Possibility for construction of realistic 6D gauge-Higgs unification models

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Abstract

Gauge-Higgs unification models are studied as candidates for new physics beyond the standard model, which give interesting suggestions about the origin of the Higgs field. In these models, we identify extra components of higher dimensional gauge fields as Higgs fields so that higher dimensional gauge symmetry protects the Higgs mass against quantum corrections. I research 6-dimensional (6D) gauge-Higgs unification models especially.

First, I review the simple models of the gauge-Higgs unification. Then, I investigate the 6D models that have the custodial symmetry. We constrain gauge groups, orbifold compactifying the extra dimensions, gauge group representations of matter fields by requiring the theory to be realistic. Furthermore, I also investigate models that have the magnetic fluxes penetrating the compactified space as a background to realize the three generations of the matters and the hierarchical structure of the Yukawa couplings. Finally, I discuss a possibility of building realistic 6D gauge-Higgs unification models from the results we obtained.

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Chapter 1

Introduction

The standard model (SM) of the particle physics well-describes our world, but it still has many theoretical or experimental problems. For example, we do not know the origins of the Higgs boson, which causes the electro-weak (EW) symmetry breaking, nor the three generations of the matter fields, nor the hierarchical Yukawa couplings. We need new physics to solve these problems. In this chapter, I introduce the gauge-Higgs unification models as candidates for new physics after giving a brief review of the SM.

1.1 Standard model

It is known that there are four fundamental interactions in our world: the electromagnetic, the weak, the strong, and the gravitational interactions. Among them, the first three are described in the SM. The gauge symmetry of SM is $SU(3)_C \times SU(2)_L \times U(1)_Y$, and $SU(2)_L \times U(1)_Y$ is spontaneously broken to $U(1)_{\rm EM}$ by the Higgs mechanism.

The Lagrangian of SM is

$$
\begin{aligned}
\mathcal{L}_{\rm SM} = & -\frac{1}{4}{\rm tr}G_{\mu\nu}G^{\mu\nu} - \frac{1}{4}{\rm tr}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \\
& + i\bar{q}^i_L\left(\partial_\mu - ig_G\frac{\lambda_\alpha}{2}G^\alpha_\mu - ig_A\frac{\sigma_a}{2}A^a_\mu - i\frac{g_B}{6}B_\mu\right)\gamma^\mu q^i_L \\
& + i\bar{d}^i_R\left(\partial_\mu - ig_G\frac{\lambda_\alpha}{2}G^\alpha_\mu + i\frac{g_B}{3}B_\mu\right)\gamma^\mu d^i_R \\
& + i\bar{u}^i_R\left(\partial_\mu - ig_G\frac{\lambda_\alpha}{2}G^\alpha_\mu - i\frac{2g_B}{3}B_\mu\right)\gamma^\mu u^i_R \\
& + i\bar{l}^i_L\left(\partial_\mu - ig_A\frac{\sigma_a}{2}A^a_\mu + i\frac{g_B}{2}B_\mu\right)\gamma^\mu l^i_L \\
& + i\bar{e}^i_R\left(\partial_\mu + ig_B B_\mu\right)\gamma^\mu e^i_R \\
& - \left(y^d_{ij}\bar{q}^j_L H d^i_R + y^u_{ij}\bar{u}^i_R\epsilon H q^j_L + y^e_{ij}\bar{l}^j_L H e^i_R + \text{h.c.}\right) \\
& - \left|\left(\partial_\mu - ig_A\frac{\sigma_a}{2}A^a_\mu - i\frac{g_B}{2}B_\mu\right)H\right|^2 - V(H),
\end{aligned} \tag{1.1.1}
$$

where

$$
\begin{aligned}
& G_{\mu\nu} \equiv \partial_\mu G_\nu - \partial_\nu G_\mu - ig_G\left[G_\mu, G_\nu\right], \\
& F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - ig_A\left[A_\mu, A_\nu\right], \\
& B_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu, \\
& q^i_L \equiv \begin{pmatrix} u^i_L \\ d^i_L \end{pmatrix}, \quad l^i_L \equiv \begin{pmatrix} \nu^i_L \\ e^i_L \end{pmatrix}, \quad (i = 1, 2, 3)
\end{aligned} \tag{1.1.2}
$$

and G_μ, A_μ, B_μ are the $SU(3)_C, SU(2)_L, U(1)_Y$ gauge fields, g_G, g_A, g_B are the $SU(3)_C$, $SU(2)_L$, $U(1)_Y$ gauge couplings, respectively. The coupling constants $y^u_{ij}, y^d_{ij}, (i, j = 1, 2, 3)$ are the up- and the down-type Yukawa couplings, $\lambda_\alpha(\alpha = 1, \cdots, 8)$ and $\sigma_a(a = 1, 2, 3)$ are the Gell-Mann matrices and the Pauli matrices, respectively. H denotes the Higgs field that is a complex scalar $SU(2)_L$ doublet, $\epsilon H q^i_L \equiv \epsilon_{ab}H^a q^{ib}_L$ $(a, b = 1, 2 : SU(2)_L$ indices), and $V(H)$ is the Higgs potential. The potential that is renormalisable and breaks the EW symmetry dynamically is generally written as

$$
V(H) = -\mu^2 H^\dagger H + \lambda\left(H^\dagger H\right)^2, \tag{1.1.3}
$$

where

$$
H = \begin{pmatrix} H_+ \\ H_0 \end{pmatrix}. \tag{1.1.4}
$$

The components H_0 and H_+ are $U(1)_{\rm EM}$ neutral and positive-charged, respectively.

1.2 Higgs mechanism

The Higgs mechanism is indispensable for describing how the gauge bosons and the matter fermions get their masses through the EW symmetry breaking in SM. The EW symmetry is broken when the potential in (1.1.3) has the VEV as

$$H^\dagger H = v^2 \equiv \frac{\mu^2}{2\lambda}. \tag{1.2.1}$$

Using the $SU(2)_L \times U(1)_Y$ gauge symmetry, the VEV is always parameterized as

$$\langle H \rangle = \begin{pmatrix} 0 \\ v \end{pmatrix}. \tag{1.2.2}$$

Including the fluctuation modes, the Higgs doublet H can be written as

$$H = \exp\left(i\xi^a \frac{\sigma^a}{2}\right) \begin{pmatrix} 0 \\ v + \eta \end{pmatrix}. \quad (a = 1, 2, 3) \tag{1.2.3}$$

When we choose the unitary gauge, this becomes

$$H = \begin{pmatrix} 0 \\ v + \eta \end{pmatrix}. \tag{1.2.4}$$

The gauge bosons get the masses from the kinetic term of H in (1.1.1):

$$\mathcal{L}_{\text{mass}} = -m_W^2 \left(W_\mu^+ W^{-\mu}\right) - \frac{m_Z^2}{2} Z_\mu Z^\mu, \tag{1.2.5}$$

where

$$\begin{aligned} m_W &= \frac{g_A}{\sqrt{2}} v, \quad m_Z = \sqrt{\frac{g_A^2 + g_B^2}{2}} v, \\ W_\mu^\pm &= \frac{1}{\sqrt{2}} \left(A_\mu^1 \mp i A_\mu^2\right), \\ Z_\mu &= \cos\theta_W A_\mu^3 - \sin\theta_W B_\mu. \end{aligned} \tag{1.2.6}$$

Here, θ_W is the Weinberg angle defined by

$$\tan\theta_W = \frac{g_B}{g_A}. \tag{1.2.7}$$

The orthogonal combination of Z_μ, which has no mass term, is identified as the photon:

$$A_\mu^\gamma = \sin\theta_W A_\mu^3 + \cos\theta_W B_\mu. \tag{1.2.8}$$

In this way, the longitudinal components of the gauge fields absorb the unphysical degrees of freedom of 3 Nambu-Goldstone bosons ξ^a, and the gauge bosons corresponding to the broken gauge symmetries get non-vanishing masses. At the same time, the Higgs field also gets a mass, which comes from $V(H)$. The mass of the physical Higgs η is

$$m_\eta = \sqrt{2}\mu = 2\sqrt{\lambda}v. \tag{1.2.9}$$

The matter fermions also get masses through the Yukawa interactions with the Higgs fields.

The Higgs boson was discovered in 2012 by the LHC experiments [58, 59]. The discovery made the set of the particles that appear in SM complete. However, the origin of the Higgs sector is still unknown.

1.3 Gauge-Higgs Unification

The gauge-Higgs unification (GHU) models [8, 9, 10, 11] are attractive candidates for new physics beyond SM. We identify the extra dimensional components of the higher dimensional gauge field as 4D Higgs fields. In this case, the Higgs field is ruled by the gauge principle and the theories do not need any elementary scalar fields. Besides, the higher dimensional gauge symmetry forbids the Higgs mass at tree-level, and protects the Higgs mass against quantum corrections.[1] So they are expected to solve the gauge hierarchy problem. The EW symmetry is broken dynamically by one-loop effect in GHU models.

1.3.1 Fairle and Manton's model

In 1979, David Fairlie and Nicholas Manton extended the idea of Kalza-Klein theory [8, 9] and suggested the 6D gauge theory on $M^4 \times S^2$, where M^4 is the 4-dimensional (4D) Minkowski spacetime and S^2 is 2-dimensional sphere. They decomposed the 6D gauge field as

$$A_M(x, y) = (A_\mu(x, y), A_m(x, y)), \tag{1.3.1}$$

where $M = 0, 1, 2, 3, 4, 5$ is the 6D Lorentz index, x^μ $(\mu = 0, 1, 2, 3)$ is the 4D coordinate on M^4 and y^m $(m = 4, 5)$ is the extra dimensional coordinate on S^2. Then, $A_\mu(x, y)$ can be decomposed into the Kaluza-Klein (KK) modes as

$$A_\mu(x, y) = \sum_n A_\mu^{(n)}(x) f_{A,n}(y), \tag{1.3.2}$$

[1] In 6D models on an orbifold, tree-level Higgs mass terms are allowed at a fixed point of the orbifold.

where $f_{A,n}(y)$ are called the KK mode functions for $A_\mu^{(n)}(x)$. The zero-mode gauge field $A_\mu^{(0)}$ is identified as the 4D gauge field that appears at low energies. In the same way, we can decompose the extra components of the gauge field $A_m(x, y)$ into the KK modes as

$$A_m(x, y) = \sum_n A_m^{(n)}(x) f_{\varphi,n}(y). \tag{1.3.3}$$

This contains the zero-mode $A_m^{(0)}(x)$. In their setup, the background field configuration of $A_m(x, y)$ has the rotational symmetry $SO(3)$ and there is a magnetic flux on S^2. They showed that scalar fields originating from $A_m(x, y)$ play a role of the Higgs fields that break the gauge symmetry $SU(2)_L \times U(1)_Y$, whichi is obtained from a larger gauge group in six dimensions, to the electromagnetic symmetry $U(1)_{\mathrm{EM}}$. This is the first research of GHU models.

They considered the simple Lie groups $SU(3)$, $SO(5)$, G_2 as the larger gauge group that is brokn to $SU(2)_L \times U(1)_Y$. Their rank is 2 and the same as that of $SU(2)_L \times U(1)_Y$. They calculated the Weinberg angle θ_W and the mass spectrum for each gauge group. They found that the most realistic value of θ_W is predicted in the case of G_2, and all the mass scales of the W, Z and the Higgs bosons and the first KK excited mode are given by $\mathcal{O}(R^{-1})$. The latter result stems from the fact that the model has only a single scale R^{-1} and all the masses are generated at tree-level.

1.3.2 Hosotani mechanism and 6D GHU models

In 1983, Hosotani proposed a mechanism that breaks the gauge symmetry by quantum effect [10]. He indicated that the Aharanov-Bohm effect occurs when the extra dimensional spaces are not simply connected and the Aharanov-Bohm phase (or the Wilson-line phase) plays a role of the 4D Higgs field. The EW symmetry can be broken by this mechanism. In such a case, we can generate a hierarchy between the Higgs mass and the KK mass scales since the Higgs mass is suppressed by the loop factor.

The simplest models of GHU with the Hosotani mechanism are based on 5-dimensional (5D) gauge theories whose gauge groups are $U(3)$ in the flat spacetime [15, 18] and $SO(5) \times U(1)$ in the warped spacetime [19, 21, 22]. In these models, the EW symmetry is broken dynamically by the VEV of the Wilson-line phase $\theta_{\mathrm{H}} \equiv \int_C dy\ A_y$, where C is a non-contractible cycle along the extra dimension and A_y is the 5-th component of the gauge field. According to Refs.[14], the W boson mass m_W is expressed in terms of θ_H as

$$m_W = \begin{cases} \frac{|\langle\theta_H\rangle|}{2\pi R} & \text{in flat case} \\ \frac{ke^{-k\pi R}}{\sqrt{2\pi kR}}\,|\sin\langle\theta_H\rangle| & \text{in warped case} \end{cases}, \tag{1.3.4}$$

where R is a typical radius of the extra-dimensional space and k is the inverse AdS curvature radius. The KK mass scale $m_{\rm KK}$ is given by

$$m_{\rm KK} = \begin{cases} R^{-1} & \text{in flat case} \\ \pi ke^{-k\pi R} & \text{in warped case} \end{cases}. \tag{1.3.5}$$

Notice that this is independent of θ_H in contrast to m_W. Thus, we can realize the hierarchy between m_W and $m_{\rm KK}$ if the VEV of θ_H is small enough. From the experimental bounds, $m_{\rm KK}$ must be larger than a few TeV.[2] If $m_{\rm KK} \geq 4$ TeV, for example, we can see that $\langle\theta_H\rangle \leq \mathcal{O}(0.1)$ from (1.3.4) and (1.3.5).

The effective potential for θ_H is induced at one-loop level. It has a form of

$$V_{\rm eff}(\theta_H) = \frac{3}{l_6\pi^3} m_{\rm KK}^4 f(\theta_H). \tag{1.3.6}$$

where $l_6 \equiv 128\pi^3$ is the 6D loop factor, and $f(\theta_H)$ is a dimensionless periodic function of θ_H with a period 2π. An explicit form of $f(\theta_H)$ is determined by matter contents of the theory. Without any fine-tuning among the model parameters, we obtain $\langle\theta_H\rangle = \mathcal{O}(1)\pi$ from the potential (1.3.6). The Higgs mass m_H can be estimated from (1.3.6) as

$$m_H = \begin{cases} \left\{f''(\theta_H)\frac{12g_4^2}{l_6\pi}\right\}^{\frac{1}{2}} \frac{1}{2R} & \text{in flat case} \\ \left\{f''(\theta_H)\frac{3\pi g_4^2}{l_6}\right\}^{\frac{1}{2}} \sqrt{\frac{k\pi R}{2}}ke^{-k\pi R} & \text{in warped case} \end{cases}, \tag{1.3.7}$$

where $f''(\theta_H) \equiv \frac{d^2 f(\theta_H)}{d\theta_H^2}$. In the flat case, m_H is typically estimated around $\mathcal{O}(10)$ GeV, which is too light to be realistic. In the warped case, on the other hand, m_H can be heavy enough to reproduce the observed value thanks to logarithm of the warped factor.

In each case, we have to realize a small value of $\langle\theta_H\rangle$ in order to obtain the realistic mass spectrum, which is difficult to achieve without any fine-tunings. This problem arises from the fact that 5D GHU models have no Higgs potential at tree-level.

In 6D models, this problem can be solved because the Higgs quartic couplings exist at tree-level that originate from tr $\left([A_4, A_5]^2\right)$ in the 6D gauge kinetic term, while quadratic terms are induced at one-loop level.

[2] According to Refs. [60, 61], $m_{\rm KK} > 4.16$ TeV in the flat case, and $m_{\rm KK} > 2.68$ TeV for the KK graviton in the warped case.

In the flat spacetime, for example, the effective potential up to one-loop level has a form of

$$V(\theta_H) = -\frac{c_2 g^2}{l_6 R^2}\left(\frac{\theta_H}{g\pi R}\right)^2 + c_4 g^2\left(\frac{\theta_H}{g\pi R}\right)^4 + \mathcal{O}(\theta_H^6), \tag{1.3.8}$$

where $c_2, c_4 = \mathcal{O}(1)$ are numerical constants, g is the 4D $SU(2)_L$ gauge coupling constant, By minimizing this, we find that

$$\langle\theta_H\rangle \simeq \frac{g\pi\sqrt{c_2}}{\sqrt{2l_6 c_4}} \simeq \frac{0.02\sqrt{c_2}}{\sqrt{c_4}} \ll 1, \tag{1.3.9}$$

and the KK modes are estimated to be around a few TeV without tuning model parameters. Besides, we can realize the observed Higgs mass more easily than 5D case. So 6D GHU models are phenomenologically attractive. Another reason why 6D GHU models are well worth researching is a possibility of realizing the generations of matter fermions and the Yukawa hierarchy by introducing background magnetic fluxes. Such fluxes break the gauge symmetry and realize chiral fermions in 4D effective theories. We evaluated the Yukawa couplings with the magnetic fluxes that break the EW symmetry and realize the three generations of the matter fermions or one-Higgs doublet case in 6D GHU models on T^2/Z_N orbifold.

The structure of this thesis is as follows. In the Chapter 2, $SU(3)$ GHU model in the flat or warped metric is introduced as the simplest example of 5D GHU models. I explain the setup of the model and show how much Yukawa and weak gauge couplings for the quarks and leptons deviate from the experimental values in 5D GHU models. In the Chapter 3, I select the concrete 6D GHU models that have the custodial symmetry. We constrained the 6D gauge groups and the orbifolds compactifying the extra dimensions and the G representations that matter fields belong to by generalization of group theory. In the Chapter 4, I introduce magnetic fluxes penetrating the extra dimensions to realize the matter generations and the Yukawa hierarchy by overlap integrals of zero-mode wave functions. In the Chapter 5, I summarize the results of model building, and tell the problems and future prospects of 6D GHU models.

Chapter 2

Example of gauge-Higgs unification

As I told in the previous chapter, GHU approaches have been investigated as the model of new physics beyond the SM. The EW symmetry is broken by the nonvanishing VEV of the Wilson-line phase in these models. Some models have been constructed to explain the origin of the Higgs field in the SM. In the simplest models, gauge group is often taken as $SU(3)_C \times SU(3)_W$ in 5D flat metric, or $SU(3)_C \times SO(5) \times U(1)$ in 5D warped metric. In this chapter, I show the simplest example of GHU models. We will evaluate the weak gauge couplings and the Yukawa couplings for the matter fermions in the presence of the bulk fermions' mass terms and see whether they deviate from the experimental values.

2.1 $SU(3)_W$ model

The $SU(3)_W$ GHU models are considered because the gauge group is the minimum simple group that contains $SU(2)_L \times U(1)_Y$ subgroup and one $SU(2)_L$ Higgs doublet as the extra component of the gauge field. In these models, the symmetry breaking $SU(3)_W \to SU(2)_L \times U(1)_Y$ is caused by orbifold projection. In the simplest model the spacetime is 5D, and the 5th dimension is compactified with S^1/Z_2.

From the next section, I calculate the zero-mode and the KK mode wavefunction of each field and evaluate the Yukawa couplings and gauge couplings on the configuration of $SU(3)_W$ GHU model in the case of flat or warped metric and mention the the mass of the Higgs boson.

2.2 Field content

We think $SU(3)_W$ gauge theory. $SU(3)_W$ gauge field is expressed as $A_M = A_M^a T^a$ where T^a is $\frac{1}{2}\times$Gell-Mann matrices. We can decompose A_M as

$$A_M = \sum_{a=1}^{8} A_M^a \frac{\lambda^a}{2} = \frac{1}{2}\begin{pmatrix} A_M^3 + \frac{1}{\sqrt{3}}A_M^8 & A_M^1 - iA_M^2 & A_M^4 - iA_M^5 \\ A_M^1 + iA_M^2 & -A_M^3 + \frac{1}{\sqrt{3}}A_M^8 & A_M^6 - iA_M^7 \\ A_M^5 + iA_M^5 & A_M^6 + iA_M^7 & -\frac{2}{\sqrt{3}}A_M^8 \end{pmatrix}, \tag{2.2.1}$$

and 5D matter field that belongs to $SU(3)$ fundamental representation is written as Ψ^f. 5D Lagrangian is

$$\mathcal{L}_{5\mathrm{D}} = -\frac{1}{2}\mathrm{tr}\left(F^{(A)MN}F^{(A)}_{MN} - \frac{1}{\xi}f_{\mathrm{gf}}^2\right) + i\sum_f \left\{\bar{\Psi}^f\Gamma^M\mathcal{D}_M\Psi^f - iM\epsilon(y)\bar{\Psi}^f\Psi^f\right\}, \tag{2.2.2}$$

where

$$\begin{aligned}
&F^{(A)}_{MN} \equiv \partial_M A_N - \partial_N A_M - ig_A[A_M, A_N],\\
&\mathcal{D}_M \equiv \partial_M - ig_A A_M,\\
&g_A : \text{ gauge coupling of } A_M,\\
&\Gamma^\mu = \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix} \quad (\mu = 0,1,2,3), \quad \Gamma^5 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix},\\
&\sigma^\mu = (1, \sigma^i)\ ,\ \bar{\sigma}^\mu = (-1, \sigma^i)\ ,\ (i = 1,2,3)\\
&\bar{\Psi} = i\Psi^\dagger\Gamma^5, \quad M : \text{ bulk mass parameter},\\
&\epsilon : \text{ step function},
\end{aligned} \tag{2.2.3}$$

and $-\frac{1}{\xi}f_{\mathrm{gf}}^2$ is the gauge fixing term.

2.3 Compactified space

We think 5D flat metric:

$$ds^2 = G_{MN}dx^M dx^N = \eta_{\mu\nu}dx^\mu dx^\nu + (dy)^2, \tag{2.3.1}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ denotes 4D Minkowski metric ($\mu, \nu = 0,1,2,3, M, N = 0,1,2,3,4$) and compactify the 5th dimension y by S^1/Z_2 orbifold.

2.3.1 S^1/Z_2 orbifold

This is the one-dimensional orbifold of the interval. The compactified extra dimension y by S^1 is identified as

$$y \sim y + 2\pi R, \tag{2.3.2}$$

where R is the radius of S^1, and by Z_2 action, y is identified as

$$y \sim -y. \tag{2.3.3}$$

2.4 Orbifold boundary conditions

2.4.1 Gauge fields

S^1 boundary conditions of A_M are

$$A_M(x^\mu, y + 2\pi R) = T A_M(x^\mu, y) T^\dagger. \tag{2.4.1}$$

where T is a unitary matrix of the translation transformation on S^1. If we define P_0, P_π as unitary matrices of Z_2 transformation on $y = 0, \pi R$ respectively, we can write

$$P_\pi = T P_0, \tag{2.4.2}$$

so Z_2 boundary conditions of A_M are written as

$$\begin{aligned}
A_\mu(x^\mu, -y) &= P_0 A_\mu(x^\mu, y) P_0^\dagger, \\
A_y(x^\mu, -y) &= -P_0 A_y(x^\mu, y) P_0^\dagger, \\
A_\mu(x^\mu, \pi R - y) &= P_\pi A_\mu(x^\mu, \pi R + y) P_\pi^\dagger, \\
A_y(x^\mu, \pi R - y) &= -P_\pi A_y(x^\mu, \pi R + y) P_\pi^\dagger,
\end{aligned} \tag{2.4.3}$$

where A_y is the 5th component of A_M. As stated above, A_μ and A_y must have an oppsosite Z_2 parity for gauge invariance. So when A_μ has a zero-mode, A_y cannnot have. Zero-mode fields on the flat profile must have Z_2 eigenvalues as $(\lambda_0, \lambda_\pi) = (+, +)$ where λ_0, λ_π is an eigenvalue of Z_2 transformation at $y = 0, \pi$ respectively.

2.4.2 Matter fields

Next, I define boundary conditions for matter fields Ψ. If $y_i = 0, \pi R$, I can write as

$$\Psi(x, y_i - y) = \pm P_i \Gamma^5 \Psi(x, y_i + y), \tag{2.4.4}$$

where $\Gamma^5 = \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. For this boundary condition, components whose Z_2 eigenvalues are $(\lambda_0, \lambda_\pi) = (+,+)$ are restricted to one chirality. So we can realize chiral theory. This can be rewritten as

$$\begin{aligned}\Psi(x, -y) &= \eta_0 P_0 \Gamma^5 \Psi(x, y),\\ \Psi(x, \pi R + y) &= \eta_\pi P_\pi \Gamma^5 \Psi(x, \pi R - y),\end{aligned} \tag{2.4.5}$$

where $\eta_0, \eta_\pi = \pm$.

2.4.3 Zero-mode conditions

As stated above, zero-mode fields have constant profile on the flat metric and invariant for Z_2 transformation. When symmetry breaking $SU(3)_W \to SU(2)_L \times U(1)_Y$ is caused by orbifold boundary conditions, the components of A_M that should have zero-mode are as follows:

$$A_\mu^{(0)} = \frac{1}{2}\begin{pmatrix} A_\mu^{3(0)} + \frac{1}{\sqrt{3}}A_\mu^{8(0)} & A_\mu^{1(0)} - iA_\mu^{2(0)} & \\ A_\mu^{1(0)} + iA_\mu^{2(0)} & -A_\mu^{3(0)} + \frac{1}{\sqrt{3}}A_\mu^{8(0)} & \\ & & -\frac{2}{\sqrt{3}}A_\mu^{8(0)} \end{pmatrix}, \tag{2.4.6}$$

$$A_y^{(0)} = \frac{1}{2}\begin{pmatrix} & & A_y^{4(0)} - iA_y^{5(0)} \\ & & A_y^{6(0)} - iA_y^{7(0)} \\ A_y^{4(0)} + iA_y^{5(0)} & A_y^{6(0)} + iA_y^{7(0)} & \end{pmatrix}, \tag{2.4.7}$$

and for matter fields,

$$\Psi_R^{(0)} = \begin{pmatrix} \\ \\ \Psi_R^{3(0)} \end{pmatrix}, \quad \Psi_L^{(0)} = \begin{pmatrix} \Psi_L^{1(0)} \\ \Psi_L^{2(0)} \\ \\ \end{pmatrix}, \tag{2.4.8}$$

where $\Gamma^5\Psi_R = \Psi_R, \Gamma^5\Psi_L = -\Psi_L$. In order that these components have zero-modes, the Z_2 parity (λ_0, λ_π) should be as follows:

$$A_\mu = \left(\begin{array}{cc|c} (+,+) & (+,+) & (-,-) \\ (+,+) & (+,+) & (-,-) \\ \hline (-,-) & (-,-) & (+,+) \end{array}\right), \quad A_y = \left(\begin{array}{cc|c} (-,-) & (-,-) & (+,+) \\ (-,-) & (-,-) & (+,+) \\ \hline (+,+) & (+,+) & (-,-) \end{array}\right), \tag{2.4.9}$$

$$\Psi_R = \left(\begin{array}{c} (-,-) \\ (-,-) \\ \hline (+,+) \end{array}\right), \quad \Psi_L = \left(\begin{array}{c} (+,+) \\ (+,+) \\ \hline (-,-) \end{array}\right). \tag{2.4.10}$$

For example, if we take P_0, P_π as

$$P_0 = P_\pi = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}, \tag{2.4.11}$$

we can realize (2.4.9) and (2.4.10).

Here, in (2.4.7) we select $(A_y^4 + iA_y^5, A_y^6 + iA_y^7)$ as the $SU(2)_L$ Higgs doublet whose VEV breaks $SU(2)_L \times U(1)_Y$ to $U(1)_{\rm EM}$ from the components of A_y that have zero-modes, so we identify $(A_y^4 - iA_y^5, A_y^6 - iA_y^7)^t$ as the Hermite conjugate field of the Higgs doublet.

2.5 Mode functions and Mass eigenvalues

2.5.1 Gauge sector

In GHU models, the nonzero VEV of the Wilson-line phase breaks the EW symmetry. Here, we decompose A_M as

$$A_M = \langle A_M \rangle + \tilde{A}_M, \tag{2.5.1}$$

where $\langle A_M \rangle$ is the background part and $\tilde{A}_M$ is the fluctuation part of A_M. I select $\xi = 1$ and the function of the gauge-fixing term as

$$f_{\rm gf} = D^M \tilde{A}_M, \tag{2.5.2}$$

where

$$D_M \tilde{A}_N \equiv \partial_M \tilde{A}_N - ig_A \left[\langle A_M \rangle, \tilde{A}_N \right]. \tag{2.5.3}$$

From (2.2.2), we can derive the linearized equation of motion of $\tilde{A}_M$ as

$$D^M D_M \tilde{A}_N - ig_A \left[\langle F_{NM} \rangle, \tilde{A}^M \right] = 0. \tag{2.5.4}$$

To derive the mode functions of gauge fields with the nonzero Wilson-line phase θ_H, I change the basis as

$$\tilde{A}_M \to \hat{A}_M \equiv \Omega \tilde{A}_M \Omega^{-1}, \tag{2.5.5}$$

$$\Omega(y) \equiv \exp\left\{ -ig_A \int_0^y dy' \langle A_y \rangle (y') \right\}, \tag{2.5.6}$$

where $\Omega(y)$ is the gauge transformation matrix. Due to this, D_M changes into ∂_M and $\langle F_{NM} \rangle$ vanishes , so (2.5.4) becomes

$$\partial^M \partial_M \hat{A}_N = 0, \tag{2.5.7}$$

$$\therefore\ \partial^M \partial_M \hat{A}^a_N = 0. \tag{2.5.8}$$

We substitute the KK expansion for $\tilde{A}_M$ on this equation:

$$\hat{A}^a_\mu(x,y) = \sum_n \hat{f}^a_n(y) \hat{A}^n_\mu(x), \tag{2.5.9}$$

$$\hat{A}^a_y(x,y) = \sum_n \hat{g}^a_n(y) \hat{A}^n_y(x). \tag{2.5.10}$$

We apply the on-shell condition for $\hat{A}^n_M(x)$: $(\Box - m_n^2)\hat{A}^n_M(x) = 0$, where $\Box \equiv \partial^\mu \partial_\mu$, then we obtain the eigenequations for m_n (KK mode equations):

$$\partial_y^2 \hat{f}^a_n(y) = -m_n^2 \hat{f}^a_n(y), \tag{2.5.11}$$

$$\partial_y^2 \hat{g}^a_n(y) = -m_n^2 \hat{g}^a_n(y). \tag{2.5.12}$$

For $m_n > 0$, the solutions of these equations are

$$\hat{f}^a_n(y) = \mathcal{A}^a_n \cos(m_n y) + \mathcal{B}^a_n \sin(m_n y), \tag{2.5.13}$$

$$\hat{g}^a_n(y) = \mathcal{C}^a_n \cos(m_n y) + \mathcal{D}^a_n \sin(m_n y), \tag{2.5.14}$$

where $\mathcal{A}^a_n, \mathcal{B}^a_n, \mathcal{C}^a_n, \mathcal{D}^a_n$ are y-independent constants.

Now, we derive KK mode functions (containing zero-mode functions) and KK mass eigenvalues of gauge sector. The boundary conditions for the zero-modes of each components read from (2.4.9) are

$$\begin{aligned} &\partial_y A^a_\mu|_{y=0,\pi R} = 0 \ \ (a = 1,2,3,8), \quad A^a_\mu|_{y=0,\pi R} = 0 \ \ (a = 4,5,6,7), \\ &A^a_y|_{y=0,\pi R} = 0 \ \ (a = 1,2,3,8), \quad \partial_y A^a_y|_{y=0,\pi R} = 0 \ \ (a = 4,5,6,7). \end{aligned} \tag{2.5.15}$$

We must rewrite these conditions by the new basis (2.5.5).

The non-zero Wilson-line phase breaks the symmetry $SU(2)_L \times U(1)_Y$ down to $U(1)_{\rm EM}$. When the EW simmetry is broken, we can take the zero-mode of A_y as $A_y^{(0)} = \frac{1}{2}A_y^7\lambda^7$ and the classical solution of gauge field is $\langle A_y \rangle = \frac{1}{2}a\lambda^7$ (a: constant of mass dimension 1). So the Wilson-line phase θ_H can be written as

$$\begin{aligned}\theta_H &\equiv \frac{1}{2}g_A \int_0^{\pi R} dy A_y^7(y) \\ &= \frac{1}{2}g_A \pi R a. \end{aligned} \tag{2.5.16}$$

This value is determined dynamically (not by hand) at one-loop level. $\Omega(y)$ in (2.5.6) is rewritten as

$$\begin{aligned}\Omega(y) &= \exp\left\{-i\theta(y)\lambda^7\right\} \\ &= \begin{pmatrix} 1 & & \\ & \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ & \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}, \end{aligned} \tag{2.5.17}$$

where

$$\begin{aligned}\theta = \theta(y) &\equiv \frac{g_A}{2}\int_0^y dy' A_y^7(y') \\ &= \frac{g_A}{2} a y \\ &= \frac{y}{\pi R}\theta_H. \end{aligned} \tag{2.5.18}$$

The gauge transformation induced by $\Omega(y)$ preserves the boundary conditions (2.4.3) and (2.4.5), but shifts θ_H by $2n\pi$. So the θ_H is a variable by 2π. The EW symmetry is broken dynamically when θ_H has a nonzero VEV.

Then, A_M^a are mixed by θ as

$$\begin{aligned}\begin{pmatrix} \hat{A}_M^1 \\ \hat{A}_M^4 \end{pmatrix} &= \begin{pmatrix} \cos\frac{1}{2}\theta & -\sin\frac{1}{2}\theta \\ \sin\frac{1}{2}\theta & \cos\frac{1}{2}\theta \end{pmatrix} \begin{pmatrix} A_M^1 \\ A_M^4 \end{pmatrix}, \\ \begin{pmatrix} \hat{A}_M^2 \\ \hat{A}_M^5 \end{pmatrix} &= \begin{pmatrix} \cos\frac{1}{2}\theta & -\sin\frac{1}{2}\theta \\ \sin\frac{1}{2}\theta & \cos\frac{1}{2}\theta \end{pmatrix} \begin{pmatrix} A_M^2 \\ A_M^5 \end{pmatrix}, \\ \begin{pmatrix} \hat{A}'^3_M \\ \hat{A}_M^6 \end{pmatrix} &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} A'^3_M \\ A_M^6 \end{pmatrix}, \\ \hat{A}_M^7 = A_M^7 \quad &, \quad \hat{A}'^8_M = A'^8_M \end{aligned} \tag{2.5.19}$$

where

$$\begin{pmatrix} A'^3_M \\ A'^8_M \end{pmatrix} \equiv \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} A^3_M \\ A^8_M \end{pmatrix}. \tag{2.5.20}$$

For example, the boundary conditions for (A^1_μ, A^4_μ):(2.5.15) change to

$$\begin{aligned} \partial_y f^1_n \Big|_{y=0,\pi R} &= \partial_y \left(\cos\frac{\theta}{2} \cdot \hat{f}^1_n + \sin\frac{\theta}{2} \cdot \hat{f}^4_n \right) \Big|_{y=0,\pi R} = 0, \\ f^4_n \Big|_{y=0,\pi R} &= -\sin\frac{\theta}{2} \cdot \hat{f}^1_n + \cos\frac{\theta}{2} \cdot \hat{f}^4_n \Big|_{y=0,\pi R} = 0. \end{aligned} \tag{2.5.21}$$

We find $\hat{f}^1_{A,n}(y) = \mathcal{A}^1_n \cos(m_n y)$, $\hat{f}^4_{A,n}(y) = \mathcal{B}^4_n \sin(m_n y)$ quickly, so the condition (2.5.21) is rewritten as

$$\begin{pmatrix} -\cos(m_n \pi R) \sin\frac{\theta_H}{2} & \sin(m_n\pi R)\cos\frac{\theta_H}{2} \\ -m_n \sin(m_n\pi R)\cos\frac{\theta_H}{2} & m_n \cos(m_n \pi R)\sin\frac{\theta_H}{2} \end{pmatrix} \begin{pmatrix} \mathcal{A}^1_n \\ \mathcal{B}^4_n \end{pmatrix} = 0. \tag{2.5.22}$$

We obtain when det of the left hand side is 0:

$$\begin{aligned} \tan^2(m_n \pi R) &= \tan^2\left(\frac{\theta_H}{2}\right), \\ \therefore \quad m_n &= \left| \pm\frac{\theta_H}{2\pi R} + \frac{n}{R} \right| . \end{aligned} \tag{2.5.23}$$

From (2.5.22) and the ortho-normalization condition:

$$\int_0^{\pi R} dy \left\{ \hat{f}^1_n(y) \hat{f}^1_l(y) + \hat{f}^4_n(y)\hat{f}^4_l(y) \right\} = \delta_{n,l}, \tag{2.5.24}$$

$\mathcal{A}^1_n$ and $\mathcal{B}^4_n$ are determined.

$$\begin{aligned} \therefore \ \hat{f}^1_n &= \frac{1}{\sqrt{\pi R}} \cos(m_n y), \\ \hat{f}^4_n &= \frac{1}{\sqrt{\pi R}} \sin(m_n y), \quad \left(m_n = \left| \frac{\theta_H}{2\pi R} + \frac{n}{R} \right|, \quad n: \text{ an integer} \right) \end{aligned} \tag{2.5.25}$$

We find $\hat{f}^2 = \hat{f}^1$, $\hat{f}^5 = \hat{f}^4$. The other mode functions are

$$\begin{aligned} \hat{f}'^3_n &= \frac{1}{\sqrt{\pi R}} \cos(m_n y), \\ \hat{f}^6_n &= \frac{1}{\sqrt{\pi R}} \sin(m_n y), \quad \left(m_n = \left| \frac{\theta_H}{\pi R} + \frac{n}{R} \right|, \quad n: \text{ an integer} \right) \end{aligned} \tag{2.5.26}$$

$$\hat{f}^7_n = \sqrt{\frac{2}{\pi R}} \sin(m_n y), \quad \left(m_n = \left| \frac{n}{R} \right|, \ n \neq 0 \right) \tag{2.5.27}$$

$$\hat{f}^8_n = \sqrt{\frac{2}{\pi R}} \cos(m_n y). \quad \left(m_n = \left| \frac{n}{R} \right| \right) \tag{2.5.28}$$

Here, notice that $d\hat{f}_n^a/dy$ whose f_n^a have the massless mode satisfies the mode function of $\hat{g}_n^a$ which does not have the masless mode. from (2.5.15). So $\hat{g}_n^a \propto d\hat{f}_n^a/dy$ is valid for such a, and they have the same mass eigenvalue m_n. Similarly, $d\hat{g}_n^a/dy$ whose g_n^a have the massless mode satisfies the mode function of $\hat{f}_n^a$ which does not have the masless mode, and $\hat{f}_n^a \propto d\hat{g}_n^a/dy$ for such a. The mode functions of A_y^a are

$$
\begin{aligned}
\hat{g}_n^1 &= \frac{1}{\sqrt{\pi R}}\sin(m_n y), \\
\hat{g}_n^4 &= \frac{1}{\sqrt{\pi R}}\cos(m_n y), \quad \left(m_n = \left|\frac{\theta_H}{2\pi R} + \frac{n}{R}\right|, \quad n: \text{ an integer}\right).
\end{aligned} \tag{2.5.29}
$$

We find $\hat{g}^2 = \hat{g}^1$, $\hat{g}^5 = \hat{g}^4$, and

$$
\begin{aligned}
\hat{g}_n^{\prime 3} &= \frac{1}{\sqrt{\pi R}}\sin(m_n y), \\
\hat{g}_n^6 &= \frac{1}{\sqrt{\pi R}}\cos(m_n y), \quad \left(m_n = \left|\frac{\theta_H}{\pi R} + \frac{n}{R}\right|, \quad n: \text{ an integer}\right)
\end{aligned} \tag{2.5.30}
$$

$$
\hat{g}_n^7 = \sqrt{\frac{2}{\pi R}}\cos(m_n y), \quad \left(m_n = \left|\frac{n}{R}\right|\right) \tag{2.5.31}
$$

$$
\hat{g}_n^8 = \sqrt{\frac{2}{\pi R}}\sin(m_n y). \quad \left(m_n = \left|\frac{n}{R}\right|, \ n \neq 0\right) \tag{2.5.32}
$$

When we assign the bosonic fields that appear in the SM to these mode functions, they are expressed as

$$
\begin{aligned}
&\hat{A}_\mu^1 = \sum_{n=0}^{\infty} \hat{f}_n^1(y) W_{\mu,n}(x), \quad \hat{A}_\mu^2 = \sum_{n=0}^{\infty} \hat{f}_n^2(y) W_{\mu,n}(x), \\
&\hat{A}_\mu^4 = \sum_{n=0}^{\infty} \hat{f}_n^4(y) W_{\mu,n}(x), \quad \hat{A}_\mu^5 = \sum_{n=0}^{\infty} \hat{f}_n^5(y) W_{\mu,n}(x), \\
&\hat{A}_\mu^{3'} = \sum_{n=0}^{\infty} \hat{f}_n^{3'}(y) Z_{\mu,n}, \quad \hat{A}_\mu^6 = \sum_{n=0}^{\infty} \hat{f}_n^6(y) Z_{\mu,n}(x) \\
&\hat{A}_\mu^{8'} = \sum_{n=0}^{\infty} \hat{f}_n^{8'}(y) \gamma_{\mu,n}(x), \\
&\hat{A}_y^4 = \sum_{n=0}^{\infty} \hat{f}_n^4(y) \varphi_n(x), \quad \hat{A}_y^5 = \sum_{n=0}^{\infty} \hat{f}_n^5(y) \varphi_n(x), \\
&\hat{A}_y^6 = \sum_{n=0}^{\infty} \hat{f}_n^6(y) \varphi_n(x), \quad \hat{A}_y^7 = \sum_{n=0}^{\infty} \hat{f}_n^7(y) \varphi_n(x),
\end{aligned} \tag{2.5.33}
$$

where $W_{\mu,n}(x)$, $Z_{\mu n}$, $\gamma_{\mu,n}(x)$, $\varphi_n(x)$ means the 4D sector for the KK mode of W boson, Z boson, photon, the Higgs boson, respectively.

2.5.2 Fermion sector

We can derive the equation of motion for Ψ^f from (2.2.2):

$$i\Gamma^N \left(\partial_N - ig_A \langle A_N \rangle\right) \Psi^f - iM\epsilon\Psi^f = 0. \tag{2.5.34}$$

We take $\epsilon = 1$ in the region $0 \le y \le 1$.

By the gauge transformation with $\Omega(y)$,

$$\hat{\Psi} = \Omega(y)\Psi. \tag{2.5.35}$$

(2.5.34) is rewritten as

$$\begin{aligned}
&\gamma^\mu \partial_\mu \hat{\Psi}^f_{\mathrm{R}} - \left(\partial_y + M\right) \hat{\Psi}^f_{\mathrm{L}} = 0, \\
&\gamma^\mu \partial_\mu \hat{\Psi}^f_{\mathrm{L}} + \left(\partial_y - M\right) \hat{\Psi}^f_{\mathrm{R}} = 0,
\end{aligned} \tag{2.5.36}$$

where γ^μ is the 4D γ matrices, $N = \mu = 0, 1, 2, 3$ component of Γ^N, and $\hat{\Psi}_{\mathrm{R}} = \frac{1+\gamma_5}{2}\Psi$, $\hat{\Psi}_{\mathrm{L}} = \frac{1-\gamma_5}{2}\hat{\Psi}$ ($\hat{\Psi} = \hat{\Psi}_{\mathrm{R}} + \hat{\Psi}_{\mathrm{L}}$, $\gamma_5\hat{\Psi}_{\mathrm{R}} = +\hat{\Psi}_{\mathrm{R}}$, $\gamma_5\hat{\Psi}_{\mathrm{L}} = +\hat{\Psi}_{\mathrm{L}}$). We decompose $\hat{\Psi}$ into the KK modes, and substitute the on-shell conditions for $\hat{\Psi}^f_n(x)$, the 4D sector of the KK mode for $\hat{\Psi}^f(x, y)$: $(\gamma^\mu\partial_\mu - m_n)\hat{\Psi}^f(x) = 0$ into (2.5.36), the we obtain the mode equations for $\hat{\Psi}^f$:

$$\begin{aligned}
&D_\pm(M)\hat{h}^f_{\mp n}(y) = -m_n \hat{h}^f_{\pm n}(y), \\
&D_\pm(M) \equiv \pm\partial_y + M,
\end{aligned} \tag{2.5.37}$$

where the double signs correspond and $+, -$ means R, L. When $m_n \ge M$, the solutions are

$$\begin{aligned}
&\hat{h}^f_{\mathrm{R}n}(y) = A^f_n \cos(\lambda_n y) + B^f_n \sin(\lambda_n y), \\
&\hat{h}^f_{\mathrm{L}n}(y) = -\frac{1}{m_n}\{(MA^f_n - \lambda_n B^f_n)\cos(\lambda_n y) + (MB^f_n + \lambda_n A^f_n)\sin(\lambda_n y)\}, \\
&\lambda_n \equiv \sqrt{m_n^2 - M^2},
\end{aligned} \tag{2.5.38}$$

where A^f_n, B^f_n are constants. The boundary conditions for $\hat{h}^f$ are

$$\begin{aligned}
&D_+\Psi^1_{\mathrm{L}} = D_+\Psi^2_{\mathrm{L}} = D_-\Psi^3_{\mathrm{R}} = 0, \\
&\Psi^1_{\mathrm{R}} = \Psi^2_{\mathrm{R}} = \Psi^3_{\mathrm{L}} = 0, \quad (\text{at } y = 0, \pi R) \\
&D_\pm \equiv D_\pm(M).
\end{aligned} \tag{2.5.39}$$

The relation between Ψ^f and $\hat{\Psi}^f$ is

$$\begin{aligned}
\Psi^1 &= \hat{\Psi}^1, \\
\Psi^2 &= \left\{ \cos\frac{\theta}{2}\cdot\hat{\Psi}^2 + \sin\frac{\theta}{2}\cdot\hat{\Psi}^3 \right\}, \\
\Psi^3 &= \left\{ -\sin\frac{\theta}{2}\cdot\hat{\Psi}^2 + \cos\frac{\theta}{2}\cdot\hat{\Psi}^3 \right\}.
\end{aligned} \tag{2.5.40}$$

The ortho-normalization conditions of $\hat{h}^i$ are

$$\begin{aligned}
&\int_0^{\pi R} dy \hat{h}^1_{\chi_4 n}(y)\hat{h}^1_{\chi_4 l}(y) = \delta_{nl}, \\
&\int_0^{\pi R} dy \left\{ \hat{h}^2_{\chi_4 n}(y)\hat{h}^2_{\chi_4 l}(y) + \hat{h}^3_{\chi_4 n}(y)\hat{h}^3_{\chi_4 l}(y) \right\} = \delta_{nl},
\end{aligned} \tag{2.5.41}$$

where χ_4 means the 4D chirality. A^f_n, B^f_n are determined from the conditions, so the solutions are

$$\begin{aligned}
&\hat{h}^1_{\mathrm{R}n} = \sqrt{\frac{2}{\pi R}}\sin(\lambda_n y), \qquad \hat{h}^1_{\mathrm{L}n} = \sqrt{\frac{2}{\pi R}}\cos(\lambda_n y + \alpha), \\
&m_n = \sqrt{M^2 + \frac{n^2}{R^2}}, \qquad \lambda_n = \frac{n}{R},
\end{aligned}$$

when $m_n \geq M$, where $\cos\alpha \equiv \frac{\lambda_n}{m_n}$, $\sin\alpha \equiv \frac{M}{m_n}$. Also,

$$\begin{aligned}
&\hat{h}^2_{\mathrm{R}n} = B^2_n \sin(\lambda_n y), \qquad \hat{h}^3_{\mathrm{R}n} = B^3_n \left\{ \frac{\lambda_n}{M}\cos(\lambda_n y) + \sin(\lambda_n y) \right\}, \\
&\hat{h}^2_{\mathrm{L}n} = B^2_n \cos(\lambda_n y + \alpha), \qquad \hat{h}^3_{\mathrm{L}n} = -\frac{m_n}{M} B^3_n \sin(\lambda_n y), \\
&\sin(\lambda_n \pi R) = \frac{\lambda_n}{m_n}\sin\left(\frac{\theta_H}{2}\right), \qquad \lambda_n \equiv \sqrt{m_n^2 - M^2}.
\end{aligned} \tag{2.5.42}$$

When $0 < m_n < M$, the mode functions are

$$\begin{aligned}
&\hat{h}^f_{\mathrm{R}n}(y) = A^f_n e^{\lambda_n y} + B^f_n e^{-\lambda_n y}, \\
&\hat{h}^f_{\mathrm{L}n}(y) = -\frac{1}{m_n}\left\{ (M-\lambda_n)A^f_n e^{\lambda y} + (M+\lambda_n)B^f_n e^{-\lambda_n y} \right\}, \\
&\lambda_n \equiv \sqrt{M^2 - m_n^2}.
\end{aligned} \tag{2.5.43}$$

Then, there is no solution for $\hat{h}^1_{\mathrm{R,L}n}$, and

$$\begin{aligned}
&\hat{h}^2_{\mathrm{R}n} = A^2_n(e^{\lambda_n y} - e^{-\lambda_n y}), \qquad \hat{h}^3_{\mathrm{R}n} = B^3_n(\frac{\lambda_n + M}{\lambda_n - M}e^{\lambda_n y} + e^{-\lambda_n y}), \\
&\hat{h}^2_{\mathrm{L}n}(y) = -\frac{A^2_n}{m_n}\left\{ (M-\lambda_n)e^{\lambda_n y} - (M+\lambda_n)e^{-\lambda_n y} \right\}, \qquad \hat{h}^3_{\mathrm{L}n} = \frac{M+\lambda_n}{m_n}B^3_n(e^{\lambda_n y} - e^{-\lambda_n y}), \\
&m_n^2 = \frac{2\sin^2\frac{\theta_H}{2}}{\cosh(2\lambda_n \pi R) + \sin^2\frac{\theta_H}{2} - \cos^2\frac{\theta_H}{2}}M^2, \quad (\lambda_n \equiv \sqrt{M^2 - m_n^2}).
\end{aligned} \tag{2.5.44}$$

When $m_n = 0$, the mode functions are

$$\begin{aligned}\hat{h}^f_{\mathrm{R}0} &= A^f_0 e^{My},\\ \hat{h}^f_{\mathrm{L}0} &= B^f_0 e^{-My}.\end{aligned} \tag{2.5.45}$$

Then, the massless modes are

$$\begin{aligned}&\hat{h}^1_{\mathrm{R}0} = 0, \quad \hat{h}^1_{\mathrm{L}0} = \sqrt{\frac{2M}{1-e^{-2M\pi R}}}e^{-My},\\ &\hat{h}^2_{\mathrm{R}0} = 0, \quad \hat{h}^2_{\mathrm{L}0} = \begin{cases} 0, & (\theta_H \neq 0 \bmod 2\pi) \\ \sqrt{\frac{2M}{1-e^{-2M\pi R}}}e^{-My}, & (\theta_H = 0 \bmod 2\pi) \end{cases}\\ &\hat{h}^3_{\mathrm{R}0} = \begin{cases} 0, & (\theta_H \neq 0 \bmod 2\pi) \\ \sqrt{\frac{2M}{e^{2M\pi R}-1}}e^{My}, & (\theta_H = 0 \bmod 2\pi) \end{cases} \quad \hat{h}^3_{\mathrm{L}0} = 0.\end{aligned} \tag{2.5.46}$$

After obtaining the mode functions of fermions, we assign the fermions that appear in the SM to them. The mass spectrums of fermions depend on the bulk mass M^f, so we can realize the mass of the fermions by specifying M^f for the lightest mode of each $\hat{\Psi}^f$. From the interaction between $\hat{A}^8_M$ and $\hat{\Psi}^f$, we find the ratio of the hypercharge among $SU(2)_L$ doublet $(\hat{\Psi}^1_{\mathrm{L}}, \hat{\Psi}^2_{\mathrm{L}})$ and $\hat{\Psi}_{\mathrm{R}}$ is $1 : 1 : -2$. For example, we can assign $(u_{\mathrm{L}}, d_{\mathrm{L}})$ and d_{R} to them, and can assign $(\nu_{\mathrm{L}}, e_{\mathrm{L}})$ and e_{R} by an additional $U(1)$ group.

2.6 Gauge couplings

Here, we derive the 4D effective gauge couplings of fermions from the 5D interaction between $\hat{A}_M$ and $\hat{\Psi}$. The 5D gauge interaction is

$$\begin{aligned}I^{\mathrm{5D}}_{\mathrm{gc}} &= \int d^5x g_A \bar{\Psi}^f \Gamma^M A_M \Psi^f\\ &= \int d^4x \int_0^{\pi R} dy g_A \left\{ \bar{\hat{\Psi}}^f \gamma^\mu \hat{A}_\mu \hat{\Psi}^f + \cdots \right\}.\end{aligned} \tag{2.6.1}$$

Inserting (2.5.33) into this, we obtain

$$\mathcal{L}^{\mathrm{4D}}_{\mathrm{gc}} = \frac{g_{(0)}}{\sqrt{2}} \bar{e}_{\mathrm{L}0} \gamma^\mu W_{\mu 0} \nu_{\mathrm{L}0} + \mathrm{h.c.} + \cdots, \tag{2.6.2}$$

where the ellipsis means the contribution of the KK modes of relevant fields. The 4D gauge coupling is expressed as

$$g_{(0)}(\theta_H, M) \equiv g_A \int_0^{\pi R} dy \left(\hat{h}^2_{\mathrm{L}0} \hat{f}^1_0 + \hat{h}^3_{\mathrm{L}0} \hat{f}^4_0 \right) \hat{h}^1_{\mathrm{L}}. \tag{2.6.3}$$

For the limit $\theta_H \to 0$, the EW symmetry is unbroken and the gauge coupling for all the fermions become universal. The coupling is easily calculated as

$$g_{(0)}(0, M) = g_A \int_0^{\pi R} dy \left(\hat{h}^2_{\mathrm{L}0} \cdot \frac{1}{\sqrt{\pi R}} + 0 \cdot 0 \right) \hat{h}^1_{\mathrm{L}0}. \tag{2.6.4}$$

Since $\hat{h}^2_{\mathrm{L}0} = \sqrt{\frac{2M}{1-e^{-2M\pi R}}} e^{-My} = \hat{h}^1_{\mathrm{L}0}$,

$$g_{(0)}(0, M) = \frac{g_A}{\sqrt{\pi R}} \int_0^{\pi R} dy (\hat{h}^1_{\mathrm{L}0})^2 = \frac{g_A}{\sqrt{\pi R}} = g_4, \tag{2.6.5}$$

where we used the normalization condition for $\hat{h}^1_{\mathrm{L}0}$, and g_4 means the 4D $SU(2)_L$ gauge coupling. This has no dependence on M. After the EW symmetry breaking, θ_H become the nonzero value by quantum effect, and $g_{(0)}$ have dependence on M.

For simplicity, we consider the case that $M = 0$ and $\theta_H \neq 0$. In this case, the relevant mode functions become

$$\begin{aligned}
\hat{h}^1_{\mathrm{L}0} &= \sqrt{\frac{2}{\pi R}}, \\
\hat{h}^2_{\mathrm{L}0} &= \frac{1}{\sqrt{\pi R}} \cos\left(\frac{\theta_H}{2\pi R} y\right), \qquad \hat{h}^3_{\mathrm{L}0} = \frac{1}{\sqrt{\pi R}} \sin\left(\frac{\theta_H}{2\pi R} y\right), \\
\hat{f}^1_0 &= \frac{1}{\sqrt{\pi R}} \cos\left(\frac{\theta_H}{2\pi R} y\right), \qquad \hat{f}^4_0 = \frac{1}{\sqrt{\pi R}} \sin\left(\frac{\theta_H}{2\pi R} y\right),
\end{aligned} \tag{2.6.6}$$

so $g_{(0)}$ is written as

$$\begin{aligned}
g_{(0)} &= \frac{g_A}{\pi R} \sqrt{\frac{2}{\pi R}} \int_0^{\pi R} dy \sin\left(\frac{\theta_H}{2\pi R} y\right) \\
&= g_A \sqrt{\frac{2}{\pi R}} \frac{1}{\theta_H} \left(1 - \cos\theta_H\right).
\end{aligned} \tag{2.6.7}$$

So the deviation from g_4 is evaluated by the ratio:

$$\frac{g_{(0)}}{g_4} = \frac{\sqrt{2}}{\theta_H} \left(1 - \cos\theta_H\right). \tag{2.6.8}$$

This becomes 1 when $\theta_H \sim 2.3$.

Now, we change our policy to the models with the warped spacetime. The mass spectrums and the couplings in the case of warped metric $ds^2 = e^{-2ky}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2$ are calculated in [21]. In this paper, the Bessel functions of the zero-mode fermion are approximated as

$$
\begin{aligned}
(\alpha>1)\quad & \hat{h}^2_{\mathrm{L0}}(z)\sim\hat{h}^1_{\mathrm{R0}}(z)\sim\sqrt{2k(\alpha-1)}z^{1/2-\alpha}, \\
& \hat{h}^3_{\mathrm{L0}}(z)\sim\sqrt{\frac{k(\alpha-1)}{2}}\frac{\sin\theta_H}{z_\pi^1\alpha}z^{1/2+\alpha}, \qquad (2.6.9) \\
(\alpha<0)\quad & \hat{h}^2_{\mathrm{L0}}(z)\sim\left|\cos\frac{\theta_H}{2}\right|\frac{\sqrt{2k(1-\alpha)}}{z_\pi^{1-\alpha}}z^{1/2-\alpha} \\
& \hat{h}^3_{\mathrm{L0}}(z)\sim\mathrm{sgn}\left(\cos\frac{\theta_H}{2}\right)\sin\frac{\theta_H}{2}\frac{\sqrt{2k(1-\alpha)}}{z_\pi^{1-\alpha}}z^{1/2-\alpha}, \qquad (2.6.10)
\end{aligned}
$$

where $z=e^{ky}$, k is the Ads curvature radius, $z_\pi=e^{k\pi R}$, and $\alpha=M/k+1/2$. The Bessel functions of the $A^{1,4}_\mu$ zero-mode are approximated as

$$
\begin{aligned}
& \hat{f}^1_0\sim\frac{1}{\sqrt{\pi R}}, \\
& \frac{\hat{f}^1_0(z_\pi)}{\hat{f}^1_0(1)}\sim\cos^2\frac{\theta_H}{2},\quad \frac{\hat{f}^4_0(z_\pi)}{\hat{f}^1_0(1)}\sim\sin\frac{\theta_H}{2}\cos\frac{\theta_H}{2}. \qquad (2.6.11)
\end{aligned}
$$

Using these mode functions, the 4D gauge coupling $g_{(0)}$ is evaluated as

$$
\begin{aligned}
& g_{(0)}\left(\frac{M}{k}>\frac{1}{2}\right)\sim g_4, \\
& g_{(0)}\left(\frac{M}{k}<-\frac{1}{2}\right)\sim g_4\left|\cos\frac{\theta_H}{2}\right|. \qquad (2.6.12)
\end{aligned}
$$

The zero-mode function of each fermion is characterized by M/k, and they evaluated the deviation of (2.6.12) from g_4 of each flavor. They concluded the deviations are very small and become larger for heavy fermions. Such deviations mean that the universality of the weak interaction is slightly broken in the presence of the non-zero θ_H.

2.7 Yukawa couplings

As with the previous case, we derive the 4D effective Yukawa couplings. In GHU models, the Yukawa couplings stem from the 5D gauge interaction. The corresponding term is

$$
\begin{aligned}
\mathcal{L}^{\mathrm{5D}}_{\mathrm{yukawa}} &= g_A\bar{\psi}\Gamma^4A_y\psi \qquad (2.7.1) \\
&= g_A\left(\bar{\hat{\Psi}}^1,\bar{\hat{\Psi}}^2,\bar{\hat{\Psi}}^3\right)\Gamma^4\frac{1}{2}\begin{pmatrix} & & \\ & & -i\hat{A}^7_y \\ & i\hat{A}^7_y & \end{pmatrix}\begin{pmatrix}\hat{\Psi}^1\\ \hat{\Psi}^2\\ \hat{\Psi}^3\end{pmatrix}. \qquad (2.7.2)
\end{aligned}
$$

The 4D Yukawa coupling is expressed as

$$\mathcal{L}^{5D}_{\text{yukawa}} = y_e \phi_0 \bar{e}_{\text{L}0} e_{\text{R}0} + \text{h.c.} \cdots . \tag{2.7.3}$$

From (2.7.2),

$$y_e = \frac{g_A}{2} \int_0^{\pi R} dy \hat{f}_0^7 \left(\hat{h}^2_{\text{L}0} \hat{h}^3_{\text{R}0} - \hat{h}^2_{\text{R}0} \hat{h}^3_{\text{L}0} \right) . \tag{2.7.4}$$

In the flat case with the limit $\theta_H \to 0$,

$$\begin{aligned} y_e &= \frac{g_A}{2} \int_0^{\pi R} \sqrt{\frac{\pi R}{2}} \left(\sqrt{\frac{2M}{1 - e^{-2M\pi R}}} e^{-My} \cdot \sqrt{\frac{2M}{1 - e^{-2M\pi R}}} e^{-My} - 0 \cdot 0 \right) \\ &= g_A \sqrt{2\pi R} \frac{M e^{-M\pi R}}{1 - e^{-2M\pi R}} . \end{aligned} \tag{2.7.5}$$

This can realize the 4D Yukawa coupling by tuning M for each fermion.

After the EW symmetry breaking, y_e has the dependence of $\theta_H (\neq 0)$. When $M = 0$,

$$\hat{h}^2_{\text{R}0} = \frac{1}{\sqrt{\pi R}} \sin \left(\frac{\theta_H}{2\pi R} \right), \qquad \hat{h}^3_{\text{R}0} = \frac{1}{\sqrt{\pi R}} \cos \left(\frac{\theta_H}{2\pi R} \right). \tag{2.7.6}$$

Then,

$$\begin{aligned} y_e &= \frac{g_A}{2} \int_0^{\pi R} \sqrt{\frac{2}{\pi R}} \left\{ \frac{1}{\pi R} \cos^2 \left(\frac{\theta_H}{2\pi R} \right) - \frac{1}{\pi R} \sin^2 \left(\frac{\theta_H}{2\pi R} \right) \right\} \\ &= \frac{g_A}{2} \sqrt{\frac{2}{\pi R}} \frac{\sin \theta_H}{\theta_H} \end{aligned} \tag{2.7.7}$$

This is a decreasing function of θ_H from $\theta_H = 0$ to $\theta_H \sim 4.5$. Now, we define the Higgs VEV v in the SM as

$$v \equiv \frac{2m_W}{g_4} \sim 246 \text{ GeV}. \tag{2.7.8}$$

The deviation of y_e from the real 4D Yukawa coupling is evaluated by the ratio

$$r \equiv \frac{|y_e| \, v}{m_e}. \tag{2.7.9}$$

In the present case,this ratio is

$$r(M = 0) = \frac{\sqrt{2} m_W}{m_e} \cdot \frac{\sin \theta_H}{\theta_H}. \tag{2.7.10}$$

The deviation is small for e when $\theta_H \sim \pi$.

Next, we see the case with the warped metric. According to [21], the ratio r with $M = 0$ is approximated as

$$r(M=0) \sim \frac{2}{\theta_H} \sin\frac{\theta_H}{2}, \tag{2.7.11}$$

for $-\pi \leq \theta_H \leq \pi$ with the Bessel functions.

For $|M|/k > 1/2$,

$$r\left(|M|/k > 1/2\right) \sim \left|\cos\frac{\theta_H}{2}\right| \tag{2.7.12}$$

In the case, r is almost independent on M/k and vanishes as θ_H approaches π for all quarks and leptons. I conclude that it is difficult to realize the Yukawa coupling for each flavor, so the Yukawa hierarchy by the mass terms of the bulk fermions in the present setup.

2.8 The mass of the Higgs boson

In the present case, the EW symmetry is broken by the VEV of the zero-mode of $A^7_{y(z)}$. The Higgs potential is flat in the 5D GHU at the classical level, and the flat direction is determined by θ_H. This flatness is lifted at the one-loop level. Such potential $V_{\text{eff}}(\theta_H)$ determines $\langle\theta_H\rangle$. Then, the Higgs field obtain a finite mass.

According to the [14], the general (flat or warped) form of the one-loop effective potential is expressed as

$$V_{\text{eff}}(\theta_H) = \frac{3}{l_6\pi^3} m_{\text{KK}}^4 f(\theta_H). \tag{2.8.1}$$

where $f(\theta_H)$ is a dimensionless periodic function of θ_H with a 2π period. The Higgs mass can be obtained by expanding $V_{\text{eff}}(\theta_H)$ around the $\langle\theta_H\rangle = \theta_H^{\text{min}}$, that gives the global minimum of $V_{\text{eff}}(\theta_H)$:

$$m_H^2 = f''(\theta_H^{\text{min}}) \frac{3\pi g_4^2}{l_6^2} \frac{R(e^{2k\pi R}-1)}{k} m_{\text{KK}}^4 \tag{2.8.2}$$

$$\begin{aligned} \therefore m_H &= \left\{ f''(\theta_H^{\text{min}}) \frac{3g_4^2}{256\pi^3} \right\}^{\frac{1}{2}} \sqrt{kR}\, m_{\text{KK}} \\ &= \left\{ f''(\theta_H^{\text{min}}) \frac{3g_4^2}{128\pi^2} \right\}^{\frac{1}{2}} \frac{k\pi R}{2} \frac{m_W}{\sin\frac{\theta_H^{\text{min}}}{2}}, \end{aligned} \tag{2.8.3}$$

in the warped case. In [13], the factor $\frac{k\pi R}{2} \sim 19$ and $\theta_H = \frac{\pi}{2}$ gives $m_H = 125$ GeV with $f''(\theta_H)^{\frac{1}{2}} \sim 1.9$.

Chapter 3

6D gauge-Higgs Unification with custodial symmetry

3.1 Motivations and purposes

In extra-dimensional models, coupling constants in 4D effective theories generally deviate from the standard model values even at tree level due to the mixing with the KK modes [26, 27, 28]. Unless m_{KK} is very high, models need some mechanisms to suppress such deviations. Especially a requirement that the rho parameter and the $Z\bar{b}b$-coupling do not deviate too much often imposes severe constraints on the model building. It is known that the custodial symmetry can protect them against the corrections induced by the mixing with the KK modes [19, 29]. Hence we focus on 6D GHU models that has the custodial symmetry in this section.

The purpose of this chapter is to select candidates for realistic 6D GHU models by making use of the group theoretical analysis. The analysis is useful to investigate the GHU models because the Higgs sector is determined by the gauge group structure. There are some works along this direction. 5D models are analyzed in Ref. [31], the tree-level Higgs potentials in 6D models are calculated in Ref. [17], and models in arbitrary dimensions are discussed in Ref. [32]. In these works, the custodial symmetry is not considered and the electroweak gauge symmetry $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_Y$ is embedded into a simple group. Thus the Weinberg angle θ_W is determined only by the group structure, and they found that no simple group realizes the observed value of θ_W. However, this assumption is not indispensable because the color symmetry $\mathrm{SU}(3)_C$ is not unified anyway. Besides, any brane localized terms allowed by the symmetries are not introduced in those works. In fact, the realistic models constructed so far have allowed both an extra U(1) gauge symmetry, which is relevant to the realization of the experimental value of θ_W, and various terms and fields localized at the fixed points of the orbifolds [15, 18, 22, 24]. Therefore, we include both ingredients in our analysis. Since larger gauge groups contain more unwanted exotic particles, we restrict the 6D gauge group to $\mathrm{SU}(3)_C \times G \times \mathrm{U}(1)_Z$, where G is a simple group whose rank is less than four.

This chapter is organized as follows. In the next section, we explain our setup and derive conditions for zero-modes. In Sec. 3.3, we list the zero-modes in the bosonic sector for all the rank-two and the rank-three groups. In Sec. 3.4, we find a condition to preserve the custodial symmetry, and provide explicit expressions of the W and the Z boson masses. In Sec. 3.5, we discuss embeddings of quarks into 6D fermions, and search for appropriate representations of G that the 6D fermions should belong to. In Sec. 3.6, we calculate the Higgs potential at tree level. We summarize at Sec. 5 . In Appendix A, we collect formulae in the Cartan-Weyl basis of the gauge group generators. In Appendix B, general forms of the orbifold boundary conditions are shown. In Appendix C, we list irreducible decompositions of various G representations into the $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}$ multiplets.

3.2 Setup

3.2.1 Compactified space

The 6D spacetime is assumed to be flat, and the metric is given by

$$ds^2 = \eta_{MN} dx^M dx^N = \eta_{\mu\nu} dx^\mu dx^\nu + (dx^4)^2 + (dx^5)^2, \tag{3.2.1}$$

where $M, N = 0, 1, \cdots, 5$, $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ is the 4D Minkowski metric, and the coordinates of the extra-dimensions (x^4, x^5) are identified as

$$\begin{pmatrix} x^4 \\ x^5 \end{pmatrix} \sim \begin{pmatrix} x^4 \\ x^5 \end{pmatrix} + 2\pi n_1 R_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\pi n_2 R_2 \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \tag{3.2.2}$$

where n_1 and n_2 are integers, and $R_1, R_2 > 0$ and $0 < \theta < \pi$ are constants. In order to obtain a chiral 4D theory at low energies, we compactify the extra space on a two-dimensional orbifold. All possible orbifolds are T^2/Z_N $(N = 2, 3, 4, 6)$. It is convenient to use a complex (dimensionless) coordinate $z \equiv \frac{1}{2\pi R_1}(x^4 + ix^5)$. Then, the orbifold obeys the identification,

$$z \sim \omega z + n_1 + n_2 \tau, \tag{3.2.3}$$

where $\omega = e^{2\pi i/N}$ and $\tau \equiv \frac{R_2}{R_1} e^{i\theta}$ [33]. Note that an arbitrary value of τ is allowed when $N = 2$ while it must be equal to ω when $N \neq 2$.

The orbifold T^2/Z_N has the following fixed points in the fundamental domain [35, 39].

$$z = z_{\mathrm{f}} \equiv \begin{cases} 0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2} & (\text{on } T^2/Z_2) \\ 0, \frac{2+\tau}{3}, \frac{1+2\tau}{3} & (\text{on } T^2/Z_3) \\ 0, \frac{1+\tau}{2} & (\text{on } T^2/Z_4) \\ 0 & (\text{on } T^2/Z_6) \end{cases} \tag{3.2.4}$$

4D fields or interactions are allowed to be introduced on these fixed points.

3.2.2 Field content

We consider a 6D gauge theory whose gauge group is $\mathrm{SU}(3)_C \times G \times \mathrm{U}(1)_Z$, where G is a simple group. Since G must include $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}}$, its rank r is greater than one. In this paper, we investigate cases in which $r = 2, 3$. In the following, we omit $\mathrm{SU}(3)_C$ since it is irrelevant to the discussion. The 6D gauge fields for G and $\mathrm{U}(1)_Z$ are denoted as A_M and B^Z_M, and the field strengths and the covariant derivative are defined as $F^{(A)}_{MN} \equiv \partial_M A_N - \partial_N A_M - i[A_M, A_N]$, $F^{(Z)}_{MN} \equiv \partial_M B^Z_N - \partial_N B^Z_M$, and $\mathcal{D}_M \equiv \partial_M - iA_M - iq_Z B^Z_M$, where q_Z is a $\mathrm{U}(1)_Z$ charge. The 6D Lagrangian is expressed as

$$\begin{aligned} \mathcal{L} = &- \frac{1}{4g_A^2} \mathrm{tr}\left(F^{(A)MN} F^{(A)}_{MN}\right) - \frac{1}{4g_Z^2} F^{(Z)MN} F^{(Z)}_{MN} + i \sum_f \bar{\Psi}^f \Gamma^M \mathcal{D}_M \Psi^f \\ &+ \sum_{z_{\mathrm{f}}} \mathcal{L}^{(z_{\mathrm{f}})} \delta^{(2)}(z - z_{\mathrm{f}}), \end{aligned} \tag{3.2.5}$$

where g_A and g_Z are the 6D gauge coupling constants for G and $\mathrm{U}(1)_Z$, Γ^M are the 6D gamma matrices, and $\mathcal{L}^{(z_{\mathrm{f}})}$ are 4D Lagrangians localized at the fixed points $z = z_{\mathrm{f}}$.

The G gauge field A_M is decomposed as

$$A_M = \sum_i C^i_M H_i + \sum_\alpha W^\alpha_M E_\alpha, \tag{3.2.6}$$

where $\{H_i, E_\alpha\}$ are the generators in the Cartan-Weyl basis, i.e., H_i $(i = 1, \cdots, r)$ are the Cartan generators and α runs over all the roots of G. Since A_M is Hermitian, C^i_M are real and $W^{-\alpha}_M = (W^\alpha_M)^*$. In the complex coordinate (x^μ, z), the extra-dimensional components are expressed as

$$\begin{aligned} A_z &= \pi R_1 \left(A_y - iA_5\right), \qquad A_{\bar{z}} = A_z^\dagger, \\ B^Z_z &= \pi R_1 \left(B^Z_4 - iB^Z_5\right), \qquad B^Z_{\bar{z}} = B^{Z\dagger}_z. \end{aligned} \tag{3.2.7}$$

3.2.3 Orbifold conditions for gauge fields

As shown in Appendix B, the general orbifold boundary conditions for the gauge fields can be expressed as

$$
\begin{aligned}
&A_M(x, z+1) = A_M(x, z), \\
&B^Z_\mu(x, z+1) = B^Z_\mu(x, z), \qquad B^Z_z(x, z+1) = B^Z_z(x, z), \\
&A_M(x, z+\tau) = A_M(x, z), \\
&B^Z_\mu(x, z+\tau) = B^Z_\mu(x, z), \qquad B^Z_z(x, z+\tau) = B^Z_z(x, z), \\
&A_\mu(x, \omega z) = P A_\mu(x, z) P^{-1}, \qquad A_z(x, \omega z) = \omega^{-1} P A_z(x, z) P^{-1}, \\
&B^Z_\mu(x, \omega z) = B^Z_\mu(x, z), \qquad B^Z_z(x, \omega z) = \omega^{-1} B^Z_z(x, z),
\end{aligned}
\tag{3.2.8}
$$

where P are elements of G. The orbifold conditions for 6D fermions are provided in (3.5.2).

Since the zero-modes of the gauge fields have flat profiles over the extra dimensional space, the condition for A_M to have zero-modes are determined by the choice of the matrix P in (3.2.8). It is always possible to choose the generators so that P is expressed in terms of the Cartan generators as

$$
P = \exp\left(ip \cdot H\right), \tag{3.2.9}
$$

where $p \cdot H \equiv \sum_i p_i H_i$ and p_i are real constants. Thus $P H_i P^{-1} = H_i$ and $P E_\alpha P^{-1} = e^{ip\cdot\alpha} E_\alpha$, and the relevant conditions in (3.2.8) are rewritten as

$$
\begin{aligned}
&C^i_\mu(x, \omega z) = C^i_\mu(x, z), \qquad C^i_z(x, \omega z) = \omega^{-1} C^i_z(x, z), \\
&W^\alpha_\mu(x, \omega z) = e^{ip\cdot\alpha} W^\alpha_\mu(x, z), \qquad W^\alpha_z(x, \omega z) = e^{i(p\cdot\alpha - \frac{2\pi}{N})} W^\alpha_z(x, z).
\end{aligned}
\tag{3.2.10}
$$

This indicates that C^i_μ always have zero-modes while C^i_z do not irrespective of the choice of the matrix P. Therefore the orbifold boundary conditions cannot reduce the rank of G as mentioned in Ref. [34]. Besides, B^Z_μ has a zero-mode while B^Z_z does not. Namely $\mathrm{U}(1)_Z$ is unbroken by the orbifold conditions. In contrast, whether W^α_μ and W^α_z have zero-modes depend on the choice of P. Since (3.2.10) is the Z_N transformation, p_i must satisfy $e^{iNp\cdot\alpha} = \mathbf{1}$. Thus possible values of $p \cdot \alpha$ are

$$
p \cdot \alpha = \frac{2 n_\alpha \pi}{N}, \tag{3.2.11}
$$

where n_α is an integer.

From (3.2.10), the conditions for W_μ^α and W_z^α to have zero-modes are expressed as

$$p \cdot \alpha = \begin{cases} 0, & (\text{for } W_\mu^\alpha) \\ \frac{2\pi}{N}, & (\text{for } W_z^\alpha) \end{cases} \tag{3.2.12}$$

where the equalities hold modulo 2π.

Now we focus on P such that the orbifold bondary conditions break G to $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R} \times \mathrm{U}(1)^{r-2}$. We denote the positive roots that specify $\mathrm{SU}(2)_\mathrm{L}$ and $\mathrm{SU}(2)_\mathrm{R}$ as α_L and α_R, respectively. Then they must satisfy $\alpha_L \cdot \alpha_R = 0$, and $\alpha_L + \alpha_R$ is not a root, and p_i must satisfy

$$\begin{aligned} p \cdot \alpha_L = p \cdot \alpha_R = 0, \qquad & (\mathrm{mod}\ 2\pi) \\ p \cdot \beta = \frac{2n_\beta \pi}{N}, \qquad & (n_\beta \in \mathbb{Z}, \quad n_\beta \notin N\mathbb{Z}) \end{aligned} \tag{3.2.13}$$

where β is the root of G other than α_L and α_R.

3.3 Zero-modes of gauge and Higgs fields

In this section, we investigate the field content of the zero-modes from the 6D gauge fields.

3.3.1 Rank two groups

First we consider a case that $r = 2$, i.e., $G = \mathrm{SO}(5), \mathrm{G}_2$. In this case, the unbroken gauge group by the orbifold conditions is $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R} \times \mathrm{U}(1)_Z$. We do not consider G =SU(3) because it does not contain $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}$ as a subgroup. The roots of G can be expressed as linear combinations of two-dimensional basis vectors $\boldsymbol{e}^i$ $(i = 1, 2)$.

SO(5)

The roots are $\{\pm \boldsymbol{e}^i \pm \boldsymbol{e}^j, \pm \boldsymbol{e}^i\}$ $(1 \le i \ne j \le 2)$. We can choose the unbroken subgroup $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}$ as

$$(\alpha_L, \alpha_R) = (\boldsymbol{e}^1 + \boldsymbol{e}^2, \boldsymbol{e}^1 - \boldsymbol{e}^2). \tag{3.3.1}$$

The other possible choices are essentially equivalent to this case.[1] Then the adjoint representation of G is decomposed into the irreducible representations of $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}$ as

$$\mathbf{10} = (\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) + (\mathbf{2}, \mathbf{2}). \tag{3.3.2}$$

[1] We cannot choose them as $(\alpha_L, \alpha_R) = (\boldsymbol{e}^1, \boldsymbol{e}^2)$ because $\alpha_L + \alpha_R$ is a root in such a case.

The candidate for the Higgs scalars is a bidoublet $\mathbf{(2,2)}$, which consists of $\pm\boldsymbol{e}^1$ and $\pm\boldsymbol{e}^2$. The conditions in (3.2.13) are now expressed as

$$\begin{aligned} p_1 + p_2 = p_1 - p_2 = 0, & \qquad (\mathrm{mod}\ 2\pi) \\ p_1 = \frac{2n_P\pi}{N}. & \qquad (n_P \in \mathbb{Z}, \quad n_P \notin N\mathbb{Z}) \end{aligned} \tag{3.3.3}$$

It is enough to search the solution in a range $0 \leq p_1, p_2 < 2\pi$. The solution exists when $N \neq 3$, and it is

$$(p_1, p_2) = (\pi, \pi), \tag{3.3.4}$$

or

$$P = \exp\{i\pi(H_1 + H_2)\}. \tag{3.3.5}$$

Therefore the zero-mode condition for $\mathbf{(2,2)}$ is expressed as

$$\pi = \frac{2\pi}{N}. \tag{3.3.6}$$

Namely, we have a bidoublet Higgs when $N = 2$, while no Higgs exists in the other cases.

$\mathbf{G_2}$

The roots are $\{\pm(\boldsymbol{e}^1 \pm \sqrt{3}\boldsymbol{e}^2)/2, \pm(\boldsymbol{e}^1 \pm \frac{1}{\sqrt{3}}\boldsymbol{e}^2)/2, \pm\boldsymbol{e}^1, \pm\boldsymbol{e}^2/\sqrt{3}\}$. We can choose the $\mathrm{SU(2)_L \times SU(2)_R}$ subgroup as

$$(\alpha_L, \alpha_R) = \left(\boldsymbol{e}^1, \frac{\boldsymbol{e}^2}{\sqrt{3}}\right), \quad \left(\frac{\boldsymbol{e}^2}{\sqrt{3}}, \boldsymbol{e}^1\right). \tag{3.3.7}$$

The other possible choices are essentially equivalent to these cases.

Let us first consider the case of $(\alpha_L, \alpha_R) = (\boldsymbol{e}^1, \boldsymbol{e}^2/\sqrt{3})$. The irreducible decomposition of the adjoint representation of G is

$$\mathbf{14} = \mathbf{(3,1)} + \mathbf{(1,3)} + \mathbf{(2,4)}. \tag{3.3.8}$$

The candidate for the Higgs scalars is $\mathbf{(2,4)}$. The conditions in (3.2.13) are expressed as

$$\begin{aligned} p_1 = \frac{p_2}{\sqrt{3}} = 0, & \qquad (\mathrm{mod}\ 2\pi) \\ \frac{p_1}{2} + \frac{p_2}{2\sqrt{3}} = \frac{2n_P\pi}{N}. & \qquad (n_P \in \mathbb{Z}, \quad n_P \notin N\mathbb{Z}) \end{aligned} \tag{3.3.9}$$

It is enough to search the solution in a range $0 \leq p_1, \frac{p_2}{\sqrt{3}} < 2\pi$. The solution exists when $N \neq 3$, and it is

$$P = \exp\left(2\sqrt{3}\pi i H_2\right). \tag{3.3.10}$$

Therefore the zero-mode condition for $\mathbf{(2,4)}$ is expressed as

$$\pi = \frac{2\pi}{N}. \tag{3.3.11}$$

Namely, we have a $\mathbf{(2,4)}$ multiplet as the Higgs scalar zero-modes when $N = 2$, while no Higgs exists in the other cases.

In the case of $(\alpha_L, \alpha_R) = (\boldsymbol{e}^2/\sqrt{3}, \boldsymbol{e}^1)$, the results are obtained by exchanging $\mathrm{SU(2)_L}$ and $\mathrm{SU(2)_R}$ in the above resuts. Hence we do not have $\mathrm{SU(2)_L}$-doublet Higgses.

3.3.2 Rank three groups

Next we consider a case that $r = 3$, i.e., G =SU(4),SO(7),Sp(6). In this case, the unbroken gauge group by the orbifold conditions is $\mathrm{SU(2)_L} \times \mathrm{SU(2)_R} \times \mathrm{U(1)}_X \times \mathrm{U(1)}_Z$. The roots of G can be expressed as linear combinations of three-dimensional basis vectors $\boldsymbol{e}^i$ $(i = 1, 2, 3)$.

SU(4)

The roots are $\{\sqrt{2}\boldsymbol{e}^1, \sqrt{2}\boldsymbol{e}^2, \pm\frac{\boldsymbol{e}^1}{\sqrt{2}}\pm\frac{\boldsymbol{e}^2}{\sqrt{2}}+\boldsymbol{e}^3\}$.[2] We can choose the $\mathrm{SU(2)_L} \times \mathrm{SU(2)_R}$ subgroup as

$$(\alpha_L, \alpha_R) = (\sqrt{2}\boldsymbol{e}^1, \sqrt{2}\boldsymbol{e}^2). \tag{3.3.12}$$

The other choices are essentially equivalent to this case. The $U(1)_X$ generator Q_X is identified as

$$Q_X = 2\boldsymbol{e}_3 \cdot H = 2H_3. \tag{3.3.13}$$

The irreducible decomposition of the adjoint representation of G is

$$\mathbf{15} = \mathbf{(3,1)_0} + \mathbf{(1,3)_0} + \mathbf{(2,2)_{+2}} + \mathbf{(2,2)_{-2}} + \mathbf{(1,1)_0}, \tag{3.3.14}$$

where $\mathbf{(3,1)_0}$, $\mathbf{(1,3)_0}$ and $\mathbf{(1,1)_0}$ correspond to $\mathrm{SU(2)_L}$, $\mathrm{SU(2)_R}$ and $\mathrm{U(1)}_X$ generators, respectively. Thus the candidates for the Higgs scalars are two bidoublets. Independent conditions in (3.2.13) are expressed as

$$\begin{aligned} &\sqrt{2}p_1 = \sqrt{2}p_2 = 0, \qquad (\mathrm{mod}\ 2\pi) \\ &\frac{p_1}{\sqrt{2}} + \frac{p_2}{\sqrt{2}} + p_3 = \frac{2n_P\pi}{N}, \qquad (n_P \in \mathbb{Z},\ \ n_P \notin N\mathbb{Z}) \end{aligned} \tag{3.3.15}$$

[2]It is sometimes convenient to embed these roots into a four-dimensional vector space. Then they are expressed as $\hat{\boldsymbol{e}}^I - \hat{\boldsymbol{e}}^J$ $(1 \le I \ne J \le 4)$, where $\hat{\boldsymbol{e}}^I$ are the basis vectors of the embeded space. The original basis vectors are expressed as $\boldsymbol{e}^1 = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{e}}^1 - \hat{\boldsymbol{e}}^2)$, $\boldsymbol{e}^2 = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{e}}^3 - \hat{\boldsymbol{e}}^4)$ and $\boldsymbol{e}^3 = \frac{1}{2}(\hat{\boldsymbol{e}}^1 + \hat{\boldsymbol{e}}^2 - \hat{\boldsymbol{e}}^3 - \hat{\boldsymbol{e}}^4)$.

The solution is

$$P = \exp\left(\frac{2n_P\pi i}{N}H_3\right), \tag{3.3.16}$$

where $n_P = 1, \cdots, N-1$. Therefore the zero-mode conditions for $\mathbf{(2,2)_{\pm 2}}$ are

$$\pm\frac{2n_P\pi}{N} = \frac{2\pi}{N}. \quad (\mathrm{mod}\ 2\pi) \tag{3.3.17}$$

Therefore, the scalar zero-modes we have are

$$\begin{aligned}
\mathbf{(2,2)_{+2}},\ \ \mathbf{(2,2)_{-2}} &: (\text{when } N=2)\\
\mathbf{(2,2)_{+2}} &: (\text{when } N=3,4,6 \text{ and } n_P=1)\\
\mathbf{(2,2)_{-2}} &: (\text{when } N=3,4,6 \text{ and } n_P=N-1)\\
\text{Nothing} &: (\text{in the other cases})
\end{aligned} \tag{3.3.18}$$

SO(7)

The roots are $\{\pm\boldsymbol{e}^i \pm \boldsymbol{e}^j, \pm\boldsymbol{e}^i\}$ $(1 \leq i \neq j \leq 3)$. Essentially inequivalent choices of the $\mathrm{SU(2)_L} \times \mathrm{SU(2)_R}$ subgroup are

$$(\alpha_L, \alpha_R) = (\boldsymbol{e}^1+\boldsymbol{e}^2, \boldsymbol{e}^1-\boldsymbol{e}^2),\ \ (\boldsymbol{e}^1+\boldsymbol{e}^2, \boldsymbol{e}^3),\ \ (\boldsymbol{e}^3, \boldsymbol{e}^1+\boldsymbol{e}^2). \tag{3.3.19}$$

(I) $(\boldsymbol{\alpha_L}, \boldsymbol{\alpha_R}) = (\boldsymbol{e^1}+\boldsymbol{e^2}, \boldsymbol{e^1}-\boldsymbol{e^2})$

The $U(1)_X$ generator is

$$Q_X = \boldsymbol{e}^3 \cdot H = H_3. \tag{3.3.20}$$

The irreducible decomposition of the adjoint representation of G is

$$\begin{aligned}
\mathbf{21} = \mathbf{(3,1)_0} + \mathbf{(1,3)_0} + \mathbf{(2,2)_{+1}} + \mathbf{(2,2)_{-1}} + \mathbf{(2,2)_0}\\
+\mathbf{(1,1)_{+1}} + \mathbf{(1,1)_{-1}} + \mathbf{(1,1)_0},
\end{aligned} \tag{3.3.21}$$

where $\mathbf{(3,1)_0}$, $\mathbf{(1,3)_0}$ and $\mathbf{(1,1)_0}$ correspond to $\mathrm{SU(2)_L}$, $\mathrm{SU(2)_R}$ and $\mathrm{U(1)_X}$ generators, respectively. Thus the candidates for the scalar zero-modes are three bidoublets and two singlets. Independent conditions in (3.2.13) are expressed as

$$\begin{aligned}
&p_1+p_2 = p_1-p_2 = 0, \quad (\mathrm{mod}\ 2\pi)\\
&p_1+p_3, p_1, p_3 = \frac{2n_P\pi}{N}. \quad (n_P \in \mathbb{Z}, \quad n_P \notin N\mathbb{Z})
\end{aligned} \tag{3.3.22}$$

It is enough to search the solution in a range $0 \leq p_1, p_2, p_3 < 2\pi$. The solution exists only when $N = 4, 6$, and it is

$$P = \exp\left\{ i\pi \left(H_1 + H_2 + \frac{2n_P}{N} H_3 \right) \right\}, \tag{3.3.23}$$

where $n_P \neq 0, N/2$. Therefore the zero-mode conditions for $\mathbf{(2,2)_{\pm 1}}$, $\mathbf{(2,2)_0}$ and $\mathbf{(1,1)_{\pm 1}}$ are

$$\pi \pm \frac{2n_P\pi}{N} = \frac{2\pi}{N}, \qquad \pi = \frac{2\pi}{N}, \qquad \pm\frac{2n_P\pi}{N} = \frac{2\pi}{N}, \tag{3.3.24}$$

respectively. The double-signs correspond.

When $N = 4$, the scalar zero-modes we have are

$$\begin{aligned} &\mathbf{(2,2)_{-1}}, \ \ \mathbf{(1,1)_{+1}} : (\text{when } n_P = 1) \\ &\mathbf{(2,2)_{+1}}, \ \ \mathbf{(1,1)_{-1}} : (\text{when } n_P = 3) \end{aligned} \tag{3.3.25}$$

When $N = 6$, they are

$$\begin{aligned} &\mathbf{(1,1)_{+1}} : (\text{when } n_P = 1) \\ &\mathbf{(2,2)_{-1}} : (\text{when } n_P = 2) \\ &\mathbf{(2,2)_{+1}} : (\text{when } n_P = 4) \\ &\mathbf{(1,1)_{-1}} : (\text{when } n_P = 5) \end{aligned} \tag{3.3.26}$$

(II) $(\boldsymbol{\alpha_L}, \boldsymbol{\alpha_R}) = (\boldsymbol{e^1} + \boldsymbol{e^2}, \boldsymbol{e^3})$

The $U(1)_X$ generator is

$$Q_X = (\boldsymbol{e}^1 - \boldsymbol{e}^2) \cdot H = H_1 - H_2. \tag{3.3.27}$$

The irreducible decomposition of the adjoint representation of G is

$$\begin{aligned} \mathbf{21} = \mathbf{(3,1)_0} + \mathbf{(1,3)_0} + \mathbf{(2,3)_{+1}} + \mathbf{(2,3)_{-1}} \\ + \mathbf{(1,1)_{+2}} + \mathbf{(1,1)_{-2}} + \mathbf{(1,1)_0}. \end{aligned} \tag{3.3.28}$$

The candidates for the scalar zero-modes are $\mathbf{(2,3)_{\pm 1}}$ and $\mathbf{(1,1)_{\pm 1}}$. Independent conditions in (3.2.13) are expressed as

$$\begin{aligned} &p_1 + p_2 = p_3 = 0, \qquad (\text{mod } 2\pi) \\ &p_1 + p_3, p_2 + p_3, p_1 - p_2 = \frac{2n_P\pi}{N}. \qquad (n_P \in \mathbb{Z}, \ \ n_P \notin N\mathbb{Z}) \end{aligned} \tag{3.3.29}$$

It is enough to search the solution in a range $0 \leq p_1, p_2, p_3 < 2\pi$. The solution exists when $N = 3, 4, 6$, and it is

$$P = \exp\left\{\frac{2n_P\pi i}{N}(H_1 - H_2)\right\}, \tag{3.3.30}$$

where $n_P \neq 0, N/2$. Therefore the zero-mode conditions for $\mathbf{(2,3)_{\pm 1}}$ and $\mathbf{(1,1)_{\pm 1}}$ are

$$\pm\frac{2n_P\pi}{N} = \frac{2\pi}{N}, \qquad \pm\frac{4n_P\pi}{N} = \frac{2\pi}{N}, \tag{3.3.31}$$

respectively.

When $N = 3$, the scalar zero-modes we have are

$$\begin{aligned} &\mathbf{(2,3)_{+1}},\ \ \mathbf{(1,1)_{-2}} : (\text{when } n_P = 1) \\ &\mathbf{(2,3)_{-1}},\ \ \mathbf{(1,1)_{+2}} : (\text{when } n_P = 2) \end{aligned} \tag{3.3.32}$$

When $N = 4$, they are

$$\begin{aligned} &\mathbf{(2,3)_{+1}} : (\text{when } n_P = 1) \\ &\mathbf{(2,3)_{-1}} : (\text{when } n_P = 3) \end{aligned} \tag{3.3.33}$$

When $N = 6$, they are

$$\begin{aligned} &\mathbf{(2,3)_{+1}} : (\text{when } n_P = 1) \\ &\text{Nothing} : (\text{when } n_P = 2, 4) \\ &\mathbf{(2,3)_{-1}} : (\text{when } n_P = 5) \end{aligned} \tag{3.3.34}$$

(III) $(\boldsymbol{\alpha_L}, \boldsymbol{\alpha_R}) = (\boldsymbol{e^1} + \boldsymbol{e^2}, \boldsymbol{e^3})$

The results are obtained by exchanging $\mathrm{SU(2)_L}$ and $\mathrm{SU(2)_R}$ in the case (II). Hence we do not have $\mathrm{SU(2)_L}$-doublet Higgses.

Sp(6)

The roots are $\{\pm\boldsymbol{e}^i \pm \boldsymbol{e}^j, \pm 2\boldsymbol{e}^i\}$ $(1 \leq i \neq j \leq 3)$. Essentially inequivalent choices of the $\mathrm{SU(2)_L} \times \mathrm{SU(2)_R}$ are

$$(\alpha_L, \alpha_R) = (2\boldsymbol{e}^1, 2\boldsymbol{e}^2),\ \ (\boldsymbol{e}^1 + \boldsymbol{e}^2, 2\boldsymbol{e}^3),\ \ (2\boldsymbol{e}^3, \boldsymbol{e}^1 + \boldsymbol{e}^2). \tag{3.3.35}$$

(I) $(\boldsymbol{\alpha_L}, \boldsymbol{\alpha_R}) = (2\boldsymbol{e^1}, 2\boldsymbol{e^2})$

The $U(1)_X$ generator is

$$Q_X = \boldsymbol{e}^3 \cdot H = H_3. \tag{3.3.36}$$

The irreducible decomposition of the adjoint representation of G is

$$\begin{aligned}\mathbf{21} = &(\mathbf{3},\mathbf{1})_\mathbf{0} + (\mathbf{1},\mathbf{3})_\mathbf{0} + (\mathbf{2},\mathbf{2})_\mathbf{0} + (\mathbf{2},\mathbf{1})_{\mathbf{+1}} + (\mathbf{2},\mathbf{1})_{\mathbf{-1}} \\ &+(\mathbf{1},\mathbf{2})_{\mathbf{+1}} + (\mathbf{1},\mathbf{2})_{\mathbf{-1}} + (\mathbf{1},\mathbf{1})_{\mathbf{+2}} + (\mathbf{1},\mathbf{1})_{\mathbf{-2}} + (\mathbf{1},\mathbf{1})_\mathbf{0}.\end{aligned} \tag{3.3.37}$$

The candidates for the scalar zero-modes are four $\mathrm{SU(2)_L}$-doublets and six $\mathrm{SU(2)_L}$-singlets. Independent conditions in (3.2.13) are expressed as

$$\begin{aligned}&2p_1 = 2p_2 = 0, \qquad (\mathrm{mod}\ 2\pi) \\ &p_1 + p_2, p_1 \pm p_3, p_2 \pm p_3, 2p_3 = \frac{2n\pi}{N}, \qquad (n \in \mathbb{Z}, \ \ n \notin N\mathbb{Z})\end{aligned} \tag{3.3.38}$$

The solutions exist only when $N = 4, 6$. They are

$$P = \begin{cases} P_n^{(1)} \equiv \exp\left\{i\pi\left(H_2 + \frac{2n\pi}{N}H_3\right)\right\}, \\ P_n^{(2)} \equiv \exp\left\{i\pi\left(H_1 + \frac{2n\pi}{N}H_3\right)\right\}, \end{cases} \tag{3.3.39}$$

where $n \neq 0, N/2$.

When $N = 4$, the scalar zero-modes we have are

$$\begin{aligned}&(\mathbf{2},\mathbf{1})_{\mathbf{+1}}, \ \ (\mathbf{1},\mathbf{2})_{\mathbf{-1}} : (\text{for } P_1^{(1)} \text{ or } P_3^{(2)}) \\ &(\mathbf{2},\mathbf{1})_{\mathbf{-1}}, \ \ (\mathbf{1},\mathbf{2})_{\mathbf{+1}} : (\text{for } P_3^{(1)} \text{ or } P_1^{(2)})\end{aligned} \tag{3.3.40}$$

When $N = 6$, they are

$$\begin{aligned}&(\mathbf{2},\mathbf{1})_{\mathbf{+1}} : (\text{for } P_1^{(1)} \text{ or } P_4^{(2)}) \\ &(\mathbf{1},\mathbf{2})_{\mathbf{-1}} : (\text{for } P_2^{(1)} \text{ or } P_5^{(2)}) \\ &(\mathbf{1},\mathbf{2})_{\mathbf{+1}} : (\text{for } P_4^{(1)} \text{ or } P_1^{(2)}) \\ &(\mathbf{2},\mathbf{1})_{\mathbf{-1}} : (\text{for } P_5^{(1)} \text{ or } P_2^{(2)})\end{aligned} \tag{3.3.41}$$

(II) $(\boldsymbol{\alpha_L}, \boldsymbol{\alpha_R}) = (\boldsymbol{e^1} + \boldsymbol{e^2}, 2\boldsymbol{e^3})$

The $U(1)_X$ generator is

$$Q_X = \left(\boldsymbol{e}^1 - \boldsymbol{e}^2\right) \cdot H = H_1 - H_2. \tag{3.3.42}$$

The irreducible decomposition of the adjoint representation of G is

$$\mathbf{21} = (\mathbf{3},\mathbf{1})_{\mathbf{0}} + (\mathbf{1},\mathbf{3})_{\mathbf{0}} + (\mathbf{3},\mathbf{1})_{+\mathbf{2}} + (\mathbf{3},\mathbf{1})_{-\mathbf{2}} + (\mathbf{2},\mathbf{2})_{+\mathbf{1}} + (\mathbf{2},\mathbf{2})_{-\mathbf{1}} + (\mathbf{1},\mathbf{1})_{\mathbf{0}}. \tag{3.3.43}$$

The conditions in (3.2.13) are expressed as

$$\begin{aligned} &p_1 + p_2 = 2p_3 = 0, \qquad (\text{mod } 2\pi) \\ &p_1 + p_3, p_2 + p_3, 2p_1, 2p_2 = \frac{2n\pi}{N}, \qquad (n \in \mathbb{Z}, \ \ n \notin N\mathbb{Z}) \end{aligned} \tag{3.3.44}$$

where $n \neq 0, N/2$. The solutions exist only when $N = 3, 4, 6$. They are

$$P = \begin{cases} P_n^{(1)} \equiv \exp\left\{\frac{2n\pi i}{N}(H_1 - H_2)\right\}, \\ P_n^{(2)} \equiv \exp\left\{i\pi\left(\frac{2n-N}{N}(H_1 - H_2) + H_3\right)\right\}. \end{cases} \tag{3.3.45}$$

When $N = 3$, the scalar zero-modes we have are

$$\begin{aligned} &(\mathbf{3},\mathbf{1})_{-\mathbf{2}}, \ \ (\mathbf{2},\mathbf{2})_{+\mathbf{1}} : (\text{for } P_1^{(1)} \text{ or } P_1^{(2)}) \\ &(\mathbf{3},\mathbf{1})_{+\mathbf{2}}, \ \ (\mathbf{2},\mathbf{2})_{-\mathbf{1}} : (\text{for } P_2^{(1)} \text{ or } P_2^{(2)}) \end{aligned} \tag{3.3.46}$$

When $N = 4$, they are

$$\begin{aligned} &(\mathbf{2},\mathbf{2})_{+\mathbf{1}} : (\text{for } P_1^{(1)} \text{ or } P_1^{(2)}) \\ &(\mathbf{2},\mathbf{2})_{-\mathbf{1}} : (\text{for } P_3^{(1)} \text{ or } P_3^{(2)}) \end{aligned} \tag{3.3.47}$$

When $N = 6$, they are

$$\begin{aligned} &(\mathbf{2},\mathbf{2})_{+\mathbf{1}} : (\text{for } P_1^{(1)} \text{ or } P_1^{(2)}) \\ &(\mathbf{2},\mathbf{2})_{-\mathbf{1}} : (\text{for } P_5^{(1)} \text{ or } P_5^{(2)}) \\ &\text{Nothing} : (\text{in the other cases}) \end{aligned} \tag{3.3.48}$$

(III) $(\boldsymbol{\alpha_L}, \boldsymbol{\alpha_R}) = (\boldsymbol{2e^3}, \boldsymbol{e^1 + e^2})$

The results are obtained by exchanging $\mathrm{SU}(2)_{\mathrm{L}}$ and $\mathrm{SU}(2)_{\mathrm{R}}$ in the case (II).

3.4 Custodial symmetry and Weinberg angle

3.4.1 Custodial symmetry

Here we consider a condition that the custodial symmetry is preserved after the electroweak symmetry is broken.

The $\mathrm{SU(2)_L}$ and $\mathrm{SU(2)_R}$ generators are

$$(T_L^{\pm}, T_L^3) = \left(\frac{E_{\pm\alpha_L}}{|\alpha_L|}, \frac{\alpha_L \cdot H}{|\alpha_L|^2}\right), \qquad (T_R^{\pm}, T_R^3) = \left(\frac{E_{\pm\alpha_R}}{|\alpha_R|}, \frac{\alpha_R \cdot H}{|\alpha_R|^2}\right), \tag{3.4.1}$$

respectively. Thus we can rewrite (3.2.6) as

$$\begin{aligned} A_\mu =& W_{L\mu}^+ T_L^+ + W_{L\mu}^- T_L^- + W_{L\mu}^3 T_L^3 + W_{R\mu}^+ T_R^+ + W_{R\mu}^- T_R^- + W_{R\mu}^3 T_R^3 \\ &+ B_\mu^X \mathrm{x} \cdot H + \cdots , \end{aligned} \tag{3.4.2}$$

where

$$\begin{aligned} W_{L\mu}^{\pm} &\equiv |\alpha_L|\, W_\mu^{\pm\alpha_L}, \qquad W_{L\mu}^3 \equiv \alpha_L \cdot C_\mu, \\ W_{R\mu}^{\pm} &\equiv |\alpha_R|\, W^{\pm\alpha_R}, \qquad W_{R\mu}^3 \equiv \alpha_R \cdot C_\mu, \end{aligned} \tag{3.4.3}$$

and $B_\mu^X \equiv \frac{\mathrm{x}\cdot C_\mu}{|\mathrm{x}|^2}$ is the $\mathrm{U(1)}_X$ gauge field that does not exist when $r = 2$. The ellipses denote components that do not have zero-modes. Since the generators in (3.4.1) are normalized as

$$\mathrm{tr}\left(T_L^+ T_L^-\right) = \mathrm{tr}\left((T_L^3)^2\right) = \frac{1}{|\alpha_L|^2}, \qquad \mathrm{tr}\left(T_R^+ T_R^-\right) = \mathrm{tr}\left((T_R^3)^2\right) = \frac{1}{|\alpha_R|^2}, \tag{3.4.4}$$

the canonically normalized zero-mode gauge fields are

$$\hat{W}_{L\mu}^{\pm,3} \equiv \frac{\sqrt{\mathcal{A}}}{g_A\, |\alpha_L|} W_{L\mu}^{\pm,3}, \qquad \hat{W}_{R\mu}^{\pm,3} \equiv \frac{\sqrt{\mathcal{A}}}{g_A\, |\alpha_R|} W_{R\mu}^{\pm,3}, \qquad \hat{B}_\mu^Z \equiv \frac{\sqrt{\mathcal{A}}}{g_Z} B_\mu^Z, \tag{3.4.5}$$

where $\mathcal{A}$ is the area of the fundamental domain of T^2/Z_N.

Since we have assumed that $\mathrm{SU(2)_R} \times \mathrm{U(1)}_Z$ is unbroken by the orbifold boundary conditions, we introduce some 4D scalar fields at one of the fixed points of T^2/Z_N in order to break it to $\mathrm{U(1)}_Y$. We demand that the custodial symmetry $\mathrm{SU(2)}_V \subset \mathrm{SU(2)_L} \times \mathrm{SU(2)_R}$ remains unbroken after the Higgs fields have VEVs. The generators of $\mathrm{SU(2)}_V$ are

$$\begin{aligned} T_V^{\pm} &\equiv T_L^{\pm} + T_R^{\pm} = \frac{E_{\pm\alpha_L}}{|\alpha_L|} + \frac{E_{\pm\alpha_R}}{|\alpha_R|}, \\ T_V^3 &\equiv T_L^3 + T_R^3 = \frac{\alpha_L \cdot H}{|\alpha_L|^2} + \frac{\alpha_R \cdot H}{|\alpha_R|^2}. \end{aligned} \tag{3.4.6}$$

Thus the conditions for $\mathrm{SU}(2)_V$ to be unbroken are

$$\begin{aligned}\left[T_V^{\pm},\langle A_z\rangle\right] &= \sum_\beta \langle W_z^\beta\rangle\left(\frac{N_{\pm\alpha_L,\beta}E_{\beta\pm\alpha_L}}{|\alpha_L|}+\frac{N_{\pm\alpha_R,\beta}E_{\beta\pm\alpha_R}}{|\alpha_R|}\right)=0,\\ \left[T_V^3,\langle A_z\rangle\right] &= \sum_\beta \langle W_z^\beta\rangle\left(\frac{\alpha_L\cdot\beta}{|\alpha_L|^2}+\frac{\alpha_R\cdot\beta}{|\alpha_R|^2}\right)E_\beta=0,\end{aligned} \tag{3.4.7}$$

since C_z^i do not have zero-modes and thus $\langle C_z^i\rangle=0$.

Rank two groups

Let us first consider the rank two groups. We introduce the following Lagrangian at $z=0$.[3]

$$\mathcal{L}_{\rm bd}=\left\{-\mathcal{D}_\mu\phi^\dagger\mathcal{D}^\mu\phi-V(\phi)\right\}\delta(z), \tag{3.4.8}$$

where ϕ is a complex scalar field belonging to $\mathbf{(1,2)_{+1/2}}$ under $\mathrm{SU(2)_L\times SU(2)_R}\times \mathrm{U}(1)_Z$, and $V(\phi)$ is a potential that force ϕ to have a nonvanishing VEV. After ϕ gets a VEV, $\mathrm{SU(2)_R}\times\mathrm{U}(1)_Z$ is broken to $\mathrm{U}(1)_Y$, and the corresponding zero-mode gauge field is expressed as

$$\hat{B}_\mu^Y\equiv\sin\theta_Z\hat{W}_{R\mu}^3+\cos\theta_Z\hat{B}_\mu^Z, \tag{3.4.9}$$

where the mixing angle θ_Z is determined by $\tan\theta_Z=g_Z/(g_A|\alpha_R|)$. The hypercharge operator Y is identified as

$$Y=T_R^3+Q_Z=\frac{\alpha_R\cdot H}{|\alpha_R|^2}+Q_Z. \tag{3.4.10}$$

After W_z^β have nonvanishing VEVs, $\mathrm{SU(2)_L}\times\mathrm{U}(1)_Y$ is broken to the electromagnetic symmetry $\mathrm{U(1)_{em}}$. Since W_z^β is $\mathrm{U}(1)_Z$ neutral and only $U(1)_{\rm em}$ neutral W_z^β can have nonvanishing VEVs, the root β must satisfy

$$\frac{\alpha_L\cdot\beta}{|\alpha_L|^2}+\frac{\alpha_R\cdot\beta}{|\alpha_R|^2}=0, \tag{3.4.11}$$

if $\langle W_z^\beta\rangle\neq0$. Thus the second condition in (3.4.7) is automatically satisfied. The roots that satisfy (3.4.11) are $\pm\boldsymbol{e}^2\in\mathbf{(2,2)}$ in SO(5), and $\pm\left(\frac{\boldsymbol{e}^1}{2}-\frac{\boldsymbol{e}^2}{2\sqrt{3}}\right)\in\mathbf{(2,4)}$ in G_2. Then, from the first condition in (3.4.7), we obtain a condition,

$$\left|\langle W_z^{e^2}\rangle\right|=\left|\langle W_z^{-e^2}\rangle\right|,\qquad \langle W_z^\beta\rangle=0,\ \ (\beta\neq\pm\boldsymbol{e}^2) \tag{3.4.12}$$

for SO(5), while no nonvanishing VEV is allowed for G_2.

[3]Of course, we can assume that $\mathcal{L}_{\rm bd}$ is localized at another fixed point.

Rank three groups

Next consider the rank three groups. Since the unbroken gauge symmetry is $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R} \times \mathrm{U}(1)_X \times \mathrm{U}(1)_Z$ by the orbifold conditions, let us first assume that ϕ in (3.4.8) also has a nonzero $\mathrm{U}(1)_X$ charge in order to obtain $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_Y$ at low energies. Then the $\mathrm{U}(1)_Y$ gauge field B^Y_μ becomes a linear combination of $W^3_{R\mu}$, B^X_μ and B^Z_μ, and the hypercharge is identified as

$$Y = T^3_R + Q_X + Q_Z = \frac{\alpha_R \cdot H}{|\alpha_R|^2} + \mathrm{x} \cdot H + Q_Z. \tag{3.4.13}$$

Thus the condition (3.4.11) now becomes

$$\frac{\alpha_L \cdot \beta}{|\alpha_L|^2} + \frac{\alpha_R \cdot \beta}{|\alpha_R|^2} + \mathrm{x} \cdot \beta = 0. \tag{3.4.14}$$

Then the second condition in (3.4.7) requires that both (3.4.11) and $\mathrm{x} \cdot \beta = 0$ must be satisfied if $\langle W^\beta_z \rangle \neq 0$. This meas that scalar components have nonzero VEVs only if the corresponding roots β satisfy both (3.4.11) and $Q_X = 0$. Such roots do not exist among those satisfying the zero-modes listed up in Sec. 3.3.2. Therefore we introduce two complex scalar fields ϕ_1 and ϕ_2 instead of ϕ on the fixed point,

$$\mathcal{L}_\mathrm{bd} = \Big\{ -\mathcal{D}_\mu \phi_1^\dagger \mathcal{D}^\mu \phi_1 - \mathcal{D}_\mu \phi_2^\dagger \mathcal{D}^\mu \phi_2 - V(\phi_1, \phi_2) \Big\} \delta(z), \tag{3.4.15}$$

where ϕ_1 and ϕ_2 are complex scalars belonging to $\mathbf{(1,2)_{0,+1/2}}$ and $\mathbf{(1,1)_{+1,0}}$ respectively under $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R} \times \mathrm{U}(1)_X \times \mathrm{U}(1)_Z$, and $V(\phi_1, \phi_2)$ is a potential for them. Since ϕ_1 is neutral for $\mathrm{U}(1)_X$, the $\mathrm{U}(1)_Y$ gauge field B^Y_μ is independent of B^X_μ. Hence the hypercharge is now identified as (3.4.10). The $\mathrm{U}(1)_X$ charges are no longer relevant to the $\mathrm{U}(1)_Y$ and $\mathrm{U}(1)_\mathrm{em}$ charges because $\mathrm{U}(1)_X$ is completely broken by a VEV of another scalar ϕ_2. Thus the $\mathrm{U}(1)_Y$ gauge field is given by (3.4.9). In this case, the $\mathrm{U}(1)_\mathrm{em}$ neutral condition becomes (3.4.11), which is consistent with the second condition in (3.4.7). As a result, possible nonvanishing VEVs are as follows.

$$\begin{aligned}
&\left|\langle W_z^{\pm(e^1-e^3)} \rangle\right| = \left|\langle W_z^{\pm(e^2-e^4)} \rangle\right| \in \mathbf{(2,2)_{\pm 2}}, \quad \text{in SU(4)} \\
&\left|\langle W_z^{\pm(e^2+e^3)} \rangle\right| = \left|\langle W_z^{\pm(-e^2+e^3)} \rangle\right| \in \mathbf{(2,2)_{\pm 1}}, \quad \left|\langle W_z^{\pm e^3} \rangle\right| \in \mathbf{(1,1)_{\pm 1}} \quad \text{in SO(7) (I)}, \\
&\left|\langle W_z^{\pm(e^1-e^3)} \rangle\right| = \left|\langle W_z^{\pm(-e^2+e^3)} \rangle\right| \in \mathbf{(2,2)_{\pm 1}} \quad \text{in Sp(6) (II), Sp(6) (III)},
\end{aligned} \tag{3.4.16}$$

where the double-signs correspond.

In summary, fields that can have nonzero VEVs are the neutral components of a bidoublet $\mathbf{(2,2)}$ or a singlet $\mathbf{(1,1)}$. The above conditions indicate that a bidoublet $\mathcal{H}_a$ must have a VEV:

$$\langle \mathcal{H}_a \rangle = \begin{pmatrix} v_a & \\ & v_a \end{pmatrix}, \tag{3.4.17}$$

where $v_a > 0$, if we define a phase of each field component appropriately.

3.4.2 Weinberg angle and W and Z boson masses

In the approximation that the W and Z bosons have constant profiles over the extra dimensions, the 4D $\mathrm{SU(2)_L}$ and $\mathrm{U(1)}_Y$ gauge coupling constants are read off from the couplings to the matter zero-modes, and are identified as

$$g = \frac{g_A |\alpha_L|}{\sqrt{\mathcal{A}}}, \qquad g' = \frac{g_A g_Z |\alpha_R|}{\sqrt{\mathcal{A}(g_A^2 |\alpha_R|^2 + g_Z^2)}}. \tag{3.4.18}$$

Thus the Weinberg angle is calculated as

$$\tan^2\theta_W \equiv \frac{g'^2}{g^2} = \frac{g_Z^2 |\alpha_R|^2}{|\alpha_L|^2 (g_A^2 |\alpha_R|^2 + g_Z^2)}. \tag{3.4.19}$$

We can obtain the experimental value $\tan^2\theta_W \simeq 0.30$ by tuning the ratio g_Z/g_A.

Next we derive the expressions of the W and Z boson masses. From (3.4.5) and (3.4.9), the expression (3.4.2) becomes

$$A_\mu = W_{L\mu}^+ T_L^+ + W_{L\mu}^- T_L^- + W_{L\mu}^3 T_L^3 + \sin\theta_Z B_\mu^Y T_R^3 + \cdots, \tag{3.4.20}$$

where $B_\mu^Y \equiv \frac{g_A|\alpha_R|}{\sqrt{\mathcal{A}}} \hat{B}_\mu^Y$, after the breaking $\mathrm{SU(2)_R} \times \mathrm{U(1)}_Z \to \mathrm{U(1)}_Y$. Then it follows that

$$\begin{aligned}[A_\mu, \langle A_z \rangle] = \sum_\beta W_z^\beta \Bigg\{ & W_{L,\mu}^+ \frac{N_{\alpha_L,\beta}}{|\alpha_L|} E_{\beta+\alpha_L} + W_{L\mu}^- \frac{N_{-\alpha_L,\beta}}{|\alpha_L|} E_{\beta-\alpha_L} \\ & + \left(W_{L\mu}^3 \frac{\alpha_L \cdot \beta}{|\alpha_L|^2} + B_\mu^Y \sin\theta_Z \frac{\alpha_R \cdot \beta}{|\alpha_R|^2} \right) E_\beta \Bigg\}. \end{aligned} \tag{3.4.21}$$

From the results in the previous subsections, the only components that contribute to the W and Z boson masses are the neutral components of bidoublets. Since the roots that form a bidoublet are expressed as

$$\begin{pmatrix} \gamma_a + \alpha_L & \xrightarrow{\alpha_R} & \gamma_a + \alpha_L + \alpha_R \\ \uparrow_{\alpha_L} & & \uparrow_{\alpha_L} \\ \gamma_a & \xrightarrow{\alpha_R} & \gamma_a + \alpha_R \end{pmatrix}, \tag{3.4.22}$$

where a labels bidoublets, (3.4.21) are rewritten as

$$[A_\mu, \langle A_z\rangle] = \sum_{\gamma_a}\left[\langle W_z^{\gamma_a+\alpha_L}\rangle\left\{\frac{e^{i\zeta}}{\sqrt{2}}W_{L\mu}^- E_{\gamma_a} + \left(\frac{1}{2}W_{L\mu}^3 - \frac{\sin\theta_Z}{2}B_\mu^Y\right)E_{\gamma_a+\alpha_L}\right\}\right.$$
$$\left. + \langle W_z^{\gamma_a+\alpha_R}\rangle\left\{\frac{e^{i\eta}}{\sqrt{2}}W_{L\mu}^+ E_{\gamma_a+\alpha_L+\alpha_R} - \left(\frac{1}{2}W_{L\mu}^3 - \frac{\sin\theta_Z}{2}B_\mu^Y\right)E_{\gamma_a+\alpha_R}\right\}\right], \tag{3.4.23}$$

where γ_a runs over the $T_L^3 = T_R^3 = -\frac{1}{2}$ components of the zero-mode bidoublets. We have used that $|N_{-\alpha_L,\gamma_a+\alpha_L}|^2 = |N_{\alpha_L,\gamma_a+\alpha_R}|^2 = \frac{|\alpha_L|^2}{2}$, and $\zeta \equiv \arg(N_{-\alpha_L,\gamma_a+\alpha_L})$ and $\eta \equiv \arg(N_{\alpha_L,\gamma_a+\alpha_R})$. Thus the relevant terms in 6D Lagrangian are calculated as

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4g_A^2}\mathrm{tr}(F^{(A)MN}F^{(A)}_{MN}) + \cdots = -\frac{1}{2g_A^2\pi^2R_1^2}\mathrm{tr}\left([A^\mu, \langle A_z\rangle]\,[A_\mu, \langle A_z\rangle]^\dagger\right) + \cdots \\
&= -\sum_a \frac{|\langle W_z^{\gamma_a+\alpha_L}\rangle|^2 + |\langle W_z^{\gamma_a+\alpha_R}\rangle|^2}{2g_A^2\pi^2R_1^2}\left\{\frac{1}{2}W_L^{+\mu}W_{L\mu}^- + \left(\frac{1}{2}W_{L\mu}^3 - \frac{\sin\theta_Z}{2}B_\mu\right)^2\right\} \\
&= -\frac{g^2\sum_a v_a^2}{\mathcal{A}}\left\{\hat{W}_L^{+\mu}\hat{W}_{L\mu}^- + \frac{1}{2}\left(\hat{W}_{L\mu}^3 - \frac{|\alpha_R|\sin\theta_Z}{|\alpha_L|}\hat{B}_\mu\right)^2\right\}.
\end{aligned} \tag{3.4.24}$$

At the last step, we have used that (3.4.5), and $|\langle W_z^{\gamma_a+\alpha_L}\rangle| = |\langle W_z^{\gamma_a+\alpha_R}\rangle| \equiv \frac{\sqrt{2}}{|\alpha_L|}g\pi R_1 v_a$ (g: 4D $\mathrm{SU(2)_L}$ gauge coupling), which follows from (3.4.12) or (3.4.16). We obtain the W and the Z boson mass terms by integrating (3.4.24) over the extra dimensions, and their masses are read off as

$$\begin{aligned}
m_W &= g\sum_a v_a, \\
m_Z &= \left(1 + \frac{|\alpha_R|^2\sin^2\theta_Z}{|\alpha_L|^2}\right)^{1/2} m_W = \left(1 + \frac{g_Z^2|\alpha_R|^2}{|\alpha_L|^2(g_A^2|\alpha_R|^2 + g_Z^2)}\right)^{1/2} m_W.
\end{aligned} \tag{3.4.25}$$

From these and (3.4.19), we find that $\rho \equiv m_W^2/(m_Z^2\cos^2\theta_W) = 1$. This is expected because we have assumed that only $\mathrm{SU(2)_L}$ doublets and singlets have nonzero VEVs and neglected the z-dependence of the mode functions for the W and the Z bosons. The custodial symmetry plays a crucial role when such z-dependence is taken into account.

3.5 Matter field

We consider a case that quarks and leptons live in the bulk. This case is interesting because the hierarchical structure of the Yukawa coupling constants can be realized by the wave function localization [36, 37], and the generation structure can also be obtained by a background magnetic flux [39]. In the following, we focus on the quark sector, but a similar argument is also applicable to the lepton sector.

3.5.1 Zero-mode conditions

A 6D Weyl fermion Ψ_{χ_6} with the 6D chirality $\chi_6 = \pm$ is decomposed as

$$\Psi_{\chi_6} = \sum_{\chi_4=\pm} \Psi_{\chi_6,\chi_4}, \tag{3.5.1}$$

where $\chi_4 = \pm$ is the 4D chirality. The orbifold boundary conditions for Ψ_{χ_6,χ_4} are given by [51]

$$\begin{aligned} \Psi_{\chi_6,\chi_4}(x, z+1) &= \Psi_{\chi_6,\chi_4}(x,z), \\ \Psi_{\chi_6,\chi_4}(x, z+\tau) &= \Psi_{\chi_6,\chi_4}(x,z), \\ \Psi_{\chi_6,\chi_4}(x, \omega z) &= \omega^{-\frac{\chi_4\chi_6}{2}} e^{i\varphi_\omega} P \Psi_{\chi_6,\chi_4}(x,z). \end{aligned} \tag{3.5.2}$$

A factor $\omega^{-\frac{\chi_4\chi_6}{2}}$ appears because a 6D spinor is charged under a rotation in the extra-dimensional space. The phase φ_ω satisfies (B.0.4).

As pointed out in Ref. [38], the generations and the hierarchy among the Yukawa couplings can be obtained by introducing an extra gauge symmetry G_F and assuming a magnetic flux on T^2/Z_N and the Wilson-line phases for it. The zero-modes are contained in Ψ_{χ_6,χ_4} as

$$\Psi_{\chi_6,\chi_4}(x,z) = \sum_{j=1}^{j_{\max}} \sum_{\mu} f_{\chi_6}^{(j)\mu}(z) |\mu\rangle \psi_{\chi_4}^{(j)\mu}(x) + \cdots , \tag{3.5.3}$$

where μ runs over the weights of the zero-mode states,[4] and the ellipsis denotes the nonzero KK modes. The number of the zero-modes $j_{\max}$ is determined by the magnetic flux [39]. The zero-mode functions $f_{\chi_6}^{(j)\mu}(z)$ are determined so that (3.5.3) satisfies the first two conditions in (3.5.2). From the last condition in (3.5.2), we obtain

$$\psi_{\chi_4}^{(j)\mu}(x) = \omega^{-\frac{\chi_4\chi_6}{2}} e^{i\varphi_\omega} P \psi_{\chi_4}^{(j)\mu}(x). \tag{3.5.4}$$

Namely, the zero-mode is an eigenvector of $\omega^{-\frac{\chi_4\chi_6}{2}} e^{i\varphi_\omega} P$ with an eigenvalue 1. Denote the highest weight of a representation $\mathcal{R}$ that Ψ_{χ_4,χ_6} belongs to as $\mu_{\max}$. Then μ is expressed as

$$\mu = \mu_{\max} - \sum_i k_i \alpha_i, \tag{3.5.5}$$

[4] Do not confuse it with the 4D Lorentz index.

where k_i are non-negative integers. Since $P^N|\mu\rangle = e^{iNp\cdot\mu}|\mu\rangle = e^{iNp\cdot\mu_{\rm max}}|\mu\rangle$,[5] the phase φ_ω is determined by (B.0.4) as $\varphi_\omega = \frac{\pi}{N}(2m_\omega+1) - p\cdot\mu_{\rm max}$, where $m_\omega = 0, 1, \cdots, N-1$. Thus we find that

$$\omega^{-\frac{\chi_4\chi_6}{2}} e^{i\varphi_\omega} P|\mu\rangle = e^{-\frac{2\pi i}{N}\cdot\frac{\chi_4\chi_6}{2}} \exp\left(\frac{(2m_\omega+1)\pi i}{N} - ip\cdot\mu_{\rm max}\right) e^{ip\cdot\mu}|\mu\rangle$$
$$= \exp\left(\frac{\pi i(2m_\omega+1-\chi_4\chi_6)}{N} - i\sum_i k_i(p\cdot\alpha_i)\right)|\mu\rangle. \qquad (3.5.6)$$

Namely, the zero-mode condition for the state $|\mu\rangle$ is

$$\frac{\pi(2m_\omega+1-\chi_4\chi_6)}{N} - \sum_i k_i(p\cdot\alpha_i) = 0. \quad (\mathrm{mod}\ 2\pi) \qquad (3.5.7)$$

3.5.2 $Z\bar{b}_L b_L$ coupling

When the quarks live in the bulk, the $Z\bar{b}_L b_L$ coupling can receive a large correction induced by the diagrams exchanging of the KK gauge and fermion modes. The authors of Ref. [29] pointed out that the custodial symmetry plays an important role to suppress the deviation of this coupling from the standard model value. The $Z\bar{b}_L b_L$ coupling is protected if the theory has a parity symmetry $\mathcal{P}_{\rm LR}$ that exchanges $\mathrm{SU}(2)_{\rm L}$ and $\mathrm{SU}(2)_{\rm R}$, and b_L is a component of $T_L^3 = T_R^3 = -\frac{1}{2}$ in a bidoublet $(\mathbf{2},\mathbf{2})$ for $\mathrm{SU}(2)_{\rm L}\times\mathrm{SU}(2)_{\rm R}$. Since the Higgs field also belongs to $(\mathbf{2},\mathbf{2})$, the right-handed quarks should belong to $(\mathbf{1},\mathbf{1})$ or $(\mathbf{1},\mathbf{3})+(\mathbf{3},\mathbf{1})$.

Cases in which the bosonic sector has the parity symmetry $\mathcal{P}_{\rm LR}$ and a scalar bidoublet are SO(5), SU(4) and SO(7) (I) in Sec. 3.3. In Appendix C, we list the irreducible representations of these groups whose dimensions are less than 30, and their decomposition into the $\mathrm{SU}(2)_{\rm L}\times\mathrm{SU}(2)_{\rm R}(\times\mathrm{U}(1)_X)$ multiplets. There is no $(\mathbf{1},\mathbf{3})+(\mathbf{3},\mathbf{1})$ multiplets included in the list. Hence the left-handed and the right-handed quarks should be embedded into $(\mathbf{2},\mathbf{2})$ and $(\mathbf{1},\mathbf{1})$, respectively.

3.5.3 Yukawa coupling constants

General expression

The Yukawa couplings originate from the 6D minimal couplings in the kinetic term, $i\bar{\Psi}_{\chi_6}\Gamma^M\mathcal{D}_M\Psi_{\chi_6} = -\frac{i\chi_6}{\pi R_1}\bar{\Psi}_{\chi_6,\chi_4=\chi_6}A_z\Psi_{\chi_6,\chi_4=-\chi_6} + \text{h.c.} + \cdots$. The canonically normalized Higgs zero-mode H^β is contained in A_z as $A_z = \sum_\beta \frac{\sqrt{2}}{|\alpha_L|} g\pi R_1 H^\beta E_\beta + \cdots$, where g

[5] We have used (3.2.11) at the second equality.

is the $\mathrm{SU}(2)_{\mathrm{L}}$ gauge coupling constant (see (3.6.5)). Then the Yukawa couplings in 4D effective Lagrangian are expressed as

$$\mathcal{L}_{\mathrm{yukawa}} = \begin{cases} \displaystyle\sum_{i,j} \left(\sum_{\beta,\mu_L} y_{(+)ij}^{\mathcal{R}_H\mathcal{R}_L\mathcal{R}_R} \left(H^\beta\right)^* \bar{\psi}_L^{(i)\mu_L} \psi_R^{(j)\mu_L+\beta} + \mathrm{h.c.} \right) & (\chi_6 = +) \\ \displaystyle\sum_{i,j} \left(\sum_{\beta,\mu_L} y_{(-)ij}^{\mathcal{R}_H\mathcal{R}_L\mathcal{R}_R} H^\beta \bar{\psi}_L^{(i)\mu_L} \psi_R^{(j)\mu_L-\beta} + \mathrm{h.c.} \right) & (\chi_6 = -) \end{cases}, \tag{3.5.8}$$

where $\mathcal{R}_H$, $\mathcal{R}_L$ and $\mathcal{R}_R$ are irreducible representations of $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}} (\times \mathrm{U}(1)_X) \times \mathrm{U}(1)_Z$ that $|\beta\rangle$, $|\mu_L\rangle$ and $|\mu_R\rangle = |\mu_L + \chi_6\beta\rangle$ belong to, and

$$\begin{aligned} y_{(+)ij}^{\mathcal{R}_H\mathcal{R}_L\mathcal{R}_R} &\equiv i\sqrt{2}g\langle\mu_L|E_{-\beta}|\mu_L+\beta\rangle \int d^2z\; f_{+,0}^{(i)\mu_L*}(z) f_{+,0}^{(j)\mu_L+\beta}(z), \\ y_{(-)ij}^{\mathcal{R}_H\mathcal{R}_L\mathcal{R}_R} &\equiv i\sqrt{2}g\langle\mu_L|E_{\beta}|\mu_L-\beta\rangle \int d^2z\; f_{-,0}^{(i)\mu_L*}(z) f_{-,0}^{(j)\mu_L-\beta}(z). \end{aligned} \tag{3.5.9}$$

Note that these coupling constants only depend on the representations $\{\mathcal{R}_H, \mathcal{R}_L, \mathcal{R}_R\}$, and take common values for $\beta \in \mathcal{R}_H$ and $\mu_L \in \mathcal{R}_L$.

Embedding of quarks

Exponentially small Yukawa couplings can be obtained by using the wave function localization in the extra dimensions [36, 37].[6] For the third generation, we assume that the overlap integrals in (3.5.9) do not provide any suppression factors, i.e., equal one. Then the Yukawa couplings is determined only by the group-theoretical factors. In the following, we focus on the third generation quarks.

Consider a 6D Dirac fermions $\Psi = \Psi_+ + \Psi_-$ that belongs to the representation $\mathcal{R}$. The theory is assumed to be symmetric under an exchange: $\Psi_+ \leftrightarrow -\Psi_-$ so that a 6D mass term $M_\Psi(\bar{\Psi}_+\Psi_- + \bar{\Psi}_-\Psi_+)$ is prohibited. We also assume that $\Psi_{\chi_6,-}$ and $\Psi_{\chi_6,+}$ have zero-modes $\mathcal{Q}_L^{(\chi_6)} \in \mathbf{(2,2)}$ and $\lambda_R^{(\chi_6)} \in \mathbf{(1,1)}$. The Higgs fields H^β that couple to them form bidoublets $\mathcal{H}_a$. Then, from (3.5.8), the Yukawa couplings from $i\sum_{\chi_6=\pm} \bar{\Psi}_{\chi_6}\Gamma^M\mathcal{D}_M\Psi_{\chi_6}$ before the breaking of $\mathrm{SU}(2)_{\mathrm{R}} \times \mathrm{U}(1)_Z$ at the fixed point are expressed as

$$\mathcal{L}_{\mathrm{yukawa}} = \sum_a \left\{ y_a^{(+)} \mathrm{tr}\left(\bar{\mathcal{Q}}_L^{(+)}\tilde{\mathcal{H}}_a\right)\lambda_R^{(+)} + y_a^{(-)} \mathrm{tr}\left(\bar{\mathcal{Q}}_L^{(-)}\mathcal{H}_a\right)\lambda_R^{(-)} + \mathrm{h.c.} \right\}, \tag{3.5.10}$$

[6] This can be realized by the Wilson-line phases for the extra flavor gauge symmetry G_F in the presence of a magnetic flux [38].

where $\tilde{\mathcal{H}}_a \equiv \sigma_2 \mathcal{H}_a^* \sigma_2$ and

$$y_a^{(+)} = i\sqrt{2}g\langle \mu_L | E_{-\beta} | \mu_L + \beta \rangle = i\sqrt{2}g N^*_{\beta,\mu_L},$$
$$y_a^{(-)} = i\sqrt{2}g\langle \nu_L | E_{\beta} | \nu_L - \beta \rangle = i\sqrt{2}g N^*_{-\beta,\nu_L}. \tag{3.5.11}$$

Here $|\mu_L\rangle, |\nu_L\rangle \in (\mathbf{2},\mathbf{2})$, $|\mu_L+\beta\rangle, |\nu_L-\beta\rangle \in (\mathbf{1},\mathbf{1})$, and a complex constant $N_{\beta,\mu}$ is defined below (A.0.3). Note that $\mathcal{Q}_L^{(+)}$ and $\mathcal{Q}_L^{(-)}$ ($\lambda_R^{(+)}$ and $\lambda_R^{(-)}$) belong to different $(\mathbf{2},\mathbf{2})$ ($(\mathbf{1},\mathbf{1})$) in $\mathcal{R}$ because the same $(\mathbf{2},\mathbf{2})$ ($(\mathbf{1},\mathbf{1})$) cannot satisfy (3.5.7) for $\chi_6 = \pm$ simultaneously. We denote them as $\mathcal{Q}_L^{(\chi_6)} \in (\mathbf{2},\mathbf{2})_{\boldsymbol{\chi_6}}$ and $\lambda_R^{(\chi_6)} \in (\mathbf{1},\mathbf{1})_{\boldsymbol{\chi_6}}$. The Yukawa couplings depend on how the quark fields are contained in $\mathcal{Q}_L^{(\pm)}$ and $\lambda_R^{(\pm)}$.

As we will see in Sec. 3.6, the Higgs potential at tree level only contains quartic terms. The electroweak symmetry breaking occurs at one-loop level, and the top Yukawa coupling provides a dominant contribution to the one-loop Higgs potential. In general, such one-loop potential breaks $\mathrm{SU(2)_L} \times \mathrm{SU(2)_R}$, and thus the Higgs VEVs are not aligned as (3.4.17). Namely the custodial symmetry is broken. A simple way to avoid this difficulty is to assume that the quark fields couple to the Higgs fields only through a combination $\mathcal{H}_a + \tilde{\mathcal{H}}_a$. In order to achieve this, the quark fields must be equally contained in both Ψ_+ and Ψ_-. Specifically we introduce 4D fermions $\zeta_R \in (\mathbf{2},\mathbf{2})$ and $\eta_L \in (\mathbf{1},\mathbf{1})$ localized at a fixed point, which transform as $\zeta_R \to -\zeta_R$ and $\eta_L \to -\eta_L$ under $\Psi_\pm \to -\Psi_\mp$. Then combinations $\mathcal{Q}'_L \equiv (-\mathcal{Q}_L^{(+)} + \mathcal{Q}_L^{(-)})/\sqrt{2}$ and $\lambda'_R \equiv (-\lambda_R^{(+)} + \lambda_R^{(-)})/\sqrt{2}$ have masses with them at the fixed point and are decoupled at low energies. Then we obtain the desired form of the Yukawa couplings,[7]

$$\mathcal{L}_{\text{yukawa}} = \frac{y_\lambda}{2} \sum_a \operatorname{tr}\left\{ \bar{\mathcal{Q}}_L \left(\mathcal{H}_a + \tilde{\mathcal{H}}_a \right) \right\} \lambda_R + \text{h.c.} + \cdots, \tag{3.5.12}$$

where $y_\lambda \equiv y_a^{(+)} = y_a^{(-)}$, $\mathcal{Q}_L \equiv (\mathcal{Q}_L^{(+)} + \mathcal{Q}_L^{(-)})/\sqrt{2}$ and $\lambda_R \equiv (\lambda_R^{(+)} + \lambda_R^{(-)})/\sqrt{2}$.

Now we will see how the quark fields should be embedded into 6D fields. For simplicity, we consider a case that there is one Higgs bidoublet $\mathcal{H}$ as a zero-mode for a while. We introduce two 6D Dirac fermions $\Psi^{(2/3)} = \Psi_+^{(2/3)} + \Psi_-^{(2/3)}$ and $\Psi^{(-1/3)} = \Psi_+^{(-1/3)} + \Psi_-^{(-1/3)}$, whose $\mathrm{U}(1)_Z$ charges are $2/3$ and $-1/3$, respectively. Let us assume that $\Psi^{(q_Z)}$ ($q_Z = 2/3, -1/3$) contain $\mathcal{Q}_L^{(q_Z)} \in (\mathbf{2},\mathbf{2})$ and $\lambda_R^{(q_Z)} \in (\mathbf{1},\mathbf{1})$ as zero-modes. The bidoublets are decomposed as

$$\mathcal{Q}_L^{(2/3)} = (Q_L^{(1)}, Q_L^{(2)}), \qquad \mathcal{Q}_L^{(-1/3)} = (Q_L^{(3)}, Q_L^{(4)}), \qquad \mathcal{H} = (\tilde{H}_2, H_1), \tag{3.5.13}$$

[7]Notice that $\mathcal{Q}_R^{(\mp)}$ and $\lambda_L^{(\mp)}$ also satisfy the zero-mode condition (3.5.7) when $\mathcal{Q}_L^{(\pm)}$ and $\lambda_R^{(\pm)}$ are zero-modes. So we need additional 4D localized fermions to decouple them.

where $\tilde{H}_2^i \equiv \epsilon_{ij} H_1^{j*}$, and $\{Q_L^{(1)}, Q_L^{(3)}, \tilde{H}_2\}$ and $\{Q_L^{(2)}, Q_L^{(4)}, H_1\}$ are $\mathrm{SU(2)_L}$ doublets whose T_R^3 eigenvalues are $-1/2$ and $1/2$, respectively. Then the Yukawa couplings in the form of (3.5.12) are expressed as

$$\begin{aligned}
\mathcal{L}_{\mathrm{yukawa}} &= \frac{y_t}{2} \mathrm{tr} \left\{ \mathcal{Q}_L^{(2/3)\dagger} \left(\mathcal{H} + \tilde{\mathcal{H}} \right) \right\} t_R + \frac{y_b}{2} \mathrm{tr} \left\{ \mathcal{Q}_L^{(-1/3)\dagger} \left(\mathcal{H} + \tilde{\mathcal{H}} \right) \right\} b_R + \mathrm{h.c.} \\
&= \frac{y_t}{2} \left\{ Q_L^{(1)\dagger} \left(\tilde{H}_2 + \tilde{H}_1 \right) + Q_L^{(2)\dagger} \left(\tilde{H}_2 + \tilde{H}_1 \right) \right\} t_R \\
&+ \frac{y_b}{2} \left\{ Q_L^{(3)\dagger} \left(H_1 + H_2 \right) + Q_L^{(4)\dagger} \left(H_1 + H_2 \right) \right\} b_R + \mathrm{h.c.},
\end{aligned} \tag{3.5.14}$$

where y_t and y_b are calculated from (3.5.11). Only a combination $H_1 + H_2$ couples to the quarks. Thus this combination obtains a tachyonic mass while the other combination $H_1 - H_2$ does not at one-loop level. Therefore the latter does not have a nonzero VEV, and $\langle H_1 \rangle = \langle H_2 \rangle$ is realized. Namely, the alignment (3.4.17) is achieved. Since $Q_L^{(1)}$ and $Q_L^{(4)}$ have the same quantum numbers for $\mathrm{SU(2)_L} \times \mathrm{U(1)}_Y$, they are mixed with each other after the breaking $\mathrm{SU(2)_R} \times \mathrm{U(1)}_Z \to \mathrm{U(1)}_Y$ occurs at the fixed point. The left-handed quark is identified as a linear combination,

$$q_L = \cos\theta_q Q_L^{(1)} + \sin\theta_q Q_L^{(4)}, \tag{3.5.15}$$

where θ_q is a mixing angle. The orthogonal combination and $Q_L^{(2)}$ and $Q_L^{(3)}$ are exotic fields that must be decoupled at low energies. Hence we need to introduce 4D localized fermions that couple with those exotic components. As a result, the following Yukawa couplings are obtained at low energies.

$$\mathcal{L}_{\mathrm{yukawa}}^{SU(2)_L \times U(1)_Y} = \frac{y_t}{2} \cos\theta_q q_L^\dagger \left(\tilde{H}_2 + \tilde{H}_1 \right) t_R + \frac{y_b}{2} \sin\theta_q q_L^\dagger \left(H_1 + H_2 \right) b_R + \mathrm{h.c.}. \tag{3.5.16}$$

We can extend this to the two-Higgs-bidoublet case straightforwardly. When $y_t = y_b$, the large ratio of the top quark mass m_t to the bottom quark mass m_b is obtained if $\theta_q = \mathcal{O}(m_b/m_t)$.[8] In such a case, m_t is calculated as

$$m_t \simeq \left| y_t \sum_a v_a \right| = \sqrt{2} g \left| N_{\beta,\mu_L} \right| \sum_a v_a = \sqrt{2} \left| N_{\beta,\mu_L} \right| m_W, \tag{3.5.17}$$

where v_a is defined in (3.4.17). We have used that $\cos\theta_q \simeq 1$, (3.4.25) and (3.5.11). Therefore, the observed top quark mass is obtained if $|N_{\beta,\mu_L}| = \sqrt{2}$.[9]

[8] In contrast to the mixing between $\mathcal{Q}_L^{(+)}$ and $\mathcal{Q}_L^{(-)}$, the mixing angle θ_q can take arbitrary values because there is no symmetry to fix it.

[9] A small deviation from the observed value of m_t is expected to be explained by quantum correction. (See Ref. [45].)

Available representations for matter fermions

In summary, the quark multiplets should be embedded into two 6D Dirac fermions $\Psi^{(2/3)}$ and $\Psi^{(-1/3)}$ whose $\mathrm{U}(1)_Z$ charges are $2/3$ and $-1/3$, respectively. Irreducible representations $\mathcal{R}$ which they belong to must satisfy the following conditions.

1. $\mathcal{R}$ includes two bidoublets and two singlets, which are denoted as $\mathbf{(2,2)}_\pm$ and $\mathbf{(1,1)}_\pm$, respectively.

2. There are weights that satisfy $|\mu_L\rangle \in \mathbf{(2,2)}_+$, $|\mu_L+\beta\rangle \in \mathbf{(1,1)}_+$, $|\nu_L\rangle \in \mathbf{(2,2)}_-$ and $|\nu_L-\beta\rangle \in \mathbf{(1,1)}_-$, and $|N_{\beta,\mu_L}| = |N_{-\beta,\nu_L}| = \sqrt{2}$.

3. All the states in $\mathbf{(2,2)}_\pm$ and $\mathbf{(1,1)}_\pm$ satisfy the zero-mode condition (3.5.7).

We will search for $\mathcal{R}$ that satisfies these conditions from the list in Appendix C.

SO(5)

There is no irreducible representation that satisfies the condition 1 among the list in Appendix C.1.

SU(4)

Only $\mathbf{20'}$ satisfies the condition 1 among the list in Appendix C.2. The weights of $\mathbf{20'}$ that form $\mathbf{(2,2)}$ and $\mathbf{(1,1)}$ are

$$\mathbf{(2,2)_{\pm 2}} : \begin{pmatrix} \frac{\boldsymbol{e}^1-\boldsymbol{e}^2}{\sqrt{2}} \pm \boldsymbol{e}^3 & \xrightarrow{\alpha_R} & \frac{\boldsymbol{e}^1+\boldsymbol{e}^2}{\sqrt{2}} \pm \boldsymbol{e}^3 \\ \uparrow\alpha_L & & \uparrow\alpha_L \\ \frac{-\boldsymbol{e}^1-\boldsymbol{e}^2}{\sqrt{2}} \pm \boldsymbol{e}^3 & \xrightarrow{\alpha_R} & \frac{-\boldsymbol{e}^1+\boldsymbol{e}^2}{\sqrt{2}} \pm \boldsymbol{e}^3 \end{pmatrix},$$
$$\mathbf{(1,1)_{\pm 4}} : \pm 2\boldsymbol{e}^3, \qquad \mathbf{(1,1)_0} \;:\; \mathbf{0}. \tag{3.5.18}$$

where the double-signs correspond. Notice that the weights that form bidoublets are the same as the roots that form the Higgs bidoublets.

When the Higgs bidoublet $\mathbf{(2,2)_{\pm 2}}$ appears as a zero-mode, one example of (β,μ_L,ν_L) is chosen as

$$(\beta,\mu_L,\nu_L) = \left(\frac{\boldsymbol{e}^1-\boldsymbol{e}^2}{\sqrt{2}} \pm \boldsymbol{e}^3, \frac{-\boldsymbol{e}^1+\boldsymbol{e}^2}{\sqrt{2}} \pm \boldsymbol{e}^3, \frac{\boldsymbol{e}^1-\boldsymbol{e}^2}{\sqrt{2}} \mp \boldsymbol{e}^3\right), \tag{3.5.19}$$

where the double-signs correspond. Then $\{\mu_L-\beta, \mu_L, \mu_L+\beta\}$ and $\{\nu_L+\beta, \nu_L, \nu_L+\beta\}$ are the weights, but $\mu_L \pm 2\beta$ and $\nu_L \pm 2\beta$ are not. Therefore, from (A.0.3), the condition 2 is satisfied.

Since $(p_1, p_2, p_3) = (0, 0, 2n_P\pi/3)$ and the simple roots are $\alpha_1 = \sqrt{2}\boldsymbol{e}^1$, $\alpha_2 = -\frac{\boldsymbol{e}^1}{\sqrt{2}} - \frac{\boldsymbol{e}^2}{\sqrt{2}} + \boldsymbol{e}^3$ and $\alpha_3 = \sqrt{2}\boldsymbol{e}^2$, the zero-mode condition (3.5.7) becomes

$$\frac{\pi(2m_\omega + 1 - \chi_4\chi_6)}{N} - \frac{2n_P k_2 \pi}{N} = 0, \quad (\text{mod } 2\pi) \tag{3.5.20}$$

where $m_\omega = 0, 1, 2$. The decomposition of $\mathbf{20'}$ is given by (C.2.6), and

$$\begin{aligned}
k_2 &= 0 : \mathbf{(1,1)_{+4}}, \\
k_2 &= 1 : \mathbf{(2,2)_{+2}}, \\
k_2 &= 2 : \mathbf{(3,3)_0}, \quad \mathbf{(1,1)_0}, \\
k_2 &= 3 : \mathbf{(2,2)_{-2}}, \\
k_2 &= 4 : \mathbf{(1,1)_{-4}}.
\end{aligned} \tag{3.5.21}$$

Thus the condition 3 is satisfied only when the model is compactified on T^2/Z_3. In fact, when $(N, n_P, m_\omega) = (3, 1, 0)$, the fermionic zero-modes from each 6D Dirac fermion are

$$\begin{aligned}
&\mathcal{Q}_L^{(+)} \in \mathbf{(2,2)_{+2}}, \quad \mathcal{Q}_R^{(+)} \in \mathbf{(2,2)_{-2}}, \quad \lambda_L^{(+)} \in \mathbf{(1,1)_{-4}}, \quad \lambda_R^{(+)} \in \mathbf{(1,1)_{+4}}, \\
&\mathcal{Q}_L^{(-)} \in \mathbf{(2,2)_{-2}}, \quad \mathcal{Q}_R^{(-)} \in \mathbf{(2,2)_{+2}}, \quad \lambda_L^{(-)} \in \mathbf{(1,1)_{+4}}, \quad \lambda_R^{(-)} \in \mathbf{(1,1)_{-4}},
\end{aligned} \tag{3.5.22}$$

and when $(N, n_P, m_\omega) = (3, 2, 2)$, they are

$$\begin{aligned}
&\mathcal{Q}_L^{(+)} \in \mathbf{(2,2)_{-2}}, \quad \mathcal{Q}_R^{(+)} \in \mathbf{(2,2)_{+2}}, \quad \lambda_L^{(+)} \in \mathbf{(1,1)_{+4}}, \quad \lambda_R^{(+)} \in \mathbf{(1,1)_{-4}}, \\
&\mathcal{Q}_L^{(-)} \in \mathbf{(2,2)_{+2}}, \quad \mathcal{Q}_R^{(-)} \in \mathbf{(2,2)_{-2}}, \quad \lambda_L^{(-)} \in \mathbf{(1,1)_{-4}}, \quad \lambda_R^{(-)} \in \mathbf{(1,1)_{+4}}.
\end{aligned} \tag{3.5.23}$$

By introducing 4D localized fermions with appropriate quantum numbers to decouple unwanted zero-modes, the desired Yukawa coupling (3.5.12) are obtained. For the other choices of (N, n_P, m_ω), we cannot obtain the necessary multiplets.

SO(7) (I)

The irreducible representations that satisfy the condition 1 among the list in Appendix C.3 are **21** and **27**. These also satisfy the condition 2. However, they cannot satisfy the condition 3 regardless of a choice of (N, n_P, m_ω).

3.6 Higgs potential

In contrast to the 5D GHU models, we have quartic couplings of the Higgs fields at tree level. The relevant terms in the 6D Lagrangian are

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4g_A^2}\mathrm{tr}\left(F^{(A)MN}F^{(A)}_{MN}\right) + \cdots \\ &= -\frac{1}{2g_A^2(\pi R_1)^2}\mathrm{tr}\left((\partial^\mu A_z)^\dagger \partial_\mu A_z\right) - \frac{1}{8g_A^2(\pi R_1)^4}\mathrm{tr}\left([A_z, A_{\bar{z}}]^2\right) + \cdots . \end{aligned} \tag{3.6.1}$$

In this section, we calculate the classical Higgs potential V_{tree} focusing on the Higgs bidoublets, which are relevant to the electroweak symmetry breaking. In the previous section, we have shown that only a model of G =SU(4) compactified on T^2/Z_3 has required zero-mode spectrum for the quarks. For the sake of completeness, however, we will also calculate V_{tree} in the other cases that have Higgs bidoublets. We have one Higgs bidoublet in the cases of SO(5) on T^2/Z_2, SU(4) on T^2/Z_N ($N = 3, 4, 6$), SO(7) (I) on T^2/Z_N ($N = 4, 6$), and Sp(6) (II) or (III) on T^2/Z_3, and we have two Higgs bidoublets in the case of SU(4) on T^2/Z_2.

3.6.1 SO(5) case

First we consider the SO(5) case. In this case, the roots that form the bidoublet are

$$\begin{pmatrix} \boldsymbol{e}^2 & \xrightarrow{\alpha_R} & \boldsymbol{e}^1 \\ \uparrow_{\alpha_L} & & \uparrow_{\alpha_L} \\ -\boldsymbol{e}^1 & \xrightarrow{\alpha_R} & -\boldsymbol{e}^2 \end{pmatrix}. \tag{3.6.2}$$

From (3.6.1), the kinetic terms of the zero-modes W_z^β in the 4D effective Lagrangian are

$$\mathcal{L}_{\mathrm{eff}} = -\frac{\mathcal{A}}{2(g_A\pi R_1)^2}\sum_\beta (\partial^\mu W_z^\beta)^* \partial_\mu W_z^\beta + \cdots . \tag{3.6.3}$$

We have used (A.0.4), and $\mathcal{A}$ is the area of T^2/Z_N. Thus the canonically normalized Higgs bidoublet is defined as

$$\mathcal{H} = \begin{pmatrix} H_2^{2*} & H_1^1 \\ -H_2^{1*} & H_1^2 \end{pmatrix} \equiv \frac{\sqrt{\mathcal{A}}}{\sqrt{2}g_A\pi R_1}\begin{pmatrix} W_z^{e^2} & W_z^{e^1} \\ -W_z^{-e^1} & W_z^{-e^2} \end{pmatrix}. \tag{3.6.4}$$

Then it follows that

$$\begin{aligned} A_z &= \frac{\sqrt{2}g_A\pi R_1}{\sqrt{\mathcal{A}}}\left(H_1^1 E_{e^1} + H_2^{2*}E_{e^2} + H_1^2 E_{-e^2} + H_2^{1*}E_{-e^1}\right), \\ [A_z, A_{\bar{z}}] &= \frac{2(g_A\pi R_1)^2}{\mathcal{A}}\Big[\left(\left|H_1^1\right|^2 - \left|H_2^1\right|^2\right)H_1 + \left(\left|H_2^2\right|^2 - \left|H_1^2\right|^2\right)H_2 \\ &\qquad + \left\{N_{e^1,e^2}\left(H_1^1 H_1^{2*} - H_2^1 H_2^{2*}\right)E_{\alpha_L}\right. \\ &\qquad\quad \left.+N_{e^1,-e^2}\left(H_1^1 H_2^2 - H_1^2 H_2^1\right)E_{\alpha_R} + \mathrm{h.c.}\right\}\Big], \end{aligned} \tag{3.6.5}$$

where we have used (A.0.2). Hence, from (3.6.1), $V_{\rm tree}$ is calculated as

$$\begin{aligned}
V_{\rm tree} &= \frac{\mathcal{A}}{8g_A^2(\pi R_1)^4}\mathrm{tr}\left([A_z, A_{\bar{z}}]^2\right) \\
&= \frac{g_A^2}{2\mathcal{A}}\left[\left(\left|H_1^1\right|^2 - \left|H_2^1\right|^2\right)^2 + \left(\left|H_2^2\right|^2 - \left|H_1^2\right|^2\right)^2\right. \\
&\qquad\qquad \left. +2\left|N_{e^1,e^2}\right|^2\left|H_1^1 H_1^{2*} - H_2^1 H_2^{2*}\right|^2 + 2\left|N_{e^1,-e^2}\right|^2\left|H_1^1 H_2^2 - H_1^2 H_2^1\right|^2\right] \\
&= \frac{g^2}{4}\left\{\left(H_2^\dagger H_2 - H_1^\dagger H_1\right)^2 + 4\left|\tilde{H}_2^\dagger H_1\right|^2\right\} \\
&= \frac{g^2}{4}\left[\left\{\mathrm{tr}\left(\mathcal{H}^\dagger\mathcal{H}\right)\right\}^2 - 4\det\left(\mathcal{H}^\dagger\mathcal{H}\right)\right], \qquad (3.6.6)
\end{aligned}$$

where $H_2 \equiv (H_2^1, H_2^2)^t$ and $H_1 \equiv (H_1^1, H_1^2)^t$ are the $\mathrm{SU(2)_L}$ doublets with the hypercharge $Y = \frac{1}{2}$, and $\tilde{H}_2^i \equiv \epsilon_{ij}H_2^{j*}$. We have used that (3.4.18) with $|\alpha_L|^2 = 2$, and $|N_{e^1,e^2}|^2 = |N_{e^1,-e^2}|^2 = 1$. The above result agrees with Eq.(7) in Ref. [17]. The final expression in (3.6.6) is manifestly invariant under the transformation: $\mathcal{H} \to U_L\mathcal{H}U_R^\dagger$ ($U_L \in \mathrm{SU(2)_L}$ and $U_R \in \mathrm{SU(2)_R}$).

3.6.2 Cases of rank three groups

Next we consider the cases of the rank three groups. In these cases, the candidates for the Higgs bidoublets consist of the following roots.

$$\begin{pmatrix} \gamma+\alpha_L & \xrightarrow{\alpha_R} & \gamma+\alpha_L+\alpha_R \\ \uparrow_{\alpha_L} & & \uparrow_{\alpha_L} \\ \gamma & \xrightarrow{\alpha_R} & \gamma+\alpha_R \end{pmatrix}, \quad \begin{pmatrix} -\gamma-\alpha_R & \xrightarrow{\alpha_R} & -\gamma \\ \uparrow_{\alpha_L} & & \uparrow_{\alpha_L} \\ -\gamma-\alpha_L-\alpha_R & \xrightarrow{\alpha_R} & -\gamma-\alpha_L \end{pmatrix}, \qquad (3.6.7)$$

where $\gamma = -\frac{\boldsymbol{e}^1}{\sqrt{2}} - \frac{\boldsymbol{e}^2}{\sqrt{2}} + \boldsymbol{e}^3$ for SU(4), $\gamma = -\boldsymbol{e}^1 + \boldsymbol{e}^3$ for SO(7) (I) and $\gamma = -\boldsymbol{e}^2 - \boldsymbol{e}^3$ for Sp(6) (II) or (III). The canonically normalized Higgs bidoublets are defined as

$$\begin{aligned}
\mathcal{H}_+ &= \begin{pmatrix} H_{2+}^{2*} & H_{1+}^1 \\ -H_{2+}^{1*} & H_{1+}^2 \end{pmatrix} \equiv \frac{\sqrt{\mathcal{A}}}{\sqrt{2}g_A\pi R_1}\begin{pmatrix} W_z^{\gamma+\alpha_L} & W_z^{\gamma+\alpha_L+\alpha_R} \\ -W_z^{\gamma} & W_z^{\gamma+\alpha_R} \end{pmatrix}, \\
\mathcal{H}_- &= \begin{pmatrix} H_{2-}^{2*} & H_{1-}^1 \\ -H_{2-}^{1*} & H_{1-}^2 \end{pmatrix} \equiv \frac{\sqrt{\mathcal{A}}}{\sqrt{2}g_A\pi R_1}\begin{pmatrix} W_z^{-\gamma-\alpha_R} & W_z^{-\gamma} \\ -W_z^{-\gamma-\alpha_L-\alpha_R} & W_z^{-\gamma-\alpha_L} \end{pmatrix}, \qquad (3.6.8)
\end{aligned}$$

where the signs in the suffixes denote the signs of the $U(1)_X$ charges. Then it follows that

$$\begin{aligned}
A_z &= \frac{\sqrt{2}g_A\pi R_1}{\sqrt{\mathcal{A}}}\left(H^1_{1+}E_{\gamma_{LR}} + H^{2*}_{2+}E_{\gamma_L} + H^2_{1+}E_{\gamma_R} + H^{1*}_{2+}E_{\gamma}\right.\\
&\qquad \left. + H^1_{1-}E_{-\gamma} + H^{2*}_{2-}E_{-\gamma_R} + H^2_{1-}E_{-\gamma_L} + H^{1*}_{2-}E_{-\gamma_{LR}}\right) + \cdots ,\\
[A_z, A_{\bar{z}}] &= \frac{2(g_A\pi R_1)^2}{\mathcal{A}}\left[\left(\left|H^1_{2+}\right|^2 - \left|H^1_{1-}\right|\right)\gamma\cdot H + \left(\left|H^2_{2+}\right|^2 - \left|H^2_{1-}\right|^2\right)\gamma_L\cdot H\right.\\
&\qquad + \left(\left|H^2_{1+}\right|^2 - \left|H^2_{2-}\right|^2\right)\gamma_R\cdot H + \left(\left|H^1_{1+}\right|^2 - \left|H^1_{2-}\right|^2\right)\gamma_{LR}\cdot H\\
&\qquad + \left\{N_{\gamma_{LR},-\gamma_R}\left(-H^1_{1+}H^{2*}_{1+} + H^1_{2-}H^{2*}_{2-}\right)E_{\alpha_L}\right.\\
&\qquad\quad + N_{\gamma_{LR},-\gamma_L}\left(-H^1_{1+}H^2_{2+} + H^2_{1-}H^1_{2-}\right)E_{\alpha_R}\\
&\qquad\quad + N_{\gamma_L,-\gamma}\left(-H^1_{2+}H^{2*}_{2+} + H^1_{1-}H^{2*}_{1-}\right)E_{\alpha_L}\\
&\qquad\quad \left.\left. + N_{\gamma_R,-\gamma}\left(-H^2_{1+}H^1_{2+} + H^1_{1-}H^2_{2-}\right)E_{\alpha_R} + \text{h.c.}\right\}\right] + \cdots , \qquad (3.6.9)
\end{aligned}$$

where $\gamma_L \equiv \gamma + \alpha_L$, $\gamma_R \equiv \gamma + \alpha_R$ and $\gamma_{LR} \equiv \gamma + \alpha_L + \alpha_R$, and the ellipses denote fields belonging to other multiplets, if any. After some calculations, we obtain

$$\begin{aligned}
V_{\text{tree}} &= \frac{g^2}{2}\left[\left(|H_{1+}|^2 - |H_{2-}|^2\right)^2 + \left(|H_{2+}|^2 - |H_{1-}|^2\right)^2\right.\\
&\qquad + \left|H^\dagger_{1+}\tilde{H}_{2+}\right|^2 + \left|H^\dagger_{1+}\tilde{H}_{2-}\right|^2 + \left|H^\dagger_{1-}\tilde{H}_{2+}\right|^2 + \left|H^\dagger_{1-}\tilde{H}_{2-}\right|^2\\
&\qquad \left. - \left|\tilde{H}^t_{2+}H_{2-}\right|^2 - \left|\tilde{H}^t_{1+}H_{1-}\right|^2 + \left|\tilde{H}^\dagger_{2+}H_{1+} + \tilde{H}^\dagger_{2-}H_{1-}\right|^2\right]\\
&= \frac{g^2}{2}\left[\left\{\text{tr}\left(\mathcal{H}^\dagger_+\mathcal{H}_+\right)\right\}^2 + \left\{\text{tr}\left(\mathcal{H}^\dagger_-\mathcal{H}_-\right)\right\}^2 - \text{tr}\left(\tilde{\mathcal{H}}^\dagger_+\tilde{\mathcal{H}}_+\mathcal{H}^\dagger_-\mathcal{H}_-\right)\right.\\
&\qquad \left. - \text{tr}\left(\mathcal{H}^\dagger_-\tilde{\mathcal{H}}_+\tilde{\mathcal{H}}^\dagger_+\mathcal{H}_-\right) - 2\det\left(\mathcal{H}^\dagger_+\mathcal{H}_+\right) - 2\det\left(\mathcal{H}^\dagger_-\mathcal{H}_-\right)\right] + \cdots , \qquad (3.6.10)
\end{aligned}$$

where $\tilde{H}^i_{1,2+} \equiv \epsilon_{ij}H^{j*}_{1,2+}$, and $\tilde{\mathcal{H}}_\pm \equiv \sigma_2\mathcal{H}^*_\pm\sigma_2$ We have used that

$$\begin{aligned}
&\gamma\cdot\gamma_{LR} = \gamma_L\cdot\gamma_R = 0,\\
&|\gamma_{LR}|^2 = |\gamma_L|^2 = |\gamma_R|^2 = |\gamma|^2 = 2,\\
&\gamma\cdot\gamma_L = \gamma\cdot\gamma_R = \gamma_L\cdot\gamma_{LR} = \gamma_R\cdot\gamma_{LR} = \frac{|\gamma|^2}{2} = 1,\\
&|N_{\gamma_{LR},-\gamma_L}|^2 = |N_{\gamma_{LR},-\gamma_R}|^2 = |N_{\gamma_L,-\gamma}|^2 = |N_{\gamma_R,-\gamma}|^2 = \frac{|\gamma|^2}{2} = 1,\\
&\frac{N_{\gamma_L,-\gamma}}{N_{\gamma_{LR},-\gamma_R}} = \frac{N^*_{\gamma,\alpha_L}}{N^*_{\gamma_R,\alpha_L}} = \frac{N^*_{\gamma,\alpha_R}}{N^*_{\gamma_L,\alpha_R}} = \frac{N_{\gamma_R,-\gamma}}{N_{\gamma_{LR},-\gamma_L}}, \qquad (3.6.11)
\end{aligned}$$

which are followed by (A.0.2), (A.0.3) and the fact that $\alpha_L\cdot\alpha_R = 0$ and $[E_{\alpha_L}, E_{\alpha_R}] = 0$. We have also chosen the phases of the Higgs fields so that $N_{\gamma_L,-\gamma}/N_{\gamma_{LR},-\gamma_R} = -1$.

The final expression in (3.6.10) is manifestly invariant under the transformation: $\mathcal{H}_\pm \to U_L \mathcal{H}_\pm U_R$ ($U_L \in \mathrm{SU(2)_L}$ and $U_R \in \mathrm{SU(2)_R}$). Except for the case of SU(4) on T^2/Z_2, one of the bidoublets $\mathcal{H}_\pm$ is absent due to the orbifold boundary conditions. In such cases, the model becomes a two-Higgs-doublet model. In contrast to the SO(5) case, the potential (3.6.10) with $\mathcal{H}_+ = 0$ or $\mathcal{H}_- = 0$ does not agree with (7) of Ref. [17]. This is because they have assumed $\gamma + \alpha_L + \alpha_R = -\gamma$, which only holds in the SO(5) case.

Finally we comment on the Higgs mass. We consider a case of SU(4) on T^2/Z_3. The tree-level Higgs potential (3.6.10) becomes

$$\begin{aligned} V_{\text{tree}} &= \frac{g^2}{2}\left[\left\{\mathrm{tr}\left(\mathcal{H}^\dagger\mathcal{H}\right)\right\}^2 - 2\det\left(\mathcal{H}^\dagger\mathcal{H}\right)\right] \\ &= \frac{g^2}{2}\left\{\left(H_1^\dagger H_1\right)^2 + \left(H_2^\dagger H_2\right)^2 + 2\left|\tilde{H}_2^\dagger H_1\right|^2\right\}, \end{aligned} \tag{3.6.12}$$

where $\mathcal{H} = (\tilde{H}_2, H_1)$ is one of $\mathcal{H}_\pm$. Since only the $\mathrm{U(1)_{em}}$ neutral components H_1^2 and H_2^2 can have nonzero VEVs, we focus on them. As discussed in Sec. 3.5.3, we expect that $h_+ \equiv (H_1^2 + H_2^2)/\sqrt{2}$ has a tachyonic mass while $h_- \equiv (H_1^2 - H_2^2)/\sqrt{2}$ does not at one-loop level. Including such mass terms, the potential becomes

$$V = -m_+^2 |h_+|^2 + m_-^2 |h_-|^2 + \frac{g^2}{4}\left(|h_+|^2 + |h_+|^2\right)^2 + \cdots, \tag{3.6.13}$$

where $m_\pm^2 > 0$, and the ellipsis denotes terms involving the charged components. By minimizing this potential, we obtain

$$\langle h_+\rangle^2 = \frac{2m_+^2}{g^2}, \qquad \langle h_-\rangle = 0. \tag{3.6.14}$$

Therefore, the alignment (3.4.17) is actually achieved. The mass of the lightest neutral Higgs boson is

$$m_H = \frac{g}{\sqrt{2}}\left|\langle h_+\rangle\right| = gv = m_W, \tag{3.6.15}$$

where v is defined as $\langle H_1^2\rangle = \langle H_2^2\rangle = v$, and we have used (3.4.25) at the last equality. This is lighter than the observed value $m_H \simeq 125$ GeV, but we should note that there is a sizable quantum correction just like in the supersymmetric models [46].

3.7 Discussion

We have investigated 6D GHU models compactified on T^2/Z_N ($N = 2, 3, 4, 6$) that have the custodial symmetry. The gauge group is assumed to be $\mathrm{SU(3)}_C \times G \times \mathrm{U(1)}_Z$, where

G is a simple group. Since G includes $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}$, its rank must be more than one. The Higgs fields originate from the extra-dimensional components of the G gauge field. In contrast to 5D models [19, 21, 22], we have at least two Higgs doublets. Thus their VEVs need to be aligned as (3.4.17) to preserve the custodial symmetry. This severely constrains the structure of models.

In order to select candidates for realistic models, we demanded the following requirements.

- The model has a scalar bidoublet zero-mode as the Higgs fields.
- The bosonic sector has a symmetry under a parity $\mathcal{P}_{\mathrm{LR}}$ that exchanges $\mathrm{SU}(2)_\mathrm{L}$ and $\mathrm{SU}(2)_\mathrm{R}$ in order to protect the $Z\bar{b}_L b_L$ coupling against a large deviation induced by mixing with the KK modes.
- The quark fields are embedded into 6D fermions so that they couple to the Higgs bidoublet $\mathcal{H}$ only through a combination $\mathcal{H} + \sigma_2 \mathcal{H}^* \sigma_2$.
- The representation $\mathcal{R}$ that the 6D fermions belong to provides a large group factor to realize the top Yukawa coupling constant.

The third requirement is demanded in order for the Higgs VEVs to be aligned as (3.4.17). The third and fourth requirements can be achieved if $\mathcal{R}$ satisfies the three conditions in Sec. 3.5.3.

There is only one candidate that satisfies the above requirements if we restrict ourselves to the cases that $\mathrm{rank}\, G \leq 3$ and $\dim \mathcal{R} \leq 30$. It is the case of $G = \mathrm{SU}(4)$, $N = 3$ and $\mathcal{R} = \mathbf{20'}$. Namely, the model is 6D $\mathrm{SU}(3)_C \times \mathrm{U}(4)$ gauge theory compactified on T^2/Z_3, and the top and the bottom quarks are embedded into the symmetric tensor of SO(6). Our results are summarized in Table I. In the cases with blank, there is no choice of the orbifold boundary conditions so that G is broken to $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R} (\times \mathrm{U}(1)_X)$. We have focused on the third generation quarks to restrict G, N and $\mathcal{R}$. Embeddings of other fermions are much less constrained.

There are many issues that we have not discussed in this chapter. We have approximated the mode functions of the W and the Z bosons as constants. However, after the electroweak symmetry is broken, they are no longer constant and depend on z. This z-dependence causes the deviation of the ρ parameter and the $Z\bar{b}_L b_L$ coupling from the standard model values. We have to check that the custodial symmetry actually suppresses

	SO(5)	G_2	SU(4)	SO(7)		Sp(6)	
				(I)	(II), (III)	(I)	(II), (III)
T^2/Z_2	1 (S)	0	2 (S)				
T^2/Z_3			1 (S) ✓		0		1
T^2/Z_4	0 (S)	0	1 (S)	1 (S)	0	0 (S)	1
T^2/Z_6	0 (S)	0	1 (S)	1 (S)	0	0 (S)	1

Table I: Summary of the results. The numbers denote those of the Higgs bidoublets. (I), (II) and (III) represent three different ways of choosing the $SU(2)_L \times SU(2)_R$ subgroup in Sec. 3.3.2. "(S)" indicates that the spectrum is symmetric under $SU(2)_L \leftrightarrow SU(2)_R$. The check mark is added to a case that there is an appropriate embedding of quarks into 6D fermions.

these deviations by solving the mode equations in a specific model. We should also calculate the one-loop effective potential to check that the vacuum alignment (3.4.17) is actually achieved, and to evaluate the Higgs mass spectrum. The moduli stabilization in the gauge-Higgs unification is also an important subject [47, 25]. These issues are left for future works.

Chapter 4

Generations and Yukawa hierarchy in 6D gauge-Higgs unification

In this chapter, we introduce constant magnetic fluxes as backgrounds of gauge field strengths to realize the hierarchy of the Yukawa coupling constants among matter flavors on 6D GHU models. The 4D effective Yukawa couplings on GHU models originate from the higher dimensional gauge coupling, so the Yukawa couplings on respective flavors need some mechanisms in order to have respective different values.

4.1 Introduction

In the previous chapter, we saw that the top Yulawa coupling is realized with the group factor of a large representation, such as $SU(4)$ $\mathbf{20'}$, in 6D GHU models with the custodial symmetry. In GHU models, as mentioned above, the Yukawa couplings become flavor-universal with the flat profile of the zero-mode wave functions of the fields that are relevant to the Yukawa interactions. One concrete way to avoid such a situation is to change the values of the overlap integrals of the Yukawa couplings by localizing the zero-mode wave functions at the extra dimensions.

In 5D models, kink mass terms of the bulk fermions are introduced for controling the Yukawa couplings. However, we cannot introduce them because of the double periodicity in 6D models. Instead of them, we introduce constant magnetic fluxes penetrating the extra dimensions as backgrounds of gauge field strengths. At the extra dimensions, the zero-mode wave functions are shifted by the constant parts of background gauge fields, the Wilson-line phases (or Scherk-Schwarz phases), and the possible values of them are restricted by the orbifold compactifying the extra dimensions and the values of the magnetic fluxes that respective fields feel. These wave functions are called "the Jacobi-theta functions".

The zero-modes of the feilds feeling the mgnetic fluxes degenerate depending on the values of fluxes and the Z_N twist phases of the orbifold wave functions. This degeneration can be regarded as an origin of the matter generations. We will see whether we can realize the hierarchical Yukawa structure and the three generations of the matter fields in the SM by the magnetic fluxes that cause desired symmetry breaking in 6D GHU models and the Wilson-line phases. We'll also check the effect for zero-modes of introducing the non-kink mass terms of the bulk fermions.

4.2 Setup

4.2.1 Compactified space

The setup of spacetime is the same as Subsec. 3.2.1 .

4.2.2 Field content

We consider a 6D gauge group is $SU(3)_C \times G \times U(1)_X$, where G is a simple group that has $SU(2)_L \times U(1)_Z$ subgroup. As is the same as Chapter 3, we don't purpose to unify $SU(3)_C$, so it is irrelevant to the following discussion. The field content consists of the G gauge field A_M and the $U(1)_X$ gauge field B_M, and 6D (8-component) Dirac fermion Ψ^f (f=$1, 2, \cdots, n_f$). We define the field strengths and covariant derivatives as

$$\begin{aligned} F_{MN} &\equiv \partial_M A_N - \partial_N A_M - i\left[A_M, A_N\right], \\ B_{MN} &\equiv \partial_M B_N - \partial_N B_M, \\ \mathcal{D}_M \Psi^f &\equiv \left(\partial_M - iA_M - iq_f B_M\right)\Psi^f, \end{aligned} \tag{4.2.1}$$

where q_f is the $U(1)_X$ charge of Ψ^f. Now, 6D Lagrangian is written as

$$\begin{aligned} \mathcal{L} = &-\frac{1}{2g_A^2}\mathrm{tr}\left(F^{MN}F_{MN}\right) - \frac{1}{4g_B^2}B^{MN}B_{MN} \\ &+\sum_f \left(i\bar{\Psi}^f\Gamma^M\mathcal{D}_M\Psi^f + M_f\bar{\Psi}^f\Psi^f\right), \end{aligned} \tag{4.2.2}$$

where Γ^M are 6D Gamma matrices and a real constant M_f is a 6D bulk mass parameter of Ψ^f. g_A and g_B are 6D gauge couplings of G and $U(1)_X$, respectively. The Lagrangian is invariant under the transformation:

$$
\begin{aligned}
A_M &\to U A_M U^{-1} + iU\partial_M U^{-1}, \\
B_M &\to B_M + \partial_M \chi, \\
\Psi^f &\to e^{iq_f\chi} U \Psi^f,
\end{aligned}
\tag{4.2.3}
$$

where $U \in G$ and χ is a real function. The G gauge field is decomposed as

$$
A_M = \sum_i C_M^i H_i + \sum_{\boldsymbol{\alpha}} W_M^{\boldsymbol{\alpha}} E_{\boldsymbol{\alpha}},
\tag{4.2.4}
$$

where H_i and $E_{\boldsymbol{\alpha}}$ are the generators inthe Cartan-Weyl basis. In the complex coordinate (x^μ, z), the wxtra components of the gauge fields are expressed as

$$
\begin{aligned}
A_z &= \pi R_1(A_4 - iA_5), \qquad A_{\bar{z}} = A_z^\dagger, \\
B_z &= \pi R_1(B_4 - iB_5), \qquad B_{\bar{z}} = B_z^\dagger.
\end{aligned}
\tag{4.2.5}
$$

The fields satisfy the torus boundary conditions:

$$
\begin{aligned}
A_M(x, z+s) &= U_s(z) A_M(x,z) U_s^{-1}(z) + i(U_s \partial_M U_s^{-1})(z), \\
B_M(x, z+s) &= U_s(z) B_M(x,z) U_s^{-1}(z) + i\partial_M \chi_s(z), \\
\Psi^f(x, z+s) &= e^{iq_f\chi_s(z)} U_s(z) \Psi^f(x,z), \\
(s &= 1, \tau)
\end{aligned}
\tag{4.2.6}
$$

and the orbifold boundary conditions:

$$
\begin{aligned}
A_\mu(x, \omega z) &= PM(x,z)P^{-1}, \quad A_\mu(x,\omega z) = \omega^{-1} PM(x,z)P^{-1}, \\
B_\mu(x, \omega z) &= B_M(x,z), \quad B_\mu(x,\omega z) = \omega^{-1} B_M(x,z), \\
\Psi^f(x, \omega z) &= \omega^{-\frac{\chi_4\chi_6}{2}} e^{iq_f\varphi_\omega} P \Psi^f(x,z),
\end{aligned}
\tag{4.2.7}
$$

where $\chi_4,\ \chi_6$ are the 4D, 6D chiralities, and $\varphi_\omega,\ P \in G$ are a real constant, a unitary constant matrix, respectively. We can choose always P as

$$
P = \exp(ip \cdot H),
\tag{4.2.8}
$$

where $p \cdot H = \Sigma_i p^i H^i$ and p^i are real constants. Since (4.2.7) is Z_N transformation, p and α must satisfy

$$
\begin{aligned}
e^{ip\cdot\boldsymbol{\alpha}} &= \exp\left(\frac{2n_{\boldsymbol{\alpha}}\pi i}{N}\right), \\
\omega^{-\frac{\chi_4\chi_6}{2}} e^{iq_f\varphi_f} e^{ip\cdot\boldsymbol{\mu}} &= \exp\left(\frac{2n_{\boldsymbol{\mu}f}^{\chi_4\chi_6}}{N}\right),
\end{aligned}
\tag{4.2.9}
$$

where $n_{\boldsymbol{\alpha}}\pi i$ and $n_{\boldsymbol{\mu f}}^{\chi_4\chi_6}$ are integers.

P is chosen in such a way as to break G to $SU(2)_L \times U(1)_Z \times U(1)^{r-2}$ (r: rank of G). We assume that $U(1)_Z \times U(1)_X$ is broken to the hypercharge group $U(1)_Y$ by some interactions localized at an orbifold fixed point. The generators of $SU(2)_L \times U(1)_Z$ subgroups are expressed as

$$(T_L^{\pm}, T_L^3) = \left(\frac{E_{\pm\boldsymbol{\alpha}_L}}{|\boldsymbol{\alpha}_L|}, \frac{\boldsymbol{\alpha}_L \cdot H}{|\boldsymbol{\alpha}_L|^2}\right), \tag{4.2.10}$$

$$\mathcal{Q}_Z = \boldsymbol{\zeta} \cdot H, \tag{4.2.11}$$

where $\boldsymbol{\alpha}_L$ is a root of G, and $\boldsymbol{\zeta}$ is a constant real vector that satisfies $\boldsymbol{\zeta}\cdot\boldsymbol{\alpha}_L = 0$. Hypercharge Y is expressed as

$$Y = \mathcal{Q}_Z + \mathcal{Q}_X, \tag{4.2.12}$$

where $\mathcal{Q}_X$ is $U(1)_X$ generator.

4.2.3 Magnetic fluxes

We introduce constant magnetic fluxes penetrating the compact space as backgrounds of gauge field strengths. For simplicity, we assume $U(1)_X$ gauge field and the Cartan components of the G gauge field have nonvanishing backgrounds, and those of field strengths are constants. The nonvanishing fluxes are

$$\mathcal{C} \equiv \int_{T^2/Z_N} dx^4 dx^5 \langle C_{45}^i \rangle = \mathcal{A}\langle C_{45}^i \rangle = -\frac{2i\mathrm{Im}\tau}{N}\langle C_{z\bar{z}}^i \rangle \tag{4.2.13}$$

$$\mathcal{B} \equiv \int_{T^2/Z_N} dx^4 dx^5 \langle B_{45} \rangle = \mathcal{A}\langle B_{45} \rangle = -\frac{2i\mathrm{Im}\tau}{N}\langle B_{z\bar{z}} \rangle \tag{4.2.14}$$

where

$$C_{z\bar{z}} \equiv \partial_z C_{\bar{z}}^i - \partial_{\bar{z}} C_z^i \tag{4.2.15}$$

$$B_{z\bar{z}} \equiv \partial_z B_{\bar{z}} - \partial_{\bar{z}} B_z \tag{4.2.16}$$

$$\mathcal{A} \equiv (2\pi R_1)^2 \mathrm{Im}\tau/N = 4\pi^2 R_1 R_2 \sin\theta/N \tag{4.2.17}$$

: the area of the fundamental region of T^2/Z_N

Then, backgrounds values of vector potentials are

$$\langle C_z^i \rangle = -\frac{iN(\mathcal{C}^i\bar{z} + \bar{c}^i)}{4\mathrm{Im}\tau},$$
$$\langle B_z \rangle = -\frac{iN(\mathcal{B}\bar{z} + \bar{b})}{4\mathrm{Im}\tau}, \tag{4.2.18}$$

where c^i and b are complex constants, which we call "Wilson-line phases". We can always make these values absorbed into "Scherk-Schwarz phases" by redefinition of fields. From the values of $\langle C_z^i \rangle$ and $\langle B_z \rangle$, we identify $U_s(z)$ and $\chi_s(z)$ $(s = 1, \tau)$:

$$U_s(z) = \exp\left\{ i \sum_i \left(\frac{N\mathcal{C}^i \mathrm{Im}(\bar{s}z)}{2\mathrm{Im}\tau} + \Phi_s^i \right) H_i \right\},$$
$$\chi_s(z) = \frac{N\mathcal{B}\mathrm{Im}(\bar{s}z)}{2\mathrm{Im}\tau} + \varphi_s, \tag{4.2.19}$$

where Φ_s^i and φ_s are real constants, the Scherk-Schwarz phases. These values take only the discrete values when we compactify the extra dimensions on T^2/Z_N as is shown at Appendix E.

From substituting (4.2.19) for (4.2.6), and the single-valuedness of W_z^α and Ψ^f, we get the quantization conditions of the fluxes :

$$N\mathcal{C} \cdot \boldsymbol{\alpha} = 2k_{\boldsymbol{\alpha}}\pi,$$
$$N\mathcal{C} \cdot \boldsymbol{\mu} + q_f N\mathcal{B} = 2k_{\boldsymbol{\mu}f}\pi, \tag{4.2.20}$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\mu}$ a root and a weight of G, and $k_{\boldsymbol{\alpha}}$, $k_{\boldsymbol{\mu}f} \in \mathbb{Z}$. Using these conditions, the background gauge fields are expressed as

$$\langle C_z \cdot \boldsymbol{\alpha} \rangle = -\frac{i\pi k_{\boldsymbol{\alpha}}(\bar{z} + \bar{\zeta}_{\boldsymbol{\alpha}})}{2\mathrm{Im}\tau},$$
$$\langle C_z \cdot \boldsymbol{\mu} + q_f B_z \rangle = -\frac{i\pi k_{\boldsymbol{\mu}f}(\bar{z} + \bar{\zeta}_{\boldsymbol{\mu}f})}{2\mathrm{Im}\tau}. \tag{4.2.21}$$

where

$$\zeta_{\boldsymbol{\alpha}} \equiv \frac{c \cdot \boldsymbol{\alpha}}{\mathcal{C} \cdot \boldsymbol{\alpha}}, \qquad \zeta_{\boldsymbol{\mu}f} \equiv \frac{c \cdot \boldsymbol{\mu} + q_f b}{\mathcal{C} \cdot \boldsymbol{\mu} + q_f \mathcal{B}}. \tag{4.2.22}$$

4.2.4 Equations of motion and KK expansion

We decompose the gauge fields into the background and the fluctuation parts:

$$A_M = \langle A_M \rangle + \tilde{A}_M, \tag{4.2.23}$$
$$B_M = \langle B_M \rangle + \tilde{B}_M, \tag{4.2.24}$$

and derive the linearized equations of $\tilde{A}_M$ and $\tilde{B}_M$. We choose the gauge-fixing term as

$$\mathcal{L}_{\mathrm{gf}} = -\frac{1}{g_A^2}\mathrm{tr}\left\{ \left(D^M \tilde{A}_M \right)^2 \right\} - \frac{1}{2g_B^2}\left(\partial^M \tilde{B}_M \right)^2, \tag{4.2.25}$$

where

$$D_M \tilde{A}_N \equiv \partial_M - i\left[\langle A_M \rangle, \tilde{A}_N\right]. \tag{4.2.26}$$

The 6D Dirac fermion Ψ^f is decomposed into 2-component spinors as

$$\Psi^f = \begin{pmatrix} \hat{\Psi}^f_+ \\ \hat{\Psi}^f_- \end{pmatrix}, \tag{4.2.27}$$

$$\hat{\Psi}^f_{\pm\hat{\alpha}} = \begin{pmatrix} \psi^f_{\pm\alpha} \\ \bar{\lambda}^{f\dot{\alpha}}_\pm \end{pmatrix}, \tag{4.2.28}$$

where the signs are 6D chiralities. $\hat{\alpha}$ and $(\alpha, \dot{\alpha})$ are the 4-component and the 2-component spinor indices, respectively.

The Lagrangians are written as

$$\begin{aligned}
\mathcal{L} + \mathcal{L}_{\rm gf} = &- \frac{1}{g_A^2}\mathrm{tr}\left\{D^M \tilde{A}^N D_M \tilde{A}_N - i\langle F^{MN}\rangle \left[\tilde{A}_M, \tilde{A}_N\right]\right\} - \frac{1}{2g_B^2}\partial^M \tilde{B}^N \partial_M \tilde{B}_N \\
&+ \sum_f \Bigg\{ - i\lambda^f_+ \sigma^\mu D_\mu \bar{\lambda}^f_+ - i\bar{\psi}^f_+ \bar{\sigma}^\mu D_\mu \psi^f_+ + \frac{1}{\pi R_1}\left(\lambda_+ D_{\bar{z}} \psi^f_+ - \bar{\psi}_+ D_{\bar{z}} \bar{\lambda}^f_+\right) \\
&\quad - i\lambda^f_- \sigma^\mu D_\mu \bar{\lambda}^f_- - i\bar{\psi}^f_- \bar{\sigma}^\mu D_\mu \psi^f_- + \frac{1}{\pi R_1}\left(\lambda_- D_{\bar{z}} \psi^f_- - \bar{\psi}_- D_{\bar{z}} \bar{\lambda}^f_-\right) \\
&\quad - M_f \left(\lambda^f_+ \psi^f_- + \lambda^f_- \psi^f_+ + \mathrm{h.c.}\right) \Bigg\} + \cdots\ ,
\end{aligned} \tag{4.2.29}$$

where the ellipsis is higher order terms in the fluctuation fields, and

$$D_M \psi_\pm \equiv (\partial_M - i\langle A_M \rangle - iq_f \langle B_M \rangle)\psi^f_\pm, \tag{4.2.30}$$

$$D_M \bar{\lambda}_\pm \equiv (\partial_M - i\langle A_M \rangle - iq_f \langle B_M \rangle)\bar{\lambda}^f_\pm. \tag{4.2.31}$$

We have dropped total derivative terms. From (4.2.29), we obtain the linearized equations of motion for $\tilde{A}_M$ and Ψ^fas

$$D^M D_M \tilde{A}_N - i\left[\langle F_{NM}\rangle, \tilde{A}^M\right] + \cdots = 0, \tag{4.2.32}$$

$$i\Gamma^M D_M \Psi^f + M_f \Psi^f + \cdots = 0. \tag{4.2.33}$$

These are rewritten as

$$
\begin{aligned}
&\left(\pi^2 R_1^2 \Box_4 + \partial_z \partial_{\bar{z}}\right) \tilde{C}^i_\mu = 0, \qquad \left(\pi^2 R_1^2 \Box_4 + \mathcal{O}_{\boldsymbol{\alpha}}\right) \tilde{W}^\alpha_\mu = 0, \\
&\left(\pi^2 R_1^2 \Box_4 + \partial_z \partial_{\bar{z}}\right) \tilde{B}_\mu = 0, \\
&\left(\pi^2 R_1^2 \Box_4 + \partial_z \partial_{\bar{z}}\right) \tilde{C}^i_z = 0, \qquad \left(\pi^2 R_1^2 \Box_4 + \mathcal{O}_{\boldsymbol{\alpha}} + \delta\mathcal{O}_{\boldsymbol{\alpha}}\right) \tilde{W}^\alpha_z = 0, \\
&\left(\pi^2 R_1^2 \Box_4 + \partial_z \partial_{\bar{z}}\right) \tilde{B}_z = 0, \\
&- i\sigma^\mu \partial_\mu \bar{\lambda}^f_+ + \frac{1}{\pi R_1} D_{\bar{z}} \psi^f_+ - M_f \psi^f_- = 0, \\
&- i\sigma^\mu \partial_\mu \bar{\lambda}^f_- + \frac{1}{\pi R_1} D_{\bar{z}} \psi^f_- - M_f \psi^f_+ = 0, \\
&- i\sigma^\mu \partial_\mu \bar{\psi}^f_+ + \frac{1}{\pi R_1} D_{\bar{z}} \lambda^f_+ - M_f \lambda^f_- = 0, \\
&- i\sigma^\mu \partial_\mu \bar{\psi}^f_- + \frac{1}{\pi R_1} D_{\bar{z}} \lambda^f_- - M_f \lambda^f_+ = 0,
\end{aligned}
\tag{4.2.34}
$$

where

$$
\Box_4 \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu, \tag{4.2.35}
$$

$$
\begin{aligned}
\mathcal{O}_{\boldsymbol{\alpha}} &\equiv \{\partial_{\bar{z}} - i\langle C_{\bar{z}} \cdot \boldsymbol{\alpha}\rangle\} \{\partial_{\bar{z}} - i\langle C_z \cdot \boldsymbol{\alpha}\rangle\} - \mathrm{Im}\langle \partial_{\bar{z}} C_z \cdot \boldsymbol{\alpha}\rangle \\
&= \{\partial_{\bar{z}} - i\langle C_z \cdot \boldsymbol{\alpha}\rangle\} \{\partial_{\bar{z}} - i\langle C_{\bar{z}} \cdot \boldsymbol{\alpha}\rangle\} + \mathrm{Im}\langle \partial_{\bar{z}} C_z \cdot \boldsymbol{\alpha}\rangle, \\
\delta\mathcal{O}_{\boldsymbol{\alpha}} &\equiv -\frac{i}{2} \langle \partial_z C_{\bar{z}} - \partial_{\bar{z}} C_z\rangle \cdot \boldsymbol{\alpha}.
\end{aligned}
\tag{4.2.36}
$$

The KK expansions are written as

$$
\begin{aligned}
&C^i_\mu(x,z) = \frac{g_A}{\sqrt{2}\pi R_1} \sum_n f^i_n(z) C^{i(n)}_\mu(x), \qquad W^{\boldsymbol{\alpha}}_\mu = \frac{g_A}{\sqrt{2}\pi R_1} \sum_n f^{\boldsymbol{\alpha}}_n(z) W^{\boldsymbol{\alpha}(n)}_\mu(x), \\
&B_\mu(x,z) = \frac{g_A}{\sqrt{2}\pi R_1} \sum_n f^B_n(z) B^{(n)}_\mu(x), \\
&\tilde{C}^i_z(x,z) = \langle C^i_z\rangle(z) + g_A \sum_n g^i_n(z) \varphi^i_n(x), \qquad \tilde{W}^{\boldsymbol{\alpha}}_z = g_A \sum_n g^{\boldsymbol{\alpha}}_n(z) \varphi^{\boldsymbol{\alpha}}_n(x), \\
&B_\mu(x,z) = \langle B_z\rangle(z) + g_B \sum_n g^B_n(z) \varphi^B_n(x), \\
&\psi^f_\pm(x,z) = \frac{1}{\sqrt{2}\pi R_1} \sum_n \sum_{\boldsymbol{\mu}} h^{(\pm)\boldsymbol{\mu}}_{Rn}(z) |\boldsymbol{\mu}\rangle \psi^{(\pm)}_n(x) \\
&\lambda^f_\pm(x,z) = \frac{1}{\sqrt{2}\pi R_1} \sum_n \sum_{\boldsymbol{\mu}} h^{(\pm)\boldsymbol{\mu}}_{Rn}(z) |\boldsymbol{\mu}\rangle \lambda^{(\pm)}_n(x),
\end{aligned}
\tag{4.2.37}
$$

where $|\boldsymbol{\mu}\rangle$ is a state of the weight vector $\boldsymbol{\mu}$ of the G representation. All the mode functions are dimensionless, and normalized as

$$
\int_{T^2/Z_N} dz d\bar{z} F^*_n(z) F_m(z) = \delta_{nm}, \tag{4.2.38}
$$

where $F_n(z)$ denotes the mode functions. The coefficients in the KK expansion are determined so that the 4D KK modes have canonically normalized kinetic terms[1]. The mode equations are writen as

$$\begin{aligned}
&\partial_z\partial_{\bar{z}} f_n^i = -\tilde{m}_n^2 f_n^i, \quad \mathcal{O}_{\boldsymbol{\alpha}} f_n^{\boldsymbol{\alpha}} = -\tilde{m}_n^2 f_n^{\boldsymbol{\alpha}}, \quad \partial_z\partial_{\bar{z}} f_n^B = -\tilde{m}_n^2 f_n^B, \\
&\partial_z\partial_{\bar{z}} g_n^i = -\tilde{m}_n^2 g_n^i, \quad (\mathcal{O}_{\boldsymbol{\alpha}} + \delta\mathcal{O}_{\boldsymbol{\alpha}})\, g_n^{\boldsymbol{\alpha}} = -\tilde{m}_n^2 g_n^{\boldsymbol{\alpha}}, \quad \partial_z\partial_{\bar{z}} g_n^B = -\tilde{m}_n^2 g_n^B, \\
&D_{\bar{z}}^{(f)} h_{Rn}^{(+)\boldsymbol{\mu}} - \tilde{M}_f h_{Rn}^{(-)\boldsymbol{\mu}} = -\tilde{m}_n h_{Ln}^{(+)\boldsymbol{\mu}}, \\
&D_{\bar{z}}^{(f)} h_{Rn}^{(-)\boldsymbol{\mu}} - \tilde{M}_f h_{Rn}^{(+)\boldsymbol{\mu}} = -\tilde{m}_n h_{Ln}^{(-)\boldsymbol{\mu}}, \\
&D_{\bar{z}}^{(f)} h_{Rn}^{(+)\boldsymbol{\mu}} + \tilde{M}_f h_{Rn}^{(-)\boldsymbol{\mu}} = \tilde{m}_n^* h_{Ln}^{(+)\boldsymbol{\mu}}, \\
&D_{\bar{z}}^{(f)} h_{Rn}^{(-)\boldsymbol{\mu}} + \tilde{M}_f h_{Rn}^{(+)\boldsymbol{\mu}} = \tilde{m}_n^* h_{Ln}^{(-)\boldsymbol{\mu}},
\end{aligned} \tag{4.2.39}$$

where $\tilde{M}_f \equiv \pi R_1 M_f$, $\tilde{m}_n \equiv \pi R_1 m_n$ and m_n is the KK mass eigenvalue. The values of $\tilde{m}_n$ are complex in general for fermions, while they are real for the bosons because the differential operators of the bosons are hermite.

We can rewrite (4.2.36) using (4.2.21) as

$$\begin{aligned}
\mathcal{O}_{\boldsymbol{\alpha}} &= \left(\partial_{\bar{z}} + \frac{\pi k_{\boldsymbol{\alpha}} z}{2\mathrm{Im}\tau}\right)\left(\partial_z - \frac{k_{\boldsymbol{\alpha}}\pi}{2\mathrm{Im}\tau}\right) + \frac{k_{\boldsymbol{\alpha}}\pi}{2\mathrm{Im}\tau} \\
&= \left(\partial_z - \frac{k_{\boldsymbol{\alpha}}\pi}{2\mathrm{Im}\tau}\right)\left(\partial_{\bar{z}} + \frac{\pi k_{\boldsymbol{\alpha}} z}{2\mathrm{Im}\tau}\right) - \frac{k_{\boldsymbol{\alpha}}\pi}{2\mathrm{Im}\tau}
\end{aligned} \tag{4.2.40}$$

$$\delta\mathcal{O} = \frac{k_{\boldsymbol{\alpha}}\pi}{2\mathrm{Im}\tau}, \tag{4.2.41}$$

Also we can write

$$D_z^{(f)} = \partial_z - i\langle C_z \cdot \boldsymbol{\mu}\rangle - iq_f\langle B_z\rangle = \partial_z - \frac{\pi k_{\boldsymbol{\mu} f}\bar{z}}{2\mathrm{Im}\tau}, \tag{4.2.42}$$

$$D_{\bar{z}}^{(f)} = \partial_{\bar{z}} - i\langle C_{\bar{z}} \cdot \boldsymbol{\mu}\rangle - iq_f\langle B_{\bar{z}}\rangle = \partial_{\bar{z}} + \frac{\pi k_{\boldsymbol{\mu} f} z}{2\mathrm{Im}\tau}. \tag{4.2.43}$$

[1] $\int dx^4 dx^5 = 2(\pi R_1)\int dz d\bar{z}$.

The torus boundary conditions (4.2.6) can be rewritten by (4.2.19) as

$$
\begin{aligned}
f_n^i(z+s) &= f_n^i(z), \quad f_n^B(z+s) = f_n^B(z), \\
f_n^{\boldsymbol{\alpha}}(z+s) &= \exp\left\{\frac{ik_{\boldsymbol{\alpha}}\pi}{\mathrm{Im}\tau}\mathrm{Im}(\bar{s}z) + 2\pi i\phi_s^{\boldsymbol{\alpha}}\right\} f_n^{\boldsymbol{\alpha}}, \\
g_n^i(z+s) &= g_n^i(z), \quad g_n^B(z+s) = g_n^B(z), \\
g_n^{\boldsymbol{\alpha}}(z+s) &= \exp\left\{\frac{ik_{\boldsymbol{\alpha}}\pi}{\mathrm{Im}\tau}\mathrm{Im}(\bar{s}z) + 2\pi i\phi_s^{\boldsymbol{\alpha}}\right\} g_n^{\boldsymbol{\alpha}}, \\
h_{Rn}^{(\pm)\boldsymbol{\mu}}(z+s) &= \exp\left\{\frac{ik_{\boldsymbol{\mu}f}\pi}{\mathrm{Im}\tau}\mathrm{Im}(\bar{s}z) + 2\pi i\phi_s^{\boldsymbol{\mu}f}\right\} h_{Rn}^{(\pm)\boldsymbol{\mu}}, \\
h_{Ln}^{(\pm)\boldsymbol{\mu}}(z+s) &= \exp\left\{\frac{ik_{\boldsymbol{\mu}f}\pi}{\mathrm{Im}\tau}\mathrm{Im}(\bar{s}z) + 2\pi i\phi_s^{\boldsymbol{\mu}f}\right\} h_{Ln}^{(\pm)\boldsymbol{\mu}}
\end{aligned}
\tag{4.2.44}
$$

where

$$
\phi_s^{\boldsymbol{\alpha}} \equiv \frac{\Phi_s \cdot \boldsymbol{\alpha}}{2\pi}, \qquad \phi_s^{\boldsymbol{\mu}f} \equiv \frac{\Phi_s \cdot \boldsymbol{\mu} + q_f\varphi_s}{2\pi}, \qquad (s = 1, \tau) \tag{4.2.45}
$$

and the orbifold boundary condition (4.2.7) can be rewritten as

$$
\begin{aligned}
f_n^i(\omega z) &= f_n^i(z), \quad f_n^{\boldsymbol{\alpha}}(\omega z) = e^{ip\cdot\boldsymbol{\alpha}} f_n^{\boldsymbol{\alpha}} \quad f_n^B(\omega z) = f_n^B(z), \\
g_n^i(\omega z) &= \omega^{-1} g_n^i(z), \quad g_n^{\boldsymbol{\alpha}}(\omega z) = \omega^{-1} e^{ip\cdot\boldsymbol{\alpha}} g_n^{\boldsymbol{\alpha}} \quad g_n^B(\omega z) = \omega^{-1} g_n^B(z), \\
h_{Rn}^{(\pm)\boldsymbol{\mu}} &= \omega^{\mp\frac{1}{2}e^{iq_f\varphi}} e^{ip\cdot\boldsymbol{\mu}} h_{Rn}^{(\pm)\boldsymbol{\mu}}, \quad h_{Ln}^{(\pm)\boldsymbol{\mu}} = \omega^{\pm\frac{1}{2}e^{iq_f\varphi}} e^{ip\cdot\boldsymbol{\mu}} h_{Ln}^{(\pm)\boldsymbol{\mu}}
\end{aligned}
\tag{4.2.46}
$$

4.3 Mode functions and KK masses

We derive the mode functions of relevant fields on T^2 and T^2/Z_N, and the KK masses accompanying with them from mode equations (4.2.39) as the solutions satisfying the boundary conditions (4.2.44) and (4.2.46). The mode functions with the tilde mean the T^2 wave functions, and we derive the T^2/Z_N wave functions from T^2 ones. In this section, we see the zero-mode conditions for magnetic fluxes from the KK masses.

4.3.1 T^2 wave functions

4D gauge sector

First, we derive $\tilde{f}_n^i(z)$, the T^2 wave functions of C_μ^i, the Cartan components of the 4D G gauge field. that satisfy (4.2.44) are

$$
\tilde{f}_{n,l}^i(z) = \mathcal{N}_{n,l}^{ci} \cos\left\{\frac{2\pi}{\mathrm{Im}\tau}(n\mathrm{Im}z + l\mathrm{Im}(\bar{\tau}z))\right\} + \mathcal{N}_{n,l}^{si} \sin\left\{\frac{2\pi}{\mathrm{Im}\tau}(n\mathrm{Im}z + l\mathrm{Im}(\bar{\tau}z))\right\}, \tag{4.3.1}
$$

where $\mathcal{N}^{ci}_{n,l}$, $\mathcal{N}^{si}_{n,l}$ are normalization factors of real constants, and the KK masses are

$$\tilde{m}_{n,l}\,(=\pi R_1 m_{n,l}) = \frac{\pi\,|n+l\tau|}{\mathrm{Im}\tau}. \tag{4.3.2}$$

The functions $\tilde{f}^B_{n,l}(z)$, the KK mode wave functions of 4D $U(1)_X$ gauge field on T^2, have the same forms as (4.3.1):

$$\tilde{f}^B_{n,l}(z) = \mathcal{N}^{cB}_{n,l}\cos\left\{\frac{2\pi}{\mathrm{Im}\tau}(n\mathrm{Im}z + l\mathrm{Im}(\bar{\tau}z))\right\} + \mathcal{N}^{sB}_{n,l}\sin\left\{\frac{2\pi}{\mathrm{Im}\tau}(n\mathrm{Im}z + l\mathrm{Im}(\bar{\tau}z))\right\}, \tag{4.3.3}$$

where $\mathcal{N}^{cB}_{n,l}$ and $\mathcal{N}^{sB}_{n,l}$ are real constants, and their KK masses are the same as $\tilde{f}^i_{n,l}(z)$.

Then, we derive $\tilde{f}^{\boldsymbol{\alpha}}_{n,l}$, the T^2 wave functions of $W^{\boldsymbol{\alpha}}_\mu$. The mode equations of them in (4.2.39) are rewritten as

$$D^{(\boldsymbol{\alpha})}_{\bar{z}} D^{(\boldsymbol{\alpha})}_{z} \tilde{f}^{\boldsymbol{\alpha}}_n = -\left(\tilde{m}^2_n + \frac{\pi k_{\boldsymbol{\alpha}}}{2\mathrm{Im}\tau}\right)\tilde{f}^{\boldsymbol{\alpha}}_n \tag{4.3.4}$$

$$\left(D^{(\boldsymbol{\alpha})}_{z} D^{(\boldsymbol{\alpha})}_{\bar{z}} \tilde{f}^{\boldsymbol{\alpha}}_n = -\left(\tilde{m}^2_n - \frac{\pi k_{\boldsymbol{\alpha}}}{2\mathrm{Im}\tau}\right)\tilde{f}^{\boldsymbol{\alpha}}_n\right) \tag{4.3.5}$$

where

$$D^{(\boldsymbol{\alpha})}_{z} = \partial_z - i\langle\boldsymbol{\alpha}\cdot C_z\rangle = \partial_z - \frac{\pi k_{\boldsymbol{\alpha}}}{2\mathrm{Im}\tau}\bar{z} \tag{4.3.6}$$

$$D^{(\boldsymbol{\alpha})}_{\bar{z}} = \partial_{\bar{z}} - i\langle\boldsymbol{\alpha}\cdot C_z\rangle = \partial_{\bar{z}} + \frac{\pi k_{\boldsymbol{\alpha}}}{2\mathrm{Im}\tau}z \tag{4.3.7}$$

For these mode equations, the zero-mode solutions must satisfy

$$D^{(\boldsymbol{\alpha})}_{z}\tilde{f}^{\boldsymbol{\alpha}}_0 = 0, \tag{4.3.8}$$

or

$$D^{(\boldsymbol{\alpha})}_{\bar{z}}\tilde{f}^{\boldsymbol{\alpha}}_0 = 0. \tag{4.3.9}$$

(i) $k_{\boldsymbol{\alpha}} > 0$

Only (4.3.9) has the zero-mode solution that satisfies (4.2.44).

$$\tilde{f}^{\boldsymbol{\alpha}}_0(z) \equiv \mathcal{N}^{\boldsymbol{\alpha}}_0 \exp\left(i\pi k_{\boldsymbol{\alpha}} z \frac{\mathrm{Im}z}{\mathrm{Im}\tau}\right)\vartheta\begin{bmatrix}(j+\phi^{\boldsymbol{\alpha}}_1)/k_{\boldsymbol{\alpha}} \\ -\phi^{\boldsymbol{\alpha}}_\tau\end{bmatrix}(k_{\boldsymbol{\alpha}}z, k_{\boldsymbol{\alpha}}\tau), \tag{4.3.10}$$

where $j = 1, 2, \cdots, k_{\boldsymbol{\alpha}}$, and $\mathcal{N}^{\boldsymbol{\alpha}}_0 \equiv (2k_{\boldsymbol{\alpha}}\mathrm{Im}\tau)^{1/4}$ is the normalization factor. ϑ means the Jacobi-theta function. The difinition is

$$\vartheta\begin{bmatrix}a \\ b\end{bmatrix}(kz, k\tau) \equiv \sum_{l=-\infty}^{+\infty} e^{i\pi(l+a)^2 k\tau} e^{2\pi i(l+a)(kz+b)}. \tag{4.3.11}$$

This solution has the mass eigenvalue from (4.3.5) = 0:

$$\tilde{m}_0^2 = \frac{\pi k_{\boldsymbol{\alpha}}}{2\mathrm{Im}\tau}. \tag{4.3.12}$$

These wave functions satisfy

$$\int_{T^2} d^2z \left\{ \tilde{f}^{\boldsymbol{\alpha}(i)}(z) \right\}^* \tilde{f}^{\boldsymbol{\alpha}(j)}(z) = \delta_{ij} \tag{4.3.13}$$

Here, we define the two-dimensional Laplace operator $\Delta^{(\boldsymbol{\alpha})}$ as

$$\Delta^{(\boldsymbol{\alpha})} \equiv -\frac{1}{2}\left(D_z^{(\boldsymbol{\alpha})} D_{\bar{z}}^{(\boldsymbol{\alpha})} + D_{\bar{z}}^{(\boldsymbol{\alpha})} D_z^{(\boldsymbol{\alpha})} \right), \tag{4.3.14}$$

and these satisfy

$$\begin{aligned} &\left[D_z^{(\boldsymbol{\alpha})}, D_{\bar{z}}^{(\boldsymbol{\alpha})} \right] = \frac{\pi k_{\boldsymbol{\alpha}}}{\mathrm{Im}\tau}, \\ &\left[\Delta^{(\boldsymbol{\alpha})}, D_z^{(\boldsymbol{\alpha})} \right] = \frac{\pi k_{\boldsymbol{\alpha}}}{\mathrm{Im}\tau} D_z^{(\boldsymbol{\alpha})}, \quad \left[\Delta^{(\boldsymbol{\alpha})}, D_{\bar{z}}^{(\boldsymbol{\alpha})} \right] = -\frac{\pi k_{\boldsymbol{\alpha}}}{\mathrm{Im}\tau} D_{\bar{z}}^{(\boldsymbol{\alpha})}. \end{aligned} \tag{4.3.15}$$

This algebra is similar to the one-dimensional harmonic oscillator in quantum mechanics. These operators are rewritten as

$$\begin{aligned} &\Delta^{(\boldsymbol{\alpha})} = \frac{\pi k_{\boldsymbol{\alpha}}}{\mathrm{Im}\tau}\left(\hat{N}^{(\boldsymbol{\alpha})} + \frac{1}{2} \right), \quad \hat{N}^{(\boldsymbol{\alpha})} \equiv \hat{a}^{(\boldsymbol{\alpha})\dagger} \hat{a}^{(\boldsymbol{\alpha})}, \\ &\hat{a}^{(\boldsymbol{\alpha})\dagger} \equiv i\sqrt{\frac{\mathrm{Im}\tau}{\pi k_{\boldsymbol{\alpha}}}} D_z^{(\boldsymbol{\alpha})}, \quad \hat{a}^{(\boldsymbol{\alpha})} \equiv i\sqrt{\frac{\mathrm{Im}\tau}{\pi k_{\boldsymbol{\alpha}}}} D_{\bar{z}}^{(\boldsymbol{\alpha})}, \quad \left[\hat{a}^{(\boldsymbol{\alpha})}, \hat{a}^{(\boldsymbol{\alpha})\dagger} \right] = 1. \end{aligned} \tag{4.3.16}$$

From (4.3.6) and (4.3.7), we get

$$\Delta^{(\boldsymbol{\alpha})} \tilde{f}_n^{\boldsymbol{\alpha}} = \tilde{m}_n^2 \tilde{f}_n^{\boldsymbol{\alpha}}, \tag{4.3.17}$$

and the KK excited modes are gained by operating $D_z^{(\boldsymbol{\alpha})}$ on (4.3.10) as

$$\tilde{f}_n^{\boldsymbol{\alpha}(j)}(z) \propto \left(D_z^{(\boldsymbol{\alpha})} \right)^n \tilde{f}_0^{\boldsymbol{\alpha}(j)}(z), \tag{4.3.18}$$

with the KK mass:

$$\tilde{m}_n^2 = \frac{\pi k_{\boldsymbol{\alpha}}}{\mathrm{Im}\tau}\left(n + \frac{1}{2} \right) \geq \frac{\pi}{2\mathrm{Im}\tau} > 0 \tag{4.3.19}$$

This means there is no zero-mode in this case.

(ii) $\boldsymbol{k_{\alpha} = 0}$

(4.3.6) ((4.3.7)) becomes

$$\partial_{\bar{z}} \partial_z \tilde{f}_{n,l}^{\boldsymbol{\alpha}} = -\tilde{m}_{n,l}^2 \tilde{f}_{n,l}^{\boldsymbol{\alpha}} \tag{4.3.20}$$

Solutions that satisfy (4.2.44) are

$$\tilde{f}^{\boldsymbol{\alpha}}_{n,l}(z) = \mathcal{N}^{\boldsymbol{\alpha}}_{n,l} \exp\left\{ 2\pi i(n+\phi^{\boldsymbol{\alpha}}_{\tau})\frac{\mathrm{Im}z}{\mathrm{Im}\tau} + 2\pi i(l-\phi^{\boldsymbol{\alpha}}_{1})\frac{\mathrm{Im}(\bar{\tau}z)}{\mathrm{Im}\tau} \right\}, \tag{4.3.21}$$

where $\mathcal{N}^{\boldsymbol{\alpha}}_{n,l}$ is a normalization factor and the KK masses are

$$\tilde{m}_{n,l} = \frac{\pi\left|(n+\phi^{\boldsymbol{\alpha}}_{\tau}) + (l-\phi^{\boldsymbol{\alpha}}_{1})\tau\right|}{\mathrm{Im}\tau}. \tag{4.3.22}$$

Nortice it have the zero-mode only when $\phi^{\boldsymbol{\alpha}}_1,\ \phi^{\boldsymbol{\alpha}}_{\tau} = 0$.

(iii) $\boldsymbol{k_{\alpha} < 0}$

Only (4.3.8) has the zero-mode solutions that satisfy (4.2.44).

$$\tilde{f}^{\boldsymbol{\alpha}}_{0}(z) \equiv \mathcal{N}^{\boldsymbol{\alpha}}_{0} \exp\left(i\pi k_{\boldsymbol{\alpha}}\bar{z}\frac{\mathrm{Im}\bar{z}}{\mathrm{Im}\bar{\tau}} \right) \vartheta \begin{bmatrix} (j+\phi^{\boldsymbol{\alpha}}_{1})/k_{\boldsymbol{\alpha}} \\ -\phi^{\boldsymbol{\alpha}}_{\tau} \end{bmatrix} (k_{\boldsymbol{\alpha}}\bar{z}, k_{\boldsymbol{\alpha}}\bar{\tau}), \tag{4.3.23}$$

This solution has the mass eigenvalue from (4.3.4) $= 0$:

$$\tilde{m}^2_0 = \frac{\pi\left|k_{\boldsymbol{\alpha}}\right|}{2\mathrm{Im}\tau}. \tag{4.3.24}$$

As well as the case **(i)**, we can define the operators like the one-dimensional harmonic oscillator in quantum mechanics by the Laplace operator:

$$\begin{aligned} \Delta^{(\boldsymbol{\alpha})} &= \frac{\pi k_{\boldsymbol{\alpha}}}{\mathrm{Im}\tau}\left(\hat{N}^{(\boldsymbol{\alpha})} + \frac{1}{2} \right), \quad \hat{N}^{(\boldsymbol{\alpha})} \equiv \hat{a}^{(\boldsymbol{\alpha})\dagger}\hat{a}^{(\boldsymbol{\alpha})}, \\ \hat{a}^{(\boldsymbol{\alpha})\dagger} &\equiv i\sqrt{\frac{\mathrm{Im}\tau}{\pi k_{\boldsymbol{\alpha}}}}D^{(\boldsymbol{\alpha})}_{\bar{z}}, \quad \hat{a}^{(\boldsymbol{\alpha})} \equiv i\sqrt{\frac{\mathrm{Im}\tau}{\pi k_{\boldsymbol{\alpha}}}}D^{(\boldsymbol{\alpha})}_{z}, \quad \left[\hat{a}^{(\boldsymbol{\alpha})}, \hat{a}^{(\boldsymbol{\alpha})\dagger}\right] = 1. \end{aligned} \tag{4.3.25}$$

The KK excited modes are gained by operating $D^{(\boldsymbol{\alpha})}_{\bar{z}}$ on (4.3.23) as

$$\tilde{f}^{\boldsymbol{\alpha}(j)}_{n}(z) \propto \left(D^{(\boldsymbol{\alpha})}_{\bar{z}}\right)^n \tilde{f}^{\boldsymbol{\alpha}(j)}_{0}(z), \tag{4.3.26}$$

with the KK mass:

$$\tilde{m}^2_n = \frac{\pi k_{\boldsymbol{\alpha}}}{\mathrm{Im}\tau}\left(n+\frac{1}{2}\right) \geq \frac{\pi}{2\mathrm{Im}\tau} > 0 \tag{4.3.27}$$

This means there is no zero-mode solution in this case.

Higgs sector

Here, we derive the T^2 wave functions of the extra components of the 6D gauge fields.

The KK modes of C_z^i and B_z ($g_n^i(z)$ and $g_n^B(z)$) that satisfy (4.2.44) are the same as the solutions of (4.3.1) and the KK mass eigenvalues are the same as (4.3.2). T^2 zero-mode wave functions of C_z^i and B_z are flat.

Next, we derive the KK modes of $W_z^{\boldsymbol{\alpha}}$ ($g_n^{\boldsymbol{\alpha}}(z)$). The equation of $g_n^{\boldsymbol{\alpha}}(z)$ in (4.2.39) is rewritten as

$$D_{\bar{z}}^{(\boldsymbol{\alpha})} D_z^{(\boldsymbol{\alpha})} \tilde{g}_n^{\boldsymbol{\alpha}} = -\left(\tilde{m}_n^2 + \frac{\pi k_{\boldsymbol{\alpha}}}{\mathrm{Im}\tau}\right) \tilde{g}_n^{\boldsymbol{\alpha}} \tag{4.3.28}$$

$$\left(D_z^{(\boldsymbol{\alpha})} D_{\bar{z}}^{(\boldsymbol{\alpha})} \tilde{g}_n^{\boldsymbol{\alpha}} = -\tilde{m}_n^2 \tilde{g}_n^{\boldsymbol{\alpha}}\right). \tag{4.3.29}$$

For the zero-mode, the solutions of (4.3.28) or (4.3.29) must satisfy

$$D_z^{(\boldsymbol{\alpha})} \tilde{g}_0^{\boldsymbol{\alpha}} = 0, \tag{4.3.30}$$

or

$$D_{\bar{z}}^{(\boldsymbol{\alpha})} \tilde{g}_0^{\boldsymbol{\alpha}} = 0. \tag{4.3.31}$$

(i) $\boldsymbol{k_{\alpha} > 0}$

Only (4.3.29)$= 0$ has a solution, and zero-mode wavefunction is

$$\tilde{g}_0^{\boldsymbol{\alpha}}(z) \equiv \mathcal{N}_0^{\boldsymbol{\alpha}} \exp\left(i\pi k_{\boldsymbol{\alpha}} z \frac{\mathrm{Im}z}{\mathrm{Im}\tau}\right) \vartheta \begin{bmatrix} (j+\phi_1^{\boldsymbol{\alpha}})/k_{\boldsymbol{\alpha}} \\ -\phi_\tau^{\boldsymbol{\alpha}} \end{bmatrix} (k_{\boldsymbol{\alpha}} z, k_{\boldsymbol{\alpha}} \tau), \tag{4.3.32}$$

where $j = 1, 2, \cdots, k_{\boldsymbol{\alpha}}$. This solution is gained from (4.3.31), so has massless mode:

$$\tilde{m}_0^2 = 0, \tag{4.3.33}$$

and KK mode functions are obtained by operating creation operator $D_z^{(\boldsymbol{\alpha})}$ ((4.3.6)) on (4.3.32):

$$\tilde{g}_n^{\boldsymbol{\alpha}(j)}(z) \propto D_z^{(\boldsymbol{\alpha})} \tilde{g}_0^{\boldsymbol{\alpha}(j)}(z), \tag{4.3.34}$$

and mass eigenvalue is

$$\tilde{m}_n^2 = \frac{n\pi k_{\boldsymbol{\alpha}}}{\mathrm{Im}\tau}. \tag{4.3.35}$$

(ii) $\boldsymbol{k_\alpha = 0}$

This case is the same as gauge sector, so KK wavefunction is (4.3.21) and KK mass eigenvalue is (4.3.22). There is massless mode only when $\phi_1^{\boldsymbol{\alpha}}, \phi_\tau^{\boldsymbol{\alpha}} = 0$.

(iii) $\boldsymbol{k_\alpha < 0}$

Only (4.3.28)= 0 has a solution, and zero-mode wavefunction is

$$\tilde{g}_0^{\boldsymbol{\alpha}}(z) \equiv \mathcal{N}_0^{\boldsymbol{\alpha}} \exp\left(i\pi k_{\boldsymbol{\alpha}} \bar{z} \frac{\mathrm{Im}\bar{z}}{\mathrm{Im}\bar{\tau}}\right) \vartheta \begin{bmatrix} (j+\phi_1^{\boldsymbol{\alpha}})/k_{\boldsymbol{\alpha}} \\ -\phi_{\bar{\tau}}^{\boldsymbol{\alpha}} \end{bmatrix} (k_{\boldsymbol{\alpha}}\bar{z}, k_{\boldsymbol{\alpha}}\bar{\tau}), \tag{4.3.36}$$

where $j = 1, 2, \cdots, k_{\boldsymbol{\alpha}}$. This solution is gained from (4.3.30), so mass eigenvalue is

$$\tilde{m}_0^2 = -\frac{\pi k_{\boldsymbol{\alpha}}}{\mathrm{Im}\tau}, \tag{4.3.37}$$

so does not have massless mode. KK mode functions are obtained by operating $D_{\bar{z}}^{(\boldsymbol{\alpha})}$ on (4.3.36):

$$\tilde{g}_n^{\boldsymbol{\alpha}(j)}(z) \propto D_{\bar{z}}^{(\boldsymbol{\alpha})} \tilde{g}_0^{\boldsymbol{\alpha}(j)}(z), \tag{4.3.38}$$

and mass eigenvalue is

$$\tilde{m}_n^2 = -\frac{(n+1)\pi k_{\boldsymbol{\alpha}}}{\mathrm{Im}\tau}. \tag{4.3.39}$$

Fermion sector

$$\left(D_z^{(\boldsymbol{\mu}f)} D_{\bar{z}}^{(\boldsymbol{\mu}f)} - \tilde{M}_f^2\right) \tilde{h}_{\mathrm{R}n}^{(+)\boldsymbol{\mu}} = -\left|\tilde{m}_n^2\right| \tilde{h}_{\mathrm{R}n}^{(+)\boldsymbol{\mu}}, \tag{4.3.40}$$

$$\left(D_{\bar{z}}^{(\boldsymbol{\mu}f)} D_z^{(\boldsymbol{\mu}f)} - \tilde{M}_f^2\right) \tilde{h}_{\mathrm{R}n}^{(-)\boldsymbol{\mu}} = -\left|\tilde{m}_n^2\right| \tilde{h}_{\mathrm{R}n}^{(-)\boldsymbol{\mu}}, \tag{4.3.41}$$

$$\left(D_{\bar{z}}^{(\boldsymbol{\mu}f)} D_z^{(\boldsymbol{\mu}f)} - \tilde{M}_f^2\right) \tilde{h}_{\mathrm{L}n}^{(+)\boldsymbol{\mu}} = -\left|\tilde{m}_n^2\right| \tilde{h}_{\mathrm{L}n}^{(+)\boldsymbol{\mu}}, \tag{4.3.42}$$

$$\left(D_z^{(\boldsymbol{\mu}f)} D_{\bar{z}}^{(\boldsymbol{\mu}f)} - \tilde{M}_f^2\right) \tilde{h}_{\mathrm{L}n}^{(-)\boldsymbol{\mu}} = -\left|\tilde{m}_n^2\right| \tilde{h}_{\mathrm{L}n}^{(-)\boldsymbol{\mu}}. \tag{4.3.43}$$

The first (second) and forth (third) conditions are the same with the same boundary condition, so solutions satisfy

$$\tilde{h}_{\mathrm{R}n}^{(+)\boldsymbol{\mu}} = e^{i\delta_1} \tilde{h}_{\mathrm{L}n}^{(-)\boldsymbol{\mu}}, \tag{4.3.44}$$

$$\tilde{h}_{\mathrm{R}n}^{(-)\boldsymbol{\mu}} = e^{i\delta_2} \tilde{h}_{\mathrm{L}n}^{(+)\boldsymbol{\mu}} \tag{4.3.45}$$

Where δ_1, δ_2 are phases of real constants. The zero-mode solutions of these equations must satisfy

$$D_{\bar{z}}^{(\boldsymbol{\mu}f)} \tilde{h}_{\mathrm{R}0}^{(+)\boldsymbol{\mu}} = 0, \tag{4.3.46}$$

$$D_{z}^{(\boldsymbol{\mu}f)} \tilde{h}_{\mathrm{R}0}^{(-)\boldsymbol{\mu}} = 0, \tag{4.3.47}$$

$$D_{z}^{(\boldsymbol{\mu}f)} \tilde{h}_{\mathrm{L}0}^{(+)\boldsymbol{\mu}} = 0, \tag{4.3.48}$$

$$D_{\bar{z}}^{(\boldsymbol{\mu}f)} \tilde{h}_{\mathrm{L}0}^{(+)\boldsymbol{\mu}} = 0, \tag{4.3.49}$$

and corresponding mass eigenvalue is $|\tilde{m}_0| = \tilde{M}_f^2$. Thus, there is no zero-mode when $M_f^2 \neq 0$. So, I will not introduce fermion bulk mass term when discussing zero-modes.

(i) $\boldsymbol{k_{\mu f} > 0}$

(4.3.40) and (4.3.43) have solutions:

$$\begin{aligned}\tilde{h}_{\mathrm{R}0}^{(+)\boldsymbol{\mu}(j)}(z) &= e^{i\delta_1} \tilde{h}_{\mathrm{L}0}^{(-)\boldsymbol{\mu}(j)}(z) \\ &= \mathcal{N}_0^{\boldsymbol{\mu}f} \exp\left(i\pi k_{\boldsymbol{\mu}f} z \frac{\mathrm{Im} z}{\mathrm{Im}\tau} \right) \vartheta \begin{bmatrix} (j + \phi_1^{\boldsymbol{\mu}f})/k_{\boldsymbol{\mu}f} \\ -\phi_\tau^{\boldsymbol{\mu}f} \end{bmatrix} (k_{\boldsymbol{\mu}f} z, k_{\boldsymbol{\mu}f}\tau),\end{aligned} \tag{4.3.50}$$

where $j = 1, 2, \cdots, k_{\boldsymbol{\mu}f}$, and $\mathcal{N}_0^{\boldsymbol{\mu}f}$ is a normalization factor.

KK excited modes are obtained by operating $D_z^{(\boldsymbol{\mu}f)}$ ((4.2.42)) on (4.3.57):

$$\tilde{h}_{\mathrm{R}n}^{(+)\boldsymbol{\mu}(j)}(z) = e^{i\delta_1} \tilde{h}_{\mathrm{L}n}^{(-)\boldsymbol{\mu}(j)}(z) \propto \left(D_z^{(\boldsymbol{\mu}f)} \right)^n \tilde{h}_{\mathrm{R}0}^{(+)\boldsymbol{\mu}(j)}(z), \tag{4.3.51}$$

with the KK mass:

$$|\tilde{m}_n|^2 = \tilde{M}_f^2 + \frac{n\pi k_{\boldsymbol{\mu}f}}{\mathrm{Im}\tau}. \tag{4.3.52}$$

From (4.2.39), we find

$$\begin{aligned}\tilde{h}_{\mathrm{R}n}^{(-)\boldsymbol{\mu}(j)}(z) &= \frac{1}{\tilde{M}_f - e^{-i\delta_2}\tilde{m}_n} D_{\bar{z}}^{(\boldsymbol{\mu}f)} \tilde{h}_{\mathrm{R}n}^{(+)\boldsymbol{\mu}(j)}(z) \\ &= \frac{1}{-e^{-i\delta_2}\tilde{M}_f + \tilde{m}_n} D_{\bar{z}}^{(\boldsymbol{\mu}f)} \tilde{h}_{\mathrm{L}n}^{(-)\boldsymbol{\mu}(j)}(z).\end{aligned} \tag{4.3.53}$$

(ii) $\boldsymbol{k_{\mu f} = 0}$

In this case, tequations (4.3.40) $\sim$ (4.3.43) are same:

$$\partial_{\bar{z}} \partial_z \tilde{h}_{n,l} = \left(\tilde{M}_{n,l}^2 - |\tilde{m}_n|^2 \right), \tag{4.3.54}$$

where $\tilde{h}_{n,l} = \tilde{h}^{(\pm)\boldsymbol{\mu}}_{\mathrm{R(L)}n,l}$. The solutions that satisfy (4.2.44) are

$$\tilde{h}^{\boldsymbol{\mu}f}_{n,l} = \mathcal{N}^{\boldsymbol{\mu}}_{n,l} \exp\left\{2\pi i(n+\phi^{\boldsymbol{\mu}f}_{\tau})\frac{\mathrm{Im}z}{\mathrm{Im}\tau} + 2\pi i(l-\phi^{\boldsymbol{\mu}f}_{1})\frac{\mathrm{Im}(\bar{\tau}z)}{\mathrm{Im}\tau}\right\}, \tag{4.3.55}$$

where $\mathcal{N}^{\boldsymbol{\mu}}_{n,l}$ are normalization factors with the KK masses:

$$|\tilde{m}_{n,l}|^2 = \tilde{M}_f^2 \left|\frac{\pi(n+\phi^{\boldsymbol{\mu}f}_{\tau}) + \pi(l-\phi^{\boldsymbol{\mu}f}_{1})\tau}{\mathrm{Im}\tau}\right|^2. \tag{4.3.56}$$

Nortice it have the zero-mode only when $\phi^{\boldsymbol{\mu}}_1, \phi^{\boldsymbol{\mu}}_{\tau} = 0$.

(iii) $\boldsymbol{k_{\mu f} < 0}$

(4.3.41) and (4.3.42) have solutions:

$$\begin{aligned}\tilde{h}^{(-)\boldsymbol{\mu}(j)}_{\mathrm{R}0}(z) &= e^{i\delta_2}\tilde{h}^{(+)\boldsymbol{\mu}(j)}_{\mathrm{L}0}(z) \\ &= \mathcal{N}^{\boldsymbol{\mu}f}_0 \exp\left(i\pi k_{\boldsymbol{\mu}f}\bar{z}\frac{\mathrm{Im}\bar{z}}{\mathrm{Im}\bar{\tau}}\right)\vartheta\begin{bmatrix}(j+\phi^{\boldsymbol{\mu}f}_1)/k_{\boldsymbol{\mu}f} \\ -\phi^{\boldsymbol{\mu}f}_{\tau}\end{bmatrix}(k_{\boldsymbol{\mu}f}\bar{z}, k_{\boldsymbol{\mu}f}\bar{\tau}),\end{aligned} \tag{4.3.57}$$

where $j = 1, 2, \cdots, k_{\boldsymbol{\mu}f}$, and $\mathcal{N}^{\boldsymbol{\mu}f}_0$ is a normalization factor.

KK excited modes are obtained by operating $D^{(\boldsymbol{\mu}f)}_{z}$ ((4.2.42)) on (4.3.57):

$$\tilde{h}^{(-)\boldsymbol{\mu}(j)}_{\mathrm{R}n}(z) = e^{i\delta_1}\tilde{h}^{(+)\boldsymbol{\mu}(j)}_{\mathrm{L}n}(z) \propto \left(D^{(\boldsymbol{\mu}f)}_{\bar{z}}\right)^n \tilde{h}^{(-)\boldsymbol{\mu}(j)}_{\mathrm{R}0}(z), \tag{4.3.58}$$

with the KK mass:

$$|\tilde{m}_n|^2 = \tilde{M}_f^2 + \frac{n\pi k_{\boldsymbol{\mu}f}}{\mathrm{Im}\tau}. \tag{4.3.59}$$

From (4.2.39), we find

$$\begin{aligned}\tilde{h}^{(+)\boldsymbol{\mu}(j)}_{\mathrm{R}n}(z) &= \frac{1}{\tilde{M}_f - e^{-i\delta_1}\tilde{m}_n} D^{(\boldsymbol{\mu}f)}_{z}\tilde{h}^{(-)\boldsymbol{\mu}(j)}_{\mathrm{R}n}(z) \\ &= \frac{1}{-e^{-i\delta_1}\tilde{M}_f + \tilde{m}_n} D^{(\boldsymbol{\mu}f)}_{z}\tilde{h}^{(+)\boldsymbol{\mu}(j)}_{\mathrm{L}n}(z).\end{aligned} \tag{4.3.60}$$

4.3.2 T^2/Z_N wave functions

In this theory, we compactify the extra dimensions by T^2/Z_N, so we need to evaluate the 4D effective theory by the T^2/Z_N orbifold wave functions. The orbifold wave functions that satisfy the boundary conditions

$$\hat{F}_n(\omega z) = \eta' \hat{F}_n(z), \tag{4.3.61}$$

can be obtained by the T^2 wave functions as

$$\hat{F}_n(z) = \frac{1}{N}\sum_{k=0}^{N-1} \eta'^{-k} \tilde{F}_n(\omega^k z), \tag{4.3.62}$$

where $\tilde{F}_n(z)$ are T^2 wave functions. Now we introduce magnetic fluxes, so the zero-mode wave functions are not always flat, and we can choose η', the phase of Z_N transformation, for the zero-modes in the region: $\eta' = 1, \omega, \omega^2, \cdots \omega^{N-1}$ generally. This determines the twisted boundary conditions of zero-modes. Zero-mode wave functions that satisfy the boundary condition $\hat{F}_0(\omega z) = \eta \hat{F}_0(z)$ $(\eta = 1, \omega, \cdots, \omega^{N-1})$ are also expressed as

$$\hat{F}_0(z) = \frac{1}{N}\sum_{k=0}^{N-1} \eta^{-k} \tilde{F}_0(\omega^k z). \tag{4.3.63}$$

Orbifold zero-mode wave functions are also expressed as a linear combination of torus zero-mode wave functions:

$$\hat{F}_0^{(j)}(z) = \sum_{i=1}^{|K|} C_{ji}^{(\eta)} \tilde{F}_0^{(i)}(z), \tag{4.3.64}$$

where i, j mean physical state indices of $\hat{F}_0$ or $\tilde{F}_0$ that run from 1 to $|K|$, for example correspond to flavors of fermions, and $C^{(\eta)}$ is a constant $|K| \times |K|$ matrix that mix $|i\rangle, |j\rangle$ states. The constants $C_{ji}^{(\eta)}$ are evaluated as

$$C_{ji}^{(\eta)} = \int_{T^2} d^2z \left\{ \tilde{F}_0^{(j)}(z) \right\}^* \hat{F}_0^{(i)}(z). \tag{4.3.65}$$

$\tilde{F}_0^{(j)}(\omega^l z)$ is a solution of mode equations of $\tilde{F}_0^{(j)}(z)$ that satisfies T^2 boundary conditions of $\tilde{F}_0^{(j)}(z)$, so it can be also expressed as a linear combination of $\tilde{F}_0^{(j)}(z)$:

$$\tilde{F}_0^{(j)}(\omega^l z) = \sum_{i=1}^{|K|} D_{ji}^{(\omega^l)} \tilde{F}_0^{(i)}(z), \tag{4.3.66}$$

where $D_{ji}^{(\omega^l)}$ are constants. Thus, $C_{ji}^{(\eta)}$ are expressed as

$$C_{ji}^{(\eta)} \equiv \frac{1}{N}\sum_{l=0}^{N-1} \eta^{-l} D_{ji}^{(\omega^l)} \tag{4.3.67}$$

At a glance, $|K|$ physical states of $\hat{F}_0$ seem to degenerate in (4.3.64), but not all of them are independent. In fact, the matrix $C^{(\eta)}$ includes zero eigenvalues generally. The number of physical states of $\hat{F}_0$ equals to the rank of $C^{(\eta)}$ that is evaluated analytically by the method of the quantum mechanics (described in detail by [40][2]) or numerically. Numerlically, we can check the number of nonzero eigenvalues of $C^{(\eta)}$ Here, the matrix $C^{(\eta)}$ is hermitian because

$$C^{(\eta)\dagger} = \frac{1}{N}\sum_{l=0}^{N-1} \eta^l D^{(\bar{\omega}^l)\dagger} = \frac{1}{N}\sum_{l=0}^{N-1} \eta^{-l'} D^{(\omega^{l'})} = C^{(\eta)}, \tag{4.3.68}$$

where $l' = -l$ Thus, we can diagonalize $C^{(\eta)}$ with a unitary matrix $V^{(\eta)}$ as

$$V^{(\eta)}C^{(\eta)}V^{(\eta)\dagger} = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_r, 0, \cdots, 0), \tag{4.3.69}$$

where $|\lambda_1| \geq |\lambda_1| \geq \cdots |\lambda_r| \geq 0$ and r is the rank of $C^{(\eta)}$. Then, we find

$$\sum_{i=1}^{|K|} V_{ji}^{(\eta)} \hat{F}_0^{(i)}(z) = \begin{cases} \lambda_j \sum_i V_{ji}^{(\eta)} \tilde{F}_0^{(i)}(z), & (1 \leq j \leq r) \\ 0. & (r+1 \leq j \leq |K|) \end{cases} \tag{4.3.70}$$

So it is convenient if we change the T^2/Z_N wavefunction's basis to those which is linearly independent for all j:

$$F_0^{(j)}(z) \equiv \sqrt{N}\sum_{i=1}^{|K|} V_{ji}^{(\eta)} \tilde{F}_0^{(j)}, \tag{4.3.71}$$

where $j = 1, 2, \cdots, r$. We find this basis satisfies the orthonormal condition:

$$\begin{aligned} \int_{T^2/Z_N} d^2z \left\{F_0^{(i)}(z)\right\}^* F_0^{(j)}(z) &= \frac{1}{N}\int_{T^2} d^2z \left\{F_0^{(i)}\right\}^* F_0^{(j)} \\ &= \int_{T^2} d^2z \left\{\sum_k V_{ik}^{(\eta)} \tilde{F}_0^{(k)}\right\}^* \left\{\sum_l V_{jl}^{(\eta)} \tilde{F}_n^{(l)}\right\} \\ &= \sum_{k,l} V_{ik}^{(\eta)*} V_{jl}^{(\eta)} \delta_{kl} = (V^{(\eta)}V^{(\eta)\dagger})_{ji} = \delta_{ij}, \end{aligned} \tag{4.3.72}$$

if T^2 wave functions satisfy the orthonormal condition:

$$\int_{T^2} d^2z \left\{\tilde{F}_0^{(k)}(z)\right\}^* \tilde{F}_0^{(l)}(z) = \delta_{kl}. \tag{4.3.73}$$

The T^2/Z_N wave functions of the KK modes are obtained by operating $D_z = D_z^{(\boldsymbol{\alpha})}, D_z^{(\boldsymbol{\mu} f)}$ or $D_{\bar{z}} = D_{\bar{z}}^{(\boldsymbol{\alpha})}, D_{\bar{z}}^{(\boldsymbol{\mu} f)}$ on $F_0^{(j)}(z)$, in the same with those of T^2 wave functions. Here, the first excited mode is expressed as

$$\begin{aligned} D_z\left(\tilde{F}_0^{(j)}(\omega^l z)\right) &= \omega^l \left(D_z \tilde{F}_0^{(j)}\right)(\omega^l z) \propto \omega^l \tilde{F}_1(\omega^l z), \quad (\text{for } K > 0) \\ D_{\bar{z}}\left(\tilde{F}_0^{(j)}(\omega^l z)\right) &= \bar{\omega}^l \left(D_{\bar{z}} \tilde{F}_0^{(j)}\right)(\omega^l z) \propto \omega^l \tilde{F}_1(\bar{\omega}^l z), \quad (\text{for } K < 0) \end{aligned} \tag{4.3.74}$$

[2]Analytic forms of $C^{(\eta)}$ are derived using operator foamlism in [40].

so the Z_N twist phase η in $C^{(\eta)}$ changes into $\eta\omega^{-1}$ for $K>0$, or $\eta\omega$ for $K<0$. Therefore, the n-th excited modes are expressed as

$$F_n^{(j)}(z) = \begin{cases} \sqrt{N}\sum_{k=1}^{K} V_{ji}^{(\eta\omega^{-n})}\tilde{F}_n^{(k)}(z) & (\text{for } K>0) \\ \sqrt{N}\sum_{k=1}^{|K|} V_{ji}^{(\eta\omega^{n})}\tilde{F}_n^{(k)}(z) & (\text{for } K<0) \end{cases}. \tag{4.3.75}$$

The number of mass eigenstates at the n-th KK level is the rank of $C^{(\eta\omega^{-n})}$ for $K>0$, or the rank of $C^{(\eta\omega^{n})}$ for $K<0$.

The constants $D_{ji}^{(\omega^l)}$ are functions of K and $\zeta = \frac{2}{K}(\tau\phi_1 - \phi_\tau)$, and satisfy

$$D_{ji}^{(\omega^l)}[-K,\zeta] = D_{ij}^{(\bar{\omega}^l)}[K,\zeta]. \tag{4.3.76}$$

Thus, we find

$$\begin{aligned} C_{ji}^{(\eta)}[-K,\zeta] &= \frac{1}{N}\sum_{l=0}^{N-1}\bar{\eta}^l D_{ji}^{(\omega^l)}[-K,\zeta] = \frac{1}{N}\sum_{l=0}^{N-1}\bar{\eta}^l D_{ij}^{(\bar{\omega}^l)}[K,\zeta] \\ &= \frac{1}{N}\sum_{l'=0}^{N-1}\bar{\eta}^{-l'} D_{ji}^{(\omega^{-l'})}[K,\zeta] = C_{ij}^{(\bar{\eta})}[K,\zeta], \end{aligned} \tag{4.3.77}$$

where $l' \equiv -l$. This indicates that the degeneration number of the zero-modes of a field that feels the flux $K<0$ and the Z_N twist phase η are equal to that of a field that feels a flux $|K|$ and the Z_N twist phase $\bar{\eta}$.

In this section, we discuss the T^2/Z_N wave functions of the zero-modes (masless) and the KK modes (massive) for 4D gauge, Higgs, fermion, and the flux conditions for zero-modes.

4D gauge sector

We discuss the zero-mode wave functions of the 4D gauge fields on T^2/Z_N. The gauge fields C_μ^i, B_μ do not feel the magnetic fluxes and the zero-mode wave functions are flat, so we can obtain their zero-mode wave functions $f_{0,0}^i(z)$ and $f_{0,0}^B(z)$ on T^2/Z_N from (4.3.63):

$$\begin{aligned} \hat{f}_{0,0}^i(z) &= \frac{1}{N}\sum_{l'=0}^{N-1}\left\{1^{-l'}\cdot\tilde{f}_{0,0}^i(z)\right\} = \mathcal{N}_{0,0}^{ci}, \\ \hat{f}_{0,0}^B(z) &= \frac{1}{N}\sum_{l'=0}^{N-1}\left\{1^{-l'}\cdot\tilde{f}_{0,0}^B(z)\right\} = \mathcal{N}_{0,0}^{cB}, \end{aligned} \tag{4.3.78}$$

where $\tilde{f}^i_{0,0}(z)$, $\tilde{f}^B_{0,0}(z)$ is the n and $l = 0$ wave function in (4.3.1), respectively.

On the other hand, the gauge field $W^{\boldsymbol{\alpha}}_\mu$ may feel the magnetic fluxes. However, as we have seen in (4.3.19) or (4.3.27), $W^{\boldsymbol{\alpha}}_\mu$ have no zero-mode solution if the values of the magnetic fluxes they feel are not zero ($k_{\boldsymbol{\alpha}} \neq 0$). Then, we must consider such the background magnetic fluxes $\mathcal{C}^i$ and $\mathcal{B}$ that the fields $W^{\boldsymbol{\alpha}_L}_\mu$ do not feel in the following discussion for the gauge symmetry breaking $G \to SU(2)_L \times U(1)_Z$, where $\boldsymbol{\alpha}_L$ is one of the G root vectors corresponding to the generators (of the non-Cartan components) for the $SU(2)_L$ subgroup. We can obtain the zero-mode wave functions from (4.3.63):

$$\hat{f}^{\boldsymbol{\alpha}}_{0,0}(z) = \frac{1}{N} \sum_{l'=0}^{N-1} 1^{-l'} \cdot \tilde{f}^{\boldsymbol{\alpha}}_{0,0}(z) = \mathcal{N}^{\boldsymbol{\alpha}}_{0,0}, \tag{4.3.79}$$

where $\tilde{f}^{\boldsymbol{\alpha}}_{0,0}(z)$ is the n and $l = 0$ wave function in (4.3.21).

Higgs sector

As we have seen in (4.2.46), $\eta = \omega^{-1}$ for the orbifold wave functions of C^i_z. T^2 zero-mode functions of C^i_z is flat , so T^2/Z_N zero-mode function of it is written as

$$\begin{aligned} g^i_{0,0}(z) &= \tilde{g}^i_{0,0} \sum_{k=0}^{N-1} (\omega^{-1})^{-k} \\ &= 0. \end{aligned} \tag{4.3.80}$$

Thus orbifold wave functions of C^i_z cannot have a zero-mode solution. The same is the case with B_z.

The Higgs field is included in the zero-mode of $W^{\boldsymbol{\alpha}}_z$. Now, we see whether $W^{\boldsymbol{\alpha}}_z$ has a zero-mode when $k_{\boldsymbol{\alpha}}$ is positive, zero, or negative.

(i) $\boldsymbol{k_{\alpha} > 0}$

The orbifold wavefunction (defined by (4.3.62)) that satisfies (4.2.46) is written as

$$\hat{g}^{\boldsymbol{\alpha}(j)}_n(z) \equiv \frac{1}{N} \sum_{k=0}^{N-1} \left(\omega^{-1} e^{ip\cdot\boldsymbol{\alpha}}\right)^{-k} \tilde{g}^{\boldsymbol{\alpha}(j)}_n(\omega^k z), \tag{4.3.81}$$

where $\tilde{g}^{\boldsymbol{\alpha}(j)}_n(z)$ is defined by (4.3.34). When we rewrite $\hat{g}^{\boldsymbol{\alpha}(j)}_n(z)$ as

$$\hat{g}^{\boldsymbol{\alpha}(j)}_n(z) = \sum_{i=1}^{k_{\boldsymbol{\alpha}}} C^{(\eta\omega^{-n})}_{ji} \tilde{g}^{\boldsymbol{\alpha}(i)}_n, \tag{4.3.82}$$

we can change to the independent basis

$$g_n^{\boldsymbol{\alpha}(j)} \equiv \sqrt{N} \sum_{i=1}^{k_{\boldsymbol{\alpha}}} V_{ji}^{(\eta\omega^{-n})} \tilde{g}_n^{\boldsymbol{\alpha}(i)}, \tag{4.3.83}$$

where $V^{(\eta)}$ is the diagonalizing matrix of $C^{(\eta)}$. As we have seen, $\tilde{g}_n^{\boldsymbol{\alpha}(j)}$ satisfies the orthonoramal condition:

$$\int_{T^2} d^2z \left\{\tilde{g}_n^{\boldsymbol{\alpha}(k)}\right\}^* \tilde{g}_n^{\boldsymbol{\alpha}(l)} = \delta_{kl}, \tag{4.3.84}$$

so $g_n^{\boldsymbol{\alpha}(j)}$ also satisfies the orthonormal condition from (4.3.72). The zero-mode wavefunction on T^2/Z_N is rewritten as

$$g_0^{\boldsymbol{\alpha}(j)}(z) = \mathcal{N}_0^{\boldsymbol{\alpha}} \sqrt{N} \sum_{i=1}^{k_{\boldsymbol{\alpha}}} V_{ji}^{(\eta)} \exp\left(i\pi k_{\boldsymbol{\alpha}} z \frac{\mathrm{Im}z}{\mathrm{Im}\tau}\right) \vartheta \begin{bmatrix} (i+\phi_1^{\boldsymbol{\alpha}})/k_{\boldsymbol{\alpha}} \\ -\phi_\tau^{\boldsymbol{\alpha}} \end{bmatrix} (k_{\boldsymbol{\alpha}} z, k_{\boldsymbol{\alpha}} \tau). \tag{4.3.85}$$

(ii) $\boldsymbol{k_{\alpha} = 0}$

In this case, we have seen $\tilde{g}_n^{\boldsymbol{\alpha}}(z)$ have the zero-mode only when $\phi_1^{\boldsymbol{\alpha}}$, $\phi_\tau^{\boldsymbol{\alpha}} = 0$, and the zero-mode wavefunction is a constant. The T^2/Z_N zero-mode wave function is

$$\begin{aligned} \hat{g}_{0,0}^{\boldsymbol{\alpha}}(z) &= \frac{\mathcal{N}_{0,0}^{\boldsymbol{\alpha}}}{N} \sum_{k=0}^{N-1} (\omega^{-1} e^{ip\cdot\boldsymbol{\alpha}})^{-k} \\ &= \frac{\mathcal{N}_{0,0}^{\boldsymbol{\alpha}}}{N} \sum_{k=0}^{N-1} \exp\left\{\frac{2(1-n_{\boldsymbol{\alpha}})\pi i}{N} k\right\}. \end{aligned} \tag{4.3.86}$$

The value of this function become zero unless $n_{\boldsymbol{\alpha}} = 1 \pmod N$. Thus it has the zero-mode only when $n_{\boldsymbol{\alpha}} = 1 \pmod N$:

$$\hat{g}_{0,0}^{\boldsymbol{\alpha}}(z) = \mathcal{N}_{0,0}^{\boldsymbol{\alpha}}. \tag{4.3.87}$$

(iii) $\boldsymbol{k_{\alpha} < 0}$

There is no zero-mode in this case.

Fermion sector

(i) $\boldsymbol{k_{\mu f}>0}$

The zero-mode wavefunction is expressed as

$$\begin{aligned}\hat{h}_{\mathrm{R0}}^{(+)\boldsymbol{\mu}(j)} &= e^{i\delta_1}\hat{h}_{\mathrm{L0}}^{(-)\boldsymbol{\mu}(j)}(z)\\ &\equiv \frac{1}{N}\sum_{k=0}^{N-1}\left(\omega^{-\frac{1}{2}}e^{iq_f}\varphi_\omega e^{ip\cdot\boldsymbol{\mu}}\right)^{-k}\tilde{h}_{\mathrm{R0}}^{(+)\boldsymbol{\mu}(j)}(\omega^k z)\\ &= \frac{1}{N}\sum_{k=0}^{N-1}\exp\left(-\frac{2n_{\boldsymbol{\mu}f}^{+}\pi i}{N}k\right)\tilde{h}_{\mathrm{R0}}^{(+)\boldsymbol{\mu}(j)}(\omega^k z),\end{aligned} \tag{4.3.88}$$

where $j=1,2,\cdots,k_{\boldsymbol{\mu}f}$ and $n_{\boldsymbol{\mu}f}^{+}$ is defined in (4.2.9).

We change the orthonormal basis:

$$\begin{aligned}h_{\mathrm{R0}}^{(+)\boldsymbol{\mu}(j)}(z) &= e^{i\delta_1}h_{\mathrm{L0}}^{(-)\boldsymbol{\mu}(j)}(z)\\ &\equiv \sqrt{N}\sum_{i=1}^{k_{\boldsymbol{\mu}f}}V_{ji}^{(\eta)}\tilde{h}_{\mathrm{R0}}^{(+)\boldsymbol{\mu}(i)}\\ &= \mathcal{N}_0^{\boldsymbol{\mu}}\sqrt{N}\sum_{i=1}^{k_{\boldsymbol{\alpha}}}V_{ji}^{(\eta)}\exp\left(i\pi k_{\boldsymbol{\mu}f}z\frac{\mathrm{Im}z}{\mathrm{Im}\tau}\right)\vartheta\begin{bmatrix}(j+\phi_1^{\boldsymbol{\mu}f})/k_{\boldsymbol{\mu}f}\\ -\phi_\tau^{\boldsymbol{\mu}f}\end{bmatrix}(k_{\boldsymbol{\mu}f}z,k_{\boldsymbol{\mu}f}\tau).\end{aligned} \tag{4.3.89}$$

where $V^{(\eta)}$ is the diagonlizing matrix of

$$C_{ij}^{(\eta)} \equiv \int_{T^2}d^2z\left\{\tilde{h}_{\mathrm{R0}}^{(+)\boldsymbol{\mu}(j)}(z)\right\}^*\hat{h}_{\mathrm{R0}}^{(+)\boldsymbol{\mu}(i)}(z). \tag{4.3.90}$$

j in (4.3.89) runs from 1 to r (rank of $C^{(\eta)}$). The mass eigenvalue is $|\tilde{m}_0|=\left|\tilde{M}_f\right|$, so there is the massless mode only when $\tilde{M}_f=0$.

(ii) $\boldsymbol{k_{\mu f}=0}$

The zero-mode function satisfies

$$\hat{h}_{\mathrm{R0},0}^{(\pm)\boldsymbol{\mu}}(z),\hat{h}_{\mathrm{L0},0}^{\mp\boldsymbol{\alpha}}(z)\propto\sum_{k=0}^{N-1}\exp-\frac{2n_{\boldsymbol{\mu}f}^{\pm}\pi i}{N}k. \tag{4.3.91}$$

This mode exists only when $n_{\boldsymbol{\mu}f}^{\pm}=0$ (mod N).

(iii) $\boldsymbol{k_{\mu f} < 0}$

The zero-mode function is expressed as

$$\begin{aligned}
\hat{h}_{\mathrm{L0}}^{(+)\boldsymbol{\mu}(j)} &= e^{-i\delta_2}\hat{h}_{\mathrm{R0}}^{(-)\boldsymbol{\mu}(j)}(z) \\
&\equiv \frac{1}{N}\sum_{k=0}^{N-1}\left(\omega^{-\frac{1}{2}}e^{iq_f}\varphi_\omega e^{ip\cdot\boldsymbol{\mu}}\right)^{-k}\tilde{h}_{\mathrm{L0}}^{(+)\boldsymbol{\mu}(j)}(\omega^k z) \\
&= \frac{1}{N}\sum_{k=0}^{N-1}\exp\left(-\frac{2n_{\boldsymbol{\mu}f}^{-}\pi i}{N}k\right)\tilde{h}_{\mathrm{L0}}^{(+)\boldsymbol{\mu}(j)}(\omega^k z).
\end{aligned} \tag{4.3.92}$$

where $j = 1, 2, \cdots, k_{\boldsymbol{\mu}f}$ and $n_{\boldsymbol{\mu}f}^{-}$ is defined in (4.2.9).

We change the orthonormal basis:

$$\begin{aligned}
h_{\mathrm{L0}}^{(+)\boldsymbol{\mu}(j)}(z) &= e^{-i\delta_2}h_{\mathrm{R0}}^{(-)\boldsymbol{\mu}(j)}(z) \\
&\equiv \sqrt{N}\sum_{i=1}^{k_{\boldsymbol{\mu}f}}V_{ji}^{(\eta)}\tilde{h}_{\mathrm{L0}}^{(+)\boldsymbol{\mu}(i)}, \\
&= \mathcal{N}_0^{\boldsymbol{\mu}}\sqrt{N}\sum_{i=1}^{k_{\boldsymbol{\alpha}}}V_{ji}^{(\eta)}\exp\left(i\pi k_{\boldsymbol{\mu}f}z\frac{\mathrm{Im}\bar{z}}{\mathrm{Im}\bar{\tau}}\right)\vartheta\begin{bmatrix}(j+\phi_1^{\boldsymbol{\mu}f})/k_{\boldsymbol{\mu}f} \\ -\phi_\tau^{\boldsymbol{\mu}f}\end{bmatrix}(k_{\boldsymbol{\mu}f}\bar{z}, k_{\boldsymbol{\mu}f}\bar{\tau}),
\end{aligned} \tag{4.3.93}$$

where $V^{(\eta)}$ is the diagonlizing matrix of

$$C_{ij}^{(\eta)} \equiv \int_{T^2} d^2z\left\{\tilde{h}_{\mathrm{L0}}^{(+)\boldsymbol{\mu}(j)}(z)\right\}^*\hat{h}_{\mathrm{L0}}^{(+)\boldsymbol{\mu}(i)}(z), \tag{4.3.94}$$

j in (eq:posiferindbasis) runs from 1 to r (rank of $C^{(\eta)}$). The mass eigenvalue is $|\tilde{m}_0| = \left|\tilde{M}_f\right|$, so there is the massless mode only when $\tilde{M}_f = 0$.

4.4 Yukawa coupling constants

In this section, we derive the expressions of the Yukawa coupling constants. As we have seen in the previous section, the bulk fermion's mass forbids the zero-mode solution as massless mode. Then, instead of 6D Dirac fermions Ψ^f, we introduce only 6D Weyl fermions $\Psi_\pm^f$ defined as

$$\Psi_+^f = \begin{pmatrix}\hat{\Psi}_+^f \\ \mathbf{0}_4\end{pmatrix}, \qquad \Psi_-^f = \begin{pmatrix}\mathbf{0}_4 \\ \hat{\Psi}_-^f\end{pmatrix}. \tag{4.4.1}$$

where $\hat{\Psi}_\pm$ are 4-component spinors defined in (4.2.28).

4.4.1 General expression

Now, the Yukawa couplings stem from the 6D gauge interaction:

$$
\begin{aligned}
S &= \int dx^6 \left(\sum_{f_+} i\bar{\Psi}_+^{f_+}\gamma^M \mathcal{D}_M \Psi_+^{f_+} + \sum_{f_-} i\bar{\Psi}_- \gamma^M \mathcal{D}_M \Psi_-^{f_-} \right) + \cdots \\
&= \int dx^4 \int d^2z 2\pi R_1 \left(-\sum_{f_+} i\bar{\psi}_+^{f_+} A_z \bar{\lambda}_+^{f_+} + \sum_{f_-} i\lambda_-^{f_-} A_z \psi_-^{f_-} \right) + \text{h.c.} + \cdots , \qquad (4.4.2)
\end{aligned}
$$

where $d^2z = dzd\bar{z}$. The 4D effective Lagrangian of the Yukawa interactions is expressed as

$$
\begin{aligned}
\mathcal{L}^{\text{4D}}_{\text{yukawa}} =& \sum_{\boldsymbol{\mu}} \sum_{f_+} \sum_{i,j,k} y^{(+)\boldsymbol{\mu} f_+}_{ijk} \bar{\psi}^{(\boldsymbol{\mu}+\boldsymbol{\alpha})f_+(i)}_{+0} \varphi_0^{\boldsymbol{\alpha}(k)} \bar{\lambda}^{\boldsymbol{\mu} f_+(j)}_{+0} + \text{h.c.} \\
&+ \sum_{\boldsymbol{\mu}} \sum_{f_-} \sum_{i,j,k} y^{(-)\boldsymbol{\mu} f_-}_{ijk} \lambda^{\boldsymbol{\mu} f_-(j)}_{-0} \varphi_0^{\boldsymbol{\alpha}(k)} \psi^{(\boldsymbol{\mu}+\boldsymbol{\alpha})f_-(i)}_{-0} + \text{h.c.}, \qquad (4.4.3)
\end{aligned}
$$

where

$$
\begin{aligned}
y^{(+)\boldsymbol{\mu} f_+}_{ijk} \equiv& -\frac{ig_A \langle \boldsymbol{\mu}_L + \boldsymbol{\alpha} | E_{\boldsymbol{\alpha}} | \boldsymbol{\mu}_L \rangle}{\pi R_1} \int_{T^2/Z_N} d^2z \left\{ h^{(+)(\boldsymbol{\mu}_L+\boldsymbol{\alpha})f_+(i)}_{\text{R}0}(z) \right\}^* g_0^{\boldsymbol{\alpha}(k)}(z) h^{(+)\boldsymbol{\mu}_L f_+(j)}_{\text{L}0}(z) \\
=& -\frac{2ig_4\sqrt{\text{Im}\tau}}{N^{\frac{3}{2}}} \langle \boldsymbol{\mu}_L + \boldsymbol{\alpha} | E_{\boldsymbol{\alpha}} | \boldsymbol{\mu}_L \rangle \int_{T^2/Z_N} d^2z \left\{ h^{(+)(\boldsymbol{\mu}_L+\boldsymbol{\alpha})f_+(i)}_{\text{R}0}(z) \right\}^* g_0^{\boldsymbol{\alpha}(k)}(z) h^{(+)\boldsymbol{\mu}_L f_+(j)}_{\text{L}0}(z) \\
=& -2ig_4\sqrt{\text{Im}\tau} \langle \boldsymbol{\mu}_L + \boldsymbol{\alpha} | E_{\boldsymbol{\alpha}} | \boldsymbol{\mu}_L \rangle \sum_{i'=1}^{|K_1|} \sum_{j'=1}^{|K_2|} \sum_{k'=1}^{|K_3|} V^{(\eta_1)*}_{ii'} V^{(\eta_2)}_{jj'} V^{(\eta_3)}_{kk'} \\
& \times \int_{T^2} d^2 \mathcal{F}^{(i')*}(z; K_1, \xi_1) \mathcal{F}^{(j')}(z; K_2, \xi_2) \mathcal{F}^{(k')}(z; K_3, \xi_3), \qquad (4.4.4) \\
y^{(-)}_{ijk} \equiv& -2ig_4\sqrt{\text{Im}\tau} \langle \boldsymbol{\mu}_L + \boldsymbol{\alpha} | E_{\boldsymbol{\alpha}} | \boldsymbol{\mu}_L \rangle \sum_{i'=1}^{|K_1|} \sum_{j'=1}^{|K_2|} \sum_{k'=1}^{|K_3|} V^{(\eta_1)}_{ii'} V^{(\eta_2)*}_{jj'} V^{(\eta_3)}_{kk'} \\
& \times \int_{T^2} d^2 \mathcal{F}^{(i')}(z; K_1, \xi_1) \mathcal{F}^{(j')*}(z; K_2, \xi_2) \mathcal{F}^{(k')}(z; K_3, \xi_3), \qquad (4.4.5)
\end{aligned}
$$

where $g_4 \equiv \frac{g_A}{\sqrt{\mathcal{A}}} = \frac{\sqrt{N}g_A}{2\pi R_1 \sqrt{\text{Im}\tau}}$ is the 4D gauge coupling constant, $\boldsymbol{\mu}_L$ & $\boldsymbol{\alpha}$ mean the weight vector of the Left handed fermion and the root vector of the Higgs neutral component respectively, $K_1 \equiv k_{(\boldsymbol{\mu}_L+\boldsymbol{\alpha})f_\pm}$, $\xi_1 \equiv \xi_{(\boldsymbol{\mu}_L+\boldsymbol{\alpha})f_\pm}$, $K_2 \equiv k_{\boldsymbol{\mu}_L f_\pm}$, $\xi_2 \equiv \xi_{\boldsymbol{\mu}_L f_\pm}$, $K_3 \equiv k_{\boldsymbol{\alpha}}$, $\xi_3 \equiv \xi_{\boldsymbol{\alpha}}$, $\{\eta_1, \eta_2, \eta_3\}$ are the Z_N transformation phases of $\{|\boldsymbol{\mu}_L + \boldsymbol{\alpha}\rangle, |\boldsymbol{\mu}_L\rangle, |\boldsymbol{\alpha}\rangle\}$ fields, the indices

i, j, k run over the degenerate zero-modes, $f_\pm$ mean flavor indices, $z \equiv w_1 + \tau w_2$, and

$$\mathcal{F}^{(j)}(z; K, \xi) \equiv \begin{cases} (2K\mathrm{Im}\tau)^{\frac{1}{4}} e^{K\pi i(z+\xi)\frac{\mathrm{Im}(z+\xi)}{\mathrm{Im}\tau}} \vartheta \begin{bmatrix} \frac{j}{K} \\ 0 \end{bmatrix} (K(z+\xi), K\tau), & (K > 0) \\ (2\,|K|\,\mathrm{Im}\tau)^{\frac{1}{4}} e^{K\pi i(\bar{z}+\bar{\xi})\frac{\mathrm{Im}(\bar{z}+\bar{\xi})}{\mathrm{Im}\bar{\tau}}} \vartheta \begin{bmatrix} \frac{j}{K} \\ 0 \end{bmatrix} (K(\bar{z}+\bar{\xi}), K\bar{\tau}). & (K < 0) \end{cases} \tag{4.4.6}$$

4.4.2 Couplings to $\chi_6 = +$ fermions

From the gauge symmetry of the Yukawa Lagrangian, the following relations are satisfied:

$$K_1 = K_2 + K_3, \quad K_1\xi_1 = K_2\xi_2 + K_3\xi_3, \tag{4.4.7}$$

where K_1, K_2, K_3 satisfies

$$K_1 > 0, \quad K_2 < 0, \quad K_3 > 0, \tag{4.4.8}$$

from the zero-mode conditions. We find that

$$\begin{aligned} &\mathcal{F}^{(i')*}(z; K_1, \xi_1)\mathcal{F}^{(j')}(z; K_2, \xi_2) \\ &= \frac{1}{\sqrt{K_3}} \sum_{m=1}^{K_3} \mathcal{F}^{(i'-j'+K_1 m)*}(z; K_3, \xi_3)\mathcal{F}^{(|K_2|i'+K_1 j'+K_1|K_2|m)*}\left(0; |K_1K_2K_3|\,, \frac{\xi_1 - \xi_2}{K_3}\right), \end{aligned} \tag{4.4.9}$$

which comes from the relations:

$$\begin{aligned} &\vartheta \begin{bmatrix} \frac{i'}{K_1} \\ 0 \end{bmatrix} (K_1(z+\xi_1), K_1\tau) \cdot \vartheta \begin{bmatrix} -\frac{j'}{|K_2|} \\ 0 \end{bmatrix} (|K_2|\,(z+\xi_2), |K_2|\,\tau) \\ &= \sum_{l=1}^{K_1+|K_2|} \vartheta \begin{bmatrix} \frac{i'-j'+K_1 l}{K_1+|K_2|} \\ 0 \end{bmatrix} \left((K_1 + |K_2|) \left(z + \frac{K_1\xi_1 + |K_2|\,\xi_2}{K_1 + |K_2|} \right), (K_1 + |K_2|\,\tau) \right) \\ &\quad \times \vartheta \begin{bmatrix} \frac{|K_2|i'+K_1 j'+K_1|K_2|l}{K_1|K_2|(K_1+|K_2|)} \\ 0 \end{bmatrix} (K_1\,|K_2|\,(\xi_1 - \xi_2), K_1\,|K_2|\,(K_1 + |K_2|\,\tau)), \end{aligned} \tag{4.4.10}$$

$$\mathcal{F}^{(j)*}(z; K, \xi) = \mathcal{F}^{(-j)}(z; -K, \xi), \tag{4.4.11}$$

and (4.4.7). Then, from the orthonormal condition of $\mathcal{F}^{(j)}(z; K, \xi)$

$$\int_{T^2} d^2z \mathcal{F}^{(j)*}(z; K, \xi)\mathcal{F}^{(k)}(z; K, \xi) = \delta_{jk}, \tag{4.4.12}$$

we get

$$
\begin{aligned}
&\int_{T^2} d^2z \mathcal{F}^{(i')*}(z;K_1,\xi_1)\mathcal{F}^{(j')}(z;K_2.\xi_2)\mathcal{F}^{(k')}(z;K_3,\xi_3) \\
=&\frac{1}{\sqrt{K_3}}\sum_{m=1}^{K_3}\mathcal{F}^{(K_2i'-K_1j'+K_1K_2m)}\left(0,K_1K_2K_3,\frac{\xi_1-\xi_2}{K_3}\right)\delta_{i'-j'+K_1m,k'},
\end{aligned}
\tag{4.4.13}
$$

where

$$
\delta_{i'-j'+K_1m,k'}=\begin{cases}1 & (i'-j'+K_1m=k' \text{ mod } K_3) \\ 0 & (\text{other cases})\end{cases}. \tag{4.4.14}
$$

Therefore, (4.4.4) becomes

$$
\begin{aligned}
y_{ijk}^{(+)}=&-\frac{2ig_4\sqrt{\mathrm{Im}\tau}}{\sqrt{K_3}}\langle\boldsymbol{\mu}_L+\boldsymbol{\alpha}|E_{\boldsymbol{\alpha}}|\boldsymbol{\mu}_L\rangle\sum_{i'=1}^{K_1}\sum_{j'=1}^{|K_2|}\sum_{k'=1}^{K_3}V_{ii'}^{(\eta_1)*}[K_1,\zeta_1]V_{jj'}^{(\eta_2)}[K_2,\zeta_2]V_{kk'}^{(\eta_3)}[K_3,\zeta_3] \\
&\times\sum_{m=1}^{K_3}\mathcal{F}^{(K_2i'-K_1j'+K_1K_2m)}\left(0,K_1K_2K_3,\frac{\zeta_1-\zeta_2}{K_3}\right)\delta_{i'-j'+K_1m,k'}.
\end{aligned}
\tag{4.4.15}
$$

The matrix $V^{(\eta)}$ depends on the flux and the Wilson-line phase. The indices i,j,k runs from 1 to the rank of $C^{(\eta_1)},C^{(\eta_2)},C^{(\eta_3)}$, respectively.

4.4.3 Couplings to $\chi_6=-$ fermions

As with the case of $^chi_6=+$, K_a and ζ_a satisfy

$$
K_2=K_1+K_3,\qquad K_2\zeta_2=K_1\zeta_1+K_3\zeta_3, \tag{4.4.16}
$$

where

$$
K_1<0,\quad K_2>0,\quad K_3>0. \tag{4.4.17}
$$

The yukawa coupling constants (4.4.5) are expressed as

$$
\begin{aligned}
y_{ijk}^{(-)}=&\frac{2ig_4\sqrt{\mathrm{Im}\tau}}{\sqrt{K_3}}\langle\boldsymbol{\mu}_L+\boldsymbol{\alpha}|E_{\boldsymbol{\alpha}}|\boldsymbol{\mu}_L\rangle\sum_{i'=1}^{|K_1|}\sum_{j'=1}^{K_2}\sum_{k'=1}^{K_3}V_{ii'}^{(\eta_1)}[K_1,\zeta_1]V_{jj'}^{(\eta_2)*}[K_2,\zeta_2]V_{kk'}^{(\eta_3)}[K_3,\zeta_3] \\
&\times\sum_{m=1}^{K_3}\mathcal{F}^{(K_1j'-K_2i'+K_1K_2m)}\left(0,K_1K_2K_3,\frac{\zeta_2-\zeta_1}{K_3}\right)\delta_{j'-i'+K_1m,k'}.
\end{aligned}
\tag{4.4.18}
$$

4.5 Model

4.5.1 $SU(3)\times U(1)$ model on T^2/Z_3

Now, we consider a specific model. We choose $G = SU(3)$, $N = 3$, $n_f = 4$, and the matter fermions as two $\chi_6 = \chi_6^{\mathbf{3}}$ spinors $(\Psi^1_{\chi_6^{\mathbf{3}}}, \Psi^3_{\chi_6^{\mathbf{3}}})$ that belong to $\mathbf{3}$ of $SU(3)$ and two $\chi_6 = \chi_6^{\bar{\mathbf{3}}}$ fermions $(\Psi^2_{\chi_6^{\bar{\mathbf{3}}}}, \Psi^4_{\chi_6^{\bar{\mathbf{3}}}})$ that belong to $\bar{\mathbf{3}}$ of $SU(3)$[3]. The $U(1)_X$ charges are assigned as $(q_1, q_2, q_3, q_4) = (0, 1/3, -1/3, -2/3)$. The roots of $SU(3)$ are

$$\boldsymbol{\alpha}_1 = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right), \quad \boldsymbol{\alpha}_2 = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right), \quad \boldsymbol{\alpha}_3 = \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 = (1, 0),$$
$$-\boldsymbol{\alpha}_1, -\boldsymbol{\alpha}_2, -\boldsymbol{\alpha}_3. \tag{4.5.1}$$

The weights of $\mathbf{3}$ are

$$\begin{aligned}
\boldsymbol{\mu}_1 &= \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right), \\
\boldsymbol{\mu}_2 = \boldsymbol{\mu}_1 - \boldsymbol{\alpha}_1 &= \left(0, -\frac{1}{\sqrt{3}}\right), \\
\boldsymbol{\mu}_3 = \boldsymbol{\mu}_2 - \boldsymbol{\alpha}_2 &= \left(-\frac{1}{2}, \frac{1}{2\sqrt{3}}\right).
\end{aligned} \tag{4.5.2}$$

The weights of $\bar{\mathbf{3}}$ are $-\boldsymbol{\mu}_1, -\boldsymbol{\mu}_2, -\boldsymbol{\mu}_3$.

To break G to $SU(2)_L \times U(1)_Z$, we choose the parameter of P matrix in (4.2.8) as

$$p = \frac{2\pi n_p}{N}\left(1, -\frac{1}{\sqrt{3}}\right), \quad (n_p = 0, 1, 2) \tag{4.5.3}$$

After symmetry breaking, $SU(2)_L$ root $\boldsymbol{\alpha}_L$ and the vector of $U(1)_X$ charge are identified as

$$\boldsymbol{\alpha}_L = \boldsymbol{\alpha}_1, \quad \boldsymbol{\zeta} = \left(\frac{1}{2}, -\frac{1}{2\sqrt{3}}\right), \tag{4.5.4}$$

respectively. When we assign the hypercharge $\pm 1/2$ to the Higgs doublet, the normalization for $\boldsymbol{\zeta}$ is determined. From (4.2.9) and (4.5.3),

$$n_{\pm\boldsymbol{\alpha}_1} = 0, \quad n_{\pm\boldsymbol{\alpha}_2} = n_{\pm\boldsymbol{\alpha}_3} = \pm n_p - 1.$$
(double signs correspond.)

[3] Unless we specify $\chi_6 = +$ or $-$ for $\Psi^f_{\chi_6}$, we use the character Ψ^f as $\Psi^f_{\chi_6}$ $(f = 1, 2, 3, 4)$ in the following pages.

Under the unbroken $SU(2)_L$, the $SU(3)$ adjoint representation is decomposed as

$$\begin{aligned}
|-\boldsymbol{\alpha}_1\rangle,\ |\mathbf{0}\rangle_T,\ |\boldsymbol{\alpha}_1\rangle &: \text{triplet } (q_z=0)\\
|\boldsymbol{\alpha}_2\rangle,\ |\boldsymbol{\alpha}_3\rangle &: \text{doublet } (q_z=1/2)\\
|-\boldsymbol{\alpha}_2\rangle,\ |-\boldsymbol{\alpha}_3\rangle &: \text{doublet } (q_z=-1/2)\\
|\mathbf{0}\rangle_S &: \ (q_z=0)
\end{aligned}\tag{4.5.5}$$

where q_Z is the $U(1)_Z$ charge (the eigenvalue of $\mathcal{Q}_Z$), and $|\mathbf{0}\rangle_S$ and $|\mathbf{0}\rangle_S$ are the states of Cartan generators. These states do not have $U(1)_X$ charge, so each q_Z in (4.5.5) equals to the hypercharge Y. As you see, $\{|\boldsymbol{\alpha}_2\rangle, |\boldsymbol{\alpha}_3\rangle\}$ or $\{|-\boldsymbol{\alpha}_2\rangle, |-\boldsymbol{\alpha}_3\rangle\}$ corresponds to the Higgs doublet.

Next, we discuss the quantum number of matter fields. We defined ζ in $\mathcal{Q}_Z$ as (4.5.4), so the hypercharges of $\mathbf{3}$ are

$$\begin{aligned}
(Y(\boldsymbol{\mu}_1), Y(\boldsymbol{\mu}_2), Y(\boldsymbol{\mu}_3)) &= (\zeta\cdot\boldsymbol{\mu}_1, \zeta\cdot\boldsymbol{\mu}_2, \zeta\cdot\boldsymbol{\mu}_3) + (q_f, q_f, q_f)\\
&= (1/6+q_f, 1/6+q_f, -1/3+q_f)\\
&= \begin{cases}(1/6, 1/6, -1/3) & (f=1)\\ (-1/2, -1/2, -1) & (f=3)\end{cases},
\end{aligned}\tag{4.5.6}$$

and the hypercharges of $\bar{\mathbf{3}}$ are

$$\begin{aligned}
(Y(-\boldsymbol{\mu}_1), Y(-\boldsymbol{\mu}_2), Y(-\boldsymbol{\mu}_3)) &= (\zeta\cdot\boldsymbol{\mu}_1, \zeta\cdot\boldsymbol{\mu}_2, \zeta\cdot\boldsymbol{\mu}_3) + (q_f, q_f, q_f)\\
&= (-1/6+q_f, -1/6+q_f, 1/3+q_f)\\
&= \begin{cases}(1/6, 1/6, 2/3) & (f=2)\\ (-1/2, -1/2, 0) & (f=4)\end{cases},
\end{aligned}\tag{4.5.7}$$

so $SU(2)_L$ doublets $\{|\boldsymbol{\mu}_1\rangle, |\boldsymbol{\mu}_2\rangle\}$ & $\{|-\boldsymbol{\mu}_1\rangle, |-\boldsymbol{\mu}_2\rangle\}$ are identified as the reft-handed doublets, and $|\boldsymbol{\mu}_3\rangle$ & $|-\boldsymbol{\mu}_3\rangle$ are identified as the right-handed singlets in SM. Now, we can assign one generation quarks and leptons to Ψ^f:

$$\begin{aligned}
&Q_L(\mathbf{2_{1/6}}),\quad d_R(\mathbf{1_{-1/3}}),\quad (\Psi^1)\\
&Q'_L(\mathbf{2_{1/6}}),\quad u_R(\mathbf{1_{2/3}}),\quad (\Psi^2)\\
&L_L(\mathbf{2_{1/6}}),\quad e_R(\mathbf{1_{-1}}),\quad (\Psi^3)\\
&L'_L(\mathbf{2_{-1/2}}),\quad \nu_R(\mathbf{1_0}),\quad (\Psi^4)
\end{aligned}\tag{4.5.8}$$

where the **1** or **2** means the $SU(2)$ representation, and the subscript of it means the $U(1)_Y$ hypercharge.

We require the magnetic fluxes to break in the same way as the orbifold conditions do. Then, the direction of $G(=SU(3))$ flux is determined as

$$(\mathcal{C}^1, \mathcal{C}^2) = \mathcal{C}^1 \left(1, -\frac{1}{\sqrt{3}}\right). \tag{4.5.9}$$

The $\mathcal{C}^1$ and $\mathcal{B}$ are determined by the quantization conditions (4.2.20) for all the fields. All the conditions are discribed as

$$\begin{aligned}
&0 = 2k_{\pm\boldsymbol{\alpha}_1}\pi \ \ (SU(2)_L \text{ gauge fields}), \quad \pm N\mathcal{C}^1 = 2k_{\pm\boldsymbol{\alpha}_2}\pi = 2k_{\pm\boldsymbol{\alpha}_3}\pi \ \ (\text{Higgs doublet}),\\
&\frac{N\mathcal{C}^1}{3} = 2k_{\boldsymbol{\mu}_1 1}\pi = 2k_{\boldsymbol{\mu}_2 1}\pi \ \ (\Psi^1_{\mathrm{L}}), \quad -\frac{2N\mathcal{C}^1}{3} = 2k_{\boldsymbol{\mu}_3 1\pi} \ \ (\Psi^1_{\mathrm{R}}),\\
&N\left(-\frac{\mathcal{C}^1}{3} + \frac{\mathcal{B}}{3}\right) = 2k_{-\boldsymbol{\mu}_1 2}\pi = 2k_{-\boldsymbol{\mu}_2 2}\pi \ \ (\Psi^2_{\mathrm{L}}), \ \ N\left(\frac{2\mathcal{C}^1}{3} + \frac{\mathcal{B}}{3}\right) = 2k_{-\boldsymbol{\mu}_3 2}\pi \ \ (\Psi^2_{\mathrm{R}}),\\
&N\left(-\frac{\mathcal{C}^1}{3} - \frac{2\mathcal{B}}{3}\right) = 2k_{\boldsymbol{\mu}_1 3}\pi = 2k_{-\boldsymbol{\mu}_2 3}\pi \ \ (\Psi^3_{\mathrm{L}}), \ \ N\left(\frac{2\mathcal{C}^1}{3} + \frac{\mathcal{B}}{3}\right) = 2k_{\boldsymbol{\mu}_3 3}\pi \ \ (\Psi^3_{\mathrm{R}}),\\
&N\left(-\frac{\mathcal{C}^1}{3} - \frac{\mathcal{B}}{3}\right) = 2k_{-\boldsymbol{\mu}_1 4}\pi = 2k_{-\boldsymbol{\mu}_2 4}\pi \ \ (\Psi^4_{\mathrm{L}}), \ \ N\left(\frac{2\mathcal{C}^1}{3} - \frac{\mathcal{B}}{3}\right) = 2k_{\boldsymbol{\mu}_3 4}\pi \ \ (\Psi^4_{\mathrm{R}}).
\end{aligned} \tag{4.5.10}$$

$$\begin{aligned}
\therefore \qquad\qquad & k_{\pm\boldsymbol{\alpha}_1} = 0, \ \ k_{\pm\boldsymbol{\alpha}_2} = k_{\pm\boldsymbol{\alpha}_3} = \pm 3k,\\
& N\mathcal{C}^1 = 6k\pi, \ \ N\mathcal{B} = 6k'\pi,\\
& k_{\boldsymbol{\mu}_1 1} = k_{\boldsymbol{\mu}_2 1} = k, \ \ k_{\boldsymbol{\mu}_3 1} = -2k,\\
& k_{-\boldsymbol{\mu}_1 2} = k_{-\boldsymbol{\mu}_2 2} = -k + k', \ \ k_{-\boldsymbol{\mu}_3 2} = -2k + k',\\
& k_{\boldsymbol{\mu}_1 3} = k_{\boldsymbol{\mu}_2 3} = k - 2k', \ \ k_{\boldsymbol{\mu}_3 3} = -2k - k',\\
& k_{-\boldsymbol{\mu}_1 4} = k_{-\boldsymbol{\mu}_2 4} = -k - k', \ \ k_{-\boldsymbol{\mu}_3 4} = 2k - k',
\end{aligned} \tag{4.5.11}$$

where $\Psi^f_{\mathrm{R,L}}$ means the right-, the left-handed fermion contained in Ψ^f respectively, and $k, \ k'$ are integers.

We know η (the eigenvalue of Z_N transformation) for the Higgs field as

$$\omega^{-1} \exp\left(\frac{2n_p\pi i}{N}\right) = \omega^{n_p - 1}, \quad (\{|\boldsymbol{\alpha}_2\rangle, |\boldsymbol{\alpha}_3\rangle\} \text{ Higgs}) \tag{4.5.12}$$

$$\omega^{-1} \exp\left(\frac{-2n_p\pi i}{N}\right) = \omega^{-(n_p + 1)}. \quad (\{|-\boldsymbol{\alpha}_2\rangle, |-\boldsymbol{\alpha}_3\rangle\} \text{ Higgs}) \tag{4.5.13}$$

η for matter fields are

$$\omega^{-\frac{\chi_6}{2}} e^{iq_f\varphi_f} e^{ip\cdot\boldsymbol{\mu}_{\mathrm{R}}}, \quad (\text{right-handed fermion}) \tag{4.5.14}$$

$$\omega^{\frac{\chi_6}{2}} e^{iq_f\varphi_f} e^{ip\cdot\boldsymbol{\mu}_{\mathrm{L}}}. \quad (\text{left-handed fermion}) \tag{4.5.15}$$

The Z_N twist phases for the Yukawa terms must be always equal to 1. Then, when we choose $H = \{|\boldsymbol{\alpha}_2\rangle, |\boldsymbol{\alpha}_3\rangle\} \equiv H_+$ as the Higgs doublet, the following relations are satisfied:

- **3** representation

$$\left(\omega^{\frac{\chi_6}{2}} e^{iq_f\phi_f} e^{ip\cdot\boldsymbol{\mu}_{\rm L}}\right)^{-1} \cdot \omega^{n_p-1} \cdot \omega^{-\frac{\chi_6}{2}} e^{iq_f\varphi_f} e^{ip\cdot\boldsymbol{\mu}_{\rm R}} = 1,$$
$$\therefore \chi_6 = -. \tag{4.5.16}$$

- $\bar{\mathbf{3}}$ representation

$$\omega^{\frac{\chi_6}{2}} e^{iq_f\phi_f} e^{ip\cdot\boldsymbol{\mu}_{\rm L}} \cdot \omega^{n_p-1} \cdot \left(\omega^{-\frac{\chi_6}{2}} e^{iq_f\varphi_f} e^{ip\cdot\boldsymbol{\mu}_{\rm R}}\right)^{-1} = 1,$$
$$\therefore \chi_6 = +. \tag{4.5.17}$$

When we choose $H = \{|-\boldsymbol{\alpha}_2\rangle, |-\boldsymbol{\alpha}_3\rangle\} \equiv H_-$ as the Higgs doublet,

- **3** representation

$$\omega^{\frac{\chi_6}{2}} e^{iq_f\phi_f} e^{ip\cdot\boldsymbol{\mu}_{\rm L}} \cdot \omega^{-(n_p+1)} \cdot \left(\omega^{-\frac{\chi_6}{2}} e^{iq_f\varphi_f} e^{ip\cdot\boldsymbol{\mu}_{\rm R}}\right)^{-1} = 1,$$
$$\therefore \chi_6 = +. \tag{4.5.18}$$

- $\bar{\mathbf{3}}$ representation

$$\left(\omega^{\frac{\chi_6}{2}} e^{iq_f\phi_f} e^{ip\cdot\boldsymbol{\mu}_{\rm L}}\right)^{-1} \cdot \omega^{-(n_p+1)} \cdot \omega^{-\frac{\chi_6}{2}} e^{iq_f\varphi_f} e^{ip\cdot\boldsymbol{\mu}_{\rm R}} = 1,$$
$$\therefore \chi_6 = -. \tag{4.5.19}$$

4.5.2 Numbers of zero-modes

When we choose $H = H_+$, we can assign χ_6 to each Ψ^f as

$$\begin{aligned}(\Psi^1, \Psi^3) &: -\\ (\Psi^2, \Psi^4) &: +\end{aligned} \tag{4.5.20}$$

The Z_N twist phases η for the fields feeling the magnetic fluxes are as follows:

$$\eta = \begin{cases} \omega^{n_p-1}, \quad (\{|\boldsymbol{\alpha}_2\rangle, |\boldsymbol{\alpha}_3\rangle\} \ \text{Higgs}) \\ (\chi_6 = +) \begin{cases} \omega^{n_f+1}, & \text{(left-handed fermion)} \\ \omega^{n_f+n_p}, & \text{(right-handed fermion)} \end{cases} \\ (\chi_6 = -) \begin{cases} \omega^{n_f+n_p}, & \text{(left-handed fermion)} \\ \omega^{n_f+1}, & \text{(right-handed fermion)} \end{cases} \end{cases} \tag{4.5.21}$$

where n_f is an integer for each f. In the subsection 4.3, we saw the conditions for the magnetic fluxes such that each field have zero-modes. All the matter fields in (4.5.8) and the Higgs field have the zero-modes when the values of the magnetic fluxes satisfy

$$\begin{aligned} k, 2k+k', k-2k', 2k-k' &\geq 1, \\ -2k, -k+k', -2k-2k', -k-k' &\leq -1, \qquad (4.5.22) \\ \therefore k \geq 1, \quad -k+1 \leq k' &\leq \frac{k-1}{2}. \qquad (4.5.23) \end{aligned}$$

On the other hand, when we choose $H = H_-$, we can assign χ_6 to each Ψ^f as

$$\begin{aligned} (\Psi^1, \Psi^3) &: + \\ (\Psi^2, \Psi^4) &: - \qquad (4.5.24) \end{aligned}$$

In this case, η are as follows:

$$\eta = \begin{cases} \omega^{-(n_p+1)}, & (\{|-\boldsymbol{\alpha}_2\rangle, |-\boldsymbol{\alpha}_3\rangle\} \text{ Higgs}) \\ (\chi_6 = +) \begin{cases} \omega^{n_f+n_p}, & (\text{left-handed fermion}) \\ \omega^{n_f-1}, & (\text{right-handed fermion}) \end{cases} \\ (\chi_6 = -) \begin{cases} \omega^{n_f-1}, & (\text{left-handed fermion}) \\ \omega^{n_f+n_p}. & (\text{right-handed fermion}) \end{cases} \end{cases} \qquad (4.5.25)$$

All the matter fields in (4.5.8) and the Higgs field have the zero-modes when the values of the magnetic fluxes satisfy

$$\begin{aligned} k, 2k+k', k-2k', 2k-k' &\leq -1, \\ -2k, -k+k', -2k-2k', -k-k' &\geq 1, \qquad (4.5.26) \\ \therefore k \leq -1, \quad \frac{k+1}{2} \leq k' &\leq -k-1. \qquad (4.5.27) \end{aligned}$$

Next, we discuss the Scherk-Schwarz phases. In the case of T^2/Z_3 the possible Scherk-Schwarz phases are

$$\begin{aligned} \phi_1^{\boldsymbol{\alpha}} = \phi_\tau^{\boldsymbol{\alpha}} &= \frac{l_{\boldsymbol{\alpha}}}{3} + \frac{1}{4}\left\{1 - (-1)^{k_{\boldsymbol{\alpha}}}\right\}, \\ \phi_1^{\boldsymbol{\mu} f} = \phi_\tau^{\boldsymbol{\mu} f} &= \frac{l_{\boldsymbol{\mu} f}}{3} + \frac{1}{4}\left\{1 - (-1)^{k_{\boldsymbol{\mu} f}}\right\}, \qquad (4.5.28) \end{aligned}$$

where $l_{\boldsymbol{\alpha}}$, $l_{\boldsymbol{\mu} f} = 0, 1, 2$ (See Appendix.E.). When $SU(2)_L$ is unbroken, $\phi_s^{\boldsymbol{\alpha}_1} = 0$. So the Scherk-Schwarz phases Φ_s^i and φ_s in (4.2.19) should be

$$(\Phi_s^1, \Phi_s^2) = 2\pi l \left(1, -\frac{1}{\sqrt{3}}\right), \quad \varphi_s = \pi l', \qquad (4.5.29)$$

	H	Q_L	d_R	Q'_L	u_R	L_L	e_R	L'_L	ν_R
K	$\pm 3k$	k	$-2k$	$-k+k'$	$2k+k'$	$k-2k'$	$-2k-2k'$	$-k-k'$	$2k-k'$
$\eta : H = H_+$	ω^{n_p-1}	$\omega^{n_1+n_p}$	ω^{n_1+1}	ω^{n_2+1}	$\omega^{n_2+n_p}$	$\omega^{n_3+n_p}$	ω^{n_3+1}	ω^{n_4+1}	$\omega^{n_4+n_p}$
$\eta : H = H_-$	$\omega^{-(n_p+1)}$	$\omega^{n_1+n_p}$	ω^{n_1-1}	ω^{n_2-1}	$\omega^{n_2+n_p}$	$\omega^{n_3+n_p}$	ω^{n_3-1}	ω^{n_4-1}	$\omega^{n_4+n_p}$
ϕ	l	$\frac{l}{3}$	$-\frac{2l}{3}$	$-\frac{2l-l'}{6}$	$\frac{4l+l'}{6}$	$\frac{l-l'}{3}$	$-\frac{2l+l'}{3}$	$-\frac{2l+l'}{6}$	$\frac{4l-l'}{6}$

Table I: The values of the magnetic fluxes K, Z_N twist phases η, the Scherk-Schwarz phases $\phi \equiv K\zeta/2(\tau-1)$ (ζ: Wilson-line phase). The constant $2l$ (l') is even for even k (k'), and odd for odd k (k'). The K for the Higgs doublet $= +3k$ corresponds to $H_+ \equiv \{|\boldsymbol{\alpha}_2\rangle, |\boldsymbol{\alpha}_3\rangle\}$, and the K for the Higgs doublet $= -3k$ corresponds to $H_- \equiv \{|-\boldsymbol{\alpha}_2\rangle, |-\boldsymbol{\alpha}_3\rangle\}$.

where l, l' are real constants. The Scherk-Schwarz phases that respective fields feel are expressed as

$$\begin{aligned}
&\phi_s^{\boldsymbol{\alpha}_1} = 0, \quad \phi_s^{\boldsymbol{\alpha}_2} = \phi_s^{\boldsymbol{\alpha}_3} = l, \\
&\phi_s^{\boldsymbol{\mu}_1 1} = \phi_s^{\boldsymbol{\mu}_2 1} = \frac{l}{3}, \quad \phi_s^{\boldsymbol{\mu}_3 1} = \frac{2l}{3}, \\
&\phi_s^{-\boldsymbol{\mu}_1 2} = \phi_s^{-\boldsymbol{\mu}_2 2} = \frac{l}{3} + \frac{l'}{6}, \quad \phi_s^{-\boldsymbol{\mu}_3 2} = \frac{2l}{3} + \frac{l'}{6}, \\
&\phi_s^{-\boldsymbol{\mu}_1 3} = \phi_s^{-\boldsymbol{\mu}_2 3} = \frac{l}{3} - \frac{l'}{3}, \quad \phi_s^{-\boldsymbol{\mu}_3 3} = -\frac{2l}{3} - \frac{l'}{3}, \\
&\phi_s^{-\boldsymbol{\mu}_1 4} = \phi_s^{-\boldsymbol{\mu}_2 4} = -\frac{l}{3} + \frac{l'}{6}, \quad \phi_s^{-\boldsymbol{\mu}_3 4} = \frac{2l}{3} - \frac{l'}{6},
\end{aligned} \tag{4.5.30}$$

where $s = 1, \tau$. These phases $\phi = \phi_s^{\boldsymbol{\alpha}}, \phi_s^{\boldsymbol{\mu} f}$ are derined mod $|k_{\boldsymbol{\alpha}}|$, $|k_{\boldsymbol{\mu} f}|$, respectively. We find that $2l$ (l') is even for even k k', and odd for odd k (k').

Then, the Yukawa sector of this model is determined by the nine integers: k, k', l, l', n_p, and n_f ($n_f = 1,2,3,4$). The numbers of the zero-modes of the fields feeling the magnetic fluxes are determined by the values of the fluxes K, the Z_N twist phases η, and the Wilson-line phases ζ (or the Scherk-Schwarz phases ϕ) that each field feels. We surmmalized them in Table I. The number of the zero-modes are read off from Table 4$\sim$7 in [39][4]. The detail classification of the zero-mode number is explained by [42].

[4]The definition of η in [39] is different from ours when $K < 0$. In this paper, it should be described as $\bar{\eta}$.

4.5.3 Realization of three generations

From Table 1∼4 in [39], the three generations are realized when $k = 6$ $(H = H_+)$, $k' = 0$, $n_p = 0$, $n_{1,3} = 0$, $n_{2,4} = 2$, $l = l' = 0$[5]. In this case, the 4D effective Yukawa Lagrangian in (4.4.3) becomes

$$\mathcal{L}^{\rm 4D}_{\rm yukawa} = -\sum_{i,j=1}^{3}\sum_{k=1}^{5}\Big(y^{(k)D}_{ij}\bar{Q}^j_L H_k d^i_R + y^{(k)U}_{ij}\bar{u}^i_R \epsilon H_k {Q'}^j_L$$
$$+y^{(k)E}_{ij}\bar{L}^j_L H_k e^i_R + y^{(k)EN}_{ij}\bar{\nu}^i_R\epsilon H_k {L'}^j_L + \text{h.c.}\Big), \tag{4.5.31}$$

where $\epsilon H_k Q'^j \equiv \epsilon_{ab} H^a_k {Q'}^{jb}_L$ and $\epsilon H_k L'^j \equiv \epsilon_{ab} H^a_k {L'}^{jb}_L$ $(a, b = 1, 2 : SU(2)_L$ indices), and

$$y^{(k)D}_{ij} = y^{(k)E}_{ij} = \frac{ig_4}{\sqrt{2}\cdot 3^{\frac{3}{4}}}\sum_{i'=1}^{12}\sum_{j'=1}^{6}\sum_{k'=1}^{18} V^{(\omega)}_{ii'}[-12,0]V^{(1)*}_{jj'}[6,0]V^{(\omega^2)}_{kk'}[18,0]$$
$$\times\sum_{m=1}^{18}\mathcal{F}^{(-12j'-6i'-72m)}(0,-1296,0)\delta_{j'-i'+6m,k'},$$
$$y^{(k)U}_{ij} = y^{(k)N}_{ij} = \frac{ig_4}{\sqrt{2}\cdot 3^{\frac{3}{4}}}\sum_{i'=1}^{12}\sum_{j'=1}^{6}\sum_{k'=1}^{18} V^{(\omega^2)*}_{ii'}[12,0]V^{(1)}_{jj'}[-6,0]V^{(\omega^2)}_{kk'}[18,0]$$
$$\times\sum_{m=1}^{18}\mathcal{F}^{(-6i'-12j'-72m)}(0,-1296,0)\delta_{i'-j'+12m,k'}, \tag{4.5.32}$$

where $g_4 \sim 0.652$ is the 4D $SU(2)_L$ gauge coupling. We have used

$$\langle\boldsymbol{\mu}_1|E_{\boldsymbol{\alpha}_3}|\boldsymbol{\mu}_3\rangle = \langle\boldsymbol{\mu}_2|E_{\boldsymbol{\alpha}_2}|\boldsymbol{\mu}_3\rangle = \frac{1}{\sqrt{2}},$$
$$\langle-\boldsymbol{\mu}_3|E_{\boldsymbol{\alpha}_3}|-\boldsymbol{\mu}_1\rangle = \langle\boldsymbol{\mu}_3|E_{\boldsymbol{\alpha}_2}|\boldsymbol{\mu}_2\rangle = -\frac{1}{\sqrt{2}}. \tag{4.5.33}$$

Now, the matter contents that appear as the zero-modes are

$$Q^i_L,\ d^i_R,\ {Q'}^i_L,\ u^i_R,\ L^i_L,\ e^i_R,\ {L'}^i_L,\ \nu^i_R, \tag{4.5.34}$$

where $i = 1, 2, 3$. In this case, we must remove extra $SU(2)_L$ doublets. Then, we introduce the brane-localized mass terms:

$$\mathcal{L}_{\rm brane} = \sum_{i=1}^{3}\Big[\bar{\tilde{Q}}^i_R(x)\Big\{c^i_Q\boldsymbol{Q}^i_L(x,z) + {c'}^i_Q{\boldsymbol{Q}'}^i_L(x,z)\Big\}$$
$$+\bar{\tilde{L}}^i_R(x)\Big\{c^i_L\boldsymbol{L}^i_L(x,z) + {c'}^i_L{\boldsymbol{L}'}^i_L(x,z)\Big\} + \text{h.c.}\Big]\delta^{(2)}(z), \tag{4.5.35}$$

[5]This realization of the three generaions occurs only when $N = 3$.

where $\tilde{Q}_R^i$ and $\tilde{L}_R^i$ are brane localized 4D fermions, c_Q, c'_Q, c_L, c'_L are the brane mass parameters that are dimensionless constants, and $\boldsymbol{Q}_L, \boldsymbol{Q'}_L, \boldsymbol{L}_L, \boldsymbol{L'}_L$ are $SU(2)_L$ doublet components of $\Psi_-^1, \Psi_+^2, \Psi_-^3, \Psi_+^4$, respectively. Focusing on the zero-modes, (4.5.35) is rewritten as

$$\begin{aligned}\mathcal{L}_{\text{brane}} = \sum_{i=1}^{3} \Bigg[&\bar{\tilde{Q}}_R^j(x) \left\{ m_{Q0}^{ij} Q_L^i(x) + m'^{ij}_{Q0} Q'^j_L(x) \right\} \\ &+ \bar{\tilde{L}}_R^i(x) \left\{ m_{L0}^{ij} L_L^i(x) + m'^{ij}_{L0} L'^i_L(x) \right\} + \text{h.c.} + \cdots \Bigg] \delta^{(2)}(z), \end{aligned} \tag{4.5.36}$$

where the ellipsis means the terms including the massive KK modes, and

$$\begin{aligned} m_{Q0}^{ij} &\equiv \frac{c_Q^i h_{\text{L0}}^{(-)\boldsymbol{\mu}_1 1(j)}(0)}{\sqrt{2}R_1}, \qquad m'^{ij}_{Q0} \equiv \frac{c'^i_Q h_{\text{L0}}^{(+)\boldsymbol{\mu}_1 2(j)}(0)}{\sqrt{2}R_1}, \\ m_{L0}^{ij} &\equiv \frac{c_Q^i h_{\text{L0}}^{(-)\boldsymbol{\mu}_1 3(j)}(0)}{\sqrt{2}R_1}, \qquad m'^{ij}_{L0} \equiv \frac{c'^i_Q h_{\text{L0}}^{(+)\boldsymbol{\mu}_1 4(j)}(0)}{\sqrt{2}R_1}, \end{aligned} \tag{4.5.37}$$

are effective mass parameters. When these parameters are large enough, the following linear combinations remain in the 4D effective theory:

$$\begin{aligned} q_L^i &\equiv V_Q^{i+3,j} Q_L^j + V_Q^{i+3,j+3} Q'^j_L, \\ l_L^i &\equiv V_L^{i+3,j} L_L^j + V_Q^{i+3,j+3} L'^j_L, \end{aligned} \tag{4.5.38}$$

where $i = 1, 2, 3$, and V_Q and V_L are 6×6 matrices that satisfy

$$\begin{aligned} U_Q(m_{Q0}, m'_{Q0}) V_Q^{-1} &= \begin{pmatrix} \lambda_Q^1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_Q^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_Q^3 & 0 & 0 & 0 \end{pmatrix}, \\ U_L(m_{L0}, m'_{L0}) V_L^{-1} &= \begin{pmatrix} \lambda_L^1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_L^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_L^3 & 0 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{4.5.39}$$

with 3×3 unitary matrices U_Q and U_L. After the extra doublets decoupled, the Lagrangian (4.5.31) is rewritten as

$$\begin{aligned} \mathcal{L}_{\text{yukawa}}^{\text{4D}} = -\sum_{i,j=1}^{3} \sum_{k=1}^{5} \Big(&\tilde{y}_{ij}^{(k)D} \bar{q}_L^j H_k d_R^i + \tilde{y}_{ij}^{(k)U} \bar{u}_R^i \epsilon H_k q_L^j \\ &+ \tilde{y}_{ij}^{(k)E} \bar{l}_L^j H_k e_R^i + \tilde{y}_{ij}^{(k)EN} \bar{\nu}_R^i \epsilon H_k l_L^j + \text{h.c.} \Big), \end{aligned} \tag{4.5.40}$$

where

$$\tilde{y}_{ij}^{(k)D} \equiv y_{ij'}^{(k)D}(V_Q^{-1})^{j',j+3}, \qquad \tilde{y}_{ij}^{(k)U} \equiv y_{ij'}^{(k)U}(V_Q^{-1})^{j'+3,j+3},$$
$$\tilde{y}_{ij}^{(k)E} \equiv y_{ij'}^{(k)E}(V_L^{-1})^{j',j+3}, \qquad \tilde{y}_{ij}^{(k)N} \equiv y_{ij'}^{(k)N}(V_L^{-1})^{j'+3,j+3}. \tag{4.5.41}$$

In order to avoid large flavor-changing process, only one-Higgs-doublet H_{k_0} get a nonvanishing VEV$\langle H_{k_0}\rangle = v$. Then, the fermion masses are eigen-values of the mass matrices:

$$M_{ij}^D = \tilde{y}_{ij}^{(k_0)D}v, \quad M_{ij}^U = \tilde{y}_{ij}^{(k_0)U}v, \quad M_{ij}^E = \tilde{y}_{ij}^{(k_0)E}v, \quad M_{ij}^E = \tilde{y}_{ij}^{(k_0)E}v, \tag{4.5.42}$$

The masse can be controled by tuning c_Q^i, c'^i_Q, c_L^i c'^i_L through the unitary matrices V_Q and V_L. If $V_Q \sim \mathbf{1}_6$, we can realize the hierarchy between m_t and m_b. In this case, the eigen-values of $\tilde{y}_{ij}^{(k_0)U}$ approximate those of $y_{ij}^{(k_0)U}$, $\left|\lambda_i^{(k_0)U}\right|$ $(i = 1, 2, 3)$ we calculated. We found that we can realize the top quark Yukawa coupling, which is 0.921, 0.945 ~ 1, when $k_0 = 2, 5$, respectively. This result means that an enhancement factor $\sqrt{2}$ for the Yukawa couplings can be obtained with the background magnetic fluxes from the overlap integrals, compared to the cases that the zero-mode wave functions have the flat profiles. However we cannot realize large hierarchy among the Yukawa couplings.

Besides, in this case, the five zero-modes of the Higgs doublets degenerate. The situation may be problematic because such large number of Higgs doublets seem difficult to be discovered in the present experiment. These zero-modes originate from the same 6D gauge field. So if they existed, they might well have similar masses. Therefore, we change our focus to the case that only one zero-mode of Higgs doublet appears in the next subsection, ignoring the realization of the matter generations by magnetic fluxes.

4.5.4 One-Higgs-doublet case

When we focus on one-Higgs-doublet, the possible choices of $(k, n_p) = (1, 2)$ or $(2, 0)$. We choose the case $(k, n_p) = (2, 0)$ because the Yukawa couplings are more restricted in the other case. The possible values of k' are -1 or 0 from (4.5.23). In these cases, each component in (4.5.8) has at most one zero-mode.

(i) $\boldsymbol{k' = 0}$

$$y^D = Y^{(-)}\left(n_1, \frac{2l}{3}, \frac{l}{3}\right), \qquad y^U = Y^{(+)}\left(n_2, \frac{4l+l'}{6}, \frac{2l-l'}{6}\right),$$
$$y^E = Y^{(-)}\left(n_3, -\frac{2l+l'}{3}, \frac{l-l'}{3}\right), \qquad y^N = Y^{(+)}\left(n_4, \frac{4l-l'}{6}, -\frac{2l+l'}{6}\right), \tag{4.5.43}$$

where l is an integer, l' is an even number, and

$$\begin{aligned}
Y^{(+)} \equiv & \frac{ig_4}{\sqrt{2}\cdot 3}\sum_{i'=1}^{4}\sum_{j'=1}^{2}\sum_{k'=1}^{6} V_{1i'}^{(\omega^n)*}[4,\phi_1]V_{1j'}^{(\omega^{n+1})}[-2,\phi_2]V_{1k'}^{(\omega^{-1})}[6,\phi_1-\phi_2] \\
& \times \sum_{m=1}^{6}\mathcal{F}^{(-2i'-4j'-8m)}\left(0,-48,\frac{(\phi_1+\phi_2)(\tau-1)}{12}\right)\delta_{i'-j'+4m,k'} \\
Y^{(+)} \equiv & \frac{ig_4}{\sqrt{2}\cdot 3}\sum_{i'=1}^{4}\sum_{j'=1}^{2}\sum_{k'=1}^{6} V_{1i'}^{(\omega^{n+1})}[-4,\phi_1]V_{1j'}^{(\omega^{n})*}[2,\phi_2]V_{1k'}^{(\omega^{-1})}[6,\phi_2-\phi_1] \\
& \times \sum_{m=1}^{6}\mathcal{F}^{(-2i'-4j'-8m)}\left(0,-48,\frac{(\phi_1+\phi_2)(\tau-1)}{12}\right)\delta_{i'-j'+4m,k'}, \qquad (4.5.44)
\end{aligned}$$

where ϕ_a $(a=1,2)$ are defined by $\zeta_a = \frac{2\phi_a}{K_a}(\tau-1)$ (We used them instead of ζ_a.). The possible values of n,ϕ_1,ϕ_2 are

$$\begin{aligned}
& n = 0,\ 1,\ 2, \quad (\text{mod } n=3) \\
& \phi_1 = \phi_2 - \text{floor}(\phi_2) + u, \quad (\text{mod } n=4) \\
& \phi_2 = 0,\ 1/3,\ 2/3,\ 1,\ 4/3,\ 5/3, \quad (\text{mod } n=2) \qquad (4.5.45)
\end{aligned}$$

where $u=0,1,2,3$. The possible numerical values of the Yukawa couplings are

$$\left|y^{D,U,E,N}\right| = 0.191,\ 0.270,\ 0.369,\ 0.522,\ 0.573,\ 0.811. \qquad (4.5.46)$$

(ii) $\boldsymbol{k'=1}$

$$\begin{aligned}
y^D =& Y^{(-)}\left(n_1,\frac{2l}{3},\frac{l}{3}\right), \\
y^U =& \frac{ig_4}{\sqrt{2}\cdot 3^{\frac{1}{4}}}\sum_{i'=1}^{3}\sum_{j'=1}^{3}\sum_{k'=1}^{6} V_{1i'}^{(\omega^{n_2})*}\left[3,\frac{4l+l'}{6}\right]V_{1j'}^{(\omega^{n_2+1})}\left[-3,\frac{2l-l'}{6}\right]V_{1k'}^{(\omega^{-1})}[6,l] \\
& \times \sum_{m=1}^{6}\mathcal{F}^{(-3i'-3j'-9m)}\left(0,-54,\frac{2(l+l')(\tau-1)}{54}\right)\delta_{i'-j'+3m,k'} \\
y^E =& Y^{(+)}\left(n_3,\frac{l-l'}{3},-\frac{2l+l'}{3}\right), \\
y^N =& \frac{ig_4}{\sqrt{2}\cdot 3^{\frac{1}{4}}}\sum_{i'=1}^{5}\sum_{k'=1}^{6} V_{1i'}^{(\omega^{n_4})*}\left[5,\frac{4l-l'}{6}\right]V_{11}^{(\omega^{n_4+1})}\left[-1,-\frac{2l+l'}{6}\right]V_{1k'}^{(\omega^{-1})}[6,l] \\
& \times \sum_{m=1}^{6}\mathcal{F}^{(-i'-5-5m)}\left(0,-30,-\frac{(l-l')(\tau-1)}{45}\right)\delta_{i'-1+5m,k'} \qquad (4.5.47)
\end{aligned}$$

The possible numerical values of the Yukawa couplings are

$$\begin{aligned} \left|y^{D,E}\right| &= 0.191,\ 0.270,\ 0.369,\ 0.522,\ 0.573,\ 0.811, \\ \left|y^{U}\right| &= 0.365,\ 0.430,\ 0.461,\ 0.667,\ 0.798, \\ \left|y^{N}\right| &= 0.101,\ 0.176,\ 0.188,\ 0.288,\ 0.533,\ 0.541,\ 0.559,\ 0.924. \end{aligned} \tag{4.5.48}$$

We found that the region of the numerical values of the Yukawa coupling constants is [0.1, 1] in both the case with the three generations and the case with one-Higgs-doublet. So we conclude that we cannot realize the Yukawa hierarchy only with the backgound mgnetic fluxes and the Wilsn-line phases.

4.6 Discussion

We introduced the constant magnetic fluxes penetrating the compactified space as backgounds of gauge field strengths to realize the matter generations and the Yukawa hierarchy. The overlap integrals of the Yukawa couplings deviate from the constant profile due to the shifts for zero-mode wave functions (the Jacobi-theta functions), induced by the Wilson-line phases (or the Scherk-Schwarz phases) they feel.

We considered the 6D GHU models whose gauge groups are $G \times U(1)_X$ (G: simple Lie group) and extra dimensions are compactified by T^2/Z_N ($N = 2, 3, 4, 6$). Magnetic fluxes are introduced for the $U(1)_X$ and the Cartan components of G. The zero-modes of the fields feeling the magnetIc fluxes degenerate with the number depending on the values of fluxes and the Wilson-line phases they feel, and the zero-mode orbifold boundary conditions. These parameters are discrete and the available Wilson-line phases are constrained by the values of N.

As a simplest example, we selected the $SU(3)\times U(1)_X$ model with four 6D Weyl fermions that belong to $\mathbf{3}$ or $\bar{\mathbf{3}}$ of $SU(3)$. In this model, the Yukawa sector is determined by nine integers. From Refs.[39], the matter field content that appear in SM can be realized with three generations only on T^2/Z_3, also using the brane localized mass terms of 4D heavy fermions to decople the extra $SU(2)_L$ doublet fermions. However, we faced a problem.

The signs of the flux values that the left- and the right-handed fermions in one 6D flavor feel are reverse because the 6D chiralities for the left- and the right-handed fermions in one gauge multiplet are the same, and they feel the common background magnetic fluxes. Then, the absolute value of the flux that the Higgs doublet feel equals to the sum of the absolute values of the fluxes that the left- and the right-handed fermions coupling to the Higgs field feel, for the gauge symmetry of the Yukawa terms, and the sum value is often large. Therefore, in GHU models with magnetic fluxes, the degeneration number of the zero-mode Higgs fields become large. We found at least five Higgs doublets are needed to realize the three generations of the matter fields in the simplest example on T^2/Z_3. Therefore, we changed our focus to a case with only one-Higgs-doublet.

The numerical results for the values of the Yukawa couplings are as follows: We could realize the large number such as 1. The result indicates that we can realize the top quark mass without a large representation by magnetic fluxes in the case with the three generations of matter fields. However, we could not realize the values for the Yukawa coupling constants of the matter fields other than the top quark because the smallest value is $\mathcal{O}(0.1)$ in either case of three generation and one-Higgs-doublet case. The shifts of that zero-mode wave functions on T^2/Z_N with magnetic fluxes are restricted to some discrete values. The mode functions on T^2/Z_N ($N = 3, 4, 6$) are given by the mixtures of T^2 mode functions. This fact makes the profiles of mode functions complicated. So we conclude that we cannot realize the large Yukawa hierarchy only with magnetic fluxes and the Wilson-line phases.

I think the fact that the Wilson-line phases are not entirely free parameters on T^2/Z_N and the patterns of the shifts of the zero-mode wave functions are constrained by the orbifold is desirable. However, this restriction also makes it too difficult to realize all the Yukawa couplings for the matter fields other than the top quark. This problem can be solved by compactifying with the other manifold such as T^2 since it enables to take the Wilson-line phases entirely freely by hand, but we cannot consider the interactions localized on the fixed points in this case.

We should check the effects of 4D localized mass terms to realize the Yukawa hierarchy because these are expected to help the realization of the desirable values of the Yukawa coupling constants by tuning their mass parameters. And we must check the effects of KK mixing induced by the 4D localized terms, and the backgrounds of $W_z^{\boldsymbol{\alpha}}$. Such effects are closely related to the deviations of 4D effective couplings from the values of SM. The realization of mixing angles for matter fields by magnetic fluxes in GHU models is subject to investigate, too. These issues are left for our future works.

Chapter 5

Summary

6D GHU models are phenomenologocally attractive because the existence of Higgs quartic couplings at tree level make it easier to reproduce the experimental value of the Higgs mass and large KK masses above the experimental lower bound and background magnetic fluxes can be introduced to realize the matter generations from a single bulk fermion. In this paper, we have mainly discussed 6D GHU models on a T^2/Z_N orbifold.

We selected the gauge groups, the orbifold compactifying the extra dimensions, and the representation of the 6D fermion that the 3rd generation quarks are emmbeded into by imposing the requirements for models to have the custodial symmetry and the experimental value of the top quark mass. We find that the best candidate is a $U(4)$ gauge theory on T^2/Z_3, and the 3rd generation quarks are emmbeded into $SU(4)$ $\mathbf{20}'$.

I also discussed a case that the magnetic fluxes are present. In this case, there is a possibility to realize the generations of matter fields from a single 6D fermion, and the hierarchy among the Yukawa couplings. Especially we can realize the top quark Yukawa coupling without introducing a large representation of the matter fields thanks to nontrivial profiles of the zero-mode wave functions. However, we found that it is difficult to realize a hierarchical structure of the Yukawa couplings in cases that the three generations of matter fermions or one-Higgs-doublet are realized. This difficulty stems from the fact that the profiles of the mode functions become complicated on T^2/Z_N $(N = 3, 4, 6)$ compared with the cases of T^2 or T^2/Z_2. The 4D localized mass terms may help to realize the Yukawa hierarchy in the former case.

It is known that the realization of the Yukawa sector is one of challenging issues of GHU models since all the Yukawa couplings originate from 6D gauge couplings and thus become universal in the simplest setup. The background magnetic fluxes and the Wilson-line phases can save these problems. We should check how the results of the Chapter 3 changes when they are introduced into the models. As I mentioned in the previous chapter, we neglected the effects of background of the Higgs when we calculate the KK (zero-)mode wave functions. Originally, we must consider the effects of the VEV of the Wilson-line phase θ_H after the EW symmetry is broken. And we should consider the case that the non-diagonal parts of the extra dimensional components in the G gauge field strength have constant backgrounds. When we calculate the Yukawa couplings with the background of the Higgs, we will get some different results about the Yukawa sector, I think. We must also consider the one-loop effective potential of the Higgs with the magnetic fluxes and the Wilson-line phases in 6D case in order to evaluate the Higgs mass spectrum exactly. I hope these future attempts will help to construct realistic 6D GHU models.

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Appendix A

Cartan-Weyl basis

The generators of a simple group G whose rank is r in the Cartan-Weyl basis are H_i $(i = 1, \cdots, r)$ and E_α, which satisfy

$$\begin{aligned} & H_i^\dagger = H_i, \qquad E_\alpha^\dagger = E_{-\alpha}, \\ & [H_i, H_j] = 0, \qquad [H_i, E_\alpha] = \alpha_i E_\alpha, \\ & [E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha,\beta}, \qquad [E_\alpha, E_{-\alpha}] = \alpha \cdot H, \end{aligned} \tag{A.0.1}$$

where α, β are the root vectors, and $\alpha \neq \beta$. A complex constant $N_{\alpha,\beta}$ is nonzero only when $\alpha + \beta$ is a root, and satisfies the following equations.

$$N_{\alpha,\beta} = -N_{\beta,\alpha} = -N^*_{-\alpha,-\beta} = N_{\beta,-\alpha-\beta} = N_{-\alpha-\beta,\alpha}. \tag{A.0.2}$$

For a series of the weights $\{\mu - q\alpha, \cdots, \mu - \alpha, \mu, \mu + \alpha, \cdots, \mu + p\alpha\}$, where neither $\mu - (q+1)\alpha$ nor $\mu + (p+1)\alpha$ is a weight, it follows that

$$\frac{2\alpha \cdot \mu}{|\alpha|^2} = q - p, \qquad |N_{\alpha,\mu}|^2 = \frac{p(q+1)\,|\alpha|^2}{2}, \tag{A.0.3}$$

where a complex constant $N_{\alpha,\mu}$ is defined as $E_\alpha|\mu\rangle = N_{\alpha,\mu}|\mu + \alpha\rangle$. The generators are normalized as

$$\mathrm{tr}(H_i H_j) = \delta_{ij}, \qquad \mathrm{tr}(H_i E_\alpha) = 0, \qquad \mathrm{tr}(E_\alpha E_\beta) = \delta_{\alpha,-\beta}. \tag{A.0.4}$$

Appendix B

T^2/Z_N orbifold boundary conditions

The orbifold T^2/Z_N is defined by identifying points of $\mathbb{R}^2$ by a discrete group Γ which is generated by three descrete transformations $\mathcal{O}_1$: $z \to z+1$, $\mathcal{O}_\tau$: $z \to z+\tau$ and $\mathcal{O}_\omega$: $z \to \omega z$. Field values of a 6D field at Γ-equivalent points must be related to each other through gauge transformations [1] in order for the Lagrangian to be single-valued on T^2/Z_N. Thus the most general orbifold boundary conditions are given by [51]

$$\begin{aligned}
&A_M(x, z+1) = T_1 A_M(x,z) T_1^{-1}, \\
&B_\mu^Z(x, z+1) = B_\mu^Z(x,z), \qquad B_z^Z(x,z+1) = B_z^Z(x,z), \\
&\Psi_{\chi_6}(x, z+1) = e^{i\varphi_1} T_1 \Psi_{\chi_6}(x,z),
\end{aligned} \tag{B.0.1}$$

for the translation $\mathcal{O}_1$,

$$\begin{aligned}
&A_M(x, z+\tau) = T_\tau A_M(x,z) T_\tau^{-1}, \\
&B_\mu^Z(x, z+\tau) = B_\mu^Z(x,z), \qquad B_z^Z(x,z+\tau) = B_z^Z(x,z), \\
&\Psi_{\chi_6}(x, z+\tau) = e^{i\varphi_\tau} T_\tau \Psi_{\chi_6}(x,z),
\end{aligned} \tag{B.0.2}$$

for the translation $\mathcal{O}_\tau$, and

$$\begin{aligned}
&A_\mu(x, \omega z) = P A_\mu(x,z) P^{-1}, \qquad A_z(x,\omega z) = \omega^{-1} P A_z(x,z) P^{-1}, \\
&B_\mu^Z(x, \omega z) = B_\mu^Z(x,z), \qquad B_z^Z(x,\omega z) = \omega^{-1} B_z^Z(x,z), \\
&\Psi_{\chi_4,\chi_6}(x, \omega z) = \omega^{-\frac{\chi_4\chi_6}{2}} e^{i\varphi_\omega} P \Psi_{\chi_4,\chi_6},
\end{aligned} \tag{B.0.3}$$

[1] More properly, they are related through automorphisms of the Lie algebra of G. For simplicity, we do not consider a case of outer automorphisms [34].

for the Z_N twist $\mathcal{O}_\omega$. Matrices T_1, T_τ and P are elements of G, and φ_1 and φ_τ are the Scherk-Schwarz phases. A factor ω^{-1} and $\omega^{-\frac{\chi_4\chi_6}{2}}$ in (B.0.3) appears because B_z^Z and Ψ_{χ_4,χ_6} are charged under the rotation in the extra-dimensional space. Since $(\omega^{-\frac{\chi_4\chi_6}{2}})^N = -1$, the phase φ_ω is determined so that

$$e^{iN\varphi_\omega} P^N = -\mathbf{1}. \tag{B.0.4}$$

The matrices T_1, T_τ and P satisfy the relations,

$$[T_1, T_\tau] = 0, \qquad P^N = \mathbf{1},$$
$$P^{-1}T_1P = \begin{cases} T_1^{-1} & (N=2) \\ T_\tau^{-1}T_1^{-1} & (N=3) \\ T_\tau^{-1} & (N=4) \\ T_\tau^{-1}T_1 & (N=6) \end{cases}, \qquad P^{-1}T_\tau P = \begin{cases} T_\tau^{-1} & (N=2) \\ T_1 & (N=3,4,6) \end{cases}, \tag{B.0.5}$$

which reflect the properties of $\mathcal{O}_1$, $\mathcal{O}_\tau$ and $\mathcal{O}_\omega$.

Here we perform a gauge transformation,

$$A_M \to U A_M U^{-1} + iU\partial_M U^{-1}, \qquad \Psi \to U\Psi, \tag{B.0.6}$$

where

$$U(z) \equiv \exp\left\{ -\frac{\mathrm{Im}\,(\tau\bar{z})}{\mathrm{Im}\,\tau} \ln T_1 - \frac{\mathrm{Im}\,z}{\mathrm{Im}\,\tau} \ln T_\tau \right\}, \tag{B.0.7}$$

Using (B.0.5), we can show that

$$U(z+1) = U(z)T_1^{-1}, \qquad U(z+\tau) = U(z)T_\tau^{-1},$$
$$P^{-1}U(\omega z)P = U(z), \qquad P^{-1}\left(iU\partial_z U^{-1}\right)P = \omega^{-1}\left(iU\partial_z U^{-1}\right). \tag{B.0.8}$$

Thus, the matrices T_1 and T_τ in (B.0.1) and (B.0.2) can be absorbed by this gauge transformation, while the conditions in (B.0.3) are unchanged. Since we need the fermionic zero-modes, we assume that $\varphi_1 = \varphi_\tau = 0$ for the fermion that the quarks are embedded. Then the orbifold boundary conditions are reexpressed as (3.2.8) and (3.5.2).

Appendix C

Decomposition of representation of G

Here we list various representations of G =SO(5),SU(4),SO(7), and their irreducible decompositions to multiplets of the $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R} (\times \mathrm{U}(1)_X)$ subgroup.

Each representation is specified by the Dynkin coefficients m_i $(i = 1, \cdots, r)$, and the highest weight is expressed as $\mu_{\max} = \sum_i m_i \mu_i$, where μ_i are fundamental weights. The dimension of the representation is calculated by the Weyl dimension formula:

$$\dim \mathcal{R} = \prod_l \frac{\sum_i (m_i + 1) l_i \left|\alpha_i\right|^2}{\sum_i l_i \left|\alpha_i\right|^2}, \tag{C.0.1}$$

where α_i are simple roots, and l_i are numbers such that $\sum_i l_i \alpha_i$ are positive roots. We consider irreducible representations whose dimensions are less than 30 in the following.[1]

C.1 SO(5)

The dimension formula (C.0.1) becomes

$$\dim \mathcal{R} = \frac{1}{6}(m_1 + 1)(m_2 + 1)(m_1 + m_2 + 2)(2m_1 + m_2 + 3). \tag{C.1.1}$$

The decompositions to the irreducible representation of $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}$ are as follows.

$[\boldsymbol{m_1}, \boldsymbol{m_2}] = [\mathbf{1}, \mathbf{0}]$

$$\mathbf{5} = (\mathbf{2}, \mathbf{2}) + (\mathbf{1}, \mathbf{1}). \tag{C.1.2}$$

$[\boldsymbol{m_1}, \boldsymbol{m_2}] = [\mathbf{0}, \mathbf{1}]$

$$\mathbf{4} = (\mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{2}). \tag{C.1.3}$$

[1] The irreducible decompositions of other representations and the weights of each representation are easily obtained by using LieART [52].

$[\boldsymbol{m_1}, \boldsymbol{m_2}] = [\mathbf{2}, \mathbf{0}]$

$$\mathbf{14} = (\mathbf{3}, \mathbf{3}) + (\mathbf{2}, \mathbf{2}) + (\mathbf{1}, \mathbf{1}). \tag{C.1.4}$$

$[\boldsymbol{m_1}, \boldsymbol{m_2}] = [\mathbf{1}, \mathbf{1}]$

$$\mathbf{16} = (\mathbf{3}, \mathbf{2}) + (\mathbf{2}, \mathbf{3}) + (\mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{2}). \tag{C.1.5}$$

$[\boldsymbol{m_1}, \boldsymbol{m_2}] = [\mathbf{0}, \mathbf{2}]$

This is the adjoint representation and decomposed as (3.3.2).

$[\boldsymbol{m_1}, \boldsymbol{m_2}] = [\mathbf{0}, \mathbf{3}]$

$$\mathbf{20} = (\mathbf{4}, \mathbf{1}) + (\mathbf{3}, \mathbf{2}) + (\mathbf{2}, \mathbf{3}) + (\mathbf{1}, \mathbf{4}). \tag{C.1.6}$$

C.2 SU(4)

The dimension formula (C.0.1) becomes

$$\begin{aligned} \dim \mathcal{R} = \frac{1}{12}&(m_1+1)(m_2+1)(m_3+1)(m_1+m_2+2) \\ &\times(m_2+m_3+2)(m_1+m_2+m_3+3). \end{aligned} \tag{C.2.1}$$

The decompositions to the irreducible representation of $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}} \times \mathrm{U}(1)_X$ are as follows.

$[\boldsymbol{m_1}, \boldsymbol{m_2}, \boldsymbol{m_3}] = [\mathbf{1}, \mathbf{0}, \mathbf{0}]$

$$\mathbf{4} = (\mathbf{2}, \mathbf{1})_{+\mathbf{1}} + (\mathbf{1}, \mathbf{2})_{-\mathbf{1}}. \tag{C.2.2}$$

$[\boldsymbol{m_1}, \boldsymbol{m_2}, \boldsymbol{m_3}] = [\mathbf{0}, \mathbf{1}, \mathbf{0}]$

$$\mathbf{6} = (\mathbf{2}, \mathbf{2})_{\mathbf{0}} + (\mathbf{1}, \mathbf{1})_{+\mathbf{2}} + (\mathbf{1}, \mathbf{1})_{-\mathbf{2}}. \tag{C.2.3}$$

$[\boldsymbol{m_1}, \boldsymbol{m_2}, \boldsymbol{m_3}] = [\mathbf{0}, \mathbf{0}, \mathbf{1}]$

$$\bar{\mathbf{4}} = (\mathbf{2}, \mathbf{1})_{-\mathbf{1}} + (\mathbf{1}, \mathbf{2})_{+\mathbf{1}}. \tag{C.2.4}$$

$[\boldsymbol{m_1}, \boldsymbol{m_2}, \boldsymbol{m_3}] = [\mathbf{1}, \mathbf{0}, \mathbf{1}]$

This the adjoint representation and decomposed as (3.3.14).

$[\boldsymbol{m_1}, \boldsymbol{m_2}, \boldsymbol{m_3}] = [\mathbf{0}, \mathbf{1}, \mathbf{1}]$

$$\mathbf{20} = (\mathbf{3}, \mathbf{2})_{-\mathbf{1}} + (\mathbf{2}, \mathbf{3})_{+\mathbf{1}} + (\mathbf{2}, \mathbf{1})_{+\mathbf{1}} + (\mathbf{2}, \mathbf{1})_{-\mathbf{3}} + (\mathbf{1}, \mathbf{2})_{+\mathbf{3}} + (\mathbf{1}, \mathbf{2})_{-\mathbf{1}}. \tag{C.2.5}$$

$$[\boldsymbol{m_1}, \boldsymbol{m_2}, \boldsymbol{m_3}] = [\mathbf{0}, \mathbf{2}, \mathbf{0}]$$
$$\mathbf{20'} = (\mathbf{3}, \mathbf{3})_{\mathbf{0}} + (\mathbf{2}, \mathbf{2})_{\mathbf{+2}} + (\mathbf{2}, \mathbf{2})_{\mathbf{-2}} + (\mathbf{1}, \mathbf{1})_{\mathbf{+4}} + (\mathbf{1}, \mathbf{1})_{\mathbf{-4}} + (\mathbf{1}, \mathbf{1})_{\mathbf{0}}. \tag{C.2.6}$$

$$[\boldsymbol{m_1}, \boldsymbol{m_2}, \boldsymbol{m_3}] = [\mathbf{1}, \mathbf{1}, \mathbf{0}]$$
$$\overline{\mathbf{20}} = (\mathbf{3}, \mathbf{2})_{\mathbf{+1}} + (\mathbf{2}, \mathbf{3})_{\mathbf{-1}} + (\mathbf{2}, \mathbf{1})_{\mathbf{+3}} + (\mathbf{2}, \mathbf{1})_{\mathbf{-1}} + (\mathbf{1}, \mathbf{2})_{\mathbf{+1}} + (\mathbf{1}, \mathbf{2})_{\mathbf{-3}}. \tag{C.2.7}$$

$$[\boldsymbol{m_1}, \boldsymbol{m_2}, \boldsymbol{m_3}] = [\mathbf{0}, \mathbf{0}, \mathbf{3}]$$
$$\mathbf{20''} = (\mathbf{4}, \mathbf{1})_{\mathbf{-3}} + (\mathbf{3}, \mathbf{2})_{\mathbf{-1}} + (\mathbf{2}, \mathbf{3})_{\mathbf{+1}} + (\mathbf{1}, \mathbf{4})_{\mathbf{+3}}. \tag{C.2.8}$$

$$[\boldsymbol{m_1}, \boldsymbol{m_2}, \boldsymbol{m_3}] = [\mathbf{3}, \mathbf{0}, \mathbf{0}]$$
$$\overline{\mathbf{20}}'' = (\mathbf{4}, \mathbf{1})_{\mathbf{+3}} + (\mathbf{3}, \mathbf{2})_{\mathbf{+1}} + (\mathbf{2}, \mathbf{3})_{\mathbf{-1}} + (\mathbf{1}, \mathbf{4})_{\mathbf{-3}}. \tag{C.2.9}$$

C.3 SO(7)

The dimension formula (C.0.1) becomes

$$\begin{aligned}\dim \mathcal{R} = \frac{1}{720}&(m_1+1)(m_2+1)(m_3+1)(m_1+m_2+2)(m_2+m_3+2)(2m_2+m_3+3)\\ &\times(m_1+m_2+m_3+3)(m_1+2m_2+m_3+4)(2m_1+2m_2+m_3+5).\end{aligned} \tag{C.3.1}$$

The $\mathrm{SU(2)_L} \times \mathrm{SU(2)_R}$ subgroup is chosen as $(\alpha_L, \alpha_R) = (\boldsymbol{e}^1 + \boldsymbol{e}^2, \boldsymbol{e}^1 - \boldsymbol{e}^2)$. The decompositions to the irreducible representation of $\mathrm{SU(2)_L} \times \mathrm{SU(2)_R} \times \mathrm{U(1)}_X$ are as follows.

$$[\boldsymbol{m_1}, \boldsymbol{m_2}, \boldsymbol{m_3}] = [\mathbf{1}, \mathbf{0}, \mathbf{0}]$$
$$\mathbf{7} = (\mathbf{2}, \mathbf{2})_{\mathbf{0}} + (\mathbf{1}, \mathbf{1})_{\mathbf{+1}} + (\mathbf{1}, \mathbf{1})_{\mathbf{-1}} + (\mathbf{1}, \mathbf{1})_{\mathbf{0}}. \tag{C.3.2}$$

$[\boldsymbol{m_1}, \boldsymbol{m_2}, \boldsymbol{m_3}] = [\mathbf{0}, \mathbf{1}, \mathbf{0}]$ This is the adjoint representation and decomposed as (3.3.21).

$$[\boldsymbol{m_1}, \boldsymbol{m_2}, \boldsymbol{m_3}] = [\mathbf{0}, \mathbf{0}, \mathbf{1}]$$
$$\mathbf{8} = (\mathbf{2}, \mathbf{1})_{\mathbf{+1/2}} + (\mathbf{2}, \mathbf{1})_{\mathbf{-1/2}} + (\mathbf{1}, \mathbf{2})_{\mathbf{+1/2}} + (\mathbf{1}, \mathbf{2})_{\mathbf{-1/2}}. \tag{C.3.3}$$

$$[\boldsymbol{m_1}, \boldsymbol{m_2}, \boldsymbol{m_3}] = [\mathbf{2}, \mathbf{0}, \mathbf{0}]$$
$$\begin{aligned}\mathbf{27} = &(\mathbf{3}, \mathbf{3})_{\mathbf{0}} + (\mathbf{2}, \mathbf{2})_{\mathbf{+1}} + (\mathbf{2}, \mathbf{2})_{\mathbf{-1}} + (\mathbf{2}, \mathbf{2})_{\mathbf{0}} + (\mathbf{1}, \mathbf{1})_{\mathbf{0}}\\ &+(\mathbf{1}, \mathbf{1})_{\mathbf{+2}} + (\mathbf{1}, \mathbf{1})_{\mathbf{+1}} + (\mathbf{1}, \mathbf{1})_{\mathbf{0}} + (\mathbf{1}, \mathbf{1})_{\mathbf{-1}} + (\mathbf{1}, \mathbf{1})_{\mathbf{-2}}.\end{aligned} \tag{C.3.4}$$

Appendix D

Absorption of Wilson-line phases

We show that "the Wilson-line phases" c^i and b in (4.2.18) can be absorbed into "the Scherk-Schwarz phases". We redefine the field as

$$\begin{aligned} A'_M &= V A_M V^{-1} + iV\partial_M V^{-1}, \\ B'_M &= B_M + \partial_M \Lambda, \\ \Psi'^f &= e^{iq_f\Lambda}\Psi^f, \end{aligned} \tag{D.0.1}$$

where

$$\begin{aligned} V &\equiv \exp\left(i\sum_i \mathrm{Im}(\bar{v}^i z)H_i\right), \\ \Lambda &\equiv \mathrm{Im}(\bar{\lambda}z), \end{aligned} \tag{D.0.2}$$

with complex constants v^i and λ. (D.0.1) is rewritten as

$$\begin{aligned} C'^i_M &= C^i_M - \frac{i\bar{v}}{2}\delta_{Mz} + \frac{iv}{2}\delta_{M\bar{z}}, \\ W'^{\boldsymbol{\alpha}}_M &= \exp\left\{i\sum_i \mathrm{Im}(\bar{v}^i z)\boldsymbol{\alpha}_i W^{\boldsymbol{\alpha}}_M\right\}, \\ B'_M &= B_M - \frac{i\bar{\lambda}}{2}\delta_{Mz} + \frac{i\lambda}{2}\delta_{M\bar{z}}, \\ \Psi'^f &= \exp\left\{iq_f\left(\mathrm{Im}(\bar{\lambda}z) + \sum_i \mathrm{Im}(\bar{v}^i z)H_i\right)\right\}\Psi^f. \end{aligned} \tag{D.0.3}$$

The Lagrangian density is invariant under the field redefinition: $(A, B, \Psi) \to (A', B', \Psi')$. By this redefinition, the background values of the vector potential in (4.2.18) are shifted as

$$\langle C_z^i \rangle \to \left\langle C_z^i - \frac{i\bar{v}^i}{2} \right\rangle = -\frac{iN(\mathcal{C}^i \bar{z} + \bar{c}^i)}{4\mathrm{Im}\tau} - \frac{iv^i}{2},$$
$$\langle B_z \rangle \to \left\langle B_z - \frac{i\bar{\lambda}}{2} \right\rangle = -\frac{iN(\mathcal{B}\bar{z} + b)}{4\mathrm{Im}\tau} - \frac{i\lambda}{2}. \tag{D.0.4}$$

Then, we can cancel c^i and b if we choose V and Λ in (D.0.1) as

$$v^i = -\frac{Nc^i}{2\mathrm{Im}\tau}, \qquad \lambda = -\frac{Nb}{2\mathrm{Im}\tau}. \tag{D.0.5}$$

The torus boundary conditions (4.2.6) become

$$\begin{aligned}
C'^i_M(x, z+s) &= C'^i_M(x,z) + \frac{N\mathcal{C}^i}{4\mathrm{Im}\tau}(-i\bar{s}\delta_{Mz} + is\delta_{M\bar{z}}), \\
W'^{\boldsymbol{\alpha}}_M(x, z+s) &= \exp\left\{ i \left(\frac{N\mathcal{C}\mathrm{Im}(\bar{s}z)}{2\mathrm{Im}\tau} + \Phi_s + \mathrm{Im}(\bar{v}s) \right) \cdot \boldsymbol{\alpha} \right\} W'^{\boldsymbol{\alpha}}_M(x,z), \\
B'_M(x, z+s) &= B'_M(x,z) + \frac{N\mathcal{B}}{4\mathrm{Im}\tau}(-i\bar{s}\delta_{Mz} + is\delta_{M\bar{z}}), \\
\Psi'^f(x, z+s) &= e^{iq_f(\varphi_s + \mathrm{Im}(\bar{\lambda}s))} \exp\left\{ i\phi_s + \mathrm{Im}(\bar{v}s) \cdot H \right\} \\
&\quad \times e^{iq_f\chi_s(z)} U_s(z) \Psi'^f(x,z),
\end{aligned} \tag{D.0.6}$$

where $s = 1, \tau$. This means the Scherk-Schwarz phases are shifted as

$$\Phi_s^i \to \Phi_s^i - \mathrm{Im}(\bar{s}v^i) = \Phi_s^i + \mathrm{Im}\left(\bar{s}\frac{Nc^i}{2\mathrm{Im}\tau} \right),$$
$$\varphi_s^i \to \varphi_s^i - \mathrm{Im}(\bar{s}\lambda) = \varphi_s + \mathrm{Im}\left(\bar{s}\frac{Nb}{2\mathrm{Im}\tau} \right). \tag{D.0.7}$$

In this way, the Wilson-line phases are absorbed into the Scherk-Schwarz phases.

Appendix E

Possible values of the Scherk-Schwarz phases

As referred in [39], the Scherk-Schwarz phases (in the absence of the Wilson-line phases) are restricted to the discrete values on T^2/Z_N.

We define

$$\mathcal{F}^{\omega}(x,z) \equiv \mathcal{F}(x,\omega z), \tag{E.0.1}$$

where $\mathcal{F} = C^i_M, W^{\boldsymbol{\alpha}}_M, B_M, \hat{\Psi}^f$. $\mathcal{F}^{\omega}$ satisfies the same boundary conditions as those of $\mathcal{F}(x,z)$ from (4.2.7). Here, we express the torus boundary condition as

$$\mathcal{F}(x,z+s) = \mathcal{U}_s(z)\mathcal{F}(x,z), \tag{E.0.2}$$

where $s = 1, \tau$ and $\mathcal{U}_s(z)$ is an operator that acts on $\mathcal{F}(x,z)$. Then,

$$\begin{aligned}
\mathcal{F}^{\omega}(x,z+s) &= \mathcal{U}_s(z)\mathcal{F}^{\omega}(x,z) = \mathcal{U}_s(z)\mathcal{F}(x,\omega z) \qquad (s=1,\tau), \\
\mathcal{F}^{\omega}(x,z+1) &= \begin{cases} \mathcal{F}(x,-z-1) = \mathcal{U}^{-1}(-z-1)\mathcal{F}(x,-z) & (N=1) \\ \mathcal{F}(x,\omega z+\tau) = \mathcal{U}_{\tau}(\omega z)\mathcal{F}(x,\omega z) & (N\neq 2) \end{cases}, \\
\mathcal{F}(x,z+\tau) &= \mathcal{F}(x,\omega z+\omega\tau) \\
&= \begin{cases} \mathcal{U}_{\tau}^{-1}(-z-\tau)\mathcal{F}(x,-z) & (N=2) \\ \mathcal{U}_{\tau}^{-1}(\omega z-1-\tau)\mathcal{U}_1^{-1}(\omega z-1)\mathcal{F}(x,\omega z) & (N=3) \\ \mathcal{U}_1^{-1}(\omega z-1)\mathcal{F}(x,\omega z) & (N=4) \\ \mathcal{U}_{\tau}(\omega z-1)\mathcal{U}_1^{-1}(\omega z-1)\mathcal{F}(x,\omega z) & (N=6) \end{cases}.
\end{aligned} \tag{E.0.3, E.0.4}$$

Here, we used the relation between ω and τ:

$$\omega\tau = \begin{cases} -\tau & (N=2) \\ -1-\tau & (N=3) \\ -1 & (N=4) \\ -1+\tau & (N=6) \end{cases} . \tag{E.0.5}$$

From (E.0.3) and (E.0.4), we find

$$\mathcal{U}_1(z) = \begin{cases} \mathcal{U}_1^{-1}(-z-1) & (N=2) \\ \mathcal{U}_\tau(\omega z) & (N\neq 2) \end{cases} ,$$

$$\mathcal{U}_\tau(z) = \begin{cases} \mathcal{U}_\tau^{-1}(-z-\tau) & (N=2) \\ \mathcal{U}_\tau^{-1}(\omega z-1-\tau)\mathcal{U}_1^{-1}(\omega z-1) & (N=3) \\ \mathcal{U}_1(\omega z-1) & (N=4) \\ \mathcal{U}_\tau(\omega z-1)\mathcal{U}_1^{-1}(\omega z-1) & (N=6) \end{cases} . \tag{E.0.6}$$

These conditions are rewritten with the Scherk-Schwarz phases in (4.2.45) as

$$\begin{aligned} \phi_s^{\boldsymbol{\alpha}} &= -\phi_s^{\boldsymbol{\alpha}}, \quad \phi_s^{\boldsymbol{\mu}f} = -\phi_s^{\boldsymbol{\mu}f}, \qquad (N=2) \\ \phi_1^{\boldsymbol{\alpha}} = \phi_\tau^{\boldsymbol{\alpha}} &= \begin{cases} -k_{\boldsymbol{\alpha}}/2 - \phi_\tau^{\boldsymbol{\alpha}} - \phi_1^{\boldsymbol{\alpha}} & (N=3) \\ -\phi_1^{\boldsymbol{\alpha}} & (N=4) \\ k_{\boldsymbol{\alpha}}/2 + \phi_\tau^{\boldsymbol{\alpha}} - \phi_1^{\boldsymbol{\alpha}} & (N=6) \end{cases} , \\ \phi_1^{\boldsymbol{\mu}f} = \phi_\tau^{\boldsymbol{\mu}f} &= \begin{cases} -k_{\boldsymbol{\mu}f}/2 - \phi_\tau^{\boldsymbol{\mu}f} - \phi_1^{\boldsymbol{\mu}f} & (N=3) \\ -\phi_1^{\boldsymbol{\mu}f} & (N=4) \\ k_{\boldsymbol{\mu}f}/2 + \phi_\tau^{\boldsymbol{\mu}f} - \phi_1^{\boldsymbol{\mu}f} & (N=6) \end{cases} , \end{aligned} \tag{E.0.7}$$

mode 1. Solving these equations, we find the possible values of the Scherk-Schwarz phases are

$$(\phi_1^{\boldsymbol{\alpha}}, \phi_\tau^{\boldsymbol{\alpha}}),\ (\phi_1^{\boldsymbol{\mu}f}, \phi_\tau^{\boldsymbol{\mu}f}) = \begin{cases} (0,0),\ (\frac{1}{2},0),\ (0,\frac{1}{2}),\ (\frac{1}{2},\frac{1}{2}) & (N=2) \\ (0,0),\ (\frac{1}{3},\frac{1}{3}),\ (\frac{2}{3},\frac{2}{3}) & (N=3,\ k=\text{even}) \\ (\frac{1}{6},\frac{1}{6}),\ (\frac{1}{2},\frac{1}{2}),\ (\frac{5}{6},\frac{5}{6}) & (N=3,\ k=\text{odd}) \\ (0,0),\ (\frac{1}{2},\frac{1}{2}) & (N=4) \\ (0,0) & (N=6,\ k=\text{even}) \\ (\frac{1}{2},\frac{1}{2}) & (N=6,\ k=\text{odd}) \end{cases} , \tag{E.0.8}$$

mod 1. Here, $k=k_{\boldsymbol{\alpha}}$ for $(\phi_1^{\boldsymbol{\alpha}}, \phi_\tau^{\boldsymbol{\alpha}})$, and $k=k_{\boldsymbol{\mu}f}$ for $(\phi_1^{\boldsymbol{\mu}f}, \phi_\tau^{\boldsymbol{\mu}f})$, respectively.

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