$Z = \sum_{i=1}^{2N_f} e^{2\pi i \zeta m_i} \prod_{j \neq i}^{2N_f} s_b\Big(E_{ji} + \frac{iQ}{2}\Big) \cdot Z_V^{(i)} \cdot \bar{Z}_V^{(i)}, \tag{5.1.6}$$

where $E_{ji} = -(m_j - m_i)$, and $Z_V^{(i)}$ and $\bar{Z}_V^{(i)}$ are expected as 3d vortex and anti-vortex partition functions on $S^1 \times \mathbb{R}^2$ [73],

$$Z_V^{(i)} = = \sum_{n=0}^{\infty} \frac{(-1)^{nN_f} e^{-2\pi\zeta b^{-1} n}}{\prod_{l=1}^{n} 2 \sinh \pi i b^{-2}(l - 1 - n) \prod_{l=1}^{n} \prod_{j \neq i}^{2N_f} 2 \sinh \pi b^{-1}(E_{ji} + ilb^{-1})}, \tag{5.1.7}$$

$$\bar{Z}_V^{(i)} = = \sum_{m=0}^{\infty} \frac{(-1)^{mN_f} e^{-2\pi\zeta bm}}{\prod_{l=1}^{m} 2 \sinh \pi i b^{2}(l - 1 - m) \prod_{l=1}^{m} \prod_{j \neq i}^{2N_f} 2 \sinh \pi b(E_{ji} + ilb)}. \tag{5.1.8}$$

Note that $Z_V^{(i)}|_{b^{-1} \to b} = \bar{Z}_V^{(i)}$ and $\bar{Z}_V^{(i)}|_{b \to b^{-1}} = Z_V^{(i)}$. In fact we used the relation $(-1)^{2N_f mn} = 1$ $(m, n \in \mathbb{Z}_{\geq 0})$ to obtain the above factorization form.

Furthermore even if we add a CS term, we can also evaluate it using the above contour [31]. The contribution at each pole is

$$e^{i\pi\kappa(-m_i-imb-inb^{-1})^2} = e^{i\pi\kappa m_i^2}e^{-2\pi\kappa m_i(mb+nb^{-1})}e^{-i\pi\kappa(m^2b^2+n^2b^{-2})}(-1)^{2\kappa mn}. \tag{5.1.9}$$

From this, if we set the bare CS level κ as an integer, the above factorization property is not spoiled. In addition to the above condition for the number of matters $2N_f$, this fact is associated with the condition that the effective CS level must be an integer (2.2.1):

$$\kappa_{\text{eff}} = \kappa + \frac{1}{2}\cdot 2N_f \in \mathbb{Z}. \tag{5.1.10}$$

We emphasize that the condition for the factorization corresponds to the parity anomaly cancellation condition.

$U(1)$ theory with N_f-flavors

In the same way, let us consider a $U(1)$ theory with N_f-flavors (a pair of fundamental and anti-fundamental representations for each) on S_b^3. First the partition function obtained by the Coulomb branch localization is

$$\begin{aligned} Z &= \int_{-\infty}^{\infty} dx\, e^{i\pi\kappa x^2-2\pi i\zeta x}\prod_{j=1}^{N_f} s_b\Big(-x-m_j+\frac{iQ}{2}\Big)\cdot s_b\Big(x+\tilde{m}_j+\frac{iQ}{2}\Big) \\ &= \int_{-\infty}^{\infty} dx\, e^{i\pi\kappa x^2-2\pi i\zeta x}\prod_{j=1}^{N_f}\frac{s_b\big(-x-(m_j^{(v)}+m_j^{(a)})+\frac{iQ}{2}\big)}{s_b\big(-x-(m_j^{(v)}-m_j^{(a)})-\frac{iQ}{2}\big)}, \end{aligned} \tag{5.1.11}$$

where in the second line, we defined $m^{(v)} = \frac{1}{2}(m+\tilde{m})$, $m^{(a)} = \frac{1}{2}(m-\tilde{m})$ as the vector mass and axial mass (2.2.4), and we used the relation $s_b(x) = 1/s_b(-x)$. Then, poles are,

$$\text{(fundamental:)} \qquad x = -(m_j^{(v)}+m_j^{(a)}) - i(mb+nb^{-1}), \tag{5.1.12}$$

$$\text{(anti-fundamental:)} \qquad x = -(m_j^{(v)}-m_j^{(a)}) + i(mb+nb^{-1}). \tag{5.1.13}$$

Note that the poles for the anti-fundamentals are in the upper-plane. So as we found in the chiral theory, evaluating the poles for the fundamentals, we can obtain the following result,

$$Z = \sum_{i=1}^{N_f}\frac{e^{i\pi\kappa(m_i^{(v)}+m_i^{(a)})^2+2\pi i\zeta(m_i^{(v)}+m_i^{(a)})}}{s_b(C_{ii}-\frac{iQ}{2})}\prod_{A\neq i}^{N_f}\frac{s_b(D_{Ai}+\frac{iQ}{2})}{s_b(C_{Ai}-\frac{iQ}{2})}\cdot Z_{\text{V}}^{(i)}\cdot\bar{Z}_{\text{V}}^{(i)}, \tag{5.1.14}$$

where $D_{ji} = -(m_j^{(v)} - m_i^{(v)}) - (m_j^{(a)} - m_i^{(a)})$, $C_{ji} = -(m_j^{(v)} - m_i^{(v)}) + (m_j^{(a)} + m_i^{(a)})$, and

$$\begin{aligned} & Z_{\rm V}^{(i)} \\ = & \sum_{n=0}^{\infty} \frac{(-1)^{N_f n} e^{-i\pi\kappa b^{-2} n^2} e^{-2\pi\{\kappa(m_i^{(v)}+m_i^{(a)})+\zeta\}b^{-1}n} \prod_{l=1}^{n}\prod_{j=1}^{N_f} 2\sinh \pi b^{-1}(C_{ji}+(l-1)ib^{-1})}{\prod_{l=1}^{n} 2\sinh \pi i b^{-2}(l-1-n) \prod_{l=1}^{n}\prod_{j\neq i}^{N_f} 2\sinh \pi b^{-1}(D_{ji}+ilb^{-1})}, \end{aligned} \tag{5.1.15}$$

and $\bar{Z}_{\rm V}^{(i)} = Z_{\rm V}^{(i)}|_{b^{-1}\to b}$. As before, we used the parity anomaly cancellation condition to obtain the factorization form,

$$\kappa_{\rm eff} = \kappa + \frac{1}{2}(N_f - N_f) = \kappa \in \mathbb{Z}. \tag{5.1.16}$$

Non-Abelian theory

In the non-Abelian case, we can apply the Cauchy formula,

$$\prod_{i<j}^{N} 2\sinh(x_i - x_i) = \frac{1}{\prod_{i<j}^{N} 2\sinh(\chi_i - \chi_i)} \sum_{\sigma\in S^N} (-1)^\sigma \prod_{i=1}^{N}\prod_{j\neq\sigma(i)}^{N} 2\cosh(x_i - \chi_j), \tag{5.1.17}$$

where χ_i is an auxiliary field such that $\chi_i \neq \chi_j$, (mod πi). Applying it to the 1-loop determinant for the $U(N)$ vector multiplet, the N-multiple integral simply reduces to the one-dimensional one. Therefore we can evaluate the non-Abelian theories similarity to the Abelian case [33]. We summarize just these results:

$U(N)$ theory with $2N_f$ fundamental multiplets

The partition function we obtained using the Coulomb branch localization (4.1.44) is

$$\begin{aligned} Z &= \frac{1}{N!}\int d^N x \; e^{i\pi\kappa\sum_{i=1}^N x_i^2 - 2\pi i\zeta\sum_{i=1}^N x_i} \prod_{i<j}^{N}\Big[\sinh \pi b(x_i - x_j)\sinh \pi b^{-1}(x_i - x_j)\Big] \\ & \quad \times \prod_{i=1}^{N}\prod_{a=1}^{2N_f} s_b\Big(-x_i - m_a + \frac{iQ}{2}\Big). \end{aligned} \tag{5.1.18}$$

Evaluating poles, the result is

$$Z = \sum_{(l_1,\cdots,l_N)\subset(1,\cdots,2N_f)} e^{i\pi\kappa\sum_{i=1}^N m_{l_i}^2 + 2\pi i\zeta\sum_{i=1}^N m_{l_i}} \prod_{i<j}^{N}\Big[\sinh(\pi b E_{l_i l_j})\sinh(\pi b^{-1} E_{l_i l_j})\Big]$$

$$\times \prod_{i=1}^{N} \prod_{A \neq \{l_i\}}^{2N_f} s_b\Big(E_{A\,l_i} + \frac{iQ}{2}\Big) \cdot Z_{\rm V}^{\{l_i\}} \cdot \bar{Z}_{\rm V}^{\{l_i\}}, \tag{5.1.19}$$

where $E_{AB} = -(m_A - m_B)$, and the summation over $(l_1, \cdots, l_N) \subset (1, \cdots, 2N_f)$ means ${}_{2N_f}C_N$ combinations, and

$$\begin{aligned} &\bar{Z}_{\rm V}^{\{l_i\}} \\ =& \sum_{\vec{k}=0}^{\infty} \frac{\prod_{i=1}^{N} (-1)^{(N+N_f)k_i} e^{-i\pi\kappa b^2 k_i^2} e^{-2\pi b(\kappa m_{l_i} + \zeta)k_i}}{\prod_{i,j}^{N} \prod_{l=1}^{k_i} 2\sinh \pi b(E_{l_j l_i} + (l-1-k_j)ib) \prod_{i=1}^{N} \prod_{A \neq \{l_i\}}^{2N_f} \prod_{l=1}^{k_i} 2\sinh \pi b(E_{A l_i} + ilb)}, \end{aligned} \tag{5.1.20}$$

where $\vec{k} = (k_1, \cdots, k_N)$, and $Z_{\rm V}^{\{l_i\}} = \bar{Z}_{\rm V}^{\{l_i\}}|_{b \to b^{-1}}$. Note that we also used the parity anomaly cancellation condition to obtain the above result.

$U(N)$ theory with N_f-flavors

The partition function we obtained using the Coulomb branch localization (4.1.44) is

$$\begin{aligned} Z &= \frac{1}{N!} \int d^N x \; e^{i\pi\kappa \sum_{i=1}^{N} x_i^2 - 2\pi i\zeta \sum_{i=1}^{N} x_i} \prod_{i<j}^{N} \Big[\sinh \pi b(x_i - x_j) \sinh \pi b^{-1}(x_i - x_j)\Big] \\ &\quad \times \prod_{i=1}^{N} \prod_{A=1}^{N_f} \frac{s_b(-x_i - (m_A^{(v)} + m_A^{(a)}) + \frac{iQ}{2})}{s_b(-x_i - (m_A^{(v)} - m_A^{(a)}) - \frac{iQ}{2})}. \end{aligned} \tag{5.1.21}$$

As we have seen in the Abelian case, this contour integral has the contribution from only the poles of the fundamental multiplets. The result is

$$\begin{aligned} Z &= \sum_{(l_1, \cdots, l_N) \subset (1, \cdots, N_f)} e^{i\pi\kappa \sum_{i=1}^{N} m_{l_i}^2 + 2\pi i\zeta \sum_{i=1}^{N} m_{l_i}} \\ &\quad \times \prod_{i<j}^{N} \Big[\sinh(\pi b D_{l_i l_j}) \sinh(\pi b^{-1} D_{l_i l_j})\Big] \prod_{i=1}^{N} \frac{\prod_{A \neq \{l_i\}}^{N_f} s_b(D_{A l_i} + \frac{iQ}{2})}{\prod_{B=1}^{N_f} s_b(C_{B l_i} - \frac{iQ}{2})} \cdot Z_{\rm V}^{\{l_i\}} \cdot \bar{Z}_{\rm V}^{\{l_i\}}, \end{aligned} \tag{5.1.22}$$

where $D_{AB} = -(m_A^{(v)} - m_B^{(v)}) - (m_A^{(a)} - m_B^{(a)})$, $C_{AB} = -(m_A^{(v)} - m_B^{(v)}) + (m_A^{(a)} + m_B^{(a)})$, and

$$\begin{aligned} \bar{Z}_{\rm V}^{\{l_i\}} = \sum_{\vec{k}=0}^{\infty} \Big\{ & \Big(\prod_{i=1}^{N} (-1)^{(N+N_f)k_i} e^{-i\pi\kappa b^2 k_i^2} e^{-2\pi b(\kappa m_{l_i} + \zeta)k_i}\Big) \\ \times & \frac{\prod_{i=1}^{N} \prod_{\beta=1}^{N_f} \prod_{l=1}^{k_i} 2\sinh \pi b(C_{\beta, l_i} + (l-1)ib)}{\prod_{i,j=1}^{N} \prod_{l=1}^{k_i} 2\sinh \pi b\big\{D_{l_j, l_i} + (l-1-k_j)ib\big\} \prod_{i=1}^{N} \prod_{\alpha \neq \{l_i\}}^{N_f} \prod_{l=1}^{k_i} 2\sinh \pi b(D_{\alpha, l_i} + ilb)} \Big\}, \end{aligned}$$

(5.1.23)

where $Z_{\rm V}^{\{l_i\}} = \bar{Z}_{\rm V}^{\{l_i\}}|_{b\to b^{-1}}$. In order to obtain the above result, we used the parity anomaly cancellation condition, $\kappa_{\rm eff} = \kappa + \frac{N_f - N_f}{2} = \kappa \in \mathbb{Z}$.

5.2 Superconformal index

In this section we stand just some results for the superconformal index. We should evaluate poles as we did in the ellipsoid case, and can obtain these results as follows [32].

$U(1)$ theory with $2N_f$ fundamental chiral multiplets

Let us consider a $U(1)$ theory with $2N_f$ fundamental chiral multiplets[2]. Here we do not include the CS term, and set R-charge $\Delta = 0$ for simplicity. Then using (4.2.30), the superconformal index is

$$\mathcal{I} = \sum_{m\in\mathbb{Z}} \oint \frac{dz}{2\pi i z}\, z^{-2\pi\zeta}\, \omega^m\, (xz)^{\frac{-m}{2}(2N_f)} \prod_{i=1}^{2N_f} \frac{(x^{-m+2}\, z\, \xi_i\, ; x^2)_\infty}{(x^{-m} z^{-1} \xi_i^{-1}\, ; x^2)_\infty}, \tag{5.2.1}$$

where ξ_i is fugacity for $SU(2N_f)$ flavor symmetry. Also since this theory has $U(1)_A$ flavor symmetry, we can rescale the flavor fugacity freely by introducing a $U(1)_A$ fugacity (but we do not consider it here). So even if we set the R-charge to zero, we reproduce a result for the nonzero R-charge by an appropriate rescaling [32].

Note that if the number of the chiral multiplets is not even, the above integrand becomes multi-valued. That is to say, the parity anomaly cancellation ($\kappa_{\rm eff} = 0 + \frac{1}{2}\cdot 2N_f = N_f \in \mathbb{Z}$) enables us to compute it in this case.

Evaluating the contour integral, if we identify the fugacity for the flavor symmetry as

$$\xi = e^{i\beta M}, \tag{5.2.2}$$

the the superconformal index is

$$\mathcal{I} \;=\; \sum_{i=1}^{2N_f} \prod_{j\neq i}^{2N_f} e^{2i\zeta\beta M_j} \frac{(e^{-i\beta(M_j - M_i)} x^2\, ; x^2)_\infty}{(e^{i\beta(M_j - M_i)}\, ; x^2)_\infty} \cdot Z_V^{(i)} \cdot \bar{Z}_V^{(i)}, \tag{5.2.3}$$

where

$$Z_V^{(i)} \;=\; \sum_{n=0}^{\infty} \left((-1)^{-nN_f} (-\omega)^n \prod_{k=1}^{n} \frac{1}{2\sinh \beta_2(k-1-n) \prod_{j\neq i}^{2N_f} 2\sinh \frac{i\beta(M_j - M_i) + 2\beta_2 k}{2}} \right), \tag{5.2.4}$$

[2] Note that our notation is slightly different from [32].

$$\bar{Z}_V^{(i)} = \sum_{n=0}^{\infty}\left((-1)^{-nN_f}(-\omega)^{-n}\prod_{k=1}^{n}\frac{1}{2\sinh\beta_2(k-1-n)\prod_{j\neq i}^{2N_f}2\sinh\frac{i\beta(M_j-M_i)+2\beta_2 k}{2}}\right). \quad (5.2.5)$$

Note that we defined $n \in \mathbb{Z}_{\geq 0}$ in the vortex partition function by also including a flux $m \in \mathbb{Z}$ in an appropriate way. The difference between the vortex and anti-vortex partition functions is only the contribution from the topological symmetry.

$U(N)$ theory with N_f fundamental and $\tilde{N}_f$ anti-fundamental chiral multiplets

Let us consider a $U(N)$ theory with N_f fundamental and $\tilde{N}_f$ anti-fundamental chiral multiplets. Here we also set R-charge $\Delta = 0$. Then using (4.2.30), the superconformal index is

$$\begin{aligned}\mathcal{I} &= \frac{1}{N!}\sum_{\vec{m}\in\mathbb{Z}^N}\oint\Big(\prod_{i=1}^{N}\frac{dz_i}{2\pi i z_i}\,z_i^{\pi\kappa m_i-2\pi\zeta}\,\omega^{\sum_{i=1}^{N}m_i}\Big)\prod_{\substack{i,j=1\\ i\neq j}}^{N}e^{-\frac{|m_i-m_j|}{2}}\left(1-z_i^{-1}z_j x^{|m_i-m_j|}\right)\\ &\quad\times\prod_{i=1}^{N}\prod_{A=1}^{N_f}(x\,z_i\,\xi_A)^{-\frac{m_i}{2}}\frac{(x^{-m_i+2}\,z_i\,\xi_A\,;x^2)_\infty}{(x^{-m_i}z_i^{-1}\xi_A^{-1}\,;x^2)_\infty}\prod_{B=1}^{\tilde{N}_f}(x\,z_i^{-1}\tilde{\xi}_B^{-1})^{\frac{m_i}{2}}\frac{(x^{m_i+2}\,z_i^{-1}\,\tilde{\xi}_B^{-1}\,;x^2)_\infty}{(x^{m_i}z_i\,\tilde{\xi}_B\,;x^2)_\infty}, \quad (5.2.6)\end{aligned}$$

where ξ_A and $\tilde{\xi}_B$ are fugacities for $SU(N_f)\times SU(\tilde{N}_f)$ flavor symmetry, respectively. As we mentioned above, using the $U(1)_A$ symmetry, we reproduce a nonzero R-charge result. In this case, the above integrand becomes also single-valued thanks to the parity anomaly cancellation condition $\kappa_{\text{eff}} = \kappa + \frac{N_f-\tilde{N}_f}{2} \in \mathbb{Z}$.

As the result, if we identify the fugacities for the flavor symmetry as[3]

$$\xi = e^{i\beta M}, \quad \tilde{\xi} = e^{-i\beta\tilde{M}}, \quad (5.2.7)$$

then the superconformal index becomes

$$\begin{aligned}\mathcal{I} &= \sum_{(l_1,\cdots,l_N)\subset(1,\cdots,N_f)}\Big(\prod_{i=1}^{N}e^{2\pi i\zeta\beta M_{l_i}}\Big)\prod_{\substack{i,j=1\\ i\neq j}}^{N}\Big[2\sinh\frac{-i\beta(M_{l_i}-M_{l_j})}{2}\Big]\\ &\quad\times\prod_{i=1}^{N}\left[\prod_{A\neq\{l_i\}}^{N_f}\frac{(e^{-i\beta(M_{l_i}-M_A)}x^2;x^2)_\infty}{(e^{i\beta(M_{l_i}-M_A)};x^2)_\infty}\prod_{B=1}^{\tilde{N}_f}\frac{(e^{i\beta(M_{l_i}-\tilde{M}_B)}x^2;x^2)_\infty}{(e^{-i\beta(M_{l_i}-\tilde{M}_B)};x^2)_\infty}\right]\cdot Z_V^{\{l_i\}}\cdot\bar{Z}_V^{\{l_i\}}, \quad (5.2.8)\end{aligned}$$

where

$$Z_V^{\{l_i\}} = \sum_{\vec{k}=0}^{\infty}(-1)^{\left(\kappa-\frac{N_f-\tilde{N}_f}{2}\right)\sum_{i=1}^{N}k_i}\,e^{-i\pi\kappa\sum_{j=1}^{N}(\beta M_{l_j}k_j+\beta_2 k_j^2)}\,(-\omega)^{\sum_{i=1}^{n}k_i}$$

[3] Note that the fugacity for the anti-fundamental matters has opposite sign differently from [32].

$$\times \frac{\prod_{i=1}^{N}\prod_{B=1}^{\tilde{N}_f}\prod_{l=1}^{k_i} 2\sinh\frac{i\beta(M_{l_i}-\tilde{M}_B)+2\beta_2(l-1)}{2}}{\prod_{i,j=1}^{N}\prod_{l=1}^{k_i} 2\sinh\frac{i\beta(M_{l_i}-M_{l_j})+2\beta_2(l-1-k_j)}{2}\prod_{i=1}^{N}\prod_{A\neq\{l_i\}}^{N_f}\prod_{l=1}^{k_i} 2\sinh\frac{i\beta(M_A-M_{l_i})+2\beta_2 l}{2}}, \tag{5.2.9}$$

$$\bar{Z}_V^{\{l_i\}} = \sum_{\vec{k}=0}^{\infty}(-1)^{-\left(\kappa+\frac{N_f-\tilde{N}_f}{2}\right)\sum_{i=1}^{N}k_i} e^{i\pi\kappa\sum_{i=1}^{N}(\beta M_{l_i}k_i+\beta_2 k_i^2)}\,(-\omega)^{-\sum_{i=1}^{n}k_i}$$
$$\times \frac{\prod_{i=1}^{N}\prod_{B=1}^{\tilde{N}_f}\prod_{l=1}^{k_i} 2\sinh\frac{i\beta(\tilde{M}_B-M_{l_i})+2\beta_2(l-1)}{2}}{\prod_{i,j=1}^{N}\prod_{l=1}^{k_i} 2\sinh\frac{i\beta(M_{l_i}-M_{l_j})+2\beta_2(l-1-k_j)}{2}\prod_{i=1}^{N}\prod_{A\neq\{l_i\}}^{N_f}\prod_{l=1}^{k_i} 2\sinh\frac{i\beta(M_A-M_{l_i})+2\beta_2 l}{2}}. \tag{5.2.10}$$

Note that the difference between the vortex and anti-vortex partition functions is only the contributions from the CS term and the topological symmetry.

5.3 Some questions

In the above discussion, we have shown that for a certain class of 3d $\mathcal{N}=2$ theories on S_b^3 and $S^1\times S^2$, these partition functions factorize into products of 3d vortex and anti-vortex partition functions as well as the other factors. We express that formally as

$$Z = \sum_i Z_{\text{cl}}^{(i)}\cdot Z_{\text{1-loop}}^{(i)}\cdot Z_{\text{V}}^{(i)}\cdot \bar{Z}_{\text{V}}^{(i)}. \tag{5.3.1}$$

We have some questions:

- Why do these partition functions include the vortex partition functions?
- Why do the same building block (the 3d vortex partition functions) appear in S_b^3 and $S^1\times S^2$? Since the building block might be more fundamental quantity than the original partition function, we expect that their analysis would give us significant insights in a supersymmetric gauge theory on a curved space.
- How about theories with generic R-charge? For generic R-charge, we cannot evaluate them using the above contour. For the superconformal index, as we mentioned in section 5.2, we can reproduce a result for generic R-charge using $U(1)_A$ symmetry. How about the ellipsoid case?
- How about the other matter content? In the above examples, we have seen that theories with fundamental and anti-fundamental matters have the factorization structure. It is interesting to investigate whether theories with other representation matters have the same structures.

Next chapter, we will answer these questions using an idea, "the Higgs branch localization".

Holomorphic block

Before starting to consider the Higgs branch localization, let us comment on an idea of holomorphic block [74]. First we note that the geometries S_b^3 and $S^1 \times S^2$ consist of two $D^2 \times S^1$ (solid torus topologically), so-called the Heegaard decomposition. The partition function of $\mathcal{N}=2$ theories is expected as

$$Z_{\mathcal{M}} = \sum_{\alpha} B^{\alpha}(x;q) B^{\alpha}(\tilde{x};\tilde{q}), \tag{5.3.2}$$

where roughly speaking $B^{\alpha}(x;q)$ is a partition function on $\mathbb{R}^2 \times S^1$ (so-called "holomorphic block"), which counts the number of BPS states in a discrete vacuum labeled by α, and q is a fugacity for the angular momentum, and x is a fugacity for the flavor symmetry. This is analogue of the topological/anti-topological fusion in two dimensions [75]. It is expected that this quantity is associated with the 3d vortex partition function in the argument of [74].

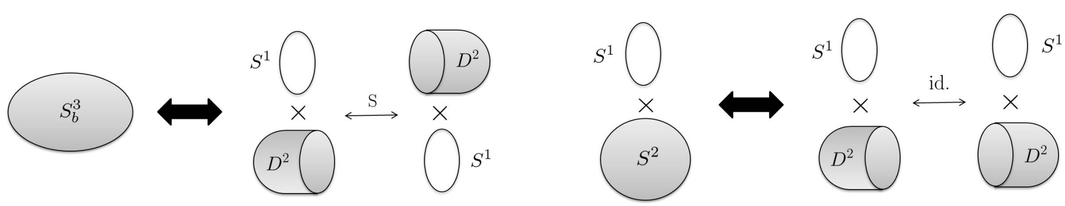


Figure 5.2: Heeggard decomposition of S_b^3 and $S^1 \times S^2$

For example, let us consider $U(1)$ theory with $2N_f$-fundamental matters on S_b^3. We can rewrite the vortex and anti-vortex partition functions (5.1.7), (5.1.8) in the following way,

$$Z_V^{(i)} = \sum_{m=0}^{\infty} \frac{\left\{(-1)^m \tilde{q}^{\frac{1}{2}m(m+1)}\right\}^{N_f} \tilde{x}_i^m}{\prod_{l=1}^m (1-\tilde{q}^l) \prod_{l=1}^m \prod_{j\neq i}^{2N_f} \left\{1-(\tilde{x}_j \tilde{x}_i^{-1})^{1/N_f}\, \tilde{q}^l\right\}}, \tag{5.3.3}$$

$$\bar{Z}_V^{(i)} = \sum_{n=0}^{\infty} \frac{\left\{(-1)^n q^{\frac{1}{2}n(n+1)}\right\}^{N_f} x_i^n}{\prod_{l=1}^n (1-q^l) \prod_{l=1}^n \prod_{j\neq i}^{2N_f} \left\{1-(x_j x_i^{-1})^{1/N_f}\, q^l\right\}}, \tag{5.3.4}$$

where $x = e^{-2\pi b\mu}$, $\tilde{x} = e^{-2\pi b^{-1}\mu}$, $q = e^{-2\pi i b^2}$ and $\tilde{q} = e^{-2\pi i b^{-2}}$, and we set a mass parameter $\mu_i := \zeta - N_f m_i$. Note that the vortex and anti-vortex partition functions change each other for a replacement $(x,q) \leftrightarrow (\tilde{x},\tilde{q})$. In the above correspondence, the partition function is expected as a S-fusion of the corresponding holomorphic blocks,

$$Z_{S^3_b}(\mu,b) = \sum_{\alpha} B^{\alpha}(x;q) B^{\alpha}(\tilde{x};\tilde{q}) = ||B^{\alpha}(x;q)||^2_S, \tag{5.3.5}$$

where each term is expected to correspond to anti-vortex or vortex partition function part. Also the superconformal index is expected as an identity fusion of the corresponding holomorphic blocks.

Chapter 6

Higgs branch localization

In this chapter we give a natural interpretation to the factorization using "the Higgs branch localization", in which the saddle point is characterized by the value of the chiral multiplet scalar field. We also find that a large class of $\mathcal{N} = 2$ theories have the same factorization structures. The content in this chapter is based on [36].

6.1 Partition function on the three-ellipsoid

In this section we reconsider the partition function on the ellipsoid using the idea of Higgs branch localization.

6.1.1 Localized configurations

BPS configurations

First we note from (4.1.3) that there is no BPS configuration like the BPS vortex configuration. Recall that we have required all the fields to satisfy the reality condition (2.1.13). However, one can show that by relaxing the reality condition for D in (4.1.11) , one can find the wider BPS configurations

$$\begin{aligned} &F_{12} = 0. \quad F_{23} + \mathrm{Im}D\cos\vartheta = 0, \quad F_{31} - \mathrm{Im}D\sin\vartheta = 0, \\ &D_\mu\sigma = 0, \quad \mathrm{Re}D = -\frac{\sigma}{Rf(\vartheta)}, \end{aligned} \tag{6.1.1}$$

which allow BPS vortex configurations[1]. As we will see later, an appropriate choice of a deformation term leads us to a natural change of the integral contour of D from real to complex, and giving rise to a nontrivial $\mathrm{Im}D$.

[1] The same procedure has been performed in the 2-dimensional case [34, 35].

Deformation term

Next let us consider a choice of the deformation term[2]. We add a new deformation term to the Coulomb branch deformation terms (4.1.11) and (4.1.14),

$$\mathcal{L}_{\rm YM} + \mathcal{L}_{\psi} \quad \to \quad \mathcal{L}_{\rm YM} + \mathcal{L}_{\psi} + \mathcal{L}_{H}, \tag{6.1.2}$$

where $\mathcal{L}_H$ is the new deformation term for our Higgs branch localization defined by

$$\mathcal{L}_H = \mathcal{Q}V_H = i^{-1}\,\mathcal{Q}\,\mathrm{Tr}\Big[\ \frac{(\epsilon^\dagger\lambda - \bar{\epsilon}^\dagger\bar{\lambda})\,h}{4i}\ \Big], \tag{6.1.3}$$

as in the two dimensional case [35]. Here h is a function of the scalar fields. For example, if we consider $\mathcal{N}=2$ theory with a fundamental chiral multiplet including scalar field ϕ, then we choose h as[3]

$$h = \phi\phi^\dagger - \chi\cdot \mathbb{1}_N, \tag{6.1.4}$$

where χ is a parameter taken as $\chi \to \pm\infty$ later[4]. When we further add an anti-fundamental chiral multiplet including scalar field $\tilde{\phi}$, and an adjoint chiral multiplet with its scalar X, we choose h as

$$h = \phi\phi^\dagger - \tilde{\phi}^\dagger\tilde{\phi} + [X, X^\dagger] - \chi\cdot\mathbb{1}_N. \tag{6.1.5}$$

Although we do not discuss why we choose the above deformation term yet, let us consider how this affects the localization specifically.

In fact we can write the bosonic part of $\mathcal{L}_H$ as

$$\mathcal{L}_H|_{\rm Bos.} \;=\; \mathrm{Tr}\Bigg[\bigg(-\frac{1}{2}\cos\vartheta F_{23} + \frac{1}{2}\sin\vartheta F_{31} + \frac{i}{2}D + \frac{i}{2f(\vartheta)}\sigma\bigg)\,h\Bigg]. \tag{6.1.6}$$

Combined with $\mathcal{L}_{\rm YM}$, completing the square leads to

$$\begin{aligned}
&\mathcal{L}_{\rm YM}|_{\rm bos.} + \mathcal{L}_H|_{\rm bos.}\\
&= \mathrm{Tr}\Bigg[\ \frac{1}{2}F_{12}^2 + \frac{1}{2}\left(\sin\vartheta F_{23} + \cos\vartheta F_{31}\right)^2 + \frac{1}{2}\left(\cos\vartheta F_{23} - \sin\vartheta F_{31} - \frac{1}{2}h\right)^2\\
&\qquad + \frac{1}{2}(D_\mu\sigma)^2 + \frac{1}{2}\left(D + \frac{1}{f(\vartheta)}\sigma + \frac{i}{2}h\right)^2\ \Bigg].
\end{aligned} \tag{6.1.7}$$

[2] Below we take $R=1$ for simplicity.

[3] Recall that V_H must satisfy $\mathcal{Q}^2V_H = 0$, where $\mathcal{Q}^2$ generates (4.1.5). In fact we find that this choice is consistent.

[4] To be precise, we should take $h = \pm\phi\phi^\dagger - \chi\cdot\mathbb{1}_N$, whose sign depends on one of χ. Also note that χ vanishes under the trace if the gauge group does not include an Abelian subgroup.

Although the final parentheses include an imaginary part, the auxiliary field D can be trivially integrated out and it becomes semi-positive definite after that. Since $\mathcal{L}_\psi$ is also semi-positive definite itself, we obtain the following localized configurations:

$$\begin{aligned}
&F_{12}=0, \quad \sin\vartheta F_{23}+\cos\vartheta F_{31}=0, \quad \cos\vartheta F_{23}-\sin\vartheta F_{31}-\frac{1}{2}h=0,\\
&D_\mu\sigma=0, \quad D+\frac{1}{f(\vartheta)}\sigma+\frac{i}{2}h=0, \quad \sin\vartheta D_1\phi+\cos\vartheta D_2\phi+iD_3\phi=0,\\
&\cos\vartheta D_1\phi-\sin\vartheta D_2\phi+\frac{i\Delta}{f(\vartheta)}\phi=0, \quad \sigma\phi=0, \quad F=0.
\end{aligned} \tag{6.1.8}$$

Although we usually take the integral contour of D to be real, in the present case, we take it as

$$D=-\frac{1}{f(\vartheta)}\sigma-\frac{i}{2}h. \tag{6.1.9}$$

That is to say, in order to make the action semi-positive definite, we have to change the integral contour of D from $\mathbb{R}$ to $\mathbb{R}-ih/2$. As we briefly mentioned before, this gives the imaginary part of D, and hence we can obtain the BPS configuration (6.1.1).

Away from the north and south poles

Let us consider an $\mathcal{N}=2$ $U(N)$ gauge theory with N_f fundamental chiral multiplets with real mass M. Here we take the R-charge $\Delta=0$, then it allows us to give nontrivial localized values (constant value) to the chiral multiplet scalar field[5]. Note that even if we take $\Delta=0$, we can reproduce the result for general Δ by an analytic continuation of the real mass M, since $\hat{\mathcal{Q}}^2$ is holomorphic with respect to $M+i(b+b^{-1})\frac{\Delta}{2}$ in (4.1.5)[6]. For this case, the localized configuration is

$$\begin{aligned}
&F_{12}=0, \quad \sin\vartheta F_{23}+\cos\vartheta F_{31}=0, \quad \cos\vartheta F_{23}-\sin\vartheta F_{31}-\frac{1}{2}(\phi\phi^\dagger-\chi\cdot\mathbb{1}_N)=0,\\
&D_\mu\sigma=0, \quad D+\frac{1}{f(\vartheta)}(\sigma+M)+\frac{i}{2}(\phi\phi^\dagger-\chi\cdot\mathbb{1}_N)=0,\\
&\sin\vartheta D_1\phi+\cos\vartheta D_2\phi+iD_3\phi=0, \quad \cos\vartheta D_1\phi-\sin\vartheta D_2\phi=0,\\
&(\sigma+M)\phi=0, \quad F=0,
\end{aligned} \tag{6.1.10}$$

where we omit the flavor indices, and choose h as (6.1.4).

First, we consider only configurations in a region except the north and south poles, i.e. $\vartheta\neq 0,\pi/2$. In this case ($\Delta=0$), we find that ϕ should be constant. Although ϕ can

[5]See appendix for details.

[6]In our notation, note that the lowest component ϕ in the chiral multiplet takes $\mathcal{R}=-\Delta$.

take arbitrary constant values as long as $(\sigma+M)\phi=0$ is satisfied, we can show that only those such that $\phi\phi^\dagger=\chi\cdot\mathbb{1}_N$ has a nonzero contribution in the limit $\chi\to\pm\infty$ as follows. From eq. (6.1.10), we can write the field strength explicitly as

$$F_{\mu\nu}\,dx^\mu\wedge dx^\nu \;=\; d\left[\frac{(\phi\phi^\dagger-\chi\cdot\mathbb{1}_N)}{6(b^2-b^{-2})}f^3(\vartheta)(bd\varphi_1-b^{-1}d\varphi_2)\right]. \qquad (6.1.11)$$

We find that the gauge field then satisfies $v^\mu A_\mu=0$ up to a gauge choice (recall that $v^\mu=(0,b^{-1},b)$ (4.1.37)). Since the one-loop determinant is the determinant of $\hat{\mathcal{Q}}^2$ as (4.1.32), the one-loop determinant should be χ-independent from (4.1.24). Therefore χ-dependence appears only in the classical contribution. If we have the FI or CS term,

$$Z_{\rm cl}\big|_{\text{at (6.1.10)}}=e^{-(S_{\rm CS}+S_{\rm FI})}\big|_{\text{at (6.1.10)}}\sim\exp\Big[-\frac{\pi}{2}(\kappa\sigma^{(0)}-\zeta){\rm Tr}(\phi\phi^\dagger-\chi\cdot\mathbb{1}_N)\Big], \quad (6.1.12)$$

where $\sim$ means an equality up to a phase factor. That is to say, this gives an exponential suppression factor $\sim e^{-|\chi|}$ which vanishes in the $|\chi|\to\infty$ limit, except for ϕ such that $\phi\phi^\dagger=\chi\cdot\mathbb{1}_N$[7].

Thus we conclude that non-vanishing configuration is only the Higgs branch solution,

$$F_{\mu\nu}=0,\quad D_\mu\sigma=0,\quad (\sigma+M)\phi=0,\quad \phi\phi^\dagger-\chi\cdot\mathbb{1}_N=0,\quad D+\frac{1}{f(\vartheta)}(\sigma+M)=0. \quad (6.1.13)$$

With explicit indices, the third equation is

$$\sigma_{ij}\phi_{jA}+\phi_{iB}M_{BA}=0,\quad i,j=1,\cdots,N,\quad A,B=1,\cdots,N_f, \qquad (6.1.14)$$

where i,j are the gauge indices and A,B are flavor indices. Let us suppose that $N_f\geq N$. We can always diagonalize $\sigma_{ij}={\rm diag}(\sigma_1,\cdots,\sigma_N)$ and $M_{AB}={\rm diag}(m1,\cdots,m_{N_f})$ using the gauge transformation and flavor rotation, respectively. Then this equation implies

$$\Big((\sigma_i+m_1)\phi_{i1},\ (\sigma_i+m_2)\phi_{i2},\cdots,(\sigma_i+m_{N_f})\phi_{iN_f}\Big)=0, \qquad (6.1.15)$$

in terms of the $(N\times N_f)$ matrix form. Therefore, up to gauge and flavor rotations, we find

$$\sigma_i=-m_{l_i},\quad \phi_{iA}=\sqrt{\chi}\,\delta_{l_iA}, \qquad (6.1.16)$$

where $(l_1,\cdots,l_N)$ implies a set of N integers in $(1,\cdots,N_f)$, i.e. the localized configurations are labeled by ${}_{N_f}C_N$ discrete values.

[7] Note that if we take $\chi\to-\infty$, we should take $h=-\phi\phi^\dagger-\chi\cdot\mathbb{1}_N$, as mentioned in footnote 4. We have considered this case as if χ was positive, for simplicity.

One-loop determinant

Let us compute the one-loop determinant around the saddle point (6.1.13). Now we have three types of fields:

- Vector multiplet,
- Chiral multiplets with vanishing VEV,
- Chiral multiplets with nonzero VEV.

The first two types of one-loop determinants are obtained by just substituting $\sigma_i = -m_{l_a}$ into the Coulomb branch result, since the new deformation term V_H does not have any derivative terms. The last one-loop determinant coming from the chiral multiplet with a non-vanishing VEV is nontrivial for an arbitrary value of χ since we have to consider the combined system of the vector and chiral multiplets. However, since we can ignore any derivative terms in the limit $\chi \to \pm\infty$, we expect that the one-loop determinant should give no contribution in this limit. As we will see later, we will find that this expectation is true by comparing the final result with the Coulomb branch result. Thus we conclude that the one-loop determinants on the Higgs branch are

$$Z_{\text{vec}}^{\text{(1-loop)}} = \prod_{i<j}^{N} \sinh \pi b(m_{l_i} - m_{l_j}) \sinh \pi b^{-1}(m_{l_i} - m_{l_j}), \tag{6.1.17}$$

$$Z_{\text{chi}}^{\text{(1-loop)}} = \prod_{A \neq \{l_i\}}^{N_f} \prod_{i=1}^{N} s_b\left(\frac{iQ}{2} + m_{l_i} - m_A\right). \tag{6.1.18}$$

For general R-charge Δ, the analytic continuation of the real masses $m_A \to m_A + iQ/2\Delta$ induces the following result,

$$Z_{\text{chi}}^{\text{(1-loop)}} = \prod_{A \neq \{l_i\}}^{N_f} \prod_{i=1}^{N} s_b\left(\frac{iQ}{2}(1-\Delta) + m_{l_i} - m_A\right). \tag{6.1.19}$$

At the north and south poles

Next let us consider the configurations (6.1.10) on the north and south poles ($\vartheta = 0, \pi/2$). At these poles, we find the following configurations:
At the north pole ($\vartheta = 0$),

$$F_{12} = 0, \quad F_{31} = 0, \quad F_{23} - \frac{1}{2}(\phi\phi^\dagger - \chi \cdot \mathbb{1}_N) = 0, \quad D_\mu \sigma = 0, \quad (\sigma + M)\phi = 0,$$

$$D+b^{-1}\sigma+\frac{i}{2}(\phi\phi^\dagger-\chi\cdot\mathbb{1}_N)=0,\quad D_2\phi+iD_3\phi=0,\quad D_1\phi=0,\quad F=0. \qquad (6.1.20)$$

Note that the vortex equations appear above. At the south pole ($\vartheta=\pi/2$),

$$\begin{aligned} &F_{12}=0,\quad F_{23}=0,\quad -F_{31}-\frac{1}{2}(\phi\phi^\dagger-\chi\cdot\mathbb{1}_N)=0,\quad D_\mu\sigma=0,\quad (\sigma+M)\phi=0,\\ &D+b\sigma+\frac{i}{2}(\phi\phi^\dagger-\chi\cdot\mathbb{1}_N)=0,\quad D_1\phi+iD_3\phi=0,\quad D_2\phi=0,\quad F=0, \end{aligned} \qquad (6.1.21)$$

where the anti-vortex equations appear. As we mentioned, we note that the free parameter χ plays a role of the FI parameter in the ordinary vortex equations. Since the vortex size is $\sim 1/\sqrt{\chi}$, the vortex becomes point-like in the $\chi\to\pm\infty$ limit (however note that the vortex size is not a modulus). Thus we find that vortex and anti-vortex localize on the north and south poles.

Here we would like to identify the S^1 fiber length β and the Ω-deformation parameter ε, which characterize the three-dimensional vortex partition function on $\mathbb{R}^2_\varepsilon\times S^1_\beta$. First recall that the metric on the ellipsoid is

$$\begin{aligned} ds^2 &= R^2\Big[f(\vartheta)^2d\vartheta^2+b^2\sin^2\vartheta d\varphi_1^2+b^{-2}\cos^2\vartheta d\varphi_2^2\Big]\\ &= \frac{R^2}{4}\Big[\ f(\theta/2)^2d\theta^2+\frac{2\cos^2(\theta/2)\sin^2(\theta/2)}{b^{-2}\cos^2(\theta/2)+b^2\sin^2(\theta/2)}d\varphi^2\\ &\quad +\big\{b^{-2}\cos^2(\theta/2)+b^2\sin^2(\theta/2)\big\}\Big(d\psi+\frac{b^{-2}\cos^2(\theta/2)-b^2\sin^2(\theta/2)}{b^{-2}\cos^2(\theta/2)+b^2\sin^2(\theta/2)}d\varphi\Big)^2\ \Big], \end{aligned} \qquad (6.1.22)$$

where we have switched to the Hopf fibration coordinates in the second line. In particular, the metrics for the north and south pole neighborhood are

$$ds_N^2=\frac{R^2}{4}\Big[\ b^2d\theta^2+b^2\theta^2d\varphi^2+b^{-2}(d\psi+d\varphi)^2\ \Big],\qquad (\theta\sim 0), \qquad (6.1.23)$$

$$ds_S^2=\frac{R^2}{4}\Big[\ b^{-2}d\theta^2+b^{-2}(\pi-\theta)^2d\varphi^2+b^2(d\psi-d\varphi)^2\ \Big],\qquad (\theta\sim\pi). \qquad (6.1.24)$$

Identifying the fiber directions as $\varphi_2=\frac{1}{2}(\psi+\varphi)$ and $\varphi_1=\frac{1}{2}(\psi-\varphi)$ on the north and south poles respectively, we can read off the S^1 fiber lengths,

$$\beta_N=2\pi b^{-1}R,\quad \beta_S=2\pi bR. \qquad (6.1.25)$$

Next we consider the Ω-deformation parameter. The Ω-bachground parameter is a rotational parameter on the base S^2 generated by the vector field v in $\hat{\mathcal{Q}}^2$ in this case. Since each fiber direction is φ_2 and φ_1 on the north and south poles, we can identify it from (4.1.5) as (c.f. [10, 34, 69])

$$\varepsilon_N = \frac{ib^{-1}}{R}, \quad \varepsilon_S = \frac{ib}{R}, \tag{6.1.26}$$

respectively.

Result

As a result of the Higgs branch localization, it turns out that the partition function takes formally the following factorized form

$$Z = \sum_{\text{Higgs branch}} Z_{\text{cl}} \cdot Z_{\text{vec}}^{(\text{1-loop})} \cdot Z_{\text{chi}}^{(\text{1-loop})} \cdot Z_{\text{V}} \cdot \bar{Z}_{\text{V}}, \tag{6.1.27}$$

where Z_{cl} is the contributions from the CS and FI terms at the Higgs branch solution. This is the result we desired. That is to say, we identify the vortices appearing in the Coulomb branch result as ones coming from the localized contributions from the north and south poles on the base S^2 in the Higgs branch.

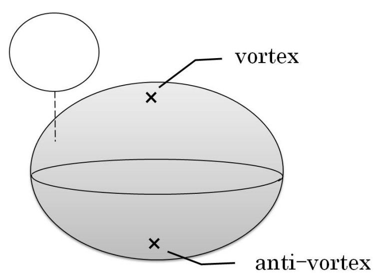


Figure 6.1: Localization of the vortex and anti-vortex on the base S^2

Finally, let us comment on the case where anti-fundamental chiral multiplets and one adjoint chiral multiplet are added (6.1.5). One can show that fundamental, anti-fundamental and adjoint scalar fields cannot have VEV simultaneously for generic masses. This reflects the fact that anti-fundamental and adjoint scalar fields can contribute only to the fermionic moduli [76]. Therefore, away from the north and south poles, the one-loop determinant for each anti-fundamental or adjoint chiral multiple is just

$$Z_{\text{chi}}^{(\text{1-loop})} = \prod_{\omega \in R} s_b\Big(\frac{iQ}{2} - \omega(\sigma)\Big)\Big|_{\omega_i = -m_{l_i}}. \tag{6.1.28}$$

As we will see in the next section, however, these multiplets nontrivially contribute to vortex partition functions.

In the next section, we will explicitly calculate Z_{V} and $\bar{Z}_{\mathrm{V}}$ by evaluating the vortex world line theories, and we will compare the result with the Coulomb branch one.

Remarks

In section 5.1, we have found that the condition for the factorization corresponds to the parity anomaly cancellation condition. In fact this is also the same in the Higgs branch localization since the localization procedure itself would not hold due to the gauge symmetry breaking.

We have seen that the new deformation term $\mathcal{L}_H$ (6.1.3) induces the (anti-)vortex configuration at the north (south) pole. However if we replace the function h with $-h$, then we find the vortex at the south pole and anti-vortex at the north pole. Note that the whole partition function is invariant under this replacement since h appears only in the $\mathcal{Q}$-exact term $\mathcal{L}_H$. So which pole the (anti-)vortex appears on is not physical in this situation.

6.1.2 Vortex partition function and localization

The vortex that we have encountered above is the half BPS vortex in 3d $\mathcal{N}=2$ SUSY gauge theory. As we have seen in section 2.4, the vortex moduli space is described by a one-dimensional theory with $\mathcal{N}=(0,2)$ type supersymmetry. In this section we compute the vortex partition function using the one-dimensional theory for the vortex.

Vortex partition function of 3d $\mathcal{N}=2$ with N_f fundamental and $\tilde{N}_f$ anti-fundamental chiral multiplets

We take the vortex number as k for the gauge field strength along $\mathbb{R}^2 \subset \mathbb{R}^2 \times S^1$:

$$k = \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{Tr}_N F_A. \tag{6.1.29}$$

As we have seen in section 2.4, the vortex quantum mechanics for 3d $\mathcal{N}=2$ with N_f fundamental and $\tilde{N}_f$ anti-fundamental chiral multiplets consists of the following one-dimensional multiplets:[8]

[8]The derivation of these multiplets in our notation are summarized in appendix D.

- $U(k)$ vector multiplet: $(A_t, \varphi, D, \lambda^+, \bar{\lambda}^+)$

$$\begin{aligned}
&\mathcal{Q}A_\omega = -\frac{i}{2}(\bar{\epsilon}^+\lambda^+ + \bar{\lambda}^+\epsilon^+), \quad \mathcal{Q}A_{\bar{\omega}} = 0, \\
&\mathcal{Q}\bar{\lambda}^+ = \bar{\epsilon}^+(D - iF_{12}), \qquad \mathcal{Q}(D - iF_{12}) = -2i\epsilon^+ D_{\bar{\omega}}\bar{\lambda}^+, \\
&\mathcal{Q}\lambda^+ = \epsilon^+(-D - iF_{12}), \quad \mathcal{Q}(-D - iF_{12}) = -2i\bar{\epsilon}^+ D_{\bar{\omega}}\lambda^+,
\end{aligned} \tag{6.1.30}$$

- an adjoint chiral multiplet: $(B, \bar{B}, \psi^-, \bar{\psi}^-)$

$$\begin{aligned}
&\mathcal{Q}B = -\bar{\epsilon}^+\lambda^-, \quad \mathcal{Q}\lambda^- = i\epsilon^+(2D_{\bar{\omega}}B - \varepsilon B), \\
&\mathcal{Q}\bar{B} = \epsilon^+\bar{\lambda}^-, \qquad \mathcal{Q}\bar{\lambda}^- = -i\bar{\epsilon}^+(2D_{\bar{\omega}}\bar{B} + \varepsilon\bar{B}),
\end{aligned} \tag{6.1.31}$$

- fundamental chiral multiplets with N flavors:
  $(I^i, \bar{I}^i, \psi_I^{i-}, \bar{\psi}_I^{i-}), \quad i = 1, \cdots, N,$

$$\begin{aligned}
&\mathcal{Q}I = -\bar{\epsilon}^+\psi_I^-, \quad \mathcal{Q}\psi_I^- = i\epsilon^+(2D_{\bar{\omega}}I - Im), \\
&\mathcal{Q}\bar{I} = -\epsilon^+\bar{\psi}_I^-, \quad \mathcal{Q}\bar{\psi}_I^- = i\bar{\epsilon}^+(2D_{\bar{\omega}}\bar{I} + m\bar{I}),
\end{aligned} \tag{6.1.32}$$

- anti-fundamental chiral multiplets with $N_f - N$ flavors:
  $(J^j, \bar{J}^j, \psi_J^{j-}, \bar{\psi}_J^{j-}), \quad j = N+1, \cdots, N_f,$

$$\begin{aligned}
&\mathcal{Q}J = -\bar{\epsilon}^+\psi_J^-, \quad \mathcal{Q}\psi_J^- = i\epsilon^+(2D_{\bar{\omega}}J + J\tilde{m}), \\
&\mathcal{Q}\bar{J} = -\epsilon^+\bar{\psi}_J^-, \quad \mathcal{Q}\bar{\psi}_J^- = i\bar{\epsilon}^+(2D_{\bar{\omega}}\bar{J} - \tilde{m}\bar{J}),
\end{aligned} \tag{6.1.33}$$

- fundamental Fermi multiplets with $\tilde{N}_f$ flavors:
  $(\psi^{p+}, \bar{\psi}^{p+}, F^p, F^p), \quad p = 1, \cdots, \tilde{N}_f$

$$\begin{aligned}
&\mathcal{Q}\psi^+ = -\bar{\epsilon}^+E + \bar{\epsilon}^+F, \quad \mathcal{Q}F = i\epsilon^+(-2D_{\bar{\omega}}\psi^+ + \tilde{M}\psi^+ + \psi_E^-), \\
&\mathcal{Q}\bar{\psi}^+ = -\bar{\epsilon}^+\bar{E} + \bar{\epsilon}^+\bar{F}, \quad \mathcal{Q}\bar{F} = i\epsilon^+(-2D_{\bar{\omega}}\bar{\psi}^+ - \psi^+\tilde{M} + \bar{\psi}_E^-),
\end{aligned} \tag{6.1.34}$$

where $\mathcal{Q}$ is a Grassmann-odd, and ε is the Ω-background parameter introduced to regularize the flat direction of the adjoint fields, m ($\tilde{m}$) and $\tilde{M}$ are the twisted masses of the (anti-)chiral multiplets and Fermi multiplet, respectively. Here we have omitted the flavor indices for simplicity. We also defined $A_\omega = \frac{1}{2}(A_1 - iA_2)$, $A_{\bar{\omega}} = \frac{1}{2}(A_1 + iA_2)$, $D_\omega = \frac{1}{2}(D_1 - iD_2)$ and $D_{\bar{\omega}} = \frac{1}{2}(D_1 + iD_2)$ with $A_1 = A_\tau$, $A_2 = \varphi$. Roughly speaking, the mass parameters m, $\tilde{m}$ and $\tilde{M}$ in the 3d language are as follows.

- The N twisted masses m:
  the real masses $(m_{l_1}, \cdots, m_{l_N})$ of the 3d fundamental chiral multiplet satisfying (6.1.16). For simplicity, we take $(l_1, \cdots, l_N) = (1, \cdots, N)$ in this section.

- The $(N_f - N)$ twisted masses $\tilde{m}$:
  the real masses for the 3d fundamental chiral multiplets which are not $(m_{l_1}, \cdots, m_{l_N})$.

- The $\tilde{N}_f$ twisted masses $\tilde{M}$:
  the real masses for the 3d anti-fundamental chiral multiplets.

We set $\epsilon^+ = 1, \bar{\epsilon}^+ = 1$ in the rest of this section. Then the Lagrangian of the vortex quantum mechanics is written in the following $\mathcal{Q}$-exact manner:

$$\begin{aligned}
\mathcal{L}_{\text{vec}} &= \frac{1}{2}\mathcal{Q}\,\mathrm{Tr}_k\,\bar{\lambda}^+(D + iF_{12}), && (6.1.35)\\
\mathcal{L}_B &= \mathcal{Q}\,\mathrm{Tr}_k \left(i\bar{B}(2D_\omega + \varepsilon)\lambda^- - i\bar{B}[\lambda^+, B] \right), && (6.1.36)\\
\mathcal{L}_I &= \mathcal{Q} \left(i\bar{I}(2D_\omega + m)\psi_I^- - i\bar{I}\lambda^+ I \right), && (6.1.37)\\
\mathcal{L}_J &= \mathcal{Q} \left(iJ(2D_\omega - \tilde{m})\bar{\psi}_J^- - iJ\lambda^+ \bar{J} \right), && (6.1.38)\\
\mathcal{L}_{\text{Fermi}} &= \frac{1}{2}\mathcal{Q} \left(\overline{\mathcal{Q}\psi^+} \cdot \psi^+ + \psi^+ \mathcal{Q}\overline{\bar{\psi}^+} \right). && (6.1.39)
\end{aligned}$$

The FI term is also written in a $\mathcal{Q}$-exact form as[9]

$$\mathcal{L}_{\text{FI}} = -\frac{ir}{2}\mathcal{Q}\left(\bar{\lambda}^+ - \lambda^+\right). \tag{6.1.40}$$

The CS term is written as [77]

$$\mathcal{L}_{\text{CS}} = 2i\kappa \mathrm{Tr}_k A_{\bar{\omega}}. \tag{6.1.41}$$

Note that this is not $\mathcal{Q}$-exact but is $\mathcal{Q}$-closed. Here κ corresponds to the bare CS level in three dimensions.

When we set all the mass parameters to zero, the D-term condition of the vortex quantum mechanics gives the k-vortex moduli space:

$$\mathcal{M}^k_{N,N_f} = \left\{ (B, I, J) \middle| \, [B, B^\dagger] + I\bar{I} - \bar{J}J = r \cdot \mathbb{1}_k \right\} / U(k). \tag{6.1.42}$$

Here we assume that r is positive. The partition function of the vortex world line is defined as

$$Z^k_{\text{V}} = \int \mathcal{D}\Psi \exp\left(-\int_0^\beta d\tau \left(\mathcal{L}_{CS} + t\mathcal{Q}V\right) \right), \tag{6.1.43}$$

with

$$\mathcal{Q}V = (\mathcal{L}_{\text{vec}} + \mathcal{L}_B + \mathcal{L}_I + \mathcal{L}_J + \mathcal{L}_{\text{Fermi}} + \mathcal{L}_{\text{FI}}). \tag{6.1.44}$$

[9]As we have seen in section 2.4, the FI parameter r is different from χ, which corresponds to that of the 3d $\mathcal{N} = 2$ theory.

Here Ψ denotes the collection of the fields in the vortex quantum mechanics, and β is the length of the compactified circle of the world line and is identified with the length of the circle fiber on the points supporting the point-like (anti-)vortex in the three-ellipsoid.

Let us evaluate the vortex partition function. First we drop the CS term; we will add this later. Since the action for this case is written as the $\mathcal{Q}$-exact form, we can perform the path integral exactly via localization. Taking $t \to \infty$, the saddle points for bosonic fields are given by zeros of the supersymmetric variations (6.1.31), (6.1.32) and (6.1.33) as

$$2D_{\bar{\omega}}B - \varepsilon B = 0, \quad 2D_{\bar{\omega}}I - mI = 0, \quad 2D_{\bar{\omega}}J + \tilde{m}J = 0. \tag{6.1.45}$$

If we take the gauge fixing condition as $\partial_t A_{\bar{\omega}} = 0$, then (6.1.45) reduces to the constant matrix valued equations, namely

$$[2iA_{\bar{\omega}}, B] - \varepsilon B = 0, \quad 2iA_{\bar{\omega}}I - mI = 0, \quad -2iJA_{\bar{\omega}} + \tilde{m}J = 0. \tag{6.1.46}$$

These equations are the fixed point equations for the vortex moduli space under the equivariant rotation with respect to $U(1)_m^{N_f-1} \times U(1)_\varepsilon$ [73, 78, 79, 76, 77]. By taking the diagonal gauge for the constant mode for $A_{\bar{\omega}}$, the solutions are given by[10]

$$2iA_{\bar{\omega},(l,i)} = m_i + (l-1)\varepsilon, \quad i = 1 \cdots N, \quad l = 1, \cdots, k_i. \tag{6.1.47}$$

The fixed points are classified by N-tuple non-negative integers $(k_1, \cdots, k_N)$ with $\sum_{i=1}^N k_i = k$, where k_i is the vorticity for the i-th diagonal $U(1)_i \subset U(N)$. By applying the localization formula [80], the one-loop determinant around the fixed point labeled by $(k_1, \cdots, k_N)$ is given by

$$\begin{aligned} Z_{\mathrm{V}}^{(k_1,\cdots,k_N)} =& \frac{\prod_{i,j=1}^N \prod_{l=1}^{k_i} \prod_{\tilde{l}=1}^{k_j} \mathrm{Det}(\partial_t + 2iA_{\bar{\omega},(l,i)} - 2iA_{\bar{\omega}(\tilde{l},j)})}{\prod_{i,j=1}^N \prod_{l=1}^{k_i} \prod_{\tilde{l}=1}^{k_j} \mathrm{Det}(\partial_t + 2iA_{\bar{\omega},(l,i)} - 2iA_{\bar{\omega},(\tilde{l},j)} - \varepsilon)} \\ &\times \frac{\prod_{i=1}^N \prod_{p=1}^{\tilde{N}_f} \prod_{l=1}^{k_i} \mathrm{Det}(\partial_t - 2iA_{\bar{\omega},(l,i)} - M_p)}{\prod_{i',j'=1}^N \prod_{l'=1}^{k_{i'}} \mathrm{Det}(\partial_t + 2iA_{\bar{\omega},(l',i')} + m_{j'}) \prod_{i'=1}^N \prod_{j=N+1}^{N_f} \prod_{l'=1}^{k_{i'}} \mathrm{Det}(\partial_t - 2iA_{\bar{\omega},(l',i')} - \tilde{m}_j)}. \end{aligned} \tag{6.1.48}$$

Here the denominator, the numerators in the first and second lines come from the one-loop determinants of (6.1.35)-(6.1.38), ghost and Fermi multiplet, respectively. The functional

[10] We can show that $J = 0$ for $r > 0$. Note that I^i is an eigenvector of the operator $2iA_{\bar{\omega}}$ with eigenvalue m_i, and B is a ladder operator of $2iA_{\bar{\omega}}$. Therefore the space of the fixed points is expanded by generators constructed by successive actions of B on the eigenvector I^i ("one-dimensional N-colored Young diagrams with total box number k").

determinant on the circle with radius β is evaluated as

$$\mathrm{Det}(\partial_t + a) = 2\sinh\frac{\beta a}{2}. \tag{6.1.49}$$

Then, (6.1.48) becomes

$$\begin{aligned}
Z_{\mathrm{V}}^{(k_1,\cdots,k_N)} &= \frac{\prod_{i,j=1}^{N}\prod_{l=1}^{k_i}\prod_{\tilde{l}=1}^{k_j} 2\sinh\frac{\beta}{2}\big(m_{j,i}+(l-\tilde{l})\varepsilon\big)}{\prod_{i,j=1}^{N}\prod_{l=1}^{k_i}\prod_{\tilde{l}=1}^{k_j} 2\sinh\frac{\beta}{2}\big(m_{j,i}+(l-\tilde{l}-1)\varepsilon\big)} \\
&\times \frac{\prod_{i=1}^{N}\prod_{j=1}^{\tilde{N}_f}\prod_{l=1}^{k_i} 2\sinh\frac{\beta}{2}\big(m_{j,i}+(l-1)\varepsilon\big)}{\prod_{i,j=1}^{N}\prod_{l=1}^{k_i}\sinh\frac{\beta}{2}\big(m_{j,i}+(l-1)\varepsilon\big)\prod_{i=1}^{N}\prod_{j=N+1}^{N_f}\prod_{l=1}^{k_i} 2\sinh\frac{\beta}{2}\big(m_{i,j}-(l-1)\varepsilon\big)} \\
&= \frac{\prod_{i=1}^{N}\prod_{j=1}^{\tilde{N}_f}\prod_{l=1}^{k_i} 2\sinh\frac{\beta}{2}\big(m_{j,i}+(l-1)\varepsilon\big)}{\prod_{i,j=1}^{N}\prod_{l=1}^{k_i} 2\sinh\frac{\beta}{2}\big(m_{j,i}+(l-1-k_j)\varepsilon\big)\prod_{i=1}^{N}\prod_{j=N+1}^{N_f}\prod_{l=1}^{k_i} 2\sinh\frac{\beta}{2}\big(m_{i,j}-(l-1)\varepsilon\big)},
\end{aligned} \tag{6.1.50}$$

where we define m_{ij} as

$$m_{j,i} = \begin{cases} m_i - m_j, & (i,j\in\{1,\cdots N\}), \\ m_i - \tilde{m}_j, & (i\in\{1,\cdots N\},\quad j\in\{N+1,\cdots N_f\}), \\ m_i - \tilde{M}_j, & (i\in\{1,\cdots N\},\quad j\in\{1,\cdots \tilde{N}_f\}). \end{cases} \tag{6.1.51}$$

Next we consider the CS term contribution. The CS term at the fixed point labeled by $(k_1,\cdots,k_N)$ is evaluated as

$$e^{-2i\kappa\int \mathrm{Tr}A_{\bar{\omega}}}\Big|_{\text{fixed point}} = e^{-\beta\kappa\sum_{i=1}^{N}\sum_{l=1}^{k_i} 2iA_{\bar{\omega},(l,i)}} = e^{-\beta\kappa\sum_{i=1}^{N}\left(k_i m_i + \varepsilon\frac{k_i(k_i-1)}{2}\right)}. \tag{6.1.52}$$

Therefore, up to an overall sign, the vortex partition function for 3d $\mathcal{N}=2$ $U(N)$ Chern-Simons-matter theory with N_f fundamental and $\tilde{N}_f$ anti-fundamental chiral multiplets is given by[11]

$$\begin{aligned}
Z_{\mathrm{V}} &= \sum_{\vec{k}=0}^{\infty}\left(\prod_{i=1}^{N} z_i^{k_i}\right) e^{-\beta\kappa\sum_{i=1}^{N}\left(k_i m_i+\varepsilon\frac{k_i(k_i-1)}{2}\right)} \\
&\times \frac{\prod_{i=1}^{N}\prod_{j=1}^{\tilde{N}_f}\prod_{l=1}^{k_i} 2\sinh\frac{\beta}{2}\big(m_{j,i}+(l-1)\varepsilon\big)}{\prod_{i,j=1}^{N}\prod_{l=1}^{k_i} 2\sinh\frac{\beta}{2}\big(m_{j,i}+(l-1-k_j)\varepsilon\big)\prod_{i=1}^{N}\prod_{j=N+1}^{N_f}\prod_{l=1}^{k_i} 2\sinh\frac{\beta}{2}\big(m_{j,i}+(l-1)\varepsilon\big)},
\end{aligned} \tag{6.1.53}$$

where we denoted the 3d complexified FI-parameters as z_i.

[11] While we have computed the vortex partition function using the vortex world line theory, one might expect that we can also compute them in terms of the equivariant character. However, one can show that there is a difference between the results obtained by the vortex world line theory and equivariant character, and in fact (6.1.53) matches the vortex partition function appeared in the factorization [36].

Vortex partition function of 3d $\mathcal{N} = 2^*$ with fundamental hyper multiplets

Next we consider a vortex partition function of a real mass deformation of 3d $\mathcal{N} = 4$ $U(N)$ supersymmetric gauge theory, whose supersymmetry is often referred to as $\mathcal{N} = 2^*$. We take the matter multiplets as N_f fundamental hypermultiplets. The vortex world line theory preserves four supercharges which are the dimensional reduction of 2d $\mathcal{N} = (2,2)$ type SUSY to one dimension. The Lagrangian consists of $\mathcal{N} = (2,2)$ $U(k)$ vector multiplet, an adjoint chiral multiplet, whose lowest component is given by B, fundamental chiral multiplets with N-flavors, whose lowest component is I, and anti-fundamental chiral multiplets with N_f-flavors whose lowest component is J. We take the real mass parameter for three-dimensional $\mathcal{N} = 2$ adjoint chiral multiplet as m^*. Then the $\mathcal{N} = (2,2)$ multiplets split into $\mathcal{N} = (0,2)$ multiplets.

The supersymmetry transformations and Lagrangians of these multiplets can be written in similar manner as the last subsection. Since the fixed point condition for this theory is the same as (6.1.46), the vortex partition function is

$$\begin{aligned} Z_{\mathrm{V}}^{(k_1,\cdots,k_N)} &= \frac{\prod_{i,j=1}^{N}\prod_{l=1}^{k_i} 2\sinh\frac{\beta}{2}\left(m^* + m_{j,i} + (l-1-k_j)\varepsilon\right)}{\prod_{i,j=1}^{N}\prod_{l=1}^{k_i} 2\sinh\frac{\beta}{2}\left(m_{j,i} + (l-1-k_j)\varepsilon\right)} \\ &\quad\times \frac{\prod_{i=1}^{N}\prod_{j=N+1}^{N_f}\prod_{l=1}^{k_i} 2\sinh\frac{\beta}{2}\left(m^* + m_{i,j} - (l-1)\varepsilon\right)}{\prod_{i=1}^{N}\prod_{j=N+1}^{N_f}\prod_{l=1}^{k_i} 2\sinh\frac{\beta}{2}\left(m_{i,j} - (l-1)\varepsilon\right)}. \end{aligned} \tag{6.1.54}$$

In the 2d limit $\beta \to 0$, the leading behavior reproduces the vortex partition function for $\mathcal{N} = (2,2)^*$ supersymmetric gauge theory considered in [35] as

$$\begin{aligned} \lim_{\beta\to 0} Z_{\mathrm{V}}^{(k_1,\cdots,k_N)} &\sim \frac{\prod_{i,j}^{N}\prod_{l=1}^{k_i}(m^* + m_{j,i} + (l-1-k_j)\varepsilon)}{\prod_{i,j}^{N}\prod_{l=1}^{k_i}(m_{j,i} + (l-1-k_j)\varepsilon)} \\ &\quad\times \frac{\prod_{i=1}^{N}\prod_{j=N+1}^{N_f}\prod_{l=1}^{k_i}(m^* + m_{i,j} - (l-1)\varepsilon)}{\prod_{i=1}^{N}\prod_{j=N+1}^{N_f}\prod_{l=1}^{k_i}(m_{i,j} - (l-1)\varepsilon)}. \end{aligned} \tag{6.1.55}$$

When the real mass parameter m^* goes to zero, the supersymmetry of the 3d theory enhances to $\mathcal{N} = 4$. Then the one-loop determinant (6.1.54) becomes

$$Z_{\mathrm{V}}^{(k_1,\cdots,k_N)}\Big|_{m^*=0} = 1. \tag{6.1.56}$$

The k-vortex partition function Z_{V}^{k} is given by

$$Z_{\mathrm{V}}^{k}\Big|_{m^*=0} = \sum_{k_1+\cdots+k_N=k} 1. \tag{6.1.57}$$

This agrees with the number of possible configurations, where k D1-branes ended on the N D3-branes in the type IIB brane construction of vortices in the 3d $\mathcal{N}=4$ gauge theory [47].

On the other hand when m^* goes to infinity, the leading asymptotic behavior becomes

$$\begin{aligned} Z_{\mathrm{V}}^{(k_1,\cdots,k_N)} \sim \; & \frac{1}{\prod_{i,j=1}^{N}\prod_{l=1}^{k_i} 2\sinh\frac{\beta}{2}\big(m_{j,i}+(l-1-k_j)\varepsilon\big)} \\ & \times\frac{1}{\prod_{i=1}^{N}\prod_{j=N+1}^{N_f}\prod_{l=1}^{k_i} 2\sinh\frac{\beta}{2}\big(m_{i,j}-(l-1)\varepsilon\big)} . \end{aligned} \tag{6.1.58}$$

This agrees with the one-loop determinant of the vortex partition function without the adjoint and anti-fundamental chiral multiplets. Note that the factorization in 3d theory with any adjoint matter have not been derived from the Coulomb branch localization yet. Hence we conjecture that the partition function of the mass deformed $\mathcal{N}=4$ gauge theory is factorized into the product of vortex partition function (6.1.54) and its anti-vortex partner as well as other factors.

6.1.3 Results

We summarize the partition function on the ellipsoid which is obtained by the Higgs branch localization. First, as we have discussed above, we can identify the S^1 length β and the Ω background parameter ε in the vortex partition functions with the fiber radius (6.1.25) and the rotational parameter (6.1.26) respectively:[12]

$$\beta = 2\pi b^{-1}, \quad \varepsilon = ib^{-1}, \quad \text{at the north pole } (\theta=0)\ , \tag{6.1.59}$$

$$\beta = 2\pi b, \qquad \varepsilon = ib, \qquad \text{at the south pole } (\theta=\pi)\ . \tag{6.1.60}$$

Furthermore, we have to take the equivariant masses in the vortex partition function differently from the ones in the 3d $\mathcal{N}=2$ theories as

$$\begin{cases} m_i \to m_i + \frac{\varepsilon}{2}, & (i=1,\cdots,N), \\ \tilde{m}_i \to m_i - \frac{\varepsilon}{2}, & (i=N+1,\cdots,N_f), \\ \tilde{M}_i \to \tilde{M}_i + \frac{\varepsilon}{2}, & (i=1,\cdots,\tilde{N}_f), \end{cases} \tag{6.1.61}$$

to be consistent with the Coulomb branch result. Then the vortex partition function that we have obtained (6.1.53) becomes

$$Z_{\mathrm{V}} = \sum_{\vec{k}=0}^{\infty}\left(\prod_{i=1}^{N} z_i^{k_i}\right) e^{-\beta\kappa\sum_{i=1}^{N}\left(k_i m_i+\varepsilon\frac{k_i^2}{2}\right)}$$

[12] Recall that we set $R=1$.

$$\times \frac{\prod_{i=1}^{N}\prod_{j=1}^{\tilde{N}_f}\prod_{l=1}^{k_i} 2\sinh\frac{\beta}{2}\big(m_{j,i}+(l-1)\varepsilon\big)}{\prod_{i,j=1}^{N}\prod_{l=1}^{k_i} 2\sinh\frac{\beta}{2}\big(m_{j,i}+(l-1-k_j)\varepsilon\big)\prod_{i=1}^{N}\prod_{j=N+1}^{N_f}\prod_{l=1}^{k_i} 2\sinh\frac{\beta}{2}\big(m_{j,i}+l\varepsilon\big)}. \tag{6.1.62}$$

Although similar mass shifts have been observed in an instanton partition function in 4d $\mathcal{N}=2^*$ theory on S^4 [81], we have not found its physical interpretation yet. It is interesting if we could find any physical origin of this shift.

In fact, using the above parameter identification and the mass shift, we can confirm that our results are equivalent to the ones which are obtained in section 5.1. That is to say, although the vortices appear in the Coulomb branch and we did not know why, we can give an interpretation that they are coming from contributions on the north and south poles on the base S^2 in the Higgs branch localization. Furthermore we also stress that the Higgs branch localization gives us new information:

- $\mathcal{N}=2$ theories with any R-charge have the factorization properties.
- Whether a theory has the factorization property does not depend on the matter content if there is no anomaly, and are enough matters not to break the supersymmetries (SUSY vacua).

Let us see an example which confirms an agreement with the Coulomb branch result.

$U(N)$ gauge theory with N_f-flavors

Let us consider a $U(N)$ gauge theory with N_f-flavors. Then note that the parity anomaly cancellation condition is

$$\kappa_{\text{eff}} = \kappa + \frac{N_f - N_f}{2} = \kappa \in \mathbb{Z}. \tag{6.1.63}$$

We can reproduce a result with nonzero R-charge by the analytic continuation, $M \to M + \frac{iQ}{2}\Delta$. Then we obtain the following result,

$$\begin{aligned} Z \;=\; & \sum_{(l_1,\cdots,l_N)\subset(1,\cdots,N_f)} e^{i\pi\kappa\sum_{i=1}^{N} m_{l_i}^2 + 2\pi i\zeta\sum_{i=1}^{N}(m_{l_i}+\frac{iQ}{2}\Delta)} \\ & \times \prod_{i<j}^{N}\Big[\sinh(\pi b D_{l_i l_j})\sinh(\pi b^{-1} D_{l_i l_j})\Big]\prod_{i=1}^{N}\frac{\prod_{A\neq\{l_i\}}^{N_f} s_b(D_{A l_i}+\frac{iQ}{2})}{\prod_{B=1}^{N_f} s_b(C_{B l_i}-\frac{iQ}{2}(1-2\Delta))}\cdot Z_{\text{V}}^{\{l_i\}}\cdot \bar{Z}_{\text{V}}^{\{l_i\}}, \end{aligned} \tag{6.1.64}$$

where $D_{AB}=-(m_A^{(v)}-m_B^{(v)})-(m_A^{(a)}-m_B^{(a)})$, $C_{AB}=-(m_A^{(v)}-m_B^{(v)})+(m_A^{(a)}+m_B^{(a)})$, and by identifying the complexified FI parameter as

$$z_i=e^{-2\pi bk_i\zeta}, \tag{6.1.65}$$

then the vortex partition functions (6.1.62) are

$$\begin{aligned}\bar{Z}_{\mathrm{V}}^{\{l_i\}}=&\sum_{\vec{k}=0}^{\infty}\Bigg\{\Big(\prod_{i=1}^{N}e^{-2\pi b\kappa\left(m_{l_i}k_i+\frac{ib}{2}k_i^2\right)}e^{-2\pi b\zeta k_i}\Big)\\&\times\frac{\prod_{i=1}^{N}\prod_{B=1}^{N_f}\prod_{l=1}^{k_i}2\sinh\pi b(C_{B,l_i}+iQ\Delta+(l-1)ib)}{\prod_{i,j=1}^{N}\prod_{l=1}^{k_i}2\sinh\pi b\big\{D_{l_j,l_i}+(l-1-k_j)ib\big\}\prod_{i=1}^{N}\prod_{A\neq\{l_i\}}^{N_f}\prod_{l=1}^{k_i}2\sinh\pi b(D_{A,l_i}+ilb)}\Bigg\},\end{aligned} \tag{6.1.66}$$

and $Z_{\mathrm{V}}^{\{l_i\}}=\bar{Z}_{\mathrm{V}}^{\{l_i\}}|_{b\to b^{-1}}$. Note that this result is the generalization of a result in section 5.1 with any R-charge Δ. If we take $\Delta=0$, we find that these results are equivalent to (5.1.22) and (5.1.23), up to an overall sign. For the other examples in section 5.1, we can also confirm the equivalence.

6.1.4 Supersymmetric Wilson loop

Let us comment on a BPS Wilson loop on the ellipsoid. We define the supersymmetric Wilson loop in the representation R as

$$W_R(C)=\mathrm{Tr}_R\,\mathcal{P}\exp\left(\oint_C d\tau(iA_\mu\dot{x}^\mu+\sigma|\dot{x}|)\right), \tag{6.1.67}$$

where C is the contour of the Wilson loop parametrized by τ and $\dot{x}^\mu=dx^\mu/d\tau$. In [82], the author has argued that the Wilson loop preserves two supercharges when the contour C is

$$\varphi_2(\tau)=b^{-2}\varphi_1(\tau)+\text{const.},\quad \vartheta=\text{const.}(\neq 0,\pi/2). \tag{6.1.68}$$

Note that this contour becomes a closed loop with a torus knot if and only if b^2 is a rational number. For the Coulomb branch localization, the VEV of the Wilson loop is given by

$$\langle W_R(C)\rangle=\langle\mathrm{Tr}_R\,U\rangle,\quad \text{with } U=\mathrm{diag}(e^{2\pi\sigma_1^{(0)}},\cdots,e^{2\pi\sigma_N^{(0)}}), \tag{6.1.69}$$

where $\sigma_i^{(0)}$ is the saddle point value in the Coulomb branch localization (4.1.13).

For the Higgs branch localization, when the contour C cycles the north and south poles respectively, we have additional contributions,

$$\oint_{\text{around pole}}A_\mu dx^\mu=\beta n,\qquad n\in\mathbb{Z}, \tag{6.1.70}$$

so we find that from (6.1.16), (6.1.59) and (6.1.60), the VEVs are insertions of

$$\langle \mathrm{Tr}_R\, U\rangle \quad \text{with } U = \mathrm{diag}(e^{-2\pi m_{l_1}+2\pi i b^{-1}n_1+2\pi i b\bar{n}_1}, \cdots, e^{-2\pi m_{l_N}+2\pi i b\bar{n}_N+2\pi i b^{-1}n_N}), \tag{6.1.71}$$

to the vortex and anti-vortex partition functions, respectively. We find that the Wilson loop expectation value are symmetric under the interchange between the north and south pole values: $(ib^{-1}, n) \leftrightarrow (ib, \bar{n})$. This is consistent with the observation that the Wilson loops act on holomorphic blocks and anti-holomorphic blocks [74].

6.2 Partition function on $S^1 \times S^2$

We consider the partition function on $S^1 \times S^2$ using the Higgs branch localization. The procedure is the same as the ellipsoid case.

We add a new deformation term,

$$\mathcal{L}_H = \mathcal{Q}\,\mathrm{Tr}\Big[\frac{(\epsilon^\dagger\lambda - \bar{\epsilon}^\dagger\bar{\lambda})h}{2i}\Big], \tag{6.2.1}$$

where h is a function of scalar fields like (6.1.4) and (6.1.5). Again, combined with $\mathcal{L}_{\mathrm{YM}}$ (4.2.14), completing the square for the bosonic part leads to

$$\begin{aligned}
&\mathcal{L}_{\mathrm{YM}}|_{\mathrm{bos.}} + \mathcal{L}_H|_{\mathrm{bos.}} \\
&= \mathrm{Tr}\Big[\ \frac{1}{4}(V_1+\cos\theta\, h)^2 + \frac{1}{4}(V_2 - \sin\theta\, h)^2 + \frac{1}{4}V_3^2 \\
&\qquad + \frac{1}{4}(\bar{V}_1+\cos\theta\, h)^2 + \frac{1}{4}(\bar{V}_2+\sin\theta\, h)^2 + \frac{1}{4}\bar{V}_3^2 + \frac{1}{2}(D+ih)^2\ \Big],
\end{aligned} \tag{6.2.2}$$

where

$$V_a = \frac{1}{2}\epsilon_{abc}F^{bc} - D_a\sigma + \delta_{a1}\sigma, \quad \bar{V}_a = \frac{1}{2}\epsilon_{abc}F^{bc} + D_a\sigma + \delta_{a1}\sigma, \tag{6.2.3}$$

where a, b and c are orthogonal frame indices. We also note that this becomes semi-positive definite by integrating out the auxiliary field D:

$$D = -ih. \tag{6.2.4}$$

Since D has an imaginary part, we find the vortex type BPS configuration like the ellipsoid case. Combining with $\mathcal{L}_\psi$ (4.2.19), we find that the localized configuration is given by

$$\begin{aligned}
&F_{23} + \sigma + \cos\theta\, h = 0, \quad F_{31} = F_{12} = 0, \\
&D_2\sigma + \sin\theta\, h = 0, \quad D_1\sigma = D_3\sigma = 0, \quad D + ih = 0, \\
&D_1\phi = 0, \quad \sin\frac{\theta}{2}D_-\phi + \cos\frac{\theta}{2}(\sigma+M)\phi = 0,
\end{aligned} \tag{6.2.5}$$

$$F = 0, \qquad \cos\frac{\theta}{2} D_+ \phi + \sin\frac{\theta}{2}(\sigma + M)\phi = 0,$$

where we set the R-charge $\Delta = 0$. Also note that we shift the τ direction to (4.2.10). If we take the R-charge as zero, we find that ϕ has only constant solution such that[13]

$$\Big(\sum_A \gamma_A F_A + a\Big)\phi = 0. \tag{6.2.6}$$

where a is a holonomy along the τ direction, and F_A are Cartan generators of the flavor symmetry. Also if ϕ is constant, we find $\sigma = 0$.

As we did in the ellipsoid case, we can show that there is no contribution of ϕ except for $h = 0$ by taking an appropriate limit of the free parameter χ. Therefore we have the following localized configurations except $\theta = 0, \pi$:

$$\begin{aligned} &F_{\mu\nu} = 0, \quad \sigma = 0, \quad D = 0, \\ &F = 0, \quad \Big(\sum_A \gamma_A F_A + a\Big)\phi = 0, \quad h = 0. \end{aligned} \tag{6.2.7}$$

For simplicity, let us consider a $U(N)$ theory with N_f fundamental chiral multiplets. Then, we take $h = (\phi\phi^\dagger - \chi \cdot \mathbb{1}_N)$ [14]. We identify the flavor fugacities as

$$\xi_A^{F_A} = e^{i\gamma_A F_A} = e^{i\beta M_A}. \tag{6.2.8}$$

Then the second equation in the second line of (6.2.7) is

$$\beta \cdot \phi_{iB} M_{BA} + a_{ij}\phi_{jA} = 0, \quad i, j = 1, \cdots, N, \quad A, B = 1, \cdots, N_f. \tag{6.2.9}$$

Let us assume $N_f \geq N$. Then we have ${}_{N_f}C_n$ choices of the vacua in the same way as the ellipsoid case:

$$a_i = -\beta M_{l_i}, \quad \phi_{iA} = \sqrt{\chi}\delta_{l_i A}, \tag{6.2.10}$$

where $(l_1, \cdots, l_N)$ is a set of N integers in $(1, \cdots, N_f)$. Then we have also three types one-loop determinants:

- Vector multiplet,
- Chiral multiplets with vanishing VEV,

[13] See appendix B.2.2 for details.

[14] Note that if we take $\chi \to -\infty$, we should take $h = -\phi\phi^\dagger - \chi \cdot \mathbb{1}_N$, as mentioned in footnote 4. We have considered this case as if χ was positive, for simplicity.

- Chiral multiplets with nonzero VEV.

However, we expect that the one-loop determinant for $\phi \neq 0$ affects no contribution as mentioned in the ellipsoid case. Therefore we have only to compute the one-loop determinants around $a_i = -\beta M_{l_i}$. The one-loop determinants are from (4.2.25) and (4.2.26),

$$\begin{aligned} Z^{\text{(1-loop)}}_{\text{vec}} &= \prod_{i<j}^{N} 2\sinh\Big[-\frac{i}{2}\beta(M_{l_i}-M_{l_j})\Big] 2\sinh\Big[-\frac{i}{2}\beta(M_{l_i}-M_{l_j})\Big] \\ &= \prod_{i\neq j}^{N} 2\sinh\Big[-\frac{i}{2}\beta(M_{l_i}-M_{l_j})\Big], \end{aligned} \tag{6.2.11}$$

$$Z^{\text{(1-loop)}}_{\text{chi}} = \prod_{i=1}^{N}\prod_{A\neq\{l_i\}}^{N_f} \frac{(x^2 e^{-i\beta(M_{l_i}-M_A)};x^2)_\infty}{(e^{i\beta(M_{l_i}-M_A)};x^2)_\infty}. \tag{6.2.12}$$

Note that they are equivalent to a part of (5.2.8).

Finally let us consider the configurations at the north and south poles. At the north pole ($\theta = 0$), we have

$$\begin{aligned} &F_{12}=0, \quad F_{31}=0, \quad F_{23}+h=0, \quad \sigma=0, \quad (a+i\beta M)\phi=0, \\ &D+ih=0, \quad D_2\phi - iD_3\phi=0, \quad D_1\phi=0, \quad F=0, \end{aligned} \tag{6.2.13}$$

which correspond to the anti-vortex equations, while we have at the south pole ($\theta = \pi$)

$$\begin{aligned} &F_{12}=0, \quad F_{31}=0, \quad F_{23}-h=0, \quad \sigma=0, \quad (a+i\beta M)\phi=0, \\ &D+ih=0, \quad D_2\phi + iD_3\phi=0, \quad D_1\phi=0, \quad F=0, \end{aligned} \tag{6.2.14}$$

which are the vortex equations.

We should apply the vortex partition function, which we have computed in the last section, to the present case, too. Since we know the S^1 length as β, what we need to do is identifying the Ω background parameter. We read off the Ω background parameter from (4.2.24):[15]

$$\varepsilon = 2\beta_2\beta^{-1}. \tag{6.2.15}$$

Then, the vortex partition function for a $U(N)$ theory with N_f fundamental and anti-fundamental chiral multiplets is from (6.1.62)

$$Z^{\{l_i\}}_{\text{V}} = \sum_{\vec{k}=0}^{\infty} \frac{1}{\prod_{i,j=1}^{N}\prod_{l=1}^{k_i} 2\sinh\frac{i\beta M_{l_i,l_j}+(l-1-k_j)2\beta_2}{2} \prod_{i=1}^{N}\prod_{A\neq\{i_i\}}^{N_f}\prod_{l=1}^{k_i} 2\sinh\frac{i\beta M_{A,l_i}+2l\beta_2}{2}}.$$

[15]Note that we set an S^2 radius as $R=1$.

(6.2.16)

where we shift $M \to iM$ in (6.1.62), and ignore the CS term for simplicity. We find that this vortex partition function is equivalent to a part of (5.2.9) ((5.2.10)), up to an overall sign. In fact, if we consider a $U(N)$ theory with N_f fundamental and $\tilde{N}_f$ anti-fundamental chiral multiplets, we find that the vortex partition function is equivalent to (5.2.9) ((5.2.10)).

Substituting the localized configurations for (4.2.28), the classical contribution is given as

$$Z_{\mathrm{cl}} = e^{2\pi i \zeta \beta \sum_{i=1}^{N} M_{l_i}}. \tag{6.2.17}$$

This also matches the classical part of (5.2.8).

Summarizing the above, we have obtained the factorization form (5.2.8) of the partition function on $S^1 \times S^2$ in section 5.2 using the Higgs branch localization.

Chapter 7

Conclusion

In this paper, we have shown that the partition functions of some $\mathcal{N}=2$ theories on S_b^3 and $S^1 \times S^2$ factorize into the three-dimensional vortex and anti-vortex partition functions as well as the other factors using the Higgs branch localization. We have given a natural interpretation of this factorization in terms of contributions coming from the north and south poles on the base S^2.

Furthermore we stress that one can also apply the technique of the Higgs branch localization to other various theories. While we have just considered $U(N)$ gauge theories with fundamental and anti-fundamental matters explicitly, it it possible to consider more general theories: theories with more complicated gauge group, quiver gauge theories or gauge theories with more complicated representations of gauge groups, etc. As an interesting application, if we understand a vortex partition function of theories with a bi-fundamental matter, we could apply the idea of the Higgs branch localization to the ABJM model.

Instead of considering different gauge groups and matter representations, it is also possible to investigate whether the factorization occurs on a different curved space. As we mentioned above, while we have considered the partition function on an ellipsoid, it is possible to perform different deformations of the three-sphere. However, it has been known that in many cases their partition functions amount to the same result as that for the ellipsoid by appropriate parameter identifications [60, 62, 63, 64]. Therefore, while we have shown that the partition function on the ellipsoid has the factorization structure, it implies that the result is also applicable to a class of squashed three-spheres directly. In fact recent studies have made it clear [16], and the authors have studied that how deformations for a geometry affects to the partition function systematically.

As mentioned in the last part of section 5.3, the factorization is expected to be related to the decomposition in terms of the holomorphick blocks [74]. In terms of a decomposition of three-manifolds, we expect that the partition functions on other geometry also factorize to the holomorphick blocks, and we can construct various partition functions by gluing the holomorphick blocks appropriately. For example, the authors in [83] have applied the factorization properties on $S^3/\mathbb{Z}_n$ to determine relative phase factors coming from different holonomies. Such idea could give us understanding of the supersymmetic gauge theory on a curved manifold. We expect that it would be very useful to give a factorization form exactly using the Higgs branch localization.

Also we might develop our understanding of vortex partition functions using the factorization structure. While we have not understood the vortices for various theories well yet, the vortices are related with many situations, in particular, the three-dimensional mirror symmetry, in which it is expected that particles and vortices are mapped each other [43]. Of course, it is important to consider them since they encode non-perturbative information of quantum field theories. It could be attractive to study the vortices through the factorization structure.

Furthermore we expect that the partition functions on $S^1 \times S^3$ and $T^2 \times S^2$ have the same factorization properties, since their geometries are one-dimensional uplifts of S^3 and $S^1 \times S^2$. In fact recent studies have shown that the partition function on $S^1 \times S^3$ has the factorization structure [84, 85]. Then, as the three-dimensional case, vortex and anti-vortex partition functions appear, more precisely the elliptic lifts of the 2d vortex partition functions. In terms of the Higgs branch localization, we understand them as contributions coming from vortex-membranes, which wrap torus fibers on north and south poles on the base S^2 of S^3. Also there is an application of the Higgs branch localization to five-dimensional theories [86].

While exact results have been obtained in supersymmetric field theories, the gravity duals for the $\mathcal{N} = 2$ theories on the squashed three-sphere also have been proposed [87, 88, 89, 90, 91]. It would be very interesting to understand what structure in the gravity side corresponds to the factorization structure in the field theory side. Although it might be challenging, it would shed more light on the understanding of the superstring theory.

Finally we expect that our studies will be useful for future researches.

Appendix A

Convention

We summarize our convention in this paper.

A.1 Spinors

The Clifford algebra is given by

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}, \tag{A.1.1}$$

where μ, ν are spacetime indices and γ_μ is defined as $\gamma_\mu := \gamma_a\, e^a{}_\mu$ where $e^a{}_\mu$ is a vielbein, a is a local Lorentz index and $\{\gamma_a\}$ is the Pauli matrix. The contraction of the spinor indices is defined by

$$\bar\psi\psi := \bar\psi^\alpha C_{\alpha\beta}\, \psi^\beta, \quad \bar\psi\gamma^\mu\psi := \bar\psi^\alpha C_{\alpha\beta}\, (\gamma^\mu)^\beta{}_\gamma\, \psi^\gamma, \tag{A.1.2}$$

where C is a charge conjugation matrix given by $C = -i\gamma_2$. For Grassmann-odd spinors, the following relations are satisfied,

$$\bar\psi\psi = \psi\bar\psi, \quad \bar\psi\gamma^\mu\psi = -\psi\gamma^\mu\bar\psi, \quad (\gamma^\mu\bar\psi)\psi = -\bar\psi\gamma^\mu\psi. \tag{A.1.3}$$

Remarks

- Note that some of the signs in the above relations change for Grassmann-even spinors:

$$\bar\psi\psi = -\psi\bar\psi, \quad \bar\psi\gamma^\mu\psi = \psi\gamma^\mu\bar\psi, \quad (\gamma^\mu\bar\psi)\psi = -\bar\psi\gamma^\mu\psi. \tag{A.1.4}$$

- We also use the matrix notation; for example,

$$\bar\psi\psi = \bar\psi^T C\psi, \quad \bar\psi\gamma^\mu\psi = \bar\psi^T C\gamma^\mu\psi. \tag{A.1.5}$$

A.2 Three-sphere

In this section we summarize the basic properties of S^3, following [66]. We also use them to compute the partition function on the ellipsoid.

First we define the three-sphere with the radius R as a pair of complex coordinates $(u, v) \in \mathbb{C}^2$ such that

$$u\bar{u} + v\bar{v} = R^2. \tag{A.2.1}$$

The isometry is $SO(4) \cong SU(2)_L \times SU(2)_R$.

Hopf fibration coordinates

We parametrize the three-sphere by the following coordinates (θ, ϕ, ψ),

$$u = R \sin \frac{\theta}{2} e^{i\frac{(\psi - \phi)}{2}}, \quad v = R \cos \frac{\theta}{2} e^{i\frac{(\psi + \phi)}{2}}, \tag{A.2.2}$$

where $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$ and $0 \le \psi \le 4\pi$. Then the metric is

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu = du d\bar{u} + dv d\bar{v} \\ &= \frac{R^2}{4}(d\theta^2 + d\varphi^2 + d\psi^2 + 2\cos\theta d\varphi d\psi) \\ &= \frac{R^2}{4}\left\{d\theta^2 + \sin^2\theta d\varphi^2 + (d\psi + \cos\theta d\varphi)^2\right\}. \end{aligned} \tag{A.2.3}$$

Using these coordinates, the three-sphere can be regarded as an S^1-fibration on the base S^2 locally, so-called "the Hopf fibration".

Torus fibration coordinates

We also parametrize the three-sphere in an alternative coordinates $(\vartheta, \varphi_1, \varphi_2)$,

$$u = R \sin\vartheta\, e^{i\varphi_1}, \quad v = R\cos\vartheta\, e^{i\varphi_2}, \tag{A.2.4}$$

where $0 \le \vartheta \le \pi/2$, $0 \le \varphi_1, \varphi_2 \le 2\pi$. Comparing this with (A.2.2), we find

$$\vartheta = \frac{\theta}{2}, \quad \varphi_1 = \frac{1}{2}(\psi - \varphi), \quad \varphi_2 = \frac{1}{2}(\psi + \varphi). \tag{A.2.5}$$

The metric is written as

$$ds^2 = R^2(d\vartheta^2 + \sin^2\vartheta d\varphi_1^2 + \cos^2\vartheta d\varphi_2^2). \tag{A.2.6}$$

Using these coordinates, the three-sphere can be viewed as a T^2-fibration on a line segment locally, so-called "the torus fibration".

A.3 Supersymmetries on S^3, S_b^3 and $\mathbb{R}\times S^2$

The supersymmetry transformation is generated by supercharges with the Killing spinors on each geometry satisfying

$$D_\mu\epsilon = \gamma_\mu\tilde{\epsilon}, \quad D_\mu\bar{\epsilon} = \gamma_\mu\tilde{\bar{\epsilon}}. \tag{A.3.1}$$

Note that only for the S_b^3 case, the covariant derivative contains a background $U(1)$ gauge field,

$$D := d + \frac{1}{4}\gamma^{ab}\omega^{ab} - i\mathcal{R}V, \quad V = \frac{1}{2}\Big(1 - \frac{b}{f(\vartheta)}\Big)d\varphi_1 + \frac{1}{2}\Big(1 - \frac{b^{-1}}{f(\vartheta)}\Big)d\varphi_2, \tag{A.3.2}$$

These expressions for S^3, S_b^3 and $\mathbb{R}\times S^2$ are explicitly given by[1]

$$\tilde{\epsilon} = \frac{i}{2R}\epsilon, \qquad \tilde{\bar{\epsilon}} = \frac{i}{2R}\bar{\epsilon}, \qquad \text{for } S^3, \tag{A.3.3}$$

$$\tilde{\epsilon} = \frac{i}{2Rf(\vartheta)}\epsilon, \qquad \tilde{\bar{\epsilon}} = \frac{i}{2Rf(\vartheta)}\bar{\epsilon}, \qquad \text{for } S_b^3, \tag{A.3.4}$$

$$\tilde{\epsilon} = -\frac{1}{2R}\gamma_1\epsilon, \qquad \tilde{\bar{\epsilon}} = \frac{1}{2R}\gamma_1\bar{\epsilon}, \qquad \text{for } \mathbb{R}\times S^2. \tag{A.3.5}$$

Using the Killing spinors, we can construct supersymmetric theories on each space. In the following we summarize the supersymmetry transformations on their curved space. Note that δ is a Grassmann-even, i.e. parameters ϵ and $\bar{\epsilon}$ are Grassmann-odd, below.

A.3.1 Vector multiplet

The supersymmetry transformation is given by

$$\begin{aligned}
\delta A_\mu &= \frac{i}{2}(\bar{\epsilon}\gamma_\mu\lambda - \bar{\lambda}\gamma_\mu\epsilon),\\
\delta\sigma &= \frac{1}{2}(\bar{\epsilon}\lambda - \bar{\lambda}\epsilon),\\
\delta\lambda &= -\frac{1}{2}\gamma^{\mu\nu}\epsilon F_{\mu\nu} + i\gamma^\mu\epsilon D_\mu\sigma - D\epsilon + \frac{2i}{3}\sigma\gamma^\mu D_\mu\epsilon,\\
\delta\bar{\lambda} &= -\frac{1}{2}\gamma^{\mu\nu}\bar{\epsilon}F_{\mu\nu} - i\gamma^\mu\bar{\epsilon}D_\mu\sigma + D\bar{\epsilon} - \frac{2i}{3}\sigma\gamma^\mu D_\mu\bar{\epsilon},\\
\delta D &= -\frac{i}{2}\bar{\epsilon}\gamma^\mu D_\mu\lambda - \frac{i}{2}D_\mu\bar{\lambda}\gamma^\mu\epsilon + \frac{i}{2}[\bar{\epsilon}\lambda, \sigma] + \frac{i}{2}[\bar{\lambda}\epsilon, \sigma] - \frac{i}{6}(D_\mu\bar{\epsilon}\gamma^\mu\lambda + \bar{\lambda}\gamma^\mu D_\mu\epsilon),
\end{aligned} \tag{A.3.6}$$

[1] We have defined each space in section 3.3.

When we decompose it into $\delta = \delta_\epsilon + \delta_{\bar\epsilon}$, these commutators generate the following algebra:

$$\begin{aligned}
[\delta_\epsilon, \delta_{\bar\epsilon}]\, A_\mu &= iv^\nu \partial_\nu A_\mu + i\partial_\mu v^\nu A_\nu - D_\mu \Lambda, \\
[\delta_\epsilon, \delta_{\bar\epsilon}]\, \sigma &= iv^\mu \partial_\mu \sigma + i[\Lambda, \sigma] + \rho\sigma, \\
[\delta_\epsilon, \delta_{\bar\epsilon}]\, \lambda &= iv^\mu \partial_\mu \lambda + \frac{i}{4}\Theta_{\mu\nu}\gamma^{\mu\nu}\lambda + i[\Lambda, \lambda] + \frac{3}{2}\rho\lambda + \alpha\lambda + \alpha\lambda, \\
[\delta_\epsilon, \delta_{\bar\epsilon}]\, \bar\lambda &= iv^\mu \partial_\mu \bar\lambda + \frac{i}{4}\Theta_{\mu\nu}\gamma^{\mu\nu}\bar\lambda + i[\Lambda, \bar\lambda] + \frac{3}{2}\rho\bar\lambda - \alpha\bar\lambda, -\alpha\bar\lambda, \\
[\delta_\epsilon, \delta_{\bar\epsilon}]\, D &= iv^\mu \partial_\mu D + i[\Lambda, D] + 2\rho D + \mathcal{W},
\end{aligned} \tag{A.3.7}$$

and $[\delta_\epsilon, \delta_{\epsilon'}] = [\delta_{\bar\epsilon}, \delta_{\bar\epsilon'}] = 0$ for any fields, where

$$\begin{aligned}
& v^\mu = \bar\epsilon\gamma^\mu\epsilon, \quad \Theta^{\mu\nu} = D^{[\mu} v^{\nu]} + v^\lambda \omega_\lambda^{\mu\nu}, \\
& \Lambda = v^\mu i A_\mu + \sigma\bar\epsilon\epsilon, \quad \rho = \frac{i}{3}(\bar\epsilon\gamma^\mu D_\mu \epsilon + D_\mu\bar\epsilon\gamma^\mu\epsilon), \\
& \alpha = \frac{i}{3}(D_\mu\bar\epsilon\gamma^\mu\epsilon - \bar\epsilon\gamma^\mu D_\mu\epsilon) + v^\mu V_\mu, \quad \mathcal{W} = \frac{1}{3}\sigma(\bar\epsilon\gamma^\mu\gamma^\nu D_\mu D_\nu\epsilon - \epsilon\gamma^\mu\gamma^\nu D_\mu D_\nu \bar\epsilon).
\end{aligned} \tag{A.3.8}$$

Here V_μ is the background $U(1)$ gauge field and $\omega_\lambda^{\mu\nu}$ is the spin connection. As long as consider S^3, S^3_b and $S^1 \times S^2$, we finds that $\mathcal{W}$ and ρ vanish. Since the algebra generates the translation, Lorentz rotation, R-symmetry rotation and gauge transformation, the algebra closes off-shell.

The FI term is on each geometry,

$$\mathcal{L}_{\text{FI}} = D - \frac{\sigma}{R}, \qquad \text{for } S^3, \tag{A.3.9}$$

$$\mathcal{L}_{\text{FI}} = D - \frac{\sigma}{Rf(\vartheta)}, \qquad \text{for } S^3_b, \tag{A.3.10}$$

$$\mathcal{L}_{\text{FI}} = D - \frac{A_1}{R}, \qquad \text{for } \mathbb{R} \times S^2. \tag{A.3.11}$$

We confirm that their SUSY transformations for the Abelian subgroup give just total derivatives.

A.3.2 Chiral multiplet

The supersymmetry transformation is given by

$$\begin{aligned}
\delta\phi &= \bar\epsilon\psi,\\
\delta\bar\phi &= \epsilon\bar\psi,\\
\delta\psi &= i\gamma^\mu\epsilon D_\mu\phi + i\epsilon\sigma\phi + \frac{2\Delta i}{3}\gamma^\mu D_\mu\epsilon\phi + \bar\epsilon F,\\
\delta\bar\psi &= i\gamma^\mu\bar\epsilon D_\mu\bar\phi + i\bar\phi\sigma\bar\epsilon + \frac{2\Delta i}{3}\bar\phi\gamma^\mu D_\mu\bar\epsilon + \bar F\epsilon,\\
\delta F &= \epsilon(i\gamma^\mu D_\mu\psi - i\sigma\psi - i\lambda\phi) + \frac{i}{3}(2\Delta-1)D_\mu\epsilon\gamma^\mu\psi,\\
\delta\bar F &= \bar\epsilon(i\gamma^\mu D_\mu\bar\psi - i\bar\psi\sigma + i\bar\phi\bar\lambda) + \frac{i}{3}(2\Delta-1)D_\mu\bar\epsilon\gamma^\mu\bar\psi,
\end{aligned} \tag{A.3.12}$$

We have assigned R-charges: $(-\Delta,\ \Delta,\ 1-\Delta,\ \Delta-1,\ 2-\Delta,\ \Delta-2)$ to $(\phi,\bar\phi,\psi,\bar\psi,F,\bar F)$, respectively. The commutators generate the following algebra:

$$\begin{aligned}
[\delta_\epsilon,\delta_{\bar\epsilon}]\,\phi &= iv^\mu\partial_\mu\phi + i\Lambda\phi + \Delta\rho\phi - \Delta\alpha\phi,\\
[\delta_\epsilon,\delta_{\bar\epsilon}]\,\bar\phi &= iv^\mu\partial_\mu\bar\phi - i\bar\phi\Lambda + \Delta\rho\bar\phi + \Delta\alpha\bar\phi,\\
[\delta_\epsilon,\delta_{\bar\epsilon}]\,\psi &= iv^\mu\partial_\mu\psi + \frac{1}{4}\Theta_{\mu\nu}\gamma^{\mu\nu}\psi + i\Lambda\psi + \Big(\Delta+\frac{1}{2}\Big)\rho\psi + (1-\Delta)\alpha\psi,\\
[\delta_\epsilon,\delta_{\bar\epsilon}]\,\bar\psi &= iv^\mu\partial_\mu\bar\psi + \frac{1}{4}\Theta_{\mu\nu}\gamma^{\mu\nu}\bar\psi - i\bar\psi\Lambda + \Big(\Delta+\frac{1}{2}\Big)\rho\bar\psi + (\Delta-1)\alpha\bar\psi,\\
[\delta_\epsilon,\delta_{\bar\epsilon}]\,F &= iv^\mu\partial_\mu F + i\Lambda F + (\Delta+1)\rho F + (2-\Delta)\alpha F,\\
[\delta_\epsilon,\delta_{\bar\epsilon}]\,\bar F &= iv^\mu\partial_\mu\bar F - i\bar F\Lambda + (\Delta+1)\rho\bar F + (\Delta-2)\alpha\bar F,
\end{aligned} \tag{A.3.13}$$

and $[\delta_\epsilon,\delta_{\epsilon'}] = [\delta_{\bar\epsilon},\delta_{\bar\epsilon'}] = 0$ except for fields F and $\bar F$. In fact the commutator of F is

$$[\delta_\epsilon,\delta_{\epsilon'}]F = \epsilon\gamma^{\mu\nu}\epsilon'(2D_\mu D_\nu\phi + iF_{\mu\nu}\phi) + \frac{2\Delta}{3}\phi(\epsilon\gamma^\mu\gamma^\nu D_\mu D_\nu\epsilon' - \epsilon'\gamma^\mu\gamma^\nu D_\mu D_\nu\epsilon). \tag{A.3.14}$$

However, for S^3, S^3_b and $S^1\times S^2$, one finds that $[\delta_\epsilon,\delta_{\epsilon'}]F = 0$ [58, 59]. For the commutator on $\bar F$, it is the same, $[\delta_\epsilon,\delta_{\epsilon'}]\bar F = 0$.

Since $\rho = 0$ for all spaces that we consider, we find that this algebra closes off-shell.

Appendix B

Analysis of localized configurations

B.1 Ellipsoid

B.1.1 Chiral multiplet

The BPS equations of the chiral multiplet on S_b^3 are given by (4.1.16),

$$\begin{aligned}
&\cos\vartheta D_1\phi - \sin\vartheta D_2\phi + \frac{i\Delta}{Rf(\vartheta)}\phi = 0, \quad \sigma\phi = 0,\\
&\sin\vartheta D_1\phi + \cos\vartheta D_2\phi + iD_3\phi = 0, \qquad F = 0.
\end{aligned} \tag{B.1.1}$$

First, let us consider the case of $A_\mu = 0$. Then the first column of the above equations become

$$\cos\vartheta\,\partial_1\phi - \sin\vartheta\,\partial_2\phi + \frac{i\Delta}{Rf(\vartheta)}\phi = 0, \quad \sin\vartheta\,\partial_1\phi + \cos\vartheta\,\partial_2\phi + i\partial_3\phi = 0. \tag{B.1.2}$$

We can expand the field ϕ due to the periodicity,

$$\phi(\varphi_1, \varphi_2, \vartheta) = \sum_{m,n\in\mathbb{Z}} \tilde{\phi}_{m,n}(\vartheta) e^{im\varphi_1 + in\varphi_2}. \tag{B.1.3}$$

Then, the first one of (B.1.2) is

$$\Big(nb + mb^{-1} + \frac{\Delta}{f(\vartheta)}\Big)\phi_{m,n}(\vartheta) = 0. \tag{B.1.4}$$

We assume that there is (m, n) such that $\Delta = -f(nb + mb^{-1})$. Then the second one of (B.1.2) becomes

$$\Big\{\sin 2\vartheta\,\partial_\vartheta + f(\vartheta)(nb - mb^{-1}) + \Delta\cos 2\vartheta\Big\}\,\phi_{m,n}(\vartheta) = 0. \tag{B.1.5}$$

The solution is

$$\phi_{m,n} = g_{m,n}(\vartheta) \cdot (\sin 2\vartheta)^{-\Delta/2}, \tag{B.1.6}$$

where we denote only the dependence on Δ explicitly, and $g_{m,n}(\vartheta)$ is a smooth function for ϑ. Here we have assumed that $\Delta \geq 0$. For $\Delta > 0$, we find that this solution is not smooth. For $\Delta = 0$, since the reality condition $(\bar{\phi})^\dagger = \phi$ is satisfied, only $(m,n) = (0,0)$ of the modes survives. From (B.1.4), the solution is $\phi_{0,0} = \text{const}$. If we consider the case of $A_\mu \neq 0$, the result does not change.

B.2 $S^1 \times S^2$

B.2.1 Vector multiplet

The localized configurations of the vector multiplet on $S^1 \times S^2$ in the Coulomb branch localization are given by (4.2.15),

$$F_{23} + \sigma = 0, \quad F_{31} = F_{12} = 0, \quad D_\mu \sigma = 0, \quad D = 0. \tag{B.2.1}$$

If we diagonalize σ, then $F_{23} = -\sigma$ is constant. Furthermore we can use the quantization condition for the flux on S^2,

$$m = \frac{1}{2\pi} \int_{S^2} F = \frac{1}{2\pi} \int_{S^2} F_{23}\, e^2 \wedge e^3 = \frac{F_{23}}{2\pi} \int_{S^2} \sin\theta d\theta d\varphi = 2F_{23}, \tag{B.2.2}$$

where m is an element of the Cartan subalgebra and quantized. Therefore, we obtain

$$F_{23} = -\sigma = \frac{m}{2}. \tag{B.2.3}$$

From this,

$$F_{23} = \frac{m}{2} \quad \Leftrightarrow \quad \partial_\theta A_\varphi = \frac{m}{2} \sin\theta \quad \Leftrightarrow \quad A_\varphi = C - \frac{m}{2} \cos\theta, \tag{B.2.4}$$

where C is an integration constant. Since A_φ needs to vanish on the north and south poles, we obtain the following result,

$$A_\varphi^\pm = \frac{m}{2}(\pm 1 - \cos\theta), \tag{B.2.5}$$

where $A_\varphi^\pm$ denote the sections on the patches including the north(+) and south($-$) poles, respectively.

B.2.2 Chiral multiplet

The BPS equations of the chiral multiplet on $S^1 \times S^2$ are given by (4.2.20),

$$\sin\frac{\theta}{2} D_- \phi + \cos\frac{\theta}{2}(\sigma + \Delta)\phi = 0, \quad D_1\phi = 0,$$
$$\cos\frac{\theta}{2} D_+ \phi + \sin\frac{\theta}{2}(\sigma - \Delta)\phi = 0, \quad F = 0, \tag{B.2.6}$$

where $D_\pm = D_2 \mp iD_3$. We rewrite the first column of them in the following way,

$$\sin\theta\, D_2\phi + \sigma\phi + \Delta\cos\theta\,\phi = 0, \quad i\sin\theta\, D_3\phi + \cos\theta\,\sigma\phi + \Delta\phi = 0. \tag{B.2.7}$$

Recall that the localized configurations of the vector multiplet (4.2.17) are

$$A = -\frac{a}{\beta}d\tau + \frac{m}{2}(\pm 1 - \cos\theta)d\varphi, \quad \sigma = -\frac{m}{2}, \quad D = 0. \tag{B.2.8}$$

By solving the first equation of (B.2.7), we find the dependence on θ,

$$\phi = C(\tau,\varphi)\cdot\left(\tan\frac{\theta}{2}\right)^{m/2}(\sin\theta)^{-\Delta}, \tag{B.2.9}$$

where $C(\tau,\varphi)$ is an integration constant and we have treated $\sigma^{(0)}$ as if it was a number for simplicity. Since we have assumed that $\Delta \geq 0$, we find that there is no smooth solution of ϕ for $\Delta > 0$.

Next let us analyze $D_1\phi = 0$. Recall that (4.2.10),

$$\partial_\tau \quad \to \quad \partial_\tau + \frac{1}{\beta}\Big\{\beta_1(-\mathcal{R} - j_3) + \beta_2 j_3 - i\sum_i \gamma_i F_i\Big\}. \tag{B.2.10}$$

Also we expand ϕ using the monopole harmonics in the following way [71],

$$\phi = \sum_{n\in\mathbb{Z}}\sum_{l,k}\phi_{n,l,k}\, e^{\frac{2\pi i n\tau}{\beta}} Y_{\frac{m}{2},l,k}, \tag{B.2.11}$$

where $l \in \frac{|m|}{2} + \mathbb{N}$ and $k = -l, -l+1, \cdots, +l$. If we take the R-charge to be zero, we find that only modes $(l,k) = (0,0)$ survive from (B.2.7) due to the reality condition. Since $j_3\, Y_{\frac{m}{2},l,k} = k\, Y_{\frac{m}{2},l,k}$, the equation $D_1\phi = 0$ implies

$$\Big(2\pi i n - i\sum_i \gamma_i F_i - ia\Big)\phi_{n,0,0} = 0. \tag{B.2.12}$$

Furthermore we always set $n = 0$ by absorbing it into the holonomy. Finally, for $\Delta = 0$, we have a constant solution ϕ which is satisfied with

$$\Big(\sum_i \gamma_i F_i + a\Big)\phi = 0. \tag{B.2.13}$$

Furthermore if ϕ is constant, (B.2.7) implies $\sigma = 0$.

Appendix C

Computations of the one-loop determinants

In this chapter we consider the one-loop determinants on S_b^3 and $S^1 \times S^2$ using the index theorem following [66]. As we mentioned in section 4.1.4, the one-loop determinant and the equivariant index are related with

$$\mathrm{ind} D_{10} = \sum_j c_j e^{w_j} \quad \leftrightarrow \quad Z_{\text{1-loop}} = \prod_j w_j^{-\frac{c_j}{2}}. \tag{C.0.1}$$

C.1 Index theorem

Let us stand an index theorem that we use in the one-loop computations following [11, 68] (c.f. the original reference [67]). Let (E_0, E_1) be a pair of vector bundles on a manifold M, and V_i be a set of the sections of E_i. Let D be a differential operator $D : V_0 \to V_1$, and let π be a map $\pi : T^*M \to M$, then the pullback $\pi^* E_i$ becomes a vector bundle over T^*M. The symbol of the differential operator D is $\sigma(D) : \pi^* E_0 \to \pi^* E_1$. If we use the local coordinate x^i, the symbol is defined by replacing all derivatives in the highest order with the momenta, $\frac{\partial}{\partial x^i} \to i p_i$. If the symbol $\sigma(D)$ is invertible over $T^*M \setminus \{0\}$, then the differential operator D is elliptic.

Let $T = U(1)^n$ be a maximal torus of a compact Lie group G on M and E_i, and let D be a differential operator which commutes with the G-action. Then an equivariant index of G is defined by

$$\mathrm{ind} D(t) = \mathrm{Tr}_{\mathrm{Ker} D}\, t - \mathrm{Tr}_{\mathrm{CoKer} D}\, t, \quad t = (t_1, \cdots, t_n) \in T. \tag{C.1.1}$$

The Atiyah-Singer index theorem states that if there is a discrete set F of fixed points on

M by the G-action, the index is

$$\mathrm{ind}D(t) = \sum_{p\in F} \frac{\mathrm{Tr}_{E_0(p)}\, t - \mathrm{Tr}_{E_1(p)}\, t}{\det_{TM_p}(1-t)}. \tag{C.1.2}$$

If D is an elliptic operator on a compact manifold M, it is known that the summation over the fixed points yields a finite Laurent polynomial in t.

Incidentally, we consider a subspace of T^*M such that, for $\forall x \in M$,

$$T^*_G M_x = \{p \in T^*M_x \mid p \cdot v(g) = 0,\ \forall g \in \mathfrak{g}\}, \tag{C.1.3}$$

where $v(g)$ is a vector field generated by the Lie algebra $\mathfrak{g}$ of G. If a symbol $\sigma(D)$ is invertible on $T^*_G M \setminus \{0\}$, the differential operator is called transversally elliptic with respect to the G action. In this case, it is also known that the above index contribution over fixed points is well-defined.

Example: Dolbeault operator

We consider the index of the Dolbeault operator on $\mathbb{C}$ under $T = U(1)$ action, $\bar\partial : \Omega^{0,0}(\mathbb{C}) \to \Omega^{0,1}(\mathbb{C})$, $z \to tz$ ($z \in \mathbb{C}$, $t \in U(1)$).

First let us consider it without using the index theorem. We should evaluate the $U(1)$ character on the space of holomorphic functions. Since the transformation of the function under $z \to \tilde z = tz$ is

$$\tilde f(\tilde z) = f(z), \quad \to \quad \tilde f(z) = f(t^{-1}z). \tag{C.1.4}$$

Since any holomorphic function is also expressed by $f(z) = \sum_{k=0}^{\infty} c_k z^k$, the transformation $z \to \tilde z = tz$ is

$$\tilde f(\tilde z) = f(z) = \sum_{k=0}^{\infty} c_k z^k = \sum_{k=0}^{\infty} c_k (t^{-k}\tilde z^k), \quad \to \quad \tilde c_k = t^{-k} c_k. \tag{C.1.5}$$

Therefore the index is

$$\mathrm{ind}\,\bar\partial(t) = \sum_{k=0}^{\infty} t^{-k} = \frac{1}{1-t^{-1}}. \tag{C.1.6}$$

Next let us consider the same index using the index theorem. Since the fixed point under the $U(1)$ is the origin $z = 0$, we should evaluate the $U(1)$ action at the origin. The transformation of the $E_0 = \Omega^{0,0}(\mathbb{C})$ under the $U(1)$ is trivial, while the transformation of the $E_1 = \Omega^{0,1}(\mathbb{C})$ is

$$\tilde f_{\bar z} = f_{\bar z} \frac{d\bar z}{d\bar{\tilde z}} = f_{\bar z}\, \bar t^{-1} = f_{\bar z}\, t, \tag{C.1.7}$$

where we have used $\bar{t} = t^{-1}$. Also since $\det_{T_pM}(1-t) = (1-t)(1-t^{-1})$, we obtain the index using the index theorem in the following way,

$$\operatorname{ind}\bar{\partial}(t) = \frac{1-t}{(1-t)(1-t^{-1})} = \frac{1}{1-t^{-1}}. \tag{C.1.8}$$

This result equals to (C.1.6).

C.2 Ellipsoid

For S_b^3, the Killing spinors (3.3.7) satisfy $\bar{\epsilon} = C\epsilon^*$. We summarize some convenient relations:[1]

$$\begin{aligned}
&\epsilon^\dagger\lambda = (C\epsilon^*)^T C\lambda = (\bar{\epsilon}\lambda),\\
&\bar{\epsilon}^\dagger\bar{\lambda} = (C\epsilon^*)^\dagger\bar{\lambda} = \epsilon^T C^T\lambda = -(\epsilon\bar{\lambda}),\\
&\epsilon^\dagger\gamma^\mu\lambda = (\epsilon^*)^T\gamma^\mu\lambda = (C\epsilon^*)^T C\gamma^\mu\lambda = (\bar{\epsilon}\gamma^\mu\lambda),\\
&\bar{\epsilon}^\dagger\gamma^\mu\bar{\lambda} = (C\epsilon^*)^\dagger\gamma^\mu\bar{\lambda} = \epsilon^T C^T\gamma^\mu\bar{\lambda} = -(\epsilon\gamma^\mu\bar{\lambda}),
\end{aligned} \tag{C.2.1}$$

$$\begin{aligned}
&\epsilon^\dagger\epsilon = 1 = \bar{\epsilon}^\dagger\bar{\epsilon}, \quad \epsilon^\dagger\gamma_1\epsilon = \cos\vartheta = -\bar{\epsilon}^\dagger\gamma_1\bar{\epsilon}, \quad \epsilon^\dagger\gamma_2\epsilon = -\sin\vartheta = -\bar{\epsilon}^\dagger\gamma_2\bar{\epsilon},\\
&\epsilon^\dagger\gamma_3\epsilon = 0 = \bar{\epsilon}^\dagger\gamma_3\bar{\epsilon}, \quad \epsilon^\dagger\bar{\epsilon} = 0.
\end{aligned} \tag{C.2.2}$$

Note that in (C.2.1), we have switched the matrix notation to the component one.

Recall that we define the supercoordinate:

$$X_0 = (X_0^{\text{vec}}; X_0^{\text{chi}}) = (\tilde{A}_\mu\ ; \phi, \bar{\phi}), \quad X_1 = (X_1^{\text{vec}}; X_1^{\text{chi}}) = (\Lambda, c, \bar{c}\ ; \epsilon\psi, \bar{\epsilon}\bar{\psi}), \tag{C.2.3}$$

where

$$\Lambda = \bar{\epsilon}\lambda + \epsilon\bar{\lambda}. \tag{C.2.4}$$

Also we define

$$\Lambda_\mu = \bar{\epsilon}\gamma_\mu\lambda + \epsilon\gamma_\mu\bar{\lambda}. \tag{C.2.5}$$

C.2.1 Chiral multiplet

We consider the 1-loop determinant of the chiral multiplet around $\phi = F = 0$.

$$V_{\text{chi}} \;=\; (\mathcal{Q}\psi)^\dagger\psi + (\mathcal{Q}\bar{\psi})^\dagger\bar{\psi}$$

[1] Note that ϵ, $\bar{\epsilon}$ are Grassmann-even spinors.

$$
\begin{aligned}
&= -D_\mu\bar{\phi}(\bar{\epsilon}\gamma^\mu\psi) - \bar{\phi}\sigma(\bar{\epsilon}\psi) + \frac{i\Delta}{Rf}\bar{\phi}(\bar{\epsilon}\psi) + i\bar{F}(\epsilon\psi) \\
&\quad + D_\mu\phi(\epsilon\gamma^\mu\bar{\psi}) + \sigma\phi(\epsilon\bar{\psi}) - \frac{i\Delta}{Rf}\phi(\epsilon\bar{\psi}) - iF(\bar{\epsilon}\bar{\psi}),
\end{aligned} \tag{C.2.6}
$$

Also we can rewrite this in the following way,

$$
\begin{aligned}
V_{\rm chi} &= -D_\mu\bar{\phi}(\bar{\epsilon}\gamma^\mu\psi)_\perp - D_\mu\bar{\phi}v^\mu(\bar{\epsilon}\psi) - \bar{\phi}\sigma(\bar{\epsilon}\psi) + \frac{i\Delta}{Rf}\bar{\phi}(\bar{\epsilon}\psi) + i\bar{F}(\epsilon\psi) \\
&\quad + D_\mu\phi(\epsilon\gamma^\mu\bar{\psi})_\perp + D_\mu\phi v^\mu(\epsilon\bar{\psi}) + \sigma\phi(\epsilon\bar{\psi}) - \frac{i\Delta}{Rf}\phi(\epsilon\bar{\psi}) - iF(\bar{\epsilon}\bar{\psi}),
\end{aligned} \tag{C.2.7}
$$

where $\perp$ means orthogonal to v^μ. Also we used the following formula,

$$
\begin{aligned}
\bar{\epsilon}\gamma_\mu\psi &= v_\mu(\bar{\epsilon}\psi) + (\bar{\epsilon}\gamma_\mu\psi)_\perp = v_\mu(\bar{\epsilon}\psi) - (\epsilon\psi)(\bar{\epsilon}\gamma_\mu\bar{\epsilon}) \\
\epsilon\gamma_\mu\bar{\psi} &= -v_\mu(\epsilon\bar{\psi}) + (\epsilon\gamma_\mu\bar{\psi})_\perp = -v_\mu(\epsilon\bar{\psi}) + (\bar{\epsilon}\bar{\psi})(\epsilon\gamma_\mu\epsilon).
\end{aligned} \tag{C.2.8}
$$

Since all the fields of the chiral multiplet vanish at the saddle point, the quadratic fluctuations become

$$
\begin{aligned}
V^{(2)}_{\rm chi} &= -D^{(0)}_\mu\bar{\phi}(\bar{\epsilon}\gamma^\mu\psi)_\perp - D^{(0)}_\mu\bar{\phi}v^\mu(\bar{\epsilon}\psi) - \bar{\phi}\sigma^{(0)}(\bar{\epsilon}\psi) + \frac{i\Delta}{Rf}\bar{\phi}(\bar{\epsilon}\psi) + i\bar{F}(\epsilon\psi) \\
&\quad + D^{(0)}_\mu\phi(\epsilon\gamma^\mu\bar{\psi})_\perp + D^{(0)}_\mu\phi v^\mu(\epsilon\bar{\psi}) + \sigma^{(0)}\phi(\epsilon\bar{\psi}) - \frac{i\Delta}{Rf}\phi(\epsilon\bar{\psi}) - iF(\bar{\epsilon}\bar{\psi}),
\end{aligned} \tag{C.2.9}
$$

where $D^{(0)}_\mu = \partial_\mu + iA^{(0)}_\mu$ and $\sigma^{(0)}$ mean the quantities at the saddle point, and from now on we will omit the notation (0) for simplicity. Here from (C.2.3), extracting terms associated with D_{10},

$$
\begin{aligned}
X^{\rm chi}_1 D^{\rm chi}_{10} X^{\rm chi}_0 &= -D_\mu\phi(\epsilon\gamma^\mu\bar{\psi})_\perp + D_\mu\bar{\phi}(\bar{\epsilon}\gamma^\mu\psi)_\perp \\
&= -(\bar{\epsilon}\bar{\psi})(\epsilon\gamma^\mu\epsilon)\, D_\mu\phi + \text{c.c.}\,,
\end{aligned} \tag{C.2.10}
$$

where the charge conjugation is defined by $\psi^* = -C\bar{\psi}$. Here, let us consider the differential operator $(\epsilon\gamma^\mu\epsilon)D_\mu$. First we note the following fact, for $(\vartheta, \varphi_1, \varphi_2)$,

$$
\epsilon\gamma^\mu\epsilon = e^{i(\varphi_1+\varphi_2)}\Big(\frac{i}{Rb}\cot\vartheta,\ -\frac{ib}{R}\tan\vartheta,\ \frac{1}{Rf}\Big), \tag{C.2.11}
$$

$$
\epsilon\gamma_\mu\epsilon = e^{i(\varphi_1+\varphi_2)}\Big(iRb\cos\vartheta\sin\vartheta,\ -\frac{iR}{b}\cos\vartheta\sin\vartheta,\ Rf\Big). \tag{C.2.12}
$$

Also we define the following quantities,

$$
\omega^\mu = (0, 0, \frac{1}{Rf}), \quad \omega_\mu = (0, 0, Rf),
$$

$$u^\mu = (\frac{1}{Rb}\cot\vartheta,\, -\frac{b}{R}\tan\vartheta,\, 0), \quad u_\mu = (Rb\cos\vartheta\sin\vartheta,\, -\frac{R}{b}\cos\vartheta\sin\vartheta,\, 0). \quad \text{(C.2.13)}$$

These satisfy $\omega^\mu\omega_\mu = u^\mu u_\mu = 1$, $\omega^\mu u_\mu = 0$. Also,

$$v^\mu = \bar{\epsilon}\gamma^\mu\epsilon = \Big(\frac{1}{Rb},\, \frac{b}{R},\, 0\Big), \quad v_\mu = (Rb\sin^2\vartheta,\, \frac{R}{b}\cos^2\vartheta,\, 0), \tag{C.2.14}$$

so these also satisfy $v^\mu\omega_\mu = v^\mu u_\mu = 0$ and $v^\mu v_\mu = 1$. Therefore (u, v, w) is orthogonal normal basis of the tangent space. The symbol σ of differential operator $(\epsilon\gamma^\mu\epsilon)D_\mu$ becomes

$$\sigma = i(\epsilon\gamma^\mu\epsilon)ip_\mu = -e^{i(\varphi_1+\varphi_2)}(\omega\cdot p + iu\cdot p). \tag{C.2.15}$$

Therefore,

$$|\sigma|^2 = |-e^{i(\varphi_1+\varphi_2)}(\omega\cdot p + iu\cdot p)|^2 = (\omega\cdot p)^2 + (u\cdot p)^2 = p^2 - (v\cdot p)^2, \tag{C.2.16}$$

where we used $p^2 = (v\cdot p)^2 + (\omega\cdot p)^2 + (u\cdot p)^2$ in the last line. We find that the differential operator is not elliptic, but restricting the momentum to the orthogonal direction of v, the symbol becomes invertible. Therefore this differential operator is transversally elliptic with respect to the action generated by $\hat{\mathcal{Q}}^2$ (4.1.24).

Since we would like to apply the index theorem, next let us consider the indices on the north and south poles, which are fixed points on the base S^2 for the $\hat{\mathcal{Q}}^2$-action. At the north and south poles, the metric becomes respectively,

$$ds^2\big|_{\theta\sim 0,\, S^2} \sim \frac{R^2}{4}b^2(d\theta^2 + \theta^2 d\varphi^2), \tag{C.2.17}$$

$$ds^2\big|_{\theta\sim \pi,\, S^2} \sim \frac{R^2}{4}b^{-2}(d\theta^2 + (\theta-\pi)^2 d\varphi^2). \tag{C.2.18}$$

Then, the relevant operator at the north pole on the base S^2 becomes

$$(\epsilon\gamma^\mu\epsilon)D_\mu\big|_{\theta\sim 0,\, S^2} = R\,b\,e^{i(\psi+\varphi)}D_{\bar{z}}, \tag{C.2.19}$$

where we have chosen $z \sim \theta e^{-i\varphi}$ as the local complex coordinate. Then $D_{\bar{z}} = \frac{1}{2}e^{-i\varphi}(D_\theta - i\theta^{-1}D_\varphi)$. Also we note that $\psi+\varphi$ is the fiber direction at the north pole. Likewise,

$$(\epsilon\gamma^\mu\epsilon)D_\mu\big|_{\theta\sim \pi,\, S^2} = R\,b^{-1}\,e^{i(\psi-\varphi)}D_{\bar{z}}, \tag{C.2.20}$$

where we have chosen $z \sim (\pi-\theta)e^{i\varphi}$ as the local complex coordinate, and $\psi-\varphi$ is the fiber direction at the south pole. Note that $D_{\bar{z}} = -\frac{1}{2}e^{i\varphi}(D_\theta - \frac{i}{\pi-\theta}D_\varphi)$. Therefore D_{10}^{chi} acts as a twisted Dolbeault operator at the north and south pole on the base S^2, which are directions orthogonal to v.

The index for the untwisted Dolbeault operator is $\mathrm{ind}\,\bar{\partial}(t) = \frac{1}{1-t^{-1}}$ (C.1.6). Taking account of the gauge bundle, the rotation of the fiber direction and R-symmetry rotation from (4.1.5), then the indices are[2]

$$\mathrm{ind}\, D_{10}^{\mathrm{chi}}\big|_{\text{N-pole}} = \sum_{n\in\mathbb{Z}} e^{inb}\, e^{\frac{i}{2}\Delta(b+b^{-1})}\frac{1}{1-e^{-i(b-b^{-1})}}\sum_{\omega} e^{\omega(\hat{\sigma})}, \tag{C.2.21}$$

$$\mathrm{ind}\, D_{10}^{\mathrm{chi}}\big|_{\text{S-pole}} = \sum_{n\in\mathbb{Z}} e^{inb^{-1}}\, e^{\frac{i}{2}\Delta(b+b^{-1})}\frac{1}{1-e^{i(b-b^{-1})}}\sum_{\omega} e^{\omega(\hat{\sigma})}, \tag{C.2.22}$$

where we identify t with $e^{i(b-b^{-1})}$ at the north pole, and $e^{-i(b-b^{-1})}$ at the south pole from (4.1.37), and define $\hat{\sigma} = R\sigma$. Note that we have to distinguish the complex index from the real index. Therefore the total contribution is given by

$$\begin{aligned}
&\mathrm{ind}\, D_{10}^{\mathrm{chi}} = 2\Big(\mathrm{ind}\, D_{10}^{\mathrm{chi}}\big|_{\text{N-pole}} + \mathrm{ind}\, D_{10}^{\mathrm{chi}}\big|_{\text{S-pole}}\Big)\\
= \;& 2\Big(\exp\Big[\prod_{\omega\in R}\prod_{m=0}^{\infty}\prod_{n=0}^{\infty} i\Big\{mb+nb^{-1}+\frac{Q}{2}-i\omega(\hat{\sigma})-\frac{Q}{2}(1-\Delta)\Big\}\Big]\\
&\qquad - \exp\Big[\prod_{\omega\in R}\prod_{m=0}^{\infty}\prod_{n=0}^{\infty} -i\Big\{mb+nb^{-1}+\frac{Q}{2}+i\omega(\hat{\sigma})+\frac{Q}{2}(1-\Delta)\Big\}\Big]\Big).
\end{aligned} \tag{C.2.23}$$

According to the rule (4.1.36), up to an overall sign, the determinant is

$$\begin{aligned}
Z_{\mathrm{chi}}^{(\text{1-loop})} &= \prod_{\omega\in R}\prod_{m=0}^{\infty}\prod_{n=0}^{\infty}\frac{\Big(mb+nb^{-1}+\frac{Q}{2}+i\omega(\hat{\sigma})+\frac{Q}{2}(1-\Delta)\Big)}{\Big(mb+nb^{-1}+\frac{Q}{2}-i\omega(\hat{\sigma})-\frac{Q}{2}(1-\Delta)\Big)}\\
&= \prod_{w\in R} s_b\left(\frac{iQ}{2}(1-\Delta)-w\cdot\hat{\sigma}\right).
\end{aligned} \tag{C.2.24}$$

C.2.2 Vector multiplet

Let us consider the one-loop determinant of the vector multiplet around $A_\mu = 0$, $\sigma =$ const., $D = 0$.

$$\begin{aligned}
&V_{\mathrm{vec}} = \mathrm{Tr}\Big[(\mathcal{Q}\lambda)^\dagger\lambda + (\mathcal{Q}\bar{\lambda})^\dagger\bar{\lambda}\Big]\\
= \;& \mathrm{Tr}\Big[\frac{1}{2}\epsilon_{\mu\nu\rho}F^{\nu\rho}(\bar{\epsilon}\gamma^\mu\lambda-\epsilon\gamma^\mu\bar{\lambda}) - D_\mu\sigma(\bar{\epsilon}\gamma^\mu\lambda+\epsilon\gamma^\mu\bar{\lambda}) + iD(\bar{\epsilon}\lambda+\epsilon\bar{\lambda}) + \frac{i}{Rf}(\bar{\epsilon}\lambda+\epsilon\bar{\lambda})\Big].
\end{aligned} \tag{C.2.25}$$

[2]We consider contributions of $U(1)$ actions in $iR\hat{\mathcal{Q}}^2$. Note that the coefficient(iR) affects only an overall factor in the one-loop determinant.

Note that the following formula,

$$\bar{\epsilon}\lambda = \frac{1}{2}(\Lambda + v^\mu \Lambda_\mu), \quad \epsilon\bar{\lambda} = \frac{1}{2}(\Lambda - v^\mu \Lambda_\mu),$$
$$\bar{\epsilon}\gamma_\mu\lambda = \frac{1}{2}(v_\mu \Lambda + \Lambda_\mu - i\epsilon_{\mu\nu\rho} v^\nu \Lambda^\rho), \quad \epsilon\gamma_\mu\bar{\lambda} = \frac{1}{2}(-v_\mu \Lambda + \Lambda_\mu + i\epsilon_{\mu\nu\rho} v^\nu \Lambda^\rho), \tag{C.2.26}$$

Then,

$$V_{\text{vec}} = \text{Tr}\Big[\frac{1}{2}\epsilon_{\mu\nu\rho}F^{\nu\rho}v^\mu\Lambda - iF^{\mu\nu}v_\mu\Lambda_\nu - D_\mu\sigma\Lambda^\mu + \frac{i}{2}D\Lambda + \frac{i}{Rf}\sigma\Lambda\Big]. \tag{C.2.27}$$

Also taking account of the gauge fixing term, the quadratic fluctuations become

$$\begin{aligned}\hat{V}^{(2)}_{\text{vec}} &= V^{(2)}_{\text{vec}} + \text{Tr}\Big(\bar{c}G(\tilde{A}) + \frac{\xi}{2}B\Big)\\ &= \text{Tr}\Big[\ \epsilon^{\mu\nu\rho}D^{(0)}_\nu\tilde{A}_\rho v_\mu\Lambda - iD^{(0)}_\mu\tilde{A}_\nu\Big(v^\mu\Lambda^\nu - v^\nu\Lambda^\mu\Big) - D^{(0)}_\mu\tilde{\sigma}\Lambda^\mu\\ &\qquad - i[\tilde{A}_\mu, \sigma^{(0)}]\Lambda^\mu + \frac{i}{2}\tilde{D}\Lambda + \frac{1}{Rf}\tilde{\sigma}\Lambda + \bar{c}\Big(G(\tilde{A}) + \frac{\xi}{2}B\Big)\ \Big].\end{aligned} \tag{C.2.28}$$

From now on we will omit (0) for simplicity. We also note the following fact,

$$\begin{aligned}\Lambda_\mu &= \bar{\epsilon}\gamma_\mu\lambda + \epsilon\gamma_\mu\bar{\lambda}\\ &= -2\hat{\mathcal{Q}}\tilde{A}_\mu + 2Q_B\tilde{A}_\mu = -2\hat{\mathcal{Q}}\tilde{A}_\mu + 2D_\mu c,\end{aligned} \tag{C.2.29}$$

Furthermore, we divide the fluctuation of the gauge field into the components parallel and orthogonal to the vector field v^μ: $\tilde{A}_\mu = a_\mu + v_\mu b$, where $v^\mu a_\mu = 0$. Then from (C.2.3), the terms related with D^{vec}_{10} are, up to a total derivative,

$$\begin{aligned}X^{\text{vec}}_1 D^{\text{vec}}_{10} X^{\text{vec}}_0 &= \epsilon^{\mu\nu\rho}D_\nu(a_\rho + v_\rho b)v_\mu\Lambda - 2iD^{mu}(a_\nu + v_\nu b)(v^\mu D^\nu c - v^\nu D^\mu c) + \bar{c}G(a_\mu + v_\mu b),\\ &= \Big(2ic\ \bar{c}\ \Lambda\Big)\begin{pmatrix}2D^{[\mu}v^{\nu]}D_\nu v_\mu & 2D^{[\mu}v^{\nu]}D_\nu\\ G^\mu v_\mu & G^\mu\\ \epsilon^{\mu\nu\rho}v_\rho D_\nu v_\mu & -\epsilon^{\mu\nu\rho}v_\rho D_\nu\end{pmatrix}\begin{pmatrix}b\\ a_\mu\end{pmatrix},\end{aligned} \tag{C.2.30}$$

We take the following gauge fixing term concretely,

$$G(\tilde{A}) = G^\mu\tilde{A}_\mu = \big\{D^\mu + (D^\nu\mathcal{D}_\nu - v^\nu D_\nu)\big\}\tilde{A}_\mu, \tag{C.2.31}$$

where we introduce

$$\mathcal{D}_\mu := \frac{Rf}{2}\epsilon_{\mu\nu\rho}v^\nu D^\rho. \tag{C.2.32}$$

Then we can obtain a block-diagonalized form in the following way,

$$X^{\text{vec}}_1 D^{\text{vec}}_{10} X^{\text{vec}}_0$$

$$= \left(2ic,\ \bar{c} - iv^\nu D_\nu c,\ -2\Lambda\right) \begin{pmatrix} -D^{[\mu}v^{\nu]}D_\nu v_\mu - \frac{1}{2}D_\nu v^\nu D^\mu v_\mu & 0 \\ 0 & D^\mu \\ 0 & -\epsilon^{\mu\nu\rho}v_\rho D_\nu \end{pmatrix} \begin{pmatrix} b \\ a_\mu + \mathcal{D}_\mu b \end{pmatrix}. \tag{C.2.33}$$

Since when we consider the symbol, we take account of only the differential operators which are of the highest order. Therefore we consider only the upper left corner in the matrix. The symbol has $|\sigma|^2 = p^2 - (v \cdot p)^2$. Therefore the differential operator is not elliptic, but restricting the momentum to the orthogonal direction of v, the symbol becomes invertible. So the differential operator is transversally elliptic with respect to actions generated by $\hat{\mathcal{Q}}^2$ (4.1.24).

The relevant differential operator is the differential in the twisted de Rham complex, $D_{\mathrm{dR}} : \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2$. First we consider an index of the untwisted de Rahm complex [35]. The index is defined by

$$\mathrm{ind}\, D_{dR}(t) = \mathrm{Tr}_{\mathrm{Ker}\, d_0} t - \mathrm{Tr}_{\mathrm{CoKer}\, d_0} t - \mathrm{Tr}_{\mathrm{Ker}\, d_1} - \mathrm{Tr}_{\mathrm{CoKer}\, d_1} t, \tag{C.2.34}$$

where d_p is a differential which acts on Ω^p. This complex is equivalent to one of the following complex; $\mathrm{ind}\, D_{\mathrm{dR}} = -\mathrm{ind}(d^\dagger \oplus d)$ where

$$d^\dagger \oplus d : \ \Omega^1 \ \to \Omega^0 \oplus \Omega^2. \tag{C.2.35}$$

We use the index theorem (C.1.2) by performing a complexification of the complex. Let us introduce a complex variable z, and consider the $U(1)$ action, $z \to tz$ ($\bar{z} \to t^{-1}\bar{z}$). First Ω^0 is generated by 1, so $\mathrm{Tr}_{\Omega^0} t = 1$. Ω^1 is generated by dz and $d\bar{z}$, so $\mathrm{Tr}_{\Omega^1} t = t + t^{-1}$. Ω^2 is generated by $dz \wedge d\bar{z}$, so $\mathrm{Tr}_{\Omega^2} t = 1$. Finally, TM is generated by ∂_z and $\partial_{\bar{z}}$, so $\det_{TM}(1-t) = (1-t)(1-t^{-1})$. Therefore, using (C.1.2), the index of the untwisted de Rham complex is

$$\mathrm{ind} D_{\mathrm{dR}} = -\frac{t + t^{-1} - 1 - 1}{(1-t)(1-t^{-1})} = 1, \tag{C.2.36}$$

Taking account of the gauge bundle and the rotation of the fiber direction, the contribtion to the index on the north pole is[3]

$$\mathrm{ind} D^{\mathrm{vec}}_{10}\big|_{\text{N-pole}} = -\sum_{n\in\mathbb{Z}} e^{inb} \sum_{\alpha} e^{\alpha(\hat{\sigma})}, \tag{C.2.37}$$

[3] Here we consider contributions of $U(1)$ actions in $iR\hat{\mathcal{Q}}^2$.

where α denotes the root, and the contribution to the index on the south pole is

$$\mathrm{ind}D_{10}^{\mathrm{vec}}\big|_{\text{S-pole}} = -\sum_{n\in\mathbb{Z}} e^{inb^{-1}} \sum_{\alpha} e^{\alpha(\hat{\sigma})}. \tag{C.2.38}$$

Therefore applying the rule (4.1.36), up to an overall factor, the one-loop determinant is

$$Z_{\mathrm{vec}}^{(\text{1-loop})} = \prod_{\alpha>0} \sinh(\pi b\alpha(\hat{\sigma}))\sinh(\pi b\alpha(\hat{\sigma})), \tag{C.2.39}$$

where we have used the following formula,

$$\prod_{n=1}^{\infty}\left(1+\frac{a^2}{n^2}\right) = \frac{\sinh a\pi}{a\pi}. \tag{C.2.40}$$

Note that this one-loop determinant does not include a contribution from the ghosts.

C.3 $S^1 \times S^2$

For $S^1 \times S^2$, the Killing spinors (4.2.8) and (4.2.9) satisfy $\bar{\epsilon} = C\gamma_1\epsilon^*$. We summarize convenient relations:[4]

$$\begin{aligned}
\epsilon^\dagger\lambda &= (\epsilon^*)^T\lambda = (C\gamma_1\epsilon^*)^T C\gamma_1\lambda = (\bar{\epsilon}\gamma_1\lambda),\\
\bar{\epsilon}^\dagger\bar{\lambda} &= (C\gamma_1\epsilon^*)^\dagger\bar{\lambda} = \epsilon^T\gamma_1 C^T\lambda = (\epsilon\gamma_1\bar{\lambda}),\\
\epsilon^\dagger\gamma^\mu\lambda &= (\epsilon^*)^T\gamma^\mu\lambda = (C\gamma_1\epsilon^*)^T C\gamma_1\gamma^\mu\lambda = (\bar{\epsilon}\gamma_1\gamma^\mu\lambda),\\
\bar{\epsilon}^\dagger\gamma^\mu\bar{\lambda} &= (C\gamma_1\epsilon^*)^\dagger\gamma^\mu\bar{\lambda} = \epsilon^T\gamma_1 C^T\gamma^\mu\bar{\lambda} = (\epsilon\gamma_1\gamma^\mu\bar{\lambda}),
\end{aligned} \tag{C.3.1}$$

$$\begin{aligned}
&\epsilon^\dagger\epsilon = 1 = \bar{\epsilon}^\dagger\bar{\epsilon}, \quad \epsilon^\dagger\gamma^1\epsilon = -\cos\theta = -\bar{\epsilon}^\dagger\gamma^1\bar{\epsilon}, \quad \epsilon^\dagger\gamma^2\epsilon = \sin\theta = \bar{\epsilon}^\dagger\gamma^2\bar{\epsilon}, \quad \epsilon^\dagger\gamma^3\epsilon = 0 = \bar{\epsilon}^\dagger\gamma^3\bar{\epsilon},\\
&\epsilon^\dagger\bar{\epsilon} = ie^{i\phi}\sin\theta, \quad \epsilon^\dagger\gamma^1\bar{\epsilon} = 0, \quad \epsilon^\dagger\gamma^2\bar{\epsilon} = ie^{i\phi}, \quad \epsilon^\dagger\gamma^3\bar{\epsilon} = -e^{i\phi}\cos\theta.
\end{aligned} \tag{C.3.2}$$

Note that in (C.3.1), we have switched the matrix notation to the component one.

We introduce

$$\varepsilon_0 := \begin{pmatrix}\epsilon\\ -\bar{\epsilon}\end{pmatrix}, \quad \varepsilon_1 := \begin{pmatrix}\gamma_1\epsilon\\ -\gamma_1\bar{\epsilon}\end{pmatrix}, \quad \varepsilon_j := \begin{pmatrix}\gamma_j\epsilon\\ \gamma_j\bar{\epsilon}\end{pmatrix}, \quad (j=2,3), \tag{C.3.3}$$

then they satisfy $\varepsilon_m^\dagger\varepsilon_m = 0$, $(m=0,1,2,3)$. We also define

$$\Lambda_m := \varepsilon_m^\dagger\begin{pmatrix}\gamma_1\lambda\\ -\gamma_1\bar{\lambda}\end{pmatrix}, \tag{C.3.4}$$

and choose supercoordinates as

$$X_0 = (X_0^{\mathrm{vec}}; X_0^{\mathrm{chi}}) = (\tilde{A}_j, \tilde{\sigma}\ ; \phi, \bar{\phi}), \quad X_1 = (X_1^{\mathrm{vec}}; X_1^{\mathrm{chi}}) = (\Lambda_1, c, \bar{c}\ ; \epsilon\gamma_1\psi, \bar{\epsilon}\gamma_1\bar{\psi}). \tag{C.3.5}$$

[4]Note that ϵ, $\bar{\epsilon}$ are Grassmann-even spinors.

C.3.1 Chiral multiplet

Let us consider the one-loop determinant of the chiral multiplet.

$$\begin{aligned} V_{\text{chi}} &= (\mathcal{Q}\psi)^\dagger\psi + (\mathcal{Q}\bar{\psi})^\dagger\psi \\ &= -D_\mu\bar{\phi}(\bar{\epsilon}\gamma_1\gamma^\mu\psi) - D_\mu\phi(\epsilon\gamma_1\gamma^\mu\bar{\psi}) - \bar{\phi}\sigma(\bar{\epsilon}\gamma_1\psi) - \sigma\phi(\epsilon\gamma_1\bar{\psi}) \\ &\quad +\Delta\bar{\phi}(\bar{\epsilon}\psi) - \Delta\phi(\epsilon\bar{\psi}) - i\bar{F}(\epsilon\gamma_1\psi) - iF(\bar{\epsilon}\gamma_1\bar{\psi}), \end{aligned} \tag{C.3.6}$$

Note that the following relations,

$$\bar{\epsilon}\gamma^\mu\psi = (\bar{\epsilon}\gamma^\mu\gamma_1\epsilon)(\bar{\epsilon}\psi) + (\bar{\epsilon}\gamma^\mu\bar{\epsilon})(\epsilon\gamma_1\psi), \tag{C.3.7}$$

$$\epsilon\gamma^\mu\bar{\psi} = (\epsilon\gamma^\mu\epsilon)(\bar{\epsilon}\gamma_1\bar{\psi}) + (\epsilon\gamma_\mu\gamma_1\bar{\epsilon})(\epsilon\bar{\psi}). \tag{C.3.8}$$

Using these formulas,

$$\begin{aligned} V_{\text{chi}} &= -D^a\bar{\phi}\Big[i\epsilon_{1ab}\Big\{(\bar{\epsilon}\gamma^b\gamma_1\epsilon)(\bar{\epsilon}\psi) + (\bar{\epsilon}\gamma^b\bar{\epsilon})(\epsilon\gamma_1\psi)\Big\}\Big] \\ &\quad -D^a\phi\Big[i\epsilon_{1ab}\Big\{(\epsilon\gamma^b\epsilon)(\bar{\epsilon}\gamma_1\bar{\psi}) + (\epsilon\gamma^b\gamma_1\bar{\epsilon})(\epsilon\bar{\psi})\Big\}\Big] \\ &\quad -\bar{\phi}\sigma\Big\{(\bar{\epsilon}\epsilon)(\bar{\epsilon}\psi) + (\bar{\epsilon}\gamma_1\bar{\epsilon})(\epsilon\gamma_1\psi)\Big\} - \sigma\phi\Big\{(\epsilon\gamma_1\epsilon)(\bar{\epsilon}\gamma_1\bar{\psi}) + (\epsilon\bar{\epsilon})(\epsilon\bar{\psi})\Big\} \\ &\quad +\Delta\bar{\phi}(\bar{\epsilon}\psi) - \Delta\phi(\epsilon\bar{\psi}) - i\bar{F}(\epsilon\gamma_1\psi) - iF(\bar{\epsilon}\gamma_1\bar{\psi}). \end{aligned} \tag{C.3.9}$$

Therefore from (C.3.5), extracting terms related with D_{10}^{chi},

$$\begin{aligned} X_1^{\text{chi}}D_{10}^{\text{chi}}X_0^{\text{chi}} &= D_j\bar{\phi}(\bar{\epsilon}\gamma^j\gamma_1\bar{\epsilon})(\epsilon\gamma_1\psi) + D_j\phi(\epsilon\gamma^j\gamma_1\epsilon)(\bar{\epsilon}\gamma_1\bar{\psi}) \\ &\quad +\frac{m}{2}\bar{\phi}(\bar{\epsilon}\gamma_1\bar{\epsilon})(\epsilon\gamma_1\psi) + \frac{m}{2}(\epsilon\gamma_1\epsilon)(\bar{\epsilon}\gamma_1\bar{\psi}), \end{aligned} \tag{C.3.10}$$

where $j = 2, 3$, and we used $\sigma = \frac{m}{2}$. From this, we can find the differential operator, which is associated with the symbol, is

$$(\epsilon\gamma^j\gamma_1\epsilon)D_j + \text{c.c.}. \tag{C.3.11}$$

Since we have the relations explicitly,

$$(\epsilon\gamma^\theta\gamma_1\epsilon) = ie^{-i\varphi}, \quad (\epsilon\gamma^\varphi\gamma_1\epsilon) = e^{-i\varphi}\cot\theta, \tag{C.3.12}$$

the symbol σ is

$$|\sigma|^2 = p_\theta^2 + p_\varphi^2\cot^2\theta, \tag{C.3.13}$$

so the differential operator is not elliptic since the symbol vanishes at $\theta = \pi/2$ even if $p_\varphi \neq 0$. However, restricting p to the transverse direction of the vector field ∂_τ and ∂_φ, the symbol becomes invertible. Therefore the differential operator is transversally elliptic with respect to the actions generated by $\hat{\mathcal{Q}}^2$ (4.2.22).

At the north and south poles, the relevant operator becomes

$$(\epsilon\gamma^j\gamma_1\epsilon)D_j\Big|_{\theta\sim 0,\, S^2} = 2iD_{\bar{z}}, \tag{C.3.14}$$
$$(\epsilon\gamma^j\gamma_1\epsilon)D_j\Big|_{\theta\sim \pi,\, S^2} = 2ie^{-2i\varphi}D_{\bar{z}}, \tag{C.3.15}$$

where we have taken $z_N \sim \theta e^{-i\varphi}$ and $z_S \sim (\pi-\theta)e^{i\varphi}$ as the local complex coordinates at the north and south poles, respectively. That is to say, they act as the Dolbeault operators on the north ans south poles.

The indices on the north and south poles are[5]

$$\mathrm{ind}\, D_{10}^{\mathrm{chi}}\Big|_{\text{N-pole}} = \sum_{n\in\mathbb{Z}}\sum_{\rho\in R} e^{-2\pi in}\, e^{-(\rho(m)-\Delta)\beta_2} e^{i\rho(a)} e^{i\sum_i \gamma_i F_i}\frac{1}{1-e^{2\beta_2}}, \tag{C.3.16}$$
$$\mathrm{ind}\, D_{10}^{\mathrm{chi}}\Big|_{\text{S-pole}} = \sum_{n\in\mathbb{Z}}\sum_{\rho\in R} e^{-2\pi in}\, e^{-(-\rho(m)-\Delta)\beta_2} e^{i\rho(a)} e^{i\sum_i \gamma_i F_i}\frac{1}{1-e^{2\beta_2}}, \tag{C.3.17}$$

Therefore the total index is

$$\begin{aligned}\mathrm{ind}\, D_{10}^{\mathrm{chi}} &= 2\Big(\mathrm{ind}\, D_{10}^{\mathrm{chi}}\Big|_{\text{N-pole}} + \mathrm{ind}\, D_{10}^{\mathrm{chi}}\Big|_{\text{S-pole}}\Big)\\ &= 2\sum_{n\in\mathbb{Z}}\sum_{\rho\in R}\sum_{r=0}^{\infty}\Big(e^{(2r-\rho(m)+\Delta)\beta_2-2\pi in+i\sum_i\gamma_iF_i+i\rho(a)} - e^{-(2r+2-\rho(m)-\Delta)\beta_2-2\pi in+i\sum_i\gamma_iF_i+i\rho(a)}\Big).\end{aligned} \tag{C.3.18}$$

According to the rule (4.1.36), the one-loop determinant is, up to an overall factor,

$$\begin{aligned}Z_{\mathrm{chi}}^{(\text{1-loop})} &= \prod_{n\in\mathbb{Z}}\prod_{\rho\in R}\prod_{r=0}^{\infty}\frac{\pi in-(r+1-\frac{\rho(m)}{2}+\frac{\Delta}{2})\beta_2+\frac{i}{2}\sum_i\gamma_iF_i+\frac{i}{2}\rho(a)}{\pi in-(r-\frac{\rho(m)}{2}+\frac{\Delta}{2})\beta_2-\frac{i}{2}\sum_i\gamma_iF_i-\frac{i}{2}\rho(a)}\\ &= \prod_{\rho\in R}\prod_{r=0}^{\infty}\frac{\sinh\Big[-(r+1-\frac{\rho(m)}{2}+\frac{\Delta}{2})\beta_2+\frac{i}{2}\sum_i\gamma_iF_i+\frac{i}{2}\rho(a)\Big]}{\sinh\Big[-(r-\frac{\rho(m)}{2}+\frac{\Delta}{2})\beta_2-\frac{i}{2}\sum_i\gamma_iF_i-\frac{i}{2}\rho(a)\Big]}\\ &= \prod_{\rho\in R}\left(\prod_{r=0}^{\infty}\frac{e^{(r+1-\frac{\rho(m)}{2}-\frac{\Delta}{2})\beta_2-\frac{i}{2}\sum_i\gamma_iF_i-\frac{i}{2}\rho(a)}}{e^{(r-\frac{\rho(m)}{2}+\frac{\Delta}{2})\beta_2+\frac{i}{2}\sum_i\gamma_iF_i+\frac{i}{2}\rho(a)}}\times\prod_{r=0}^{\infty}\frac{1-x^{2r+2-\rho(m)-\Delta}e^{i\sum_i\gamma_iF_i}e^{i\rho(a)}}{1-x^{2r-\rho+\Delta}e^{-i\sum_i\gamma_iF_i}e^{-i\rho(a)}}\right),\end{aligned} \tag{C.3.19}$$

[5]We consider contributions of $U(1)$ actions in $i\beta\hat{\mathcal{Q}}^2$.

where $x = e^{-\beta_2}$. After regularizing this infinite product[6], we finally obtain the following result,

$$Z_{\text{chi}}^{(\text{1-loop})} = \prod_{\rho\in R} \left(x^{(1-\Delta)} e^{i\sum_i \gamma_i F_i} e^{i\rho(a)}\right)^{-\frac{\rho(m)}{2}} \prod_{r=0}^{\infty} \frac{1 - x^{2r+2-\rho(m)-\Delta} e^{i\sum_i \gamma_i F_i} e^{i\rho(a)}}{1 - x^{2r-\rho(m)+\Delta} e^{-\sum_i \gamma_i F_i} e^{-i\rho(a)}}. \quad \text{(C.3.20)}$$

We also introduce a flavor fugacity $\xi_i = e^{i\gamma_i}$, and the q-Pochhammer symbol,

$$(a;q)_n = \prod_{k=0}^{n-1} (1 - aq^k). \quad \text{(C.3.21)}$$

Then the one-loop determinant is

$$Z_{\text{chi}}^{(\text{1-loop})} = \prod_{\rho\in R} \left(x^{(1-\Delta)} e^{i\rho(a)} \prod_i \xi_i^{F_i} \right)^{-\frac{\rho(m)}{2}} \frac{(x^{2-\rho(m)-\Delta} e^{i\rho(a)} (\prod_i \xi_i^{F_i})\,; x^2)_\infty}{(x^{-\rho(m)+\Delta} e^{-i\rho(a)} (\prod_i \xi_i^{-F_i})\,; x^2)_\infty}. \quad \text{(C.3.22)}$$

C.3.2 Vector multiplet

Let us consider the one-loop determinant of the vector multiplet. We can also confirm that the differential operator D_{10}^{vec} is a transversally elliptic operator with respect to the actions generated by $\hat{\mathcal{Q}}^2$ (4.2.22) [66], although we will not review it here.

In this case, the relevant differential is also the differential in the twisted de Rham complex. Therefore, the indices on the north and south poles are[7]

$$\text{ind}\, D_{10}^{\text{vec}}\big|_{\text{N-pole}} = -\sum_{n\in\mathbb{Z}} e^{-2\pi i n} \sum_{\alpha} e^{i\alpha(a)} e^{-\alpha(m)\beta_2}, \quad \text{(C.3.23)}$$

$$\text{ind}\, D_{10}^{\text{vec}}\big|_{\text{S-pole}} = -\sum_{n\in\mathbb{Z}} e^{-2\pi i n} \sum_{\alpha} e^{i\alpha(a)} e^{\alpha(m)\beta_2}. \quad \text{(C.3.24)}$$

According to the rule (4.1.36), the one-loop determinant is, up to an overall factor,

$$\begin{aligned} Z_{\text{vec}}^{(\text{1-loop})} &= \prod_{\alpha>0} \left[2\sinh\left(\frac{i}{2}\alpha(a) - \frac{1}{2}\alpha(m)\beta_2 \right) \right] \left[2\sinh\left(\frac{i}{2}\alpha(a) + \frac{1}{2}\alpha(m)\beta_2 \right) \right] \\ &= \prod_{\alpha\in\,\text{adj}} x^{-\frac{|\alpha(m)|}{2}} \left(1 - e^{-i\alpha(a)} x^{|\alpha(m)|}\right). \end{aligned} \quad \text{(C.3.25)}$$

[6] We should regularize the logarithm of the first factor in (C.3.19). See [70, 66] for details.

[7] We consider contributions of $U(1)$ actions in $i\beta\hat{\mathcal{Q}}^2$.

Appendix D

Contents of the $\mathcal{N} = (0, 2)$ vortex world line theory

In this chapter we derive the field contents of the vortex world line theory in this paper in the manner of the appendix in [92]. We can obtain them if we reduce the dimensions by one from the 2d $\mathcal{N} = (0, 2)$ theory (c.f. [93]). Note that we take the supersymmetry charge $\mathcal{Q}$ as Grassmann-odd, i.e. ϵ and $\bar{\epsilon}$ are Grassmann-even spinors below.

From the 3d $\mathcal{N} = 2$ vector multiplet

First we consider the 2d $\mathcal{N} = (2, 2)$ theory by dimensionally reducing along the third direction of the 3d $\mathcal{N} = 2$ theory on the flat space: from (A.3.6),

$$
\begin{aligned}
&\mathcal{Q}A_\mu = \frac{i}{2}(\bar{\epsilon}\gamma_\mu\lambda + \bar{\lambda}\gamma_\mu\epsilon), \quad \mathcal{Q}\sigma = (\bar{\epsilon}P_-\lambda + \bar{\lambda}P_-\epsilon), \quad \mathcal{Q}\bar{\sigma} = (\bar{\epsilon}P_+\lambda + \bar{\lambda}P_+\epsilon), \\
&\mathcal{Q}\lambda = -\gamma^{12}\epsilon F_{12} + i\gamma^\mu P_+\epsilon D_\mu\sigma + i\gamma^\mu P_-\epsilon D_\mu\bar{\sigma} - D\epsilon + \frac{i}{2}\gamma^3\epsilon[\sigma, \bar{\sigma}], \\
&\mathcal{Q}\bar{\lambda} = -\gamma^{12}\bar{\epsilon}F_{12} - i\gamma^\mu P_-\bar{\epsilon}D_\mu\sigma - i\gamma^\mu P_+\bar{\epsilon}D_\mu\bar{\sigma} + D\bar{\epsilon} - \frac{i}{2}\gamma^3\bar{\epsilon}[\sigma, \bar{\sigma}], \\
&\mathcal{Q}D = -\frac{i}{2}\bar{\epsilon}\gamma^\mu D_\mu\lambda + \frac{i}{2}D_\mu\bar{\lambda}\gamma^\mu\epsilon + \frac{i}{2}[\bar{\epsilon}P_+\lambda, \sigma] + \frac{i}{2}[\bar{\epsilon}P_-\lambda, \bar{\sigma}] - \frac{i}{2}[\bar{\lambda}P_+\epsilon, \sigma] - \frac{i}{2}[\bar{\lambda}P_-\epsilon, \bar{\sigma}],
\end{aligned}
\tag{D.0.1}
$$

where $\mu = 1, 2$, $A_3 = \eta$, $P_\pm = \frac{1\pm\gamma^3}{2}$. We also set $\sigma + i\eta \to \sigma$ as a complex scalar field. Next we can obtain the $\mathcal{N} = (0, 2)$ by constraining ϵ and $\bar{\epsilon}$ to $P_-\epsilon = P_-\bar{\epsilon} = 0$.

$$
\begin{aligned}
\mathcal{Q}A_1 &= -\tfrac{i}{2}(\bar{\epsilon}^+\lambda^+ + \bar{\lambda}^+\epsilon^+), & \mathcal{Q}A_2 &= \frac{1}{2}(\bar{\epsilon}^+\lambda^+ + \bar{\lambda}^+\epsilon^+), \\
\mathcal{Q}\sigma &= -\bar{\epsilon}^+\lambda^-, & \mathcal{Q}\bar{\sigma} &= \bar{\lambda}^-\epsilon^+, \\
\mathcal{Q}\lambda^+ &= -i\epsilon^+F_{12} - D\epsilon^+ + \tfrac{i}{2}\epsilon^+[\sigma, \bar{\sigma}], & \mathcal{Q}\lambda^- &= i\epsilon(D_1 + iD_2)\sigma,
\end{aligned}
\tag{D.0.2}
$$

$$\mathcal{Q}\bar{\lambda}^+ = -i\bar{\epsilon}^+ F_{12} + D\bar{\epsilon}^+ - \tfrac{i}{2}\bar{\epsilon}^+[\sigma,\bar{\sigma}], \qquad \mathcal{Q}\bar{\lambda}^- = -i\bar{\epsilon}^+(D_1 + iD_2)\bar{\sigma},$$

$$\mathcal{Q}D = \frac{i}{2}\bar{\epsilon}^+(D_1 + iD_2)\lambda^+ - \frac{i}{2}(D_1 + iD_2)\bar{\lambda}^+\epsilon^+ - \frac{i}{2}[\bar{\epsilon}^+\lambda^-,\bar{\sigma}] - \frac{i}{2}[\bar{\lambda}^-\epsilon^+,\sigma].$$

Finally we reduce the second direction, and also have to shift $D \to D + \frac{i}{2}[\sigma,\bar{\sigma}]$ in order to close the Fermi-multiplet algebra. Then, we can obtain the following multiplets:

- adj. fermi multiplet: $(\bar{\lambda}^+, \lambda^+, D - iF_{12}, -D - iF_{12})$

$$\begin{aligned} \mathcal{Q}\bar{\lambda}^+ &= \bar{\epsilon}^+(D - iF_{12}), & \mathcal{Q}(D - iF_{12}) &= -2i\epsilon^+ D_{\bar{\omega}}\bar{\lambda}^+, \\ \mathcal{Q}\lambda^+ &= \epsilon^+(-D - iF_{12}), & \mathcal{Q}(-D - iF_{12}) &= -2i\bar{\epsilon}^+ D_{\bar{\omega}}\lambda^+, \end{aligned} \tag{D.0.3}$$

- adj. chiral multiplet: $(\sigma, \bar{\sigma}, \lambda^-, \bar{\lambda}^-)$

$$\begin{aligned} \mathcal{Q}\sigma &= -\bar{\epsilon}^+\lambda^-, & \mathcal{Q}\lambda^- &= 2i\epsilon^+ D_{\bar{\omega}}\sigma, \\ \mathcal{Q}\bar{\sigma} &= \epsilon^+\bar{\lambda}^-, & \mathcal{Q}\bar{\lambda}^- &= -2i\bar{\epsilon}^+ D_{\bar{\omega}}\bar{\sigma}, \end{aligned} \tag{D.0.4}$$

where we set $\omega = x^1 + ix^2$ and $F_{12} = D_\tau\varphi$ with $A_1 = A_\tau$ and $A_2 = \varphi$.

We can obtain the Lagrangian in the same manner from (2.1.14):

$$\begin{aligned} \mathcal{L}^{1d}_{\rm YM} \;=\; & \frac{1}{2}\mathrm{Tr}\Big[\, \Big(D_\tau\varphi D_\tau\varphi + D^2 - i\bar{\lambda}^+ D_\tau\lambda^+ + i\bar{\lambda}^+[\varphi,\lambda^-]\Big) + \Big(D_\tau\bar{\sigma}D_\tau\sigma + [\bar{\sigma},\varphi][\varphi,\sigma] \\ & + i\bar{\lambda}^- D_\tau\lambda^- + i\bar{\lambda}^-[\varphi,\lambda^-] + i\bar{\lambda}^-[\sigma,\lambda^+] - i\bar{\lambda}^+[\bar{\sigma},\lambda^-] + iD[\sigma,\bar{\sigma}]\Big)\,\Big], \end{aligned} \tag{D.0.5}$$

where the terms in the first parentheses imply the fermi multiplet action, and the second ones imply the chiral multiplet one. We find that each of them is $\mathcal{Q}$-exact:

$$\frac{1}{2}\,\mathcal{Q}\,\mathrm{Tr}\Big(\bar{\lambda}^+(D + iF_{12})\Big) = \frac{1}{2}\mathrm{Tr}\Big(D_\tau\varphi D_\tau\varphi + D^2 - i\bar{\lambda}^+ D_\tau\lambda^+ + i\bar{\lambda}^+[\varphi,\lambda^-]\Big), \tag{D.0.6}$$

$$\begin{aligned} \frac{1}{2}\mathcal{Q}\,\mathrm{Tr}\Big(2i\bar{\sigma}D_\omega\lambda^- - i\bar{\sigma}[\lambda^+,\sigma]\Big) \;=\; & \frac{1}{2}\mathrm{Tr}\Big(D_\tau\bar{\sigma}D_\tau\sigma + [\bar{\sigma},\varphi][\varphi,\sigma] + i\bar{\lambda}^- D_\tau\lambda^- + i\bar{\lambda}^-[\varphi,\lambda^-] \\ & + i\bar{\lambda}^-[\sigma,\lambda^+] - i\bar{\lambda}^+[\bar{\sigma},\lambda^-] + iD[\sigma,\bar{\sigma}]\Big), \end{aligned} \tag{D.0.7}$$

where we set $\epsilon = \bar{\epsilon} = 1$.

From the 3d $\mathcal{N}=2$ chiral multiplet

We can obtain the content in the same way as the vector multiplet from (A.3.12):

- Chiral multiplet: $(\phi, \bar{\phi}, \psi^-, \bar{\psi}^-)$:

$$\begin{aligned} \mathcal{Q}\phi &= -\bar{\epsilon}^+\psi^-, \quad \mathcal{Q}\psi^- = 2i\epsilon^+ D_{\bar{\omega}}\phi, \\ \mathcal{Q}\bar{\phi} &= -\epsilon^+\bar{\psi}^-, \quad \mathcal{Q}\bar{\psi}^- = 2i\bar{\epsilon}^+ D_{\bar{\omega}}\bar{\phi}. \end{aligned} \tag{D.0.8}$$

- Fermi multiplet: $(\psi^+, \bar{\psi}^+, F, \bar{F})$ (with chiral multiplet $(E, \bar{E}, \psi_E^-, \bar{\psi}_E^-)$) :

$$\begin{aligned} \mathcal{Q}\psi^+ &= i\epsilon^+\sigma\phi + \bar{\epsilon}^+ F, \quad \mathcal{Q}F = -2i\epsilon^+ D_{\bar{\omega}}\psi^+ + i\epsilon^+\sigma\psi^- + i\epsilon^+\lambda^-\phi, \\ \mathcal{Q}\bar{\psi}^+ &= i\bar{\epsilon}^+\bar{\phi}\bar{\sigma} + \bar{F}\epsilon^+, \quad \mathcal{Q}\bar{F} = -2i\bar{\epsilon}^+ D_{\bar{\omega}}\bar{\psi}^+ + i\bar{\epsilon}^+\bar{\psi}^-\bar{\sigma} - i\bar{\phi}\bar{\epsilon}^+\bar{\lambda}^-. \end{aligned} \tag{D.0.9}$$

We define $E(\sigma,\phi) := \sigma\phi$, then the fermionic partner is defined as $\psi_E^- := \sum_i \frac{\partial E(\phi_i)}{\partial \phi_i}\psi_i = \phi\lambda + \sigma\psi^-$. Then, we can rewrite the above fermi multiplet in the following way,

$$\begin{aligned} &\mathcal{Q}\psi^+ = i\epsilon^+ E + \bar{\epsilon}^+ F, \quad \mathcal{Q}F = -2i\epsilon^+ D_{\bar{\omega}}\psi^+ + i\epsilon^+\psi_E^-, \\ &\mathcal{Q}\bar{\psi}^+ = i\bar{\epsilon}^+\bar{E} + \epsilon^+\bar{F}, \quad \mathcal{Q}\bar{F} = -2i\bar{\epsilon}^+ D_{\bar{\omega}}\bar{\psi}^+ + i\bar{\epsilon}^+\bar{\psi}_E^-, \\ &\mathcal{Q}E = -\bar{\epsilon}^+\psi_E^-, \quad \mathcal{Q}\bar{E} = -\epsilon^+\bar{\psi}_E^-, \quad \mathcal{Q}\psi_E^- = 0, \quad \mathcal{Q}\bar{\psi}_E^- = 0. \end{aligned} \tag{D.0.10}$$

- Lagrangian: from (2.1.15),

$$\begin{aligned} \mathcal{L}^{1d}_{\text{chi}} = &\Big(D_\tau\bar{\phi}D_\tau\phi + \bar{\phi}\varphi^2\phi + i\bar{\phi}D\phi - i\bar{\psi}^- D_\tau\psi^- - i\bar{\psi}^-\varphi\psi^- + i\bar{\psi}^-\lambda^+\phi + i\bar{\phi}\bar{\lambda}^+\psi^-\Big) \\ &+ \Big(\bar{F}F + \bar{E}E + i\bar{\psi}^+ D_\tau\psi^+ - i\bar{\psi}^+\varphi\psi^+ - i\bar{\psi}^+\psi_E^- + i\bar{\psi}_E^-\psi^+\Big). \end{aligned} \tag{D.0.11}$$

We also find that each of them is $\mathcal{Q}$-exact:

$$\begin{aligned} \mathcal{Q}\Big(2i\bar{\phi}D_\omega\psi^- - i\bar{\phi}\lambda^+\phi\Big) \;\; = \;\; & D_\tau\bar{\phi}D_\tau\phi + \bar{\phi}\varphi^2\phi + i\bar{\phi}D\phi - i\bar{\psi}^- D_\tau\psi^- - i\bar{\psi}^-\varphi\psi^- \\ & + i\bar{\psi}^-\lambda^+\phi + i\bar{\phi}\bar{\lambda}^+\psi^-, \end{aligned} \tag{D.0.12}$$

$$\frac{1}{2}\,\mathcal{Q}\left[(\overline{\mathcal{Q}\psi^+}\psi) + (\bar{\psi}^+\overline{\mathcal{Q}\bar{\psi}^+})\right] \;\; = \;\; |E|^2 + |F|^2 + i\bar{\psi}^+ D_{\bar{\omega}}\psi^+ - iD_{\bar{\omega}}\bar{\psi}^+\psi^+ - i\bar{\psi}^+\psi_E^- + i\bar{\psi}_E^+\psi^+. \tag{D.0.13}$$

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