Multiple M2-branes and Janus Couplings

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Abstract

Recently there has been a remarkable progress in constructing $\mathcal{N} = 8$ supersymmetric three-dimensional field theory with $SO(8)$ R-symmetry by Bagger, Lambert and Gustavsson (BLG model) which can be considered as the effective action of multiple M2-branes. Another very interesting proposal for $\mathcal{N} = 6$ multiple M2-branes was also made by Aharony, Bergman, Jafferis, Maldacena (ABJM model). We clarified Lorentzian BLG model, which is one of the BLG models, could be derived from the ABJM model by taking the scaling limit. Also we found the coordinate dependent couplings was allowed in Lorentzian BLG model. This fact is important for understanding the conformal symmetry of multiple M2-branes. From the AdS/CFT point of view, we also studied the dual gravity analysis and we made a point that the gravity dual of Lorentzian BLG model was the probe branes in AdS space. We also investigate gravitational solutions in 11-dimensional supergravity with respect to the multiple M2-branes symmetry. We obtain the solutions which has basically $SU(2)$ fiber bundle over $\mathbb{C}P^2$. We squash this space and get a higher-dimensional analog of Eguchi-Hanson space. We clarify the solutions have curvature singularity at one point where base space $\mathbb{C}P^2$ shrinks to zero.

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Contents

Chapter 1

Introduction

We believe the existence of new physics beyond the Standard Model. The astrophysical observation data tells us only 4 % of the energy-momentum contributions can be interpreted by the Standard Model. The other constituents are Dark Matter and Dark Energy. Dark Matter is defined as a matter which does not couple to the photon directly (in the low-energy regime). Dark Energy is defined as a effect which comes from the cosmological constant which was originally introduced by A. Einstein. Dark Matter is expected to be clarified by the new physics beyond the Standard Model and it may appear in the Large Hadron Collider (LHC) at CERN. However there still remain problems about Dark Energy. In the Standard Model, we have succeed to construct the quantum field theory of the strong interaction, weak interaction and the electro-magnetic interaction. There remains one more interaction, "gravitational interaction". To explain the Dark Energy, quantization of gravitational force will be quite important.

There have been huge amount of investigations about the quantum gravity. People might think that the field theory for gravity can be constructed in analogy with the Standard Model. However there are some troubles in constructing quantum gravity as ordinary particle field theory, such as the nonrenormalizable ultra-violet divergences. In constructing a renormalizable quantum gravity, the most promising proposal is *String Theory*. String theory is a quantum theory and includes gravitons as oscillation modes of closed strings. Treating the gravity in terms of particle theory, we are faced with nonrenormal-

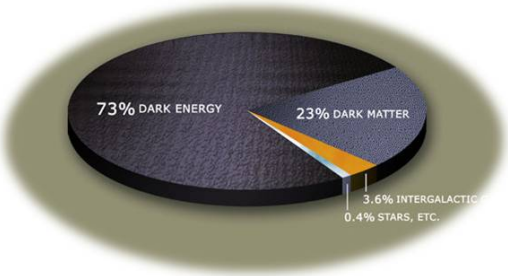

Figure 1.1: The fraction of constituents in our universe from the astorophysical observation. This suggests the matter contributes only 4% and the other part are unknown.

ization problem. The divergence in the ultra-violet regime comes from the zero-distance behavior which is obtained when we consider an integration over the whole phase space. However in the string theory, the relationship between distance and momentum is roughly like

$$\Delta L \sim \frac{\hbar}{p} + \alpha' \frac{p}{\hbar} \geq 2\sqrt{\alpha'}$$

where the parameter α' is related to the string tension as $T_s = 1/(2\pi\alpha')$. The string length are also given by $\alpha' = \ell_s^2$. Therefore we don't meet the zero-distance problem and the string theory does not seem to have the nonrenormalizable problem. Hence the string theory can become a candidate for the quantum gravity. Especially the superstring theory really includes the 10-dimensional supergravity (IIA, IIB, Hetero) as its low-energy effective actions. Therefore the string theory is a powerful model for investigating the quantum gravity. There are also interesting phenomena in the string theory. UV behavior is related to the IR behavior (UV-IR mixing) through a duality between open strings and closed strings. This can be understood as the conformal symmetry in the string theory. We expect a unified theory to exist at the UV fixed point of running coupling constants. At the fixed point the theory becomes conformal, so the string theory has been considered as a candidate for the unified theory.

The supergravity itself is also interesting. The supergravities with maximal supersymmetric construction are restricted to the four-dimensional $\mathcal{N} = 8$ or 11-dimensional $\mathcal{N} = 1$. The supersymmetry is interesting technique to cause a restriction which can constrain the fundamental theory; "Theory of Everything". If we compactify the one of the directions in 11-dimensional supergravity, we obtain the 10-dimensional supergravity which is the low-energy effective action in string theory as we mentioned. In analogy with the string theory, there seems to exist the fundamental theory which has the 11-dimensional supergravity as its low-energy action. This is so called *M-theory*. Standing on the point of view of the 11-dimensional supergravity, the M-theory should have the three-dimensional objects and six-dimensional objects, which are called M2-branes and M5-branes. The effective theory of multiple M2-branes seems to have $\mathcal{N} = 8$ supersymmetries with (maximally) an $SO(8)$ R-symmetry in $2 + 1$-dimensions since we have 32 supercharges in 11-dimensions and half of them are preserved by a world-volume parity condition. On the other hand, the effective theory for multiple M5-branes has $\mathcal{N} = 2$ supersymmetries with (maximally) an $SO(5)$ R-symmetry in $5 + 1$-dimensions. The M-theory must be interesting idea for constructing the quantum gravity because of its uniqueness protected by supersymmetry. There is also a long history about constructing M-theory. The review of these constructions (essential part only) will be reviewed in Chapter 2. We will concentrate on the M2-branes throughout this paper.

During the last year (2008), there has been remarkable progress about the effective theory of multiple M2-branes. Bagger, Lambert and Gustavsson constructed the $2 + 1$ dimensional superconformal Chern-Simons theory with the maximal $\mathcal{N} = 8$ supersymmetry and manifest $SO(8)$ R-symmetry [1–3]. In the Bagger-Lambert-Gustavsson (BLG) model, the essential idea is triple-algebras. However there are only two known realizations

of triple-algebras which are an $SO(4)$ model with a positive group metric and a Lorentzian BLG model [4–6] with a negative one.

Another important development is given by Aharony, Bergman, Jafferis and Maldacena (ABJM) [7]. The $SO(4)$ BLG model can be reformulated as an $SU(2)\times SU(2)$ bifundamental representation [8]. ABJM generalized the $SO(4)$ BLG model to a $U(N)\times U(N)$ Chern-Simons gauge theory with levels k and $-k$. This ABJM model is considered as a dual description of N multiple M2-branes placed at the orbifold singularity of $\mathbb{R}^8/\mathbb{Z}_k$. The orbifold group $\mathbb{Z}_k$ acts on a phase of complex space $\mathbb{C}^4$ and this manifold preserves only $\mathcal{N}=6$ supersymmetry for $k>2$. The ABJM model has indeed this amount of supersymmetry. For $k=1,2$ cases the theory is expected to be enhanced to $\mathcal{N}=8$, however this does not manifestly exist in the ABJM model. The explicit ABJM action is denoted in [9]. The ABJM model includes the $SO(4)$ BLG model as a special model with an $SU(2)\times SU(2)$ bi-fundamental gauge group and also the Lorentzian BLG by taking the scaling limit [10, 11]. A gravitational dual of the Lorentzian BLG was discussed with respect to the scaling limit [12].

In this paper, we emphasize the importance of coordinate dependence of the couplings in Lorentzian BLG model. The coordinate dependence was first mentioned in [13] in the context of multiple M2-branes. The meaning of the *Janus couplings* in the title is following. Originally it was considered to be a dual of supergravity solutions with a space-time dependent dilaton field [14], and it has two different "faces" at the boundary. If there are two boundaries and different coupling constants at each boundary, we should include interface terms which make gauge couplings non-constant. Supersymmetric field theories with the interface terms are constructed in [15–18]. Here we use the meaning of Janus couplings by extending the original usage to more general dependence on space-time coordinates.

Supersymmetries must be spontaneously broken in our world at low energy (Standard Model). At the TeV scale, we may have $\mathcal{N}=1$ supersymmetry because of the existence of Dark Matter. Therefore how to obtain lower supersymmetric theory from the M-theory is important. In the gravity side, the multiple M2-branes can have various seven dimensional compact Einstein manifolds as in $AdS_4\times\mathbb{X}^7$. These manifolds would be usable to obtain the lower supersymmetric theory in four-dimensions. There has been interesting progress in constructing $\mathbb{X}^7$, some of which are a squashed $\mathbb{S}^7$ of Awada, Duff and Pope [19], coset manifolds $N_{\rm I}^{p,q,r}$ of the form $SU(3)\times U(1)/U(1)\times U(1)$ by Castellani and Romans [20] and a squashed $N_{\rm I}^{0,1,0}$ geometry named as $N_{\rm II}^{0,1,0}$, by Page and Pope [21].

In particular, the squashed $\mathbb{S}^7$ has the $SO(5)\times SU(2)$ isometry group and this manifold preserves maximally $\mathcal{N}=1$ supersymmetry. To interpolate the squashed $\mathbb{S}^7$ and the round $\mathbb{S}^7$, we need to add scalar fields and potentials. These fields suggest that there is a renormalization group flow from an $SO(5)\times SU(2)$ symmetric UV fixed point to an $SO(8)$ symmetric IR fixed point [22]. There is also a development about the $2+1$ dimensional Chern-Simons theory with an $Sp(2)\times U(1)$ by Ooguri and Park [23]. The $Sp(2)$ is isomorphic to $SO(5)$. The $U(1)$ comes from an effect divided by $\mathbb{Z}_k$ as same as in the ABJM model. A dual operator was discussed, which corresponds to the renormalization

group flow from the Ooguri-Park model and the ABJM model [24]. There is also another way of discussions for squashed geometry $N_{\rm II}^{0,1,0}$. With special values for p, q, r of $N_{\rm I}^{p,q,r}$, we can obtain maximally $\mathcal{N} = 3$ supersymmetric manifold $N_{\rm I}^{0,1,0}$. The interpolation between the squashed manifold $N_{\rm II}^{0,1,0}$ and $N_{\rm I}^{0,1,0}$ was also discussed [25, 26]. The other related recent work of squashed 7-sphere is [27].

In searching for the other solution, we know that there is an interesting way to obtain more general squashed geometries in 5D supergravity by Ishihara and Matsuno [28]. This solution has a squashed $\mathbb{S}^3$ which is regarded as an $\mathbb{S}^1$ Hopf fiber bundle over $\mathbb{S}^2$ base space. They introduced a squashing function of the radius direction, which determines the level of the squashed $\mathbb{S}^3$. The solution has various faces including the Reissner-Nordström black hole, Gross-Perry-Sorkin monopole (with the Taub-NUT space) [29, 30] and also a black string. This construction is a generalization including the known solutions as firstly noted in [31]. There are some developments with respect to the Ishihara-Matsuno squashing for multi black holes [32], multi BHs with a positive cosmological constant [33], rotating BHs [34], the Kerr-Gödel BHs [35, 36] with a charge [37].

This paper is organized as follows. In chapter 2, we will review the old progresses of multiple M2-branes and new progress. In chapter 3, the BLG model will be derived from the ABJM model. In chapter 4, we will discuss the dual of the Lorentzian BLG model with respect to the derivation from the ABJM model. We will also mention that we can have coordinate dependent couplings in the Lorentzian BLG model with and without a mass term in chapter 5. In chapter 6, we will show the construction of gravitational solutions in the M-theory with the $SU(3) \times SU(2)$ isometry, which are squashed solutions in analogy with Ishihara-Matsuno solutions in five-dimension.

Chapter 2

Review

There have been a long story to investigate the M-theory. We write a introductionally review in this section, but only a essential parts for understanding the M-theory.

M-theory is defined as a theory which has the $\mathcal{N} = 1$ 11-dimensional supersymmetric gravity action as its effective theory. The $\mathcal{N} = 1$ 11-dimensional SUGRA is a highest supersymmetric gravity theory because we can have maximally $\mathcal{N} = 8$ supersymmetry in four-dimensional gravity. The constituents of 11-dimensional SUGRA are graviton $g_{\mu\nu}$, gravitino ψ_μ and 3-form gauge field $A_{\mu\nu\rho}$. The necessity of three-form fields is as follows.

The degree of freedom of gravitons is naively 11×11, and there are internal symmetries of local Lorentz transformation $1/2(11 \times 10)$, general transformation 11 and gauge transformation which can be fixed by $\partial_\mu(\sqrt{-g}g^{\mu\nu}) = 0$ as 11. So the d.o.f. of $g_{\mu\nu}$ is 44. In a case for gravitino ψ_μ (Rarita-Schwinger field), internal symmetries are supersymmetric gauge transformation which is described $\partial_\mu\chi$ as 2^5 d.o.f. and gauge transformation as 2×2^5 (which can be fixed $\partial_\mu\psi^\mu = 0$, $\Gamma_\mu\psi^\mu = 0$). There is also on-shell condition which divide the total d.o.f. by two because of first differential equation. As the result, the d.o.f. of gravitino is $1/2(11 \times 2^5 - 2^5 - 2 \times 2^5) = 128$. The result tells us lack of bosonic d.o.f. To cover this, we need three-form gauge field $A_{\mu\nu\rho}$ because its d.o.f. is 84 (transverse directions are 9 and ${}_9C_3 = 84$).

Objects which included in M-theory can be considered from the three-form field. In $(1+2)$-dimensional volume, there must exist a term which couple to this three-form field $A_{\mu\nu\rho}$ and the term is called Wess-Zumino term or Myers term. There are also six-form field $A_{\mu\nu\rho\sigma\lambda\delta}$ which is dual of three form field in 11-dimensional field theory. And we can also treat $(1 + 5)$-dimensional objects. The $(1 + 2)$-dimensional objects are called "M2-branes" and the $(1 + 5)$-dimensional ones are "M5-branes". But there had been many mysteries to understand these objects because of lack of knowledge about fundamental objects in M-theory.

2.1 Old progress of multiple M2-branes

2.1.1 Triple algebra

There have been long standing progresses about constructing a effective action of multiple M2-branes. First process for multiple M2-brane effective action is to preserve world volume diffeomorphism. Naively it can be considered that a effective action has so called *Nambu-bracket* [38] which is generalization of Poisson bracket as

$$\{X^I, X^J, X^K\} \equiv \epsilon^{ijk}\partial_i X^I \partial_j X^J \partial_k X^K. \tag{2.1.1}$$

Using this Nambu-bracket, the effective action of $(1+2)$-dimension is

$$S_{NG} = \int d^3\sigma \left(T_2 \sqrt{\{X^I, X^J, X^K\}^2} + C^{IJK}\{X^I, X^J, X^K\} \right) \tag{2.1.2}$$

where we use σ^i as world volume coordinate and the Roman indices I, J, K run from 1 to 8. This action is invariant under world-volume diffeomorphism.

In the case of Poisson brackets, we change this notation to commutators to obtain finite (matrix) representation as $\{X^I.X^J\} \to [X^I, X^J]$ and then we can construct quantum theories for multiple D-branes. For the purpose to quantize of multiple M2-branes, it seems naturally that we need to construct *triple algebras* $[X^I, X^J, X^K]$ instead of Nambu-brackets [39].

2.1.2 M2-M5 system

We know there are only two objects in M-theory which are M2-brane and M5-brane. This fact make us to cast back our strategy to construct D1-D3 system in string theory and there seems to be possible to construct M2-M5 system. First let's remind about D1-D3 system.

In D1-D3 system arguments, we can obtain a D3-brane spike solution and also multiple D1-branes solution. We see these solutions are exactly same solutions each other. We start with a D3-brane picture. The solution of a D3-brane was constructed in [40, 41] by using a D3-brane effective Dirac-Born-Infeld action.

$$S_{D3} = -T_3 \int d^4x \sqrt{-\det(\eta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu} + \partial_\mu X^I \partial_\nu X^I)}. \tag{2.1.3}$$

A half BPS solution can be obtained as

$$X^9 = \frac{N}{r}, \quad r \equiv \sqrt{(X^1)^2 + (X^2)^2 + (X^3)^2}, \quad F_{9r} = \partial_r X^9 \tag{2.1.4}$$

where gauge field was obtained to satisfy BPS equations. Note that this solution is a magnetic solution which means N is magnetic charges which represent N multiple D1-branes. If we construct a solution with electric charges N, we can obtain a D3-brane sticked with N strings. The solution is depend on three-dimensional space world volume

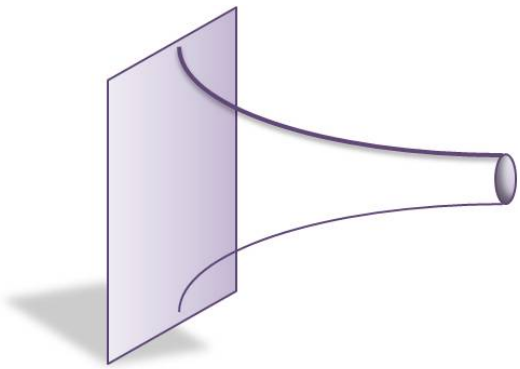

Figure 2.1: Multiple D1-branes (M2-branes) sticked in a D3-brane (M5-brane). In D3 point of view, this is a spike solution. On the other hand, we can see fuzzy $\mathbb{S}^2$ solution in D1 point of view.

of D3-brane. If r is large, the direction of X^9, a transverse direction to a D3-brane, goes to zero, but on the other hand if r goes to 0, X^9 goes to infinity. This means this solution infinitely expands as an original D3-brane does, however if we close to its central region the solution lengthens to X^9 direction. So this solution is called a *spike solution*. Note that there is a nice textbook to introduce the construction of this spike solution in Problem 20.6 and 20.7 of [42].

Let's change our eyes to D1-branes picture. In D1-branes point of view, we consider the BPS equation of multiple D1-brane. We should consider Non-Abelian generalization of DBI action, but the expanded one around the flat spacetime. The action is same as the dimensional reduced action of 10-dimensional $\mathcal{N} = 1$ super Yang-Mills action. The equations of motion are generally 2nd derivative equations and this can not be solved easily. So we concentrate on BPS equations which are 1st derivative equations. In the case for D1-branes we get [43]

$$\frac{\partial X^i}{\partial X^9} \mp \frac{i}{2}\epsilon^{ijk}[X^i, X^j] = 0 \tag{2.1.5}$$

where the indices $i, j, k = 1, 2, 3$ and X^9 direction is one of D1-branes world coordinates. This equation is called *Nahm equation*. The solution of this equation can be obtained as [44]

$$X^i = \pm\frac{1}{2X^9}\sigma^i, \quad [\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k. \tag{2.1.6}$$

The matrices σ^i obey $SU(2)$ algebra above. This solution is a fuzzy $\mathbb{S}^2$ solution which has $SO(3) \simeq SU(2)$ global symmetry in non-commutative space. The fuzzy solution has a cutoff of its rank of irreducible representation. We choose the σ^i to be in the N-dimensional irreducible representation of $SU(2)$ with quadratic Casimir $C = N^2 - 1$, we can describe a fuzzy $\mathbb{S}^2$ radius as

$$R = \sqrt{\frac{(2\pi\alpha')^2}{N}\sum_i \mathrm{Tr}(X^i)^2} = \frac{2\pi\alpha'\sqrt{N^2-1}}{2X^9} \xrightarrow{N\to\infty} \pi\alpha'\frac{N}{X^9}. \tag{2.1.7}$$

The solution we obtained in a D3-brane picture (2.1.4) has a continuum $\mathbb{S}^2$. When we compare the D1-branes fuzzy solution to the continuum one, we should take the cutoff $N \to \infty$. If we take the fuzzy sphere radius R as r, we see a perfect agreement (up to normalization) to the solution (2.1.4). This fact is an interesting consistency in string theory. Note that there is a brief review of D1-D3 system in [45].

So far we have reminded D1-D3 system in string theory, let's turn to M2-M5 system in M-theory. We learned M2-branes and M5-branes really exist in M-theory, we should consider how to make M2-M5 system since it seems to exist also in M-theory in analogy with string theory. In D1 point of view in D1-D3 system, a commutator which includes in effective D1-branes theory is essential to obtain the fuzzy $\mathbb{S}^2$ solution. However we need to construct a fuzzy $\mathbb{S}^3$ solution in M2-M5 system because of the difference of 3 space coordinates. Therefore we should take into account a triple algebra in this case. The expected BPS equation in M2-brane effective theory can be written as [45]

$$\frac{\partial X^i}{\partial s} + \frac{\lambda M_{11}^3}{8\pi}\epsilon^{ijkl}[X^i, X^j, X^k] = 0 \tag{2.1.8}$$

where we use s as one of M2 world volume coordinates, M_{11} is Plank scale in 11D and λ is a arbitrary parameter. This BPS equation is called *Basu-Harvey equation*.

The solution of this Basu-Harvey equation is

$$X^i \sim \frac{1}{\sqrt{s}}G^i, \quad [G^i, G^j, G^k] = \epsilon^{ijkl}G^L \tag{2.1.9}$$

where we use generators G^i which satisfy $SO(4)$ algebra and have structure constants ϵ^{ijkl}. This algebra is called $\mathcal{A}_4$ algebra. Reader may confuse to the usual $SO(4)$ Lie algebra. However $SO(4)$ construction in triple algebras can be obtained by using matrix representation. To realize a matrix representation, we need to reconstruct (or define) triple algebra to the form $[G_5, X^I, X^J, X^K]$ [46]. This fuzzy $\mathbb{S}^3$ solution (2.1.9) can be expanded to infinity as $s \to 0$ and can be regarded as single M5-brane.

2.1.3 Chern-Simons term

We have investigated about transverse scalars and their BPS solution. Let's concentrate on gauge fields in multiple M2-branes. Gauge fields are important with respect to supersymmetric transformation. In D-branes case, we need gauge fields for closure of SUSY transformation. And also if we write DBI type of action as a D-brane effective action, gauge fields on a D-brane take a important role to understand the T-duality.

The effective action of multiple M2-branes action may have 8 transverse scalars X^I, 3-dimensional fermions Ψ and 1-form gauge field A_μ. First let's count the d.o.f. of fermions [47]. There is 8 transverse direction, we need to keep maximally $SO(8)$ R-symmetry. Only in 3-dimensional gamma matrices suggest us to have 2-component Majorana fermions, however we also have the other gamma matrices in transverse 8 directions. So we need to consider 10 gamma matrices and they make fermions to have 2^5 components. When we consider about M2-branes we need to constraint ourselves to impose world parity on

M2-branes as $\Gamma_{012}\Psi = \Psi$. This constraint is important for the closure of any symmetry in non-Abelian case (we will meet concrete examples of multiple M2-brane later, then you can see). Also equation of motion for fermions subtract the d.o.f. of fermions. Putting these all together, M2-brane fermions have $2^5 \times 1/2 \times 1/2 = 8$ d.o.f.

To preserve SUSY, the d.o.f. of bosons should be equal to fermions. The transverse bosons are 8 and fermions are also 8. We might consider there are no need to introduce gauge fields in multiple M2-branes. However, as we will see, we really need gauge fields to close SUSY. How to realize zero d.o.f. of gauge fields ? We should introduce gauge fields as topological term. For the case for 3-dimension, we know very well it as *Chern-Simons term*.

$$\int d^3x \ \mathrm{tr}\left[A \wedge dA + A \wedge A \wedge A\right]. \tag{2.1.10}$$

Only with the Chern-Simons term, gauge fields propagate zero d.o.f. The necessity of Chern-Simons term is also important for conformal symmetry. When we have the kinetic term of gauge fields F^2, we cannot naively preserve conformal symmetry because its mass dimension is four.

We also comment in a case for D2-branes. For the fermions a situation is same as for M2-branes, so the fermions have 8 d.o.f. Since the transverse directions change to 7 for D2-brane, we only have 7 transverse scalars. There needs one more d.o.f. The lack of field can be compensate by introducing gauge fields. Gauge fields with usual kinetic terms F^2 essentially have d.o.f. of transverse direction only in world volume. So D2-brane gauge fields have one d.o.f. Putting it all together, we can show correct SUSY in D2-brane. This is different from M2-brane situation.

2.2 Remarkable progress 1; BLG model

There has been a remarkable progress in constructing $\mathcal{N} = 8$ supersymmetric three-dimensional field theory with $SO(8)$ R-symmetry by Bagger and Lambert and Gustavsson. First Bagger and Lambert tried to construct multiple M2-branes effective action by using triple algebra, however they could not close SUSY algebra [1]. Afterward Bagger, Lambert [2] and also Gustavsson [3] introduced gauge fields and they succeeded to construct multiple M2-branes effective theory [1].

2.2.1 Triple algebra

Bagger and Lambert introduced triple algebra and they investigated how to construct. To introduce triple algebra, we first define non-associative algebra.

$$< A, B, C > \equiv (A \cdot B) \cdot C - A \cdot (B \cdot C). \tag{2.2.1}$$

[1] Bagger and Lambert could construct the action, but Gustavsson wrote down only SUSY algebras. For this reason, the model sometimes has called BL model.

If product is usual, this non-associative algebra is zero. Using this non-associative algebra, we define triple algebra as

$$\begin{aligned}[A,B,C] =& < A,B,C > + < B,C,A > + < C,A,B > \\ & - < A,C,B > - < B,A,C > - < C,B,A > . \end{aligned} \tag{2.2.2}$$

We can see the importance of non-associative algebra and its product to take triple algebra meaningful.

To take into account inner product, we also define the relation of trace. Trace operator should satisfy the relations.

$$\begin{aligned} &\mathrm{tr}(A,B) = \mathrm{tr}(B,A), \quad \mathrm{tr}(A\cdot B, C) = \mathrm{tr}(A, B\cdot C), \\ &\mathrm{tr}([A,B,C],D) = -\mathrm{tr}(A,[B,C,D]). \end{aligned} \tag{2.2.3}$$

This is a bilinear map; $\mathrm{tr} : \mathcal{A}\times\mathcal{A} \to \mathbb{C}$ that is symmetric and invariant. These trace relation will be important to see invariance of gauge symmetry.

In this stage we write down the definition of structure constants of triple algebra and this can be considered also the definition of triple algebra.

$$[T^a, T^b, T^c] = f^{abcd}T_d. \tag{2.2.4}$$

With the last equation of (2.2.3), we can see the structure constant f^{abcd} should be completely anti-symmetric under exchange of indices.

We can consider gauge symmetry with triple algebra as

$$\delta X = \Lambda_{ab}[T^a, T^b, X]. \tag{2.2.5}$$

This representation for gauge symmetry is correct since a variation of trace of same scalar fields is invariant under this transformation.

$$\delta\left[\mathrm{tr}(X,X)\right] = 2f^{abcd}\Lambda_{ab}X_cX_d = 0. \tag{2.2.6}$$

Taking into account the gauge symmetry, we can introduce gauge fields and covariant derivative. The variation of gauge fields should be written as covariant derivative of gauge parameter.

$$\begin{aligned} &\delta\tilde{A}_\mu{}^b{}_a = \partial_\mu\tilde{\Lambda}^b{}_a - \tilde{\Lambda}^b{}_c\tilde{A}_\mu{}^c{}_a + \tilde{A}_\mu{}^b{}_c\tilde{\Lambda}^c{}_a \equiv D_\mu\tilde{\Lambda}^b{}_a \\ &\tilde{A}_\mu{}^b{}_a \equiv f^{cdb}{}_aA_{\mu cd}, \quad \tilde{\Lambda}^b{}_a \equiv f^{cdb}{}_a\Lambda_{cd} \end{aligned} \tag{2.2.7}$$

Then the covariant derivative can be read as

$$D_\mu X_a = \partial_\mu X_a - \tilde{A}_\mu{}^b{}_aX_b. \tag{2.2.8}$$

The gauge symmetry suggest important rule about structure constants. We need to consider "derivation" of triple algebra for gauge symmetry.

$$\delta([X,Y,Z]) = [\delta X, Y, Z] + [X, \delta Y Z] + [X, Y, \delta Z]. \tag{2.2.9}$$

From this derivation we can get

$$\begin{aligned}[T^a, T^b, [X, Y, Z]] =& [[T^a, T^b, X], Y, Z] + [X, [T^a, T^b, Y], Z] \\ &+ [X, Y, [T^a, T^b, Z]], \\ f^{efg}{}_d f^{abc}{}_g =& f^{efa}{}_g f^{bcg}{}_d + f^{efb}{}_g f^{cag}{}_b + f^{efc}{}_g f^{abg}{}_d. \end{aligned} \tag{2.2.10}$$

This equation is called *fundamental identity* in triple algebra. In the case for commutator, we have Bianchi identity for Lie-algebra and (2.2.10) is considered as an triple algebra analogy of Bianchi identity. This fundamental identity is quite important to close SUSY algebras and construct examples.

2.2.2 Supersymmetry and BLG action

Bagger, Lambert and Gustavsson considered firstly to construct SUSY algebras in analogy with D2-branes effective action. They consider the main difference from D2-brane is to take into account triple algebra in stead of Lie algebra.

$$\begin{aligned} \delta X^I_a &= i\bar{\epsilon}\Gamma^I \Psi_a \\ \delta \Psi_a &= D_\mu X^I_a \Gamma^\mu \Gamma^I \epsilon + \kappa [X^I, X^J, X^K]_a \Gamma^{IJK} \epsilon \\ \delta \tilde{A}_\mu{}^b{}_a &= i\bar{\epsilon}\Gamma_\mu \Gamma_I X^I_c \Psi_d f^{cdb}{}_a \end{aligned} \tag{2.2.11}$$

where main difference exists at second term in the variation of fermion and κ is a arbitrary constant. In this transformation, we have 16 component of fermionic fields and supersymmetric parameter ϵ (they are constrained to preserve world sheet parity as $\Gamma_{012}\epsilon = -\epsilon$ [1]).

For the closure of this SUSY algebra, we need to satisfy the relation;

$$\begin{aligned} [\delta_1 \delta_2] X^I_a &= v^\mu (D_\mu X^I)_a + \tilde{\Lambda}^b{}_a X^I_b \\ [\delta_1 \delta_2] \Psi_a &= v^\mu (D_\mu \Psi)_a + \tilde{\Lambda}^b{}_a \Psi_b \\ [\delta_1 \delta_2] \tilde{A}_\mu{}^b{}_a &= v^\mu \tilde{F}_{\mu\nu}{}^b{}_a + (D_\mu \tilde{\Lambda})^b{}_a \end{aligned} \tag{2.2.12}$$

where we use bi-spinor vector v^μ and bi-spinor scalar $\tilde{\Lambda}^b{}_a$ defined as

$$v^\mu \equiv -2i\bar{\epsilon}_2 \Gamma^\mu \epsilon_1, \quad \tilde{\Lambda}^b{}_a \equiv -i\bar{\epsilon}_2 \Gamma_{JK} \epsilon_1 X^J_c X^K_d f^{cdb}{}_a.$$

When we calculate left-hand sides of (2.2.12), we will meet extra terms of $\bar{\epsilon}_2 \Gamma_\mu \Gamma_{IJKL} \bar{\epsilon}$ bi-spinor in the closure of Ψ_a and also $\tilde{A}_\mu{}^b{}_a$. For the closure of $\tilde{A}_\mu{}^b{}_a$, happily this term vanishes as a consequence of the fundamental identity (2.2.10). On the other hand for Ψ_a, these terms cancel if we choose the arbitrary constant

$$\kappa = -\frac{1}{6}. \tag{2.2.13}$$

The other redundant terms to obtain (2.2.12) can be regarded as on-shell condition or equations of motion. These can be read from (2.2.12) as

$$\begin{aligned}
&\Gamma^\mu D_\mu \Psi_a + \frac{1}{2} f^{cdb}{}_a \Gamma_{IJ} X^I_c X^J_d \Psi_b = 0, \\
&D^2 X^I_a - \frac{i}{2} f^{cdb}{}_a \bar{\Psi}_c \Gamma^I{}_J X^J_d \Psi_b - \frac{1}{2} f^{bcd}{}_a f^{efg}{}_d X^J_b X^K_c X^I_e X^J_f X^K_g = 0, \\
&\tilde{F}_{\mu\nu}{}^b{}_a + \epsilon_{\mu\nu\lambda} f^{cdb}{}_a \left(X^J_c D^\lambda X^J_d + \frac{i}{2} \bar{\Psi}_c \Gamma^\lambda \Psi_d \right) = 0.
\end{aligned} \tag{2.2.14}$$

The bosonic equations of (2.2.14) can be obtained by taking the supervariation of the fermion equation of motion.

To derive the equations of motion (2.2.14), we can guess an action as

$$\begin{aligned}
\mathcal{L}_{BLG} = &-\frac{1}{2}\mathrm{tr}(D^\mu X^I, D_\mu X^I) + \frac{i}{2}\mathrm{tr}(\bar{\Psi}, \Gamma^\mu D_\mu \Psi) \\
&+ \frac{i}{4}\mathrm{tr}(\bar{\Psi}, \Gamma_{IJ}[X^I, X^J, \Psi]) - \frac{1}{12}\mathrm{tr}([X^I, X^J, X^K], [X^I, X^J, X^K]) \\
&+ \frac{1}{2}\epsilon^{\mu\nu\rho}(f^{abcd} A_{\mu ab} \partial_\nu A_{\lambda cd} + \frac{2}{3} f^{cda}{}_g f^{efgb} A_{\mu ab} A_{\nu cd} A_{\lambda ef}).
\end{aligned} \tag{2.2.15}$$

The potential term of scalar fields is sixth order as expected to be conformal because a mass dimension of scalar fields is 1/2 in three dimension. We can also see the existence of Chern-Simons term and the absence of gauge kinetic term F^2 as we have seen in section 2.1.3. This is surprising thing because we just order to keep supersymmetry. This action seems to preserve conformal symmetry even if quantized.

2.2.3 SO(4) BLG model

The BLG action (2.2.15) really has $\mathcal{N} = 8$ as we saw, however there are some mystery to understand the triple algebras. One example can be easily obtained by setting the structure constant as

$$f^{abcd} = \epsilon^{abcd}. \tag{2.2.16}$$

Levi-Civita symbol with four-indices is only included in $SO(4)$ algebra since it is invariant under $SO(4)$ rotations. However unfortunately there are no-go theorem which shows $SO(4)$ BLG is an only essential construction if we choose the group metric $\mathrm{tr}\, T^a T^b$ to be positive definite [48, 49].

The group $SO(4)$ can be decomposed to $SU(2) \times SU(2)$. This decomposition can

really be done in $SO(4)$ BLG model in [8]. The resultant action is

$$\begin{aligned}\mathcal{L}_{SU(2)\times SU(2)} =& -\operatorname{tr}(D^\mu X^I)^\dagger D_\mu X^I + i\operatorname{tr}\bar{\Psi}^\dagger\Gamma^\mu D_\mu\Psi \\ &+ \frac{k}{2\pi}\frac{1}{2}\epsilon^{\mu\nu\lambda}\operatorname{tr}\left(A_\mu^{(L)}\partial_\nu A_\lambda^{(L)} + \frac{2i}{3}A_\mu^{(L)}A_\nu^{(L)}A_\lambda^{(L)}\right) \\ &\quad - \frac{k}{2\pi}\frac{1}{2}\epsilon^{\mu\nu\lambda}\operatorname{tr}\left(A_\mu^{(R)}\partial_\nu A_\lambda^{(R)} + \frac{2i}{3}A_\mu^{(R)}A_\nu^{(R)}A_\lambda^{(R)}\right) \\ &- \frac{2i}{3}\frac{2\pi}{k}\operatorname{tr}\bar{\Psi}^\dagger\Gamma_{IJ}(X^IX^{J\dagger}\Psi + X^J\Psi^\dagger X^I + \Psi X^{I\dagger}X^J) \\ &- \frac{8}{3}\left(\frac{2\pi}{k}\right)^2 X^{[I}X^{J\dagger}X^{K]}X^{K\dagger}X^JX^{I\dagger}. \end{aligned} \tag{2.2.17}$$

The fields consist of two $SU(2)$ gauge fields, having Chern-Simons terms with *opposite levels*. The Chern-Simons level k should be quantized and be integer because we need to have the action invariant under non-Abelian transformation of gauge group. All the matter fields transform as bi-fundamental of $SU(2)_L \times SU(2)_R$. In this representation, we have no more triple algebras but the well known $SU(2)$ Lie algebras.

Let's consider the moduli space of the action (2.2.17) by focusing on bosonic fields [50, 51]. Generic scalar configurations for which the potential vanishes correspond (up to gauge transformations) to diagonal matrices as

$$X^I = \frac{1}{\sqrt{2}}\begin{pmatrix} z^I & 0 \\ 0 & \bar{z}^I \end{pmatrix}. \tag{2.2.18}$$

The gauge fields associated with the $U(1)$ that rotates z^I and $\bar{z}^I$ each other with the diagonal configuration (2.2.18). So we have

$$A_\mu^{(L)} = \begin{pmatrix} a_\mu^{(L)} & 0 \\ 0 & -a_\mu^{(L)} \end{pmatrix}, \quad A_\mu^{(R)} = \begin{pmatrix} a_\mu^{(R)} & 0 \\ 0 & -a_\mu^{(R)} \end{pmatrix} \tag{2.2.19}$$

with the normalization chosen to have gauge transformations

$$a_\mu^{(L)} \to a_\mu^{(L)} - \partial_\mu\theta^{(L)}, \quad a_\mu^{(R)} \to a_\mu^{(R)} - \partial_\mu\theta^{(R)}. \tag{2.2.20}$$

where $\theta^{(L,R)}$ have period 2π.

The potential term is zero and the remaining kinetic term of the action changes to be

$$S = \int d^3x \left[-\left|\partial_\mu z^I + i(a_\mu^{(L)} - a_\mu^{(R)})z^I\right|^2 + \frac{k}{2\pi}\epsilon^{\mu\nu\lambda}(a_\mu^{(L)}\partial_\nu a_\lambda^{(L)} - a_\mu^{(R)}\partial_\nu a_\lambda^{(R)})\right] \tag{2.2.21}$$

Then we combine the gauge fields linearly as

$$c_\mu = a_\mu^{(L)} + a_\mu^{(R)}, \quad b_\mu = a_\mu^{(L)} - a_\mu^{(R)}. \tag{2.2.22}$$

By using this configuration the action rewritten as

$$\mathcal{L}_{SU(2)\times SU(2)} = -\left|\partial_\mu z^I + i b_\mu z^I\right|^2 + \frac{k}{4\pi}\epsilon^{\mu\nu\lambda} b_\mu f_{\nu\lambda}, \\ f_{\mu\nu} \equiv \partial_\mu c_\nu - \partial_\nu c_\mu. \tag{2.2.23}$$

Since the new variable c_μ of gauge field is usual $U(1)$ gauge field and it should be satisfy Bianchi identity

$$\epsilon^{\mu\nu\lambda}\partial_\mu f_{\nu\lambda} = 0. \tag{2.2.24}$$

To deal with the Bianchi identity together with the action, we take it as constraint term with Lagrange multiplier field $\sigma(x)$.

$$S_{\text{const.}} = \frac{1}{8\pi}\int d^3x\, \sigma(x)\epsilon^{\mu\nu\lambda}\partial_\mu f_{\nu\lambda}. \tag{2.2.25}$$

The Lagrange multiplier field $\sigma(x)$ should be periodic because there is a monopole condition;

$$\int d^3x \frac{1}{2}\epsilon^{\mu\nu\lambda}\partial_\mu f_{\nu\lambda} = \int_M df = \int_{\partial M} f \in 4\pi\mathbb{Z}. \tag{2.2.26}$$

Taking into this fact into account, the Lagrange multiplier field $\sigma(x)$ should be periodic

$$\sigma(x) \sim \sigma(x) + 2\pi n \tag{2.2.27}$$

where we use n as an integer. Together with this periodicity, the constraint term is harmless because $e^{iS_{\text{const}}} = 1$. This periodicity will be important to investigate the moduli space.

The equation of motion for $f_{\mu\nu}$ determines

$$b_\mu = \frac{1}{2k}\partial_\mu\sigma. \tag{2.2.28}$$

Inserting the solution for b_μ, the action becomes

$$\mathcal{L}_{SU(2)\times SU(2)} = -\left|\partial_\mu z^I + \frac{i}{2k} z^I \partial_\mu \sigma\right|^2. \tag{2.2.29}$$

In the final form of action (2.2.29), we have the gauge transformation

$$z^I \to e^{i\alpha(x)} z^I, \quad \sigma \to \sigma - 2k\alpha(x) \tag{2.2.30}$$

We can now fix our gauge to set $\sigma = 0$. After doing this, we still have residual gauge transformation

$$\alpha = \frac{\pi n}{k} \tag{2.2.31}$$

which can be regarded as the periodicity for σ in (2.2.27). This residual symmetry give us a constraint for the scalar moduli should obey

$$z^I \to e^{\frac{\pi i n}{k}} z^I. \tag{2.2.32}$$

There is also a discrete symmetry which changes a scalar moduli and its complex conjugate moduli as

$$z^I \to \bar{z}^I. \tag{2.2.33}$$

(2.2.32) tells us the moduli space should be divided by $\mathbb{Z}_{2k}$ and (2.2.33) means to be divided by $\mathbb{Z}_2$. These orbifold projections do not commute with each other for $k > 1$, and the combined group is the dihedral group D_{2k}. Finally we conclude that the moduli space for the level k in $SO(4)$ BLG model is

$$(\mathbb{R}^8 \times \mathbb{R}^8)/D_{2k}. \tag{2.2.34}$$

For the commute case $k = 1$, this is just

$$(\mathbb{R}^8 \times \mathbb{R}^8)/(\mathbb{Z}_2 \times \mathbb{Z}_2). \tag{2.2.35}$$

This analysis tells us that the $SO(4)$ BLG model with Chern-Simons coefficient k is a effective theory of two M-branes living in orbifolded space.

In the supergravity picture, the orbifold moduli space $\mathbb{R}^8/\mathbb{Z}_{2k}$, except for $k = 1.2$, preserves as many as 12 supersymmetries or $\mathcal{N} = 6$, and also gives rise to an R-symmetry $SU(4) \times U(1)$ [52, 53]. However, we have 16 supersymmetries in BLG model even when we turn on the integer coefficient k in front of the Chern-Simons term. This have still remained to be a mystery of BLG model.

Fuzzy funnel solution and single M5

People might think the realization of Basu-Harvey equation and its classical solution which end on single M5-brane as we discussed section 2.1.2. The BPS equation of $SO(4)$ BLG model is [54]

$$\partial_s X^A = mX^A + \frac{1}{6}\epsilon^{ABCD}[X^B, X^C, X^D] \tag{2.2.36}$$

where we use s is one of world space coordinates of M2-branes and m is a mass. Since we can have rich structure of BPS solutions with mass term, first we turn on mass term and analyze BPS solutions.

The solution of the equation (2.2.36) can be obtained by using the solution which expand fuzzily

$$X^A = \sqrt{\frac{m}{\pi}}\frac{1}{\sqrt{1-e^{-2ms}}}T^A, \quad [T^A, T^B, T^C] = \epsilon^{ABCD}T^D. \tag{2.2.37}$$

If we wake $s \to \infty$ the radius of solution becomes a constant and the solution goes to fuzzy $\mathbb{S}^3$ solution. On the other hand, if we take $s \to 0$ the radius goes to infinity and the solution with this limit represent single M5-brane. The hole shape of this solution looks like a funnel, so this solution is called *fuzzy funnel solution*. The shape have been already depicted as figure 2.1 (p.7).

Let's evaluate the action with this BPS solution but without the mass. The fuzzy $\mathbb{S}^3$ radius can be estimated as

$$R^2 \equiv \frac{\mathrm{tr}(X^A, X^A)}{nT_2} \tag{2.2.38}$$

where n is a cutoff of irreducible representation, and T_2 is tension of M2-brane. Then the action without time-integral which can be considered as energy density of this system is

$$\begin{aligned} E &= \int d^2x \,\mathrm{tr}\left(\partial_s X^A, \partial_s X^A\right) \xrightarrow{n\to\infty} \frac{T_2^2}{2\pi}\int dx^1 \int 2\pi^2 R^3 dR \\ &= T_5 \int d^5x \end{aligned} \tag{2.2.39}$$

where in the first line we use BPS equation (2.2.36) on one-side and take the cutoff to diverge, and in the second line we use the explicit representation of tensions related as $T_5 = T_2^2/2\pi$. So we can conclude there exist BPS solution of multiple M2-brane which end on single M5-brane.

2.2.4 Lorentzian BLG model

There are no-go theorem which prevent us to construct the other model with positive metric in BLG model. $SO(4)$ BLG model is just two M2-branes effective theory. If we would like to investigate more general multiple M2-branes, there need more than two. This is important when we deal multiple M2-brane effective theory with the dual supergravity language. So let's change our eyes to allow a negative group metric. With a negative metric we can have the following structure by using the group generator T^{-1}, T^0, T^i [4–6].

$$\begin{aligned} &[T^{-1}, T^a, T^b] = 0 \\ &[T^0, T^i, T^j] = f^{ij}{}_k T^k \\ &[T^i, T^j, T^k] = f^{ijk} T^{-1} \end{aligned} \tag{2.2.40}$$

where a, b run $-1, 0, i$ and i, j, k run arbitrary. Using this algebras, we can know the metric of this system as

$$\begin{aligned} &\mathrm{tr}(T^{-1}, T^{-1}) = 0,\ \mathrm{tr}(T^{-1}, T^0) = -1,\ \mathrm{tr}(T^{-1}, T^i) = 0, \\ &\mathrm{tr}(T^0, T^0) = 0,\ \mathrm{tr}(T^0, T^i) = 0, \\ &\mathrm{tr}(T^i, T^j) = h^{ij}. \end{aligned} \tag{2.2.41}$$

We can see there is a negative metric. One can easily check that this triple algebra satisfies the fundamental identity (2.2.10). Since in (2.2.40) we only have the structure constants

as the usual Lie algebra, we have a possibility to describe multiple M2-branes. Since T^{-1} commute all the other generators in the language of triple algebra, we can regard T^{-1} as a center of the Lorentzian algebra (2.2.40).

Now we have the action with negative metric

$$\begin{aligned}\mathcal{L}_L = \Bigg[&-\frac{1}{2}(\hat{D}_\mu \hat{X}^I - B_\mu X_0^I)^2 + \frac{1}{4}(X_0^K)^2([\hat{X}^I, \hat{X}^J])^2 - \frac{1}{2}(X_0^I[\hat{X}^I, \hat{X}^J])^2 \\ &+ \frac{i}{2}\bar{\hat{\Psi}}\Gamma^\mu \hat{D}_\mu \hat{\Psi} + i\bar{\Psi}_0\Gamma^\mu B_\mu \hat{\Psi} - \frac{1}{2}\bar{\Psi}_0 \hat{X}^I[\hat{X}^J, \Gamma_{IJ}\hat{\Psi}] + \frac{1}{2}\bar{\hat{\Psi}} X_0^I[\hat{X}^J, \Gamma_{IJ}\hat{\Psi}] \\ &+ \frac{1}{2}\epsilon^{\mu\nu\lambda}\hat{F}_{\mu\nu}B_\lambda - \partial_\mu X_0^I \; B_\mu \hat{X}^I\Bigg] + \mathcal{L}_{gh}, \\ \mathcal{L}_{gh} = &(\partial_\mu X_0^I)(\partial^\mu X_{-1}^I) - i\bar{\Psi}_{-1}\Gamma^\mu \partial_\mu \Psi_0 \end{aligned} \tag{2.2.42}$$

where we have redefined the fields as

$$\hat{X} \equiv X_i T^i, \quad \hat{\Psi} \equiv \Psi_i T^i, \quad \hat{A}_\mu \equiv 2A_{\mu 0 i}T^i, \quad B_\mu \equiv A_{\mu\, ij} f^{ij}{}_k T^k.$$

The covariant derivative and the field strength

$$\hat{D}_\mu \equiv \partial_\mu \hat{X}^I + i[\hat{A}_\mu, \hat{X}^I], \quad \hat{D}_\mu \Psi \equiv \partial_\mu \hat{\Psi} + i[\hat{A}_\mu, \hat{\Psi}], \quad \hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + i[\hat{A}_\mu, \hat{A}_\nu] \tag{2.2.43}$$

are the ordinary covariant derivative and field strength for the sub-algebra $\mathcal{A}$. Since this action has a negative metric, we call this as *Lorentzian BLG model*. The specialties of Lorentzian BLG model are manifest $SO(8)$ R-symmetry and $\mathcal{N} = 8$, BF theory and usual Lie algebra $[X, Y]$. There are ghost terms because of a negative metric. And we should consider whether this model is unitary and how to eliminate this ghost term or ghost degrees of freedom. With the negative metrics, there is also a no-go theorem [55]. So we have essentially $SO(4)$ BLG model with positive metrics and Lorentzian BLG model with negative metrics.

The supersymmetry transformations for each mode are given by

$$\begin{aligned}
\delta X_0^I &= i\bar{\epsilon}\Gamma^I \Psi_0, \\
\delta X_{-1}^I &= i\bar{\epsilon}\Gamma^I \Psi_{-1}, \\
\delta \hat{X}^I &= i\bar{\epsilon}\Gamma^I \hat{\Psi}, \\
\delta \Psi_0 &= \partial_\mu X_0^I \Gamma^\mu \Gamma^I \epsilon, \\
\delta \Psi_{-1} &= \{\partial_\mu X_{-1}^I - \mathrm{tr}(B_\mu, \hat{X}^I)\}\Gamma^\mu \Gamma^I \epsilon + \frac{i}{6}\,\mathrm{tr}(\hat{X}^I, [\hat{X}^J, \hat{X}^K])\Gamma^{IJK}\epsilon, \\
\delta \hat{\Psi} &= \hat{D}_\mu \hat{X}^I \Gamma^\mu \Gamma^I \epsilon - B_\mu X_0^I \Gamma^\mu \Gamma^I \epsilon + \frac{i}{2} X_0^I [\hat{X}^J, \hat{X}^K]\Gamma^{IJK}\epsilon, \\
\delta \hat{A}_\mu &= i\bar{\epsilon}\Gamma_\mu \Gamma_I (X_0^I \hat{\Psi} - \hat{X}^I \Psi_0), \\
\delta B_\mu &= \bar{\epsilon}\Gamma_\mu \Gamma_I [\hat{X}^I, \hat{\Psi}].
\end{aligned} \tag{2.2.44}$$

The above construction of the 3-algebra contains the ordinary Lie algebra as a sub-algebra. The generators of the gauge transformation can be classified into 3 classes.

- $\mathcal{I}$=$\{T^{-1} \otimes T^a, a = 0, i\}$
- $\mathcal{A}$=$\{T^0 \otimes T^i\}$
- $\mathcal{B}$=$\{T^i \otimes T^j\}$

Then it is easy to show that

$$[\mathcal{I}, \mathcal{I}] = [\mathcal{I}, \mathcal{A}] = [\mathcal{I}, \mathcal{B}] = 0,\ [\mathcal{A}, \mathcal{A}] = \mathcal{A},\ [\mathcal{A}, \mathcal{B}] = \mathcal{B},\ [\mathcal{B}, \mathcal{B}] = \mathcal{I} \tag{2.2.45}$$

and hence the generators of $\mathcal{A}$ form a sub-algebra, which can be identified as the Lie algebra of N D2-branes. We will see concretely this gauge symmetry in section 3.2.

In Lorentzian action, we have another symmetry. The scaling of structure constants can be absorbed in redefinition of T^0, T^{-1} as we can see in (2.2.40). This means that the scaling of overall coefficient of the Lagrangian is a symmetry. To make this symmetry explicitely, we define a scaling of overall coefficient as $1/g^2$. Then the theory has the symmetry

$$\begin{aligned}
&\hat{X}^I \to g\hat{X}^I, && X_0^I \to \frac{1}{g}X_0^I, && X_{-1} \to g^3 X_{-1}^I, \\
&\hat{\Psi} \to g\hat{\Psi}, && \Psi_0 \to \frac{1}{g}\Psi_0, && \Psi_{-1} \to g^3\Psi_{-1}, \\
&\hat{A}_\mu \to \hat{A}_\mu, && B_\mu \to g^2 B_\mu. &&
\end{aligned} \tag{2.2.46}$$

So the overall coefficient of Lorentzian BLG model is irrelevant. This Lagrangian has no free parameter. The original BLG action (2.2.15) can have the integer coupling in front of Chern-Simons term as essentially same way as (2.2.17). If we scale the matter fields as

$$X^I \to \left(\frac{k}{2\pi}\right)^{\frac{1}{2}} X^I, \quad \Psi \to \left(\frac{k}{2\pi}\right)^{\frac{1}{2}} \Psi, \tag{2.2.47}$$

then we obtain the action with overall coefficients $k/2\pi$. This overall coefficient takes a important role in CS theory, but is irrelevant in BF theory.

When we focus on the fields related to T^{-1} generator, we can see such kind of fields only included linearly. Therefore we can regard these fields X_{-1}, Ψ_{-1} as Lagrange multiplier to obtain constraint equations

$$\partial^2 X_0^I = 0, \quad \Gamma^\mu \partial_\mu \Psi_0 = 0. \tag{2.2.48}$$

If we consider these constraint equations and regard solutions of these constraint equations as effective couplings of Lorentzian BLG action, then we can treat the Lorentzian BLG model as a ghost-free action. This idea is important when we deal with conformal symmetry and compare with a result from the dual gravity picture (we will see explicitly in chapter 4, p.45).

D2 reduction

If the BLG model represents effective action of multiple M2-branes correctly, we can reduce the action to the effective action of multiple D2-branes. This can be done by taking a constant vacuum expectation value of scalar fields as

$$X_0^I = v\delta^{I,8}, \quad \Psi_0 = 0. \tag{2.2.49}$$

This selection of the solution in the constraint equation (2.2.48) is compatible with the SUSY transformation and also the gauge transformation. The idea of giving a constant vev in the context of multiple M2-branes is firstly introduced by [56] in $SO(4)$ BLG and [6] in Lorentzian BLG.

Taking this vev back to the Lorentzian action (2.2.42), then we obtain

$$\begin{aligned}\mathcal{L} = \mathrm{Tr}\Bigg[& -\frac{1}{4v^2}\hat{F}_{\mu\nu}^2 - \frac{1}{2}(\hat{D}_\mu \hat{X}^A)^2 + \frac{1}{4}v^2[\hat{X}^A, \hat{X}^B]^2 \\ & + \frac{i}{2}\bar{\hat{\Psi}}\Gamma^\mu \hat{D}_\mu \hat{\Psi} + \frac{1}{2}v\bar{\hat{\Psi}}[\hat{X}^A, \Gamma_{8,A}\hat{\Psi}]\Bigg].\end{aligned} \tag{2.2.50}$$

This is exactly the same action of multiple D2-branes and breaks conformal symmetry by giving a vev.

Ghost problem

There is a discussion to avoid ghost problem [57, 58]. Basic idea is to be gauged a constant shift symmetry of X_{-1}^I, Ψ_{-1}.

At casual glance at (2.2.42), we can see a constant shift symmetry;

$$\delta_{\mathrm{sh}} X_{-1}^I = \Lambda^I, \quad \delta_{\mathrm{sh}} = \eta. \tag{2.2.51}$$

If we impose this shift symmetry to be locally gauged, we need to add a new fields C_μ^I, χ as

$$\mathcal{L}_{\mathrm{new}} = -C_\mu^I \partial^\mu X_0^I + \bar{\Psi}_0 \chi. \tag{2.2.52}$$

When the new fields transform under gauged shift symmetry as

$$\delta_{\mathrm{sh}} C_\mu^I = \partial_\mu \delta^I(x), \quad \delta_{\mathrm{sh}}\chi = i\Gamma^\mu \partial_\mu \eta(x), \tag{2.2.53}$$

we can take the Lorentzian BLG action (2.2.42) with (2.2.52) to be invariant. Note that there is another new local symmetry defined as

$$\delta_g C_\mu^I = \partial^\nu \bar{\Lambda}_{\mu\nu}^I, \quad \bar{\Lambda}_{\nu\mu}^I = -\bar{\Lambda}_{\mu\nu}^I \tag{2.2.54}$$

Under this symmetry the new term (2.2.52) itself invariant.

We can gauge fixed the gauged shift symmetry Λ^I, η to choose the field associated with T^{-1} as

$$X_{-1}^I = 0, \quad \Psi_{-1} = 0. \tag{2.2.55}$$

With this gauge, we can see there are no ghost term in Lorentzian BLG because the new additional term (2.2.52) is not a ghost term. Since the new fields C_μ^I, χ are also only included linearly, we can regard them as Lagrange multipliers and then obtain constraints

$$\partial_\mu X_0^I = 0, \quad \Psi_0 = 0. \tag{2.2.56}$$

This suggests we are allowed to have only constant solutions for effective couplings.

However this analysis seems to be spurious [10, 58, 59] [2]. In the analysis we are restricted to have only constant solutions from the beginning. This can be seen from the fact that we have used first derivative in ghost action to obtain shift symmetry. And also we can see this statement when we just simply develop the BLG model with a mass deformation [60, 61]. With the Lorentzian algebras (2.2.40), we obtain the massive ghost term [13]

$$\mathcal{L}_{\text{bosonic}-\text{gh}} = (\partial_\mu X_0^I)(\partial^\mu X_{-1}^I) + \frac{m}{2} X_0^I X_{-1}^I. \tag{2.2.57}$$

At a glance, we can see there are no shift symmetry. However, even if for this massive case, we can still consider the constraint equation method and have

$$\partial^2 X_0^I - \frac{m}{2} X_0^I = 0. \tag{2.2.58}$$

For the massive case we have rich structure of solutions which can be regarded as effective couplings of Lorentzian BLG theory. We would like to mention again that to take the constraint equations is compatible with conformal symmetry and the dual gravitational picture as we will see in chapter 4, p.45.

2.3 Remarkable progress 2; ABJM model

There has been second remarkable progress to constructing the effective action of multiple M2-branes by Aharony, Bergman, Jafferis and Maldacena (ABJM) [7]. The ABJM model has $SU(4)$ R-symmetry and $\mathcal{N} = 6$ supersymmetry. ABJM generalized the $SO(4)$ BLG model to a $U(N) \times U(N)$ bi-fundamental representation, so this model has opposite levels of Chern-Simons term. First we write down the action explicitely, then we will explain the construction of this model.

2.3.1 ABJM action

The action of ABJM model is given by (we use the convention used in [9])

$$\begin{aligned} S = \int d^3x \;\; \mathrm{tr}\,[&-(D_\mu Z_A)^\dagger D^\mu Z^A - (D_\mu W^A)^\dagger D^\mu W_A + i\zeta_A^\dagger \Gamma^\mu D_\mu \zeta^A + i\omega^{\dagger A}\Gamma^\mu D_\mu \omega_A] \\ &+ S_{CS} - S_{V_f} - S_{V_b}, \end{aligned} \tag{2.3.1}$$

[2] First, authors in [58] said Lorentzian BLG was just D2, but they retracted in v2.

with $A = 1, 2$. This is an $\mathcal{N} = 6$ superconformal $U(N) \times U(N)$ Chern-Simons theory. Z, W is a bifundamental field under the gauge group and its covariant derivative is defined by

$$D_\mu X = \partial_\mu X + iA_\mu^{(L)} X - iXA_\mu^{(R)}. \tag{2.3.2}$$

The gauge transformations $U(N) \times U(N)$ act from the left and the right on this field as $Z \to UZV^\dagger$.

The level of the Chern-Simons gauge theories is $(k, -k)$ and the coefficients of the Chern-Simons terms for the two $U(N)$ gauge groups, $A_\mu^{(L)}$ and $A_\mu^{(R)}$, are opposite. Hence the action $\mathcal{S}_{CS}$ is given by

$$S_{CS} = \int d^3x \ \frac{k}{4\pi}\epsilon^{\mu\nu\lambda} \ \mathrm{tr}\,[A_\mu^{(L)}\partial_\nu A_\lambda^{(L)} + \frac{2i}{3}A_\mu^{(L)}A_\nu^{(L)}A_\lambda^{(L)} - A_\mu^{(R)}\partial_\nu A_\lambda^{(R)} - \frac{2i}{3}A_\mu^{(R)}A_\nu^{(R)}A_\lambda^{(R)}]. \tag{2.3.3}$$

The potential term for bosons is given by

$$S_{V_b} = -\frac{4\pi^2}{3k^2}\int d^3x \ \ \mathrm{tr}\ \Big[Y^A Y_A^\dagger Y^B Y_B^\dagger Y^C Y_C^\dagger + Y_A^\dagger Y^A Y_B^\dagger Y^B Y_C^\dagger Y^C \\ +4Y^A Y_B^\dagger Y^C Y_A^\dagger Y^B Y_C^\dagger - 6Y^A Y_B^\dagger Y^B Y_A^\dagger Y^C Y_C^\dagger\Big], \tag{2.3.4}$$

and for fermions by

$$S_{V_f} = \frac{2\pi i}{k}\int d^3x \ \mathrm{tr}\ \Big[Y_A^\dagger Y^A \psi^{B\dagger}\psi_B - Y^A Y_A^\dagger \psi_B \psi^{B\dagger} + 2Y^A Y_B^\dagger \psi_A \psi^{B\dagger} - 2Y_A^\dagger Y^B \psi^{A\dagger}\psi_B \\ +\epsilon^{ABCD}Y_A^\dagger \psi_B Y_C^\dagger \psi_D - \epsilon_{ABCD}Y^A \psi^{B\dagger} Y^C \psi^{D\dagger}\Big]. \tag{2.3.5}$$

Y^A and ψ_A $(A = 1 \cdots 4)$ are defined by

$$Y^C = (Z^A, W^{\dagger A}), \quad \psi_C = (\epsilon_{AB}\zeta^B e^{i\pi/4}, \epsilon_{AB}\omega^{\dagger B}e^{-i\pi/4}), \tag{2.3.6}$$

where the index C runs from 1 to 4. Note that we will rewrite Z fields as A, $W^\dagger$ as $B^\dagger$ to take into account in superspace notation later. The fermion decomposition is useful for later convenience. The $SU(4)$ R-symmetry of the potential terms is manifest in terms of Y^A and ψ_A.

2.3.2 Construction of ABJM

What should we do is to understand how to obtain $\mathcal{N} = 6$ Chern-Simons superconformal theory as ABJM model. There are interesting works to obtain higher SUSY in CS theory. However we maximally have $\mathcal{N} = 3$ CS theory. Let's see the construction of $\mathcal{N} = 3$ CS theory [62–64].

People might be interesting at the dimensional reduction of well-known $\mathcal{N} = 4$ supersymmetric theory in four-dimension. In $\mathcal{N} = 4$ theory, we have the superpotential of the form

$$W = \tilde{\Phi}_i \varphi \Phi_i \tag{2.3.7}$$

where we use an auxiliary adjoint chiral multiplet φ in addition to the vector multiplet, and chiral multiplets come in pairs $\tilde{\Phi}_i, \Phi_i$ in conjugate representations and i, j run $1, 2$. Let's connect this superpotential with the CS term includes an additional superpotential

$$W = -\frac{k}{8\pi} \operatorname{tr} \varphi^2 \tag{2.3.8}$$

in three-dimension. Since the auxiliary field φ is included without kinetic term, it can be simply integrated out and we obtain

$$W = \frac{4\pi}{k}(\tilde{\Phi}_i T^a_{R_i} \Phi_i)(\tilde{\Phi}_j T^a_{R_j} \Phi_j). \tag{2.3.9}$$

The couple to the CS term causes to decrease SUSY to $\mathcal{N} = 3$. Note that this CS theory cannot be renormalized beyond a possible one-loop shift of k so much so that it is conformally invariant also at the quantum level [65].

Note that however if we are allowed to have coordinate dependent coupling $g(y), \theta(y)$ or complex coupling $\tau(y)$ on the compact direction y, we can still preserve $\mathcal{N} = 4$ even if the presence of CS term in three-dimension [17]. The relevant part of the action is

$$S_\theta = -\frac{1}{32\pi^2} \int d^4x \, \theta(y) \epsilon^{\mu\nu\rho\sigma} \operatorname{tr} F_{\mu\nu} F_{\rho\sigma}. \tag{2.3.10}$$

Integrating by parts for the compactified direction y and dropping any surface terms, we obtain

$$S_\theta = \frac{1}{8\pi^2} \int d^3x dy \, \frac{d\theta}{dy} \epsilon^{\mu\nu\lambda} \operatorname{tr} \left(A_\mu \partial_\nu A_\lambda + \frac{2}{3} A_\mu A_\nu A_\lambda \right). \tag{2.3.11}$$

This form of equation imply us that supersymmetrizing a four-dimensional theory with a y-dependent θ angle is somewhat similar to supersymmetrizing a three-dimensional theory with a Chern-Simons interaction. This coordinate dependence should be related to the *Janus configuration* [14] in the dual gravity. In the gravity picture, we resolve AdS_5 which describes D3-branes with coordinate dependent (on compact direction) dilaton to obtain AdS_4. The action obtained in [17] is conformal in three-dimension, this seems to be the dual of AdS_4.

In this stage, let's remind the construction of BLG model, but especially of $SO(4)$ BLG model. The $SO(4)$ BLG model can be reinterpreted as $SU(2) \times SU(2)$ bi-fundamental gauge group (2.2.17) with opposite levels of CS terms. ABJM shed some light on this construction and tried to generalize $U(N) \times U(N)$ gauge group. The key idea to do this is simple to introduce bi-fundamental chiral multiplets A_i, B_i and their pairs and follow the same manner in $\mathcal{N} = 3$. The superpotential becomes

$$W = \frac{k}{8\pi} \operatorname{tr}(\varphi^2_{(R)} - \varphi^2_{(L)}) + \operatorname{tr} B_i \varphi_{(L)} A_i + \operatorname{tr} A_i \varphi_{(R)} B_i. \tag{2.3.12}$$

We can also integrate out the auxiliary fields, the obtain

$$W = \frac{2\pi}{k} \epsilon^{ab} \epsilon^{\dot{a}\dot{b}} \operatorname{tr} \left(A_a B_{\dot{a}} A_b B_{\dot{b}} \right). \tag{2.3.13}$$

Surprisingly this superpotential is exactly same form (up to a coefficient) as Klebanov-Witten type superpotential [66].

The whole potential of ABJM comes from superpotential and also the Chern-Simons piece. The scalar potential arising from the superpotential is

$$V_{\rm sup} = |\partial W|^2 = \frac{16\pi^2}{k^2}\left(\epsilon_{\dot{a}\dot{b}}\epsilon_{\dot{c}\dot{d}}\,{\rm tr}\left[W_{\dot{b}}^\dagger Z_a^\dagger W_{\dot{a}}^\dagger W_{\dot{c}} Z_a W_{\dot{d}}\right] + \epsilon_{ab}\epsilon_{cd}\,{\rm tr}\left[Z_b^\dagger W_{\dot{a}}^\dagger Z_a^\dagger Z_c W_{\dot{a}} Z_d\right]\right) \tag{2.3.14}$$

where we use the scalar components of A as Z, and B as W without confusing. On the other hand the potential comes from the CS term is

$$V_{CS} = {\rm tr}\left[Z_c Z_c^\dagger \sigma_{(1)}^2 - 2Z_c^\dagger \sigma_{(1)} Z_c \sigma_{(2)} + Z_c^\dagger Z_c \sigma_{(2)}^2\right] + {\rm tr}\left[W_{\dot{c}} W_{\dot{c}}^\dagger \sigma_{(2)}^2 - 2W_{\dot{c}}^\dagger \sigma_{(2)} W_{\dot{c}} \sigma_{(1)} + W_{\dot{c}}^\dagger W_{\dot{c}} \sigma_{(1)}^2\right] \tag{2.3.15}$$

where $\sigma_{(1,2)}$ are the real scalar field in the vector multiplet (coming from the A_3 component of the gauge field when we dimensionally reduce from $3+1$ dimensions) as

$$S_{CS}^{\mathcal{N}=2} = \frac{k}{2\pi}\int {\rm tr}\left(A\wedge dA + \frac{2}{3}A^3 - \bar{\chi}\chi + 2D\sigma\right). \tag{2.3.16}$$

In addition we have the usual D term coupling

$${\rm tr}\left[Z_a D_{(1)} Z_a^\dagger - Z_a^\dagger D_{(2)} Z_a - W_{\dot{b}}^\dagger D_{(1)} W_{\dot{b}} + W_{\dot{b}} D_{(2)} W_{\dot{b}}^\dagger\right]. \tag{2.3.17}$$

By integrating out for $D_{(1,2)}$, we obtain the relations

$$\frac{k}{2\pi}\sigma_{(1)} = Z_a Z_a^\dagger - W_{\dot{b}}^\dagger W_{\dot{b}}, \quad \frac{k}{2\pi}\sigma_{(2)} = Z_a^\dagger Z_a - W_{\dot{a}} W_{\dot{a}}^\dagger. \tag{2.3.18}$$

Plugging this into V_{CS}, we obtain the whole potential as

$$\begin{aligned} V &= V_{\rm sup} + V_{CS} \\ &= {\rm tr}\left[Y^A Y_A^\dagger Y^B Y_B^\dagger Y^C Y_C^\dagger + Y_A^\dagger Y^A Y_B^\dagger Y^B Y_C^\dagger Y^C \right. \\ &\qquad\qquad \left. + 4Y^A Y_B^\dagger Y^C Y_A^\dagger Y^B Y_C^\dagger - 6Y^A Y_B^\dagger Y^B Y_A^\dagger Y^C Y_C^\dagger\right] \end{aligned} \tag{2.3.19}$$

where we use $Y^A \equiv (Z_1, Z_2, W_{\dot{1}}^\dagger, W_{\dot{2}}^\dagger)$. This is the $SU(4)$ invariant form of potential so much so that we have $SU(4)$ R-symmetry and $\mathcal{N}=6$. This means we have originally $SU(2)\times SU(2)$ R-symmetry as the rotation of each $A_a, B_{\dot{b}}$. However together with the CS potential, the two $SU(2)$ do not commute each other, and then rotate all together. This is the result of calculation (2.3.19). We recommend readers to see [9], who want to see explicitely with the fermion potential.

Expressions in superspace and components

We follow [9] to obtain the whole explicit ABJM constructions in superspace notation. The superspace action consists from the super Chern-Simons term, matter kinetic terms and matter potential terms. The matter potential term is the same form as the Klebanov-Witten superpotential. This is the summary and manifestation of ABJM construction which we have seen.

$$\begin{aligned}
S &= S_{\rm CS} + S_{\rm mat} + S_{\rm pot}, \\
S_{\rm CS} &= -\frac{ik}{8\pi}\int d^3x d^4\theta \int_0^1 dt\, {\rm tr}\left[\mathcal{V}\bar{D}^\alpha\left(e^{t\mathcal{V}}D_\alpha e^{-t\mathcal{V}}\right) - \hat{\mathcal{V}}\bar{D}^\alpha\left(e^{t\hat{\mathcal{V}}}D_\alpha e^{-t\hat{\mathcal{V}}}\right)\right], \\
S_{\rm mat} &= \int d^3x d^4\theta\, {\rm tr}\left[-\bar{A}_a e^{-\mathcal{V}}A^a e^{\hat{\mathcal{V}}} - \bar{B}^{\dot{b}}e^{-\hat{\mathcal{V}}}B_{\dot{b}}e^{\mathcal{V}}\right], \\
S_{\rm pot} &= \frac{8\pi}{k}\int d^3x d^2\theta W(A,B) + \frac{8\pi}{k}\int d^3x d^2\bar{\theta}\bar{W}(\bar{A},\bar{B})
\end{aligned} \tag{2.3.20}$$

with

$$\begin{aligned}
&\mathcal{V} = 2i\theta\bar{\theta}\sigma^{(L)}(x) + 2\theta\gamma_\mu\bar{\theta}A^{(L)}_\mu(x) + \sqrt{2}i\theta^2\bar{\theta}\bar{\chi}^{(L)}(x) - \sqrt{2}i\bar{\theta}^2\theta\chi^{(L)}(x) + \theta^2\bar{\theta}^2 D^{(L)}(x), \\
&A = Z + \sqrt{2}\theta\zeta + \theta^2 F_z, \quad A^\dagger = Z^\dagger - \sqrt{2}\bar{\theta}\zeta^\dagger - \bar{\theta}^2 F_z^\dagger, \\
&B = W + \sqrt{2}\theta w + \theta^2 F_w, \quad B^\dagger = W^\dagger - \sqrt{2}\bar{\theta}w^\dagger - \bar{\theta}^2 F_w^\dagger, \\
&W(A,B) = \frac{1}{4}\epsilon_{ab}\epsilon^{\dot{a}\dot{b}}\,{\rm tr}\,A^a B_{\dot{a}}A^b B_{\dot{b}}, \quad \bar{W}(\bar{A},\bar{B}) = \frac{1}{4}\epsilon^{ab}\epsilon_{\dot{a}\dot{b}}\,{\rm tr}\,\bar{A}_a\bar{B}^{\dot{a}}\bar{A}_b\bar{B}^{\dot{b}}
\end{aligned} \tag{2.3.21}$$

where σ, D, F_z and F_w are auxiliary scalars, $\chi, \bar{\chi}$ are auxiliary fermions. The $\hat{\mathcal{V}}$ has been used as a superfield of $A^{(R)}$. The action includes opposite levels of Chern-Simons terms and two complex fields each of which has $SU(2)_R$ global symmetry and Klebanov-Witten superpotential with respect to the orbifolded moduli space $\mathbb{C}^4/\mathbb{Z}_k$.

The gauge transformations rule is given by

$$\begin{aligned}
&e^{t\mathcal{V}} \to e^{i\Lambda}e^{t\mathcal{V}}e^{-i\bar{\Lambda}}, \quad e^{t\hat{\mathcal{V}}} \to e^{i\hat{\Lambda}}e^{t\hat{\mathcal{V}}}e^{-i\hat{\bar{\Lambda}}}, \\
&A \to e^{i\Lambda}Ae^{-i\bar{\Lambda}}, \quad \bar{A} \to e^{i\hat{\Lambda}}\bar{A}e^{-i\hat{\bar{\Lambda}}}, \quad B \to e^{i\bar{\Lambda}}Be^{-i\Lambda}, \quad \bar{B} \to e^{i\hat{\bar{\Lambda}}}\bar{B}e^{-i\bar{\Lambda}}
\end{aligned} \tag{2.3.22}$$

where Λ is a gauge parameter related to $A^{(L)}$ and $\hat{\Lambda}$ is with respect to $A^{(R)}$. Note that we have taken into account the bi-fundamental matter fields from $SO(4)$ BLG model, so much so that this theory has $SO(N)\times SO(N)$ gauge symmetry naturally. Since ABJM model has two bi-fundamental superfields, we have additional $U(1)$ gauge symmetry

$$A \to e^{i\alpha}A, \quad B \to e^{-i\alpha}B. \tag{2.3.23}$$

Together with this additional $U(1)$ symmetries, we have enhanced gauge symmetry $U(N)\times U(N)$ in (2.3.22).

The notation we have used is as follows.

$$\begin{aligned} g^{\mu\nu} &= \mathrm{diag}(-1,+1,+1), \\ \gamma^{\mu}{}_{\alpha}^{\ \beta} &= (i\sigma^2,\sigma^1,\sigma^3)_{\alpha}^{\ \beta}, \\ \theta^{\alpha} &= \epsilon^{\alpha\beta}\theta_{\beta}, \quad \theta_{\alpha} = \epsilon_{\alpha\beta}\theta^{\beta}, \\ & (\epsilon^{12} = -\epsilon_{12} = 1). \end{aligned} \tag{2.3.24}$$

The operator which raise or lower spinor indices is related to zero component of gamma matrices. We have used the spinor indices sum notation as

$$\begin{aligned} &\theta^{\alpha}\theta_{\alpha} \equiv \theta^2, \quad \theta^{\alpha}\bar{\theta}_{\alpha} \equiv \theta\bar{\theta}, \\ &\theta^{\alpha}\gamma^{\mu}{}_{\alpha\beta}\bar{\theta}^{\beta} \equiv \theta\gamma^{\mu}\bar{\theta}, \\ &\gamma^{\mu}{}_{\alpha\beta} = (-\mathbb{I},-\sigma^3,\sigma^1)_{\alpha\beta}. \end{aligned} \tag{2.3.25}$$

This notation is same as what we usually deal with in four-dimensional language, however only one-set of Pauli matrices is essential in three dimension. The other useful expressions, the Fierz identities, supercovariant derivatives and SUSY generators etc. are denoted in appendix of [9].

The bosonic potential term can be obtained from the F-term and D-term as

$$\begin{aligned} V_F^{\mathrm{bos}}\,(=V_{\mathrm{sup}}) &= \left|\frac{\partial W}{\partial Z^a}\right|^2 + \left|\frac{\partial W}{\partial W_{\dot{a}}}\right|^2, \\ V_D^{\mathrm{bos}}\,(=V_{CS}) &= \mathrm{tr}\left[N_a^{\dagger}N^a + M^{\dagger\dot{a}}M_{\dot{a}}\right] \end{aligned} \tag{2.3.26}$$

where $N^a = \sigma^{(L)}Z^a - Z^a\sigma^{(R)}$, $M_{\dot{a}} = \sigma^{(R)}W_{\dot{a}} - W_{\dot{a}}\sigma^{(L)}$. By integrating out the auxiliary fields, then we obtain the $SU(4)$ superpotential (2.3.19) in components. The whole action with the fermions (2.3.1) can be obtained by integrating out all auxiliary fields from the superspace expression.

2.3.3 Duality mapping from IIB

We can also say about the construction of ABJM model in terms of the duality mapping from IIB configuration. As we mentioned in the above section, we introduced usual $\mathcal{N}=4$ superpotential, but in three-dimension. This can be regarded in brane picture as introducing the N D3-brane, one of which world-coordinates (we choose this as 6 direction) is compactified. With consideration of $SO(4)$ BLG model, we need to introduce the bi-fundamental matter fields and opposite level of CS terms. This can be done in brane construction by adding two NS5-branes as in the left hand side of Figure 2.2. If we introduce a NS5-brane, there are strings which stretches over a NS5-brane as we see in Figure 2.3. In our construction we have a compact direction so much so that we should have at least two NS5-branes to obtain bi-fundamental representation.

In addition to the above construction, we should have integer k which is the coefficient of CS terms. This can be regarded as introducing the k D5 brane as in the middle picture

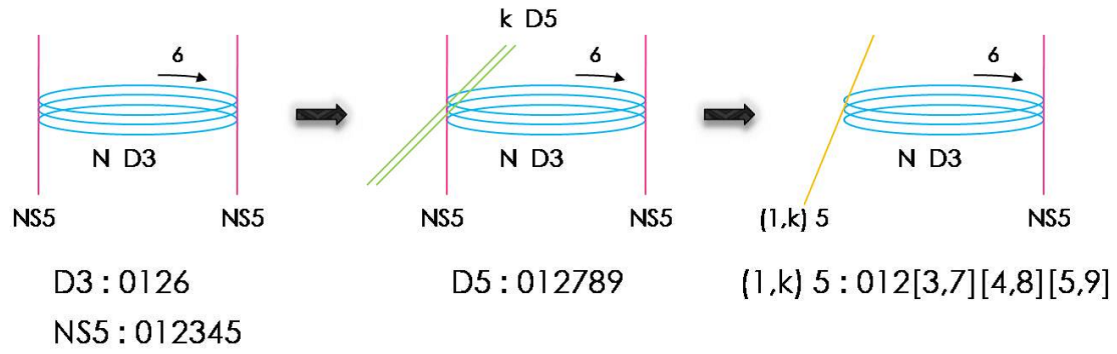

Figure 2.2: Brane construction of ABJM model. We introduced N D3-branes and two NS5-branes (left). Then we add k D5 at the cross section of D3 and NS5 (middle). k D5 and NS5 are resolved together and become $(1,k)$5-brane (right).

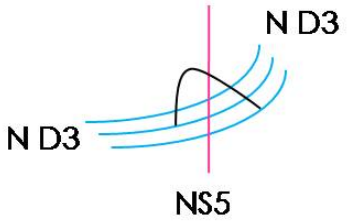

Figure 2.3: A string stretched over a NS5-brane. This string represents a bi-fundamental matter.

of Figure 2.2. The D3-branes effective action only has a usual gauge kinetic term, not the CS term. However when we introduce the k D5-branes, we also have the CS term. This can be understood as follows.

We add k D5-brane along 012789, which intersect the D3-branes along 012, as well as one of the NS5-branes along 012. Next we introduce a real mass term of equal sign for the fundamental and anti-fundamental chiral multiplets which live on D5-branes [3]. This causes a *web-deformation* in Figure 2.4 [67]. The k D5-branes and NS5-branes break along the directions 012 and dissolve into an intermediate $(1,\pm k)$5-brane which extends along three-dimensions $[3,7]_\theta, [4,8]_\theta, [5,9]_\theta$ with the angle $\tan\theta = k$. The sign of the charges of the intermediate 5-brane comes from the sign of the mass term. The coefficient of the CS term gets a contribution $+\frac{1}{2}$ from each Majorana fermion with a positive mass term, and $-\frac{1}{2}$ from each fermion with a negative mass term. Therefore we get a total k coefficient k for one of the $U(N)$ factors and $-k$ for the other. The resultant figure is showed in the right hand side of Figure 2.2.

We can also understand the existence of CS term explicitly [68]. The effective field

[3]There are also the other possible mass deformations. The separation along the 3,4,5 directions corresponds to a real mass term of equal magnitude but *opposite sign* for the fundamental chiral and anti-chiral multiplets.

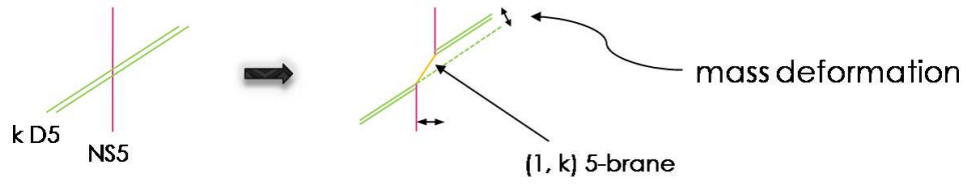

Figure 2.4: The web deformation of intersecting NS5-D5 configuration.

theory of the construction of the right hand side of Figure 2.2 has

$$S = -\frac{1}{4g_4^2}\int d^3xdx^6 F_{MN}F^{MN}, \tag{2.3.27}$$

where we use the indices M, N runs $0, 1, 2, 3$ and we concentrate only on $U(1)$ gauge field for simplicity. We need to add the boundary conditions correspond to a NS5 and $(1, k)$5-branes. The boundary condition for the NS5, D5 are

$$\begin{aligned} \text{NS5} &: F_{\mu 6} = \partial_\mu A_6 - \partial_6 A_\mu = 0, \\ \text{D5} &: F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 0 \end{aligned} \tag{2.3.28}$$

where we μ, ν are $0, 1, 2$. To obtain $(1, k)$5-brane boundary condition, we mix these boundary condition by using $SL(2, \mathbb{Z})$ transformation.

$$\begin{pmatrix} 1 & -k \\ -s & r \end{pmatrix} \begin{pmatrix} F_{\mu 6} \\ a\epsilon_{\mu\nu\rho}\partial^\nu A^\rho \end{pmatrix} = 0. \tag{2.3.29}$$

Note that this transformation matrix represents inverse transformation of usual $SL(2, \mathbb{Z})$. The resultant boundary condition is

$$(1, k)5 : \partial_\mu A_6 - \partial_6 A_\mu - ak\epsilon_{\mu\nu\rho}\partial^\nu A^\rho = 0 \tag{2.3.30}$$

where a is an arbitrary constant which cannot be determined at this stage.

We put a NS5-brane at the point $x^6 = 0$, and a $(1, k)$5-brane at $x^6 = L$ where we define the length of compact direction is $2L$. For the sake of the $(1, k)5$ boundary condition we need to add the boundary term and the action changes to

$$\begin{aligned} S = &-\frac{1}{4g_4^2}\int d^3xdx^6 \left[F_{\mu\nu}F^{\mu\nu} + 2\left(\partial_6 A_\mu\right)^2 + 2\left(\partial_\mu A_6\right)^2 - 4\left(\partial_6 A_\mu \partial_\mu A_6\right)\right] \\ &+ \frac{a}{g_4^2}k\int_{x^6=L} d^3x\epsilon_{\mu\nu\lambda}A^\mu\partial^\nu A^\lambda. \end{aligned} \tag{2.3.31}$$

We would like to focus on *massless modes* of this system. The massless modes along the 6 direction means to be the fields should satisfy the massless Klein-Gordon equation $\partial_6^2 f = 0$. First let's take the three-dimensional gauge fields to be independent of x^6 for simplicity. What we should consider is only for fields A_6 which satisfies $\partial_6^2 A_6 = 0$. The equation is second derivative equation so much so that A_6 seems to be proportional

linearly to x^6. Together with the boundary conditions of a NS5 and a $(1,k)5$, we have a massless mode for A_6 as

$$\partial_\mu A_6 = ak\frac{x^6}{L}\epsilon_{\mu\nu\rho}\partial^\nu A^\rho. \tag{2.3.32}$$

Plugging into this massless mode to the action (2.3.31) with redefinition of the constant a and the coupling

$$\frac{1}{g^2} = \frac{L}{g_4^2}, \quad \frac{a}{g_4^2} = -\frac{1}{4\pi}, \tag{2.3.33}$$

then we obtained the action for the massless mode

$$S = -\frac{1}{4g^2}\int d^3x\ F_{\mu\nu}F^{\mu\nu} - \frac{k}{4\pi}\int d^3x\ \epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda \tag{2.3.34}$$

where we integrated out along the compactified direction x^6. Therefore we can get the CS term from this construction. Note that if we allow the field A_μ to depend x^6, it can be regarded as Kaluza-Klein massive field. In order that this theory make sense, the masses of these modes must be large compared to the former one (A_μ: independent of x^6). This can be done by choosing the combination kg_4^2 to be small compared to $1/L$.

In the above section 2.3.2, we have obtained the Klebanov-Witten type superpotential (2.3.13). The reason for that can also be considered on this stage. The Klebanov-Witten superpotential was originally considered as the dual field theory of D3-branes living on Sasaki-Einstein space $T^{1,1}$ and the dual geometry is $AdS_5 \times T^{1,1}$. The $T^{1,1}$ can be regarded as $\mathbb{S}^5/\mathbb{Z}_2$ which has the global symmetry $(SU(2)\times SU(2))/U(1)$ as

$$ds^2_{T^{1,1}} = d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2\theta_2 d\phi_2^2 + (d\chi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2\,. \tag{2.3.35}$$

This space represents $U(1)$ fiber bundle over $\mathbb{S}^2\times\mathbb{S}^2$. The Klebanov-Witten superpotential also has $(SU(2)\times SU(2))/U(1)$ global symmetry because the indices i,j are $1,2$. This type of superpotential is also the general form to have this global symmetry.

By the way, now we know the coefficient k in front of CS term causes the moduli space divided by $\mathbb{Z}_k$. This will be also seen soon in the analysis of geometrical picture. When we consider the gravitational space $\mathbb{S}^5/\mathbb{Z}_k$, we can obtain almost the same form of (2.3.35) with a difference of a coefficient in front of the $U(1)$ fibration. This comes from the reason to take the regular periodicity which described as

$$d\chi^2 \text{ with } \chi \simeq \chi + \frac{2\pi}{k} \ \rightarrow\ \frac{1}{k^2}d\chi^2 \text{ with } \chi \simeq \chi + 2\pi. \tag{2.3.36}$$

However this manifolds still have the global symmetry $(SU(2)\times SU(2))/U(1)$. So in the picture of brane construction, the Klebanov-Witten superpotential is natural.

Geometrical picture

We also mention about the reason of orbifolded projection of this construction by using the geometrical picture [69] [4]. To see the geometrical picture, let's remind the Gibbons-Hawking coordinate in four-dimensional space [70]. By using the Gibbons-Hawking coordinate system, we can know the solution for vacuum Einstein equation as

$$\begin{aligned} ds_4^2 &= U d\vec{x}^2 + U^{-1}\left(d\varphi + \vec{w}d\vec{x}\right)^2 \\ \vec{\nabla}^2 U &\equiv \partial_a\partial_a U = 0, \quad \partial_a w^b - \partial_b w^a = \epsilon_{abc}\partial_c U \end{aligned} \tag{2.3.37}$$

where $\varphi \simeq \varphi + 2\pi$ and φ direction is $U(1)$ fiber bundle over base manifold.

Let's generalize to the 8-dimensional manifold by dividing $4+4$, then we obtain

$$\begin{aligned} ds_8^2 &= U_{ij} d\vec{x}^i d\vec{x}^j + U^{ij}(d\varphi_i + A_i)(d\varphi_j + A_j) \\ A_i &= d\vec{x}^j \vec{w}_{ji}, \quad \partial_{x_a^j} w_{ki}^b - \partial_{x_b^k} w_{ji}^a = \epsilon^{abc}\partial_{x_c^j} U_{ki}. \end{aligned} \tag{2.3.38}$$

The solution describing the rotating Kaluza-Klein monopole corresponding to a NS5-brane has the form

$$U = \mathbf{1} + \begin{pmatrix} h_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad h_1 = \frac{1}{2|\vec{x}_1|}. \tag{2.3.39}$$

We also have the $(1,k)$5-brane, we need to add this configuration by performing $SL(2,\mathbb{Z})$ transformation to the torus T^2 which consists from two $U(1)$ fibration. The superposition of a NS5 and the (1,k)5-brane results

$$U = \mathbf{1} + \begin{pmatrix} h_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} h_2 & kh_2 \\ kh_2 & k^2 h_2 \end{pmatrix}, \quad h_1 = \frac{1}{2|\vec{x}_1|}, \quad h_2 = \frac{1}{2|\vec{x}_1 + k\vec{x}_2|}. \tag{2.3.40}$$

At a glance, (2.3.40) has a singularity at the point $\vec{x}^{1,2} \sim 0$. We can treat this carefully by taking a near horizon limit (neglecting the $\mathbf{1}$) and $GL(2)$ transformation

$$\vec{x}'^1 = \vec{x}^1, \quad \vec{x}'^2 = \vec{x}^1 + k\vec{x}^2. \tag{2.3.41}$$

This transformation tells us to have

$$U' = \frac{1}{2}\begin{pmatrix} \frac{1}{|\vec{x}'^1|} & 0 \\ 0 & \frac{1}{|\vec{x}'^2|} \end{pmatrix} \tag{2.3.42}$$

However, we should simultaneously transform the two $U(1)$ directions to keep the Gibbons-Hawking form (2.3.38) as

$$\varphi_1' = \varphi_1 - \frac{1}{k}\varphi_2, \quad \varphi_2' = \frac{1}{k}\varphi_2. \tag{2.3.43}$$

[4] We recommend you to see the Appendix B of [7].

This form (2.3.42) tells us the superposition of two completely orthogonal KK-monopoles. And (2.3.43) shows to have the periodicities

$$(\varphi_1', \varphi_2') \simeq (\varphi_1', \varphi_2') + 2\pi \left(-\frac{1}{k}, \frac{1}{k}\right). \tag{2.3.44}$$

Thus we can conclude that KK-monopole configuration of the superposition of a NS5 and a $(1, k)$5-brane has a $\mathbb{C}^4/\mathbb{Z}_k$ singularity. And this configuration can be considered as U-dual to the brane configurations. Note that in the special case of $k = 1$ the manifold is completely non-singular, as it looks like $\mathbb{R}^8$ at the origin.

2.3.4 The dual gravity picture of ABJM

In the paper [7], it was pointed out that the ABJM model is dual to the M-theory on $AdS_4 \times \mathbb{S}^7/\mathbb{Z}_k$, which is a $d = 11$ supergravity solution of M2 branes on the orbifold $\mathbb{C}^4/\mathbb{Z}_k$. This is the correct reflection of the moduli space of the ABJM model. We first review the solution of supersymmetric M2 branes in $d = 11$ supergravity.

The $d = 11$ metric of the multiple M2-branes is given by[5],

$$\begin{aligned} ds^2 &= H^{-\frac{2}{3}} \left(\sum_{\mu,\nu=0}^{2} \eta_{\mu\nu} dx^\mu dx^\nu \right) + H^{\frac{1}{3}} \left(dr^2 + r^2 d\Omega_7^2 \right), \\ H(r) &\equiv 1 + \frac{R^6}{r^6}, \end{aligned} \tag{2.3.45}$$

where $R^6 = 32\pi^2 N' l_p^6$ and $d\Omega_7^2$ is the metric of a unit 7-sphere. N' is the number of the M2 branes and later identified with $N' = kN$.

By focusing on the near horizon region of the M2-brane, the geometry becomes $AdS_4 \times \mathbb{S}^7$ geometry. In the near horizon limit $R \gg r$, $H(r)$ is replaced by $H(r) = (R/r)^6$ and the metric becomes

$$ds^2 = \left(\frac{r}{R}\right)^4 \left(\sum_{\mu,\nu=0}^{2} \eta_{\mu\nu} dx^\mu dx^\nu \right) + \left(\frac{R}{r}\right)^2 dr^2 + R^2 d\Omega_7^2. \tag{2.3.46}$$

The first two terms is the metric of AdS_4 and the near horizon geometry of supersymmetric M2 is given by $AdS_4 \times \mathbb{S}^7$. The radius of AdS_4 is $\frac{1}{2}R$ and that of $\mathbb{S}^7$ is R. In the large $N' = kN$ limit, the radius becomes much larger than the $d = 11$ Planck length and the $d = 11$ supergravity approximation is valid.

ABJM model describes M2 branes on $\mathbb{C}_4/\mathbb{Z}_k$ orbifold and the dual geometry can be

[5]We use the convention in [71].

also obtained by dividing the transverse R^8 direction by $\mathbb{Z}_k$. The $\mathbb{S}^7$ metric is written as

$$\begin{aligned}
d\Omega_7^2 &= (d\varphi' + w)^2 + ds_{\mathbb{C}P^3}, \\
d\Omega_7^2 &= \frac{dz^i d\bar{z}^i}{\rho^2}, \quad \rho^2 \equiv \sum_{i=1}^{4} \left|z^i\right|^2, \\
ds_{\mathbb{C}}P^3 &= \frac{1}{\rho^2}\left(dz^i - z^i \frac{\bar{z}^j dz^j}{\rho^2}\right)\left(d\bar{z}^i - \bar{z}^i \frac{z^j d\bar{z}^j}{\rho^2}\right), \\
d\varphi' + w &\equiv \frac{i}{2\rho^2}\left(z^i d\bar{z}^i - \bar{z}^i dz^i\right), \quad dw = id\left(\frac{z^i}{\rho}\right) d\left(\frac{\bar{z}^i}{\rho}\right),
\end{aligned} \tag{2.3.47}$$

where we parameterized

$$\begin{aligned}
z^1 &= \rho e^{i(\phi_1+\varphi')} \cos\theta, \\
z^2 &= \rho e^{i(\phi_2+\varphi')} \sin\theta \cos\psi, \\
z^3 &= \rho e^{i(\phi_3+\varphi')} \sin\theta \sin\psi \cos\chi, \\
z^4 &= \rho e^{i\varphi'} \sin\theta \sin\psi \sin\chi.
\end{aligned} \tag{2.3.48}$$

We perform the $\mathbb{Z}_k$ quotient by dividing the overall phase of each z^i, namely the φ' direction. Then by rewriting $\varphi' = \varphi/k$ with $\varphi \sim \varphi + 2\pi$, the metric of $\mathbb{S}^7/\mathbb{Z}_k$ becomes

$$ds^2_{\mathbb{S}^7/\mathbb{Z}_k} = \frac{1}{k^2}(d\varphi + kw)^2 + ds^2_{\mathbb{C}P^3}, \tag{2.3.49}$$

where $\mathbb{C}P^3$ geometry and 1-form w are given explicitely as

$$\begin{aligned}
ds^2_{\mathbb{C}P^3} =& d\theta^2 + \sin^2\theta d\psi^2 + \sin^2\theta \sin^2\psi d\chi^2 \\
&+ \cos^2\theta \sin^2\theta \left(d\phi_1 - \cos^2\psi d\phi_2 - \sin^2\psi \cos^2\chi d\phi_3\right)^2 \\
&+ \sin^2\theta \sin^2\psi \cos^2\psi \left(d\phi_2 - \cos^2\chi d\phi_3\right)^2 \\
&+ \sin\theta^2 \sin\psi^2 \sin^2\chi \cos^2\chi d\phi_3^2, \\
w =& \cos^2\theta d\phi_1 + \sin^2\theta \cos^2\psi d\phi_2 + \sin^2\theta \sin^2\psi \cos^2\chi d\phi_3.
\end{aligned} \tag{2.3.50}$$

Before performing the $\mathbb{Z}_k$ quotient, the metric has the conformal symmetry and $SO(8)$ invariance. The orbifolding breaks the $SO(8)$ symmetry to $U(4)$ but the conformal invariance still exists. This is precisely the bosonic symmetry of the ABJM model.

The compact radius of direction φ is in Planck units

$$\frac{R_{11}}{l_p} = \frac{R}{kl_p} \sim \frac{(Nk)^{\frac{1}{k}}}{k}. \tag{2.3.51}$$

Thus, the M-theory description is valid whenever $k^5 \ll N$, however in the opposite case we should reduce to the IIA 10-dimensional language with the Kaluza-Klein reduction [72]

$$ds^2_{11} = e^{-\frac{2}{3}\phi} ds^2_{10} + e^{\frac{4}{3}\phi} (l_p)^2 (d\varphi + A)^2. \tag{2.3.52}$$

Note that this is not the usual reduction to type IIA D2-branes. The D2-branes reduction from M2-branes can be seen in Appendix D. Then we obtain in terms of IIA supergravity

$$\begin{aligned} ds_{10}^2 &= \frac{R^3}{kl_p^3}\left[\frac{1}{4}ds_{AdS_4}^2 + ds_{\mathbf{CP}^3}^2\right], \\ e^{2\phi} &= \frac{R^3}{k^3 l_p^3} \sim \frac{1}{N^2}\left(\frac{N}{k}\right)^{\frac{5}{2}}, \\ F_4 &= \frac{3}{8}\frac{R^3}{l_p^3}\hat{\epsilon}_4, \\ F_2 &= dA = kd\omega \end{aligned} \tag{2.3.53}$$

where we set $l_s = 1$ and $\hat{\epsilon}_4$ is a unit AdS_4 volume. This results shows us we can obtain AdS_4 which has a dual conformal field theory, not only with D2-branes but also D0-branes. The radius of curvature in string units is

$$\frac{R_{\text{new}}^2}{l_s^2} = \frac{R^3}{kl_p^3} = 2^{\frac{5}{2}}\pi\sqrt{\frac{N}{k}}. \tag{2.3.54}$$

This shows the IIA gravity in string units is valid if $k \ll N$, on the other hand if $N \ll k$ we can expand the dual field theory as perturbation of 'tHooft coupling $\lambda \equiv N/k$.

The validity of each theories is summarized as

$$\begin{aligned} \text{M} - \text{theory}: &\quad k^5 \ll N, \\ \text{IIA string}: &\quad N \ll k^5,\ 1 \ll \lambda, \\ \text{3D CFT}: &\quad \lambda \ll 1. \end{aligned} \tag{2.3.55}$$

The result tells us there is a well-known relation between the IIA string and 3D CFT when we take a limit $k, N \to \infty$ but keep λ finite. If we consider in terms of 10-dimension, we can compare the many result as we did in the case for D3-branes because of the existence of couplings. However in M-theory, there are some difficulties to compare through AdS/CFT correspondence.

2.3.5 Triple algebra revisited

After the ABJM appeared, Bagger and Lambert (BL) have constructed $\mathcal{N} = 6$ Chern-Simons theory by using the generalized triple algebra. The BLG model has $\mathcal{N} = 8$ with the gauge structure constants f^{abcd} which are real, completely anti-symmetric. However if we relax these condition to have complex, and not completely anti-symmetric, but to obey

$$f^{abcd} = -f^{bacd} = -f^{abdc} = f^{*cdab}, \tag{2.3.56}$$

we can also construct the $\mathcal{N} = 6$ CS theory with generalized triple algebra

$$[T^a, T^b; T^c] = f^{abc}{}_d T^d. \tag{2.3.57}$$

This new BL model was also constructed by the closure of SUSY transformations

$$\begin{aligned}
\delta Z_d^A &= i\bar{\epsilon}^{AB}\psi_{Bd},\\
\delta\psi_{Bd} &= \gamma^\mu D_\mu A_d^A \epsilon_{AB} + f_1^{abc}{}_d Z_a^C \bar{Z}_{Cb} Z_c^A \epsilon_{AB} + f_2^{abc}{}_d Z_a^C Z_b^D \bar{Z}_{Bc}\epsilon_{CD},\\
\delta\tilde{A}_\mu{}^c{}_d &= i\bar{\epsilon}_{AB}\gamma_\mu Z_a^A \psi_b^B f_3^{abc}{}_d + i\bar{\epsilon}^{AB}\gamma_\mu \bar{Z}_{Aa}\psi_{Bb} f_4^{abc}{}_d
\end{aligned} \tag{2.3.58}$$

where we use ϵ^{AB} to satisfy the reality condition $\epsilon^{AB} = \frac{1}{2}\varepsilon^{ABCD}\epsilon_{CD}$. To enclose the SUSY algebra, we need to have

$$f_1^{abc}{}_d = f_2^{acb}{}_d \equiv f^{acb}{}_d, \quad f_3^{*abcd} = -f_4^{abdc}, \quad f_3^{abc}{}_d = f^{acb}{}_d \tag{2.3.59}$$

and also the generalized fundamental identity

$$\begin{aligned}
0 &= f^{efg}{}_b f^{cba}{}_d + f^{fea}{}_b f^{cbg}{}_d + f^{*gaf}{}_b f^{ceb}{}_d + f^{*age}{}_b f^{cfb}{}_d,\\
0 &= [T^c,[T^e,T^f,T^g];T^a] + [T^c,[T^f,T^e;T^a];T^g]\\
&\quad + [T^c,T^e;[T^g,T^a;T^f]] + [T^c,T^f;[T^a,T^g;T^e]].
\end{aligned} \tag{2.3.60}$$

Together with these condition, we can close the SUSY algebra and obtain equations of motion. To obtain equations of motion, the action should be defined as

$$\begin{aligned}
\mathcal{L} =& -\mathrm{Tr}(D_\mu Z^A, D^\mu \bar{Z}_A) - i\mathrm{Tr}(\bar{\psi}^A, \gamma^\mu D_\mu \psi_A) - V + \mathcal{L}_{CS}\\
&- i\mathrm{Tr}(\bar{\psi}^A, [\psi_A, Z^B; \bar{Z}_B]) + 2i\mathrm{Tr}(\bar{\psi}^A, [\psi_B, Z^B; \bar{Z}_A])\\
&+ \frac{i}{2}\varepsilon_{ABCD}\mathrm{Tr}(\bar{\psi}^A, [Z^C, Z^D; \psi^B]) - \frac{i}{2}\varepsilon^{ABCD}\mathrm{Tr}(\bar{Z}_D, [\bar{\psi}_A, \psi_B; \bar{Z}_C])
\end{aligned} \tag{2.3.61}$$

where we use

$$\begin{aligned}
V &= \frac{2}{3}\mathrm{Tr}(\Upsilon_B^{CD}, \bar{\Upsilon}_{CD}^B);\\
\Upsilon_B^{CD} &= [Z^C, Z^D; \bar{Z}_B] - \frac{1}{2}\delta_B^C [Z^E, Z^D; \bar{Z}_E] + \frac{1}{2}\delta_B^D [Z^E, Z^C; \bar{Z}_E],
\end{aligned} \tag{2.3.62}$$

This action is invariant under $\mathcal{N} = 6$ or 12 supersymmetries.

People might wonder the relation between the new BL model and the ABJM model. The ABJM model can be obtained from the new BL model as an example

$$[X, Y; Z] = \lambda(XZ^\dagger Y - YZ^\dagger X). \tag{2.3.63}$$

This explanation of generalized triple algebra satisfies the generalized fundamental identity. This construction can be viewed as states in the bi-fundamental representation easily. If the Lie algebra acts on X by

$$\delta X = XM_R - M_L^\dagger X, \tag{2.3.64}$$

then we can check that

$$\delta[X, Y; Z] = [X, Y; Z]M_R - M_L^\dagger [X, Y; Z]. \tag{2.3.65}$$

Thus we see the action is invariant under this type of transformation. By taking (2.3.63) into (2.3.61), the ABJM action (2.3.1) can be exactly reproduced. Note that this example is only known example of this $\mathcal{N} = 6$ construction. Because there is no no-go theorem to construct new BL model, it is interesting to consider the other examples.

Chapter 3

BLG model from ABJM model

So far we have introduced BLG model and ABJM model. Since the higher supersymmetric theory cannot be easily obtained, there should be relation between BLG and ABJM. In this chapter we will discuss about the relation. This chapter in mainly devoted to explain [10, 11].

3.1 SO(4) BLG model from ABJM model

The ABJM model was the essentially generalization of $SO(4)$ BLG model, however the construction is different. So we explore the relation, especially for superpotential in the case for $SU(2) \times SU(2)$ gauge group. In the proof of $SU(4)_R$ before, we combined the bifundamental A_a with *anti-bifundamental* $B_i^\dagger$ as the complex $SU(4)$ fields Y^A. But in this section we combine with the *bifundamental* B_i as new $SU(4)$ field. We also use the $SU(2)$ gauge indices and rotation for B_i to construct new $SU(4)$ fields. More precisely we write new $SU(4)$ fields as

$$(E_I)_\alpha^{\ \beta} = (A_{i\alpha}^{\ \ \beta}, \epsilon_{\alpha\alpha'}\epsilon^{\beta\beta'} B_i^{\ \alpha'}{}_{\beta'}). \tag{3.1.1}$$

By using this notation, we can rewrite the Klebanov-Witten type superpotential as

$$\begin{aligned} W &= \epsilon^{ab}\epsilon^{\dot{a}\dot{b}} \mathrm{tr}\,(A_a B_{\dot{a}} A_b B_{\dot{b}}) \\ &= \epsilon_{mnrs} A_1^m B_1^n A_2^r B_2^s = \frac{1}{4!}\epsilon_{IJKL}\epsilon_{mnrs} E_I^m E_J^n E_K^r E_L^s \sim \det(E) \end{aligned} \tag{3.1.2}$$

where in the second line we change our expression $SU(2) \times SU(2)$ to the $SO(4)$ notation and bring A, B together to the new $SU(4)$ fields E. From the final expression, we see $SU(4)$ symmetry in superpotential. In addition we still have the original $SU(2)_R$ symmetry which exchanges the A_i with the $B_j^\dagger$ because these are not combined together. This does not commute with the new $SU(4)$ global symmetry and therefore these two symmetries combine to give an $SO(8)$ global R-symmetry. The resultant action is precisely the same as $SO(4)$ BLG model in terms of $SU(2) \times SU(2)$ bifundamental representation [8].

3.2 Lorentzian BLG gauge structures and Inönü-Wigner contraction

We first look at the gauge structures of the Lorentzian Bagger-Lambert model [4–6]. The Bagger-Lambert model [2, 3] has a gauge symmetry generated by $\tilde{T}^{ab}X = [T^a, T^b, X]$. Because of the fundamental identity

$$[T^a, T^b, [T^c, T^d, T^e]] = [[T^a, T^b, T^c], T^d, T^e] + [T^c, [T^a, T^b, T^d], T^e] + [T^c, T^d, [T^a, T^b, T^e]], \tag{3.2.3}$$

the following commutation relation holds[1];

$$\begin{aligned}[\tilde{T}^{ab}, \tilde{T}^{cd}]X &= [T^a, T^b, [T^c, T^d, X]] - [T^c, T^d, [T^a, T^b, X]] \\ &= [[T^a, T^b, T^c], T^d, X] + [T^c, [T^a, T^b, T^d], X] \\ &= (f^{abc}{}_e \tilde{T}^{ed} + f^{abd}{}_e \tilde{T}^{ce})X. \end{aligned} \tag{3.2.4}$$

The Lorentzian 3-algebra contains 2 extra generators T^{-1} and T^0 in addition to the generators of Lie algebra T^i. (Here we use the convention of [6].) The 3-algebra for them is given by

$$[T^{-1}, T^a, T^b] = 0, \tag{3.2.5}$$
$$[T^0, T^i, T^j] = f^{ij}{}_k T^k, \tag{3.2.6}$$
$$[T^i, T^j, T^k] = f^{ijk} T^{-1}, \tag{3.2.7}$$

where $a, b = \{-1, 0, i\}$. T^i are generators of the ordinary Lie algebra with the structure constant: $[T^i, T^j] = i f^{ij}{}_k T^k$. This 3-algebra satisfies the fundamental identity. The metric $h^{ab} = \mathrm{tr}\,(T^a, T^b)$ is given by

$$\begin{aligned}&\mathrm{tr}\,(T^{-1}, T^{-1}) = \mathrm{tr}\,(T^{-1}, T^i) = 0, \quad \mathrm{tr}\,(T^{-1}, T^0) = -1, \\ &\mathrm{tr}\,(T^0, T^i) = 0, \quad \mathrm{tr}\,(T^0, T^0) = 0, \quad \mathrm{tr}\,(T^i, T^j) = h^{ij}. \end{aligned} \tag{3.2.8}$$

Since the metric has a negative eigenvalue, the field associated with the generators T^{-1} and T^0 become ghost modes.

The gauge generators of the Lorentzian 3-algebra can be classified into 3 classes:

- $\mathcal{I}$=$\{T^{-1} \otimes T^a, a = 0, i\}$
- $\mathcal{A}$=$\{T^0 \otimes T^i\}$
- $\mathcal{B}$=$\{T^i \otimes T^j\}$.

[1] If we write the commutation relation as $[\tilde{T}^{ab}, \tilde{T}^{cd}] = f^{abc}{}_e \tilde{T}^{ed} + f^{abd}{}_e \tilde{T}^{ce}$, it is not always associative. But when $\tilde{T}^{ab}$ acts on a field X, associativity-violating terms (3-cocycles) vanish and it becomes an ordinary associative Lie algebra.

The generators in the class $\mathcal{I}$ vanish when they act on X, hence we set these generators 0 in the following. Since the generators in the class $\mathcal{B}$ always appear as a combination with the structure constant, we define generators $S^i \equiv f^i_{jk}\tilde{T}^{jk}$. Then they satisfy the algebra

$$[\tilde{T}^{0i}, \tilde{T}^{0j}] = if^{ij}_k \tilde{T}^{0k}, \quad [\tilde{T}^{0i}, S^j] = if^{ij}_k S^k, \quad [S^i, S^j] = 0. \tag{3.2.9}$$

The last commutator was originally proportional to the generators in the class $\mathcal{I}$. If we had kept these generators, the algebra would have become nonassociative. The algebra (3.2.9) is a semi direct sum of $SU(N)$ (or $U(N)$) and translations. In the case of $SU(2)$, it becomes $ISO(3)$ gauge group, which is the gauge group of the 3-dimensional gravity. The Lorentzian Bagger-Lambert model has the above gauge symmetries and corresponding gauge fields $\hat{A}_\mu$ and B_μ as we will see in the next section.

On the other hand, the theory proposed by Aharony et.al. [7] is a Chern-Simons gauge theory with the gauge group $U(N) \times U(N)$. They act on the bifundamental fields (e.g. X^I) from the left and the right as $X \to UXV^\dagger$. If we write the generators as T^i_L and T^i_R, the combination $T^i = T^i_L + T^i_R$ and $S^i = T^i_L - T^i_R$ satisfy the algebra

$$[T^i, T^j] = if^{ij}_k T^k, \quad [T^i, S^j] = if^{ij}_k S^k, \quad [S^i, S^j] = if^{ij}_k T^k. \tag{3.2.10}$$

By taking the Inönü-Wigner contraction, i.e. scaling the generators as $S^i \to \lambda^{-1} S^i$ and taking $\lambda \to 0$ limit, the algebra (3.2.10) becomes the algebra (3.2.9) of the Lorentzian BL model. Therefore it is tempting to think that the Lorentzian BLG model can be obtained by taking an appropriate scaling limit of the ABJM model. In the next section, we see that it is indeed the case. Interestingly, even the constraint equations in the BL model (obtained by integrating the Lagrange multiplier fields) can be derived from this scaling procedure.

3.3 Derivation of Lorentzian BLG from ABJM

3.3.1 Classical Conformal symmetry of D2 branes with dynamical coupling

Before discussing ABJM model, we investigate the symmetry properties of the Lorentzian BLG model. As was shown in [4–6], the theory can be reduced to a system of D2 branes by integrating B_μ fields. This is interpreted as giving a vev to X^I_0 field following [56], and a special solution $X^I_0 = const.$ to the constraint equation $\partial^2 X^I_0 = 0$ was considered. In our previous paper [13], we revisited the constraint equation and considered a general solution with space-time dependent $X^I_0(x)$ satisfying $\partial^2 X^I_0 = 0$. Our interpretation is slightly different from the original one, and the field X^I_0 is treated as a dynamical (but non-propagating) field. In this subsection we show that if we consider whole set of the solutions to the constraint equation the reduced action has a classical conformal symmetry as well as $SO(8)$ symmetry.

For simplicity, we neglect the fermionic field here. By integrating the B_μ gauge field the action becomes [13]

$$\mathcal{S}_0 = \int d^3x \ \ \mathrm{tr}\left[-\frac{1}{2}(\hat{D}_\mu Y^I)^2 + \frac{1}{4}X_0^2[Y^I, Y^J]^2 - \frac{1}{4(X_0)^2}\left(\hat{F}_{\mu\nu} + 2\epsilon_{\mu\nu\rho}Y_I\partial^\rho X_0^I\right)^2\right], \quad (3.3.1)$$

where $X_0^2 \equiv \sum_I X_0^I X_0^I$ and we have defined a new scalar field $Y^I = P_{IJ}\hat{X}^J$ with 7 degrees of freedom by using the projection operator

$$P_{IJ}(x) = \delta_{IJ} - \frac{X_{0I}X_{0J}}{X_0^2}. \quad (3.3.2)$$

Indices run $I, J = 0, \cdots, 8$ and Y^I transforms as a vector of $SO(8)$. The field $X_0^I(x)$ is constrained to satisfy $\partial^2 X_0^I = 0$. If we pick up a specific solution $X_0^I = v\delta_{10}^I$, the action is reduced to the familiar D2 brane effective action with a coupling constant given by v. Then $SO(8)$ symmetry is spontaneously broken to $SO(7)$. The conformal invariance is also broken[2]. However if we consider whole set of solutions, $SO(8)$ invariance is restored in the action (5.1.12) with of the background fields $X_0^I(x)$ although Y^I has only 7 degrees of freedom.

Another important symmetry of the action is a conformal symmetry. The ordinary D2 brane action with a fixed coupling constant is not conformally invariant and the near horizon limit is not described by the AdS geometry. However, as discussed in a paper by Jevicki, Kazama and Yoneya [73], Dp brane theory has a *generalized conformal symmetry* if the coupling $g(x)$ is not constant and varies with space-time. Our reduced action for D2 branes (5.1.12) has exactly the same property. The coupling constant is no longer a constant and varies with space-time. A big difference, however, is that in our case the coupling constant g is promoted to an $SO(8)$ vector X_0^I, which is a space-time dependent field satisfying the massless Klein-Gordon equation.

Under the dilation $x \to \exp(\epsilon)x$, each field transforms as $Y(x) \to Y'(x') = \exp(-\epsilon/2)Y(x)$, $X_0(x) \to X_0'(x') = \exp(-\epsilon/2)X_0(x)$ and $A_\mu(x) \to A_\mu'(x') = \exp(-\epsilon)A_\mu(x)$. It is easy to see that the action is invariant under the dilation. Special conformal transformations are more complicated. It is given by

$$\delta\ x^\mu = 2\epsilon \cdot x x^\mu - \epsilon^\mu x^2. \quad (3.3.3)$$

Writing an infinitesimal transformation for each field as $\delta Y(x) = Y'(x') - Y(x)$, we define a special conformal transformation for each field as[3]

$$\delta Y^I(x) = -\epsilon \cdot x Y^I(x) \quad (3.3.4)$$

$$\delta X_0^I(x) = -\epsilon \cdot x X_0^I(x) \quad (3.3.5)$$

$$\delta A_\mu(x) = -2\epsilon \cdot x A_\mu(x) - 2(x \cdot A\ \epsilon_\mu - \epsilon \cdot A\ x_\mu). \quad (3.3.6)$$

[2] In the paper [58], it is discussed that the conformal invariance can be restored by sending the Yang-Mills coupling to infinity or integrating it over all values.

[3] If X_0^I is replaced by a single field $g(x)$, the transformation is the same as the generalized conformal transformation in [73]. Our scalar field $Y(x)$ corresponds to their $X(x)/g(x)$.

It is straightforward to show that the action is invariant under the special conformal transformation[4]. It can be easily checked that the transformations preserve the condition $X_0 \cdot Y = 0$.

Finally we need to check that the transformation is closed within the constraint equation $\partial^2 X_0^I = 0$. From the transformation of X_0^I, we define the following transformation *at the numerically same point* as

$$\tilde{\delta} X_0(x) = X_0'(x) - X_0(x) = \delta X_0(x) - \delta x^\mu \partial_\mu X_0(x). \tag{3.3.7}$$

It is easy to see that if the original $X_0^I(x)$ satisfies the constraint equation $\partial^2 X_0(x) = 0$, then the infinitesimal variation satisfies $\partial^2(\tilde{\delta} X_0) = 0$ for both of the dilation and the special conformal transformations, which means that the transformed field also satisfies $\partial'^2 X_0'(x') = 0$. Hence the classical conformal transformation is closed within the configurations of X_0 satisfying the constraint equation [5] . If we restrict the configurations of X_0^I that satisfy $\partial X_0^I = 0$, namely, to a set of constant vectors, the above special conformal transformations cannot be defined within the set. This indicates that taking into account the whole set of the constraint equation $\partial^2 X_0^I = 0$ as adopted in [13] is important in recovering the $SO(8)$ superconformal symmetry. It is also interesting to note that $\partial^2(\tilde{\delta} X_0) = 0$ holds only when $p = 2$. (Generalized conformal transformations for general p are given in [73].)

As we see later, the D2 brane action with a space-time dependent coupling is also derived from the M2 brane theory given by Aharony et.al by taking a certain scaling limit. This scaling limit corresponds to locating the M2 branes far from the origin of the orbifold and then taking $k \to \infty$ limit. It is natural from this brane picture that the model we considered in this subsection has a classical conformal symmetry as well as $SO(8)$ symmetry.

More detailed studies of the conformal symmetries and the interpretation in the gravity side are discussed in a separate paper [12]. What we have suggested here is that if we allow the background fields X_0^I to transform under $SO(8)$ and a special conformal transformation as an $SO(8)$ vector and as in (3.3.5), the action (5.1.12) is invariant under them. Note also that the analysis here is just about the classical conformal invariance. It is interesting to see whether the conformal invariance can be preserved quantum mechanically.

[4] The kinetic term of the gauge fields is different from the ordinary one, but both of the ordinary type and ours are invariant under the same conformal transformations.

[5] In order to construct a set of solutions in which the conformal transformations are closed, it seems to be necessary to consider all the solutions to the constraint equation $\partial^2 X_0 = 0$. Instead we can consider the following set of solutions studied by Verlinde [59]

$$X_0^I(x) = \sum_i \frac{q_i^I}{|x - z_i|}$$

which satisfies the constraint equation with sources at $x = z_i$: $\partial^2 X_0^I = -4\pi \sum q_i^I \delta^3(x - z_i)$. These solutions are closed under the conformal transformation if we consider all set of q_i^I and z_i. See [12] for details.

3.3.2 ABJM model revisited

The ABJM model is similar to the Lorentzian BLG model, but different in the following points. First the gauge group is $U(N) \times U(N)$ while it is a semi direct product of $U(N)$ and translations in the BL model. Accordingly the matter fields are in the bifundamental representation in the ABJM model. Furthermore the BL model contains an extra field X_0 and Ψ_0 associated with the generator T_0, and they are required to obey the constraint equations (2.2.48).

The bosonic potential terms in both theories are sextic, but the potential in the BL theories contains two X_0^I fields and four adjoint matter fields $\hat{X}^I$ while the potential terms in the ABJM model are written in the product of six bifundamental matter fields Y. Hence it is natural to think that the trace part of Y will play a role of X_0 in the Lorentzian BLG model. We will see that, if we separate the matter field Y into a trace and a traceless part, the potential terms coincides in a certain scaling limit.

3.3.3 Scaling limit of ABJM model

In order to take a scaling limit, we first recombine the gauge fields as

$$\hat{A}_\mu = \frac{A_\mu^{(L)} + A_\mu^{(R)}}{2}, \qquad B_\mu = \frac{A_\mu^{(L)} - A_\mu^{(R)}}{2}, \tag{3.3.8}$$

then the gauge transformations corresponding to $\hat{A}_\mu$ and B_μ are $Z \to e^{i\sigma_a T^a} Z e^{-i\sigma_b T^b}$ and $Z \to e^{i\sigma_a T^a} Z e^{i\sigma_b T^b}$ respectively. They are vectorial and axial gauge transformations. Matter fields are in the adjoint representation for the $\hat{A}_\mu$ gauge fields. Hence the $U(1)$ part of $\hat{A}_\mu$ decouples from the matter sector.

The covariant derivative can be written in terms of $\hat{A}_\mu$ and B_μ as

$$\begin{aligned} D_\mu Z &= \partial_\mu Z + i[\hat{A}_\mu, Z] + i\{B_\mu, Z\} \\ &= \hat{D}_\mu Z + i\{B_\mu, Z\}, \end{aligned} \tag{3.3.9}$$

where $\hat{D}_\mu$ is the covariant derivative with respect to the gauge field $\hat{A}_\mu$. S_{CS} can be written in terms of $\hat{A}_\mu$ and B_μ as

$$S_{CS} = \int d^3x \; 4K\epsilon^{\mu\nu\rho} \; \mathrm{tr}\,[B_\mu \hat{F}_{\mu\nu} + \frac{2}{3} B_\mu B_\nu B_\rho], \tag{3.3.10}$$

where $\hat{F}_{\mu\nu}$ is field strength of $\hat{A}_\mu$.
The gauge fields $\hat{A}_\mu$, B_μ are associated with the gauge transformations generated by T^i and S^i in (3.2.10). Hence in order to take the Inönü-Wigner contraction to obtain the gauge structure of the Lorentzian BL model (3.2.9), we need to rescale the gauge field B_μ as $B^\mu \to \lambda B^\mu$ and take $\lambda \to 0$ limit. Simultaneously we need to scale the coefficient K by $\lambda^{-1}K$. Since the coefficient K is proportional to the level of the Chern-Simons theory k as $K = k/8\pi$, the scaling limit corresponds to taking the large k limit. In this scaling

limit, the cubic term of the B_μ fields vanishes and the Chern-Simons action coincides with the BF-type action in the Lorentzian BLG model:

$$S_{CS} \to \int d^3x \; 4K\epsilon^{\mu\nu\rho} \; \mathrm{tr} \; B_\mu \hat{F}_{\mu\nu}. \tag{3.3.11}$$

In order to match the covariant derivatives in the BL action (4.1.10) and in the ABJM model (3.3.9), we separate the bifundamental fields into the trace and the traceless part, and scale them differently. We write the matter fields Y^A as

$$Y^A_{ij} = Y^A_0 \delta_{ij} + \tilde{Y}^A_a T^a_{ij}, \tag{3.3.12}$$

where T^a is the generator of $SU(N)$.

Now we perform the following rescaling:

$$\begin{aligned} B_\mu &\to \lambda B_\mu, \\ Y^A_0 &\to \lambda^{-1} Y^A_0, \\ \psi_{A0} &\to \lambda^{-1} \psi_{A0}, \\ K &\to \lambda^{-1} K, \end{aligned} \tag{3.3.13}$$

where Y^A_0 and ψ_{A0} is the trace part of Y^A and ψ_A. All the other fields are kept fixed. Then take the $\lambda \to 0$ limit. If we take the scaling limit, we can show that the covariant derivatives in both theories exactly match.

In the following we consider the ABJM model with $SU(N) \times SU(N)$ gauge group. In the presence of the $U(1) \times U(1)$ group, a little more care should be taken for the scaling of the $U(1)$ part of the B_μ gauge field.

In taking the above scaling limit, many terms vanish. The kinetic term of the ABJM action becomes

$$\begin{aligned} \mathrm{tr} \; \Big[& -\frac{1}{\lambda^2} \partial_\mu Y^\dagger_{0A} \partial^\mu Y^A_0 + \frac{1}{\lambda^2} \psi^\dagger_{0A} \Gamma^\mu \partial_\mu \psi^A_0 + 2(i \partial_\mu Y^\dagger_{0A} B^\mu Y^A + h.c.) \\ & - (\hat{D}_\mu \tilde{Y}_A + 2i \tilde{B}_\mu Y_{0A})^\dagger (\hat{D}^\mu \tilde{Y}^A + 2i \tilde{B}^\mu Y^A_0) + i \tilde{\psi}^\dagger_A \Gamma^\mu \hat{D}_\mu \tilde{\psi}^A - 2 \tilde{\psi}^\dagger_A \Gamma^\mu \tilde{B}_\mu \psi^A_0 - 2 \psi^\dagger_{0A} \Gamma^\mu \tilde{B}_\mu \tilde{\psi}^A \Big]. \end{aligned} \tag{3.3.14}$$

The first and the second terms are divergent for small λ. In order to make the action finite, we need to impose that the trace part of the bifundamental fields must satisfy the constraint equations

$$\partial^2 Y^I_0 = 0, \quad \Gamma^\mu \partial_\mu \psi_{A0} = 0$$

in the $\lambda \to 0$ limit. They are precisely the same constraint equations (2.2.48) in the BL model.

In the Lorentzian BLG model, the constraints are obtained by integrating out the Lagrange multiplier fields X_{-1} and Ψ_{-1}. Here they arise from a condition that the action should be finite in the scaling limit.

The other terms in (3.3.14) are finite in the scaling limit and it can be easily shown that they are precisely the same kinetic terms as that of the Lorentzian Bagger-Lambert model (after a redefinition of the gauge field $2B_\mu \to B_\mu$ and setting $K = 1/2$). The trace part of the bifundamental fields is identified with the fields X_0 associated with one of the extra generators T^0 in the Lorentzian Bagger-Lambert model. This is the reason why we have used the same convention with subscript 0 for both of the trace part of the bifundamental fields and the field associated with the generator T^0.

Now let us check the potential terms. The potential terms of the ABJM model are invariant under the $SU(4)$ symmetries but not under full $SO(8)$. By decomposing the matter fields Y^A into the trace part Y_0^A and the traceless part $\tilde{Y}^A$, the bosonic sextic potential becomes a sum of $V_B = \sum_{n=0}^{6} V_B^{(n)}$, where $V_B^{(n)}$ contains n Y_0 fields and $(6-n)$ $\tilde{Y}$ fields. Since the coefficient of the bosonic potential is proportional to K^{-2}, $V_B^{(n)}$ term scales as λ^{2-n}. It can be easily checked that the coefficients of $V_B^{(n)}$ vanishes for $n > 3$. On the other hand, the potential terms $V_B^{(n)}$ for $n < 2$ vanish in the scaling limit of $\lambda \to 0$. Hence the only remaining term in the scaling limit is $V_B^{(2)}$. This part of the potential has the full $SO(8)$ symmetry and becomes identical with the potential in the Lorentzian BL model. In order to see that the BL potential is obtained, we assume that only the field Z^1 has the trace part for simplicity. Let us write the 4 complex scalar field Y^A by 8 real scalar fields as

$$\begin{aligned} Z^1 &= X_0^1 + iX_0^5 + i\tilde{X}_a^1 T^a - \tilde{X}_a^5 T^a, \\ Z^2 &= i\tilde{X}_a^2 T^a - \tilde{X}_a^6 T^a, \\ W_1^\dagger &= i\tilde{X}_a^3 T^a - \tilde{X}_a^7 T^a \\ W_2^\dagger &= i\tilde{X}_a^4 T^a - \tilde{X}_a^8 T^a. \end{aligned} \tag{3.3.15}$$

Substituting them into S_{V_b} and taking the scaling limit, we can obtain the following bosonic potential:

$$S_{V_b} = -\frac{1}{8K^2} \int d^3x \ \ \mathrm{tr} \left((X_0^1)^2 + (X_0^5)^2)[P_I, P_J][P^I, P^J] \right). \tag{3.3.16}$$

P^I is defined by

$$\begin{aligned} P^I &\equiv (P^1, \tilde{X}^2, \tilde{X}^3, \tilde{X}^4, \tilde{X}^6, \tilde{X}^7, \tilde{X}^8), \\ &= \left(\frac{1}{2}(\tilde{Y}^A + \tilde{Y}_A^\dagger), \frac{1}{2i}(\tilde{Y}^B - \tilde{Y}_B^\dagger) \right), \\ \tilde{Y}^A &\equiv (P^1, Z^2, W_1^\dagger, W_2^\dagger), \\ P^1 &\equiv \frac{X_0^1 \tilde{X}^5 - X_0^5 \tilde{X}^1}{\sqrt{(X_0^1)^2 + (X_0^5)^2}}. \end{aligned} \tag{3.3.17}$$

We can rewrite it as,

$$S_{V_b} = -\frac{1}{8K^2} \int d^3x \ \mathrm{tr} \left[\frac{1}{4}(X_0^K)^2 \left([\tilde{X}^I, \tilde{X}^J] \right)^2 - \frac{1}{2} \left(X_0^I [\tilde{X}^I, \tilde{X}^J] \right)^2 \right], \tag{3.3.18}$$

where we have used $X_0^I = (X_0^1, 0, 0, 0, X_0^5, 0, 0, 0)$. This is the potentials for bosons in the Lorentzian BLG model (4.1.10). It is straightforward to see that the complete potential of the BL model can be obtained by considering general X_0^I and the full $SO(8)$ invariance is restored.

It should be noted that the above potential term is written in terms of the commutators. This shows that, if we replace more than two bosons by their trace components, the potential vanishes. This assures that the would-be divergent terms $V_B^{(n)}$ for $n > 3$ vanish and the only remaining term in the scaling limit is given by the above potential.

Finally consider the fermion potential. We expand the potential as $V_f = \sum_{n=0}^{4} V_f^{(n)}$ where $V_f^{(n)}$ contains n trace parts and $(4-n)$ traceless parts. Since the coefficient of the fermion potential is proportional to $1/K$, $V_f^{(n)}$ scales as λ^{1-n}. $V_f^{(n)}$ for $n > 1$ diverges in the scaling limit and their coefficients must vanish. $V_f^{(0)}$ vanishes in the scaling limit $\lambda \to 0$. Hence the only remaining finite terms are $V_f^{(1)}$. In the following we look at the potential term with one of the bosons replaced by the trace part X_0^I. Such a term can be written as

$$\begin{aligned}
S_{V_f} = \frac{i}{2K} X_0^1 \ \mathrm{tr} \ \Big[& -\psi_1^\dagger [\tilde{X}^5, \psi_1] + \psi_2^\dagger [\tilde{X}^5, \psi_2] + \psi_3^\dagger [\tilde{X}^5, \psi_3] + \psi_4^\dagger [\tilde{X}^5, \psi_4] \\
& + \psi_1^\dagger [Y_2, \psi_2] + \psi_2^\dagger [Y_2^\dagger, \psi_1] + \psi_3^\dagger [Y_2, \psi_4^\dagger] + \psi_4 [Y_2{}^\dagger, \psi_3] \\
& + \psi_1^\dagger [Y_3, \psi_3] + \psi_3^\dagger [Y_3^\dagger, \psi_1] + \psi_4^\dagger [Y_3, \psi_2^\dagger] + \psi_2 [Y_3^\dagger, \psi_4] \\
& + \psi_1^\dagger [Y_4, \psi_4] + \psi_4^\dagger [Y_4^\dagger, \psi_1] + \psi_2^\dagger [Y_4, \psi_3^\dagger] + \psi_3 [Y_4^\dagger, \psi_2] \Big] \\
+ \frac{i}{2K} X_0^5 \ \mathrm{tr} \ \Big[& + \psi_1^\dagger [\tilde{X}^1, \psi_1] - \psi_2^\dagger [\tilde{X}^1, \psi_2] - \psi_3^\dagger [\tilde{X}^1, \psi_3] - \psi_4^\dagger [\tilde{X}^1, \psi_4] \\
& - \psi_1^\dagger [iY_2, \psi_2] + \psi_2^\dagger [iY_2^\dagger, \psi_1] + \psi_3^\dagger [iY_2, \psi_4^\dagger] - \psi_4 [iY_2^\dagger, \psi_3] \\
& - \psi_1^\dagger [iY_3, \psi_3] + \psi_3^\dagger [iY_3^\dagger, \psi_1] + \psi_4^\dagger [iY_3, \psi_2^\dagger] - \psi_2 [iY_3^\dagger, \psi_4] \\
& - \psi_1^\dagger [iY_4, \psi_4] + \psi_4^\dagger [iY_4^\dagger, \psi_1] + \psi_2^\dagger [iY_4, \psi_3^\dagger] - \psi_3 [iY_4^\dagger, \psi_2] \Big] .
\end{aligned} \tag{3.3.19}$$

Here for simplicity we have assumed that the trace part of the boson X_0^I is non vanishing for $I = 1, 5$. This can be done by using the original $SU(4)$ symmetry. Note again that these potential terms are written as a form of commutators.

To get the 3-dimensional Majorana fermion as the BL model, we rewrite the $SU(4)$ complex fermion in terms of the real variables [6].

$$\begin{aligned}
\psi_1 = i\chi_1 - \chi_5, \quad \psi_2 = i\chi_2 - \chi_6, \\
\psi_3 = i\chi_3 - \chi_7, \quad \psi_4 = i\chi_4 - \chi_8.
\end{aligned} \tag{3.3.20}$$

where χ_I are real 2-component spinors. We also expand the complex bosons as the real ones (3.3.15). Then the fermion potential (3.3.19) becomes by using the 8×8 Γ matrices

[6] When we give a vev to the X_0^4 part only, we will get 7 Γ matrices as in [74]. In our case we need 8 Γ matrices and their antisymmetrized-products because we give a vev to a more general direction.

as

$$S_{V_f} = -\frac{1}{2K}\ \mathrm{tr}\ \bar{\Psi} X_0^I [\tilde{X}^J, \Gamma_{IJ}\Psi],$$
$$\Psi \equiv (\chi_1, \chi_2, \chi_3, \chi_4, \chi_5, \chi_6, \chi_7, \chi_8)\,, \tag{3.3.21}$$

where the indices I, J run from 1 to 8 and $X_0^I = (X_0^1, 0, 0, 0, X_0^5, 0, 0, 0)$. The explicit forms of the Γ matrices are given in Appendix A. This fermion potential has the same $SO(8)$ invariant form as that of the Lorentzian BLG action (4.1.10). In the same fashion as the bosonic potential, the full $SO(8)$ invariance can be seen easily by considering the general X_0^I.

3.4 Comments on scaling limit

Recently an additional suggestion about the scaling limit in the ABJM model has been appeared by Antonyan and Tseytlin [11]. They have mentioned necessity of additional fields to obtain precise Lorentzian BLG model. The idea is simple. The ABJM model has Y^A with $U(N) \times U(N)$ gauge group so much so that it has $4 \times 2N^2 = 8N^2$ degrees of freedom. However, Lorentzian BLG model has $\hat{X}^I$ with $SU(N)$ gauge group and originally ghost fields X_{-1}^I, X_0^I so this model has $8(N^2-1) + 2 \times 8 = 8N^2 + 8$ d.o.f. They suggested to have additional ghost fields Y_{-1}^A to the ABJM model

$$\mathcal{L}_{\mathrm{ABJM}} + N \left| -\frac{1}{\lambda}\partial_\mu Y_0^A + \frac{\lambda}{N}\partial_\mu Y_{-1}^A \right|^2 . \tag{3.4.22}$$

With the additional ghost term, we can correctly reproduce the ghost term in Lorentzian BLG model (2.2.42) by taking the scaling limit $\lambda \to 0$. As we said before around (3.3.14), the divergent term should be regarded as the constraint equation. However with the additional term, we can obtain the ghost term and also the constraint equation without divergent terms.

People might think curiously to just add the ghost term. But this seems to be natural if first we consider the new BL model (2.3.61) (p.33) instead of the ABJM. In the new BL model we have generalized Lorentzian triple algebra

$$[E, T^i; T^j] = \frac{2\pi}{k} f^{ij}{}_k T^k,$$
$$[T^i, T^j; T^k] = -\frac{2\pi}{kN} f^{ijk} E + \frac{2\pi}{k} A^{ijk}{}_m T^m \tag{3.4.23}$$

where we use the generators T^i of $SU(N)$ in the general case of $U(N) \times U(N)$ Lie algebra and the coefficients $A^{ijk}{}_m$ as

$$T^i T^j = -i\delta^{ij} E + \frac{1}{2}(i f^{ij}{}_k + d^{ij}{}_k) T^k,$$
$$\mathrm{tr}\, T^i T^j = h^{ij},$$
$$A^{ijk}{}_m T^m = \frac{1}{N} h^{jk} T^i + \frac{1}{4}(i f^{kj}{}_l + d^{kj}{}_l)(i f^{il}{}_m + d^{il}{}_m) T^m - (i \leftrightarrow j), \tag{3.4.24}$$

and also we take the ABJM example of new BL model (2.3.63). By adding an extra ghost generator e which commutes with all other generators

$$[e, T^a; T^b] = 0, \tag{3.4.25}$$

and taking a similar scaling limit with

$$e = \frac{N}{\lambda} T^{-1}, \quad E = \lambda T^0 + \frac{N}{\lambda} T^{-1}, \quad k \to \frac{1}{\lambda} k, \tag{3.4.26}$$

then we obtain the Lorentzian BLG model with ghost terms precisely. Note that the generators E is not "T^0" as we considered before. It should be combined with the generators T^0, T^{-1}. This can be seen easily by considering additional field and taking the scaling limit

$$\left(\frac{1}{\lambda} Y_0\right) E + \left(-\frac{1}{\lambda} Y_0 + \frac{\lambda}{N} Y_{-1}\right) e = Y_0 T^0 + Y_{-1} T^{-1}. \tag{3.4.27}$$

Therefore E should consist from not only T^0 but also T^{-1}.

Chapter 4

Generalized Conformal Symmetry and the Gravity dual

Now we know how to obtain the Lorentzian BLG model from ABJM model. Together with the fact we also know the dual description of ABJM model, we can consider about the dual gravity description of Lorentzian BLG model by taking the scaling limit. We also clarify the conformal symmetry of Lorentzian BLG model which is expected to be conformal from the dual gravity description. We explain our work which discussed in [12].

4.1 Conformal Symmetry of ABJM and L-BLG

4.1.1 Conformal invariance of ABJM

As shown in [75], the ABJM model is invariant under the superconformal transformations. Here we study the invariance of the ABJM model under the conformal transformations, in particular the special conformal transformations.

First it is obvious that the action is invariant under the dilatation. Dilatation is defined by $x \to e^{\epsilon}x$ and simultaneously we transform each field by multiplying $e^{-n\epsilon}$ where n is the conformal weight. The scalars Y^A, fermions ψ^A and the gauge fields A_μ have weights $1/2, 1, 1$ respectively.

A little more nontrivial transformation is a special conformal transformation. It is given by

$$\delta x^\mu = 2\epsilon \cdot x x^\mu - \epsilon^\mu x^2. \tag{4.1.1}$$

If we write the infinitesimal transformation for each field $Y(x)$ as $\delta Y(x) = Y'(x') - Y(x)$, they are given by

$$\begin{aligned}
\delta Y^A(x) &= -\epsilon \cdot x Y^A(x), \\
\delta A_\mu^{(L,R)}(x) &= -2\epsilon \cdot x A_\mu^{(L,R)}(x) - 2(x \cdot A^{(L,R)}\epsilon_\mu - \epsilon \cdot A^{(L,R)} x_\mu), \\
\delta \psi^A(x) &= -2\epsilon \cdot x \psi^A(x) - \epsilon_{\mu\nu\lambda}\epsilon^\nu x^\lambda \Gamma^\mu \psi^A(x).
\end{aligned} \tag{4.1.2}$$

These transformations can be understood as follows. They look like the general coordinate transformations, but are different since the theory is restricted to live in the flat space-time with a fixed metric and the change of the metric under the general coordinate transformations must be compensated by the transformations of the fields. The first terms in each transformation reflect the conformal weight of each field. The second term in the transformation of the fermion is the local Lorentz transformation which pulls back the flat local Lorentz frame (where we use $\Gamma^{012}\psi = \psi$). The transformation for the gauge field A_μ is nothing but the general coordinate transformation with the transformation parameter (4.1.1).

The action is invariant under the above special conformal transformations. In order to see it, the following transformation rules are useful:

$$\begin{aligned}
d^3x &\to e^{6\epsilon\cdot x} d^3x, \\
\partial_\mu &\to e^{-2\epsilon\cdot x}[\partial_\mu - 2(\epsilon_\mu x^\nu \partial_\nu - x_\mu \epsilon^\nu \partial_\nu)], \\
D_\mu Y &\to e^{-3\epsilon\cdot x}\left[D_\mu Y - \{Y + 2x^\nu \partial_\nu Y + 2i(x\cdot A^{(L)} Y - Y x\cdot A^{(R)})\}\epsilon_\mu \right. \\
&\qquad\qquad \left. +\{2\epsilon^\nu \partial_\nu Y + 2i(\epsilon\cdot A^{(L)} Y - Y\epsilon\cdot A^{(R)})\}x_\mu\right], \\
F_{\mu\nu} &\to e^{-4\epsilon\cdot x}\left[F_{\mu\nu} - 2(\epsilon_\nu x^\rho F_{\mu\rho} - \epsilon_\mu x^\rho F_{\nu\rho}) + 2(x_\nu \epsilon^\rho F_{\mu\rho} - x_\mu \epsilon^\rho F_{\nu\rho})\right].
\end{aligned} \tag{4.1.3}$$

Though ϵ is an infinitesimal parameter, we write the overall factors as $e^{-2n\epsilon\cdot x}$ for convenience. They are cancelled in the action because n is the conformal weight of each field and coordinates.

Here let us check the invariance of the Chern-Simons term as an example. First the derivative part transforms as

$$\begin{aligned}
&\epsilon^{\mu\nu\lambda}\mathrm{tr} F_{\mu\nu} A_\sigma \\
&\to \epsilon^{\mu\nu\lambda} e^{-6\epsilon\cdot x}\mathrm{tr}[F_{\mu\nu}A_\lambda + 4(\epsilon_\mu x^\rho - x_\mu \epsilon^\rho)A_\lambda F_{\nu\rho} - 2F_{\mu\nu}(x\cdot A\epsilon_\lambda - \epsilon\cdot A x_\lambda)].
\end{aligned} \tag{4.1.4}$$

The pre-factor $e^{-6\epsilon\cdot x}$ is cancelled with the transformation of d^3x in (4.1.3). The rest vanishes because

$$\begin{aligned}
&\epsilon^{\mu\nu\lambda}\mathrm{tr}[2(\epsilon_\mu x^\rho - x_\mu \epsilon^\rho)A_\lambda F_{\nu\rho} - F_{\mu\nu}(x\cdot A\epsilon_\lambda - \epsilon\cdot A x_\lambda)] \\
&= \epsilon^{\mu\nu\lambda}\mathrm{tr}[2\epsilon_\mu{}^{\rho\alpha} f_\alpha F_{\nu\rho} A_\lambda - \epsilon_\lambda{}^{\rho\alpha} f_\alpha F_{\mu\nu} A_\rho] = 0.
\end{aligned} \tag{4.1.5}$$

In the second line we have defined $f^\alpha = \epsilon^{\mu\nu\alpha} x_\mu \epsilon_\nu$. Similarly the invariance of the term $\epsilon^{\mu\nu\lambda} A_\mu A_\nu A_\lambda$ can be shown by noting that the gauge field transforms as

$$A_\mu \to e^{-2\epsilon\cdot x}(A_\mu + 2\epsilon_{\mu\alpha\beta} f^\alpha A^\beta). \tag{4.1.6}$$

Hence the Chern-Simons terms are invariant under the special conformal transformation. Though we have checked it explicitly, the invariance can be naturally understood because the Chern-Simons term is independent of the metric if it is defined in a curved background space-time.

The other terms in the action are also straightforwardly shown to be invariant under the special conformal transformations.

4.1.2 ABJM to L-BLG

As shown in [10], the L-BLG model is obtained by taking a scaling limit of the ABJM model with a gauge group $SU(N)\times SU(N)$. In the gauge theory with $U(N)\times U(N)$ there is a subtlety in the scaling of the $U(1)$ part. We will discuss the issue in the Appendix B and here restrict the discussions to the $SU(N)\times SU(N)$ case.

The scaling is given as follows:

$$\begin{aligned} B_\mu &\to \lambda B_\mu, \\ X_0^I &\to \lambda^{-1} X_0^I, \\ \psi_{A0} &\to \lambda^{-1}\psi_{A0}, \\ k &\to \lambda^{-1} k \end{aligned} \tag{4.1.7}$$

where

$$Y^A = X_0^{2A-1} + iX_0^{2A} - \hat{X}^{2A} + i\hat{X}^{2A-1}, \qquad B_\mu = \frac{1}{2}(A_\mu^{(L)} - A_\mu^{(R)}) \tag{4.1.8}$$

and X_0^I and ψ_{0A} are trace components of the bifundamental matter fields, and $I = 1,\cdots,8$. When we take $\lambda \to 0$ limit and keep the other fields fixed, the action of the ABJM model is reduced to the action of the L-BLG model. Since the $k\to\infty$ limit is taken before taking the large N, our scaling corresponds to a vanishing 't Hooft coupling $N/k \to 0$. Besides the action, the same constraint equations as those in the L-BLG model can be obtained from the ABJM model:

$$\partial^2 X_0^I = 0, \quad \Gamma^\mu \partial_\mu \Psi_0 = 0, \tag{4.1.9}$$

by requiring *finiteness of the action* in the $\lambda \to 0$ limit.

In the above scaling limit we arrive at the L-BLG model:

$$\begin{aligned} \mathcal{L}_0 = \mathrm{Tr}\Bigg[&-\frac{1}{2}(\hat{D}_\mu \hat{X}^I - B_\mu X_0^I)^2 + \frac{1}{4}(X_0^K)^2([\hat{X}^I,\hat{X}^J])^2 - \frac{1}{2}(X_0^I[\hat{X}^I,\hat{X}^J])^2 \\ &+\frac{i}{2}\bar{\hat{\Psi}}\Gamma^\mu \hat{D}_\mu \hat{\Psi} + i\bar{\Psi}_0\Gamma^\mu B_\mu \hat{\Psi} - \frac{1}{2}\bar{\Psi}_0 \hat{X}^I[\hat{X}^J, \Gamma_{IJ}\hat{\Psi}] + \frac{1}{2}\bar{\hat{\Psi}} X_0^I[\hat{X}^J,\Gamma_{IJ}\hat{\Psi}] \\ &+\frac{1}{2}\epsilon^{\mu\nu\lambda}\hat{F}_{\mu\nu}B_\lambda - \partial_\mu X_0^I\ B_\mu \hat{X}^I\Bigg]. \end{aligned} \tag{4.1.10}$$

In the original formulation of the L-BLG model, the constraint equations (4.1.9) are derived by integrating the auxiliary fields X_{-1}^I and Ψ_{-1}:

$$\mathcal{L}_{gh} = (\partial_\mu X_0^I)(\partial^\mu X_{-1}^I) - i\bar{\Psi}_{-1}\Gamma^\mu\partial_\mu\Psi_0. \tag{4.1.11}$$

Since the above scaling is compatible with the conformal transformations discussed in the previous section, the action (4.1.10) is invariant under the conformal transformations (see also [76]). The action for the auxiliary fields (4.1.11) is also invariant if we define the transformations for them as

$$\begin{aligned} \delta X_{-1}^I(x) &= -\epsilon\cdot x X_{-1}^I(x), \\ \delta\Psi_{-1}(x) &= -2\epsilon\cdot x\Psi_{-1}(x) - \epsilon_{\mu\nu\lambda}\epsilon^\nu x^\lambda \Gamma^\mu \Psi_{-1}(x). \end{aligned} \tag{4.1.12}$$

4.1.3 Generalized conformal symmetry in D2 branes

Now integrate the B_μ gauge field. It has been discussed that if we pick up a specific solution to the constraint equation (4.1.9), especially a constant solution

$$X_0^I = v\ \delta^{I,8}, \quad \Psi_0 = 0, \tag{4.1.13}$$

the L-BLG model is reduced to the action of the ordinary D2 branes whose Yang-Mills coupling constant is given by $g_{YM} = v$:

$$\mathcal{L} = \mathrm{Tr}\left[-\frac{1}{4v^2}\hat{F}_{\mu\nu}^2 - \frac{1}{2}(\hat{D}_\mu \hat{X}^A)^2 + \frac{1}{4}v^2[\hat{X}^A, \hat{X}^B]^2 + \frac{i}{2}\bar{\hat{\Psi}}\Gamma^\mu \hat{D}_\mu \hat{\Psi} + \frac{1}{2}v\bar{\hat{\Psi}}[\hat{X}^A, \Gamma_{8,A}\hat{\Psi}]\right] \tag{4.1.14}$$

where $A, B = 1, \cdots, 7$. Then $SO(8)$ is spontaneously broken to $SO(7)$ because we have specialized the 8-th direction. The conformal invariance is also broken. Though the action is the same as that of the D2 branes, we see later that the interpretation of the L-BLG model as an effective theory of the ordinary D2 branes is not appropriate since the radius of curvature is much smaller than the string scale in the gravity dual.

The constraint equations (4.1.9) have more general solutions than (4.1.13) which depend on the spacetime coordinates. Then the resulting action becomes a Yang-Mills theory with a spacetime dependent coupling [13]. As we have shown [10], the action with the spacetime dependent coupling is invariant under the conformal transformations if we consider a set of spacetime dependent solutions. The conformal invariance is discussed in more details in the next section.

We here consider the simplest spacetime dependent solutions:

$$X_0^I = v(x)\,\delta^{I,8}, \quad \Psi_0 = 0, \quad \partial^2 v(x) = 0. \tag{4.1.15}$$

Then the L-BLG model is reduced to the same action as that of the D2 branes but with a spacetime varying coupling:

$$\begin{aligned}\mathcal{L} = \mathrm{Tr}\Bigg[&-\frac{1}{4v(x)^2}\hat{F}_{\mu\nu}^2 - \frac{1}{2}(\hat{D}_\mu \hat{X}^A)^2 + \frac{1}{4}v(x)^2[\hat{X}^A, \hat{X}^B]^2 \\ &+\frac{i}{2}\bar{\hat{\Psi}}\Gamma^\mu \hat{D}_\mu \hat{\Psi} + \frac{1}{2}v(x)\bar{\hat{\Psi}}[\hat{X}^A, \Gamma_{8,A}\hat{\Psi}]\Bigg].\end{aligned} \tag{4.1.16}$$

$SO(8)$ symmetry is spontaneously broken to $SO(7)$ as well, but the action with a varying $v(x)$ has a generalized conformal symmetry if the coupling transforms as

$$\delta v(x) = -(\epsilon \cdot x)\ v(x). \tag{4.1.17}$$

This transformation is originated in the special conformal transformation of the scalar field (4.1.2). The generalized conformal transformation for Dp branes were first proposed by Jevicki, Kazama and Yoneya [73]. In the present case, the transformation (4.1.17) is naturally derived since the coupling constant of the Yang-Mills action is determined by the center of mass coordinates $X_0^I(x)$ of the M2 branes.

It is worth noting that the generalized conformal transformation (4.1.17) is compatible with the constraint equations (4.1.9) only when $p = 2$. We will discuss it in the next section.

4.1.4 Conformal symmetry and $SO(8)$ invariance of L-BLG

The space-time dependent coupling $v(x)$ can be promoted to an $SO(8)$ vector $X_0^I(x)$ by considering general solutions to the constraint equations (4.1.9) as shown in [13]. Then the resultant action after integrating the B_μ gauge field becomes D2 branes effective action with space-time dependent couplings in a vector representation of the $SO(8)$. In [10] we showed that if we consider space-time dependent solutions the theory has the *generalized conformal symmetry* as well as the manifest $SO(8)$ invariance.

In this section we study more details of the generalized conformal symmetry of the L-BLG model. Especially we show that the conformal transformations are closed under the constraint equations (4.1.9).

By integrating the B_μ gauge field, we get the action $S = \int d^3x(\mathcal{L}_0 + \mathcal{L}')$:

$$\begin{aligned}\mathcal{L}_0 = \mathrm{Tr}\Bigg[&-\frac{1}{2}(\hat{D}_\mu P^I)^2 + \frac{1}{4}X_0^2[P^I,P^J]^2 + \frac{i}{2}\bar{\hat{\Psi}}\Gamma^\mu\hat{D}_\mu\hat{\Psi} + \frac{1}{2}\bar{\hat{\Psi}}[P^I,(X_0^J\Gamma_J)\Gamma_I\hat{\Psi}] \\ &+\frac{1}{2(X_0^I)^2}\Big(\frac{1}{2}\epsilon^{\mu\nu\lambda}\hat{F}_{\nu\lambda} + i\bar{\Psi}_0\Gamma^\mu\hat{\Psi} - 2P_I\partial^\mu X_0^I\Big)^2 - \frac{1}{2}\bar{\Psi}_0\Gamma_{IJ}\hat{\Psi}[P^I,P^J]\Bigg], \\ \mathcal{L}' = \frac{1}{X_0^2}\mathrm{Tr}&\left[\left(-\bar{\Psi}_0\Gamma_I(X_0^J\Gamma_J)[P^I,\hat{\Psi}] - i\bar{\Psi}_0\Gamma_\mu\hat{D}_\mu\hat{\Psi}\right)(X_0^K\hat{X}^K)\right]. \end{aligned} \tag{4.1.18}$$

where we have defined a new scalar field P_I with 7 degrees of freedom by using the projection operator

$$P_I(x) = \left(\delta_{IJ} - \frac{X_{0I}X_{0J}}{X_0^2}\right)X^J. \tag{4.1.19}$$

The $X_0^I(x)$ field is constrained to satisfy $\partial^2 X_0^I = 0$. This is a generalization of (4.1.16). We called this model as a Janus field theory of (M)2-branes since the coupling constant is varying with the space-time coordinates.

The action of the gauge field is no longer the Chern-Simons action but we can again show that it is invariant under the conformal transformations. Under the dilatation $x^\mu \to e^\epsilon x^\mu$, each field is multiplied by $e^{-n\epsilon}$ where n is the conformal weight. The weights of $P, X_0, A_\mu, \Psi, \Psi_0$ are $1/2, 1/2, 1, 1, 1$ respectively. The action is evidently invariant.

Special conformal transformation is similarly given by

$$\delta x^\mu = 2\epsilon\cdot x x^\mu - \epsilon^\mu x^2 \tag{4.1.20}$$

and the fields transform as

$$\begin{aligned}\delta P^I(x) &= -\epsilon\cdot x P^I(x), \\ \delta X_0^I(x) &= -\epsilon\cdot x X_0^I(x), \\ \delta A_\mu(x) &= -2\epsilon\cdot x A_\mu(x) - 2(x\cdot A\ \epsilon_\mu - \epsilon\cdot A\ x_\mu), \\ \delta\hat{\Psi}(x) &= -2\epsilon\cdot x\hat{\Psi}(x) - \epsilon_{\mu\nu\lambda}\epsilon^\nu x^\lambda\Gamma^\mu\hat{\Psi}(x), \\ \delta\Psi_0(x) &= -2\epsilon\cdot x\Psi_0(x) - \epsilon_{\mu\nu\lambda}\epsilon^\nu x^\lambda\Gamma^\mu\Psi_0(x). \end{aligned} \tag{4.1.21}$$

It is now straightforward to show the invariance of the action. The Lagrangian is not invariant but changes by total derivatives.

Finally we need to check that the transformation is closed within the constraint equations (4.1.9). Namely if the field $X_0^I(x)$ satisfies $\partial_x^2 X_0^I(x) = 0$, the transformed field $X_0'^I(x')$ must also satisfy $\partial_{x'}^2 X_0'^I(x') = 0$. For an infinitesimal special conformal transformation, this is equivalent to show $\partial^2 \tilde{\delta} X_0^I(x) = 0$ where $\tilde{\delta} X_0^I(x)$ is the transformation at the numerically same point defined by

$$\begin{aligned} \tilde{\delta} X_0^I(x) &= X_0'^I(x) - X_0^I(x) = \delta X_0^I(x) - \delta x^\mu \partial_\mu X_0^I(x), \\ \tilde{\delta} \Psi_0(x) &= \Psi_0'(x) - \Psi_0(x) = \delta \Psi_0(x) - \delta x^\mu \partial_\mu \Psi_0(x). \end{aligned} \tag{4.1.22}$$

In the following, in order to see the specialty for M2 (or D2)-branes, we generalize the special conformal transformation to Dp-branes [73]:

$$\tilde{\delta} X_0^I(x) = -(3-p)\epsilon \cdot x X_0^I - (2\epsilon \cdot x x^\mu - \epsilon x^2)\partial_\mu X_0^I \tag{4.1.23}$$

It is easy to show

$$\partial^2(\tilde{\delta} X_0^I(x)) = 2(p-2)\epsilon^\mu \partial_\mu X_0^I \tag{4.1.24}$$

where we have used the constraint equation $\partial^2 X_0^I = 0$. This vanishes at $p = 2$ only. Similarly, $\tilde{\delta}\Psi_0$ is given by

$$\tilde{\delta}\Psi_0(x) = -2(3-p)\epsilon \cdot x \Psi_0 - \epsilon_{\mu\nu\lambda}\epsilon^\nu x^\lambda \Gamma^\mu \Psi_0 - (2\epsilon \cdot x x^\mu - \epsilon x^2)\partial_\mu \Psi_0 \tag{4.1.25}$$

and satisfies

$$\Gamma^\alpha \partial_\alpha(\tilde{\delta}\Psi_0(x)) = 2(p-2)\Gamma^\alpha \epsilon_\alpha \Psi_0 \tag{4.1.26}$$

where we used the constraint equation $\Gamma^\alpha \partial_\alpha \Psi_0 = 0$. Again $\Gamma^\alpha \partial_\alpha(\tilde{\delta}\Psi_0(x)) = 0$ vanishes at $p = 2$ only. Both of the constraints are compatible with the generalized conformal transformations at $p = 2$. It shows a specialty of M2 (or D2) branes.

We have shown that the constraint equations are compatible with the generalized conformal transformations. If the solutions are restricted to constant ones as in (4.1.13), we no longer have the generalized conformal symmetry. It can be maintained only when we consider a set of space-time dependent solutions to the constraint equations.

Recently H. Verlinde [59] also considered space-time dependent solutions to the constraint equations and discussed the conformal symmetry of the L-BLG model. In his study the constraint equation is imposed everywhere except at z_i where a local operator $\mathcal{O}_i(z_i)$ is inserted,

$$X_0^I(x) = \sum \frac{q_i^I}{|x - z_i|}. \tag{4.1.27}$$

This is an inhomogeneous solution to the equation

$$\partial^2 X_0^I = -4\pi \sum q_i^I \delta^3(x - z_i). \tag{4.1.28}$$

We can add the homogeneous solutions to the above. If q^I and z (omitting the index i) transform as

$$\begin{aligned}\delta q^I &= \epsilon \cdot z q^I \\ \delta z_\mu &= 2(\epsilon \cdot z) z_\mu - \epsilon_\mu z^2,\end{aligned} \tag{4.1.29}$$

the transformation of X_0^I

$$\delta X_0^I(x) = -(\epsilon \cdot x) X_0^I(x) \tag{4.1.30}$$

is reproduced and the L-BLG action is invariant under the conformal transformations. Note that q^I cannot be a constant. If q^I is kept fixed, the set of solutions is not closed under the conformal transformations. In order to recover the conformal invariance, q^I should be a position z-dependent charge.

We have shown that the L-BLG model has both of the $SO(8)$ invariance and the conformal symmetry. In the next section we discuss the symmetry properties of the gravity dual of the ABJM model.

4.2 $SO(8)$ and Conformal Symmetry in Dual Gravity

4.2.1 Large k limit of ABJM geometry

In the paper [7], it was pointed out that the $U(N) \times U(N)$ ABJM model is dual to the M-theory on $AdS_4 \times \mathbb{S}^7/\mathbb{Z}_k$, which is a $d = 11$ supergravity solution of M2 branes probing the orbifold $\mathbb{C}^4/\mathbb{Z}_k$. We first review the solution of supersymmetric M2 branes in $d = 11$ supergravity.

The $d = 11$ metric of the multiple M2-branes is given by

$$\begin{aligned}ds^2 &= H^{-\frac{2}{3}} \left(\sum_{\mu,\nu=0}^{2} \eta_{\mu\nu} dx^\mu dx^\nu \right) + H^{\frac{1}{3}} \left(dr^2 + r^2 d\Omega_7^2 \right), \\ H(r) &\equiv 1 + \frac{R^6}{r^6},\end{aligned} \tag{4.2.1}$$

where $R^6 = 32\pi^2 N' l_p^6$ and $d\Omega_7^2$ is the metric of a unit 7-sphere. N' is the number of the M2 branes and identified with $N' = kN$. The three form field is also given as

$$C^{(3)} = H^{-1} dx^0 \wedge dx^1 \wedge dx^2 \tag{4.2.2}$$

and the 4-form flux normalized by the world volume is proportional to N'.

By focusing on the near horizon region of the M2-brane, the geometry becomes $AdS_4 \times \mathbb{S}^7$ geometry. In the near horizon limit $R \gg r$, $H(r)$ is replaced by $H(r) = (R/r)^6$ and the metric becomes

$$\begin{aligned}ds^2 &= \left(\frac{r}{R}\right)^4 \left(\sum_{\mu,\nu=0}^{2} \eta_{\mu\nu} dx^\mu dx^\nu \right) + \left(\frac{R}{r}\right)^2 dr^2 + R^2 d\Omega_7^2 \\ &= R^2 \left[\frac{1}{4} ds_{AdS}^2 + d\Omega_7^2 \right]\end{aligned} \tag{4.2.3}$$

where we have rescaled the M2 brane world volume coordinates by a factor $2/R^3$. Hence the near horizon geometry of the supersymmetric M2 branes is given by $AdS_4 \times \mathbb{S}^7$ with a radius R. In the large $N' = kN$ limit, the radius becomes much larger than the $d = 11$ Planck length and the $d = 11$ supergravity approximation is valid.

The ABJM model describes M2 branes on $\mathbb{C}^4/\mathbb{Z}_k$ orbifold. The dual geometry can be obtained by first specifying the polarization (choice of the complex coordinates) in $\mathbf{R}^8$ and then dividing $\mathbb{C}^4$ by $\mathbb{Z}_k$.

Since $\mathbb{S}^7$, parameterized by z^A $(A = 1, \cdots, 4)$ with $|z^A|^2 = 1$, is a $U(1)$-fibration on $\mathbb{C}P^3$, the metric of $\mathbb{S}^7$ is written as

$$d\Omega_7^2 = \left(d\varphi' + \omega\right)^2 + ds^2_{\mathbb{C}P^3} \tag{4.2.4}$$

where φ' is the overall phase of z^A. The details of the definition of coordinates are written in Appendix C.

We now perform the $\mathbb{Z}_k$ quotient by dividing the overall phase of each z^A, namely the φ' direction. By rewriting $\varphi' = \varphi/k$ with $\varphi \sim \varphi + 2\pi$, the metric of $\mathbb{S}^7/\mathbb{Z}_k$ becomes

$$ds^2_{\mathbb{S}^7/\mathbb{Z}_k} = \frac{1}{k^2}\left(d\varphi + k\omega\right)^2 + ds^2_{\mathbb{C}P^3}. \tag{4.2.5}$$

Before performing the $\mathbb{Z}_k$ quotient, the metric has the conformal symmetry associated with the AdS_4 geometry and $SO(8)$ symmetry of $\mathbb{S}^7$. The orbifolding breaks the $SO(8)$ symmetry to $SU(4) \times U(1)$ but the conformal invariance still exists. This is the bosonic symmetry of the ABJM model.

The L-BLG action can be derived by taking the scaling limit (4.1.7) of the ABJM model. In the gravity side, this scaling corresponds to locating the probe M2 branes far from the orbifold singularity and taking the large k limit. As we show in the next section, the former process recovers the $SO(8)$ if the positions of the M2 branes are considered to be dynamical variables. The latter makes the radius of the φ' circle small and $d = 11$ geometry is reduced to $d = 10$.

Now we consider the $k \to \infty$ limit of the dual geometry of the ABJM model. Following the prescription of ABJM, we shall interprete the coordinate φ as the compact direction in reducing from M-theory to type IIA superstring. Using the reduction formula [72]

$$ds^2_{11} = e^{-\frac{2}{3}\phi} ds^2_{10} + e^{\frac{4}{3}\phi} (l_p)^2 \left(d\varphi + A\right)^2 \tag{4.2.6}$$

we get the $d = 10$ metric and the dilaton field in type IIA supergravity as

$$ds^2_{10} = \frac{r}{kl_p} H^{-\frac{1}{2}} \left(\sum_{\mu,\nu=0}^{2} \eta_{\mu\nu} dx^\mu dx^\nu \right) + \frac{r}{kl_p} H^{\frac{1}{2}} \left(dr^2 + r^2 ds^2_{\mathbb{C}P^3} \right), \tag{4.2.7}$$

$$e^{2\phi} = \left(\frac{r}{kl_p}\right)^3 H^{\frac{1}{2}} = \left(\frac{R}{kl_p}\right)^3. \tag{4.2.8}$$

Hence in the $k \to \infty$ limit, the metric becomes $AdS_4 \times \mathbb{C}P^3$:

$$ds^2_{10} = \frac{R^3}{k}\left[\frac{1}{4} ds^2_{AdS_4} + ds^2_{\mathbb{C}P^3}\right] \tag{4.2.9}$$

where the radius of curvature in string units is

$$R^2_{\rm str} = \left(\frac{R}{l_s}\right)^2 = \frac{R^3}{kl_p^3} = 2^{5/2}\pi\sqrt{\frac{N}{k}}. \tag{4.2.10}$$

The dilaton is a constant and this is the reason why the $d = 10$ metric still has a conformal symmetry associated with the AdS_4 geometry. This is different from the ordinary reduction of the M2 branes to D2 branes by compactifying the 11th direction of the Cartesian coordinate (see Appendix D). Note that in the type IIA picture, in addition to the four-form RR flux F_4, there is a 2-form RR flux:

$$\begin{aligned} F_4 &= \frac{3}{8}\frac{R^3}{l_p^3}\hat{\epsilon}_4, \\ F_2 &= dA = kd\omega \end{aligned} \tag{4.2.11}$$

where $\hat{\epsilon}_4$ is the volume form in a unit radius AdS_4 space. Hence the geometry is described by the $AdS_4 \times \mathbb{C}P^3$ compactification with N units of the four form flux on AdS_4 and k units of the two-form flux on the $\mathbb{C}P^1$ in $\mathbb{C}P^3$ space.

In the $k \to \infty$ limit with N/k fixed, the compactification radius along the φ-direction R_{11} becomes very small compared to the $d = 11$ Planck length:

$$\frac{R_{11}}{l_p} = \frac{R}{kl_p} \sim \frac{(Nk)^{1/6}}{k} \to 0. \tag{4.2.12}$$

Thus the theory is reduced to a ten-dimensional type IIA superstring on $AdS_4 \times \mathbb{C}P^3$. However the scaling limit from ABJM to L-BLG is taking large k limit before taking the large N and the 't Hooft coupling N/k becomes 0 in this limit. Since $R_{11} = g_s^{2/3} l_p$, the string coupling constant $g_s = e^{\phi}$ also becomes 0:

$$g_s = e^{\phi} \sim k^{-\frac{5}{4}} N^{\frac{1}{4}} \to 0. \tag{4.2.13}$$

Since $d = 11$ Planck length l_p and $d = 10$ Planck length $l_p^{(10)}$ are related to the string length as $l_p = g_s^{1/3} l_s$ and $l_p^{(10)} = g_s^{1/4} l_s$, the ratios of the radius of the IIA geometry (4.2.9) with l_s and $l_p^{(10)}$ are given by

$$\left(\frac{R}{l_s}\right)^2 \sim \sqrt{\frac{N}{k}} \to 0, \qquad \left(\frac{R}{l_p^{(10)}}\right)^2 \sim k^{1/8} N^{3/8} \to \infty. \tag{4.2.14}$$

Therefore the Type IIA supergravity approximation itself is good but the α' expansion is not good and the theory cannot be considered as the low energy approximation of type IIA superstring. On the other hand, the radius R is much larger than the $d = 11$ Planck length and it may be more appropriately interpreted as a dimensional reduction of M2 branes in the $d = 11$ supergravity.

We summarize the various length scales in the scaling limit of the ABJM model to the L-BLG model:

$$R_{11} \ll l_p^{(11)} \ll l_p^{(10)} \ll R_{AdS} \ll l_s. \tag{4.2.15}$$

The compactification radius R_{11} is much smaller than any other scales and the theory is reduced to $d = 10$. But the radius of the $AdS_4 \times \mathbb{C}P^3$ is smaller than the string length and larger than the $d = 10$ and $d = 11$ Planck scales.

In the ordinary case of the duality between type IIB superstrings on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM in $d = 4$, the radius of curvature R is given by

$$\left(\frac{R}{l_s}\right)^4 \sim g_s N, \qquad \left(\frac{R}{l_p^{(10)}}\right)^4 \sim N. \tag{4.2.16}$$

Thus it is usually assumed that both of $g_s N$ and N are large so much so that the type IIB supergravity approximation and the α'-expansion are valid. Unless $g_s N$ is large, α' corrections cannot be neglected and the supergravity description itself is not valid. In the weak coupling limit, the dual field theory is usually considered to be more appropriate. In our case, we can consider the $d = 10$ supergravity as a dimensional reduction of $d = 11$ supergravity. However membranes wrapping the φ direction become very light strings in the unit of the radius of curvature R, and this may invalidate the supergravity approximation of the M-theory.

4.2.2 Recovery of $SO(8)$ in dual geometry of L-BLG

In taking the scaling limit $k(\gg N) \to \infty$ of the ABJM model to the L-BLG model, the eleven-dimensional geometry is reduced to the ten-dimensional $AdS_4 \times \mathbb{C}P^3$:

$$\begin{aligned} ds^2 &= H^{-\frac{2}{3}} \left(\sum \eta_{\mu\nu} dx^\mu dx^\nu \right) + H^{\frac{1}{3}} (dr^2 + r^2 ds^2_{\mathbb{C}P^3}) \\ H(r) &= \frac{R^6}{r^6}. \end{aligned} \tag{4.2.17}$$

In this section we discuss how the $SO(8)$ can be recovered in the scaling limit of the ABJM geometry to the L-BLG geometry. The L-BLG geometry is obtained by taking $k \to \infty$ limit of $AdS_4 \times \mathbb{S}^7/\mathbb{Z}_k$ and simultaneously locating the probe M2 brane far from the origin of the orbifold. In the large k limit, the geometry becomes $d = 10$ $AdS_4 \times \mathbb{C}P^3$, and there are only 7 transverse directions to the M2 brane world volume, However the radial distance in (4.2.17) is given by the distance in $d = 8$:

$$r^2 = \sum_{I=1}^{8} (X^I)^2. \tag{4.2.18}$$

It is invariant under the original $SO(8)$ rotation and the $\mathbb{Z}_k$ quotient leaves r invariant.

Now we consider a probe M2 brane in the above geometry. In the static gauge, the M2 brane world volume is identified with the coordinates x^μ ($\mu = 0, 1, 2$) and the position of the M2 brane is given by $X^I(x)$ where $I = 1, \cdots, 8$. There are only 7 independent propagating modes among 8, and the direction that is removed is the φ-direction. Remember that the φ is the overall phase of the complex coordinate z^i of the transverse R^8. Assuming that the probe M2 brane is located far from the source branes, we can

separate the probe M2 brane coordinates into the classical background fields $X_0^I(x)$ and the quantum fluctuations $\hat{X}^I(x)$. Since the M2 brane is on $\mathbb{C}^4/U(1)$, all the points on the gauge orbit generated by the φ-rotation are identified. Here the position of the M2 brane is represented by the coordinates of $\mathbf{R}^8$; a point on the gauge orbit is singled out by fixing the gauge (see Appendix C).

If the probe M2 brane is located at

$$X_0^I = v\delta^{I,8} \tag{4.2.19}$$

where v is much larger than the scale of the fluctuations, the rotation along the φ-direction is approximated by

$$\begin{aligned} &\delta X^7 = -\delta\varphi\ v, \\ &\delta X^I = 0 \quad , \quad I \neq 7. \end{aligned} \tag{4.2.20}$$

This shows that in the large v limit the φ direction can be identified with the 7th direction X^7 [1]. Since the $\mathbb{Z}_k$ orbifolding with large k corresponds to gauging away the φ-direction, the fluctuation along the 7th direction is killed and the field $\hat{X}^I$ can fluctuate only in the other 7 directions. This means that the $SO(7)$ rotation acts among the other 7 directions around the classical background of (4.2.19). If the classical background $X_0^I(x)$ takes different directions at different world volume points, the killed direction also changes locally on the world volume.

In order to get a manifest $SO(8)$ covariant formulation of this mechanism, it is convenient to separate the classical background field of the M2 brane and the fluctuations in the complex coordinates as

$$Z^A(x) = Z_0^A(x) + \hat{Z}^A(x). \tag{4.2.21}$$

If the fluctuations are much smaller than the classical background field, the φ rotation can be approximated as

$$\delta Z^A = i\delta\varphi Z_0^A. \tag{4.2.22}$$

If we write

$$\begin{aligned} Z_0^A &= X_0^{2A-1} + iX_0^{2A} \\ \hat{Z}^A &= i\hat{X}^{2A-1} - \hat{X}^{2A}, \end{aligned} \tag{4.2.23}$$

where $A = 1 \cdots 4$, the propagating degrees of freedom along the direction (4.2.22) are killed and the fluctuations are restricted to obey

$$X_0^I \hat{X}^I = 0. \tag{4.2.24}$$

[1] (4.2.19) has fixed a gauge of the φ rotation and (4.2.20) is nothing but the direction parallel to the gauge orbit. If we change a gauge,e.g. to $X_0^I = v\delta^{I,7}$, (4.2.20) is also changed accordingly.

Note that the decomposition of the complex fields into the real and the imaginary parts are different between the classical background Z_0^A and the fluctuations $\hat{Z}^A$ in (4.2.23). With this definition, if $X_0^I = v\delta^{I,8}$, the killed direction becomes the 8th direction of $\hat{X}^I$. We can write the fluctuations perpendicular to $\hat{X}^I$ in (4.2.24) as

$$P^I = \left(\delta^{IJ} - \frac{X_0^I X_0^J}{(X_0)^2}\right)\hat{X}_J. \tag{4.2.25}$$

This P^I automatically satisfies the condition (4.2.24) and 7 degrees of freedom are projected among the 8 degrees of freedom. Now everything is written in a manifestly $SO(8)$ covariant way. The $SO(8)$ covariance is recovered because we have assumed that the fluctuation is much smaller than the classical background fields of the probe M2 brane. This assumption is consistent with the scaling limit of the ABJM model to the L-BLG model.

Note here that the $SO(8)$ rotation changes the gauge choice of the φ rotation and $SO(8)$ is mixed with the $U(1)$ gauge transformation. Also note that because of the different assignments of X^I to Z^A for Z_0 and $\hat{Z}$, the $SO(8)$ is different from the original $SO(8)$ before taking the orbifolding.

The analysis here and in the previous section shows why the L-BLG model has both of the conformal symmetry and the invariance under $SO(8)$. The compactification direction along the φ direction is different from the ordinary reduction to $d = 10$ by compactifying the 11th transverse direction. The dilaton becomes constant and the AdS_4 geometry is preserved. This is the reason why there is a conformal symmetry in the effective field theory of L-BLG.

The $SO(8)$ invariance is more subtle. In the scaling limit of ABJM to L-BLG, we take $k \to \infty$ limit and simultaneously locate the probe M2 brane far from the origin of the orbifold. Then the killed direction of the fluctuations by Z_k ($k \to \infty$) orbifolding is given by the $SO(8)$ vector of the classical background fields X_0^I after specifying the gauge choice, and defining the projection operator by using X_0^I the manifest $SO(8)$ covariance is obtained.

4.2.3 Actions of probe branes in $AdS_4 \times \mathbb{C}P^3$

In this section we study possible forms of the effective field theory of probe M2 branes in the background geometry (4.2.17). The analysis in the section follows the prescription of [77] and [78] that a classical scalar field in the radial direction is interpreted as the Yang-Mills coupling. We will study probe M2 branes in a curved background while flat 11-dimensional background is used there.

By using the metric of (4.2.17), the generally covariant kinetic term can be written as

$$S_0 = -\frac{1}{2}\int d^3x \sqrt{-\det g} g^{\mu\nu} g_{IJ} \mathrm{tr}[D_\mu X^I D_\nu X^J], \tag{4.2.26}$$

where $\mu, \nu = 0, 1, 2$ are the world volume indices and $I, J = 1, \cdots, 8$ are the target space indices, and $D_\mu = \partial_\mu - iA_\mu$ is the covariant derivative to assure that X^I lies on $\mathbb{C}^4/U(1)$ (see Appendix C).

Both of the world volume metric $g^{\mu\nu}$ and the target space metric g_{IJ} are functions of the position of the M2 branes $X^I(x)$. A static gauge is taken and the world volume metric $g_{\mu\nu}$ is given by the induced metric in the curved space-time (4.2.17).

This kinetic term can be simplified as follows. The metric $g_{\mu\nu}$ and g^{IJ} are functions of the the M2 brane position through r. As we did in the previous section, we separate the 8 scalar fields $X^I(x)$ of the probe M2 branes into a classical background and quantum fluctuations. If the probe M2 branes are located far from the origin of the orbifold singularity, the position of the M2 branes is approximated by the value of the classical background fields $X_0^I(x)$ and $r \sim \sqrt{(X_0^I(x))^2}$. Inserting the explicit form of the metric, the kinetic term can be simplified (see Appendix C) as

$$S_0 = -\frac{1}{2}\int dx^3 \eta^{\mu\nu}\eta_{IJ}\mathrm{tr}[\partial_\mu P^I \partial_\nu P^J] \tag{4.2.27}$$

where $P^I(x)$ is the projected fluctuating fields (4.2.25). In deriving this action, we have used that the classical background fields X_0^I are slowly varying. Note that all the dependence of $H(r)$ vanishes and the kinetic term of the fluctuating fields does not have the explicit dependence on the position of M2 branes.

The position of the M2 branes X_0^I must satisfy the classical equation of motion on the geometry (4.2.17). Because of the cancellation of $H(r)$, it looks like a free field equation of motion. But the fields X_0^I are restricted to be on the geometry where the φ-direction is killed, and they are slightly different from the constraint equation (4.1.9) in the L-BLG model, or that in the scaling limit of the $SU(N) \times SU(N)$ ABJM model. This is related to the effect of the $U(1)$ gauge field of the ABJM model. We discuss it in Appendix B.

In the rest of this section, we dare to generalize the discussion of the kinetic term of the scalar field to the other possible terms in the the effective action of the probe M2 branes in the geometry (4.2.17). First assume that a gauge field is induced on the effective action of the probe M2 branes and its action is given by the ordinary Yang-Mills type. Then the general coordinate invariant YM action in the curved metric (4.2.17) is given by

$$-\frac{1}{4}\int d^3x \sqrt{-\det g} g^{\mu\rho}g^{\nu\sigma}\mathrm{tr}\left[F_{\mu\nu}F_{\rho\sigma}\right] = -\frac{1}{4}\int d^3x \left(\frac{R}{r}\right)^2 \eta^{\mu\rho}\eta^{\nu\sigma}\mathrm{tr}\left[F_{\mu\nu}F_{\rho\sigma}\right]. \tag{4.2.28}$$

(Since we are considering the $d = 11$ theory, there is no freedom to multiply a dilaton dependence in the action.) In this case, $H(r)$ dependence remains and the effective Yang-Mills coupling is given by the following field dependent value:

$$g_{YM}^2(x) = \frac{r^2}{R^2} = \frac{(X_0^I(x))^2}{R^2}. \tag{4.2.29}$$

Similarly if we assume that the scalar field acquires a quartic potential, the general coordinate and $SO(8)$ invariance require its form to be

$$\begin{aligned}&\frac{1}{4}\int d^3x \sqrt{-\det g} g_{IK}g_{JL}\mathrm{tr}[P^I, P^J][P^K, P^L] \\ &= \int d^3x \frac{1}{4}\frac{(X_0^I)^2}{R^2}\eta_{IK}\eta_{JL}\mathrm{tr}[P^I, P^J][P^K, P^L].\end{aligned} \tag{4.2.30}$$

Here P^I are projected scalar fields (4.2.25).

Summing up these three terms, we have the following forms of the effective action:

$$S = -\frac{1}{2}\int dx^3 \left(\mathrm{tr}[\partial_\mu P^I \partial^\mu P^I] - \frac{1}{4}\frac{R^2}{(X_0^I)^2}\mathrm{tr}\,[F_{\mu\nu}F^{\mu\nu}] + \frac{1}{4}\frac{(X_0^I)^2}{R^2}\mathrm{tr}[P^I, P^J]^2 \right). \quad (4.2.31)$$

Of course there is little justification of the above analysis but it is amusing to see that this is nothing but the bosonic part of (4.1.18). The analysis might support an interpretation that the action of L-BLG is the effective action of the probe M2 branes in the geometry of (4.2.17). The X_0^I dependence of the coefficients will be related to the conformal invariance of the M2 branes. It will be interesting to constrain possible forms of the effective action including fermions, higher derivative terms, or generic potential terms by the generalized conformal invariance.

Chapter 5

Mass deformation

In this chapter, we discussed the mass deformed Lorentzian BLG model and also explain the possibility of coordinate dependent couplings even without mass term [13]. We also clarify the BF theory and usual gauge theory which live in Lorentzian BLG model.

5.1 Bagger-Lambert-Gustavsson model

5.1.1 Comments on BLG model to D2 branes

Let's remind the Lorentzian BLG model from (2.2.42).

$$\begin{aligned}\mathcal{L}_L = \Bigg[&-\frac{1}{2}(\hat{D}_\mu \hat{X}^I - B_\mu X_0^I)^2 + \frac{1}{4}(X_0^K)^2([\hat{X}^I, \hat{X}^J])^2 - \frac{1}{2}(X_0^I[\hat{X}^I, \hat{X}^J])^2 \\ &+ \frac{i}{2}\bar{\hat{\Psi}}\Gamma^\mu \hat{D}_\mu \hat{\Psi} + i\bar{\Psi}_0 \Gamma^\mu B_\mu \hat{\Psi} - \frac{1}{2}\bar{\Psi}_0 \hat{X}^I[\hat{X}^J, \Gamma_{IJ}\hat{\Psi}] + \frac{1}{2}\bar{\hat{\Psi}} X_0^I[\hat{X}^J, \Gamma_{IJ}\hat{\Psi}] \\ &+ \frac{1}{2}\epsilon^{\mu\nu\lambda}\hat{F}_{\mu\nu}B_\lambda - \partial_\mu X_0^I \; B_\mu \hat{X}^I \Bigg] + \mathcal{L}_{gh}, \\ \mathcal{L}_{gh} = &(\partial_\mu X_0^I)(\partial^\mu X_{-1}^I) - i\bar{\Psi}_{-1}\Gamma^\mu \partial_\mu \Psi_0. \end{aligned} \tag{5.1.1}$$

X_{-1}^I and Ψ_{-1} appear only linearly in the Lorentzian BLG model and thus they are Lagrange multipliers. By integrating out these fields, we have the following constraints for the other problematic fields associated with T^0;

$$\partial^2 X_0^I = 0, \qquad \Gamma^\mu \partial_\mu \Psi_0 = 0. \tag{5.1.2}$$

This should be understood as a physical state condition $\partial^2 X_0^I |phys\rangle = 0$. In the path integral formulation, these constraints appear as a delta function $\delta(\partial^2 X_0^I)$ and those fields are constrained to satisfy the massless wave equations. In order to fully quantize the theory, we need to sum all the solutions satisfying the constraints, but we here take a special solution to the constraint equations and see what kind of field theory can be obtained.

The simplest solution is given by

$$X_0^I = v\ \delta_{10}^I, \qquad \Psi_0 = 0, \tag{5.1.3}$$

where v is some constant. This solution was considered in [4–6] and preserves all the 16 supersymmetries, the gauge symmetry generated by the subalgebra $\mathcal{A}$, and $SO(7)$ R-symmetry rotating X^A, $A = 3, ..., 9$. Another interesting solution is given by

$$X_0^I = v(x^0 + x^1)\delta_{10}^I \ , \qquad \Psi_0 = 0 \tag{5.1.4}$$

where $v(x^0 + x^1)$ is an arbitrary function on the light cone coordinate. As we see the supersymmetry transformation for Ψ_0,

$$\delta\Psi_0 = \partial_\mu X_0^I \Gamma^\mu \Gamma^I \epsilon, \tag{5.1.5}$$

the solution $X_0^I = v(x^0 + x^1)\delta_{10}^I$ preserves half of the supersymmetries.

In both cases, if we fix the fields X_0^I and Ψ_0 as above, we can integrate over the gauge field B_μ and obtain the effective action for N D2 branes[1]

$$\mathcal{L} = \mathrm{Tr}\left[-\frac{1}{2}(\hat{D}_\mu \hat{X}^A)^2 + \frac{1}{4}v^2[\hat{X}^A, \hat{X}^B]^2 + \frac{i}{2}\bar{\hat{\Psi}}\Gamma^\mu \hat{D}_\mu \hat{\Psi} - \frac{1}{4v^2}\hat{F}_{\mu\nu}^2 + \frac{1}{2}v\bar{\hat{\Psi}}[\hat{X}^A, \Gamma_{10,A}\hat{\Psi}]\right] \tag{5.1.6}$$

where $A, B = 3, \cdots, 9$. The coupling v is given by the vev of X_0^{10} and it is either a constant or an arbitrary function on the light-cone $v(x^0+x^1)$. This may be identified as the compactification radius of 11-th direction in M-theory; $v = 2\pi g_s l_s$. The supersymmetric YM theories with a space-time dependent coupling are known as Janus field theories and originally considered to be a dual of supergravity solutions with space-time dependent dilaton fields [14].

A salient feature is that the 10-th spacial fields X^{10} completely disappear from the Lagrangian by integrating out the redundant gauge field B_μ. It is interesting that Janus field theories are naturally obtained from the BLG field theories.

The $v \to 0$ limit cannot be taken after integrating the redundant gauge field B_μ. In the case of vanishing v, the Lagrangian is simply given by

$$\mathcal{L} = \mathrm{Tr}\left[-\frac{1}{2}(\hat{D}_\mu \hat{X}^I)^2 + \frac{i}{2}\bar{\hat{\Psi}}\Gamma^\mu \hat{D}_\mu \hat{\Psi}\right] \tag{5.1.7}$$

with a constraint $\hat{F}_{\mu\nu} = 0$. The action is of course invariant under the full $SO(8)$ R-symmetry.

5.1.2 Janus field theory with Dynamical coupling

In the previous subsection, we have fixed the solution of the constraint equations. But in the quantization of the BLG model, the solutions should be summed in the path integral. So we will consider more general solutions in this subsection. After integrating the modes associated with the T^{-1} generator, the partition function becomes

$$Z = \int \mathcal{D}X_0^I \mathcal{D}\Psi_o \mathcal{D}B_\mu \mathcal{D}\hat{X}^I \mathcal{D}\hat{\Psi} \mathcal{D}A_\mu \ \delta(\partial^2 X_0^I)\ \delta(\Gamma^\mu \partial_\mu \Psi_0)\ e^{iS(X_o^I, \Psi_0, B_\mu, \hat{X}^I, \hat{\Psi}, A_\mu)}. \tag{5.1.8}$$

[1]The fermion here is a 32 component spinor satisfying $\Gamma_{012}\Psi = \Psi$. In order to recover the ordinary notation for D2 branes, we rearrange it as $\tilde{\Psi} = (1 + \Gamma_{10})\Psi$. Then it satisfies $\Gamma_{10}\tilde{\Psi} = \tilde{\Psi}$ and the action is written in the usual form (no Γ_{10} in the last term).

The integrations over X_0^I and Ψ_0 are constrained to obey the massless wave equations and can be expanded as

$$X_0^I = \sum_n c_n^I f_n(x), \quad \Psi_0 = \sum_n b_n u_n(x) \tag{5.1.9}$$

where $f_n(x), u_n(x)$ are complete sets of functions satisfying the massless wave equations. Then the integration over X_0^I and Ψ_0 can be reduced to integrations over c_n^I and b_n.

Let us now choose a general solution $(X_0^I = v^I(x), \Psi_0)$ to the constraints and expand the action around it. In this case all the supersymmetries are generally broken. Inserting this general solution into the action, terms including the B_μ gauge field are given by

$$-\frac{1}{2}(\hat{D}_\mu \hat{X}^I - B_\mu X_0^I)^2 + i\bar{\Psi}_0\Gamma^\mu B_\mu \hat{\Psi} + \frac{1}{2}\epsilon^{\mu\nu\lambda}\hat{F}_{\mu\nu}B_\lambda - \partial_\mu X_0^I B_\mu \hat{X}^I. \tag{5.1.10}$$

The integration over the B_μ gauge field can be similarly performed. It is convenient to introduce the locally defined projection operator

$$P_{IJ}(x) = \delta_{IJ} - \frac{v_I v_J}{v^2}, \tag{5.1.11}$$

This operator satisfies $P^2 = P$ and $P_{IJ}v^J = 0$. In the simplest case considered in the previous subsection, $v^I = v(t+x)\delta_{10}^I$, this projects out the 10-th direction if it acts on $\hat{X}^I$. Generally, the direction removed is dependent on the space-time position.

After integrating over the B_μ field, the Lagrangian becomes $\mathcal{L}_{Janus} = \mathcal{L}_0 + \mathcal{L}'$ where

$$\begin{aligned}
\mathcal{L}_0 &= \mathrm{Tr}\left[-\frac{1}{2}(\hat{D}_\mu Y^I)^2 + \frac{1}{4}v^2[Y^I, Y^J]^2 + \frac{i}{2}\bar{\hat{\Psi}}\Gamma^\mu \hat{D}_\mu \hat{\Psi} + \frac{1}{2}\bar{\hat{\Psi}}[Y^I, (v^J\Gamma_J)\Gamma_I\hat{\Psi}]\right. \\
&\quad \left. + \frac{1}{2(v^I)^2}\left(\frac{1}{2}\epsilon^{\mu\nu\lambda}\hat{F}_{\nu\lambda} + i\bar{\Psi}_0\Gamma^\mu\hat{\Psi} - 2Y_I\partial^\mu v^I\right)^2 - \frac{1}{2}\bar{\Psi}_0\Gamma_{IJ}\hat{\Psi}[Y^I, Y^J]\right], \quad (5.1.12) \\
\mathcal{L}' &= \frac{1}{v^2}\mathrm{Tr}\left[\left(\bar{\Psi}_0\Gamma_I(v^J\Gamma_J)[Y^I, \hat{\Psi}] - i\bar{\Psi}_0\Gamma_\mu\hat{D}_\mu\hat{\Psi}\right)(v^K\hat{X}^K)\right]. \quad (5.1.13)
\end{aligned}$$

Here $I, J = 3, \cdots, 10$ and we have defined a new scalar field $Y^I = P_{IJ}\hat{X}^J$ with 7 degrees of freedom.

This is a Janus field theory whose coupling varies with space-time. The Lagrangian $\mathcal{L}_{YM}$ contains only the projected scalar field Y^I. On the other hand, in the presence of Ψ_0, the scalar field $(v^I\hat{X}^I)$ does not decouple from the Lagrangian $\mathcal{L}'$. If we can set $\Psi_0 = 0$, $\mathcal{L}'$ vanishes and the resultant Lagrangian is given by a similar form to the ordinary Super Yang-Mills Lagrangian, but the kinetic term of the gauge field $\hat{F}_{\mu\nu}$ is modified to $\hat{F}_{\mu\nu} + 2\epsilon_{\mu\nu\rho}Y_I\partial^\rho v^I$. All the supersymmetries are generally broken if we fix one solution to the constraint equations of $(X_0^I(x), \Psi_0)$ as above.

By using the above calculation, the partition function can be simply rewritten as

$$Z = \int \prod_n dc_n^I \; db_n \; W(v^I) \int \mathcal{D}\hat{X}^I \mathcal{D}\hat{\Psi}\mathcal{D}A_\mu \; e^{iS_{Janus}(\hat{X}^I, \hat{\Psi}, A_\mu; v^I(x), \Psi_0)}. \tag{5.1.14}$$

Here $W(v^I) \sim ((v^I)^2)^{-3/2}$ came from the integration over the B_μ field. It is a sum of Janus field theories. The coupling constant v^I is dynamical and varies with space-time coordinates. It is constrained to satisfy the massless equations. If we fix the "slow" variable v and perform the path integration over the other "fast" variables first, then we can get an effective action for the dynamical coupling v^I. This will determine the most stable configuration of $v^I(x)$, and accordingly one of the Janus gauge theory with the most stable coupling is determined. If the variable v^I fluctuates rapidly and cannot be considered as a slow variable, the theory becomes very different from the ordinary gauge theory with a fixed (either constant or varying) gauge coupling. This may be related to the dynamical determination of the compactification radius of 11-th direction in M-theory.

Finally we would like to comment on the unitarity of the BLG theory. If we fix one solution to the constraints, each theory behaves regularly if the coupling constant does not vary drastically. The quantization of the coupling is very difficult, but since it is not a propagating mode, it will not violate the unitarity of the theory. However the unitarity should be more carefully analyzed.

5.2 Mass deformation and Janus solutions

5.2.1 Mass deformation of BLG

The BLG model in the previous section gives a familiar effective action of N D2 branes with either a constant or a varying coupling. (For general solutions, the kinetic term of the gauge field contains a non-familiar term of $Y_I \partial^\mu v^I$.)

In this section we start from a mass deformed BLG action given by [60, 61] and show that supersymmetric Janus field theories with a Myers-term are obtained.

One parameter deformation of the BLG action preserving the full supersymmetries is given by adding the following mass and flux terms to the original Lagrangian. The mass term is given by

$$\mathcal{L}_{mass} = -\frac{1}{2}\mu^2 \mathrm{Tr}(X^I, X^I) + \frac{i}{2}\mu \mathrm{Tr}(\bar{\Psi}\Gamma_{3456}, \Psi), \tag{5.2.1}$$

and a flux term is

$$\mathcal{L}_{flux} = -\frac{1}{6}\mu\epsilon_{EFGH}\mathrm{Tr}([X^E, X^F, X^G], X^H) - \frac{1}{6}\mu\epsilon_{E'F'G'H'}\mathrm{Tr}([X^{E'}, X^{F'}, X^{G'}], X^{H'}) \tag{5.2.2}$$

Here $E, F, G, H = 3, 4, 5, 6$ and $E', F', G', H' = 7, 8, 9, 10$. This action is invariant under the original gauge transformation and the deformed SUSY transformation [2]

$$\begin{aligned} \delta X^I &= i\bar{\epsilon}\Gamma^I\Psi, \\ \delta\Psi &= (D_\mu X^I)\Gamma^\mu\Gamma_I\epsilon - \frac{1}{6}[X^I, X^J, X^K]\Gamma_{IJK}\epsilon - \mu\Gamma_{3456}\Gamma^I X^I\epsilon, \\ \delta\tilde{A}_\mu{}^b{}_a &= i\bar{\epsilon}\Gamma_\mu\Gamma_I X^I_c\Psi_d f^{cdb}{}_a. \end{aligned} \tag{5.2.3}$$

[2] To give a rigorous proof of the closure of the supersymmetry, we should check the Jacobi identity of $[Q, \{Q, Q\}]$ (appendix E of [79]) because there are non-central terms, i.e. $SO(4) \times SO(4)$ rotation term, in the algebra $\{Q, Q\}$. We thank Dr. Hai Lin for informing us of the paper [79]

This deformed theory breaks the original $SO(8)$ R-symmetry down to $SO(4) \times SO(4)$. By setting $\mu \to 0$ both the action and SUSY transformation reduce to the original BLG action. In addition there is another supersymmetry transformation:

$$\begin{aligned} &\delta X^I_a = 0, \qquad \delta \tilde{A}_\mu{}^b{}_a = 0, \\ &\delta\Psi = \exp\left(-\frac{\mu}{3}\Gamma_{3456}\Gamma_\mu x^\mu\right) T^{-1}\eta, \end{aligned} \tag{5.2.4}$$

where x^μ is the coordinates of the world volume.

5.2.2 Deformed BL to Janus

This model can be similarly investigated by expanding the fields into modes with internal indices $a = (-1, 0, i)$. The mode expansions of the mass and the flux terms become

$$\mathcal{L}_{mass} = \mu^2 X^I_{-1} X^I_0 - \frac{\mu^2}{2}\mathrm{Tr}(\hat{X}^I, \hat{X}^I) - i\mu\bar{\Psi}_{-1}\Gamma_{3456}\Psi_0 + \frac{i}{2}\mu\mathrm{Tr}(\bar{\hat{\Psi}}\Gamma_{3456}, \hat{\Psi}), \tag{5.2.5}$$

and

$$\mathcal{L}_{flux} = \frac{2i}{3}\mu\epsilon_{EFGH}X^E_0\mathrm{Tr}(\hat{X}^F, [\hat{X}^G, \hat{X}^H]) + \frac{2i}{3}\mu\epsilon_{E'F'G'H'}X^{E'}_0\mathrm{Tr}(\hat{X}^{F'}, [\hat{X}^{G'}, \hat{X}^{H'}]). \tag{5.2.6}$$

Now X^I_{-1} and Ψ_{-1} again appear linearly in the action, and they are Lagrange multipliers. Because of the mass terms, the constraint equations are modified to

$$(\partial^2 - \mu^2)X^I_0 = 0, \qquad (\Gamma^\mu\partial_\mu + \mu\Gamma_{3456})\Psi_0 = 0. \tag{5.2.7}$$

Namely the fields with the T^0 component are constrained to obey the massive wave equations. Since X^I are real fields, instead of the plane waves $\exp(ik_\mu x^\mu)$ with a time-like vector k_μ, we take the following solution to the constraint equation;

$$X^I_0 = fe^{p_\mu x^\mu}\delta^I_{10} = v(x)\delta^I_{10}, \qquad \Psi_0 = 0, \tag{5.2.8}$$

where f is an arbitrary constant and p_μ is a spacelike vector satisfying $p^2 = \mu^2$. Without loss of generality, we can take $p_\mu = (0, \mu, 0)$. This configuration preserves half of the 16 supersymmetries, since Ψ_0 transforms as:

$$\delta\Psi_0 = v(x)\mu(\Gamma^1 - \Gamma_{3456})\Gamma^{10}\epsilon. \tag{5.2.9}$$

Hence around the above configuration, we will get Janus gauge field theories with 8 supersymmetries. (For general solutions, more supersymmetries are broken.)

Inserting this configuration to the action, one can again integrate the redundant gauge field B_μ. Terms involving B_μ are given by:

$$\mathrm{Tr}\left[-\frac{1}{2}(\hat{D}_\mu\hat{X}^{10} - vB_\mu)^2 + \frac{1}{2}\epsilon^{\mu\nu\lambda}\hat{F}_{\mu\nu}B_\lambda - p^\mu vB_\mu\hat{X}^{10}\right]. \tag{5.2.10}$$

Integrating B_μ gives

$$\begin{aligned}&\mathrm{Tr}\left[\frac{1}{2v}\epsilon^{\mu\nu\lambda}\hat{F}_{\mu\nu}p_\lambda\hat{X}^{10}+\frac{1}{8v^2}(\epsilon^{\mu\nu\lambda}\hat{F}_{\mu\nu}-2v\hat{X}^{10}p^\lambda)^2\right]\\&=-\frac{1}{4v^2}\mathrm{Tr}\hat{F}_{\mu\nu}^2+\frac{\mu^2}{2}\mathrm{Tr}(\hat{X}^{10},\hat{X}^{10}).\end{aligned}\tag{5.2.11}$$

Interestingly the second term is canceled by the mass term of $\hat{X}^{10}$ and all the terms involving $\hat{X}^{10}$ have disappeared. To summarize, the resultant effective Lagrangian is given by:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}\mathrm{Tr}(\hat{D}_\mu\hat{X}^A)^2-\frac{\mu^2}{2}\mathrm{Tr}(\hat{X}^A,\hat{X}^A)+\frac{1}{4}v^2[\hat{X}^A,\hat{X}^B]^2\\&+\frac{i}{2}\mathrm{Tr}\left(\bar{\hat{\Psi}}\Gamma^\mu\hat{D}_\mu\hat{\Psi}\right)+\frac{i}{2}\mu\mathrm{Tr}(\bar{\hat{\Psi}}\Gamma_{3456},\hat{\Psi})+\frac{1}{2}v\mathrm{Tr}\left(\bar{\hat{\Psi}}[\hat{X}^A,\Gamma_{10,A}\hat{\Psi}]\right)-\frac{1}{4v^2}\mathrm{Tr}\hat{F}_{\mu\nu}^2\\&-\frac{2i}{3}v\mu\epsilon^{A'B'C'10}\mathrm{Tr}(\hat{X}^{A'},[\hat{X}^{B'},\hat{X}^{C'}]).\end{aligned}\tag{5.2.12}$$

This is a Janus field theory whose coupling constant is given by $v = f\ \exp(\mu x^1)$. The Lagrangian is invariant under the following 8 supersymmetries

$$\begin{aligned}\delta\hat{X}^A &= i\bar{\epsilon}\Gamma^A\hat{\Psi},\\\delta\hat{\Psi} &= \hat{D}_\mu\hat{X}^A\Gamma^\mu\Gamma^A\epsilon-\frac{1}{2v}\epsilon_{\mu\nu\lambda}\hat{F}^{\nu\lambda}\Gamma^\mu\Gamma^{10}\epsilon+\frac{i}{2}v[\hat{X}^A,\hat{X}^B]\Gamma^{AB}\Gamma^{10}\epsilon-\mu\Gamma_{3456}\Gamma^A\hat{X}^A\epsilon,\\\delta\hat{A}_\mu &= iv\bar{\epsilon}\Gamma_\mu\Gamma^{10}\hat{\Psi},\end{aligned}\tag{5.2.13}$$

Finally if v vanishes, i.e. for $X_0^I = 0$ and $\Psi_0 = 0$, the Lagrangian becomes

$$\mathcal{L}=-\frac{1}{2}\mathrm{Tr}(\hat{D}_\mu\hat{X}^I)^2+\frac{i}{2}\mathrm{Tr}\left(\bar{\hat{\Psi}}\Gamma^\mu\hat{D}_\mu\hat{\Psi}\right)-\frac{\mu^2}{2}\mathrm{Tr}(\hat{X}^I,\hat{X}^I)+\frac{i}{2}\mu\mathrm{Tr}(\bar{\hat{\Psi}}\Gamma_{3456},\hat{\Psi}),\tag{5.2.14}$$

with a constraint $\hat{F}_{\mu\nu} = 0$. The supersymmetry transformation is given by

$$\begin{aligned}\delta\hat{X}^I &= i\bar{\epsilon}\Gamma^I\hat{\Psi},\\\delta\hat{\Psi} &= \hat{D}_\mu\hat{X}^I\Gamma^\mu\Gamma^I\epsilon-\mu\Gamma_{3456}\Gamma^I\hat{X}^I\epsilon,\\\delta\hat{A}_\mu &= 0\end{aligned}\tag{5.2.15}$$

and the Lagrangian has the $SO(4)\times SO(4)$ R-symmetry.

Chapter 6

Gravitational instantons with Squashed $SU(3) \times SU(2)$

This chapter is organized as follows. In section 6.2, we obtain squashed 2-brane solutions with $SU(3) \times SU(2)$ isometry group in 11-dimensional supergravity [1]. These constructions are motivated by the similarity between 5D and 11D supergravity theories [80]. We use squashing functions depending on the radius coordinate as an analogy of the Ishihara-Matsuno solution. The solutions include the 11-dimensional Gross-Perry-Sorkin gravitational instanton solutions which has the asymptotic structure as the squashed manifold $N_{\rm II}^{0,1,0}$ [2]. These can be considered as a higher-dimensional analogy of the Eguchi-Hanson space.

6.1 A brief review of $SU(3) \times SU(2)$ space

In this section we will briefly review the structure of $SU(3) \times SU(2)$ Einstein space in [20, 21] which we have used in following sections. The $SU(3) \times SU(2)$ geometry was firstly considered as the special parameterization of the coset manifold $N_{\rm I}^{p,q,r}$ of the form $SU(3)\times U(1)/U(1)\times U(1)$ by Castellani and Romans [20]. The integers p, q, r characterize the embedding of each $U(1)$s. By choosing the certain combination of these $U(1)$s, we can define the maximal torus of $SU(3) \times U(1)$ Lie algebra as

$$
\begin{aligned}
Z &= -\frac{1}{\sqrt{3p^2+q^2+2r^2}}\left[p\frac{i\sqrt{3}}{2}T_8 + q\frac{i}{2}T_3 + riY\right], \\
M &= -\frac{\sqrt{2}}{\sqrt{(3p^2+q^2+2r^2)(3p^2+q^2)}}\left[rp\frac{i\sqrt{3}}{2}T_8 + rq\frac{i}{2}T_3 - (3p^2+q^2)\frac{i}{2}Y\right], \\
N &= -\frac{1}{\sqrt{3p^2+q^2}}\left[-q\frac{i}{2}T_8 + p\frac{i\sqrt{3}}{2}T_3\right]
\end{aligned}
\tag{6.1.1}
$$

[1] We use the meaning of "squashed" as twisted of the $SU(3) \times SU(2)$ space in this paper.

[2] In the original paper [30], they also mentioned about the generalization to a non-Abelian instanton.

where we use the $SU(3)$ generator of isospin T_3, the hypercharge T_8 and the generator of the separate $U(1)$. We can know the structure constants $C_{G_1G_2}{}^{G_3}$ of $SU(3)\times U(1)$ by using the (6.1.1) and also the other generators of $SU(3)$, $-i/2(T_1, T_2, T_4, T_5, T_6, T_7)$.

The metric g can be defined by using the group metric $\gamma_{G_1G_2}$ with the vielbeins E by restricting to the coset space. These are written as

$$\begin{aligned}\gamma_{G_1G_2} &= \mathrm{Tr}\left(C_{G_1G}{}^{G'}C_{G_2G'}{}^{G}\right),\\ g_{\mu\nu} &= \gamma_{\alpha\beta}E^\alpha_\mu E^\beta_\nu\end{aligned} \tag{6.1.2}$$

where we use the indices G_1, G_2 run on $\mathrm{G} = SU(3)\times U(1)$, α, β are flat indices in G/H and μ, ν are curved indices in G/H ($\mathrm{H} = U(1)\times U(1)$). We can calculate the Ricci tensor $R_{\mu\nu}$ by using these definitions.

In order to obtain Einstein space, we need four constraint equations for the parameters. However these constraints do not determine the symmetry of this space uniquely. We also add a necessary and sufficient condition for supersymmetry. With the supersymmetry condition we obtained the $N_{\mathrm{I}}^{0,1,0}$ as $\mathcal{N}=3$ supersymmetric solution, and $N_{\mathrm{I}}^{p,q,r}$ ($r\neq 0$) as $\mathcal{N}=1$ solutions. The case $r=0$ corresponds the coset space

$$\frac{\mathrm{G}}{\mathrm{H}} = \frac{SU(3)}{U(1)} \tag{6.1.3}$$

which insist that the base space is $\mathbb{C}P^2$. And also there is $SO(3)$ isometry group as the result of $\mathcal{N}=3$ supersymmetry [81]. So the $N_{\mathrm{I}}^{0,1,0}$ solution must have the isometry group $SU(3)\times SO(3)$.

To obtain explicit form of $N_{\mathrm{I}}^{0,1,0}$ solution, let us consider the $SU(2)$ Yang-Mills instanton over $\mathbb{C}P^2$ which is an indication of isometry group $SO(3)$ [21]. The $SU(2)$ Yang-Mills instanton can be regarded as the Hopf fiber bundle over $\mathbb{C}P^2$ space by generalized inverse Kaluza-Klein mechanism.

$$\begin{aligned}ds^2 &= ds^2_{\mathbb{C}P^2} + \lambda^2\left[(\rho_1^2 - A^1)^2 + (\rho_2 - A^2)^2 + (\rho_3 - A^3)^2\right],\\ ds^2_{\mathbb{C}P^2} &= d\theta^2 + \frac{1}{4}\sin^2\theta\left(\sigma_1^2 + \sigma_2^2 + \cos^2\theta\sigma_3^2\right)\end{aligned} \tag{6.1.4}$$

where the σ_i, ρ_i ($i=1,2,3$) are the $SU(2)$ Mauer-Cartan 1-forms which satisfy the $SU(2)$ algebras $d\sigma_1 = -\sigma_2\wedge\sigma_3$ etc. The explicite forms of σ_i are

$$\begin{aligned}\sigma_1 &= \cos\gamma\, d\alpha + \sin\gamma\,\sin\alpha\, d\beta,\\ \sigma_2 &= -\sin\gamma\, d\alpha + \cos\gamma\sin\alpha\, d\beta,\\ \sigma_3 &= d\gamma + \cos\alpha\, d\beta.\end{aligned} \tag{6.1.5}$$

The self-dual Yang-Mills field on any four-dimensional Einstein space can be defined by [82]

$$\begin{aligned}A^i &= -\omega_{0i} - \frac{1}{2}\epsilon_{ijk}\omega_{jk}\\ F^i &= dA^i + \frac{1}{2}\epsilon_{ijk}A^j\wedge A^k\end{aligned} \tag{6.1.6}$$

where ω_{ij} is the 1-form spin connection for four dimensional Einstein space. Applying this to $\mathbb{C}P^2$, we obtain

$$A^1 = \cos\theta\sigma_1, \quad A^2 = \cos\theta\sigma_2, \quad A^3 = \frac{1}{2}(1+\cos^2\theta)\sigma_3 \tag{6.1.7}$$

as a $SU(2)$ Yang-Mills instanton over $\mathbb{C}P^2$. Plugging this solution into (6.1.4), we get

$$\begin{aligned} ds^2 &= ds^2_{\mathbb{C}P^2} + \lambda^2 ds^2_{SU(2)}, \\ ds^2_{SU(2)} &= (\rho_1 - \cos\theta\sigma_1)^2 + (\rho_2 - \cos\theta\sigma_2)^2 + \left(\rho_3 - \frac{1}{2}(1+\cos^2\theta)\sigma_3\right)^2. \end{aligned} \tag{6.1.8}$$

There still remains an unknown constant λ in (6.1.8). If we the solution (6.1.8) to be Einstein solution, the values of λ are allowed to be

$$\lambda^2 = \frac{1}{2} \quad \text{or} \quad \lambda^2 = \frac{1}{10}. \tag{6.1.9}$$

The original $N^{0,1,0}_{\rm I}$ by Castellani and Romans is (6.1.8) with $\lambda^2 = 1/2$. The solution with $\lambda^2 = 1/10$ is a squashed $SU(3) \times SU(2)$ solution obtained by Page and Pope [21] and is named $N^{0,1,0}_{\rm II}$. There are two orientations which describe the different solutions each other. The definition of the orientation of these manifolds is the overall sign of the vielbeins E^α_μ. The positive-sign defines a left-orientation and negative-sign does a right orientation. $N^{0,1,0}_{\rm I}$ has $\mathcal{N} = 0$ supersymmetry for a left-orientation and $\mathcal{N} = 3$ for a right-orientation. $N^{0,1,0}_{\rm II}$ keep the supersymmetry $\mathcal{N} = 1$ for a left-orientation but break all supersymmetry for a right-orientation.

6.2 $SU(3) \times SU(2)$ squashed solutions

We investigate the 11-dimensional supergravity theory described by the action

$$S = \frac{1}{16\pi G}\int d^{11}x\sqrt{-g}\left(R - \frac{1}{2\cdot 4!}F_4^2\right) - \frac{1}{16\pi G}\frac{\sqrt{2}}{6}\int F_4 \wedge F_4 \wedge C_3, \tag{6.2.1}$$

where C_3 is a three-form gauge field and F_4 is its field strength. For simplicity we consider the vacuum (non-charged) solution of the above action.

$$R_{\mu\nu} = 0. \tag{6.2.2}$$

In order to get the 2-brane solution with the $SU(3) \times SU(2)$ squashed geometry, we consider the ansatz;

$$ds^2 = -dt^2 + \sum_{i=1}^{2} dx_i^2 + r(k)^2\left[u(k)^2dk^2 + kds^2_{\mathbb{C}P^2} + ds^2_{SU(2)}\right] \tag{6.2.3}$$

where the $SU(2)$ metric is a metric on the manifold of an $SO(3)$ Hopf fiber bundle over $\mathbb{C}P^2$ space as we discussed in Sec. 6.1. The explicit form of $\mathbb{C}P^2$ is in (6.1.4) and $SU(2)$

fiber bundle is described as in (6.1.8). $g(k)$ and $u(k)$ are the unknown functions of k This ansatz is the analogy of Ishihara-Matsuno solution which is the 5-dimensional squashing solutions [28]. However we rewrite the degree of freedom of the radial coordinate r as k which is the squashing function in front of the $\mathbb{C}P^2$ metric.

The solutions which satisfy (6.2.2) are (6.2.3) with,

$$u(k) = \frac{1}{|k-10|}, \quad r(k) = \frac{c}{|k-10|^{\frac{3}{10}} k^{\frac{1}{5}}} \tag{6.2.4}$$

where c is a arbitrary constants. There are 2-types of solutions, one exists $10 \leq k$ region and the other exists $0 < k \leq 10$. We investigate the property of this solutions in the following sections.

6.3 The solution for $k \geq 10$

In order to see the property easily, we rewrite the metric by using radial coordinate r as

$$ds^2 = -dt^2 + \sum_{i=1}^{2} dx_i^2 + \frac{4k^2}{(k-4)^2} dr^2 + r(k)^2 \left[k ds^2_{\mathbb{C}P^2} + ds^2_{SU(2)} \right]. \tag{6.3.1}$$

Since k cannot be solved as a functional of r analytically, we use k coordinate to interprete the property. Note that this type of radial coordinate is usual if there is no squashing function k and this form of solution is a analogy of Ishihara-Matsuno solutions. This solution can be considered as the 11-dimensional GPS gravitational instanton solution [29, 30].

First we analyze the asymptotic behavior of $r \to \infty$ which corresponds to $k \to 10$ from (6.2.4). Taking the limit of $k \to 10$ we can get the metric of 7-manifold + radial direction as

$$ds_8^2 = \frac{100}{9} \left[dr^2 + \frac{9}{10} r^2 \left(ds^2_{\mathbb{C}P^2} + \frac{1}{10} ds^2_{SU(2)} \right) \right]. \tag{6.3.2}$$

This indicates the known constant squashed Einstein manifold $N_{\mathrm{II}}^{0,1,0}$ solution [21] as we discussed below (6.1.8). Note that if we are willing to get a constant functional space solution or the harmonic function type of solution, the coefficient 9/10 in front of the Einstein 7-manifold metric is important. However for the near-brane case it is not the case.

Another considerable limit is $r \to 0$ which corresponds to $k \to \infty$. The solution goes like

$$g_{rr} \to 4, \quad g_{\mathbb{C}P^2} \to c^2, \quad g_{SU(2)} \to 0 \tag{6.3.3}$$

where we defined that the $g_{\mathbb{C}P^2}$ is the overall metric of $\mathbb{C}P^2$ base space and $g_{SU(2)}$ is that of $SU(2)$ fiber bundle space. When the solution get closer to a point $r = 0$, the fiber direction become to be compactified, it is better to interpret this solution in 8-dimensional

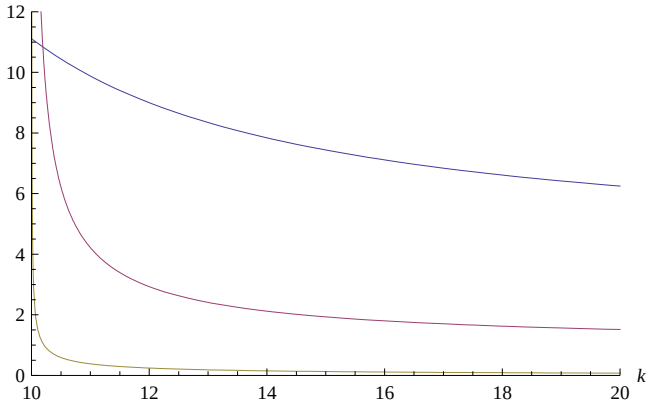

Figure 6.1: The metric for $k \geq 10$ case with $c = 1$. A blue line (top) describe g_{rr}, a purple one (middle) is $g_{\mathbb{C}P^2}$ and a yellow one (down) is $g_{SU(2)}$. The behavior of r is square root of $g_{SU(2)}$.

(super)gravity language. However the compactified direction is fibered, there is $SU(2)$ instanton solution described as (6.1.7) in 8-dimensional picture by the generalized Kaluza-Klein mechanism.

Note that the transverse directions of this solution can be transformed as the following.

$$\begin{aligned} k &= \frac{10}{1-\left(\frac{m}{R}\right)^{\frac{10}{3}}}, \quad m \equiv \frac{\sqrt{10}}{3}c, \\ ds_8^2 &= \frac{dR^2}{1-\left(\frac{m}{R}\right)^{\frac{10}{3}}} + \frac{9}{10}R^2\left[ds^2_{\mathbb{C}P^2} + \frac{1}{10}\left(1-\left(\frac{m}{R}\right)^{\frac{10}{3}}\right)ds^2_{SU(2)}\right]. \end{aligned} \tag{6.3.4}$$

This form suggests this solution is higher dimensional analog of Eguchi-Hanson space. Unfortunately this solution was already obtained in [83] [3]. However we still continue to investigate this solution.

People might think there exists a bolt type singularity. If we take the form

$$\begin{aligned} \rho^2 &= r^2\left(1-\left(\frac{m}{r}\right)^{\frac{10}{3}}\right), \\ ds^2 &\sim \frac{9}{25}\left(d\rho^2 + \frac{1}{4}\rho^2 ds^2_{SU(2)}\right) \end{aligned} \tag{6.3.5}$$

where we choose the variables of $\mathbb{C}P^2$ as constants, we can know there is no curvature singularity. This will be discussed in Sec. 6.5.

Also note that the original gravitational monopole by Sorkin is for the Taub-NUT solution. However Gross and Perry generalized this solution to have the various structures which includes Eguchi-Hanson space. The GPS monopole for Eguchi-Hanson space can

[3] I notice this thing when I have almost finished a paper. A region $k \leq 10$ of (6.2.3) with (6.2.4) is new and a investigation of curvature singularity by using Kretschmann scalar in section 6.5 is also new, but I gave up to publish because there are too little new things.

be written as

$$ds^2 = -dt^2 + \frac{dr^2}{1-\left(\frac{m}{r}\right)^4} + \frac{1}{4}r^2\left[\sigma_1^2+\sigma_2^2+\left(1-\left(\frac{m}{r}\right)^4\right)\sigma_3^2\right]. \tag{6.3.6}$$

Taub-NUT spaces are one-point source solutions in terms of Gibbons-Hawking coordinate (2.3.37), p.29. On the other hand, Eguchi-Hanson spaces are two-points source solutions.

To summarize this section, we get the new type of squashed coset solution which can be regarded as the GPS $SU(2)$ gravitational instanton solution and has the asymptotic structure as the Page-Pope $N_{\rm II}^{0,1,0}$ solution. The hole behavior of this solution is described as in Fig. 6.1.

6.4 $SO(5)\times SU(2)$ case

The case for $SO(5)\times SU(2)$ isometry as squashed $\mathbb{S}^7$ was obtained in [84] [4]. The isometry symmetry of $SO(5)\times SU(2)$ can be constructed by using the Fubini-study [19].

The Fubini-study can be written as

$$ds^2 = \left(1+\sum_n q_n\bar{q}_n\right)^{-1}\sum_m dq_m d\bar{q}_m - \left(1+\sum_n q_n\bar{q}_n\right)^{-2}\sum_{m,p}\bar{q}_m dq_m d\bar{q}_p q_p, \tag{6.4.1}$$

where m,p run $1,2$, and q_m can be taken a form

$$q_1 = \tan\chi\cos\frac{\mu}{2}U, \quad q_2 = \tan\chi\sin\frac{\mu}{2}V. \tag{6.4.2}$$

In this notation if we choose $q_{1,2}$ as complex coordinates, we can write U,V which satisfy

$$U^{-1}dU = i(d\psi+d\phi), \quad V^{-1}dV = i(d\psi-d\phi). \tag{6.4.3}$$

This precisely represents $P_2(C)$, or $\mathbb{R}^4$ with $\mathbb{S}^3$ described by $U(1)$ fiber bundle over $\mathbb{S}^2$.

In analogy with complex coordinates, we can generalize the Fubini-study to apply quaternionic coordinates which are defined by

$$U^{-1}dU = i\sigma_1+j\sigma_2+k\sigma_3, \quad V^{-1}dV = i\Sigma_1+j\Sigma_2+k\Sigma_3, \tag{6.4.4}$$

where we use σ_i,Σ_i as $SU(2)$ Maurer-Cartan 1-forms. Together with the quaternionic coordinates, we can obtain $P_2(H)$ space, or $\mathbb{R}^8$ with $\mathbb{S}^7$ described by $SU(2)$ fiber bundle over $\mathbb{S}^4$.

If we rewrite the solution of $SO(5)\times SU(2)$ case in terms of squashing function k,

$$\begin{aligned} ds_8^2 &= r(k)^2\left[u(k)^2dk^2 + k ds^2_{\mathbb{S}^4} + ds^2_{SU(2)}\right], \\ u(k) &= \frac{1}{|k-5|}, \quad r(k) = \frac{c}{|k-5|^{\frac{3}{10}}k^{\frac{1}{5}}} \end{aligned} \tag{6.4.5}$$

This solution has the additional region for $k\leq 5$. The $SU(3)\times SU(2)$ squashing solutions have quite similar forms of $SO(5)\times SU(2)$ case but the constant 10 was different from 5 in $SO(5)\times SU(2)$ case. This difference may come from the base space symmetry.

[4]This solution was also obtained by Mizoguchi-Hatsuda-Sumitomo-Tomizawa. However during we write a paper, we notice this solution was obtained by [84]. Therefore we gave up also to publish this paper.

6.5 Regularity

Let us consider whether the solutions (6.2.3) with (6.2.4) is regular or not in this section. These solution is regular at the level of Ricci scalar because of vacuum solution. However in order to search the curvature singularity, we should also consider the Kretschmann scalar

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \tag{6.5.1}$$

and its divergence points.

There are four ambiguous points which are $k = 0, 4, 10, \infty$. The subtlety points of $k = 0, 10$ come from the solution $r(k)$ (6.2.4). $k = 4$ point comes from the rewritten metric (6.3.1) at which g_{rr} diverges. Also we should investigate the $k = \infty$ point because of the metric ansatz (6.2.3). There are difficulties to write down explicitly for all k range, but we can do that at these specialized points. The result is as follows.

$$\begin{aligned} k \to \infty : &\quad R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{576}{c^4} \\ k \to 10 : &\quad R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = 0 \\ k \to 4 : &\quad R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{2^{4/5}3^{1/5}378}{c^4} \\ k \to 0 : &\quad R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \to \infty. \end{aligned} \tag{6.5.2}$$

Thus there is singularity only at $k = 0$ point, but the other points have no singularity. From this conclusion our solution considered in Sec. 6.3 is regular everywhere. This fact agree with the discussion around (6.3.5). In the next section we also consider the remaining region $k \leq 10$ although there exists the curvature singularity.

6.6 The solution for $k \leq 10$

First we analyze the solution in $4 \leq k \leq 10$ range by using the same radial reinterpretation as (6.3.1). In this region the correspond r behavior is

$$\frac{c^4}{2^{7/10}3^{3/10}} \leq r \leq \infty \quad \text{as varying } 4 \leq k \leq 10. \tag{6.6.1}$$

The metric component is described by the left hand side of Fig. 6.2. $k \to 10$ point is the same as the Page-Pope squashed solution we described in (6.3.2). On the other side $k = 4$, the metric g_{rr} diverges but this is a spurious divergence as we investigated in Sec. 6.5.

There is another range $0 \leq k \leq 4$ of this notation described by the right hand side of Fig. 6.2. In this regime, the r behavior is the opposite of (6.6.1). At the $k = 0$ point, $g_{rr}, g_{\mathbb{C}P^2}$ shrink to zero and $g_{SU(2)}$ diverges. This cannot be treated as the physical geometry, and this agree with the curvature singularity argument (6.5.2).

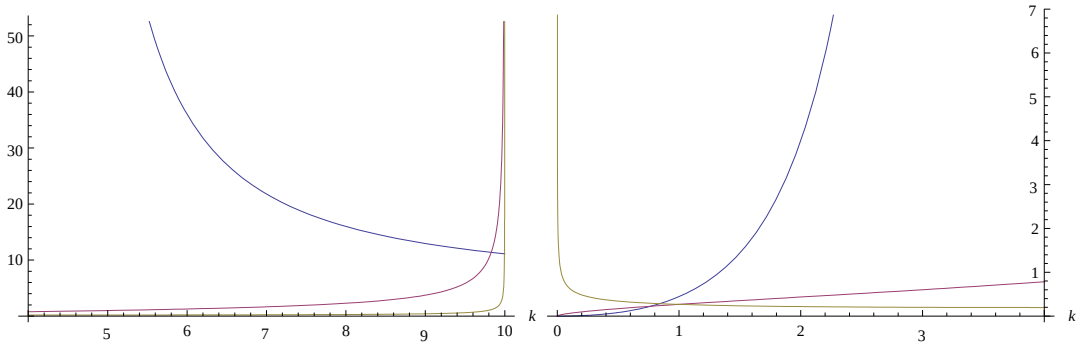

Figure 6.2: The left figure shows the metric for $4 \leq k \leq 10$ with $c = 1$ and the right figure shows the metric for $0 \leq k \leq 4$ with $c = 1$. A blue line (top) describe g_{rr}, a purple one (middle) is $g_{\mathbb{C}P^2}$ and a yellow one (down) is $g_{SU(2)}$. The behavior of r is square root of $g_{SU(2)}$.

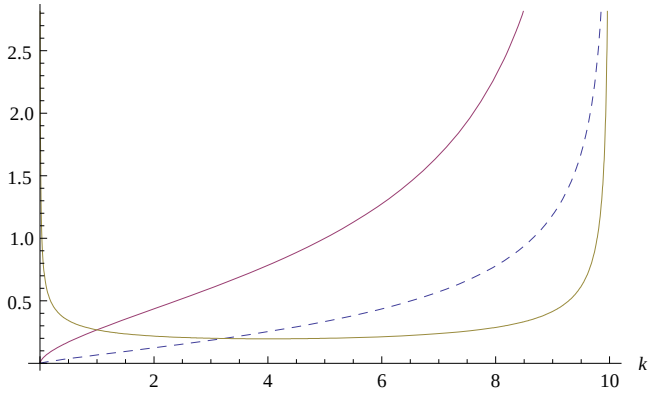

Figure 6.3: The metric in the description of new radial coordinate r' for $0 \leq k \leq 10$ with $c = 1$. A purple one (top) is $g_{\mathbb{C}P^2}$ and a yellow one (down) is $g_{SU(2)}$. The behavior of r' is described by a dashed line (blue, middle).

To avoid the spurious divergence but still define the radial coordinates, we rewrite again the initial metric (6.2.3) as

$$
\begin{aligned}
ds^2 &= -dt^2 + \sum_{i=1}^{2} dx_i^2 + dr'^2 + kr(k)^2 ds_{\mathbb{C}P^2} + r(k)^2 ds_{SU(2)} \\
r'(k) &\equiv \int^k r(t)u(t)dt \\
&= -\frac{ck^{4/5}}{48(k-10)}\left(16(10-k)^{7/10} + 10^{7/10}(k-10)\ {}_2F_1\left(\frac{4}{5}, \frac{3}{10}; \frac{9}{5}, \frac{k}{10}\right)\right). \qquad (6.6.2)
\end{aligned}
$$

where ${}_2F_1(a, b; c, z)$ is a hypergeometric function. As we can see in Fig. 6.3, r' is smoothly drown from $r' = 0$ to ∞. Of course there is a trouble at $r' = 0$ point which has the curvature singularity.

Chapter 7

Conclusions

In this paper, we discussed how to obtain Lorentzian Bagger-Lambert-Gustavsson (L-BLG) model from the Aharony-Bergman-Jafferis-Maldacena (ABJM) model. If we take the scaling limit correctly, L-BLG model is appeared from the ABJM with constraint equations. More to say, with the scaling limit, we have obtained $SO(8)_R$ from $SU(4)_R$. The scaling limit agrees with Inönü-Wigner contraction which can be realized only in group structure. By taking the scaling limit of bifundamental gauge group, then we obtained the correct gauge group of L-BLG model. Since there is a mystery to obtain $SO(8)_R$ in ABJM model, this fact should be a little bit surprising.

We also investigated the conformal symmetries of the ABJM model and L-BLG model as well as $SO(8)$ invariance . The conformal invariance, in particular, the invariance under the special conformal transformations does hold in the L-BLG model only when we consider a set of spacetime dependent solutions to the constraint equations $\partial^2 X_0^I = 0$. The conformal symmetries in the field theories are consistent with the gravity duals; $AdS_4 \times \mathbb{S}^7/\mathbb{Z}_k$ geometry for the ABJM model and $AdS_4 \times \mathbb{C}P^3$ geometry for the L-BLG.

Although the radius of AdS_4 is larger than the $d = 10$ Planck length and the type IIA supergravity approximation is good, it is much smaller than the IIA string scale and the dual geometry of the scaled theory of L-BLG cannot be interpreted as the low energy effective theory of type IIA superstring. But the radius is larger than the $d = 11$ Planck length and it can be considered as a dimensional reduction of the $d = 11$ supergravity solution. We discussed that the action of the L-BLG model could be considered as the probe M2-branes in the curved geometry $AdS_4 \times \mathbb{C}P^3$. It is amusing and also somewhat surprising that the position dependent coefficients of the coupling constant can be correctly reproduced; g_{YM}^2 is proportional to a square of the position of the M2 branes. This fact is consistent to the conformal symmetry which is expected from AdS geometry.

Now we know the ABJM model is most generalized form at this time. However the ABJM model has only $SU(4)_R$ symmetry, not $SO(8)_R$ as expected. There is a idea to obtain higher supersymmetry, which is called *Janus configuration* as its original meaning. We can take the coupling constant to be dependent to a extra dimensional coordinate which is a encircled world-volume coordinate in D3-branes. If we consider this configuration with scalars which obeys adjoint representation and one Chern-Simons term, then

we obtain $\mathcal{N} = 4$ supersymmetric Chern-Simons gauge theory [17]. A key-point to obtain higher supersymmetries in ABJM model is bi-fundamental scalars and two CS terms with opposite levels. Together with these things, in sum, the Janus configuration with bi-fundamental scalars and two CS terms expected to have higher supersymmetries than $SU(4)_R$. This should be interesting since the Janus configuration and bi-fundamental representation come from the same setup, the D3-NS5-D5 system.

We also discussed the squashed 11-dimensional solutions with $SU(3) \times SU(2)$ isometry. The solution for $k \geq 10$ is a Eguchi-Hanson type of Gross-Perry-Sorkin like $SU(2)$ gravitational instanton solution which includes the squashed manifold $N_{\rm II}^{0,1,0}$ as an asymptotic behavior. The other side $k \leq 10$ of the solution can be considered, but there is a curvature singularity at $k = 0$.

Recently warped compactification has been considered to obtain the rich structure in four-dimensional theories, with respect to phenomenological and cosmological aspects. Quite recently, there has appeared a interesting paper about this non-compact phenomena [85]. Also see a review article about flux compactifications [86]. The constructions of these noncompact extra-dimensional solutions might have interesting feature in four-dimensions.

The solution we obtained does not have a charge and a mass yet. It is interesting that the generalization of solutions to have a charge [87]. (This is the $SO(5) \times SU(2)$ case, and the $SU(3) \times SU(2)$ case has not been yet.) That solution seems to include the AdS_4 region as a certain limit. And we can also consider the AdS to non-AdS flow, because that solution has the extra dependence in front of radius direction as squashing function which runs smoothly. The solution together with a mass and a charge is also interesting in perspective of rich structures of solution itself.

There will also be an interesting developments to construct the $2 + 1$ dimensional supersymmetric Chern-Simons gauge theory which has $SU(3) \times U(1)$ R-symmetry. This model is expected to have maximally $\mathcal{N} = 3$ supersymmetry. To obtained a dual of squashed manifold $N_{\rm II}^{0,1,0}$, we need to have $\mathcal{N} = 1$ supersymmetry.

Appendix A

The Gamma matrices

The explicit forms of the antisymmetrized products of the 8×8 Γ matrices we have used in (3.3.21) are given as $\Gamma_{IJ} = \mathbb{I}_{2\times 2} \otimes \gamma_{IJ}$ where

$$\gamma_{12} = \begin{pmatrix} i\sigma^2 & & & \\ & -i\sigma^2 & & \\ & & i\sigma^2 & \\ & & & i\sigma^2 \end{pmatrix}, \qquad \gamma_{13} = \begin{pmatrix} & \mathbb{I} & & \\ -\mathbb{I} & & & \\ & & & \sigma^3 \\ & & -\sigma^3 & \end{pmatrix},$$

$$\gamma_{14} = \begin{pmatrix} & i\sigma^2 & & \\ i\sigma^2 & & & \\ & & & \sigma^1 \\ & & -\sigma^1 & \end{pmatrix}, \qquad \gamma_{15} = \begin{pmatrix} & & -\sigma^3 & \\ & & & \mathbb{I} \\ \sigma^3 & & & \\ & -\mathbb{I} & & \end{pmatrix},$$

$$\gamma_{16} = \begin{pmatrix} & & -\sigma^1 & \\ & & & -i\sigma^2 \\ \sigma^1 & & & \\ & -i\sigma^2 & & \end{pmatrix}, \qquad \gamma_{17} = \begin{pmatrix} & & & -\sigma^3 \\ & & -\mathbb{I} & \\ & \mathbb{I} & & \\ \sigma^3 & & & \end{pmatrix},$$

$$\gamma_{18} = \begin{pmatrix} & & & -\sigma^1 \\ & & i\sigma^2 & \\ & i\sigma^2 & & \\ \sigma^1 & & & \end{pmatrix}, \qquad \gamma_{52} = \begin{pmatrix} & & \sigma^1 & \\ & & & -i\sigma^2 \\ -\sigma^1 & & & \\ & -i\sigma^2 & & \end{pmatrix},$$

$$\gamma_{53} = \begin{pmatrix} & & & \mathbb{I} \\ & & \sigma^3 & \\ & -\sigma^3 & & \\ -\mathbb{I} & & & \end{pmatrix}, \qquad \gamma_{54} = \begin{pmatrix} & & & i\sigma^2 \\ & & \sigma^1 & \\ & -\sigma^1 & & \\ i\sigma^2 & & & \end{pmatrix},$$

$$\gamma_{56} = \begin{pmatrix} i\sigma^2 & & & \\ & i\sigma^2 & & \\ & & i\sigma^2 & \\ & & & -i\sigma^2 \end{pmatrix}, \qquad \gamma_{57} = \begin{pmatrix} & \sigma^3 & & \\ -\sigma^3 & & & \\ & & & \mathbb{I} \\ & & -\mathbb{I} & \end{pmatrix},$$

$$\gamma_{58} = \begin{pmatrix} & \sigma^1 & & \\ -\sigma^1 & & & \\ & & & i\sigma^2 \\ & & i\sigma^2 & \end{pmatrix} \tag{A.0.1}$$

and $\mathbb{I}_{2\times 2}$ is a 2×2 identity matrix. We have also defined

$$\Gamma^0 = i\sigma^2 \otimes \mathbb{I}_{8\times 8}. \tag{A.0.2}$$

The $i\sigma^2$ was used to contract the indices of the 2-component spinor χ and it is the 3 dimensional γ^0 matrix (see the Appendix of [9]). $\mathbb{I}_{8\times 8}$ is an 8×8 identity matrix. They satisfy the following consistency relations as $\Gamma_{12}\Gamma_{13} + \Gamma_{13}\Gamma_{12} = -(\Gamma_2\Gamma_3 + \Gamma_3\Gamma_2) = 0$. At this stage, there is an ambiguity to determine the Γ matrices, but the explicit forms of Γ_I are not necessary here. To fix the ambiguity, we need to consider more general vevs of X_0^I.

Appendix B

$U(1)$ part in ABJM model

In scaling the ABJM model to the L-BLG model, we have mainly concerned with the $SU(N) \times SU(N)$ gauge theory. In this appendix we consider the scaling limit of the $U(N) \times U(N)$ ABJM model, especially the effect of the $U(1)$ part. For simplicity we consider the bosonic terms only. In the presence of the $U(1)$ gauge field, the covariant derivative is modified to

$$D_\mu Y = \hat{D}_\mu \hat{Y} + 2iB_{0\mu}\hat{Y} + i\{\hat{B}_\mu, \hat{Y}\} + \partial_\mu Y_0 + 2i\hat{B}_\mu Y_0 + 2iB_{0\mu}Y_0, \tag{B.0.1}$$

where $B_{0\mu}$ is the axial combination of the $U(1) \times U(1)$ gauge field

$$B_{0\mu} = \frac{1}{2}(A_\mu^{(L)} - A_\mu^{(R)}). \tag{B.0.2}$$

The gauge field $B_{0\mu}$ is associated with the gauge transformation of the complex field $Y^A \to e^{i\varphi}Y^A$. Hence if the dual geometry is described by $\mathbf{C}^4/U(1)$, we need the gauge symmetry even after the scaling to L-BLG. Therefore, we do not scale the $B_{0\mu}$ field unlike B_μ. The scaling is given by

$$\hat{B}_\mu \to \lambda \hat{B}_\mu, \qquad Y_0 \to \lambda^{-1} Y_0, \qquad B_{0\mu} \to B_{0\mu} \tag{B.0.3}$$

and take the limit $\lambda \to 0$. The kinetic term of the scalar fields becomes

$$\begin{aligned} -\frac{1}{2}\mathrm{tr}|D_\mu Y_A|^2 = \mathrm{tr}\Bigg[& -\frac{1}{2}(\hat{D}_\mu \hat{Y}_A + 2i\hat{B}_\mu Y_{0A} + 2iB_{0\mu}\hat{Y}_A)^\dagger (\hat{D}^\mu \hat{Y}^A + 2i\hat{B}^\mu Y_0^A + 2iB_0^\mu \hat{Y}^A) \\ & - \frac{(\partial_\mu Y_{0A} + 2iB_{0\mu}Y_{0A})^\dagger (\partial^\mu Y_0^A + 2iB_0^\mu Y_0^A)}{2\lambda^2} \\ & -i(\partial_\mu Y_{0A} + 2iB_{0\mu}Y_{0A})^\dagger \hat{B}^\mu \hat{Y}^A + i(\partial_\mu Y_0^A + 2iB_{0\mu}Y_0^A)\hat{B}^\mu \hat{Y}_A^\dagger \Bigg]. \end{aligned} \tag{B.0.4}$$

The difference from the $SU(N) \times SU(N)$ case is that all the derivative is replaced by the covariant derivative with respect to $B_{0\mu}$. Requiring finiteness of the action, one can obtain the modified constraint

$$D_{U(1)}^2 Y_0^A \equiv (\partial_\mu + 2iB_{0\mu})(\partial^\mu + 2iB_0^\mu)Y_0^A = 0. \tag{B.0.5}$$

The gauge field $B_{0\mu}$ does not have a kinetic term and it is nothing but the auxiliary gauge field A_μ introduced in the $\mathbf{C}^4/U(1)$ gauged model discussed in Appendix C.

In the presence of the vector-like $U(1)$ gauge field

$$A_{0\mu} = \frac{1}{2}(A_\mu^{(L)} + A_\mu^{(R)}), \tag{B.0.6}$$

there is a coupling of $B_{0\mu}$ to $A_{0\mu}$ through the Chern-Simons term. If we do not scale the $A_{0\mu}$ either, it is given by

$$4\lambda^{-1}K\epsilon^{\mu\nu\rho}\mathrm{tr}B_{0\mu}F_{0\nu\rho}, \tag{B.0.7}$$

where $F_{0\mu\nu} = \partial_\mu A_{0\nu} - \partial_\nu A_{0\mu}$. Then because of the λ^{-1} coefficient this must vanish too.

If we instead scale the $A_{0\mu}$ gauge field with λ, the coefficient becomes of the order λ^0, and an integration over $B_{0\mu}$ solves it as

$$2B_{0\mu}^{(0)} = -\frac{i}{2|Y_0^A|^2}(Y_0^A\partial_\mu\bar{\hat{Y}}^A - \bar{Y_0}^A\partial_\mu\hat{Y}^A) - 2K\epsilon_{\mu\nu\rho}F_0^{\nu\rho}. \tag{B.0.8}$$

Appendix C

$SO(8)$ recovery in $\mathbb{C}^4/U(1)$ model

In Section 4.2.2 we showed the recovery of $SO(8)$ invariance in the scaling limit of $AdS_4 \times \mathbb{C}P^3$. In this appendix, we study a $\mathbb{C}^4/U(1)$ sigma model and see the recovery of $SO(8)$. This is a generalization of the equivalence of a gauged model on $\mathbb{C}P^1$ and an $O(3)$ nonlinear σ model to a higher dimensional target space.

$\mathbb{C}^4$ is parameterized by the following angular variables:

$$\begin{aligned} z^1 &= \rho e^{i(\phi_1+\varphi')}\cos\theta, \\ z^2 &= \rho e^{i(\phi_2+\varphi')}\sin\theta\cos\psi, \\ z^3 &= \rho e^{i(\phi_3+\varphi')}\sin\theta\sin\psi\cos\chi, \\ z^4 &= \rho e^{i\varphi'}\sin\theta\sin\psi\sin\chi, \\ 0 &\le \varphi' \le 2\pi, \quad 0 \le \theta, \psi, \chi, \phi_1, \phi_2, \phi_3 \le \pi. \end{aligned} \tag{C.0.1}$$

We then consider a scalar field on $\mathbb{C}^4/U(1)$ by identifying

$$z_i \sim e^{i\varphi'} z_i. \tag{C.0.2}$$

The Lagrangian of the scalar field $Z_i(x)$ on $\mathbb{C}^4/U(1)$ must be invariant under the local gauge transformation

$$Z_i(x) \to e^{i\varphi'} Z_i(x) \tag{C.0.3}$$

and the action can be written by introducing an auxiliary gauge field A_μ as

$$S = \int d^3x |(\partial_\mu - iA_\mu) Z_A|^2. \tag{C.0.4}$$

In the ABJM model, the gauge field comes from the $U(1)$ part of the axial combination of the two $U(N)$ gauge fields $B_{0\mu}$ (see Appendix B). The gauge field does not have a kinetic term and and it can be eliminated by solving the equation of motion as

$$A_\mu = \frac{i}{2|Z^A|^2}(Z^A \partial_\mu \bar{Z}^A - \bar{Z}^A \partial_\mu Z^A). \tag{C.0.5}$$

By substituting the solution to the action, we obtain a nonlinear action which depends on the Z^A fields only. The action (C.0.4) becomes

$$S = \int d^3x(|\partial Z^A|^2 - A_\mu^2|Z^A|^2). \tag{C.0.6}$$

In the case of $\mathbb{C}P^1$ model, it is well known that the model is nothing but the nonlinear σ-model on S^2. In our case, it is a nonlinear model on $\mathbb{C}^4/U(1)$.

Now we expand the field around a classical background and expand the field as

$$Z^A(x) = Z_0^A + \hat{Z}^A. \tag{C.0.7}$$

The classical background satisfies the equation of motion. Assume that the classical background is very slowly varying and much larger than the fluctuation $\hat{Z}^A$:

$$|Z_0^A| \gg |\hat{Z}^A| \ , \ |dZ_0^A|. \tag{C.0.8}$$

Under the assumption (C.0.8), the quadratic terms of the fluctuations in the action (C.0.6) become

$$S \sim \int d^3x(|\partial \hat{Z}^A|^2 - A_\mu^{(0)2}|Z_0^A|^2) \tag{C.0.9}$$

where

$$A_\mu^{(0)} = \frac{i}{2|Z_0^A|^2}(Z_0^A \partial_\mu \bar{\hat{Z}}^A - \bar{Z}_0{}^A \partial_\mu \hat{Z}^A). \tag{C.0.10}$$

If we decompose the complex fields into real components as

$$\begin{aligned} Z_0^A &= X_0^{2A-1} + iX_0^{2A} \\ \hat{Z}^A &= i\hat{X}^{2A-1} - \hat{X}^{2A}, \end{aligned} \tag{C.0.11}$$

the gauge field can be written as

$$A_\mu^{(0)} = \frac{1}{(X_0^I)^2} X_0^I \partial_\mu \hat{X}^I. \tag{C.0.12}$$

Thus the action can be written as a manifestly $SO(8)$ covariant expression:

$$S = \int d^3x\{(\partial \hat{X}^I)^2 - \frac{1}{X_0^2}(X_0^I \partial \hat{X}^I)^2\}. \tag{C.0.13}$$

In terms of the projected scalar field

$$P^I = \hat{X}^I - \frac{X_0^I X_0^J \hat{X}^J}{(X_0^I)^2}, \tag{C.0.14}$$

the action is written (under the assumption (C.0.8))

$$S = \int d^3x(\partial_\mu P^I)^2. \tag{C.0.15}$$

It is manifestly invariant under the $SO(8)$ transformations. But note that the $SO(8)$ transformation is different from the $SO(8)$ acting on the original $\mathbf{R}^8$ because of the different decompositions of the complex fields into the real components in (C.0.11).

Appendix D

Ordinary reduction of M2 to D2

In this appendix, we remind the reader of the ordinary reduction of M2 branes in $d = 11$ supergravity to D2 branes in $d = 10$ type IIA supergravity to clarify the difference from the reduction adopted in the ABJM model. By compactifying x_{11} direction and identifying $x_{11} \sim x_{11} + 2\pi R_{11}$ the M2 brane solution is given by replacing the metric (4.2.1) with a smeared harmonic function [88]

$$H(r) = \sum_{n=-\infty}^{\infty} \frac{R^6}{(r^2 + (x_{11} + 2\pi n R_{11})^2)^3}. \tag{D.0.1}$$

where r is the radial distance in the 7 non-compact transverse directions. The string coupling constant is given by $R_{11} = g_s l_s$. Then we can get the solution of the multiple D2-branes in the string frame by using the reduction rule and the Poisson resummation at distance much larger than R_{11}:

$$\begin{aligned} ds_{D2} &= H^{-\frac{1}{2}} \left(\sum_{\mu,\nu=0}^{2} \eta_{\mu\nu} dx^\mu dx^\nu \right) + H^{\frac{1}{2}} \left(dr^2 + d\Omega_6^2 \right), \\ e^\phi &= H^{\frac{1}{4}}, \\ H(r) &= \frac{6\pi^2 g_s N l_s^5}{r^5}. \end{aligned} \tag{D.0.2}$$

It is quite different from (4.2.9). Especially the dilaton is not a constant and the conformal symmetry of the M2 brane geometry is broken; it is no longer AdS_4. The transverse direction is given by the radial direction and S^6, and therefore it has the $SO(7)$ invariance.

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