Cut-elimination and Completeness in Cyclic Proof Systems

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Abstract

Cyclic proof systems, or *circular proof systems*, are proof systems which allow proof figures to contain cycles. Some cyclic proof systems are alternative proof systems for *induction*. Such systems are regarded as the formalisation of *infinite descent*. Since there is no induction rule in such proof systems, the formulas to apply induction need not to be found. This point is an advantage of cyclic proof systems for proof search. Since every cyclic proof is finite, every cyclic proof can be simulated on computers. Therefore, it can be used for automated reasoning. Indeed, cyclic proofs are useful for software verification.

The application of cyclic proof systems has been widely studied. However, more is needed to know about the theoretical property of cyclic proof systems involving predicate logic. The aim of this thesis is to investigate cyclic proof systems from a view point of *proof theory*. More precisely, this thesis focuses on the *cut-elimination property* and the *equivalence between a cyclic proof system and the corresponding ordinary proof system with induction*.

The cut-elimination property of a proof system is the following property: any provable sequent in the system is provable without the cut-rule in the system. The property is fundamental and desirable for proof systems. If the cut-elimination property of a cyclic proof system holds, formulas in each proof can be restricted. Therefore, the cut-elimination property of a cyclic proof system suggests that there is an efficient way for proof search in the cyclic proof system.

Generally, the provable sequent in the system with induction is provable in the corresponding cyclic proof system. However, the converse is not obvious when the systems do not include *Peano Arithmetic*. The known cases where a proof system with induction is equivalent to the corresponding cyclic proof system are those where the systems involve Peano Arithmetic or do not involve predicate logic.

This thesis describes three results and discusses issues around them. The first and second results are about the cut-elimination property of cyclic proof systems. The third results is about the equivalence between a cyclic proof system and the corresponding ordinary proof system with induction.

Firstly, this thesis gives a counterexample to cut-elimination in $\mathtt{CLKID}^{\omega}$, a cyclic proof system for first-order logic with inductive definitions. In other words, this thesis shows that the cut-elimination property of $\mathtt{CLKID}^{\omega}$ does not hold. It had been an open problem for 15 years whether or not the cut-elimination property of $\mathtt{CLKID}^{\omega}$ holds. The counterexample is the sequent representing that an addition predicate implies the other addition predicate with a different definition. In order to show that it is not cut-free provable, under the assumption that it is cut-free provable, an infinite sequence of nodes in a finite proof figure is constructed, which leads to a contradiction.

Secondly, this thesis gives a simpler counterexample to cut-elimination in $\mathtt{CLKID}^{\omega}$ with only unary predicates. The proof for the simpler counterexample is similar to the first one.

Thirdly, this thesis defines a cyclic proof system for *Presburger Arithmetic*, called *Cyclic Presburger Arithmetic*, and show the equivalence between Presburger Arithmetic and Cyclic Presburger Arithmetic. It is a complete and decidable theory. Since Presburger Arithmetic does not involve Peano Arithmetic, the equivalence between Presburger Arithmetic and Cyclic Presburger Arithmetic was not known. The equivalence is proved by using the completeness of Presburger Arithmetic.

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1 Introduction

1.1 Cyclic proof system

Every proof figure in ordinary proof systems is a finite tree. However, there are proof systems that prevent proof figures from being a finite tree. For example, any proof with the ω-rule is an infinite tree since the assumptions of the ω-rule are infinitely many [22].

A proof containing no infinite path is called a *well-founded proof*. Each proof in ordinary proof systems and the sequent calculus with the ω-rule is well-founded.

On the other hand, a proof containing infinite paths is called a *non-well-founded proof*. A proof figure with infinite paths seems strange since there is no axiom on the paths, and therefore, we may have the conclusion without axioms. Indeed, there is a derivation tree with infinite paths for a contradiction. For this reason, each non-well-founded proof must satisfy the condition of soundness.

Cyclic proof systems, or *circular proof systems*, are one of non-well-founded proof systems which allows any proof figure to contain cycles. Almost all cyclic proof systems are alternatives to proof systems with *induction*; they are regarded as the formalisation of *infinite descent*, a proof technique for propositions that can be proved by induction. Such a proof system is obtained by replacing induction rules with other rules and some conditions for soundness and by allowing proof figures containing cycles. Since there is no induction rule in the proof systems, we do not have to find formulas to apply induction. This point is an advantage of cyclic proof systems for proof search.

Because of the finiteness of each cyclic proof, we can simulate cyclic proofs on computers. Therefore, it can be used for automated reasoning. Indeed, cyclic proofs are useful for software verification, such as verifying properties of concurrent processes [16], termination of pointer programs [5], and decision procedures for symbolic heaps [7, 9, 20, 21].

The application of cyclic proof systems has been widely studied. However, more is needed to know about the property of cyclic proof systems involving predicate logic from the standpoint of proof theory. We aim to investigate cyclic proof systems from this standpoint. More precisely, we have researched the cut-elimination property and the equivalence between a cyclic proof system and the corresponding ordinary proof system with induction. The former clarifies a fundamental property of each cyclic proof system, whereas the latter pertains to the power of each cyclic proof system. Both issues are important from the standpoint of proof theory. We hope that our investigation will not only develop the proof theory for non-well-founded and cyclic proofs but also contributes to the study of automated inductive theorem proving.

1.2 Cut-elimination

The cut-elimination property of a proof system is the following property: if a sequent is provable in the proof system, then the sequent is provable without the cut-rule in the system. The property is fundamental and desirable for proof systems. For example, the cut-elimination theorem for first-order logic immediately implies consistency of the proof system, the subformula property, and Craig's interpolation theorem [8].

Suppose the cut-elimination property of a cyclic proof system holds. In that case, there is no logical rule in which an arbitrary formula can occur as the principal formula, and therefore, we can restrict formulas in each proof to those obtained from the goal sequent. It means there is an efficient way to search for proof in the cyclic proof system.

Despite its importance, it was an open problem whether the cut-elimination property in the cyclic proof system $\mathtt{CLKID}^{\omega}$ for first-order logic with inductive definitions holds. In Conjecture 5.2.4. of [4], Brotherston has conjectured that the cut-elimination property in the system does not hold.

This thesis provides a counterexample to cut-elimination in $\mathtt{CLKID}^{\omega}$. In other words, we show that the conjecture is correct.

The research community, including ourselves, thought that the cut-elimination property might hold if we restrict the language, such as the arity of predicates. However, we show that the cut-elimination property of $\mathtt{CLKID}^{\omega}$ does not hold even if we restrict the arity of predicates to one.

1.3 Equivalence between cyclic proof system and ordinary proof system

A provable sequent in the system with induction is generally provable in the corresponding cyclic proof system. However, whether the converse holds is not obvious when the systems do not include *Peano Arithmetic*.

Brotherston and Simpson [6] conjectured that an ordinary system for first-order logic with inductive definitions, written by $\mathtt{LKID}$, might be equivalent to the corresponding cyclic proof system, written by $\mathtt{CLKID}^{\omega}$. However, Berardi and Tatsuta [3] refuted the conjecture by giving a sequent provable in $\mathtt{CLKID}^{\omega}$ but not in $\mathtt{LKID}$. In other words, they showed that $\mathtt{CLKID}^{\omega}$ is more powerful than $\mathtt{LKID}$.

On the other hand, Berardi and Tatsuta [2] showed that the system obtained by adding Peano Arithmetic to $\mathtt{CLKID}^{\omega}$ is equivalent to that obtained by adding Peano Arithmetic to $\mathtt{LKID}$.

Presburger Arithmetic is a subsystem of Peano Arithmetic obtained by removing multiplication from Peano Arithmetic. It is a complete and decidable theory [11, 19]. We show that Presburger Arithmetic is equivalent to the corresponding cyclic proof system. The equivalence is proved by using the completeness of Presburger Arithmetic.

1.4 Our contributions

In this section, we discuss our contributions. We research cyclic proof systems from the cut-elimination property and the equivalence between a cyclic proof system and the corresponding ordinary proof system with induction. Our first and second contributions are about the cut-elimination property. Our third contribution is about the equivalence between a cyclic proof system and the corresponding ordinary proof system with induction.

1.4.1 Counterexample to cut-elimination in cyclic proof system

We provide a counterexample to cut-elimination in the cyclic proof system $\mathtt{CLKID}^{\omega}$ for first-order logic with inductive definitions.

Our counterexample is the sequent representing that an addition predicate implies another addition predicate with a different definition. It is easy to give a proof of the counterexample with the cut-rule in $\mathtt{CLKID}^{\omega}$. In order to show that it is not cut-free provable, we assume it is cut-free provable for contradiction. Then, we construct an infinite sequence of nodes in the cyclic proof, which contradicts the finiteness of occurring sequents.

1.4.2 The cut-elimination property in cyclic proof system and the arity of predicates

Our counterexample we discuss in the previous section includes ternary predicates. We investigate whether there is a simpler counterexample.

We conjectured the cut-elimination property held if we restricted the arity of predicates to one, but this conjecture is wrong; there is a counterexample with only unary predicates, which we show in this thesis.

1.4.3 Cyclic proof system for Presburger arithmetic

In this thesis, we define a cyclic proof system for Presburger Arithmetic, called Cyclic Presburger Arithmetic, and show that Presburger Arithmetic is equivalent to Cyclic Presburger Arithmetic. Since Presburger Arithmetic does not include Peano Arithmetic, we show the equivalence between Presburger Arithmetic and Cyclic Presburger Arithmetic. The equivalence is proved by the completeness of Presburger Arithmetic.

For the equivalence between Presburger Arithmetic and Cyclic Presburger Arithmetic, the completeness of Presburger Arithmetic seems to be essential. Indeed, we can show the equivalence between a proof system for the theory of successor and order, obtained by removing addition from Presburger Arithmetic, and the corresponding cyclic proof system in the same way as this thesis since the theory is complete [11, 19]. However, in some cases the equivalence holds for incomplete theories, as discussed later in Section 5.5.

1.5 Synopsis

This section outlines the remainder of this thesis.

Chapter 2: We define the syntax and semantics of the language for *first-order logic with inductive definitions* (Section 2.1). Then, we define the derivation tree (Section 2.2) and three proof systems for first-order logic with inductive definitions, LKID (Section 2.3), $\mathtt{LKID}^\omega$ (Section 2.4), and $\mathtt{CLKID}^\omega$ (Section 2.5). They are an ordinary proof system with induction, a non-well-founded infinitary proof system, and a cyclic proof system. At the end of this chapter, we show the property of $\mathtt{CLKID}^\omega$, called the *cycle-normalisation property* (Section 2.6).

Chapter 3: We give a counterexample to cut-elimination in $\mathtt{CLKID}^\omega$. The counterexample is a sequent that says an addition predicate implies the other addition predicate with a different definition. We show that there is a $\mathtt{CLKID}^\omega$-proof of the counterexample with the cut-rule (Section 3.1), We outline the proof (Section 3.2). To show the unprovability, we define $\mathtt{CLKID}^\omega_a$(Section 3.3). After the proof (Section 3.4), we discuss related work and the reason why the cut-elimination property does not hold in some cyclic proof systems.

Chapter 4: We discuss the cut-elimination property and the arity of predicates. First, we provide a simpler counterexample to cut-elimination in $\mathtt{CLKID}^\omega$ than in the previous chapter (Section 4.1). After the proof, we discuss the cut-elimination property of $\mathtt{CLKID}^\omega$and the arity of inductive predicates (Section 4.2).

Chapter 5: We discuss a cyclic proof system for Presburger Arithmetic. First, we define Presburger Arithmetic (Section 5.1) and show its completeness (Section 5.2). Then, we define two non-well-founded proof systems, Infinitary Presburger Arithmetic and Cyclic Presburger Arithmetic (Section 5.3). We show the equivalence of three systems, Presburger Arithmetic, Infinitary Presburger Arithmetic, and Cyclic Presburger Arithmetic (Section 5.4). After the proof, we discuss the equivalence between ordinary and cyclic proof systems and the cut-elimination property of Cyclic Presburger Arithmetic (Section 5.5).

Chapter 6: We conclude (Section 6.1) and give ideas for future work (Section 6.2).

2 Background: inductive definitions, non-well-founded proof system, cyclic proof system

This chapter introduces first-order logic with inductive definitions and describes its three proof systems, `LKID`, $\mathtt{LKID}^\omega$, and $\mathtt{CLKID}^\omega$. These systems are the same as `LKID`, $\mathtt{LKID}^\omega$ and $\mathtt{CLKID}^\omega$ in [4, 6].

`LKID` is an ordinary proof system obtained by adding the rules for induction to the sequent calculus for first-order logic with equality. Someone guesses that the cut-elimination property of this system does not hold because there are rules for induction, but it is wrong. In return for the cut-elimination property, the subformula property of this system does not hold.

$\mathtt{LKID}^\omega$ is a *non-well-founded infinitary proof system*. Each proof figure in this system is a possibly infinite tree where an infinite path can exist. The rules in this system are the same as in `LKID` except for the induction rule. The induction rule is replaced by the *case-split rule*. Since there is an infinite derivation tree of a contradiction, each proof in this system must satisfy the condition for soundness, *the global trace condition*. We note that the cut-elimination property of this system holds.

$\mathtt{CLKID}^\omega$ is a *cyclic proof system*. This system allows any proof figure containing cycles. The rules in this system are the same as in $\mathtt{LKID}^\omega$. We can understand $\mathtt{CLKID}^\omega$ as a subsystem of $\mathtt{LKID}^\omega$obtained by restricting proof figures to regular trees, that is to say, possibly infinite trees, each of which has finitely many subtrees. In Chapter 3, we show that the cut-elimination property of this system does not hold as opposed to $\mathtt{LKID}^\omega$.

Section 2.1 describes the language for first-order logic with inductive definitions. Section 2.2 defines the derivation tree. Section 2.3 gives an ordinary proof system `LKID`. Section 2.4 introduces $\mathtt{LKID}^\omega$. In Section 2.5, we define $\mathtt{CLKID}^\omega$. Section 2.6 shows the cycle-normalisation property for $\mathtt{CLKID}^\omega$.

2.1 Language for first-order logic with inductive definitions

This section defines the language for first-order logic with inductive definitions and its semantics. This language is the same as given in [6].

2.1.1 Syntax

We give the syntax of the language for first-order logic with inductive definitions.

In this thesis, we write $\mathbb{N}$ for the set of natural numbers and $\mathbb{N}_{>0}$ for the set of positive natural numbers.

Definition 2.1 (Ranked alphabet)**.** A *ranked alphabet* of the language for first-order logic with inductive definitions is a tuple $(\Sigma, \#)$ satisfying the following conditions:

(1) Σ denotes a set of symbols including

- an infinite set of *variable symbols* (we assume they are ordered),

- a set of *function symbols*,
- a set of *predicate symbols* constructed by
  - a set of *ordinary predicate symbols* and
  - a set of *inductive predicate symbols*,
- the set of *logical symbols* $\{\neg, \wedge, \vee, \rightarrow, \forall, \exists\}$ and
- the set of parentheses $\{(,)\}$.

(2) The set of inductive predicate symbols in Σ is finite.

(3) The ordinary predicate symbol $=$ belongs to Σ.

(4) $\#$ denotes a function from Σ to $\mathbb{N}$. $\#(a)$ is called the *arity* of the symbol a.

(5) $\#(=) = 2$.

(6) $\#(R) > 0$ for every predicate symbol R in Σ.

As usual, we call a symbol whose arity is $0, 1, 2, 3, \ldots\ldots$ a *nullary symbol*, a *unary symbol*, a *binary symbol*, a *ternary symbol*, $\ldots\ldots$ respectively. A function symbol whose arity is 0 is called a *constant symbol*.

Throughout the remainder of this section, we use “ranked alphabet” as a shorthand for “ranked alphabet of the language for first-order logic with inductive definitions.”

Definition 2.2 (Term)**.** The set of *terms* is defined inductively as follows:

(1) A variable symbols v as a string is a *term*.

(2) If t_1, t_2, ..., t_n are terms for $n \in \mathbb{N}$, then the string $ft_1t_2 \ldots t_n$ is a *term* for a function symbol f with $\#(f) = n$.

$\mathrm{Var}(t)$ denotes the set of variables occurring in t. For a tuple of terms $\mathbf{u}$, $\mathrm{Var}(\mathbf{u})$ denotes the set of variables occurring in $\mathbf{u}$.

We sometimes write $\mathbf{u}(\mathbf{x})$ for a tuple of terms, where $\mathbf{x}$ denotes a tuple of variable symbols and all variable symbols in $\mathrm{Var}(\mathbf{u})$ occur in $\mathbf{x}$.

Definition 2.3 (Atomic formula)**.** An *atomic formula* is defined as a string whose form $t_1 = t_2$ or $Rt_1t_2 \ldots t_n$, where t_1, t_2, ..., t_n denote terms and R denotes a predicate symbol except for $=$ with $\#(R) = n$.

For readability, we sometimes write $R(t_1, \ldots, t_n)$ for the atomic formula $Rt_1 \ldots t_n$. For simplicity, we sometimes write $R\mathbf{t}$ with $\mathbf{t} = (t_1, \ldots, t_n)$ for the atomic formula $Rt_1 \ldots t_n$. We call an atomic formula with an inductive predicate symbol an *I-atomic formula*.

Definition 2.4 (Formulas)**.** The set of *formulas* for first-order logic with inductive definitions is defined inductively as follows:

(1) An atomic formula is a *formula*.

(2) If φ is a formula, the string $\neg\varphi$ is a *formula*.

(3) If φ and ψ are formulas, the three strings $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, and $(\varphi \rightarrow \psi)$ are *formulas*.

(4) If φ is a formula, the two strings $\forall x\varphi$ and $\exists x\varphi$ are *formulas*.

For simplicity, we sometimes write $\varphi \star \psi$ with $\star \in \{\land, \lor, \to\}$ for the formula $(\varphi \star \psi)$. We sometimes abbreviate $(\varphi \to \psi) \land (\psi \to \varphi)$ to $\varphi \leftrightarrow \psi$.

Definition 2.5 (Bound occurrence, free occurrence)**.** Let φ be a formula in which a formula $\mathcal{Q}x\psi$ occurs as a substring with $\mathcal{Q} \in \{\forall, \exists\}$.

We call each occurrence of x in ψ a *bound occurrence*.

Each occurrence of a variable in φ which is not a bound occurrence is called a *free occurrence*. We say that a variable *occurs freely* if the occurrence of it is a free occurrence. We define a *free variable* in φ to be a variable occurring freely in φ. $\mathrm{FV}(\varphi)$ denotes the set of free variables in φ.

We define $\mathrm{FV}(\Gamma) = \bigcup_{\varphi\in\Gamma} \mathrm{FV}(\varphi)$ for a set of formulas Γ.

Definition 2.6 (Substitution of terms)**.** Let t be a term.

The term $t[x := u]$ obtained by substituting a term u for a variable symbol x is defined inductively as follows:

(1) If $t \equiv x$, then $t[x := u] \equiv u$.

(2) If $t \equiv y$ with a variable symbol y, where $y \not\equiv x$, then $t[x := u] \equiv y$.

(3) If $t \equiv ft_1 \dots t_n$ with a function symbol f and terms t_1, …, t_n, then $t[x := u] \equiv ft_1[x := u] \dots t_n[x := u]$.

The term $t[x_1 := u_1, \dots, x_n := u_n]$ obtained by substituting terms $u_1, \dots, u_n$ for variable symbols $x_1, \dots, x_n$ is defined similarly.

$\mathbf{t}[x_1 := u_1, \dots, x_n := u_n]$ with $\mathbf{t} = (t_1, \dots, t_m)$ denotes the tuple $(t_1[x_1 := u_1, \dots, x_n := u_n], \dots, t_m[x_1 := u_1, \dots, x_n := u_n])$. For a tuple of terms $\mathbf{t}(x_1, \dots, x_n)$, we sometimes write $\mathbf{t}(u_1, \dots, u_n)$ for $\mathbf{t}[x_1 := u_1, \dots, x_n := u_n]$.

We call a *substitution* a sequence of expressions whose each form is $x := t$ for a variable symbol x and a term t.

Definition 2.7 (Substitution of formulas)**.** Let φ be a formula.

The formula $\varphi[x := u]$ obtained by substituting a term u for a variable symbol x is defined inductively as follows:

(1) If $\varphi \equiv R\mathbf{t}$ with a predicate symbol R and a tuple of terms $\mathbf{t}$, then $\varphi[x := u] \equiv R\mathbf{t}[x := u]$.

(2) If $\varphi \equiv \neg\psi$ with a formula ψ, then $\varphi[x := u] \equiv \neg\psi[x := u]$.

(3) If $\varphi \equiv (\psi_1 \star \psi_2)$ with formulas ψ_1, ψ_2 and $\star \in \{\land, \lor, \to\}$, then $\varphi[x := u] \equiv (\psi_1[x := u] \star \psi_2[x := u])$.

(4) If $\varphi \equiv \mathcal{Q}y\psi$ with a variable symbol y and a formula ψ, $y \equiv x$ and $\mathcal{Q} \in \{\forall, \exists\}$, then $\varphi[x := u] \equiv \mathcal{Q}y\psi$.

(5) If $\varphi \equiv \mathcal{Q}y\psi$ with a variable symbol y and a formula ψ, where $y \notin \mathrm{Var}(u)$ and $\mathcal{Q} \in \{\forall, \exists\}$, then $\varphi[x := u] \equiv \mathcal{Q}y\psi[x := u]$.

(6) If $\varphi \equiv \mathcal{Q}y\psi$ with a variable symbol y and a formula ψ, where $y \in \mathrm{Var}(u)$, $y \not\equiv x$ and $\mathcal{Q} \in \{\forall, \exists\}$, then $\varphi[x := u] \equiv \mathcal{Q}z\psi[y := z][x := u]$, where z is the first variable symbol in the order for the set of variable symbols does not occur in φ, x, u.

The formula $\varphi[x_1 := u_1, \dots, x_n := u_n]$ obtained by substituting terms $u_1, \dots, u_n$ for variable symbols $x_1, \dots, x_n$ is defined similarly.

For a set of formulas Γ, we write $\Gamma[x_1 := u_1, \dots, x_n := u_n]$ for $\{\varphi[x_1 := u_1, \dots, x_n := u_n] \mid \varphi \in \Gamma\}$.

Definition 2.8 (Inductive definition set)**.** A *production* is defined to be a pair of a finite set of atomic formulas (empty set possibly) and an I-atomic formula.

We sometimes write

$$\frac{Q_1\mathbf{u}_1 \quad \cdots \quad Q_h\mathbf{u}_h \quad P_1\mathbf{t}_1 \quad \cdots \quad P_m\mathbf{t}_m}{P\mathbf{t}}$$

for a production $(\{Q_1\mathbf{u}_1, \ldots, Q_h\mathbf{u}_h, \ldots, P_1\mathbf{t}_1, \ldots, P_m\mathbf{t}_m\}, P\mathbf{t})$.

We call the finite set of atomic formulas of a production the *assumption* of the production. We call the I-atomic formula of a production the *conclusion* of the production. We call a production whose conclusion is an I-atomic formula with an inductive predicate symbol P a *production of* P.

An *inductive definition set* is a finite set of productions.

We defined *a language for first-order logic with inductive definitions* as a pair of a ranked alphabet and an inductive definition set.

For simplicity, with a unary function symbol s, we write $\overbrace{\mathrm{ss}\cdots\mathrm{s}}^{m}x$ for $\mathrm{s}^m x$.

Example 2.9 (Productions for N, E, and O)**.** Let N, E, and O be a unary inductive predicates. Let 0 be a constant symbol, and s be a unary function symbol.

Define the productions of N, E, and O by

$$\frac{}{\mathrm{N}(0)}\ , \qquad \frac{\mathrm{N}(x)}{\mathrm{N}(sx)}\ , \qquad \frac{}{\mathrm{E}(0)}\ , \qquad \frac{\mathrm{E}(x)}{\mathrm{O}(sx)}\ , \qquad \frac{\mathrm{O}(x)}{\mathrm{E}(sx)}\ .$$

N, E, and O intuitively represent the set of natural numbers, even numbers, and odd numbers, respectively.

Definition 2.10 (Sequent)**.** A *sequent* is a pair of finite sets of formulas denoted by $\Gamma \Rightarrow \Delta$, where Γ, Δ are the finite sets of formulas. Γ is called the antecedent of $\Gamma \Rightarrow \Delta$ and Δ is called the succedent of $\Gamma \Rightarrow \Delta$.

2.1.2 Semantics

In this section, we introduce the semantics of the language for first-order logic with inductive definitions. Like the second-order logic, there are at the least two different semantics of the language for first-order logic with inductive definitions, Standard semantics and Henkin semantics.

Complete lattice, fixed point

To introduce the semantics of the language for first-order logic with inductive definitions, we define some concepts and show a theorem.

Definition 2.11 (Complete lattice)**.** For a poset $(P, \leq_P)$ and its subset $S \subseteq P$, we define the supremum and infimum of S as the least upper bound and greatest lower bound of S, respectively.

A complete lattice $(L, \leq_L)$ is defined as a poset, where for any subset $S \subseteq L$ there exist the supremum and infimum of S.

We note that the supremum of $\emptyset$ is the minimum of $(L, \leq_L)$, and the infimum of $\emptyset$ is the maximum of $(L, \leq_L)$. Then, for each complete lattice, there exist its maximum and minimum by the definition. For simplicity, we sometimes write L for a complete lattice $(L, \leq_L)$.

For a complete lattice $(L, \leq_L)$ and a function $f \colon L \to L$, a *pre-fixed point* and a *fixed point of* f are defined as an element $x \in L$, where $f(x) \leq x$ and $f(x) = x$, respectively. *The least fixed point of a function* f is defined as the minimum of $\{x \in L \mid f(x) = x\}$. For posets $(L, \leq_L)$ and $(L', \leq_{L'})$, a function $f \colon L \to L'$ satisfying that $x \leq_L y$ implies $f(x) \leq_{L'} f(y)$ is called a *monotone function*.

We show Knaster–Tarski theorem.

Theorem 2.12 (Knaster–Tarski theorem)**.** *For a complete lattice* $(L, \leq)$ *and a monotone function* $f \colon L \to L$*, there is the least fixed point of* f*.*

Proof. Let $(L, \leq)$ be a complete lattice and $f \colon L \to L$ be a monotone function.

For a set $S \subseteq L$, we write $\bigsqcap S$ for the infimum of S i.e. the greatest lower bound of S. We define a set $\mathbf{PreFix}_f$ by $\mathbf{PreFix}_f = \{x \in L \mid f(x) \leq x\}$. We define $\mathbf{lfp}_f$ by $\mathbf{lfp}_f = \bigsqcap \mathbf{PreFix}_f$.

We show that $f(\mathbf{lfp}_f)$ is a lower bound i.e. $f(\mathbf{lfp}_f) \leq x$ holds for all $x \in \mathbf{PreFix}_f$. Let $x \in \mathbf{PreFix}_f$. By the definition of $\mathbf{lfp}_f$, we have $\mathbf{lfp}_f \leq x$. Since f is a monotone function, we have $f(\mathbf{lfp}_f) \leq f(x)$. By $x \in \mathbf{PreFix}_f$, we have $f(\mathbf{lfp}_f) \leq f(x) \leq x$.

Since $f(\mathbf{lfp}_f)$ is a lower bound, we have $f(\mathbf{lfp}_f) \leq \mathbf{lfp}_f$. Then, $\mathbf{lfp}_f \in \mathbf{PreFix}_f$.

Since $f(\mathbf{lfp}_f) \leq \mathbf{lfp}_f$ and f is a monotone function, we see $f(f(\mathbf{lfp}_f)) \leq f(\mathbf{lfp}_f)$. Hence, $f(\mathbf{lfp}_f) \in \mathbf{PreFix}_f$. Since $f(\mathbf{lfp}_f)$ is the greatest lower bound of $\mathbf{PreFix}_f$, we have $f(\mathbf{lfp}_f) = \mathbf{lfp}_f$. Therefore, $\mathbf{lfp}_f$ is a fixed point of f. Since any fixed point of f belongs to $\mathbf{PreFix}_f$, $\mathbf{lfp}_f$ is the least fixed point of f. □

Standard semantics

We give Standard semantics. In Standard semantics, the interpretation of the inductive predicates is the least fixed point of a monotone operator constructed from the inductive definition set of the considered language.

For simplicity, throughout the remainder of this section, fix $((\Sigma, L, \#), \Phi)$ be a language for first-order logic with inductive definitions with a ranked alphabet $(\Sigma, L, \#)$ and an inductive definition set. Let $P_1, \ldots, P_n$ be all inductive predicates of Σ. Let k_i be the arity of P_i.

We define $\Phi_i = \{\phi \in \Phi \mid \text{The conclusion of } \phi \text{ is an atomic formula with } P_i\}$.

For a set X, we write $\mathcal{P}(X)$ for the power set of X i.e. the set of subsets of X.

Definition 2.13 (First-order structure)**.** We define a *first-order structure* as a pair of non-emptyset $|M|$ and a function M satisfying the following conditions:

(1) For a constant symbol c, $M(c) \in |M|$.

(2) For a natural number $n > 0$ and an n-ary function symbol f, $M(f)$ is a function whose domain is $|M|^n$ and range is $|M|$.

(3) For a natural number $n > 0$ and an n-ary predicate symbol R which is not $=$, $M(R)$ is a subset of $|M|^n$.

(4) $M(=)$ is the set $\{(m, m) \mid m \in |M|\}$.

For a first-order structure $(|M|, M)$, $|M|$ is called the underlying set of $(|M|, M)$. For a symbol C and a first-order structure $(|M|, M)$, $M(C)$ is called the *interpretation of* C *in* $(|M|, M)$.

We define a *valuation* on a first-order structure $(|M|, M)$ as a function from the set of variable symbols to the underlying set of $(|M|, M)$.

For a term t, a first-order structure $(|M|, M)$, and a valuation ρ on a first-order structure $(|M|, M)$, we inductively define *the interpretation of t in $(|M|, M)$ with ρ* as follows:

(1) If t is a constant symbol, the interpretation of t in $(|M|, M)$ with ρ is $M(t)$.

(2) If t is a variable symbol, the interpretation of t in $(|M|, M)$ with ρ is $\rho(t)$.

(3) If t is the form $ft_1 \dots t_n$ with an n-ary function symbol f and terms $t_1, \dots, t_n$, and d_i is the interpretation of t_i in $(|M|, M)$ with ρ for each $i = 0, \dots, n$, then the interpretation of t in $(|M|, M)$ with ρ is $M(f)(d_1, \dots, d_n)$.

For a first-order structure M and a predicate symbol Q, we write Q^M for the interpretation of Q in M. For a first-order structure M, a valuation ρ and a term t, we write t^M_ρ for the interpretation of t in M with a valuation ρ. For a first-order structure M, a valuation ρ and a tuple of terms $\mathbf{t} = (t_1, \dots, t_m)$, we write $\mathbf{t}^M_\rho$ for $\left({t_1}^M_\rho, \dots, {t_m}^M_\rho\right)$. For a function ρ, we write $\rho[x \mapsto d]$ for the function which maps x to d and y to $\rho(y)$ with $y \neq x$.

Definition 2.14 ($\models_\rho$)**.** For a first-order structure M, a valuation ρ on M, and a formula φ, we inductively define the ternary relation $M \models_\rho \varphi$ as follows:

(1) $M \models_\rho t_1 = t_2$ holds if and only if ${\mathbf{t_1}}^M_\rho = {\mathbf{t_2}}^M_\rho$.

(2) $M \models_\rho Rt_1 \dots t_m$ holds if and only if $\left({\mathbf{t_1}}^M_\rho, {\mathbf{t_2}}^M_\rho\right) \in R^M$.

(3) $M \models_\rho \neg\psi$ holds if and only if $M \models_\rho \psi$ does not hold.

(4) $M \models_\rho \psi_1 \wedge \psi_2$ holds if and only if both $M \models_\rho \psi_1$ and $M \models_\rho \psi_2$ hold.

(5) $M \models_\rho \psi_1 \vee \psi_2$ holds if and only if either $M \models_\rho \psi_1$ or $M \models_\rho \psi_2$ holds.

(6) $M \models_\rho \psi_1 \to \psi_2$ holds if and only if either $M \models_\rho \psi_1$ does not hold or $M \models_\rho \psi_2$ holds.

(7) $M \models_\rho \forall x\psi$ holds if and only if $M \models_{\rho[x\mapsto d]} \psi$ holds for all elements $d \in |M|$, where $|M|$ is the underlying set of M.

(8) $M \models_\rho \exists x\psi$ holds if and only if $M \models_{\rho[x\mapsto d]} \psi$ holds for some elements $d \in |M|$, where $|M|$ is the underlying set of M.

Definition 2.15 (Definition set operator)**.** Let M be a first-order structure with the domain D. For a production π

$$\frac{Q_1\mathbf{u}_1 \quad \cdots \quad Q_h\mathbf{u}_h \quad P_{j_1}\mathbf{t}_1 \quad \cdots \quad P_{j_m}\mathbf{t}_m}{P_i\mathbf{t}} \ ,$$

we define a function $\varphi_\pi \colon \mathcal{P}\left(D^{k_1}\right) \times \cdots \times \mathcal{P}\left(D^{k_n}\right) \to \mathcal{P}\left(D^{k_i}\right)$ by

$$\varphi_\pi(X_1, \dots, X_n) = \left\{\mathbf{t}^M_\rho \;\middle|\; {\mathbf{u_1}}^M_\rho \in {Q_1}^M, \dots, {\mathbf{u_h}}^M_\rho \in {Q_h}^M, {\mathbf{t_1}}^M_\rho \in X_{j_1}, \dots {\mathbf{t_m}}^M_\rho \in X_{j_m}, \rho \text{ is a valuation}\right\}.$$

Then, for each $i = 1, \dots, n$, we define a function $\varphi_i \colon \mathcal{P}\left(D^{k_1}\right) \times \cdots \times \mathcal{P}\left(D^{k_n}\right) \to \mathcal{P}\left(D^{k_i}\right)$ by

$$\varphi_i(X_1, \dots, X_n) = \bigcup_{\pi \in \Phi_i} \varphi_\pi(X_1, \dots, X_n).$$

We define the *definition set operator* $\varphi_\Phi \colon \mathcal{P}\left(D^{k_1}\right) \times \cdots \times \mathcal{P}\left(D^{k_n}\right) \to \mathcal{P}\left(D^{k_1}\right) \times \cdots \times \mathcal{P}\left(D^{k_n}\right)$ by

$$\varphi_\Phi(X_1, \dots, X_n) = (\varphi_1(X_1, \dots, X_n), \dots, \varphi_n(X_1, \dots, X_n)).$$

We write $(A_1, \dots, A_n) \subseteq (B_1, \dots, B_n)$ for $A_i \subseteq B_i$ for each $i = 1, \dots, n$.

Then, we note that $\left(\mathcal{P}\left(D^{k_1}\right) \times \dots \times \mathcal{P}\left(D^{k_n}\right), \subseteq\right)$ is a complete lattice, and φ_Φ is a monotone function on $\subseteq$ i.e. $\varphi_\Phi(X_1, \dots, X_n) \subseteq \varphi_\Phi(Y_1, \dots, Y_n)$ if $(X_1, \dots, X_n) \subseteq (Y_1, \dots, Y_n)$. By Theorem 2.12, Knaster–Tarski theorem, we see that there is the least fixed point of φ_Φ. We write $\mathbf{lfp}_{\varphi_\Phi}$ for the least fixed point of φ_Φ For $i = 1, \dots, n$, we define a function $\pi_i \colon \mathcal{P}\left(D^{k_1}\right) \times \dots \times \mathcal{P}\left(D^{k_n}\right) \to \mathcal{P}\left(D^{k_i}\right)$ by $\pi_i(X_1, \dots, X_n) = X_i$.

Definition 2.16 (Standard model)**.** A first-order structure M is said to be a *standard model* for $((\Sigma, L, \#), \Phi)$ if $P_i{}^M = \pi_i\left(\mathbf{lfp}_{\varphi_\Phi}\right)$ for each $i = 1, \dots, n$.

Henkin semantics

We give Henkin semantics. In the semantics, the interpretation of the inductive predicates is the least fixed point of the definition set operator in the special class of tuples of sets, called Henkin class.

Definition 2.17 (Henkin class)**.** Let M be a first-order structure with the domain D. $\mathcal{H} = \left\{H_l \subseteq \mathcal{P}\left(D^l\right) \;\middle|\; l \in \mathbb{N}\right\}$ is called a *Henkin class for M* if $\mathcal{H}$ satisfies the following conditions:

(H1) $\{(d, d) \mid d \in D\} \in H_2$.

(H2) If Q is any predicate symbol of arity l, then $\left\{(d_1, \dots, d_l) \;\middle|\; (d_1, \dots, d_l) \in Q^M\right\} \in H_l$.

(H3) If $R \in H_{l+1}$ and $d \in D$, then $\{(d_1, \dots, d_l) \mid (d_1, \dots, d_l, d) \in R\} \in H_l$.

(H4) If $R \in H_l$ holds, $t_1, \dots, t_l$ are terms, and $x_1, \dots, x_m$ are all variable symbols, then $\left\{(\rho(x_1), \dots, \rho(x_m)) \;\middle|\; \left({t_1}_\rho^M, \dots, {t_k}_\rho^M\right) \in R, \rho \text{ is a valuation on } M\right\} \in H_m$.

(H5) If $R \in H_l$, then $D^l \setminus R \in H_l$.

(H6) If $R_1, R_2 \in H_l$, then $R_1 \cap R_2 \in H_l$.

(H7) If $R \in H_{l+1}$, then $\{(d_1, \dots, d_l) \mid \text{There exists } d \in D \text{ such that } (d_1, \dots, d_l, d) \in R\} \in H_l$.

Remark. We note that Henkin classes contain enough sets of tuples to interpret any formula of the language for first-order logic with inductive definitions [4, 6]. It means that the following statement holds: If $\mathcal{H} = \left\{H_k \subseteq \mathcal{P}\left(D^k\right) \;\middle|\; k \in \mathbb{N}\right\}$ is a Henkin class for a structure M, ρ is a valuation, F is a formula, and $x_1, \dots, x_k$ are distinct variables, then

$$\left\{(d_1, \dots, d_k) \;\middle|\; M \models_{\rho[x_1 \mapsto d_1, \dots, x_k \mapsto d_k]} F\right\} \in H_k$$

holds, where $\rho[x_1 \mapsto d_1, \dots, x_k \mapsto d_k](x_i) = d_i$ holds for $i = 1, \dots, k$ and $\rho[x_1 \mapsto d_1, \dots, x_k \mapsto d_k](y) = \rho(y)$ holds with $y \neq x_1, \dots, y \neq x_k$.

Definition 2.18 ($\mathcal{H}$-point)**.** Let M be a structure, $\mathcal{H}$ be a Henkin class for M. $(X_1, \dots, X_n) \in \mathcal{P}\left(D^{k_1}\right) \times \dots \times \mathcal{P}\left(D^{k_n}\right)$ is said to be an $\mathcal{H}$-point if $X_i \in H_{k_i}$ for each $i = 1, \dots, n$.

Remark. $\mathcal{H}$-points are under the definition set operator [4, 6]. It means that the following statement holds: If $(\mathbf{X}_1, \dots, \mathbf{X}_n)$ is an $\mathcal{H}$-point, then so is $\varphi_\Phi(\mathbf{X}_1, \dots, \mathbf{X}_n)$.

A *pre-fixed $\mathcal{H}$-point* is defined as a pre-fixed point of φ_Φ which is also an $\mathcal{H}$-point. We define the least pre-fixed $\mathcal{H}$-point as the minimum of the set of pre-fixed $\mathcal{H}$-points.

Definition 2.19 (Henkin model)**.** Let M be a structure, $\mathcal{H}$ be a Henkin class for M. $(M, \mathcal{H})$ is called a *Henkin model* if the following conditions hold:

(1) There exists the least pre-fixed $\mathcal{H}$-point $\mu_\mathcal{H}.\varphi_\Phi$.

(2) $P_i{}^M = \pi_i(\mu_\mathcal{H}.\varphi_\Phi)$ for each $i = 1, \dots, n$.

We note that a standard model is a Henkin model.

2.2 Derivation tree

In this section, we define a *derivation tree.*

Definition 2.20 (Derivation tree)**.** Let Rule be the set of names for the inference rules of each proof system. Let Seq be the set of sequents. $\mathbb{N}^*$ denotes the set of finite sequences of natural numbers. We write $\langle n_1, \dots, n_k \rangle$ for the sequence of the numbers $n_1, \dots, n_k$. We write $\sigma_1\sigma_2$ for the concatenation of σ_1 and σ_2 with σ_1, $\sigma_2 \in \mathbb{N}^*$. We write σn for $\sigma\langle n \rangle$ for $\sigma \in \mathbb{N}^*$ and $n \in \mathbb{N}$. We define a *derivation tree* to be a partial function $\mathcal{D} \colon \mathbb{N}^* \rightharpoonup \mathrm{Seq} \times (\mathrm{Rule} \cup \{(\textsc{Bud})\})$ satisfying the following conditions:

(1) $\mathrm{Dom}(\mathcal{D})$ is prefix-closed, that is to say, if $\sigma_1\sigma_2 \in \mathrm{Dom}(\mathcal{D})$ for σ_1, $\sigma_2 \in \mathbb{N}^*$, then $\sigma_1 \in \mathrm{Dom}(\mathcal{D})$.

(2) If $\sigma n \in \mathrm{Dom}(\mathcal{D})$ for $\sigma \in \mathbb{N}^*$ and $n \in \mathbb{N}$, then $\sigma m \in \mathrm{Dom}(\mathcal{D})$ for all $m \leq n$.

(3) Define $\mathcal{D}(\sigma) = (\Gamma_\sigma \Rightarrow \Delta_\sigma, R_\sigma)$.
   (a) If $R_\sigma = (\textsc{Bud})$, then $\sigma 0 \notin \mathrm{Dom}(\mathcal{D})$.
   (b) If $R_\sigma \neq (\textsc{Bud})$, then

$$\frac{\Gamma_{\sigma 0} \Rightarrow \Delta_{\sigma 0} \qquad \cdots \qquad \Gamma_{\sigma n} \Rightarrow \Delta_{\sigma n}}{\Gamma_\sigma \Rightarrow \Delta_\sigma}$$

   is a rule R_σ and $\sigma(n+1) \notin \mathrm{Dom}(\mathcal{D})$.

We write $\mathsf{conc}_{\mathcal{D}}(\sigma)$ and $\mathsf{rule}_{\mathcal{D}}(\sigma)$ for $\Gamma \Rightarrow \Delta$ and (R), respectively, where $\mathcal{D}(\sigma) = (\Gamma \Rightarrow \Delta, (\mathrm{R}))$. An element in the domain of a derivation tree is called its *node.* The empty sequence as a node is called the *root.* The node σ is called a *bud* if $\mathsf{rule}_{\mathcal{D}}(\sigma)$ is $(\textsc{Bud})$. The node σ is called a *leaf* if σ is not a bud and $\sigma 0 \notin \mathrm{Dom}(D)$. The node which is not a bud and a leaf is called an *inner node.* A derivation tree is called *infinite* if the domain of the derivation tree is infinite.

For each derivation tree $\mathcal{D}$ and each sequence $\sigma \in \mathrm{Dom}(D)$, we define a derivation tree $\mathcal{D}^{(\sigma)}$ as $\mathcal{D}^{(\sigma)}(\sigma_1) = \mathcal{D}(\sigma\sigma_1)$. It is called *the subtree of* $\mathcal{D}$ *from* σ. We say that the derivation tree $\mathcal{D}$ is *regular* if the set of subtrees $\left\{\mathcal{D}^{(\sigma)} \;\middle|\; \sigma \in \mathrm{Dom}(\mathcal{D})\right\}$ is finite.

We sometimes identify a node σ with the sequent $\mathsf{conc}_{\mathcal{D}}(\sigma)$.

Definition 2.21 (Path)**.** We define a *path* in a derivation tree $\mathcal{D}$ to be a (possibly infinite) sequence $\{\sigma_i\}_{0 \leq i < \alpha}$ of nodes in $\mathrm{Dom}(\mathcal{D})$ such that $\sigma_{i+1} = \sigma_i n$ for some $n \in \mathbb{N}$ and $\alpha \in \mathbb{N}_{>0} \cup \{\omega\}$, where $\mathbb{N}_{>0}$ is the set of positive natural numbers and ω is the least infinite ordinal. A finite path $\sigma_0, \sigma_1, \dots, \sigma_n$ is called *a path from* σ_0 *to* σ_n. The *length of a finite path* $\{\sigma_i\}_{0 \leq i < \alpha}$ is defined as α. We define *the height of a node* as the length of the path from the root to the node.

We sometimes write $\{\Gamma_i \Rightarrow \Delta_i\}_{0 \leq i < \alpha}$ for the path $\{\sigma_i\}_{0 \leq i < \alpha}$ in a derivation tree $\mathcal{D}$ if $\mathcal{D}(\sigma_i) = (\Gamma_i \Rightarrow \Delta_i, R_i)$.

Definition 2.22 (Parent, child, ancestor, descendant)**.** Let $\mathcal{D}$ be a derivation tree.

(1) For each node σ of $\mathcal{D}$, σn with $n \in \mathbb{N}$ is called a *child of* σ if $\sigma n \in \mathrm{Dom}(\mathcal{D})$. In the case, we call σ is a *parent of* σn.

(2) For nodes σ_1, σ_2 of $\mathcal{D}$, we call σ_1 an *ancestor of* σ_2 if there exists the path from σ_1 to σ_2. In the case, σ_2 is called a *descendant of* σ_1.

Logical Rules:

$$(\Gamma \cap \Delta \neq \emptyset)\ \frac{}{\Gamma \Rightarrow \Delta}\ (\textsc{Axiom}) \qquad \frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma, \neg\varphi \Rightarrow \Delta}\ (\neg\ \mathrm{L}) \qquad \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \neg\varphi, \Delta}\ (\neg\ \mathrm{R})$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta}\ (\vee\ \mathrm{L}) \qquad \frac{\Gamma \Rightarrow \varphi, \psi, \Delta}{\Gamma \Rightarrow \varphi \vee \psi, \Delta}\ (\vee\ \mathrm{R})$$

$$\frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}\ (\wedge\ \mathrm{L}) \qquad \frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \wedge \psi, \Delta}\ (\wedge\ \mathrm{R})$$

$$\frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta}\ (\to\ \mathrm{L}) \qquad \frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \to \psi, \Delta}\ (\to\ \mathrm{R})$$

$$\frac{\Gamma, \varphi[x := t] \Rightarrow \Delta}{\Gamma, \forall x\varphi \Rightarrow \Delta}\ (\forall\ \mathrm{L}) \qquad (y \notin \mathrm{FV}(\Gamma \cup \Delta \cup \{\varphi\}))\ \frac{\Gamma \Rightarrow \varphi[x := y], \Delta}{\Gamma \Rightarrow \forall x\varphi, \Delta}\ (\forall\ \mathrm{R})$$

$$(y \notin \mathrm{FV}(\Gamma \cup \Delta \cup \{\varphi\}))\ \frac{\Gamma, \varphi[x := y] \Rightarrow \Delta}{\Gamma, \exists x\varphi \Rightarrow \Delta}\ (\exists\ \mathrm{L}) \qquad \frac{\Gamma \Rightarrow \varphi[x := t], \Delta}{\Gamma \Rightarrow \exists x\varphi, \Delta}\ (\exists\ \mathrm{R})$$

$$\frac{\Gamma[x := u_1, y := u_2] \Rightarrow \Delta[x := u_1, y := u_2]}{\Gamma[x := u_2, y := u_1], u_1 = u_2 \Rightarrow \Delta[x := u_2, y := u_1]}\ (=\ \mathrm{L}) \qquad \frac{}{\Gamma \Rightarrow t = t, \Delta}\ (=\ \mathrm{R})$$

Structural Rules:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta', \Delta}\ (\textsc{Weak}) \qquad \frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (\textsc{Cut})$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma[x_1 := u_1, \ldots, x_m := u_m] \Rightarrow \Delta[x_1 := u_1, \ldots, x_m := u_m]}\ (\textsc{Sub})$$

Figure 2.1 Rules for first-order logic with equality

2.3 LKID: ordinary proof system for first-order logic with inductive definitions

In this section, we define an ordinary proof system LKID for first-order logic with inductive definitions.

The inference rules of LKID except for rules of inductive predicates are the same as that of first-order logic with equality. They are in Figure 2.1. The *principal formula* of a rule is the distinguished formula introduced by the rule in its conclusion. We use the commas in sequents for a set union. We note that the contraction rule is implicit.

We present the two inference rules for inductive predicates. Let $P_1, \ldots, P_n$ be all inductive predicates of the language we consider.

First, for each production

$$\frac{Q_1\,\mathbf{u}_1(\mathbf{x}) \quad \cdots \quad Q_h\,\mathbf{u}_h(\mathbf{x}) \quad P_{j_1}\,\mathbf{t}_1(\mathbf{x}) \quad \cdots \quad P_{j_m}\,\mathbf{t}_m(\mathbf{x})}{P_j\,\mathbf{t}(\mathbf{x})}\ ,$$

there is the inference rule

$$\frac{\Gamma \Rightarrow Q_1\,\mathbf{u}_1(\mathbf{u}), \Delta \quad \cdots \quad \Gamma \Rightarrow Q_h\,\mathbf{u}_h(\mathbf{u}), \Delta \quad \Gamma \Rightarrow P_{j_1}\,\mathbf{t}_1(\mathbf{u}), \Delta \quad \cdots \quad \Gamma \Rightarrow P_{j_m}\,\mathbf{t}_m(\mathbf{u}), \Delta}{\Gamma \Rightarrow P_j\,\mathbf{t}(\mathbf{u}), \Delta}\,(P_j\ \mathrm{R}) \quad .$$

Next, we introduce induction rule as the left introduction rule for the inductive predicate symbol. To formulate the rule, we define some concepts.

Definition 2.23 (Mutual dependency)**.** We define

$$\mathrm{Prem} = \left\{ (P_{j_1}, P_{j_2}) \;\middle|\; \begin{array}{l} P_{j_1}, P_{j_2} \text{ are inductive predicate symbols, and} \\ P_{j_2} \text{ occurs in the assumptions of a production of } P_{j_1} \text{ in } \Phi \end{array} \right\}.$$

We define a binary relation Prem^* to be the smallest reflexive-transitive closure on inductive predicate symbols including the binary relation Prem.

For two inductive predicate symbols P_{j_1} and P_{j_2}, we say that P_{j_1} is *mutually dependent* on P_{j_2} if both $\mathrm{Prem}^*(P_{j_1}, P_{j_2})$ and $\mathrm{Prem}^*(P_{j_2}, P_{j_1})$ hold.

To introduce induction rule ($\textsc{Ind}\ P_j$), for each $i = 1, \ldots, n$, we fix an arbitrary formula F_i, called an *induction hypothesis*.

For an inductive predicates P_j and induction hypotheses $\{F_i\}_{i \in \{1,\ldots,n\}}$, a *minor assumption* of $\Gamma, P_j\mathbf{u} \Rightarrow \Delta$ for a production

$$\frac{Q_1\,\mathbf{u}_1(\mathbf{x}) \quad \cdots \quad Q_h\,\mathbf{u}_h(\mathbf{x}) \quad P_{j_1}\,\mathbf{t}_1(\mathbf{x}) \quad \cdots \quad P_{j_m}\,\mathbf{t}_m(\mathbf{x})}{P_i\,\mathbf{t}(\mathbf{x})} \quad ,$$

is defined as a sequent

$$\Gamma, Q_1\,\mathbf{u}_1(\mathbf{y}), \ldots, Q_h\,\mathbf{u}_h(\mathbf{y}), G_{j_1}[\mathbf{z}_{j_1} := \mathbf{t}_1(\mathbf{y})], \ldots, G_{j_m}[\mathbf{z}_{j_m} := \mathbf{t}_m(\mathbf{y})] \Rightarrow F_i[\mathbf{z}_i := \mathbf{t}(\mathbf{y})], \Delta$$

where $\mathbf{z}_k$ is a tuple of distinct variables of the same length as the arity of P_k for all $k = 1, \ldots, n$, $\mathbf{y}$ is a tuple of distinct variables of the same length as $\mathbf{x}$, $y \in \mathrm{FV}(\Gamma \cup \Delta \cup \{P_j\mathbf{u}\})$ for all y occurring in $\mathbf{y}$, and for each $i = 1, \ldots, n$,

$$G_i = \begin{cases} F_i, & \text{if } P_i \text{ is mutually dependent on } P_j, \\ P_i\mathbf{z}_i, & \text{otherwise.} \end{cases}$$

The induction rule for P_j ($\textsc{Ind}\ P_j$) with induction hypotheses $\{F_i\}_{i \in \{1,\ldots,n\}}$ is

$$\frac{\begin{array}{l}\text{All minor assumptions of } \Gamma, P_j\mathbf{u} \Rightarrow \Delta \text{ for produc-} \\ \text{tions each of whose conclusion is an atomic formula} \\ \text{with an inductive predicate which is mutually de-} \\ \text{pendent on } P_j \end{array} \qquad \Gamma, F_j[\mathbf{z}_j := \mathbf{u}] \Rightarrow \Delta}{\Gamma, P_j\mathbf{u} \Rightarrow \Delta}\,(\textsc{Ind}\ P_j) \quad ,$$

where $\mathbf{z}_j$ is a tuple of distinct variables of the same length as the arity of P_j. We call the assumption of ($\textsc{Ind}\ P_j$) whose form is $\Gamma, F_j[\mathbf{z}_j := \mathbf{u}] \Rightarrow \Delta$ the *major assumption*.

Example 2.24 (Rules for N, E, and O)**.** Let N, E, and O be the same predicates in Example 2.9. The rules for N, E, and O are

$$
\dfrac{
\dfrac{\dfrac{}{\Rightarrow \mathrm{E}(0), \mathrm{O}(0), \mathrm{E}(x) \lor \mathrm{O}(x)}}{\Rightarrow \mathrm{E}(0) \lor \mathrm{O}(0), \mathrm{E}(x) \lor \mathrm{O}(x)}
\quad
\dfrac{
\dfrac{\dfrac{\dfrac{}{\mathrm{E}(y) \Rightarrow \mathrm{E}(sy), \mathrm{E}(y), \mathrm{E}(x) \lor \mathrm{O}(x)}}{\mathrm{E}(y) \Rightarrow \mathrm{E}(sy), \mathrm{O}(sy), \mathrm{E}(x) \lor \mathrm{O}(x)}}{\mathrm{E}(y) \Rightarrow \mathrm{E}(sy) \lor \mathrm{O}(sy), \mathrm{E}(x) \lor \mathrm{O}(x)}
\quad
\dfrac{\dfrac{\dfrac{}{\mathrm{O}(y) \Rightarrow \mathrm{O}(y), \mathrm{O}(sy), \mathrm{E}(x) \lor \mathrm{O}(x)}}{\mathrm{O}(y) \Rightarrow \mathrm{E}(sy), \mathrm{O}(sy), \mathrm{E}(x) \lor \mathrm{O}(x)}}{\mathrm{O}(y) \Rightarrow \mathrm{E}(sy) \lor \mathrm{O}(sy), \mathrm{E}(x) \lor \mathrm{O}(x)}
}{\mathrm{E}(y) \lor \mathrm{O}(y) \Rightarrow \mathrm{E}(sy) \lor \mathrm{O}(sy), \mathrm{E}(x) \lor \mathrm{O}(x)}
\quad
\dfrac{}{\mathrm{E}(x) \lor \mathrm{O}(x) \Rightarrow \mathrm{E}(x) \lor \mathrm{O}(x)}
}{\mathrm{N}(x) \Rightarrow \mathrm{E}(x) \lor \mathrm{O}(x)}
$$

Figure 2.2 LKID proof

$$
\dfrac{}{\Gamma \Rightarrow \Delta, \mathrm{N}(0)}\,(\mathrm{N\ R}_1)\ , \qquad \dfrac{\Gamma \Rightarrow \Delta, \mathrm{N}(t)}{\Gamma \Rightarrow \Delta, \mathrm{N}(st)}\,(\mathrm{N\ R}_2)\ ,
$$

$$
\dfrac{}{\Gamma \Rightarrow \Delta, \mathrm{E}(0)}\,(\mathrm{E\ R}_1)\ , \qquad \dfrac{\Gamma \Rightarrow \Delta, \mathrm{O}(t)}{\Gamma \Rightarrow \Delta, \mathrm{E}(st)}\,(\mathrm{E\ R}_2)\ , \qquad \dfrac{\Gamma \Rightarrow \Delta, \mathrm{E}(t)}{\Gamma \Rightarrow \Delta, \mathrm{O}(st)}\,(\mathrm{O\ R})\ ,
$$

$$
\dfrac{\Gamma \Rightarrow F[v := 0], \Delta \qquad \Gamma, F[v := x] \Rightarrow F[v := sx], \Delta \qquad \Gamma, F[v := t] \Rightarrow \Delta}{\Gamma, \mathrm{N}(t) \Rightarrow \Delta}\,(\textsc{Ind N})\ ,
$$

$$
\dfrac{\Gamma \Rightarrow F_{\mathrm{E}}[v := 0], \Delta \quad \Gamma, F_{\mathrm{E}}[v := y] \Rightarrow F_{\mathrm{O}}[v := sy], \Delta \quad \Gamma, F_{\mathrm{O}}[v := y] \Rightarrow F_{\mathrm{E}}[v := sy], \Delta \quad \Gamma, F_{\mathrm{E}}[v := t] \Rightarrow \Delta}{\Gamma, \mathrm{E}(t) \Rightarrow \Delta}\,(\textsc{Ind E})\ ,
$$

$$
\dfrac{\Gamma \Rightarrow F_{\mathrm{E}}[v := 0], \Delta \quad \Gamma, F_{\mathrm{E}}[v := z] \Rightarrow F_{\mathrm{O}}[v := sz], \Delta \quad \Gamma, F_{\mathrm{O}}[v := z] \Rightarrow F_{\mathrm{E}}[v := sz], \Delta \quad \Gamma, F_{\mathrm{O}}[v := t] \Rightarrow \Delta}{\Gamma, \mathrm{O}(t) \Rightarrow \Delta}\,(\textsc{Ind O})\ ,
$$

where $x \notin \mathrm{FV}(\Gamma \cup \Delta \cup \{\mathrm{N}(t)\})$, $y \notin \mathrm{FV}(\Gamma \cup \Delta \cup \{\mathrm{E}(t)\})$, and $z \notin \mathrm{FV}(\Gamma \cup \Delta \cup \{\mathrm{O}(t)\})$.

Definition 2.25 (LKID proof)**.** We define an LKID proof to be a finite derivation tree without buds.

Example 2.26. The derivation tree in Figure 2.2 is a LKID proof (labels of rules are omitted for limited space).

The soundness of LKID for the Henkin models holds. To be more accurate, if there exists an LKID proof of a sequent $\Gamma \Rightarrow \Delta$, then $\Gamma \Rightarrow \Delta$ is valid in all Henkin models. Moreover, the cut-free completeness of LKID for the Henkin models holds, but completeness of LKID for the standard models does not hold. In other words, if $\Gamma \Rightarrow \Delta$ is valid in all Henkin models, there exists an LKID cut-free *proof* of $\Gamma \Rightarrow \Delta$, and there is a sequent not provable in LKID but valid in all standard models. The soundness and cut-free completeness of LKID imply the cut-elimination property of that, i. e. all provable sequents in LKID are cut-free provable in LKID [4, 6].

2.4 LKID$^\omega$: non-well-founded infinitary proof system for first-order logic with inductive definitions

In this section, we introduce a non-well-founded infinitary proof system LKID$^\omega$ for first-order logic with inductive definitions.

The inference rules of LKID$^\omega$ are the same as that of LKID except for ($\textsc{Ind}\ P_j$). The induction rule is replaced by the *case-split rule*.

To define the case-split rule, we define some concepts. A *case distinctions* of $\Gamma, P\mathbf{u} \Rightarrow \Delta$ for a production

$$
\dfrac{Q_1\,\mathbf{u}_1(\mathbf{x}) \quad \cdots \quad Q_h\,\mathbf{u}_h(\mathbf{x}) \quad P_1\,\mathbf{t}_1(\mathbf{x}) \quad \cdots \quad P_m\,\mathbf{t}_m(\mathbf{x})}{P\,\mathbf{t}(\mathbf{x})}\ ,
$$

$$\cfrac{\cfrac{\begin{array}{c}\vdots\\ \Rightarrow\end{array}}{\Rightarrow}\ (\textsc{Weak})}{\Rightarrow}\ (\textsc{Weak})$$

Figure 2.3 Infinite derivation tree of the empty sequent

is defined as a sequent

$$\Gamma, \mathbf{u} = \mathbf{t}(\mathbf{y}), Q_1\,\mathbf{u}_1(\mathbf{y}), \ldots, Q_h\,\mathbf{u}_h(\mathbf{y}), P_1\,\mathbf{t}_1(\mathbf{y}), \ldots, P_m\,\mathbf{t}_m(\mathbf{y}) \Rightarrow \Delta,$$

where $\mathbf{y}$ is a tuple of distinct variables of the same length as $\mathbf{x}$ and $y \notin \mathrm{FV}(\Gamma \cup \Delta \cup \{P\mathbf{u}\})$ for all $y \in \mathbf{y}$.

The *case-split rule* (Case P) is

$$\frac{\text{All case distinctions of } \Gamma, P\mathbf{u} \Rightarrow \Delta \text{ for productions of } P}{\Gamma, P\mathbf{u} \Rightarrow \Delta}\ (\textsc{Case}\ P)\quad.$$

The formulas $P_1\,\mathbf{t}_1(\mathbf{y}), \ldots, P_m\,\mathbf{t}_m(\mathbf{y})$ in case distinctions are said to be *case-descendants* of the principal formula $P\mathbf{u}$.

Example 2.27 (The case-split rules for N, E, and O)**.** Let N, E, and O be the same predicates in Example 2.9. The case-split rules for N, E, and O are

$$\frac{\Gamma, t = 0 \Rightarrow \Delta \qquad \Gamma, t = sx, \mathrm{N}(x) \Rightarrow \Delta}{\Gamma, \mathrm{N}(t) \Rightarrow \Delta}\ (\textsc{Case}\ \mathrm{N})\quad,$$

$$\frac{\Gamma, t = 0 \Rightarrow \Delta \qquad \Gamma, t = sy, \mathrm{O}(y) \Rightarrow \Delta}{\Gamma, \mathrm{E}(t) \Rightarrow \Delta}\ (\textsc{Case}\ \mathrm{E})\quad,$$

$$\frac{\Gamma, t = sz, \mathrm{E}(z) \Rightarrow \Delta}{\Gamma, \mathrm{O}(t) \Rightarrow \Delta}\ (\textsc{Case}\ \mathrm{O})\quad,$$

where $x \notin \mathrm{FV}(\Gamma \cup \Delta \cup \{\mathrm{N}(t)\})$, $y \notin \mathrm{FV}(\Gamma \cup \Delta \cup \{\mathrm{E}(t)\})$, and $z \notin \mathrm{FV}(\Gamma \cup \Delta \cup \{\mathrm{O}(t)\})$.

There is an infinite derivation tree of the empty sequent, representing a contradiction (Figure 2.3), and therefore each proof in this system must satisfy the condition for soundness, *the global trace condition*. To describe the condition, we define the following concepts.

Definition 2.28 (Trace)**.** For a path $\{\Gamma_i \Rightarrow \Delta_i\}_{0 \le i < \alpha}$ in a derivation tree $\mathcal{D}$, we define a *trace following* $\{\Gamma_i \Rightarrow \Delta_i\}_{0 \le i < \alpha}$ to be a sequence of formulas $\{\tau_i\}_{0 \le i < \alpha}$ such that the following hold:

(1) τ_i is an I-atomic formula in Γ_i.

(2) If $\Gamma_i \Rightarrow \Delta_i$ is the conclusion of (Sub) with θ, then $\tau_i \equiv \tau_{i+1}[\theta]$.

(3) If $\Gamma_i \Rightarrow \Delta_i$ is the conclusion of ($=$ L) with the principal formula $t = u$ and $\tau_i \equiv F[x := t, y := u]$, then $\tau_{i+1} \equiv F[x := u, y := t]$.

(4) If $\Gamma_i \Rightarrow \Delta_i$ is the conclusion of (Case P_i), then either

- τ_i is the principal formula of the rule and τ_{i+1} is a case-descendant of τ_i, or
- τ_{i+1} is the same as τ_i.

In the former case, τ_i is said to be a *progress point* of the trace.

$$\dfrac{\dfrac{\dfrac{}{\Rightarrow \mathrm{E}(0),\mathrm{O}(0)}\,(\mathrm{E\ R}_1)}{x_n = 0 \Rightarrow \mathrm{E}(x_n),\mathrm{O}(x_n)}\,(=\mathrm{L}) \qquad \dfrac{\dfrac{\dfrac{\underline{\mathrm{N}(x_{n+1})} \Rightarrow \mathrm{E}(x_{n+1}),\mathrm{O}(x_{n+1})}{\underline{\mathrm{N}(x_{n+1})} \Rightarrow \mathrm{E}(\mathrm{s}x_{n+1}),\mathrm{E}(x_{n+1})}\,(\mathrm{E\ R}_2)}{\underline{\mathrm{N}(x_{n+1})} \Rightarrow \mathrm{E}(\mathrm{s}x_{n+1}),\mathrm{O}(\mathrm{s}x_{n+1})}\,(\mathrm{O\ R})}{x_n = \mathrm{s}x_{n+1},\underline{\mathrm{N}(x_{n+1})} \Rightarrow \mathrm{E}(x_n),\mathrm{O}(x_n)}\,(=\mathrm{L})}{\underline{\mathrm{N}(x_n)} \Rightarrow \mathrm{E}(x_n),\mathrm{O}(x_n)}\,(\textsc{Case}\ \mathrm{N})$$

Figure 2.4 A derivation tree π_n

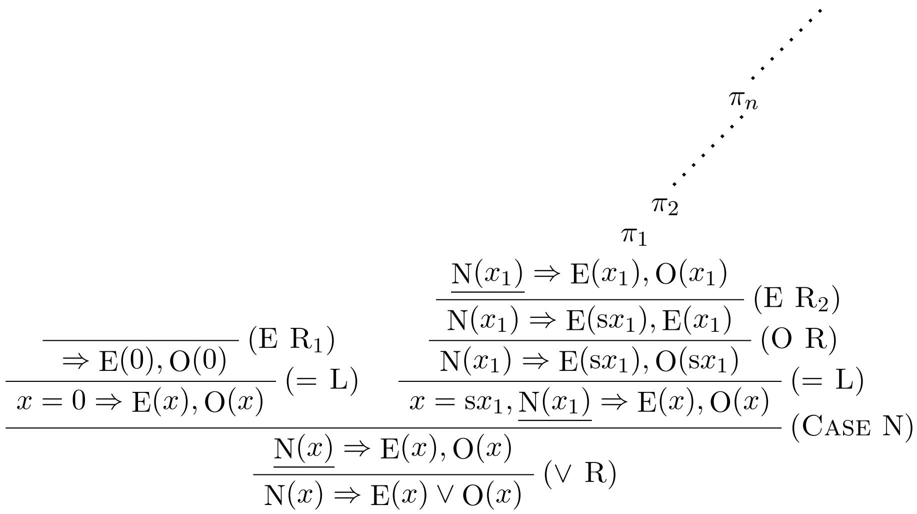


Figure 2.5 LKID$^\omega$ proof

(5) If $\Gamma_i \Rightarrow \Delta_i$ is the conclusion of any other rule, then $\tau_{i+1} \equiv \tau_i$.

Definition 2.29 (Global trace condition)**.** If a trace has infinitely many progress points, we call the trace an *infinitely progressing trace*. For a derivation tree, if, for every infinite path $\{\Gamma_i \Rightarrow \Delta_i\}_{i\geq 0}$ in the derivation tree, there exists an infinitely progressing trace following a tail of the path $\{\Gamma_i \Rightarrow \Delta_i\}_{i\geq k}$ with some $k \geq 0$, we say the derivation tree satisfies *the global trace condition*.

Definition 2.30 (LKID$^\omega$ pre-proof)**.** We define an LKID$^\omega$ *pre-proof* to be a (possibly infinite) derivation tree $\mathcal{D}$ without buds. When the root is $\Gamma \Rightarrow \Delta$, we call $\Gamma \Rightarrow \Delta$ the conclusion of the proof.

Definition 2.31 (LKID$^\omega$ proof)**.** We define an LKID$^\omega$ *proof* to be an LKID$^\omega$ pre-proof that satisfies the global trace condition.

Example 2.32. Let N, E, and O be the same predicates in Example 2.9.

The derivation tree in Figure 2.5 is a LKID$^\omega$ proof of $\mathrm{N}(x) \Rightarrow \mathrm{E}(x) \vee \mathrm{O}(x)$ (π_n in Figure 2.5 is a derivation tree in Figure 2.4). We use the underlined formulas to denote the infinitely progressing trace for some tails of any infinite path.

Because of the global trace condition, the soundness of LKID$^\omega$ for the standard models hold. In other words, if there exists an LKID$^\omega$ proof of a sequent $\Gamma \Rightarrow \Delta$, then $\Gamma \Rightarrow \Delta$ is valid in any standard model. Moreover, cut-free completeness of LKID$^\omega$ for the standard models hold.

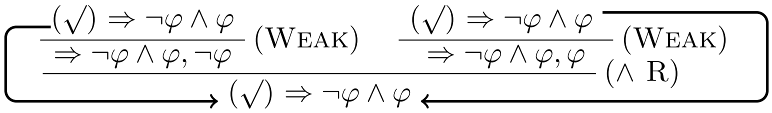


Figure 2.6 CLKID$^\omega$ pre-proof of a contradiction

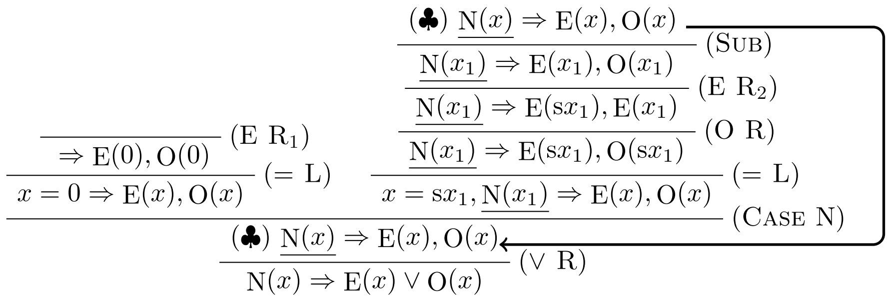


Figure 2.7 CLKID$^\omega$ proof

In other words, if $\Gamma \Rightarrow \Delta$ is valid in any standard model, there exists an LKID$^\omega$ cut-free *proof* of $\Gamma \Rightarrow \Delta$. Soundness and cut-free completeness of LKID$^\omega$ imply the cut-elimination property of it, i. e. all provable sequents in LKID$^\omega$ are cut-free provable in LKID$^\omega$ [4, 6].

2.5 CLKID$^\omega$: cyclic proof system for first-order logic with inductive definitions

This section gives a cyclic proof system CLKID$^\omega$ for first-order logic with inductive definitions.

Definition 2.33 (Companion)**.** For a finite derivation tree $\mathcal{D}$, we define the *companion* for a bud μ as an inner node σ in $\mathcal{D}$ with $\mathsf{conc}_{\mathcal{D}}(\sigma) = \mathsf{conc}_{\mathcal{D}}(\mu)$.

Definition 2.34 (CLKID$^\omega$ pre-proof)**.** We define a CLKID$^\omega$ *pre-proof* to be a pair $(\mathcal{D}, \mathcal{C})$ such that $\mathcal{D}$ is a finite derivation tree and $\mathcal{C}$ is a function mapping each bud to its companion. When the root is $\Gamma \Rightarrow \Delta$, we call $\Gamma \Rightarrow \Delta$ the conclusion of the proof.

The *graph* of a pre-proof $(\mathcal{D}, \mathcal{C})$, written $\mathcal{G}(\mathcal{D}, \mathcal{C})$, is the graph obtained from $\mathcal{D}$ by identifying each bud μ in $\mathcal{D}$ with its companion $\mathcal{C}(\mu)$.

Definition 2.35 (CLKID$^\omega$ proof)**.** We define a CLKID$^\omega$ *proof* of a sequent $\Gamma \Rightarrow \Delta$ to be a CLKID$^\omega$ pre-proof of $\Gamma \Rightarrow \Delta$ whose graph satisfies the global trace condition. If a CLKID$^\omega$ proof of $\Gamma \Rightarrow \Delta$ exists, we say $\Gamma \Rightarrow \Delta$ is *provable* in CLKID$^\omega$. A CLKID$^\omega$ proof in which (Cut) does not occur is called *cut-free*. If a cut-free CLKID$^\omega$ proof of $\Gamma \Rightarrow \Delta$ exists, we say $\Gamma \Rightarrow \Delta$ is *cut-free provable* in CLKID$^\omega$.

Since there is a CLKID$^\omega$ pre-proof of a contradiction (Figure 2.6), the global trace condition is necessary for soundness.

Example 2.36. Let N, E, and O be the same predicates in Example 2.9.
The derivation tree in Figure 2.7 is a CLKID$^\omega$ proof of $\mathrm{N}(x) \Rightarrow \mathrm{E}(x) \vee \mathrm{O}(x)$, where (♣) indicates the pairing of the companion with the bud. We use the underlined formulas to denote the infinitely progressing trace for some tails of any infinite path.

To see the relation between LKID$^\omega$ and CLKID$^\omega$, we define the following concept.

Definition 2.37 (Tree-unfolding)**.** For a $\mathtt{CLKID}^\omega$ pre-proof $(\mathcal{D},\mathcal{C})$, a *tree-unfolding* $\mathcal{T}(\mathcal{D},\mathcal{C})$ of $(\mathcal{D},\mathcal{C})$ is recursively defined by

$$\mathcal{T}(\mathcal{D},\mathcal{C})(\sigma) = \begin{cases} \mathcal{D}(\sigma), & \text{if } \sigma \in \mathrm{Dom}(\mathcal{D}) \setminus \mathrm{Bud}(\mathcal{D}), \\ \mathcal{T}(\mathcal{D},\mathcal{C})(\sigma_2\sigma_1), & \text{if } \sigma \notin \mathrm{Dom}(D) \setminus \mathrm{Bud}(\mathcal{D}) \text{ with } \sigma \equiv \sigma_0\sigma_1,\ \sigma_0 \in \mathrm{Bud}(\mathcal{D}) \text{ and } \sigma_2 \equiv \mathcal{C}(\sigma_0), \end{cases}$$

where $\mathrm{Bud}(\mathcal{D})$ is the set of buds in $\mathcal{D}$.

The tree-unfolding of a $\mathtt{CLKID}^\omega$ pre-proof is a $\mathtt{LKID}^\omega$ pre-proof whose the set of subtrees is finite. It is straightforward to show that, for a cyclic pre-proof $(\mathcal{D},\mathcal{C})$, the graph of $(\mathcal{D},\mathcal{C})$ satisfies the global trace condition if and only if the tree-unfolding of $(\mathcal{D},\mathcal{C})$ satisfies the global trace condition. Then, we can understand $\mathtt{CLKID}^\omega$ as a subsystem of $\mathtt{LKID}^\omega$.

Recall that an infinite tree is *regular* if the set of its subtrees is finite. Then, we say that $\mathtt{CLKID}^\omega$ is the subsystem $\mathtt{LKID}^\omega$ whose the underlying tree of each proof is restricted to a regular tree.

Brotherston [6] showed that each sequent provable in $\mathtt{LKID}$ is provable in $\mathtt{CLKID}^\omega$ and conjectured that the converse holds. However, Berardi and Tatsuta [3] showed that the conjecture is incorrect i.e. there is a sequent provable in $\mathtt{CLKID}^\omega$ but not in $\mathtt{LKID}$. Moreover, Berardi and Tatsuta [2] showed that the system obtained by adding Peano Arithmetic to $\mathtt{CLKID}^\omega$ is equivalent to that obtained by adding Peano Arithmetic to $\mathtt{LKID}$. The semantics, in which both soundness and completeness of $\mathtt{CLKID}^\omega$ hold, is unknown.

2.6 Cycle-normalisation

This section shows the cycle-normalisation property for $\mathtt{CLKID}^\omega$. It is proved in [4]. If each companion is an ancestor of the corresponding bud in a $\mathtt{CLKID}^\omega$ pre-proof, we say that the pre-proof is *cycle-normal*. The following proposition states that the cycle-normalisation property for $\mathtt{CLKID}^\omega$ holds.

Proposition 2.38. *For a $\mathtt{CLKID}^\omega$ pre-proof $(\mathcal{D},\mathcal{C})$, we have a $\mathtt{CLKID}^\omega$ cycle-normal pre-proof $(\mathcal{D}',\mathcal{C}')$ such that the tree-unfolding of $(\mathcal{D}',\mathcal{C}')$ is that of $(\mathcal{D},\mathcal{C})$.*

To show Proposition 2.38, we show a lemma, which is called König's lemma.

Lemma 2.39 (König's lemma)**.** *There exists an infinite path in a finitely branching infinite tree.*

Proof. Let $\mathcal{D}$ be a finitely branching infinite tree.

We show that there exists a sequence of nodes $\{\sigma_i\}_{i\in\mathbb{N}}$, which satisfies the following conditions:

(1) σ_i is a child of σ_{i-1} for $1 \leq i$.

(2) The set of descendants of σ_i is infinite.

We inductively construct $\{\sigma_i\}_{i\in\mathbb{N}}$.

We consider the case $i = 0$. Define σ_0 as the root. It satisfies (1) and (2), obviously.

We consider the case $i > 0$. Since the set of descendants of σ_{i-1} is infinite and the children of σ_{i-1} are finitely many, there exists a child σ of σ_{i-1} such that the set of descendants of σ is infinite. Let σ_i be σ. It satisfies (1) and (2), obviously.

We complete the construction. Then, we have the infinite path $\{\sigma_i\}_{i\in\mathbb{N}}$. ☐

For σ, $\sigma' \in \mathbb{N}^*$, we write $\sigma \preceq \sigma'$ when σ is an initial segment of σ'. We write $\sigma \prec \sigma'$ when $\sigma \preceq \sigma'$ and $\sigma \neq \sigma'$. We write $|\sigma|$ for the length of a sequence σ. For a set of finite sequences S, let $\overline{S} = \{\sigma \mid \sigma \preceq \sigma' \in S\}$ and $S^\circ = \{\sigma \mid \sigma \prec \sigma' \in S\}$.

Proof of Proposition 2.38. Let $(\mathcal{D}, \mathcal{C})$ be a $\mathtt{CLKID}^\omega$ pre-proof.

For simplicity, we write $\mathcal{D}_\infty$ for $\mathcal{T}(\mathcal{D}, \mathcal{C})$.

Define

$$S_1 = \left\{ \sigma \in \mathrm{Dom}(\mathcal{D}_\infty) \;\middle|\; \begin{array}{l} \text{There exists } \sigma' \prec \sigma \text{ such that } {\mathcal{D}_\infty}^{(\sigma)} = {\mathcal{D}_\infty}^{(\sigma')}, \\ \text{for all } \sigma_1 \prec \sigma \text{ and } \sigma_2 \prec \sigma,\, {\mathcal{D}_\infty}^{(\sigma_1)} \neq {\mathcal{D}_\infty}^{(\sigma_2)}, \text{ and} \\ \text{for all } n \in \mathbb{N}, \text{ there exists } \sigma_1 \succeq \sigma \text{ such that } \sigma_1 \in \mathrm{Dom}(\mathcal{D}_\infty), \text{ and } |\sigma_1| \geq n \end{array} \right\},$$

$$S_2 = \left\{ \sigma \in \mathrm{Dom}(\mathcal{D}_\infty) \;\middle|\; \sigma 0 \notin \mathrm{Dom}(\mathcal{D}_\infty), \text{ and } \sigma' \notin S_1 \text{ for all } \sigma' \preceq \sigma \right\}.$$

S_1 is the set of nodes each of which is on some infinite path, and which is of the smallest height on the path among nodes each of which has some inner node of the same subtree. S_2 is the set of leafs of which any ancestor is not in S_1.

We show that, for $\sigma \in \mathrm{Dom}(\mathcal{D}_\infty)$, either σ belongs to ${S_1}^\circ \cup \overline{S_2}$, or there exists $\sigma_0 \in S_1$ such that $\sigma_0 \preceq \sigma$.

Let $\sigma \in \mathrm{Dom}(\mathcal{D}_\infty)$. If there exists $\sigma_0 \in S_1$ such that $\sigma_0 \preceq \sigma$, we have done. Assume $\sigma_0 \not\preceq \sigma$ for all $\sigma_0 \in S_1$.

If there exists no infinite path through σ, then there exists σ' such that $\sigma\sigma' 0 \notin \mathrm{Dom}(\mathcal{D}_\infty)$. Assume there exists $\sigma'_0 \in S_1$ such that $\sigma'_0 \preceq \sigma\sigma'$. Since we assume $\sigma_0 \not\preceq \sigma$ for all $\sigma_0 \in S_1$, we have $\sigma \prec \sigma'_0$. Then, $\sigma \in {S_1}^\circ$. Assume $\sigma'_0 \notin S_1$ for all $\sigma'_0 \preceq \sigma\sigma'$, then we have $\sigma\sigma' \in S_2$. Then, $\sigma \in \overline{S_2}$.

Assume there exists an infinite path through σ. Let $\{\sigma_i\}_{i \in \mathbb{N}}$ be an infinite path in $\mathcal{D}_\infty$, where $\sigma \equiv \sigma_m$.

Since $\mathcal{D}_\infty$ is regular, the set $\left\{ {\mathcal{D}_\infty}^{(\sigma_i)} \;\middle|\; i \in \mathbb{N} \right\}$ is finite. Hence, there exist $j < k$ such that ${\mathcal{D}_\infty}^{(\sigma_j)} = {\mathcal{D}_\infty}^{(\sigma_k)}$. Let k_0 the smallest k among such k's. By the definition of S_1 and that of k_0, we have $\sigma_{k_0} \in S_1$. Since $\sigma_0 \not\preceq \sigma$ for all $\sigma_0 \in S_1$, we have $\sigma \prec \sigma_{k_0}$. Then, we have $\sigma \in {S_1}^\circ$.

Now, we see either σ belongs to ${S_1}^\circ \cup \overline{S_2}$, or there exists $\sigma_0 \in S_1$ such that $\sigma_0 \preceq \sigma$.

We define $\mathcal{D}'$ by

$$\mathcal{D}'(\sigma) = \begin{cases} \mathcal{D}_\infty(\sigma), & \text{if } \sigma \in {S_1}^\circ \cup \overline{S_2}, \\ (\Gamma \Rightarrow \Delta, (\textsc{Bud})), & \text{if } \sigma \in S_1, \mathcal{D}_\infty(\sigma) = (\Gamma \Rightarrow \Delta, (\mathrm{R})), \end{cases}$$

and $\mathcal{C}'$ by $\mathcal{C}'(\sigma) = \sigma'$ for a bud of $\mathcal{D}'$ where $\sigma' \prec \sigma$, and ${\mathcal{D}_\infty}^{(\sigma)} = {\mathcal{D}_\infty}^{(\sigma')}$.

We show that $\mathrm{Dom}(\mathcal{D}')$ is finite. Assume $\mathrm{Dom}(\mathcal{D}')$ is infinite, for contradiction. Since $\mathrm{Dom}(\mathcal{D}') = \overline{S_1} \cup \overline{S_2}$, We have $\mathrm{Dom}(\mathcal{D}') \subseteq \mathrm{Dom}(\mathcal{D}_\infty)$. Since there is no inference rule of $\mathtt{CLKID}^\omega$ whose assumptions are infinitely many, $\mathcal{D}_\infty$ is finite branching. Hence, is also finite branching. From the assumption, $\mathcal{D}'$ is a finite branching infinite tree. By Lemma 2.39, König's lemma, there exists an infinite path $\{\sigma_i\}_{i \in \mathbb{N}}$ in $\mathcal{D}'$. By $\mathrm{Dom}(\mathcal{D}') \subseteq \mathrm{Dom}(\mathcal{D}_\infty)$, the infinite path $\{\sigma_i\}_{i \in \mathbb{N}}$ is also a path in $\mathcal{D}_\infty$. Since $\mathcal{D}_\infty$ is regular, the set $\left\{ {\mathcal{D}_\infty}^{(\sigma_i)} \;\middle|\; i \in \mathbb{N} \right\}$ is finite. Hence, there exist $j < k$ such that ${\mathcal{D}_\infty}^{(\sigma_j)} = {\mathcal{D}_\infty}^{(\sigma_k)}$. Let k_0 the smallest k among such k's. By the definition of S_1 and that of k_0, we have $\sigma_{k_0} \in S_1$. Hence, $\sigma_{k_0+1} \notin \overline{S_1}$. Since σ_{k_0+2} is a child of σ_{k_0+1}, we see $\sigma_{k_0+1} \notin \overline{S_2}$. Therefore, $\sigma_{k_0+1} \notin \mathcal{D}'$. This contradicts the definition of σ_i's.

Since $\mathrm{Dom}(\mathcal{D}')$ is finite and $\mathcal{C}'(\sigma)$ is an ancestor of σ, we see that $(\mathcal{D}', \mathcal{C}')$ is a $\mathtt{CLKID}^\omega$ cycle-normal pre-proof.

We show that the tree-unfolding of $(\mathcal{D}, \mathcal{C})$ is that of $(\mathcal{D}', \mathcal{C}')$.

For simplicity, we write $\mathcal{D}'_\infty$ for $\mathcal{T}(\mathcal{D}',\mathcal{C}')$.

We show that $\mathcal{D}'_\infty(\sigma) = \mathcal{D}_\infty(\sigma)$ on $\sigma \in \mathrm{Dom}(\mathcal{D}'_\infty)$.

Let $\sigma \in \mathrm{Dom}(\mathcal{D}'_\infty)$. We show the statement by induction on $|\sigma|$.

Assume $\sigma \in S_1{}^\circ \cup \overline{S_2}$. By the definition of $\mathcal{D}'$ and $\mathcal{D}'_\infty$, $\mathcal{D}'_\infty(\sigma) = \mathcal{D}'(\sigma) = \mathcal{D}_\infty(\sigma)$.

Assume $\sigma \notin S_1{}^\circ \cup \overline{S_2}$. By the definition of $\mathcal{D}'_\infty$, there exists $\sigma_0 \preceq \sigma$ such that $\sigma_0 \in S_1$. Let $\sigma \equiv \sigma_0\sigma_1$ and $\mathcal{C}'(\sigma_0) \equiv \sigma_2$. By the definition of $\mathcal{D}'_\infty$, $\mathcal{D}'_\infty(\sigma) = \mathcal{D}'_\infty(\sigma_0\sigma_1) = \mathcal{D}'_\infty(\sigma_2\sigma_1)$. Since $\sigma_2 \preceq \sigma_0$, we have $\sigma_2\sigma_1 \preceq \sigma_0\sigma_1$. By the induction hypothesis, $\mathcal{D}'_\infty(\sigma_2\sigma_1) = \mathcal{D}_\infty(\sigma_2\sigma_1)$. Thus, $\mathcal{D}'_\infty(\sigma) = \mathcal{D}_\infty(\sigma_2\sigma_1) = \mathcal{D}_\infty(\sigma)$.

We show that $\mathrm{Dom}(\mathcal{D}'_\infty) = \mathrm{Dom}(\mathcal{D}_\infty)$. Since we have $\mathrm{Dom}(\mathcal{D}'_\infty) \subseteq \mathrm{Dom}(\mathcal{D}_\infty)$, it suffices to show $\mathrm{Dom}(\mathcal{D}'_\infty) \supseteq \mathrm{Dom}(\mathcal{D}_\infty)$.

We show that $\sigma \in \mathrm{Dom}(\mathcal{D}_\infty)$ implies $\sigma \in \mathrm{Dom}(\mathcal{D}'_\infty)$ by induction on $|\sigma|$.

Assume $\sigma \in \mathrm{Dom}(\mathcal{D}_\infty)$.

Assume $\sigma \in S_1{}^\circ \cup \overline{S_2}$. Then, $\sigma \in S_1{}^\circ \cup \overline{S_2} \subset \mathrm{Dom}(\mathcal{D}')$. Hence, $\sigma \in \mathrm{Dom}(\mathcal{D}'_\infty)$.

Assume $\sigma \notin S_1{}^\circ \cup \overline{S_2}$. Since we assume $\sigma \in \mathrm{Dom}(\mathcal{D}_\infty)$, there exists $\sigma_0 \in S_1$ such that $\sigma_0 \preceq \sigma$. Let $\sigma \equiv \sigma_0\sigma_1$ for a sequence σ_1 and $\mathcal{C}'(\sigma_0) \equiv \sigma_2$. By the induction hypothesis, $\sigma_2\sigma_1 \in \mathrm{Dom}(\mathcal{D}'_\infty)$. By the definition of $\mathcal{D}'_\infty$, $\mathcal{D}'_\infty(\sigma_0\sigma_1) = \mathcal{D}'_\infty(\sigma_2\sigma_1)$. Hence, $\sigma_0\sigma_1 \in \mathrm{Dom}(\mathcal{D}'_\infty)$.

Therefore, $\mathrm{Dom}(\mathcal{D}'_\infty) \supseteq \mathrm{Dom}(\mathcal{D}_\infty)$. Thus, $\mathrm{Dom}(\mathcal{D}'_\infty) = \mathrm{Dom}(\mathcal{D}_\infty)$.

Now, we have $\mathcal{D}_\infty = \mathcal{D}'_\infty$. □

Proposition 2.38 implies the following proposition, immediately.

Proposition 2.40. *For a* `CLKID`$^\omega$ *proof* $(\mathcal{D},\mathcal{C})$*, we have a* `CLKID`$^\omega$ *cycle-normal proof* $(\mathcal{D}',\mathcal{C}')$ *such that the tree-unfolding of* $(\mathcal{D}',\mathcal{C}')$ *is that of* $(\mathcal{D},\mathcal{C})$*.*

Proof. Let $(\mathcal{D},\mathcal{C})$ be a `CLKID`$^\omega$ proof. By Proposition 2.38, there exists a `CLKID`$^\omega$ cycle-normal pre-proof $(\mathcal{D}',\mathcal{C}')$ such that the tree-unfolding of $(\mathcal{D}',\mathcal{C}')$, is that of $(\mathcal{D},\mathcal{C})$. Since $(\mathcal{D},\mathcal{C})$ satisfies the global trace condition, the tree-unfolding of $(\mathcal{D},\mathcal{C})$ satisfies the global trace condition. Hence, $(\mathcal{D}',\mathcal{C}')$ satisfies the global trace condition. Thus, $(\mathcal{D}',\mathcal{C}')$ is a `CLKID`$^\omega$ cycle-normal. □

3 Counterexample to cut-elimination in first-order logic with inductive definitions

In this chapter, we prove the following theorem.

Theorem 3.1. *Let* 0 *be a constant symbol,* s *be a unary function symbol. Let* Add_1 *and* Add_2 *be ternary inductive predicates with the following productions:*

$$\frac{}{\mathrm{Add}_1(0,y,y)}\ , \qquad \frac{\mathrm{Add}_1(x,y,z)}{\mathrm{Add}_1(sx,y,sz)}\ , \qquad \frac{}{\mathrm{Add}_2(0,y,y)}\ , \qquad \frac{\mathrm{Add}_2(x,sy,z)}{\mathrm{Add}_2(sx,y,z)}\ .$$

(*1*) $\mathrm{Add}_2(x,y,z) \Rightarrow \mathrm{Add}_1(x,y,z)$ *is provable in* $\mathtt{CLKID}^\omega$*.*

(*2*) $\mathrm{Add}_2(x,y,z) \Rightarrow \mathrm{Add}_1(x,y,z)$ *is not cut-free provable in* $\mathtt{CLKID}^\omega$*.*

This theorem means that $\mathrm{Add}_2(x,y,z) \Rightarrow \mathrm{Add}_1(x,y,z)$ is a counterexample to cut-elimination in $\mathtt{CLKID}^\omega$.

In Section 3.1, we give a $\mathtt{CLKID}^\omega$ proof of $\mathrm{Add}_2(x,y,z) \Rightarrow \mathrm{Add}_1(x,y,z)$ with (Cut), and therefore we have Theorem 3.1 (1). Section 3.2 outline the proof of Theorem 3.1 (2). Section 3.1 gives another cyclic proof system $\mathtt{CLKID}^\omega_a$ to show Theorem 3.1 (2). Section 3.4 shows Theorem 3.1 (2). Section 3.5 discusses related work and the reason the cut-elimination property in the cyclic proof systems does not hold.

3.1 A $\mathtt{CLKID}^\omega$ proof of the counterexample with (Cut)

In this section, we show Theorem 3.1 (1).

We give the inference rules for Add_1 and Add_2 in Figure 3.1.

The derivation tree in Figure 3.3 is a $\mathtt{CLKID}^\omega$ proof of $\mathrm{Add}_2(x,y,z) \Rightarrow \mathrm{Add}_1(x,y,z)$, where $(\Diamond)$ indicates the pairing of the companion with the bud, and $\mathcal{D}_1$ is the $\mathtt{CLKID}^\omega$ proof in Figure 3.2, where $(\Diamond\Diamond)$ indicates the pairing of the companion with the bud, (some applying rules and some labels of rules are omitted for limited space). We use the underlined formulas to denote the infinitely progressing trace for some tails of any infinite path.

Since there is a $\mathtt{CLKID}^\omega$ proof of $\mathrm{Add}_2(x,y,z) \Rightarrow \mathrm{Add}_1(x,y,z)$, we have Theorem 3.1 (1). We henceforth prove Theorem 3.1 (2).

3.2 The outline of the proof

We outline our proof of Theorem 3.1 (2).

Assume there exists a cut-free $\mathtt{CLKID}^\omega$ proof of $\mathrm{Add}_2(x,y,z) \Rightarrow \mathrm{Add}_1(x,y,z)$, for contradiction. By the cycle-normalization property of $\mathtt{CLKID}^\omega$, there exists a cut-free cycle-normal $\mathtt{CLKID}^\omega$ proof of $\mathrm{Add}_2(x,y,z) \Rightarrow \mathrm{Add}_1(x,y,z)$. Let $\left(\mathcal{D}^1_{\mathrm{cf}}, \mathcal{C}^1_{\mathrm{cf}}\right)$ be the $\mathtt{CLKID}^\omega$ proof.

The key concepts for the proof are an *index sequent*, a *switching point*, and an *idling path*. To define these concepts, we define the relation $\cong_\Gamma$ for a finite set of formulas Γ to be the

$$\frac{}{\Gamma \Rightarrow \mathrm{Add}_1(0,b,b),\Delta}\ (\mathrm{Add}_1\ \mathrm{R}_1) \qquad \frac{\Gamma \Rightarrow \Delta, \mathrm{Add}_1(a,b,c)}{\Gamma \Rightarrow \Delta, \mathrm{Add}_1(sa,b,sc)}\ (\mathrm{Add}_1\ \mathrm{R}_2)$$

$$\frac{\Gamma, a=0, b=y, c=y \Rightarrow \Delta \qquad \Gamma, a=sx, b=y, c=sz, \mathrm{Add}_1(x,y,z) \Rightarrow \Delta}{\Gamma, \mathrm{Add}_1(a,b,c) \Rightarrow \Delta}\ (\textsc{Case}\ \mathrm{Add}_1)$$

$$(x,\ y,\ z \notin \mathrm{FV}(\Gamma \cup \Delta \cup \{\mathrm{Add}_1(a,b,c)\}) \text{ and } x,\ y,\ z \text{ are all distinct })$$

$$\frac{}{\Gamma \Rightarrow \mathrm{Add}_2(0,b,b),\Delta}\ (\mathrm{Add}_2\ \mathrm{R}_1) \qquad \frac{\Gamma \Rightarrow \Delta, \mathrm{Add}_2(a,sb,c)}{\Gamma \Rightarrow \Delta, \mathrm{Add}_2(sa,b,c)}\ (\mathrm{Add}_2\ \mathrm{R}_2)$$

$$\frac{\Gamma, a=0, b=y, c=y \Rightarrow \Delta \qquad \Gamma, a=sx, b=y, c=z, \mathrm{Add}_2(x,sy,z) \Rightarrow \Delta}{\Gamma, \mathrm{Add}_2(a,b,c) \Rightarrow \Delta}\ (\textsc{Case}\ \mathrm{Add}_2)$$

$$(x,\ y,\ z \notin \mathrm{FV}(\Gamma \cup \Delta \cup \{\mathrm{Add}_2(a,b,c)\}) \text{ and } x,\ y,\ z \text{ are all distinct})$$

Figure 3.1 The rules for Add_1 and Add_2

smallest congruence relation on terms containing $t_1 = t_2 \in \Gamma$ (Definition 3.4) and the *index of* $\mathrm{Add}_2(a,b,c)$ *in a sequent* $\Gamma \Rightarrow \Delta$ (Definition 3.12). If there uniquely exists $n - m$ such that $n,\ m \in \mathbb{N}$ and $s^n b \cong_\Gamma s^m b'$ for some $\mathrm{Add}_1(a',b',c') \in \Delta$, then the index of $\mathrm{Add}_2(a,b,c)$ is defined as $m - n$. If $s^n b \not\cong_\Gamma s^m b'$ for each $\mathrm{Add}_1(a',b',c') \in \Delta$ and all $n,\ m \in \mathbb{N}$, the index of $\mathrm{Add}_2(a,b,c)$ is defined as $\perp$. The index of $\mathrm{Add}_2(a,b,c)$ may be undefined, but the index is always defined in a special sequent, called an *index sequent* (Definition 3.13). A *switching point* is defined as a node that is the conclusion of ($\textsc{Case}$ Add_2) with the principal formula whose index is $\perp$ (Definition 3.15). An *idling path* is defined as a path $\{\Gamma_i \Rightarrow \Delta_i\}_{0 \le i < \alpha}$ of $\mathcal{T}\left(\mathcal{D}^1_{\mathrm{cf}}, \mathcal{C}^1_{\mathrm{cf}}\right)$ such that $\Gamma_0 \Rightarrow \Delta_0$ is an index sequent and $\Gamma_i \Rightarrow \Delta_i$ is a switching point if $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is the left assumption of $\Gamma_i \Rightarrow \Delta_i$ (Definition 3.16). Then, the following statements hold:

(1) The root is an index sequent;

(2) Every sequent in an idling path is an index sequent (Lemma 3.17);

(3) There exists a switching point on an infinite idling path (Lemma 3.19); and

(4) The rightmost path from an index sequent is infinite (Lemma 3.21).

At last, we show there exist infinite nodes in the derivation tree $\mathcal{D}^1_{\mathrm{cf}}$. Because of (1) and (4), the rightmost path from the root is an infinite idling path. By (3), there exists a switching point on the path. Let $\tilde{\sigma}_0$ be the node of the smallest height among such switching points. Let α_0 be the left assumption of $\tilde{\sigma}_0$. By (2), the sequent of α_0 is an index sequent. By (4), the rightmost path from α_0 is infinite. Therefore, there exists a bud μ_0 in the rightmost path from α_0. By (3) and the definition of $\tilde{\sigma}_0$, there exists a switching point between α_0 and μ_0. Let $\tilde{\sigma}_1$ be the node of the smallest height among such switching points. The nodes $\tilde{\sigma}_0$ and $\tilde{\sigma}_1$ are distinct by their definitions. By repeating this process as in Figure 3.4, we get a set of infinite nodes $\{\tilde{\sigma}_i \mid i \in \mathbb{N}\}$. It is a contradiction since the set of nodes of $\mathcal{D}^1_{\mathrm{cf}}$ is finite.

3.3 Another cyclic proof system $\mathtt{CLKID}^\omega_a$

We give some definitions and lemmas for proving (2) of Theorem 3.1.

We consider a cyclic proof system $\mathtt{CLKID}^\omega_a$, which is obtained by changing the left introduction rule for "=" slightly.

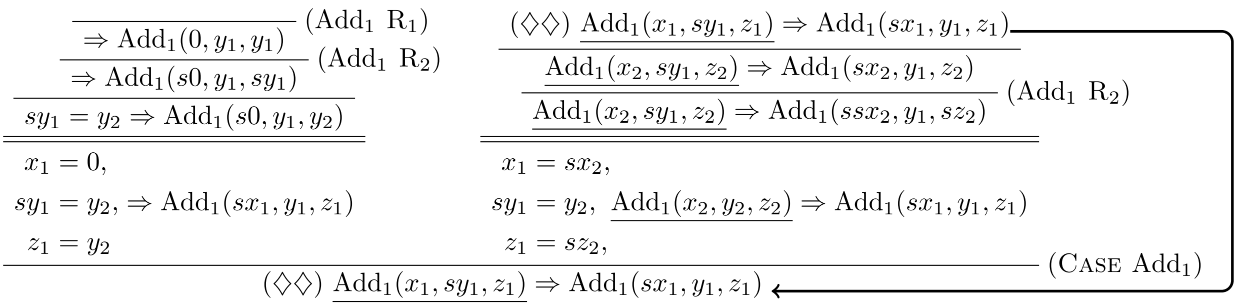


Figure 3.2 $\mathtt{CLKID}^{\omega}$ proof $\mathcal{D}_1$

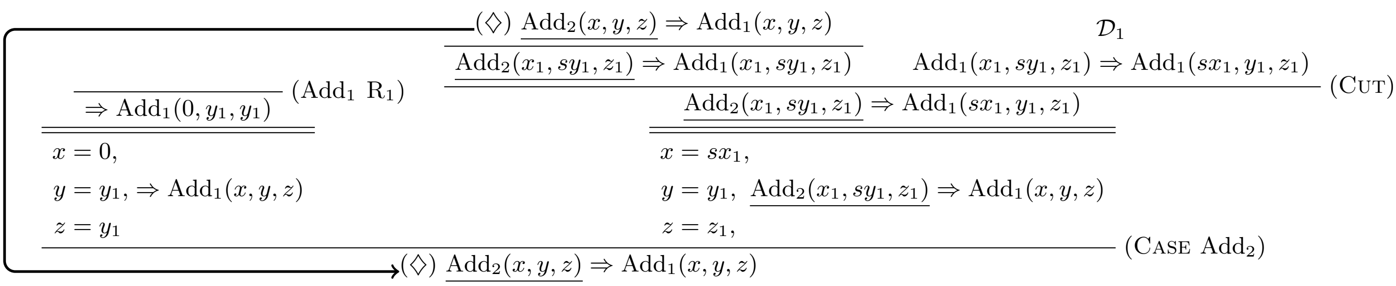


Figure 3.3 $\mathtt{CLKID}^{\omega}$ proof of $\mathrm{Add}_2(x, y, z) \Rightarrow \mathrm{Add}_1(x, y, z)$

Definition 3.2 ($\mathtt{CLKID}^{\omega}_{a}$). $\mathtt{CLKID}^{\omega}_{a}$ is the cyclic proof system obtained by replacing $(=\mathrm{L})$ with

$$\frac{\Gamma[x := u, y := t], t = u \Rightarrow \Delta[x := u, y := t]}{\Gamma[x := t, y := u], t = u \Rightarrow \Delta[x := t, y := u]}\ (=\mathrm{L}_a) \quad .$$

The principal formula of the rule of $(=\mathrm{L}_a)$ is defined as $t = u$.

Definitions of derivation trees, companions, pre-proofs, proofs for $\mathtt{CLKID}^{\omega}_{a}$ are similar to $\mathtt{CLKID}^{\omega}$.

The provability of $\mathtt{CLKID}^{\omega}$ is the same as that of $\mathtt{CLKID}^{\omega}_{a}$, since $(=\mathrm{L})$ is derivable in $\mathtt{CLKID}^{\omega}_{a}$ by

$$\cfrac{\cfrac{\Gamma[x := u, y := t] \Rightarrow \Delta[x := u, y := t]}{\Gamma[x := u, y := t], t = u \Rightarrow \Delta[x := u, y := t]}\ (\textsc{Weak})}{\Gamma[x := t, y := u], t = u \Rightarrow \Delta[x := t, y := u]}\ (=\mathrm{L}_a) \quad .$$

$\mathtt{CLKID}^{\omega}_{a}$ is necessary because of Lemma 3.18 (3). For $\mathtt{CLKID}^{\omega}$, Lemma 3.18 (3) does not hold.

Proposition 3.3. *If there exists a cut-free $\mathtt{CLKID}^{\omega}$ proof of $\Gamma \Rightarrow \Delta$, then there exists a cut-free cycle-normal $\mathtt{CLKID}^{\omega}_{a}$ proof of $\Gamma \Rightarrow \Delta$.*

Proof. Let $\mathscr{P}_0$ be a cut-free $\mathtt{CLKID}^{\omega}$ proof of $\Gamma \Rightarrow \Delta$. By Proposition 2.40, there exists a cycle-normal $\mathtt{CLKID}^{\omega}$ pre-proof $\mathscr{P}_1$ whose tree-unfolding is the same as that of $\mathscr{P}_0$. Since the

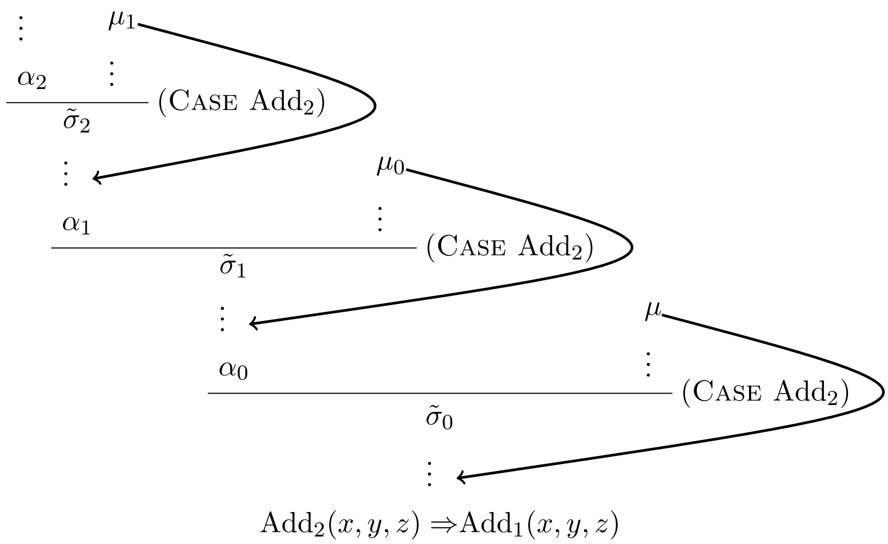


Figure 3.4 Construction of $\{\tilde{\sigma}_i\}_{i\in\mathbb{N}}$

tree-unfolding of $\mathscr{P}_1$ satisfies the global trace condition, $\mathscr{P}_1$ is a cycle-normal cut-free $\mathtt{CLKID}^\omega$ proof of $\Gamma \Rightarrow \Delta$.

A cut-free cycle-normal $\mathtt{CLKID}^\omega$ proof of $\Gamma \Rightarrow \Delta$ is transformed into the $\mathtt{CLKID}^\omega_a$ proof of $\Gamma \Rightarrow \Delta$ by replacing all applications from $(=\mathrm{L})$ to $(=\mathrm{L}_a)$and weakening. Since this replacement does not change the rules except $(=\mathrm{L})$ in the $\mathtt{CLKID}^\omega$ proof and the sequents of buds and companions, the obtained $\mathtt{CLKID}^\omega_a$ proof is cut-free and cycle-normal. □

3.4 The proof of Theorem 3.1 (2)

In this section, we prove Theorem 3.1 (2).

Throughout the remainder of this chapter, we assume there exists a cut-free $\mathtt{CLKID}^\omega$ proof of $\mathrm{Add}_2(x, y, z) \Rightarrow \mathrm{Add}_1(x, y, z)$ for contradiction. By Proposition 3.3, there exists a cut-free cycle-normal $\mathtt{CLKID}^\omega_a$ proof of $\mathrm{Add}_2(x, y, z) \Rightarrow \mathrm{Add}_1(x, y, z)$. *We write* $(\mathcal{D}^1_{\mathrm{cf}}, \mathcal{C}^1_{\mathrm{cf}})$ *for a cut-free cycle-normal* $\mathtt{CLKID}^\omega_a$ *proof of* $\mathrm{Add}_2(x, y, z) \Rightarrow \mathrm{Add}_1(x, y, z)$*.*

Remark. Let $\Gamma \Rightarrow \Delta$ be a sequent in $(\mathcal{D}^1_{\mathrm{cf}}, \mathcal{C}^1_{\mathrm{cf}})$. By induction on the height of sequents in $\mathcal{D}^1_{\mathrm{cf}}$, we can easily show the following statements:

(1) Γ consists of only atomic formulas with $=$, Add_2.

(2) Δ consists of only atomic formulas with Add_1.

(3) A term in Γ and Δ is of the form $s^n 0$ or $s^n x$ with some variable x.

(4) The possible rules in $(\mathcal{D}^1_{\mathrm{cf}}, \mathcal{C}^1_{\mathrm{cf}})$ are (Weak), (Sub), $(=\mathrm{L}_a)$, (Case Add_2), (Add_1 R_1) and (Add_1 R_2).

By (3), without loss of generality, we can assume terms are of the form $s^n 0$ or $s^n x$ with some variable x throughout the remainder of this chapter.

To define the key concept of the proof, called an index, we define the equality $\cong_\Gamma$ in a sequent $\Gamma \Rightarrow \Delta$ and show some properties.

Definition 3.4 ($\cong_\Gamma$)**.** For a set of formulas Γ, we define the relation $\cong_\Gamma$ to be the smallest congruence relation on terms which satisfies the condition that $t_1 = t_2 \in \Gamma$ implies $t_1 \cong_\Gamma t_2$.

Definition 3.5 ($\sim_\Gamma$)**.** For a set of formulas Γ and terms t_1, t_2, we define $t_1 \sim_\Gamma t_2$ by $s^n t_1 \cong_\Gamma s^m t_2$ for some $n, m \in \mathbb{N}$.

For a term t, we define $\mathrm{Var}(t)$ as a variable or a constant in t. Note that $\sim_\Gamma$ is a congruence relation and also note that $t \sim_\Gamma u$ if $\mathrm{Var}(t) = \mathrm{Var}(u)$.

Lemma 3.6. *Let Γ be a set of formulas and θ be a substitution.*

(1) *For terms t_1 and t_2, $t_1[\theta] \cong_{\Gamma[\theta]} t_2[\theta]$ if $t_1 \cong_\Gamma t_2$.*

(2) *For terms t_1 and t_2, $t_1 \not\sim_\Gamma t_2$ if $t_1[\theta] \not\sim_{\Gamma[\theta]} t_2[\theta]$.*

Proof. (1) We prove the statement by induction on the definition of $\cong_\Gamma$. We only show the base case. Assume $t_1 = t_2 \in \Gamma$. Then, $t_1[\theta] = t_2[\theta] \in \Gamma[\theta]$. Thus, $t_1[\theta] \cong_{\Gamma[\theta]} t_2[\theta]$.

(2) By Definition 3.5 and (1), we have the statement. □

Lemma 3.7. *Let Γ be a set of formulas, u_1, u_2 be terms, v_1, v_2 be variables, $\Gamma_1 \equiv (\Gamma[v_1 := u_1, v_2 := u_2], u_1 = u_2)$, and $\Gamma_2 \equiv (\Gamma[v_1 := u_2, v_2 := u_1], u_1 = u_2)$.*

(1) *For terms t_1 and t_2, $t_1[v_1 := u_1, v_2 := u_2] \cong_{\Gamma_1} t_2[v_1 := u_1, v_2 := u_2]$ if $t_1[v_1 := u_2, v_2 := u_1] \cong_{\Gamma_2} t_2[v_1 := u_2, v_2 := u_1]$.*

(2) *For terms t_1 and t_2, $t_1[v_1 := u_2, v_2 := u_1] \not\sim_{\Gamma_2} t_2[v_1 := u_2, v_2 := u_1]$ if $t_1[v_1 := u_1, v_2 := u_2] \not\sim_{\Gamma_1} t_2[v_1 := u_1, v_2 := u_2]$.*

Proof. (1) We prove the statement by induction on the definition of $\cong_{\Gamma_2}$. We only show the base case. Assume $t_1[v_1 := u_2, v_2 := u_1] = t_2[v_1 := u_2, v_2 := u_1] \in \Gamma_2$ to show $t_1[v_1 := u_1, v_2 := u_2] \cong_{\Gamma_1} t_2[v_1 := u_1, v_2 := u_2]$. If $t_1[v_1 := u_2, v_2 := u_1] = t_2[v_1 := u_2, v_2 := u_1]$ is $u_1 = u_2$, then $t_1 = t_2$ is $v_2 = v_1$, $v_2 = u_2$, $u_1 = v_1$, or $u_1 = u_2$. Therefore, $t_1[v_1 := u_1, v_2 := u_2] \cong_{\Gamma_1} t_2[v_1 := u_1, v_2 := u_2]$.

Assume $t_1[v_1 := u_2, v_2 := u_1] = t_2[v_1 := u_2, v_2 := u_1]$ is not $u_1 = u_2$. By case analysis, we have $t_1 = t_2 \in \Gamma$. Hence, $t_1[v_1 := u_1, v_2 := u_2] = t_2[v_1 := u_1, v_2 := u_2] \in \Gamma_1$. Therefore, we have $t_1[v_1 := u_1, v_2 := u_2] \cong_{\Gamma_1} t_2[v_1 := u_1, v_2 := u_2]$.

(2) By Definition 3.5 and (1), we have the statement. □

Lemma 3.8. *For a set of formulas Γ, the following statements are equivalent:*

(1) *$u_1 \cong_\Gamma u_2$.*

(2) *There exists a finite sequence of terms $\{t_i\}_{0 \le i \le n}$ with $n \ge 0$ such that $t_0 \equiv u_1$, $t_n \equiv u_2$ and $t_i = t_{i+1} \in [\Gamma]$ for $0 \le i < n$, where*

$$[\Gamma] = \{s^n t_1 = s^n t_2 \mid n \in \mathbb{N} \text{ and either } t_1 = t_2 \in \Gamma \text{ or } t_2 = t_1 \in \Gamma\}.$$

Proof. (1) ⇒ (2): Assume $u_1 \cong_\Gamma u_2$ to prove (2) by induction on the definition of $\cong_\Gamma$. We consider cases according to the clauses of the definition.

Case 1. If $u_1 = u_2 \in \Gamma$, then we have $u_1 = u_2 \in [\Gamma]$. Thus, we have (2).

Case 2. If $u_1 \equiv u_2$, then we have (2).

Case 3. We consider the case where $u_2 \cong_\Gamma u_1$. By the induction hypothesis, there exists a finite sequence of terms $\{t_i\}_{0 \le i \le n}$ such that $t_0 \equiv u_2$, $t_n \equiv u_1$ and $t_i = t_{i+1} \in [\Gamma]$ with $0 \le i < n$. Let $t'_i \equiv t_{n-i}$. The finite sequence of terms $\{t'_i\}_{0 \le i \le n}$ satisfies $t'_0 \equiv u_1$, $t'_n \equiv u_2$ and $t'_i = t'_{i+1} \in [\Gamma]$. Thus, we have (2).

Case 4. We consider the case where $u_1 \cong_\Gamma u_3$, $u_3 \cong_\Gamma u_2$. By the induction hypothesis, there exist two finite sequences of terms $\{t_i\}_{0\le i\le n}$, $\left\{t'_j\right\}_{0\le j\le m}$ such that $t_0 \equiv u_1$, $t_n \equiv t'_0 \equiv u_3$, $t'_m \equiv u_2$, $t_i = t_{i+1} \in [\Gamma]$ and $t'_j = t'_{j+1} \in [\Gamma]$ with $0 \le i < n$, $0 \le j < m$. Define $\hat{t}_k$ as t_k if $0 \le k < n$ and t'_{k-n} if $n \le k \le n+m$. The finite sequence of terms $\left\{\hat{t}_k\right\}_{0\le k\le n}$ satisfies $\hat{t}_0 \equiv u_1$, $\hat{t}_n \equiv u_2$ and $\hat{t}_k = \hat{t}_{k+1} \in [\Gamma]$. Thus, we have (2).

Case 5. We consider the case where $\hat{u}_1 \cong_\Gamma \hat{u}_2$, $u_1 \equiv u[v := \hat{u}_1]$ and $u_2 \equiv u[v := \hat{u}_2]$. By the induction hypothesis, there exists a finite sequence of terms $\{t_i\}_{0\le i\le n}$ with $n \in \mathbb{N}$ such that $t_0 \equiv \hat{u}_1$, $t_n \equiv \hat{u}_2$, $t_i = t_{i+1} \in [\Gamma]$ with $0 \le i < n$.

Assume v does not occur in u. In this case, we have $u_1 \equiv u[v := \hat{u}_1] \equiv u \equiv u[v := \hat{u}_2] \equiv u_2$. Hence, (2) holds.

Assume v occurs in u. In this case, we have $u \equiv s^m v$ for some natural numbers m. Let $t'_i = s^m t_i$ for $0 \le i \le n$. The finite sequence of terms $\{t'_i\}_{0\le i\le n}$ satisfies $t'_0 \equiv u_1$, $t'_n \equiv u_2$ and $t'_i = t'_{i+1} \in [\Gamma]$.

(2) $\Rightarrow$ (1): Assume (2) to show (1). By the assumption, there exists a finite sequence of terms $\{t_i\}_{0\le i\le n}$ with $n \in \mathbb{N}$ such that $t_0 \equiv u_1$, $t_n \equiv u_2$ and $t_i = t_{i+1} \in [\Gamma]$ with $0 \le i < n$. If $t_i = t_{i+1} \in [\Gamma]$, then $t_i = t_{i+1}$ is $s^n\hat{t}_1 = s^n\hat{t}_2$, where $\hat{t}_1 = \hat{t}_2 \in \Gamma$ or $\hat{t}_2 = \hat{t}_1 \in \Gamma$. Therefore, $t_i \cong_\Gamma t_{i+1}$. Because of the transitivity of $\cong_\Gamma$, we have $u_1 \cong_\Gamma u_2$. $\square$

Lemma 3.9. *For a set of formulas Γ_1 and $\Gamma_2 \equiv (\Gamma_1, u_1 = u'_1, \ldots, u_n = u'_n)$ with a natural number n, if $\mathrm{Var}(u'_i)$ $(i = 1, \ldots, n)$ do not occur in $\Gamma_1, u_1, \ldots, u_n, t, t'$ and are all distinct variables, then $t \cong_{\Gamma_2} t'$ implies $t \cong_{\Gamma_1} t'$.*

Proof. Let $\mathrm{Var}(u'_i) = v_i$ for each $i = 1, \ldots, n$. Assume $t \cong_{\Gamma_2} t'$, $t \not\cong_{\Gamma_1} v_i$ for all $i = 1, \ldots, n$. By Lemma 3.8, there exists a sequence $\{t_j\}_{0\le j\le m}$ with $m \in \mathbb{N}$ such that $t_0 \equiv t$, $t_m \equiv t'$ and $t_j = t_{j+1} \in [\Gamma_2]$ with $0 \le j < m$. We show $t \cong_{\Gamma_1} t'$ by induction on m.

For $m = 0$, we have $t \cong_{\Gamma_1} t'$ immediately.

We consider the case where $m > 0$.

If $t_j \not\equiv s^l u'_i$ for all $i = 1, \ldots, n$, $0 \le j \le m$ and $l \in \mathbb{N}$, then $t_j = t_{j+1} \in [\Gamma_1]$ with $0 \le i < m$. By Lemma 3.8, we have $t \cong_{\Gamma_1} t'$.

Assume that there exists j_0 with $0 \le j_0 \le m$ such that $t_{j_0} \equiv s^l u'_i$ for some $i = 1, \ldots, n$ and $l \in \mathbb{N}$. Since any formula of $[\Gamma_2]$ in which u'_i occurs is either $s^{l_0}u_i = s^{l_0}u'_i$ or $s^{l_0}u'_i = s^{l_0}u_i$ with $l_0 \in \mathbb{N}$ and $\mathrm{Var}(u'_i)$ $(i = 1, \ldots, n)$ do not occur in t, t', we have $t_{j_0-1} \equiv t_{j_0+1} \equiv s^l u_i$. Define $\bar{t}_k$ as t_k if $0 \le k < j_0$ and t_{k+1} if $j_0 \le k \le m-1$. Then, $\bar{t}_0 \equiv t$, $\bar{t}_{m-1} \equiv t'$ and $\bar{t}_k = \bar{t}_{k+1} \in [\Gamma_2]$ with $0 \le k < m-1$. By the induction hypothesis, we have $t \cong_{\Gamma_1} t'$. $\square$

Lemma 3.10. *For a set of formulas Γ_1 and $\Gamma_2 \equiv (\Gamma_1, u_1 = u'_1, \ldots, u_n = u'_n)$ with a natural number n, if $t \not\cong_{\Gamma_1} u_i$ and $t \not\cong_{\Gamma_1} u'_i$ with $i = 1, \ldots, n$, then $t \cong_{\Gamma_2} t'$ implies $t \cong_{\Gamma_1} t'$.*

Proof. Assume $t \not\cong_{\Gamma_1} u_i$, $t \not\cong_{\Gamma_1} u'_i$ for $i = 1, \ldots, n$, and $t \cong_{\Gamma_2} t'$. By Lemma 3.8, there exists a sequence $\{t_j\}_{0\le j\le m}$ with $m \in \mathbb{N}$ such that $t_0 \equiv t$, $t_m \equiv t'$ and $t_j = t_{j+1} \in [\Gamma_2]$ with $0 \le j < m$.

Assume, for all $0 \le j \le n$, $i = 1, \ldots, n$, $t_j \not\equiv s^l u_i$ and $t_j \not\equiv s^l u'_i$ with all $l \in \mathbb{N}$. Then, $t_j = t_{j+1} \in [\Gamma_1]$ with all $0 \le j < m$. By Lemma 3.8, we have $t \cong_{\Gamma_1} t'$.

Assume that there exists j with $0 \le j \le n$, such that $t_j \equiv s^l u_i$ or $t_j \equiv s^l u'_i$ for $i = 1, \ldots, n$, and some $l \in \mathbb{N}$. Let j_0 be the least number among such j's. Since j_0 is the least, we have $t_j = t_{j+1} \in [\Gamma_1]$ for all $0 \le j < j_0$. By Lemma 3.8, we have $t \cong_{\Gamma_1} s^l u_i$ or $t \cong_{\Gamma_1} s^l u'_i$. This contradicts $t \not\cong_{\Gamma_1} u_i$ and $t \not\cong_{\Gamma_1} u'_i$. $\square$

We call the assumption of (Case Add_2) whose form is

$$\Gamma, a = sx, b = y, c = z, \mathrm{Add}_2(x, sy, z) \Rightarrow \Delta$$

the right assumption of the rule. The other assumption is called *the left assumption of* the rule.

Lemma 3.11. *Let* $\Gamma \Rightarrow \Delta$ *be in* $\mathcal{D}^1_{\mathrm{cf}}$ *and*

$$\mathrm{A}(\Gamma \Rightarrow \Delta) = \{a \mid \mathrm{Add}_2(a,b,c) \in \Gamma, \mathrm{Add}_1(a,b,c) \in \Delta, \text{ or } a \equiv 0\} \text{ and}$$
$$\mathrm{BC}(\Gamma \Rightarrow \Delta) = \{b \mid \mathrm{Add}_2(a,b,c) \in \Gamma \text{ or } \mathrm{Add}_1(a,b,c) \in \Delta\} \cup \{c \mid \mathrm{Add}_2(a,b,c) \in \Gamma \text{ or } \mathrm{Add}_1(a,b,c) \in \Delta\}.$$

If $t \in \mathrm{A}(\Gamma \Rightarrow \Delta)$ *and* $u \in \mathrm{BC}(\Gamma \Rightarrow \Delta)$, *then* $t \not\sim_\Gamma u$.

Proof. We prove the statement by induction on the height of the node $\Gamma \Rightarrow \Delta$ in $\mathcal{D}^1_{\mathrm{cf}}$.

The root of $\mathcal{D}^1_{\mathrm{cf}}$ satisfies the statement.

Assume $\Gamma \Rightarrow \Delta$ is not the root. Let $\Gamma' \Rightarrow \Delta'$ be the parent of $\Gamma \Rightarrow \Delta$. We consider cases according to the rule with the conclusion $\Gamma' \Rightarrow \Delta'$.

Case 1. In the case (Weak), we have the statement by $\Gamma \subseteq \Gamma'$.

Case 2. In the case (Sub), we have the statement by Lemma 3.6 (2).

Case 3. In the case $(= \mathrm{L}_a)$, we have the statement by Lemma 3.7 (2).

Case 4. We consider the case where the rule is (Case Add_2) and $\Gamma \Rightarrow \Delta$ is the right assumption of the rule. Let $\mathrm{Add}_2(a,b,c)$ be the principal formula of the rule. There exists Π such that $\Gamma' \equiv (\Pi, \mathrm{Add}_2(a,b,c))$ and $\Gamma \equiv (\Pi, a = sx, b = y, c = z, \mathrm{Add}_2(x, sy, z))$ for fresh variables x, y, z.

Assume $t \in \mathrm{A}(\Gamma \Rightarrow \Delta)$ and $u \in \mathrm{BC}(\Gamma \Rightarrow \Delta)$ and $t \sim_\Gamma u$ for contradiction.

Define $\hat{t}$ as a if $t \equiv x$ and t otherwise. We also define $\hat{u}$ as b if $u \equiv sy$, c if $u \equiv z$ and u otherwise. Since $t \sim_\Gamma u$ holds, we have $\hat{t} \sim_\Gamma \hat{u}$. By Lemma 3.9, we have $\hat{t} \sim_{\Gamma'} \hat{u}$. Since $\hat{t} \in \mathrm{A}(\Gamma' \Rightarrow \Delta')$ and $\hat{u} \in \mathrm{BC}(\Gamma' \Rightarrow \Delta')$ hold, this contradicts the induction hypothesis.

Case 5. We consider the case where the rule is (Case Add_2) and $\Gamma \Rightarrow \Delta$ is the left assumption of the rule.

Let $\mathrm{Add}_2(a,b,c)$ be the principal formula of the rule. There exists Π such that $\Gamma' \equiv (\Pi, \mathrm{Add}_2(a,b,c))$ and $\Gamma \equiv (\Pi, a = 0, b = y, c = y)$ for a fresh variable y. Let $\Pi' \equiv (\Pi, b = y, c = y)$.

Let $t \in \mathrm{A}(\Gamma \Rightarrow \Delta)$ and $u \in \mathrm{BC}(\Gamma \Rightarrow \Delta)$. Since $\mathrm{A}(\Gamma \Rightarrow \Delta) \subseteq \mathrm{A}(\Gamma' \Rightarrow \Delta')$ holds, we have $t \in \mathrm{A}(\Gamma' \Rightarrow \Delta')$. By $\mathrm{BC}(\Gamma \Rightarrow \Delta) \subseteq \mathrm{BC}(\Gamma' \Rightarrow \Delta')$, we have $u \in \mathrm{BC}(\Gamma' \Rightarrow \Delta')$. By the induction hypothesis, $t \not\sim_{\Gamma'} u$, $t \not\sim_{\Gamma'} b$ and $t \not\sim_{\Gamma'} c$. Since the set of formulas with $=$ in Π is the same as the set of formulas with $=$ in Γ', we have $t \not\sim_\Pi u$, $t \not\sim_\Pi b$ and $t \not\sim_\Pi c$. By Lemma 3.10, $t \not\sim_{\Pi'} u$.

By the induction hypothesis, $u \not\sim_{\Gamma'} a$, $a \not\sim_{\Gamma'} b$ and $a \not\sim_{\Gamma'} c$. Since the set of formulas with $=$ in Π is the same as the set of formulas with $=$ in Γ', $u \not\sim_\Pi a$, $a \not\sim_\Pi b$ and $a \not\sim_\Pi c$. By Lemma 3.10, $u \not\sim_{\Pi'} a$.

By the induction hypothesis, $u \not\sim_{\Gamma'} 0$, $0 \not\sim_{\Gamma'} b$ and $0 \not\sim_{\Gamma'} c$. Since the set of formulas with $=$ in Π is the same as the set of formulas with $=$ in Γ', $u \not\sim_\Pi 0$, $0 \not\sim_\Pi b$ and $0 \not\sim_\Pi c$. By Lemma 3.10, $u \not\sim_{\Pi'} 0$.

By Lemma 3.10 and these three facts, $t \not\sim_\Gamma u$.

Case 6. In the case (Add_1 R_2), $\Gamma \equiv \Gamma'$ implies the statement by the induction hypothesis. □

We define a key concept, called an index.

Definition 3.12 (Index)**.** For a sequent $\Gamma \Rightarrow \Delta$ and $\mathrm{Add}_2(a,b,c) \in \Gamma$, we define *the index of* $\mathrm{Add}_2(a,b,c)$ *in* $\Gamma \Rightarrow \Delta$ as follows:

(1) If $b \not\sim_\Gamma b'$ for all $\mathrm{Add}_1(a',b',c') \in \Delta$, then the index of $\mathrm{Add}_2(a,b,c)$ in $\Gamma \Rightarrow \Delta$ is $\bot$, and

(2) if there exists uniquely $m - n$ such that n, $m \in \mathbb{N}$, $s^n b \cong_\Gamma s^m b'$ and $\mathrm{Add}_1(a',b',c') \in \Delta$, then the index of $\mathrm{Add}_2(a,b,c)$ in $\Gamma \Rightarrow \Delta$ is $m - n$ (namely the uniqueness means that $s^{n'} b \cong_\Gamma s^{m'} b''$ for m',$n' \in \mathbb{N}$ and $\mathrm{Add}_1(a'',b'',c'') \in \Delta$ imply $m - n = m' - n'$).

Note that if there exists $n, m \in \mathbb{N}$ such that $s^n b \cong_\Gamma s^m b'$ for some $\mathrm{Add}_1(a', b', c')$ and $m - n$ is not unique, then the index of $\mathrm{Add}_2(a, b, c)$ in $\Gamma \Rightarrow \Delta$ is undefined.

Definition 3.13 (Index sequent)**.** The sequent $\Gamma \Rightarrow \Delta$ is said to be an *index sequent* if the following conditions hold:

(1) If $t \in \mathrm{B}_1(\Gamma \Rightarrow \Delta)$ and $u \in \mathrm{C}(\Gamma \Rightarrow \Delta)$, then $t \not\sim_\Gamma u$, and

(2) if $s^n b \cong_\Gamma s^m b'$ with $b,\ b' \in \mathrm{B}_1(\Gamma \Rightarrow \Delta)$, then $n = m$, where

$$\mathrm{B}_1(\Gamma \Rightarrow \Delta) = \{b \mid \ \mathrm{Add}_1(a, b, c) \in \Delta\}, \text{ and}$$
$$\mathrm{C}(\Gamma \Rightarrow \Delta) = \{c \mid \ \mathrm{Add}_2(a, b, c) \in \Gamma \text{ or } \mathrm{Add}_1(a, b, c) \in \Delta\}.$$

This condition (2) guarantees the existence of an index, as shown in the following lemma. We will use (1) to calculate an index in Lemma 3.18 (1) and an infinite sequence in Lemma 3.21.

Lemma 3.14. *If $\Gamma \Rightarrow \Delta$ is an index sequent, the index of each $\mathrm{Add}_2(a, b, c) \in \Gamma$ in $\Gamma \Rightarrow \Delta$ is defined.*

Proof. If $b \not\sim_\Gamma b'$ for all $\mathrm{Add}_1(a', b', c') \in \Delta$, then the index is $\perp$.

Assume $b \sim_\Gamma b'_0$ for some $\mathrm{Add}_1(a'_0, b'_0, c'_0) \in \Delta$. By Definition 3.5, there exist n_0 and m_0 such that $s^{n_0} b \cong_\Gamma s^{m_0} b'_0$. To show the uniqueness, we fix $\mathrm{Add}_1(a'_1, b'_1, c'_1) \in \Delta$ and assume $s^{n_1} b \cong_\Gamma s^{m_1} b'_1$. Since $s^{n_0+n_1} b \cong_\Gamma s^{m_0+n_1} b'_0$ and $s^{n_1+n_0} b \cong_\Gamma s^{m_1+n_0} b'_1$, we have $s^{m_0+n_1} b'_0 \cong_\Gamma s^{m_1+n_0} b'_1$. From (2) of Definition 3.13, $m_0 + n_1 = m_1 + n_0$. Thus, $m_0 - n_0 = m_1 - n_1$. □

Definition 3.15 (Switching point)**.** A node σ in a derivation tree is called a *switching point* if the rule with the conclusion σ is (Case Add_2) and the index of the principal formula for the rule in the conclusion is $\perp$.

Definition 3.16 (Idling path)**.** A path $\{\Gamma_i \Rightarrow \Delta_i\}_{0 \le i < \alpha}$ in $\mathcal{T}\left(\mathcal{D}^1_{\mathrm{cf}}, \mathcal{C}^1_{\mathrm{cf}}\right)$ with some $\alpha \in \mathbb{N} \cup \{\omega\}$ is said to be an *idling path* if the following conditions hold:

(1) $\Gamma_0 \Rightarrow \Delta_0$ is an index sequent, and

(2) if the rule for $\Gamma_i \Rightarrow \Delta_i$ is (Case Add_2) and $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is the left assumption of the rule, then $\Gamma_i \Rightarrow \Delta_i$ is a switching point.

Lemma 3.17. *Every sequent in an idling path is an index sequent.*

Proof. Let $\{\Gamma_i \Rightarrow \Delta_i\}_{0 \le i < \alpha}$ be an idling path. We use $\mathrm{B}_1(\Gamma \Rightarrow \Delta)$ and $\mathrm{C}(\Gamma \Rightarrow \Delta)$ in Definition 3.13. We prove the statement by the induction on i.

For $i = 0$, $\Gamma_0 \Rightarrow \Delta_0$ is an index sequent by Definition 3.16.

For $i > 0$, we consider cases according to the rule with the conclusion $\Gamma_{i-1} \Rightarrow \Delta_{i-1}$.

Case 1. The case (Weak).

(1) Assume that $t \in \mathrm{B}_1(\Gamma_i \Rightarrow \Delta_i)$ and $u \in \mathrm{C}(\Gamma_i \Rightarrow \Delta_i)$. Since $\mathrm{B}_1(\Gamma_i \Rightarrow \Delta_i) \subseteq \mathrm{B}_1(\Gamma_{i-1} \Rightarrow \Delta_{i-1})$ holds, we have $t \in \mathrm{B}_1(\Gamma_{i-1} \Rightarrow \Delta_{i-1})$. By $\mathrm{C}(\Gamma_i \Rightarrow \Delta_i) \subseteq \mathrm{C}(\Gamma_{i-1} \Rightarrow \Delta_{i-1})$, we have $u \in \mathrm{C}(\Gamma_{i-1} \Rightarrow \Delta_{i-1})$. By the induction hypothesis (1), we have $t \not\sim_{\Gamma_{i-1}} u$. By $\Gamma_i \subseteq \Gamma_{i-1}$, we have $t \not\sim_{\Gamma_i} u$.

(2) Assume that $s^n b \cong_{\Gamma_i} s^m b'$ with $b,\ b' \in \mathrm{B}_1(\Gamma_i \Rightarrow \Delta_i)$ for $n,\ m \in \mathbb{N}$. By $\Gamma_i \subseteq \Gamma_{i-1}$, we have $s^n b \cong_{\Gamma_{i-1}} s^m b'$. Since $\mathrm{B}_1(\Gamma_i \Rightarrow \Delta_i) \subseteq \mathrm{B}_1(\Gamma_{i-1} \Rightarrow \Delta_{i-1})$ holds, we have $b,\ b' \in \mathrm{B}_1(\Gamma_{i-1} \Rightarrow \Delta_{i-1})$. By the induction hypothesis (2) , we have $n = m$.

Case 2. The case (Sub) with a substitution θ.

(1) Assume that $t \in \mathrm{B}_1(\Gamma_i \Rightarrow \Delta_i)$ and $u \in \mathrm{C}(\Gamma_i \Rightarrow \Delta_i)$. Since $\Gamma_{i-1} \equiv \Gamma_i[\theta]$ and $\Delta_{i-1} \equiv \Delta_i[\theta]$ hold, we have $t[\theta] \in \mathrm{B}_1(\Gamma_{i-1} \Rightarrow \Delta_{i-1})$ and $u[\theta] \in \mathrm{C}(\Gamma_{i-1} \Rightarrow \Delta_{i-1})$. By the induction hypothesis (1) , we have $t[\theta] \not\sim_{\Gamma_{i-1}} u[\theta]$. By Lemma 3.6 (2), we have $t \not\sim_{\Gamma_i} u$.

(2) Assume that $s^n b \cong_{\Gamma_i} s^m b'$ with b, $b' \in \mathrm{B}_1(\Gamma_i \Rightarrow \Delta_i)$ for n, $m \in \mathbb{N}$. By Lemma 3.6 (1), $s^n b[\theta] \cong_{\Gamma_{i-1}} s^m b'[\theta]$. Since $\Delta_{i-1} \equiv \Delta_i[\theta]$ holds, we have $b[\theta]$, $b'[\theta] \in \mathrm{B}_1(\Gamma_{i-1} \Rightarrow \Delta_{i-1})$. By the induction hypothesis (2), we have $n = m$.

Case 3. The case $(= \mathrm{L}_a)$.

Let $u_1 = u_2$ be the principal formula of the rule. There exist Γ and Δ such that

$$\begin{aligned}
\Gamma_{i-1} &\equiv (\Gamma[v_1 := u_1, v_2 := u_2], u_1 = u_2),\\
\Delta_{i-1} &\equiv (\Delta[v_1 := u_1, v_2 := u_2], u_1 = u_2),\\
\Gamma_i &\equiv (\Gamma[v_1 := u_2, v_2 := u_1], u_1 = u_2), \text{ and}\\
\Delta_i &\equiv (\Delta[v_1 := u_2, v_2 := u_1], u_1 = u_2).
\end{aligned}$$

(1) Assume that $t \in \mathrm{B}_1(\Gamma_i \Rightarrow \Delta_i)$ and $u \in \mathrm{C}(\Gamma_i \Rightarrow \Delta_i)$. From the definition of Γ_i and Δ_i, there exist terms $\hat{t}$, $\hat{u}$ such that $t \equiv \hat{t}[v_1 := u_2, v_2 := u_1]$ and $u \equiv \hat{u}[v_1 := u_2, v_2 := u_1]$. Then, $\hat{t}[v_1 := u_1, v_2 := u_2] \in \mathrm{B}_1(\Gamma_{i-1} \Rightarrow \Delta_{i-1})$ and $\hat{u}[v_1 := u_1, v_2 := u_2] \in \mathrm{C}(\Gamma_{i-1} \Rightarrow \Delta_{i-1})$. By the induction hypothesis (1), we have $\hat{t}[v_1 := u_1, v_2 := u_2] \not\sim_{\Gamma_{i-1}} \hat{u}[v_1 := u_1, v_2 := u_2]$. By Lemma 3.7 (2), we have $\hat{t}[v_1 := u_2, v_2 := u_1] \not\sim_{\Gamma_i} \hat{u}[v_1 := u_2, v_2 := u_1]$. Thus, $t \not\sim_{\Gamma_i} u$.

(2) Assume that $s^n b \cong_{\Gamma_i} s^m b'$ with b, $b' \in \mathrm{B}_1(\Gamma_i \Rightarrow \Delta_i)$ for n, $m \in \mathbb{N}$. From the definition of Γ_i and Δ_i, there exist terms $\hat{b}$, $\hat{b}' \in \Delta$ such that $b \equiv s^n\hat{b}[v_1 := u_2, v_2 := u_1]$ and $b' \equiv s^m\hat{b}'[v_1 := u_2, v_2 := u_1]$. By Lemma 3.7 (1), $s^n\hat{b}[v_1 := u_1, v_2 := u_2] \cong_{\Gamma_{i-1}} s^m\hat{b}'[v_1 := u_1, v_2 := u_2]$. From the definition of Γ_{i-1} and Δ_{i-1}, $\hat{b}[v_1 := u_1, v_2 := u_2]$, $\hat{b}'[v_1 := u_1, v_2 := u_2] \in \mathrm{B}_1(\Gamma_{i-1} \Rightarrow \Delta_{i-1})$. By the induction hypothesis (2), we have $n = m$.

Case 4. The case (Case Add_2) with the right assumption $\Gamma_i \Rightarrow \Delta_i$.

Let $\mathrm{Add}_2\left(a, \hat{b}, c\right)$ be the principal formula of the rule. There exists Π such that $\Gamma_{i-1} \equiv \left(\Pi, \mathrm{Add}_2\left(a, \hat{b}, c\right)\right)$ and $\Gamma_i \equiv \left(\Pi, a = sx, \hat{b} = y, c = z, \mathrm{Add}_2(x, sy, z)\right)$ for fresh variables x, y, z.

(1) Assume that $t \in \mathrm{B}_1(\Gamma_i \Rightarrow \Delta_i)$ and $u \in \mathrm{C}(\Gamma_i \Rightarrow \Delta_i)$. Assume that $t \sim_{\Gamma_i} u$ for contradiction. Define $\hat{u}$ as c if $u \equiv z$ and u otherwise. Since $t \sim_{\Gamma_i} u$ holds, we have $t \sim_{\Gamma_i} \hat{u}$. By Lemma 3.9, we have $t \sim_{\Gamma_{i-1}} \hat{u}$. Since $t \in \mathrm{B}_1(\Gamma_{i-1} \Rightarrow \Delta_{i-1})$ and $\hat{u} \in \mathrm{C}(\Gamma_{i-1} \Rightarrow \Delta_{i-1})$ hold, this contradicts the induction hypothesis (1).

(2) Assume that $s^n b \cong_{\Gamma_i} s^m b'$ with b, $b' \in \mathrm{B}_1(\Gamma_i \Rightarrow \Delta_i)$ for n, $m \in \mathbb{N}$. By Lemma 3.9, $s^n b \cong_{\Gamma_{i-1}} s^m b'$. Since $\Delta_{i-1} \equiv \Delta_i$ holds, we have b, $b' \in \mathrm{B}_1(\Gamma_{i-1} \Rightarrow \Delta_{i-1})$. By the induction hypothesis (2), we have $n = m$.

Case 5. The case (Case Add_2) with the left assumption $\Gamma_i \Rightarrow \Delta_i$. In this case, $\Gamma_{i-1} \Rightarrow \Delta_{i-1}$ is a switching point.

Let $\mathrm{Add}_2\left(a, \hat{b}, c\right)$ be the principal formula of the rule. There exists Π such that $\Gamma_{i-1} \equiv \left(\Pi, \mathrm{Add}_2\left(a, \hat{b}, c\right)\right)$ and $\Gamma_i \equiv \left(\Pi, a = 0, \hat{b} = y, c = y\right)$ with a fresh variable y.

(1) Assume that $t \in \mathrm{B}_1(\Gamma_i \Rightarrow \Delta_i)$ and $u \in \mathrm{C}(\Gamma_i \Rightarrow \Delta_i)$. Since $\mathrm{B}_1(\Gamma_i \Rightarrow \Delta_i) = \mathrm{B}_1(\Gamma_{i-1} \Rightarrow \Delta_{i-1})$ holds, we have $t \in \mathrm{B}_1(\Gamma_{i-1} \Rightarrow \Delta_{i-1})$. By $\mathrm{C}(\Gamma_i \Rightarrow \Delta_i) \subseteq \mathrm{C}(\Gamma_{i-1} \Rightarrow \Delta_{i-1})$, we have $u \in \mathrm{C}(\Gamma_{i-1} \Rightarrow \Delta_{i-1})$. By the induction hypothesis (1), $t \not\sim_{\Gamma_{i-1}} u$ and $t \not\sim_{\Gamma_{i-1}} c$. By Lemma 3.11, $t \not\sim_{\Gamma_{i-1}} a$ and $t \not\sim_{\Gamma_{i-1}} 0$. Since y is fresh, we have $t \not\sim_{\Gamma_{i-1}} y$. Since $\Gamma_{i-1} \Rightarrow \Delta_{i-1}$ is a switching point, we have $t \not\sim_{\Gamma_{i-1}} \hat{b}$. By Lemma 3.10, $t \not\sim_{\Gamma_i} u$.

(2) Assume that $s^n b \cong_{\Gamma_i} s^m b'$ with b, $b' \in \mathrm{B}_1(\Gamma_i \Rightarrow \Delta_i)$ for n, $m \in \mathbb{N}$ to show $n = m$. By Lemma 3.11, $s^n b \not\sim_{\Gamma_i} a$ and $s^n b \not\sim_{\Gamma_i} 0$. Since $\Gamma_{i-1} \Rightarrow \Delta_{i-1}$ is a switching point, we have $s^n b \not\sim_{\Gamma_{i-1}} \hat{b}$. By the induction hypothesis (1), $s^n b \not\sim_{\Gamma_{i-1}} c$. Since y is fresh, we have $s^n b \not\sim_{\Gamma_{i-1}}$

y. By Lemma 3.10, we have $s^n b \cong_{\Gamma_{i-1}} s^m b'$. Because of $\mathrm{B}_1(\Gamma_i \Rightarrow \Delta_i) = \mathrm{B}_1(\Gamma_{i-1} \Rightarrow \Delta_{i-1})$, we have b, $b' \in \mathrm{B}_1(\Gamma_{i-1} \Rightarrow \Delta_{i-1})$. By the induction hypothesis (2), we have $n = m$.

Case 6. The case (Add_1 R_2). Let $\mathrm{Add}_1\left(sa, \hat{b}, sc\right)$ be the principal formula of the rule.

(1) Assume that $t \in \mathrm{B}_1(\Gamma_i \Rightarrow \Delta_i)$ and $u \in \mathrm{C}(\Gamma_i \Rightarrow \Delta_i)$ and $t \sim_{\Gamma_i} u$ for contradiction. Define $\hat{u}$ as sc if $u \equiv c$ and u otherwise. Since $t \sim_{\Gamma_i} u$ holds, we have $t \sim_{\Gamma_i} \hat{u}$. Since $\Gamma_{i-1} = \Gamma_i$ holds, we have $t \sim_{\Gamma_{i-1}} \hat{u}$. Since $t \in \mathrm{B}_1(\Gamma_{i-1} \Rightarrow \Delta_{i-1})$ and $\hat{u} \in \mathrm{C}(\Gamma_{i-1} \Rightarrow \Delta_{i-1})$ hold, this contradicts the induction hypothesis (1).

(2) Assume that $s^n b \cong_{\Gamma_i} s^m b'$ with b, $b' \in \mathrm{B}_1(\Gamma_i \Rightarrow \Delta_i)$ for n, $m \in \mathbb{N}$. Because $\Gamma_{i-1} = \Gamma_i$, we have $s^n b \cong_{\Gamma_{i-1}} s^m b'$. Since the second argument of a formula with Add_1 in Δ_i is that in Δ_{i-1}, we have b, $b' \in \mathrm{B}_1(\Gamma_{i-1} \Rightarrow \Delta_{i-1})$. By the induction hypothesis (2), we have $n = m$. ☐

Lemma 3.18. *For an idling path $\{\Gamma_i \Rightarrow \Delta_i\}_{0 \le i < \alpha}$ and a trace $\{\tau_k\}_{k \ge 0}$ following $\{\Gamma_i \Rightarrow \Delta_i\}_{i \ge p}$, if d_k is the index of τ_k, the following statements holds:*

(1) *If $d_k = \bot$, then $d_{k+1} = \bot$.*

(2) *If the rule with the conclusion $\Gamma_{p+k} \Rightarrow \Delta_{p+k}$ is (Weak) or (Sub), then $d_{k+1} = d_k$ or $d_{k+1} = \bot$.*

(3) *If the rule with the conclusion $\Gamma_{p+k} \Rightarrow \Delta_{p+k}$ is $(= \mathrm{L}_a)$or (Add_1 R_2), then $d_{k+1} = d_k$.*

(4) *Assume the rule with the conclusion $\Gamma_{p+k} \Rightarrow \Delta_{p+k}$ is (Case Add_2).*

  (a) *If $\Gamma_{p+k+1} \Rightarrow \Delta_{p+k+1}$ is the left assumption of the rule, then $d_{k+1} = d_k$.*

  (b) *If $\Gamma_{p+k+1} \Rightarrow \Delta_{p+k+1}$ is the right assumption of the rule and τ_k is not a progress point of the trace, then $d_{k+1} = d_k$.*

  (c) *If $\Gamma_{p+k+1} \Rightarrow \Delta_{p+k+1}$ is the right assumption of the rule and τ_k is a progress point of the trace, then $d_{k+1} = d_k + 1$.*

Proof. Let $\tau_k \equiv \mathrm{Add}_2(a_k, b_k, c_k)$.
(1) It suffices to show that $b_{k+1} \not\sim_{\Gamma_{p+k+1}} b'$ holds for all $\mathrm{Add}_1(a', b', c') \in \Delta_{p+k+1}$ if $b_k \not\sim_{\Gamma_{p+k}} b$ holds for all $\mathrm{Add}_1(a, b, c) \in \Delta_{p+k}$. We consider cases according to the rule with the conclusion $\Gamma_{p+k} \Rightarrow \Delta_{p+k}$.

Case 1. If the rule is (Weak), we have the statement by $\Gamma_{p+k+1} \subseteq \Gamma_{p+k}$ and $\Delta_{p+k+1} \subseteq \Delta_{p+k}$.

Case 2. If the rule is (Sub), we have the statement by Lemma 3.6 (2).

Case 3. If the rule is $(= \mathrm{L}_a)$, we have the statement by Lemma 3.7 (2).

Case 4. The case (Case Add_2) with the right assumption $\Gamma_{p+k+1} \Rightarrow \Delta_{p+k+1}$. Let $\mathrm{Add}_2(a, b, c)$ be the principal formula of the rule. There exists Π such that $\Gamma_{p+k} \equiv (\Pi, \mathrm{Add}_2(a, b, c))$ and $\Gamma_{p+k+1} \equiv (\Pi, a = sx, b = y, c = z, \mathrm{Add}_2(x, sy, z))$ for fresh variables x, y, z.

We prove this case by contrapositive. To show $b_k \sim_{\Gamma_{p+k}} b'$, assume $b_{k+1} \sim_{\Gamma_{p+k+1}} b'$ for some $\mathrm{Add}_1(a', b', c') \in \Delta_{p+k+1}$. Define t as b if $b_{k+1} \equiv sy$ and b_{k+1} otherwise. Since $b_{k+1} \sim_{\Gamma_{p+k+1}} b'$ holds, we have $t \sim_{\Gamma_{p+k+1}} b'$. By Lemma 3.9, $t \sim_{\Gamma_{p+k}} b'$. By $b_k \equiv t$, we have $b_k \sim_{\Gamma_{p+k}} b'$.

Case 5. The case (Case Add_2) with the left assumption $\Gamma_{p+k+1} \Rightarrow \Delta_{p+k+1}$. In this case, $\Gamma_{p+k} \Rightarrow \Delta_{p+k}$ is a switching point. Let $\mathrm{Add}_2(a, b, c)$ be the principal formula of the rule. There exists Π such that $\Gamma_{p+k} \equiv (\Pi, \mathrm{Add}_2(a, b, c))$ and $\Gamma_{p+k+1} \equiv (\Pi, a = 0, b = y, c = y)$ with a fresh variable y.

Assume $b_k \not\sim_{\Gamma_{p+k}} b''$ for all $\mathrm{Add}_1(a'', b'', c'') \in \Delta_{p+k}$. Fix $\mathrm{Add}_1(a', b', c') \in \Delta_{p+k+1}$ to show $b_{k+1} \not\sim_{\Gamma_{p+k+1}} b'$. By $b_{k+1} \equiv b_k$ and $\Delta_{p+k} \equiv \Delta_{p+k+1}$, we have $b_{k+1} \not\sim_{\Gamma_{p+k}} b'$. From Lemma 3.11, $b' \not\sim_{\Gamma_{p+k}} a$ and $b' \not\sim_{\Gamma_{p+k}} 0$. Since y is fresh, we have $b' \not\sim_{\Gamma_{p+k}} y$. Since $\Gamma_{p+k} \Rightarrow \Delta_{p+k}$ is a switching point, $b' \not\sim_{\Gamma_{p+k}} b$. By Lemma 3.17, $\Gamma_{p+k} \Rightarrow \Delta_{p+k}$ is an index sequent. By Definition 3.13 and $\Delta_{p+k} \equiv \Delta_{p+k+1}$, $b' \not\sim_{\Gamma_{p+k}} c$. By Lemma 3.10, $b_{k+1} \not\sim_{\Gamma_{p+k+1}} b'$.

Case 6. The case (Add_1 R_2).

In this case, Γ_{p+k} is the same as Γ_{p+k+1} and the second argument of a formula with Add_2 or Add_1 in $\Gamma_{p+k} \Rightarrow \Delta_{p+k}$ is the same as that in $\Gamma_{p+k+1} \Rightarrow \Delta_{p+k+1}$. We thus have the statement.

(2) Let $d_k = n$.

Case 1. The case (Weak).

If $b_{k+1} \not\sim_{\Gamma_{p+k+1}} b$ for all $\mathrm{Add}_1(a, b, c) \in \Delta_{p+k+1}$, then $d_{k+1} = \bot$.

Assume $b_{k+1} \sim_{\Gamma_{p+k+1}} b$ for some $\mathrm{Add}_1(a, b, c) \in \Delta_{p+k+1}$. By Definition 3.5, there exist m, $l \in \mathbb{N}$ such that $s^m b_{k+1} \cong_{\Gamma_{p+k+1}} s^l b$. By $\Gamma_{p+k+1} \subseteq \Gamma_{p+k}$, we have $s^m b_{k+1} \cong_{\Gamma_{p+k}} s^l b$. Since $b_k \equiv b_{k+1}$, we have $s^m b_k \cong_{\Gamma_{p+k}} s^l b$. Since $\Delta_{p+k+1} \subseteq \Delta_{p+k}$ holds, we have $\mathrm{Add}_1(a, b, c) \in \Delta_{p+k}$. By $d_k = n$, we have $l - m = n$. Thus, $d_{k+1} = n$.

Case 2. The case (Sub) with a substitution θ. Note that $b_k \equiv b_{k+1}[\theta]$.

If $b_{k+1} \not\sim_{\Gamma_{p+k+1}} b$ for all $\mathrm{Add}_1(a, b, c) \in \Delta_{p+k+1}$, then $d_{k+1} = \bot$.

Assume that $b_{k+1} \sim_{\Gamma_{p+k+1}} b$ for some $\mathrm{Add}_1(a, b, c) \in \Delta_{p+k+1}$. By Definition 3.5, there exist m, $l \in \mathbb{N}$ such that $s^m b_{k+1} \cong_{\Gamma_{p+k+1}} s^l b$. By Lemma 3.6 (1), $s^m b_{k+1}[\theta] \cong_{\Gamma_{p+k}} s^l b[\theta]$. Since $b_k \equiv b_{k+1}[\theta]$ holds, we have $s^m b_k \cong_{\Gamma_{p+k}} s^l b[\theta]$. Since $\Delta_{p+k} \equiv \Delta_{p+k+1}[\theta]$ holds, we have $\mathrm{Add}_1(a[\theta], b[\theta], c[\theta]) \in \Delta_{p+k}$. By $d_k = n$, we have $l - m = n$. Thus, $d_{k+1} = n$.

(3) Let $d_k = n$.

Case 1. The case ($=$ L_a)with the principal formula $u_1 = u_2$.

Let $b_k \equiv b[v_1 := u_1, v_2 := u_2]$ and $b_{k+1} \equiv b[v_1 := u_2, v_2 := u_1]$ for variables v_1, v_2.

By $d_k = n$, there exist $m, l \in \mathbb{N}$ such that $s^m b[v_1 := u_1, v_2 := u_2] \cong_{\Gamma_{p+k}} s^l b[v_1 := u_1, v_2 := u_2]$ for some $\mathrm{Add}_1(a[v_1 := u_1, v_2 := u_2], b[v_1 := u_1, v_2 := u_2], c[v_1 := u_1, v_2 := u_2]) \in \Delta_{p+k}$ and $l - m = n$. From Lemma 3.7 (1), $s^m b[v_1 := u_2, v_2 := u_1] \cong_{\Gamma_{p+k+1}} s^l b[v_1 := u_2, v_2 := u_1]$. Moreover,

$\mathrm{Add}_1(a[v_1 := u_2, v_2 := u_1], b[v_1 := u_2, v_2 := u_1], c[v_1 := u_2, v_2 := u_1]) \in \Delta_{p+k+1}$. Thus, $d_{k+1} = l - m = n$.

Case 2. The case (Add_1 R_2).

Since $\tau_{p+k+1} \equiv \tau_{p+k}$ holds, Γ_{p+k} is the same as Γ_{p+k+1} and the second argument of a formula with Add_1 in Δ_{p+k} is the same as that in Δ_{p+k+1}, we have $d_{k+1} = d_k$.

(4) Let $d_k = n$. Let $\mathrm{Add}_2(a, b, c)$ be the principal formula of the rule (Case Add_2) with the conclusion $\Gamma_{p+k} \Rightarrow \Delta_{p+k}$.

(4)(a) The case where $\Gamma_{p+k+1} \Rightarrow \Delta_{p+k+1}$ is the left assumption of the rule. In this case, $\Gamma_{p+k} \Rightarrow \Delta_{p+k}$ is a switching point.

There exists Π such that $\Gamma_{p+k} \equiv (\Pi, \mathrm{Add}_2(a, b, c))$ and $\Gamma_{p+k+1} \equiv (\Pi, a = 0, b = y, c = y)$ with a fresh variable y. By $d_k = n$, there exist m, $l \in \mathbb{N}$ such that $s^m b_k \cong_{\Gamma_{p+k}} s^l b'$ for some $\mathrm{Add}_1(a', b', c') \in \Delta_{p+k}$ and $l - m = n$. Since the set of formulas with $=$ in Γ_{p+k+1} includes the set of formulas with $=$ in Γ_{p+k}, we have $s^m b_k \cong_{\Gamma_{p+k+1}} s^l b'$. By $\tau_{k+1} \equiv \tau_k$, we have $s^m b_{k+1} \cong_{\Gamma_{p+k+1}} s^l b'$. Since $\Delta_{p+k} \equiv \Delta_{p+k+1}$, we have $\mathrm{Add}_1(a', b', c') \in \Delta_{p+k+1}$. Thus, $d_{k+1} = l - m = n$.

(4)(b) The case where $\Gamma_{p+k+1} \Rightarrow \Delta_{p+k+1}$ is the right assumption of the rule and τ_k is not a progress point of the trace.

Since τ_k is not a progress point of the trace, we have $\tau_{k+1} \equiv \tau_k$. By $d_k = n$, there exist m, $l \in \mathbb{N}$ such that $s^m b_k \cong_{\Gamma_{p+k}} s^l b'$ for some $\mathrm{Add}_1(a', b', c') \in \Delta_{p+k}$ and $l - m = n$. Since the set of formulas with $=$ in Γ_{p+k} includes the set of formulas with $=$ in Γ_{p+k+1}, we have $s^m b_k \cong_{\Gamma_{p+k+1}} s^l b'$. By $\tau_{k+1} \equiv \tau_k$, we have $s^m b_{k+1} \cong_{\Gamma_{p+k+1}} s^l b'$. Since $\Delta_{p+k} \equiv \Delta_{p+k+1}$ holds, we have $\mathrm{Add}_1(a', b', c') \in \Delta_{p+k+1}$. Thus, $d_{k+1} = l - m = n$.

(4)(c) The case where $\Gamma_{p+k+1} \Rightarrow \Delta_{p+k+1}$ is the right assumption of the rule and τ_k is a progress point of the trace.

There exists Π such that $\Gamma_{p+k} \equiv (\Pi, \mathrm{Add}_2(a,b,c))$ and $\Gamma_{p+k+1} \equiv (\Pi, a = sx, b = y, c = z, \mathrm{Add}_2(x, sy, z))$ for fresh variables x, y, z. Since τ_k is a progress point of the trace, we have $\tau_k \equiv \mathrm{Add}_2(a,b,c)$ and $\tau_{k+1} \equiv \mathrm{Add}_2(x, sy, z)$. Therefore, $b_k \equiv b$ and $b_{k+1} \equiv sy$. By $d_k = n$, there exist m, $l \in \mathbb{N}$ such that $s^m b \cong_{\Gamma_{p+k}} s^l b'$ for some $\mathrm{Add}_1(a', b', c') \in \Delta_{p+k}$ and $l - m = n$. Since the set of formulas with $=$ in Γ_{p+k+1} includes the set of formulas with $=$ in Γ_{p+k}, we have $s^m b \cong_{\Gamma_{p+k+1}} s^l b'$. By $b \cong_{\Gamma_{p+k+1}} y$, we have $s^m y \cong_{\Gamma_{p+k+1}} s^l b'$. Hence, $s^m b_{k+1} \cong_{\Gamma_{p+k+1}} s^{l+1} b'$. Thus, $d_{k+1} = l + 1 - m = n + 1$. □

Lemma 3.19. *For an infinite idling path $\{\Gamma_i \Rightarrow \Delta_i\}_{i \geq 0}$ in $\mathcal{T}\left(\mathcal{D}^1_{\mathrm{cf}}, \mathcal{C}^1_{\mathrm{cf}}\right)$, there exists $l \in \mathbb{N}$ such that the following conditions hold:*

(1) *$\Gamma_l \Rightarrow \Delta_l$ is a switching point in $\mathcal{T}\left(\mathcal{D}^1_{\mathrm{cf}}, \mathcal{C}^1_{\mathrm{cf}}\right)$, and*

(2) *$\Gamma_{l+1} \Rightarrow \Delta_{l+1}$ is the right assumption of the rule with the conclusion $\Gamma_l \Rightarrow \Delta_l$.*

Proof. Since $\{\Gamma_i \Rightarrow \Delta_i\}_{i \geq 0}$ is an infinite path and $\mathcal{T}\left(\mathcal{D}^1_{\mathrm{cf}}, \mathcal{C}^1_{\mathrm{cf}}\right)$ satisfies the global trace condition, there exists an infinitely progressing trace following a tail of the path. Let $\{\tau_k\}_{k \geq 0}$ be an infinitely progressing trace following $\{\Gamma_i \Rightarrow \Delta_i\}_{i \geq p}$. Let d_k be the index of τ_k in $\Gamma_{p+k} \Rightarrow \Delta_{p+k}$.

We show that there exists $l \in \mathbb{N}$ such that $d_l = \perp$. The set $\{d_k \mid k \geq 0\}$ is finite since the set of sequents in $\{\Gamma_i \Rightarrow \Delta_i\}_{i \geq 0}$ is finite and we have a unique index of an atomic formula with Add_2 in $\Gamma_i \Rightarrow \Delta_i$. Since $\{\tau_k\}_{k \geq 0}$ is an infinitely progressing trace following $\{\Gamma_i \Rightarrow \Delta_i\}_{i \geq p}$, if there does not exist $k' \in \mathbb{N}$ such that $d_{k'} = \perp$, Lemma 3.18 implies that $\{d_k \mid k \geq 0\}$ is infinite. Thus, there exists $k' \in \mathbb{N}$ such that $d_{k'} = \perp$.

Since $\{\tau_k\}_{k \geq 0}$ is an infinitely progressing trace following $\{\Gamma_i \Rightarrow \Delta_i\}_{i \geq p}$, there exists a progress point τ_l with $l > k'$. By Lemma 3.18, $d_l = \perp$. Since τ_k is a progress point, $\Gamma_{p+k} \Rightarrow \Delta_{p+k}$ is a switching point and $\Gamma_{p+k+1} \Rightarrow \Delta_{p+k+1}$ is the right assumption of the rule. □

Definition 3.20 (Rightmost path)**.** For a derivation tree $\mathscr{D}$ and a node σ in $\mathscr{D}$, we define the *rightmost path* from the node σ as the path $\{\sigma_i\}_{0 \leq i < \alpha}$ satisfying the following conditions:

(1) The node σ_0 is σ.

(2) If σ_i is the conclusion of (Case Add_2), the node σ_{i+1} is the right assumption of the rule.

(3) If σ_i is the conclusion of the rules (Weak), (Sub), ($=$ L_a), or (Add_1 R_2), the node σ_{i+1} is the assumption of the rule.

Lemma 3.21. *The rightmost path from an index sequent in $\mathcal{T}\left(\mathcal{D}^1_{\mathrm{cf}}, \mathcal{C}^1_{\mathrm{cf}}\right)$ is infinite.*

Proof. By Definition 3.16, the rightmost path from an index sequent in $\mathcal{T}\left(\mathcal{D}^1_{\mathrm{cf}}, \mathcal{C}^1_{\mathrm{cf}}\right)$ is an idling path. By Lemma 3.17, every sequent on the path is an index sequent. By Definition 3.13, (Add_1 R_1) does not occur in the path. Thus, the path is infinite. □

Remark. For an infinite path in $\mathcal{T}\left(\mathcal{D}^1_{\mathrm{cf}}, \mathcal{C}^1_{\mathrm{cf}}\right)$, the corresponding path in $\mathcal{D}^1_{\mathrm{cf}}$ has a bud.

We have proved all lemmata for Theorem 3.1 (2).

We show that there exists a sequence $\{\tilde{\sigma}_i\}_{i \in \mathbb{N}}$ of switching points in $\mathcal{D}^1_{\mathrm{cf}}$ which satisfies the following conditions:

(i) The height of $\tilde{\sigma}_i$ is greater than the height of $\tilde{\sigma}_{i-1}$ in $\mathcal{D}^1_{\mathrm{cf}}$ for $i > 0$.

(ii) For any node σ on the path from the root to $\tilde{\sigma}_i$ in $\mathcal{D}^1_{\mathrm{cf}}$ excluding $\tilde{\sigma}_i$, σ is a switching point if and only if the child of σ on the path is the left assumption of the rule (Case Add_2).

We construct $\{\tilde{\sigma}_i\}_{i\in\mathbb{N}}$ and show (i) and (ii) by induction on i.

We consider the case $i = 0$.

The rightmost path in $\mathcal{T}\left(\mathcal{D}^1_{\mathrm{cf}}, \mathcal{C}^1_{\mathrm{cf}}\right)$ from the root is an infinite idling path since $\mathrm{Add}_2(x, y, z) \Rightarrow \mathrm{Add}_1(x, y, z)$ is an index sequent and there exists no node which is the left assumption of (Case Add_2) on the path. By Lemma 3.19, there exists a switching point on the path. Hence, there exists a switching point on the rightmost path from the root in $\mathcal{D}^1_{\mathrm{cf}}$. Let $\tilde{\sigma}_0$ be the switching point of the smallest height among such switching points. (i) and (ii) follow immediately for $\tilde{\sigma}_0$.

We consider the case $i > 0$.

Let α be the left assumption of the rule with the conclusion $\tilde{\sigma}_{i-1}$. Because of (ii), the path from the root to $\tilde{\sigma}_{i-1}$ is also an idling path. Since $\tilde{\sigma}_{i-1}$ is a switching point, the path from the root to α is also an idling path. By Lemma 3.17, α is an index sequent. By Lemma 3.21, the rightmost path from α in $\mathcal{T}\left(\mathcal{D}^1_{\mathrm{cf}}, \mathcal{C}^1_{\mathrm{cf}}\right)$ is infinite. Therefore, there is a bud on the rightmost path in $\mathcal{D}^1_{\mathrm{cf}}$ from α. Let μ be the bud.

Let π_1 be the path from the root to μ in $\mathcal{D}^1_{\mathrm{cf}}$ and π_2 be the path from $\mathcal{C}^1_{\mathrm{cf}}(b)$ to μ in $\mathcal{D}^1_{\mathrm{cf}}$. We define the path π in $\mathcal{T}\left(\mathcal{D}^1_{\mathrm{cf}}, \mathcal{C}^1_{\mathrm{cf}}\right)$ as $\pi_1{\pi_2}^{\omega}$. Let $\{\sigma_i\}_{0\leq i}$ be π. Because of (ii), π is an idling path. By Lemma 3.19, there is a switching point σ_l and σ_{l+1} is the right assumption of the rule. Hence, there is a switching point on $\pi_1\pi_2$ in $\mathcal{D}^1_{\mathrm{cf}}$ such that its child on the path is the right assumption of the rule. Define $\tilde{\sigma}_i$ as the switching point of the smallest height among such switching points.

We show $\tilde{\sigma}_i$ satisfies the conditions (i) and (ii).

(i) By the definition of $\tilde{\sigma}_i$, $\tilde{\sigma}_i$ is on the path from the root to μ. By the condition (ii), $\tilde{\sigma}_i$ is not on the path from the root to $\tilde{\sigma}_{i-1}$. Hence, the height of $\tilde{\sigma}_i$ is greater than that of $\tilde{\sigma}_{i-1}$.

(ii) Let σ be a node on the path from the root to $\tilde{\sigma}_i$ excluding $\tilde{\sigma}_i$. We can assume σ is on the path from $\tilde{\sigma}_{i-1}$ to $\tilde{\sigma}_i$ excluding $\tilde{\sigma}_i$ by the induction hypothesis.

The "only if" part: Assume that σ is a switching point. By the definition of $\tilde{\sigma}_i$, we see that σ is $\tilde{\sigma}_{i-1}$. The child of $\tilde{\sigma}_{i-1}$ on the path from the root to $\tilde{\sigma}_i$ is α, which is the left assumption of the rule.

The "if" part: Assume that the child of σ on the path is the left assumption of the rule. Since there is not the left assumption of a rule on the path from α to $\tilde{\sigma}_i$, we see that σ is $\tilde{\sigma}_{i-1}$. Thus, σ is a switching point.

We complete the construction and the proof of the properties.

Because of (i), $\tilde{\sigma}_0, \tilde{\sigma}_1, \ldots$ are all distinct in $\mathcal{D}^1_{\mathrm{cf}}$. Hence, $\{\tilde{\sigma}_i \mid i \in \mathbb{N}\}$ is infinite. This is a contradiction since the set of nodes in $\mathcal{D}^1_{\mathrm{cf}}$ is finite. Thus, we have Theorem 3.1 (2). □

3.5 Discussion

We discuss related work and why the cut-elimination property in the cyclic proof systems does not hold.

3.5.1 Related work

Kimura et al. [12] gave a counterexample to cut-elimination in cyclic proofs for separation logic. They also suggested that their proof technique cannot be applied to show a counterexample to cut-elimination in a cyclic proof system with contraction and weakening rules [12].

We discuss why we cannot apply the proof technique. In order to show that their counterexample is not provable without a cut rule, they have assumed that there exists a cut-free proof of it. Then, they have proven that the rightmost path from the root has no infinitely progressing trace following a tail of the path if the path has a companion.

$$
\begin{array}{c}
(\heartsuit)\ \mathrm{Add}_2(x_1, sy_1, z_1), \mathrm{Add}_2(x,y,z) \Rightarrow \mathrm{Add}_1(x,y,z) \\
\hline
\mathrm{Add}_2(x_2, sy_2, z_2), \mathrm{Add}_2(x,y,z) \Rightarrow \mathrm{Add}_1(x,y,z) \\
\hline
x_1 = sx_2, sy_1 = y_2, z_1 = z_2, \mathrm{Add}_2(x_2, sy_2, z_2), \mathrm{Add}_2(x,y,z) \Rightarrow \mathrm{Add}_1(x,y,z) \\
\hline
(\heartsuit)\ \mathrm{Add}_2(x_1, sy_1, z_1), \mathrm{Add}_2(x,y,z) \Rightarrow \mathrm{Add}_1(x,y,z) \quad (\textsc{Case}\ \mathrm{Add}_2) \\
\hline
x = sx_1, y = y_1, z = z_1, \mathrm{Add}_2(x_1, sy_1, z_1), \mathrm{Add}_2(x,y,z) \Rightarrow \mathrm{Add}_1(x,y,z) \\
\hline
\mathrm{Add}_2(x,y,z) \Rightarrow \mathrm{Add}_1(x,y,z) \quad (\textsc{Case}\ \mathrm{Add}_2)
\end{array}
$$

Figure 3.5 An example to which the technique in [12] cannot be applied

In contrast, the rightmost path from the root in a cut-free $\mathtt{CLKID}^\omega$ pre-proof of $\mathrm{Add}_2(x,y,z) \Rightarrow \mathrm{Add}_1(x,y,z)$ might have a companion and an infinitely progressing trace following a tail of the path that might use contraction and weakening rules. For example, pay attention to the rightmost path from the root in Figure 3.5. It has both a companion and an infinitely progressing trace following a tail of the path. Thus, we cannot use their proof technique.

Das [10] showed that the cut-elimination property of Cyclic Arithmetic, a cyclic proof system for Peano Arithmetic, does not hold, which is the conclusion of Gödel's incompleteness theorem. It seems to be unable to use providing a counterexample to cut-elimination in $\mathtt{CLKID}^\omega$ since $\mathtt{CLKID}^\omega$ does not include arithmetic. Nevertheless, since Peano Arithmetic theorems can be shown in $\mathtt{CLKID}^\omega$ by adding some finitely many axioms, the counterexample in Cyclic Arithmetic might be transformed into the one in $\mathtt{CLKID}^\omega$.

Saotome, Nakazawa, and Kimura [15] provided a counterexample to cut-elimination in the cyclic proof system for bunched logic with inductive definitions. It is extraordinary that the counterexample includes only nullary predicates. Their proof technique, called *proof unrolling*, may be used to show that our counterexample is not provable, but this topic is reserved for future work.

3.5.2 The cut-elimination property in cyclic proof systems

Why does not the cut-elimination property hold in the cyclic proof systems? The reason is not yet fully understood, but we will discuss it briefly. In common, the proofs in [12, 15] and this thesis have used the finiteness of sequents occurring in the proof figure. The more important fact is that the cut-elimination property of $\mathtt{LKID}^\omega$, the infinitary proof system, holds [6]. These facts suggest that the cut-elimination property does not hold in the cyclic proof systems because of the finiteness. This observation is interesting to compare with the following fact: the cut-elimination property of a sequent calculus for Peano Arithmetic does not hold, but the system obtained from the system by replacing the schema for induction with the ω-rule whose assumptions are infinitely many, can eliminate the cut-rule [14]. The finiteness also seems to relate to the cut-elimination property of first-order arithmetic.

4 The cut-elimination property and the arity of predicates

This chapter discusses the cut-elimination property in cyclic proof systems and the arity of predicates.

In Chapter 3, we gave the counterexample to cut-elimination of $\mathtt{CLKID}^\omega$ with ternary predicates. Is there any simpler counterexample? If there is, the counterexample may suggest something to find out why the cut-elimination property in $\mathtt{CLKID}^\omega$ does not hold.

Actually, there is a counterexample with predicates whose arity is less than three. In this chapter, we give the counterexample with only unary predicates. By the counterexample we give in this chapter, we see that we cannot eliminate the cut rule in first-order logic with inductive definitions if we restrict predicates in the language to unary predicates and $=$.

Section 4.1 provides a counterexample to cut-elimination with only unary predicates. Then, we discuss the cut-elimination property in cyclic proof systems and the arity of predicates in Section 4.2.

4.1 Counterexample to cut-elimination with only unary predicates

In this section, we prove the following theorem.

Theorem 4.1. *Let* $\mathtt{f}$, $\mathtt{1}$ *be a constant symbol,* s *be a unary function symbol. Let* $\mathrm{F}\mathtt{f}\mathrm{T}$ *and* $\mathrm{T}\mathtt{1}\mathrm{F}$ *be unary inductive predicates with the following productions:*

$$\frac{}{\mathrm{F}\mathtt{f}\mathrm{T}(\mathtt{f})}\ , \qquad \frac{\mathrm{F}\mathtt{f}\mathrm{T}(x)}{\mathrm{F}\mathtt{f}\mathrm{T}(\mathrm{s}x)}\ , \qquad \frac{}{\mathrm{T}\mathtt{1}\mathrm{F}(\mathtt{1})}\ , \qquad \frac{\mathrm{T}\mathtt{1}\mathrm{F}(\mathrm{s}x)}{\mathrm{T}\mathtt{1}\mathrm{F}(x)}\ .$$

(*1*) $\mathrm{T}\mathtt{1}\mathrm{F}(\mathtt{f}) \Rightarrow \mathrm{F}\mathtt{f}\mathrm{T}(\mathtt{1})$ *is provable in* $\mathtt{CLKID}^\omega$*.*

(*2*) $\mathrm{T}\mathtt{1}\mathrm{F}(\mathtt{f}) \Rightarrow \mathrm{F}\mathtt{f}\mathrm{T}(\mathtt{1})$ *is not cut-free provable in* $\mathtt{CLKID}^\omega$*.*

This theorem means that $\mathrm{T}\mathtt{1}\mathrm{F}(\mathtt{f}) \Rightarrow \mathrm{F}\mathtt{f}\mathrm{T}(\mathtt{1})$ is a counterexample with only unary inductive predicates to cut-elimination in $\mathtt{CLKID}^\omega$.

There are the inference rules for $\mathrm{T}\mathtt{1}\mathrm{F}$ and $\mathrm{F}\mathtt{f}\mathrm{T}$ in Figure 4.1.

A $\mathtt{CLKID}^\omega$ proof of $\mathrm{T}\mathtt{1}\mathrm{F}(\mathtt{f}) \Rightarrow \mathrm{F}\mathtt{f}\mathrm{T}(\mathtt{1})$ is the derivation tree given in Figure 4.2, where (†) indicates the pairing of the companion with the bud and the underlined formulas denotes the infinitely progressing trace for some tails of any infinite path (some applying rules and some labels of rules are omitted for limited space). Thus, Theorem 4.1 (1) is correct.

In the remainder of this section, we prove Theorem 4.1 (2) by the way similar to Theorem 3.1 (2).

Throughout the remainder of this section, we assume there exists a cut-free $\mathtt{CLKID}^\omega$ proof of $\mathrm{T}\mathtt{1}\mathrm{F}(\mathtt{f}) \Rightarrow \mathrm{F}\mathtt{f}\mathrm{T}(\mathtt{1})$, for contradiction. By Proposition 3.3, there exists a cut-free cycle-normal $\mathtt{CLKID}_a^\omega$ proof of $\mathrm{T}\mathtt{1}\mathrm{F}(\mathtt{f}) \Rightarrow \mathrm{F}\mathtt{f}\mathrm{T}(\mathtt{1})$. *We write* $(\mathcal{D}^2_{\mathrm{cf}}, \mathcal{C}^2_{\mathrm{cf}})$ *for a cut-free cycle-normal* $\mathtt{CLKID}_a^\omega$ *proof of* $\mathrm{T}\mathtt{1}\mathrm{F}(\mathtt{f}) \Rightarrow \mathrm{F}\mathtt{f}\mathrm{T}(\mathtt{1})$*.*

$$\frac{}{\Gamma \Rightarrow \Delta, \mathtt{FfT}(\mathtt{f})}\ (\mathtt{FfT}\ \mathrm{R}_1) \qquad \frac{\Gamma \Rightarrow \Delta, \mathtt{FfT}(t)}{\Gamma \Rightarrow \Delta, \mathtt{FfT}(\mathrm{s}t)}\ (\mathtt{FfT}\ \mathrm{R}_2)$$

$$\frac{\Gamma, t = \mathtt{1} \Rightarrow \Delta \qquad \Gamma, t = \mathrm{s}x, \mathtt{T1F}(x) \Rightarrow \Delta}{\Gamma, \mathtt{T1F}(t) \Rightarrow \Delta}\ (\textsc{Case}\ \mathtt{FfT})$$

$$(x \notin \mathrm{FV}(\Gamma \cup \Delta \cup \{\mathtt{T1F}(t)\}))$$

$$\frac{}{\Gamma \Rightarrow \Delta, \mathtt{T1F}(\mathtt{1})}\ (\mathtt{FfT}\ \mathrm{R}_1) \qquad \frac{\Gamma \Rightarrow \Delta, \mathtt{FfT}(\mathrm{s}t)}{\Gamma \Rightarrow \Delta, \mathtt{FfT}(t)}\ (\mathtt{FfT}\ \mathrm{R}_2)$$

$$\frac{\Gamma, t = \mathtt{1} \Rightarrow \Delta \qquad \Gamma, t = x, \mathtt{T1F}(\mathrm{s}x) \Rightarrow \Delta}{\Gamma, \mathtt{T1F}(t) \Rightarrow \Delta}\ (\textsc{Case}\ \mathtt{T1F})$$

$$(x \notin \mathrm{FV}(\Gamma \cup \Delta \cup \{\mathtt{T1F}(t)\}))$$

Figure 4.1 The rules for T1F and FfT

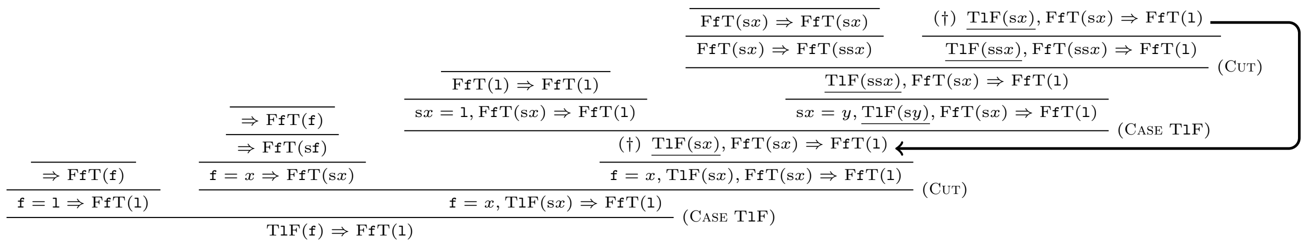


Figure 4.2 The $\mathtt{CLKID}^\omega$ proof of $\mathtt{T1F}(\mathtt{f}) \Rightarrow \mathtt{FfT}(\mathtt{1})$

Remark. Let $\Gamma \Rightarrow \Delta$ be a sequent in $\mathcal{D}^2_{\mathrm{cf}}$. By induction on the height of sequents in $\mathcal{D}^2_{\mathrm{cf}}$, we can easily show the following statements:

(1) Γ consists of only atomic formulas with $=$, T1F.

(2) Δ consists of only atomic formulas with FfT.

(3) A term in Γ and Δ is of the form $\mathrm{s}^n\mathtt{f}$, $\mathrm{s}^n\mathtt{1}$, or $\mathrm{s}^n x$ with some variable x.

(4) The possible rules in $(\mathcal{D}^2_{\mathrm{cf}}, \mathcal{C}^2_{\mathrm{cf}})$ are (Weak), (Sub), ($=$ L_a), (FfT R_1), (FfT R_2), and (Case T1F).

Without loss of generality, we can assume terms are of the form $\mathrm{s}^n\mathtt{f}$, $\mathrm{s}^n\mathtt{1}$, or $\mathrm{s}^n x$ with some variable x throughout the remainder of this section.

We define an index, which is a concept similar to Definition 3.12. $\cong_\Gamma$ and $\sim_\Gamma$ in the following definition is the same relation as Definition 3.4 and Definition 3.5, respectively.

Definition 4.2 (Index)**.** For a finite set Γ and $\mathtt{T1F}(t) \in \Gamma$, we define *the index of* $\mathtt{T1F}(t)$ *in* Γ as follows:

(1) If $t \not\sim_\Gamma \mathtt{f}$, then the index of $\mathtt{T1F}(t)$ in Γ is $\bot$, and

(2) if there uniquely exists $m - n$ such that n, $m \in \mathbb{N}$, and $\mathrm{s}^n t \cong_\Gamma \mathrm{s}^m\mathtt{f}$, then the index of $\mathtt{T1F}(t)$ in Γ is $m - n$ (namely the uniqueness means that $\mathrm{s}^{n'} t \cong_\Gamma \mathrm{s}^{m'}\mathtt{f}$ for n, $m \in \mathbb{N}$ implies $m - n = m' - n'$).

Note that if there exists n_0, m_0, n_1, $m_1 \in \mathbb{N}$ such that $\mathrm{s}^{n_0}t \cong_\Gamma \mathrm{s}^{m_0}\mathtt{f}$, $\mathrm{s}^{n_1}t \cong_\Gamma \mathrm{s}^{m_1}\mathtt{f}$ and $m_0 - n_0 \neq m_1 - n_1$, then the index of $\mathrm{T1F}(t)$ in Γ is undefined.

Definition 4.3 (Index sequent)**.** The sequent $\Gamma \Rightarrow \Delta$ is said to be an *index sequent* if the following conditions hold:

(1) $\mathtt{f} \not\sim_\Gamma \mathtt{1}$,

(2) $t \not\sim_\Gamma \mathtt{f}$ for all $\mathrm{F}\mathtt{f}\mathrm{T}(t) \in \Delta$, and

(3) if $\mathrm{s}^n\mathtt{f} \cong_\Gamma \mathrm{s}^m\mathtt{f}$, then $n = m$.

An index sequent does not occur as a conclusion of ($\mathrm{F}\mathtt{f}\mathrm{T}$ R_1) by the first and second conditions. The third condition guarantees the existence of an index, as shown in the following lemma.

Lemma 4.4. *If $\Gamma \Rightarrow \Delta$ is an index sequent, the index of all* $\mathrm{T1F}(t)$ *in Γ is defined.*

Proof. Let $\mathrm{T1F}(t) \in \Gamma$. If $t \not\sim_\Gamma \mathtt{f}$, then the index is $\perp$.

Assume $t \sim_\Gamma \mathtt{f}$. By Definition 3.5, there exist n_0 and m_0 such that $\mathrm{s}^{n_0}t \cong_\Gamma \mathrm{s}^{m_0}\mathtt{f}$. To show the uniqueness, assume $\mathrm{s}^{n_1}t \cong_\Gamma \mathrm{s}^{m_1}\mathtt{f}$ for n_1 and m_1 . Since $\mathrm{s}^{n_0+n_1}t \cong_\Gamma \mathrm{s}^{m_0+n_1}\mathtt{f}$ and $\mathrm{s}^{n_1+n_0}t \cong_\Gamma \mathrm{s}^{m_1+n_0}\mathtt{f}$, we have $\mathrm{s}^{m_0+n_1}\mathtt{f} \cong_\Gamma \mathrm{s}^{m_1+n_0}\mathtt{f}$. From (3) of Definition 4.3, $m_0 + n_1 = m_1 + n_0$. Thus, $m_0 - n_0 = m_1 - n_1$. □

Definition 4.5 (Switching point)**.** A node σ in a derivation tree is called a *switching point* if the rule with the conclusion σ is (Case T1F) and the index of the principal formula for the rule in the conclusion is $\perp$.

We call the assumption of (Case T1F) whose form is $\Gamma, t = x, \mathrm{T1F}(\mathrm{s}x) \Rightarrow \Delta$ *the right assumption of* the rule. The other assumption is called *the left assumption of* the rule.

Definition 4.6 (Idling path)**.** A path $\{\Gamma_i \Rightarrow \Delta_i\}_{0 \leq i < \alpha}$ in $\mathcal{T}\left(\mathcal{D}^2_{\mathrm{cf}}, \mathcal{C}^2_{\mathrm{cf}}\right)$ with some $\alpha \in \mathbb{N} \cup \{\omega\}$ is said to be an *idling path* if the following conditions hold:

(1) $\Gamma_0 \Rightarrow \Delta_0$ is an index sequent, and

(2) if the rule for $\Gamma_i \Rightarrow \Delta_i$ is (Case T1F) and $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ is the left assumption of the rule, then $\Gamma_i \Rightarrow \Delta_i$ is a switching point.

The following lemma corresponds to Lemma 3.17.

Lemma 4.7. *Every sequent in an idling path is an index sequent.*

Proof. Let $\{\Gamma_i \Rightarrow \Delta_i\}_{0 \leq i < \alpha}$ be an idling path. We prove the statement by the induction on i.

For $i = 0$, $\Gamma_0 \Rightarrow \Delta_0$ is an index sequent by Definition 4.6.

For $i > 0$, we consider cases according to the rule with the conclusion $\Gamma_{i-1} \Rightarrow \Delta_{i-1}$.

Case 1. The case (Weak).

(1) By the induction hypothesis (1), we have $\mathtt{f} \not\sim_{\Gamma_{i-1}} \mathtt{1}$. By $\Gamma_i \subseteq \Gamma_{i-1}$, we have $\mathtt{f} \not\sim_{\Gamma_i} \mathtt{1}$.

(2) Let $\mathrm{F}\mathtt{f}\mathrm{T}(t) \in \Delta_i$. By $\Delta_i \subseteq \Delta_{i-1}$, we have $\mathrm{F}\mathtt{f}\mathrm{T}(t) \in \Delta_{i-1}$. By the induction hypothesis (2), $t \not\sim_{\Gamma_{i-1}} \mathtt{f}$. By $\Gamma_i \subseteq \Gamma_{i-1}$, we have $t \not\sim_{\Gamma_i} \mathtt{f}$.

(3) Assume $\mathrm{s}^n\mathtt{f} \cong_{\Gamma_i} \mathrm{s}^m\mathtt{f}$. By $\Gamma_i \subseteq \Gamma_{i-1}$, we have $\mathrm{s}^n\mathtt{f} \cong_{\Gamma_{i-1}} \mathrm{s}^m\mathtt{f}$. By the induction hypothesis (3), $n = m$.

Case 2. The case (Sub) with a substitution θ.

(1) By the induction hypothesis (1), we have $\mathtt{f} \not\sim_{\Gamma_{i-1}} \mathtt{1}$. By Lemma 3.6 (2), we have $\mathtt{f} \not\sim_{\Gamma_i} \mathtt{1}$.

(2) Let $\mathrm{F}\mathtt{f}\mathrm{T}(t) \in \Delta_i$. By $\Delta_{i-1} \equiv \Delta_i[\theta]$, $\mathrm{F}\mathtt{f}\mathrm{T}(t[\theta]) \in \Delta_{i-1}$. By the induction hypothesis (2), $t[\theta] \not\sim_{\Gamma_{i-1}} \mathtt{f}$. By Lemma 3.6 (2), $t \not\sim_{\Gamma_i} \mathtt{f}$.

(3) Assume $\mathrm{s}^n\mathtt{f} \cong_{\Gamma_i} \mathrm{s}^m\mathtt{f}$. By Lemma 3.6 (1), we have $\mathrm{s}^n\mathtt{f} \cong_{\Gamma_{i-1}} \mathrm{s}^m\mathtt{f}$ By the induction hypothesis (3), $n = m$.

Case 3. The case ($=$ L_a).

Let $u_1 = u_2$ be the principal formula of the rule. There exists Γ and Δ such that

$$\begin{aligned}
\Gamma_{i-1} &\equiv (\Gamma[v_1 := u_1, v_2 := u_2], u_1 = u_2),\\
\Delta_{i-1} &\equiv (\Delta[v_1 := u_1, v_2 := u_2], u_1 = u_2),\\
\Gamma_i &\equiv (\Gamma[v_1 := u_2, v_2 := u_1], u_1 = u_2), \text{ and}\\
\Delta_i &\equiv (\Delta[v_1 := u_2, v_2 := u_1], u_1 = u_2).
\end{aligned}$$

(1) By the induction hypothesis (1), we have $\mathtt{f} \not\sim_{\Gamma_{i-1}} \mathtt{1}$. By Lemma 3.7 (2), we have $\mathtt{f} \not\sim_{\Gamma_i} \mathtt{1}$.

(2) Let $\mathrm{F\mathtt{f}T}(t) \in \Delta_i$. By the definition of Δ, there exists a term $\hat{t}$ such that $t \equiv \hat{t}[v_1 := u_2, v_2 := u_1]$. Then, $\mathrm{F\mathtt{f}T}\big(\hat{t}[v_1 := u_1, v_2 := u_2]\big) \in \Delta_{i-1}$. By the induction hypothesis (2), $\hat{t}[v_1 := u_1, v_2 := u_2] \not\sim_{\Gamma_{i-1}} \mathtt{f}$. By Lemma 3.7 (2), $\hat{t}[v_1 := u_2, v_2 := u_1] \not\sim_{\Gamma_i} \mathtt{f}$. Thus, $t \not\sim_{\Gamma_i} \mathtt{f}$.

(3) Assume $\mathrm{s}^n\mathtt{f} \cong_{\Gamma_i} \mathrm{s}^m\mathtt{f}$. By Lemma 3.7 (1), we have $\mathrm{s}^n\mathtt{f} \cong_{\Gamma_{i-1}} \mathrm{s}^m\mathtt{f}$ By the induction hypothesis (3), $n = m$.

Case 4. The case (CASE T$\mathtt{1}$F) with the right assumption $\Gamma_i \Rightarrow \Delta_i$.

Let $\mathrm{T\mathtt{1}F}(t)$ be the principal formula of the rule. There exists Π such that $\Gamma_{i-1} \equiv (\Pi, \mathrm{T\mathtt{1}F}(t))$ and $\Gamma_i \equiv (\Pi, t = x, \mathrm{T\mathtt{1}F}(\mathrm{s}x))$ for a fresh variable x.

(1) If $\mathtt{f} \sim_{\Gamma_i} \mathtt{1}$, then we have $\mathtt{f} \sim_{\Gamma_{i-1}} \mathtt{1}$ by Lemma 3.9. It contradicts the induction hypothesis (1). Thus, $\mathtt{f} \not\sim_{\Gamma_i} \mathtt{1}$.

(2) Let $\mathrm{F\mathtt{f}T}(t') \in \Delta_i$. If $t' \sim_{\Gamma_i} \mathtt{f}$, then we have $t' \sim_{\Gamma_{i-1}} \mathtt{f}$ by Lemma 3.9. It contradicts the induction hypothesis. Thus, $t' \not\sim_{\Gamma_i} \mathtt{f}$.

(3) Assume $\mathrm{s}^n\mathtt{f} \cong_{\Gamma_i} \mathrm{s}^m\mathtt{f}$. By Lemma 3.9, $\mathrm{s}^n\mathtt{f} \cong_{\Gamma_{i-1}} \mathrm{s}^m\mathtt{f}$. By the induction hypothesis (3), $n = m$.

Case 5. The case (CASE T$\mathtt{1}$F) with the left assumption $\Gamma_i \Rightarrow \Delta_i$. In this case, $\Gamma_{i-1} \Rightarrow \Delta_{i-1}$ is a switching point.

Let $\mathrm{T\mathtt{1}F}(t)$ be the principal formula of the rule. There exists Π such that $\Gamma_{i-1} \equiv (\Pi, \mathrm{T\mathtt{1}F}(t))$ and $\Gamma_i \equiv (\Pi, t = \mathtt{1})$.

Since $\Gamma_{i-1} \Rightarrow \Delta_{i-1}$ is a switching point, we have $t \not\sim_{\Gamma_{i-1}} \mathtt{f}$. By the induction hypothesis (1), $\mathtt{f} \not\sim_{\Gamma_{i-1}} \mathtt{1}$.

(1) Assume $\mathtt{f} \sim_{\Gamma_i} \mathtt{1}$ for contradiction. By $t \not\sim_{\Gamma_{i-1}} \mathtt{f}$, $\mathtt{f} \not\sim_{\Gamma_{i-1}} \mathtt{1}$ and Lemma 3.10, we have $\mathtt{f} \sim_{\Gamma_{i-1}} \mathtt{1}$. It contradicts the induction hypothesis (1). Thus, $\mathtt{f} \not\sim_{\Gamma_i} \mathtt{1}$.

(2) Let $\mathrm{F\mathtt{f}T}(t') \in \Delta_i$. Assume $t' \sim_{\Gamma_i} \mathtt{f}$ for contradiction. By $t \not\sim_{\Gamma_{i-1}} \mathtt{f}$, $\mathtt{f} \not\sim_{\Gamma_{i-1}} \mathtt{1}$ and Lemma 3.10, we have $t' \sim_{\Gamma_{i-1}} \mathtt{f}$. It contradicts the induction hypothesis (2). Thus, $t' \not\sim_{\Gamma_i} \mathtt{f}$.

(3) Assume $\mathrm{s}^n\mathtt{f} \cong_{\Gamma_i} \mathrm{s}^m\mathtt{f}$. By $t \not\sim_{\Gamma_{i-1}} \mathtt{f}$, $\mathtt{f} \not\sim_{\Gamma_{i-1}} \mathtt{1}$ and Lemma 3.10, we have $\mathrm{s}^n\mathtt{f} \cong_{\Gamma_{i-1}} \mathrm{s}^m\mathtt{f}$. By the induction hypothesis (3), $n = m$.

Case 6. The case ($\mathrm{F\mathtt{f}T}$ R_2). Let $\mathrm{F\mathtt{f}T}(\mathrm{s}t)$ be the principal formula of the rule.

(1) By the induction hypothesis (1), we have $\mathtt{f} \not\sim_{\Gamma_{i-1}} \mathtt{1}$. Since $\Gamma_{i-1} \equiv \Gamma_i$, we have $\mathtt{f} \not\sim_{\Gamma_i} \mathtt{1}$.

(2) Let $\mathrm{F\mathtt{f}T}(t') \in \Delta_i$. Define $\hat{t}$ as $\mathrm{s}t$ if $t' \equiv t$ and t' otherwise. By the induction hypothesis (2), we have $\hat{t} \not\sim_{\Gamma_i} \mathtt{f}$. Since $\Gamma_{i-1} \equiv \Gamma_i$, we have $\hat{t} \not\sim_{\Gamma_i} \mathtt{f}$. Then, $t' \not\sim_{\Gamma_i} \mathtt{f}$.

(3) Assume $\mathrm{s}^n\mathtt{f} \cong_{\Gamma_i} \mathrm{s}^m\mathtt{f}$. Since $\Gamma_{i-1} \equiv \Gamma_i$, we have $\mathrm{s}^n\mathtt{f} \cong_{\Gamma_{i-1}} \mathrm{s}^m\mathtt{f}$. By the induction hypothesis (3), $n = m$. ☐

The following lemma corresponds to Lemma 3.18.

Lemma 4.8. *For an idling path $\{\Gamma_i \Rightarrow \Delta_i\}_{0 \le i < \alpha}$ and a trace $\{\tau_k\}_{k \ge 0}$ following $\{\Gamma_i \Rightarrow \Delta_i\}_{i \ge p}$, if d_k is the index of τ_k, the following statements holds:*

(1) *If $d_k = \bot$, then $d_{k+1} = \bot$.*

(2) *If the rule with the conclusion* $\Gamma_{p+k} \Rightarrow \Delta_{p+k}$ *is* (Weak) *or* (Sub)*, then* $d_{k+1} = d_k$ *or* $d_{k+1} = \bot$.

(3) *If the rule with the conclusion* $\Gamma_{p+k} \Rightarrow \Delta_{p+k}$ *is* $(= L_a)$ *or* (FfT R_2)*, then* $d_{k+1} = d_k$.

(4) *Assume the rule with the conclusion* $\Gamma_{p+k} \Rightarrow \Delta_{p+k}$ *is* (Case T1F).

 (a) *If* $\Gamma_{p+k+1} \Rightarrow \Delta_{p+k+1}$ *is the left assumption of the rule, then* $d_{k+1} = d_k$.

 (b) *If* $\Gamma_{p+k+1} \Rightarrow \Delta_{p+k+1}$ *is the right assumption of the rule and* τ_k *is not a progress point of the trace, then* $d_{k+1} = d_k$.

 (c) *If* $\Gamma_{p+k+1} \Rightarrow \Delta_{p+k+1}$ *is the right assumption of the rule and* τ_k *is a progress point of the trace, then* $d_{k+1} = d_k + 1$.

Proof. Let $\tau_k \equiv \mathtt{T1F}(t_k)$.
(1) It suffices to show that $t_{k+1} \not\sim_{\Gamma_{p+k+1}} \mathtt{f}$ holds if $t_k \not\sim_{\Gamma_{p+k}} \mathtt{f}$. We consider cases according to the rule with the conclusion $\Gamma_{p+k} \Rightarrow \Delta_{p+k}$.

Case 1. If the rule is (Weak), we have the statement by $\Gamma_{p+k+1} \subseteq \Gamma_{p+k}$.

Case 2. If the rule is (Sub), we have the statement by Lemma 3.6 (2).

Case 3. If the rule is $(= \mathrm{L}_a)$, then we have the statement by Lemma 3.7 (2).

Case 4. The case (Case T1F) with the right assumption $\Gamma_{p+k+1} \Rightarrow \Delta_{p+k+1}$.

Let $\mathtt{T1F}(t)$ be the principal formula of the rule. There exists Π such that $\Gamma_{p+k} \equiv (\Pi, \mathtt{T1F}(t))$ and $\Gamma_{p+k+1} \equiv (\Pi, t = x, \mathtt{T1F}(\mathrm{s}x))$ with a fresh variable x.

We prove this case by contrapositive. To show $t_k \sim_{\Gamma_{p+k}} \mathtt{f}$, assume $t_{k+1} \sim_{\Gamma_{p+k+1}} \mathtt{f}$. Define $\hat{t}$ as t if $t_{k+1} \equiv \mathrm{s}x$ and t_{k+1} otherwise. Since $t_{k+1} \sim_{\Gamma_{p+k+1}} \mathtt{f}$ holds, we have $\hat{t} \sim_{\Gamma_{p+k+1}} \mathtt{f}$. By Lemma 3.9, $\hat{t} \sim_{\Gamma_{p+k}} \mathtt{f}$. By $t_k \equiv \hat{t}$, we have $t_k \sim_{\Gamma_{p+k}} \mathtt{f}$.

Case 5. The case (Case T1F) with the left assumption $\Gamma_{p+k+1} \Rightarrow \Delta_{p+k+1}$. In this case, $\Gamma_{p+k} \Rightarrow \Delta_{p+k}$ is a switching point.

Let $\mathtt{T1F}(t)$ be the principal formula of the rule. There exists Π such that $\Gamma_{p+k} \equiv (\Pi, \mathtt{T1F}(t))$ and $\Gamma_{p+k+1} \equiv (\Pi, t = \mathtt{1})$.

We prove this case by contrapositive. To show $t_k \sim_{\Gamma_{p+k}} \mathtt{f}$, assume $t_{k+1} \sim_{\Gamma_{p+k+1}} \mathtt{f}$. Since $\Gamma_{p+k} \Rightarrow \Delta_{p+k}$ is a switching point, we have $t \not\sim_{\Gamma_{p+k}} \mathtt{f}$. Since $\Gamma_{p+k} \Rightarrow \Delta_{p+k}$ is an index sequent, we have $\mathtt{f} \not\sim_{\Gamma_{p+k}} \mathtt{1}$. By Lemma 3.10, we see that $t_k \sim_{\Gamma_{p+k}} \mathtt{f}$.

Case 6. The case (FfT R_2).

In this case, since Γ_{p+k} is the same as Γ_{p+k+1}, we have the statement.

(2) Let $d_k = n$.

Case 1. The case (Weak).

If $t_{k+1} \not\sim_{\Gamma_{p+k+1}} \mathtt{f}$, then $d_{k+1} = \bot$.

Assume $t_{k+1} \sim_{\Gamma_{p+k+1}} \mathtt{f}$. By Definition 3.5, there exist m, $l \in \mathbb{N}$ such that $\mathrm{s}^{m_0} t_{k+1} \cong_{\Gamma_{p+k+1}} \mathrm{s}^{m_1}\mathtt{f}$. By $\Gamma_{p+k+1} \subseteq \Gamma_{p+k}$, we have $\mathrm{s}^{m_0} t_{k+1} \cong_{\Gamma_{p+k}} \mathrm{s}^{m_1}\mathtt{f}$. Since $t_k \equiv t_{k+1}$, we have $\mathrm{s}^{m_0} t_k \cong_{\Gamma_{p+k}} \mathrm{s}^{m_1}\mathtt{f}$. By $d_k = n$, we have $m_1 - m_0 = n$. Thus, $d_{k+1} = n$.

Case 2. The case (Sub) with a substitution θ. Note that $t_k \equiv t_{k+1}[\theta]$.

If $t_{k+1} \not\sim_{\Gamma_{p+k+1}} \mathtt{f}$, then $d_{k+1} = \bot$.

Assume that $t_{k+1} \sim_{\Gamma_{p+k+1}} \mathtt{f}$. By Definition 3.5, there exist m_0, $m_1 \in \mathbb{N}$ such that $\mathrm{s}^{m_0} t_{k+1} \cong_{\Gamma_{p+k+1}} \mathrm{s}^{m_1}\mathtt{f}$. By Lemma 3.6 (1), $\mathrm{s}^{m_0}\, t_{k+1}[\theta] \cong_{\Gamma_{p+k}} \mathrm{s}^{m_1}\mathtt{f}$. Since $t_k \equiv t_{k+1}[\theta]$ holds, we have $\mathrm{s}^{m_0} t_k \cong_{\Gamma_{p+k}} \mathrm{s}^{m_1}\mathtt{f}$. By $d_k = n$, we have $m_1 - m_0 = n$. Thus, $d_{k+1} = n$.

(3) Let $d_k = n$.

Case 1. The case $(= \mathrm{L}_a)$with the principal formula $u_1 = u_2$.

In this case, there exists a term t such that $t_k \equiv t[v_1 := u_1, v_2 := u_2]$ and $t_{k+1} \equiv t[v_1 := u_2, v_2 := u_1]$ for variables v_1, v_2.

By $d_k = n$, there exist m_0, $m_1 \in \mathbb{N}$ such that $\mathrm{s}^{m_0}\, t[v_1 := u_1, v_2 := u_2] \cong_{\Gamma_{p+k}} \mathrm{s}^{m_1}\mathtt{f}$ and $m_1 - m_0 = n$. From Lemma 3.7 (1), $\mathrm{s}^{m_0}\, t[v_1 := u_2, v_2 := u_1] \cong_{\Gamma_{p+k+1}} \mathrm{s}^{m_1}\mathtt{f}$. Thus, $d_{k+1} = m_1 - m_0 = n$.

Case 2. The case ($\mathtt{F}\mathtt{f}\mathrm{T}$ R_2).

Since $\tau_{p+k+1} \equiv \tau_{p+k}$ holds and Γ_{p+k} is the same as Γ_{p+k+1}, we have $d_{k+1} = d_k$.

(4) Let $d_k = n$. Let $\mathrm{T1F}(t)$ be the principal formula of the rule (Case T1F) with the conclusion $\Gamma_{p+k} \Rightarrow \Delta_{p+k}$.

(4)(a) The case where $\Gamma_{p+k+1} \Rightarrow \Delta_{p+k+1}$ is the left assumption of the rule. In this case, $\Gamma_{p+k} \Rightarrow \Delta_{p+k}$ is a switching point. There exists Π such that $\Gamma_{p+k} \equiv (\Pi, \mathrm{T1F}(t))$ and $\Gamma_{p+k+1} \equiv (\Pi, t = \mathtt{1})$.

By $d_k = n$, there exist m_0, $m_1 \in \mathbb{N}$ such that $\mathrm{s}^{m_0} t_k \cong_{\Gamma_{p+k}} \mathrm{s}^{m_1}\mathtt{f}$ and $m_1 - m_0 = n$.

Since the set of formulas with $=$ in Γ_{p+k+1} includes the set of formulas with $=$ in Γ_{p+k}, we have $\mathrm{s}^{m_0} t_k \cong_{\Gamma_{p+k+1}} \mathrm{s}^{m_1}\mathtt{f}$. By $\tau_k \equiv \tau_{k+1}$, we have $\mathrm{s}^{m_0} t_{k+1} \cong_{\Gamma_{p+k+1}} \mathrm{s}^{m_1}\mathtt{f}$. Thus, $d_{k+1} = m_1 - m_0 = n$.

(4)(b) The case where $\Gamma_{p+k+1} \Rightarrow \Delta_{p+k+1}$ is the right assumption of the rule and τ_k is not a progress point of the trace.

Since τ_k is not a progress point of the trace, we have $\tau_{k+1} \equiv \tau_k$. By $d_k = n$, there exist m_0, $m_1 \in \mathbb{N}$ such that $\mathrm{s}^{m_0} t_k \cong_{\Gamma_{p+k}} \mathrm{s}^{m_1}\mathtt{f}$ and $m_1 - m_0 = n$.

Since the set of formulas with $=$ in Γ_{p+k} includes the set of formulas with $=$ in Γ_{p+k+1}, we have $\mathrm{s}^{m_0} t_k \cong_{\Gamma_{p+k+1}} \mathrm{s}^{m_1}\mathtt{f}$. By $\tau_{k+1} \equiv \tau_k$, we have $\mathrm{s}^{m_0} t_{k+1} \cong_{\Gamma_{p+k+1}} \mathrm{s}^{m_1}\mathtt{f}$. Thus, $d_{k+1} = m_1 - m_0 = n$.

(4)(c) The case where $\Gamma_{p+k+1} \Rightarrow \Delta_{p+k+1}$ is the right assumption of the rule and τ_k is a progress point of the trace.

There exists Π such that $\Gamma_{p+k} \equiv (\Pi, \mathrm{T1F}(t))$ and $\Gamma_{p+k+1} \equiv (\Pi, \mathtt{f} = x, \mathrm{T1F}(\mathrm{s}x))$ for a fresh variable x. Since τ_k is a progress point of the trace, we have $\tau_k \equiv \mathrm{T1F}(t)$ and $\tau_{k+1} \equiv \mathrm{T1F}(\mathrm{s}x)$. Therefore, $t_k \equiv t$ and $t_{k+1} \equiv \mathrm{s}x$. By $d_k = n$, there exist m_0, $m_1 \in \mathbb{N}$ such that $\mathrm{s}^{m_0} t \cong_{\Gamma_{p+k}} \mathrm{s}^{m_1}\mathtt{f}$ and $m_1 - m_0 = n$. Since the set of formulas with $=$ in Γ_{p+k+1} includes the set of formulas with $=$ in Γ_{p+k}, we have $\mathrm{s}^{m_0} t \cong_{\Gamma_{p+k+1}} \mathrm{s}^{m_1}\mathtt{f}$. By $\mathtt{f} \cong_{\Gamma_{p+k+1}} x$, we have $\mathrm{s}^{m_0} x \cong_{\Gamma_{p+k+1}} \mathrm{s}^{m_1}\mathtt{f}$. Hence, $\mathrm{s}^{m}\mathrm{s}x \cong_{\Gamma_{p+k+1}} \mathrm{s}^{m_1}\mathrm{s}\mathtt{f}$. Therefore, $\mathrm{s}^{m} t_{k+1} \cong_{\Gamma_{p+k+1}} \mathrm{s}^{m_1+1}\mathtt{f}$. Thus, $d_{k+1} = m_1 + 1 - m_0 = n + 1$. □

The following lemma corresponds to Lemma 3.19.

Lemma 4.9. *For an infinite idling path $\{\Gamma_i \Rightarrow \Delta_i\}_{i \geq 0}$ in $\mathcal{T}\left(\mathcal{D}^2_{\mathrm{cf}}, \mathcal{C}^2_{\mathrm{cf}}\right)$, there exists $l \in \mathbb{N}$ such that the following conditions hold:*

(1) $\Gamma_l \Rightarrow \Delta_l$ is a switching point in $\mathcal{T}\left(\mathcal{D}^2_{\mathrm{cf}}, \mathcal{C}^2_{\mathrm{cf}}\right)$, and

(2) $\Gamma_{l+1} \Rightarrow \Delta_{l+1}$ is the right assumption of the rule with the conclusion $\Gamma_l \Rightarrow \Delta_l$.

Proof. We have the statement in the same way as proving Lemma 3.19.

□

Definition 4.10 (Rightmost path)**.** For a derivation tree $\mathscr{D}$ and a node σ in $\mathscr{D}$, we define the *rightmost path* from the node σ as the path $\{\sigma_i\}_{0 \leq i < \alpha}$ satisfying the following conditions:

(1) The node σ_0 is σ.

(2) If σ_i is the conclusion of (Case T1F), the node σ_{i+1} is the right assumption of the rule.

(3) If σ_i is the conclusion of the rules (Weak), (Sub), ($=$ L_a), or ($\mathtt{F}\mathtt{f}\mathrm{T}$ R_2), the node σ_{i+1} is the assumption of the rule.

The following lemma corresponds to Lemma 3.21.

	nullary predicates	unary predicates	N-ary ($N \geq 2$) predicates
$\mathtt{CLKID}^\omega$ with function symbols	Yes[4]	No[1]	No[1]
$\mathtt{CLKID}^\omega$ without function symbols	Yes[4]	$\underline{?}$	?
Separation Logic	No[2]	No[2]	No[3]
Bunched Logic	No[2]	No[2]	No[2]

[1]By this chapter. [2]By [15]. [3]By [12]. [4]By [13].

Table 4.1 Arity of inductive predicates and the cut-elimination property in each cyclic proof system for some logics

Lemma 4.11. *The rightmost path from an index sequent in $\mathcal{T}\left(\mathcal{D}^2_{\mathrm{cf}}, \mathcal{C}^2_{\mathrm{cf}}\right)$ is infinite.*

Proof. We have the statement in the same way as proving Lemma 3.21. □

We have proved the lemmata to show Theorem 4.1 (2).

In the same way as the construction in the last part of Section 3.4, we can construct a sequence $\{\tilde{\sigma}_i\}_{i\in\mathbb{N}}$ of switching points in $\mathcal{D}^2_{\mathrm{cf}}$ which satisfies the following conditions:

(i) The height of $\tilde{\sigma}_i$ is greater than the height of $\tilde{\sigma}_{i-1}$ in $\mathcal{D}^2_{\mathrm{cf}}$ for $i > 0$.

(ii) For any node σ on the path from the root to $\tilde{\sigma}_i$ in $\mathcal{D}^2_{\mathrm{cf}}$ excluding $\tilde{\sigma}_i$, σ is a switching point if and only if the child of σ on the path is the left assumption of the rule (Case T1F).

Because of (i), $\tilde{\sigma}_0, \tilde{\sigma}_1, \ldots$ are all distinct in $\mathcal{D}^2_{\mathrm{cf}}$. Thus, $\{\tilde{\sigma}_i \mid i \in \mathbb{N}\}$ is infinite. It is a contradiction since the set of nodes in $\mathcal{D}^2_{\mathrm{cf}}$ is finite. We have Theorem 4.1 (2). □

4.2 Discussion about the arity of predicates and the cut-elimination property

Since $\mathrm{T1F}(\mathtt{f}) \Rightarrow \mathrm{FfT}(\mathtt{1})$ is a counterexample to cut-elimination in $\mathtt{CLKID}^\omega$, the cut-elimination property in $\mathtt{CLKID}^\omega$ does not hold even if we restrict predicates in the language to unary predicates and $=$.

Table 4.1 shows the results we obtained about the cut-elimination property of each cyclic proof system for some logics. "Yes" means that the cut-elimination property holds. "No" means that the cut-elimination property does not hold. The second and third column results in the "Separation Logic" row are easily obtained from the result in [15] because the counterexample for the cyclic proof system of bunched logic also works for separation logic.

In Section 3.5, we have guessed that the reason the cut-elimination property does not hold in cyclic proof systems is the finiteness of sequents in each proof. By our observation, with "$\underline{?}$" in Table 4.1, it can restrict possible occurring sequents in each proof to finitely many. Therefore, we conjecture that the cut-elimination property of $\mathtt{CLKID}^\omega$ holds in the case.

We also conjecture that the cut-elimination property of $\mathtt{CLKID}^\omega$ does not hold with "?" in Table 4.1, i.e. if we restrict the arity of each predicate to two and restrict the term of language to variables. There is one possibility for a counterexample to cut-elimination in the case: $\mathrm{RTL}(x, y) \Rightarrow \mathrm{RTR}(x, y)$ with productions

$$\frac{}{\mathrm{RTL}(x,x)}\ , \qquad \frac{\mathrm{S}(x,z) \qquad \mathrm{RTL}(z,y)}{\mathrm{RTL}(x,y)}\ ,$$

$$\frac{}{\mathrm{RTR}(x, x)} \;, \qquad \frac{\mathrm{RTR}(x, z) \qquad \mathrm{S}(z, y)}{\mathrm{RTR}(x, y)} \;,$$

where S is a binary ordinary predicate. We guess this sequent might be a counterexample to cut-elimination in the case, but we have not proved it because we have never come up with the definition of "index" for the sequent. An issue of our proof technique is that the "index" is defined for each counterexample. Can the definition of "index" be generalised?

5 Cyclic proof system for Presburger Arithmetic

Chapters 2–4 discussed the cut-elimination property of the cyclic proof system for first-order logic with inductive definitions. This chapter discusses a first-order arithmetic, *Presburger Arithmetic*, and its cyclic proof system.

This chapter aims to define a cyclic proof system for Presburger Arithmetic and to show the equivalence between Presburger Arithmetic and the cyclic proof system.

Presburger Arithmetic is a theory obtained by removing multiplication from *Peano Arithmetic*. It is known as a complete theory [11, 19].

Generally, the cyclic proof system is more powerful than the corresponding proof system with induction, but the converse is not obvious when they do not include *Peano Arithmetic*.

In this thesis, we show that Presburger Arithmetic is equivalent to the corresponding cyclic proof system. The equivalence is proved by the completeness of Presburger Arithmetic.

Section 5.1 introduces Presburger Arithmetic. Section 5.2 shows the completeness of Presburger Arithmetic. Section 5.3 provides Infinitary Presburger Arithmetic, a non-well-founded infinitary proof system for Presburger Arithmetic. Section 5.4 defines Cyclic Presburger Arithmetic, a cyclic proof system for Presburger Arithmetic, and shows the equivalence between Presburger Arithmetic and Cyclic Presburger Arithmetic. Section 5.5 discusses related work.

5.1 Presburger Arithmetic

In this section, we introduce *Presburger Arithmetic*. Presburger Arithmetic is usually defined as the set of formulas in the language of Presburger Arithmetic which is valid in the standard interpretation in the set of natural numbers [11, 19], but, our aim is to study the proof system of Presburger Arithmetic, so we define Presburger Arithmetic as a subsystem of *Peano Arithmetic*, $\mathtt{PA}$.

We write $\mathcal{L}_{\mathtt{PA}}$ for the first-order language with equality, with signature $(0, \mathrm{s}, +, \cdot, <)$, where 0 is a constant, s is a unary function symbol, $+$, $\cdot$ are binary function symbols, and $<$ is a binary predicate symbol. We write $\mathcal{L}_{\mathtt{P}_+}$ for the first-order language obtained by removing the binary function symbol $\cdot$ from $\mathcal{L}_{\mathtt{PA}}$.

We call $\mathcal{L}_{\mathtt{P}_+}$ and $\mathcal{L}_{\mathtt{PA}}$ *the language of Presburger Arithmetic and Peano Arithmetic*, respectively. We use infix notation for $+$, $\cdot$, and $<$.

Peano Arithmetic, $\mathtt{PA}$, is the theory in the language $\mathcal{L}_{\mathtt{PA}}$ axiomatised by the following eight axioms and one axiom scheme:

(A1) $\forall x \neg(\mathrm{s}x = 0)$,

(A2) $\forall x_1 \forall x_2(\mathrm{s}x_1 = \mathrm{s}x_2 \to x_1 = x_2)$,

(A3) $\forall x(x + 0 = x)$,

(A4) $\forall x_1 \forall x_2(x_1 + \mathrm{s}x_2 = \mathrm{s}x_1 + x_2)$,

(A5) $\forall x(x \cdot 0 = 0)$,

$$\frac{\Gamma \Rightarrow \Delta, \text{One of the axioms of theory}}{\Gamma \Rightarrow \Delta} \text{ (AXIOM OF THEORY)}$$

Figure 5.1 The axiom rule for first-order theory

(A6) $\forall x_1 \forall x_2 (x_1 \cdot \mathrm{s}x_2 = x_1 + (x_1 \cdot x_2))$,

(A7) $\forall x \neg (x < 0)$,

(A8) $\forall x_1 \forall x_2 (x_1 < \mathrm{s}x_2 \leftrightarrow (x_1 < x_2 \vee x_1 = x_2))$,

(A9) $\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(\mathrm{s}x)) \to \forall x\, \varphi(x)$ for any first-order formula $\varphi(x)$.

We sometimes call (A9) induction scheme.

Presburger Arithmetic, $\mathtt{P}_+$, is the theory in the language $\mathcal{L}_{\mathtt{P}_+}$ axiomatised by six axioms, (A1), (A2), (A3), (A4), (A7) and (A8), and one axiom scheme (A9). You see that Presburger Arithmetic is a theory obtained by removing the multiplication from Peano Arithmetic.

We define a *proof in first-order theory* as a finite derivation tree with rules in Figures 2.1 and 5.1 and without buds. A proof for first-order theory whose root is assigned to $(\Gamma \Rightarrow \Delta, (\mathrm{R}))$ with a rule (R) is called a *proof of* $\Gamma \Rightarrow \Delta$. For first-order theory $\mathtt{T}$, we write $\mathtt{T} \vdash \Gamma \Rightarrow \Delta$ if there is a proof of $\Gamma \Rightarrow \Delta$ in $\mathtt{T}$. For simplicity, we write $\mathtt{T} \vdash \varphi$ for $\mathtt{T} \vdash \Rightarrow \varphi$.

The following theorem is known [11, 19]. For self-containedness, we include its proof in the next section.

Theorem 5.1 (Completeness of Presburger Arithmetic)**.** *Presburger Arithmetic is complete i.e. for any closed formula φ, either $\mathtt{P}_+ \vdash \varphi$ or $\mathtt{P}_+ \vdash \neg\varphi$.*

5.2 Completeness of Presburger Arithmetic

In this section, we show Theorem 5.1. To show the completeness of Presburger Arithmetic, we define *Presburger Arithmetic extended with modulo*, $\mathtt{P}_{+,\equiv}$, which admits the *quantifier-elimination property*. Then, we show the completeness of $\mathtt{P}_{+,\equiv}$, and it implies the completeness of Presburger Arithmetic.

5.2.1 Quantifier-elimination property

In this section, we define the *quantifier-elimination property* and introduce a condition for first-order theory admitting the quantifier-elimination property.

Definition 5.2 (Quantifier-elimination property)**.** We say that a first-order theory $\mathtt{T}$ admits *quantifier-elimination* if, for any formula φ, there exists a quantifier-free formula ψ such that $\mathtt{T} \vdash \varphi \leftrightarrow \psi$.

The following proposition gives a condition for first-order theory admitting the quantifier-elimination property. It is used to prove the quantifier-elimination property of Presburger Arithmetic extended with modulo.

Proposition 5.3. *Let $\mathtt{T}$ be a first-order theory. Assume, for any atomic formula α, there exists a quantifier-free formula α' where $\neg$ and $\to$ do not occur and $\mathtt{T} \vdash \neg\alpha \leftrightarrow \alpha'$. If, for atomic formulas $\alpha_0, \dots, \alpha_n$, where a variable x occurs freely, there exists a quantifier-free formula ψ such that $\mathtt{T} \vdash \exists x(\alpha_0 \wedge \dots \wedge \alpha_n) \leftrightarrow \psi$, then $\mathtt{T}$ admits quantifier-elimination.*

To prove this proposition, we show some lemmata.

A *literal* is defined as an atomic formula or its negation.

Lemma 5.4 (Disjunctive normal form)**.** *Let* $\mathtt{T}$ *be a first-order theory. For any quantifier-free formula* ϑ*, there exist literals* $\alpha_{i,j}$ *such that* $\mathtt{T} \vdash \vartheta \leftrightarrow (\alpha_{0,0} \wedge \cdots \wedge \alpha_{0,k_0}) \vee (\alpha_{1,0} \wedge \cdots \wedge \alpha_{1,k_1}) \vee \cdots \vee (\alpha_{m,0} \wedge \cdots \wedge \alpha_{m,k_m})$.

Proof. We show the statement by induction on construction of ϑ.

Case 1. If ϑ is an atomic formula, then $\mathtt{T} \vdash \vartheta \leftrightarrow \vartheta$. Thus, we have the statement.

Case 2. Assume ϑ is the form of $\neg\vartheta_0$. By the induction hypothesis,

$$\mathtt{T} \vdash \vartheta_0 \leftrightarrow (\alpha_{0,0} \wedge \cdots \wedge \alpha_{0,k_0}) \vee (\alpha_{1,0} \wedge \cdots \wedge \alpha_{1,k_1}) \vee \cdots \vee (\alpha_{m,0} \wedge \cdots \wedge \alpha_{m,k_m})$$

for literals $\alpha_{i,j}$. Then,

$$\mathtt{T} \vdash \neg\vartheta_0 \leftrightarrow \neg((\alpha_{0,0} \wedge \cdots \wedge \alpha_{0,k_0}) \vee (\alpha_{1,0} \wedge \cdots \wedge \alpha_{1,k_1}) \vee \cdots \vee (\alpha_{m,0} \wedge \cdots \wedge \alpha_{m,k_m})).$$

By De Morgan's law,

$$\mathtt{T} \vdash \neg\vartheta_0 \leftrightarrow (\neg\alpha_{0,0} \vee \cdots \vee \neg\alpha_{0,k_0}) \wedge (\neg\alpha_{1,0} \vee \cdots \vee \neg\alpha_{1,k_1}) \wedge \cdots \wedge (\neg\alpha_{m,0} \vee \cdots \vee \neg\alpha_{m,k_m}).$$

By the distributive law $\mathtt{T} \vdash (\varphi_0 \vee \varphi_1) \wedge \psi \leftrightarrow (\varphi_0 \wedge \psi) \vee (\varphi_1 \wedge \psi)$, we have the statement.

Case 3. Assume ϑ is the form of $\vartheta_0 \wedge \vartheta_1$. By the induction hypothesis,

$$\mathtt{T} \vdash \vartheta_0 \leftrightarrow ((\alpha_{0,0} \wedge \cdots \wedge \alpha_{0,k_0}) \vee (\alpha_{1,0} \wedge \cdots \wedge \alpha_{1,k_1}) \vee \cdots \vee (\alpha_{m,0} \wedge \cdots \wedge \alpha_{m,k_m})),$$
$$\mathtt{T} \vdash \vartheta_1 \leftrightarrow ((\alpha'_{0,0} \wedge \cdots \wedge \alpha'_{0,k_0}) \vee (\alpha'_{1,0} \wedge \cdots \wedge \alpha'_{1,k_1}) \vee \cdots \vee (\alpha'_{m,0} \wedge \cdots \wedge \alpha'_{m,k_m}))$$

for literals $\alpha_{i,j}$, $\alpha'_{i,j}$. Thus,

$$\mathtt{T} \vdash \vartheta_0 \wedge \vartheta_1 \leftrightarrow ((\alpha_{0,0} \wedge \cdots \wedge \alpha_{0,k_0}) \vee \cdots \vee (\alpha_{m,0} \wedge \cdots \wedge \alpha_{m,k_m})) \wedge ((\alpha'_{0,0} \wedge \cdots \wedge \alpha'_{0,k_0}) \vee \cdots \vee (\alpha'_{m,0} \wedge \cdots \wedge \alpha'_{m,k_m})).$$

By the distributive law $\mathtt{T} \vdash (\varphi_0 \vee \varphi_1) \wedge \psi \leftrightarrow (\varphi_0 \wedge \psi) \vee (\varphi_1 \wedge \psi)$, we have the statement.

Case 4. Assume ϑ is the form of $\vartheta_0 \vee \vartheta_1$. By the induction hypothesis,

$$\mathtt{T} \vdash \vartheta_0 \leftrightarrow ((\alpha_{0,0} \wedge \cdots \wedge \alpha_{0,k_0}) \vee (\alpha_{1,0} \wedge \cdots \wedge \alpha_{1,k_1}) \vee \cdots \vee (\alpha_{m,0} \wedge \cdots \wedge \alpha_{m,k_m})),$$
$$\mathtt{T} \vdash \vartheta_1 \leftrightarrow ((\alpha'_{0,0} \wedge \cdots \wedge \alpha'_{0,k_0}) \vee (\alpha'_{1,0} \wedge \cdots \wedge \alpha'_{1,k_1}) \vee \cdots \vee (\alpha'_{m,0} \wedge \cdots \wedge \alpha'_{m,k_m}))$$

for literals $\alpha_{i,j}$, $\alpha'_{i,j}$. Thus,

$$\mathtt{T} \vdash \vartheta_0 \vee \vartheta_1 \leftrightarrow ((\alpha_{0,0} \wedge \cdots \wedge \alpha_{0,k_0}) \vee \cdots \vee (\alpha_{m,0} \wedge \cdots \wedge \alpha_{m,k_m})) \vee ((\alpha'_{0,0} \wedge \cdots \wedge \alpha'_{0,k_0}) \vee \cdots \vee (\alpha'_{m,0} \wedge \cdots \wedge \alpha'_{m,k_m})).$$

Case 5. Assume ϑ is the form of $\vartheta_0 \to \vartheta_1$. Then, $\mathtt{T} \vdash \vartheta_0 \to \vartheta_1 \leftrightarrow \neg\vartheta_0 \vee \vartheta_1$. By the induction hypothesis,

$$\mathtt{T} \vdash \neg\vartheta_0 \leftrightarrow ((\alpha_{0,0} \wedge \cdots \wedge \alpha_{0,k_0}) \vee (\alpha_{1,0} \wedge \cdots \wedge \alpha_{1,k_1}) \vee \cdots \vee (\alpha_{m,0} \wedge \cdots \wedge \alpha_{m,k_m})),$$
$$\mathtt{T} \vdash \vartheta_1 \leftrightarrow ((\alpha'_{0,0} \wedge \cdots \wedge \alpha'_{0,k_0}) \vee (\alpha'_{1,0} \wedge \cdots \wedge \alpha'_{1,k_1}) \vee \cdots \vee (\alpha'_{m,0} \wedge \cdots \wedge \alpha'_{m,k_m}))$$

for literals $\alpha_{i,j}$, $\alpha'_{i,j}$. Thus,

$$\mathtt{T} \vdash \vartheta_0 \to \vartheta_1 \leftrightarrow ((\alpha_{0,0} \wedge \cdots \wedge \alpha_{0,k_0}) \vee \cdots \vee (\alpha_{m,0} \wedge \cdots \wedge \alpha_{m,k_m})) \vee ((\alpha'_{0,0} \wedge \cdots \wedge \alpha'_{0,k_0}) \vee \cdots \vee (\alpha'_{m,0} \wedge \cdots \wedge \alpha'_{m,k_m})).$$

□

Proposition 5.5. *Let* $\mathtt{T}$ *be a first-order theory. Assume, for any atomic formula* α*, there exists a quantifier-free formula* α' *where* $\neg$ *and* $\rightarrow$ *do not occur and* $\mathtt{T} \vdash \neg\alpha \leftrightarrow \alpha'$*. For any quantifier-free formula* ϑ*, there exist atomic formulas* $\alpha_{i,j}$ *such that* $\mathtt{T} \vdash \vartheta \leftrightarrow (\alpha_{0,0} \wedge \cdots \wedge \alpha_{0,k_0}) \vee (\alpha_{1,0} \wedge \cdots \wedge \alpha_{1,k_1}) \vee \cdots \vee (\alpha_{m,0} \wedge \cdots \wedge \alpha_{m,k_m})$*.*

Proof. Assume, for any atomic formula α, there exists a quantifier-free formula α' where $\neg$ and $\rightarrow$ do not occur and $\mathtt{T} \vdash \neg\alpha \leftrightarrow \alpha'$. Because of the distributive law $\mathtt{T} \vdash (\varphi_0 \vee \varphi_1) \wedge \psi \leftrightarrow (\varphi_0 \wedge \psi) \vee (\varphi_1 \wedge \psi)$, the form of α' is $(\alpha_{0,0} \wedge \cdots \wedge \alpha_{0,k_0}) \vee (\alpha_{1,0} \wedge \cdots \wedge \alpha_{1,k_1}) \vee \cdots \vee (\alpha_{m,0} \wedge \cdots \wedge \alpha_{m,k_m})$ for atomic formulas $\alpha_{i,j}$ without loss of generality.

Let ϑ be a quantifier-free formula. By Lemma 5.4, there exist literals $\alpha_{i,j}$ such that $\mathtt{T} \vdash \vartheta \leftrightarrow (\alpha_{0,0} \wedge \cdots \wedge \alpha_{0,k_0}) \vee (\alpha_{1,0} \wedge \cdots \wedge \alpha_{1,k_1}) \vee \cdots \vee (\alpha_{m,0} \wedge \cdots \wedge \alpha_{m,k_m})$. By the assumption, there exists a formula ϑ' where $\vee$, $\wedge$ and atomic formulas only occur and $\mathtt{T} \vdash \vartheta' \leftrightarrow (\alpha_{0,0} \wedge \cdots \wedge \alpha_{0,k_0}) \vee (\alpha_{1,0} \wedge \cdots \wedge \alpha_{1,k_1}) \vee \cdots \vee (\alpha_{m,0} \wedge \cdots \wedge \alpha_{m,k_m})$. Because of the distributive law $\mathtt{T} \vdash (\varphi_0 \vee \varphi_1) \wedge \psi \leftrightarrow (\varphi_0 \wedge \psi) \vee (\varphi_1 \wedge \psi)$, we have the statement. □

Proposition 5.6. *Let* $\mathtt{T}$ *be a first-order theory. If, for any quantifier-free formula* ϑ*, there exists a quantifier-free formula* ψ *such that* $\mathtt{T} \vdash \exists x\vartheta \leftrightarrow \psi$*, then* $\mathtt{T}$ *admits quantifier-elimination.*

Proof. Assume that, for any quantifier-free formula ϑ, there exists a quantifier-free formula ψ such that $\mathtt{T} \vdash \exists x\vartheta \leftrightarrow \psi$.

Let φ be a formula. It suffices to show that there exists a quantifier-free formula φ_0 such that $\mathtt{T} \vdash \varphi \leftrightarrow \varphi_0$. Since $\mathtt{T} \vdash \forall x\chi \leftrightarrow \neg\exists\neg\chi$ for any formula χ, we can assume $\forall$ does not occur in φ without loss of generality. We show the statement by induction on the number of occurrences of $\exists$ in φ.

If the number of occurrences of $\exists$ in φ is 0, then φ_0 is φ itself.

Let i be the number of occurrences of $\exists$ in φ and assume $i > 0$. Then, a formula whose form is $\exists x\theta$ where θ is a quantifier-free formula occurs in φ. By the assumption, there exists a quantifier-free formula ψ such that $\mathtt{T} \vdash \exists x\theta \leftrightarrow \psi$. Let φ' be the formula obtained by replacing $\exists x\theta$ with ψ. Then, $\mathtt{T} \vdash \varphi \leftrightarrow \varphi'$. The number of occurrences of $\exists$ in φ' is less than the number of occurrences of $\exists$ in φ. By the induction hypothesis, there exists a quantifier-free formula φ_0 such that $\mathtt{T} \vdash \varphi' \leftrightarrow \varphi_0$. Because of $\mathtt{T} \vdash \varphi \leftrightarrow \varphi'$, we have $\mathtt{T} \vdash \varphi \leftrightarrow \varphi_0$. □

We show Proposition 5.3.

Proof of Proposition 5.3. Assume, for atomic formulas $\alpha_0, \ldots, \alpha_n$, where a variable x occurs freely, there exists a quantifier-free formula ψ such that $\mathtt{T} \vdash \exists x(\alpha_0 \wedge \cdots \wedge \alpha_n) \leftrightarrow \psi$. By Proposition 5.6, it suffices to show that, for a quantifier-free formula ϑ, there exists a quantifier-free formula φ such that $\mathtt{T} \vdash \exists x\vartheta \leftrightarrow \varphi$.

Let ϑ be a quantifier-free formula. By Proposition 5.5, there exist atomic formulas $\alpha_{i,j}$ such that

$$\mathtt{T} \vdash \vartheta \leftrightarrow (\alpha_{0,0} \wedge \cdots \wedge \alpha_{0,k_0}) \vee (\alpha_{1,0} \wedge \cdots \wedge \alpha_{1,k_1}) \vee \cdots \vee (\alpha_{m,0} \wedge \cdots \wedge \alpha_{m,k_m}).$$

Then,

$$\mathtt{T} \vdash \exists x\vartheta \leftrightarrow \exists x((\alpha_{0,0} \wedge \cdots \wedge \alpha_{0,k_0}) \vee (\alpha_{1,0} \wedge \cdots \wedge \alpha_{1,k_1}) \vee \cdots \vee (\alpha_{m,0} \wedge \cdots \wedge \alpha_{m,k_m})).$$

Hence,

$$\mathtt{T} \vdash \exists x\vartheta \leftrightarrow (\exists x(\alpha_{0,0} \wedge \cdots \wedge \alpha_{0,k_0}) \vee \exists x(\alpha_{1,0} \wedge \cdots \wedge \alpha_{1,k_1}) \vee \cdots \vee \exists x(\alpha_{m,0} \wedge \cdots \wedge \alpha_{m,k_m})).$$

By the assumption, for $i = 0, \ldots, m$ there exists a quantifier-free formula ψ_i such that $\mathtt{T} \vdash \exists x(\alpha_{i,0} \wedge \cdots \wedge \alpha_{i,k_i}) \leftrightarrow \psi_i$. Thus,

$$\mathtt{T} \vdash \exists x\vartheta \leftrightarrow \psi_0 \vee \cdots \vee \psi_m.$$

□

5.2.2 Presburger arithmetic extended with modulo

In this section, we define Presburger arithmetic extended with modulo and show that it admits the quantifier-elimination property.

Presburger Arithmetic does not admit quantifier-elimination since there does not exist quantifier-free formula ψ such that $\mathtt{P}_+ \vdash \exists x(y = x + x) \leftrightarrow \psi$, but Presburger arithmetic extended with modulo, $\mathtt{P}_{+,\equiv}$, admits the quantifier-elimination [11].

We write $\mathcal{L}_{\mathtt{P}_{+,\equiv}}$ for the first-order language obtained by adding infinitely many binary relation symbols $\equiv_2, \equiv_3, \dots$, to $\mathcal{L}_{\mathtt{P}_+}$. We abbreviate

$$(\dots(\overbrace{z + z) + z) + \dots + z}^{m})$$

to $m \times z$. We define *Presburger Arithmetic extended with modulo*, $\mathtt{P}_{+,\equiv}$, as the theory in the language $\mathcal{L}_{\mathtt{P}_{+,\equiv}}$ axiomatised by six axioms, (A1), (A2), (A3), (A4), (A7) and (A8), and two axiom schemes (A9) and

(Mod$_m$) $\forall x_1 \forall x_2 (x_1 \equiv_m x_2 \leftrightarrow \exists z(x_1 = m \times z + x_2 \vee x_2 = m \times z + x_1))$.

We note that $\mathtt{P}_{+,\equiv}$ is a conservative extension of Presburger Arithmetic since each $\equiv_m$ is definable in Presburger Arithmetic.

For simplicity, we write $\mathrm{s}^m 0$ for $\overline{m}$.

Lemma 5.7. $\mathtt{P}_{+,\equiv}$ *admits quantifier-elimination.*

Sketch of proof. We can show

$$\begin{aligned} \mathtt{P}_+ &\vdash \neg(t = u) \leftrightarrow t < u \vee u < t, \\ \mathtt{P}_+ &\vdash \neg(t < u) \leftrightarrow t = u \vee u < t, \\ \mathtt{P}_{+,\equiv} &\vdash \neg(t \equiv_m u) \leftrightarrow t \equiv_m u + \overline{1} \vee \dots \vee t \equiv_m u + \overline{m-1} \text{ for all } m > 1. \end{aligned}$$

Then, by Proposition 5.3, it suffices to show that, for atomic formulas $\alpha_1, \dots, \alpha_l$, where a variable x occurs freely, there exists a quantifier-free formula ψ such that $\mathtt{P}_{+,\equiv} \vdash \exists x(\alpha_1 \wedge \dots \wedge \alpha_l) \leftrightarrow \psi$.

Let $\alpha_1, \dots, \alpha_l$ be atomic formulas where a variable x occurs freely.

Since we can show $\mathtt{P}_+ \vdash (t_1 + t_2) + t_3 = t_1 + (t_2 + t_3)$, $\mathtt{P}_+ \vdash t + u = u + t$, $\mathtt{P}_+ \vdash \mathrm{s}^k t = t + \overline{k}$ for all $k \in \mathbb{N}$, $\mathtt{P}_+ \vdash t_1 = t_2 \leftrightarrow t_1 + u = t_2 + u$, $\mathtt{P}_+ \vdash t_1 < t_2 \leftrightarrow t_1 + u < t_2 + u$, and $\mathtt{P}_{+,\equiv} \vdash t_1 \equiv_m t_2 \leftrightarrow t_1 + u \equiv_m t_2 + u$, we can assume that the form of each α_i is

$$n \times x + t_1 = t_2, \quad n \times x + t_1 < t_2, \quad t_1 < n \times x + t_2, \text{ or } n \times x + t_1 \equiv_m t_2,$$

where x does not occur in t_1, t_2 and $n > 0$ without loss of generality. Since $\mathtt{P}_+ \vdash t_1 = t_2 \leftrightarrow k \times t_1 = k \times t_2$, $\mathtt{P}_+ \vdash t_1 < t_2 \leftrightarrow k \times t_1 < k \times t_2$, and $\mathtt{P}_{+,\equiv} \vdash t_1 \equiv_m t_2 \leftrightarrow k \times t_1 \equiv_{km} k \times t_2$, for all $k \in \mathbb{N}_{>0}$, without loss of generality, we can assume that the number of occurrences of x in each α_i equals that of each other.

We consider the case $n > 1$. Let β_i be an atomic formula obtained by replacing the occurrence of $n \times x$ in α_i with a fresh variable y for each $i = 1, \dots, l$. Then, we can show $\mathtt{P}_{+,\equiv} \vdash \exists x(\alpha_1 \wedge \dots \wedge \alpha_l) \leftrightarrow \exists y(y \equiv_n 0 \wedge \beta_1 \wedge \dots \wedge \beta_l)$.

Thus, we can assume the form of each α_i is

$$x + t_1 = t_2, \quad x + t_1 < t_2, \quad t_1 < x + t_2, \text{ or } x + t_1 \equiv_m t_2,$$

where x does not occur in t_1, t_2 without loss of generality.

Assume there exists $j \in \{1, \dots, l\}$ such that the form of α_j is $x + u_1 = u_2$. Define $\hat{\alpha}_i$ by setting

$$\hat{\alpha}_i \equiv \begin{cases} u_1 < \mathrm{s}u_2, & i = j; \\ u_2 + t_1 = u_1 + t_2, & \text{if the form of } \alpha_i \text{ is } x + t_1 = t_2; \\ u_2 + t_1 < u_1 + t_2, & \text{if the form of } \alpha_i \text{ is } x + t_1 < t_2; \\ u_1 + t_1 < u_2 + t_2, & \text{if the form of } \alpha_i \text{ is } t_1 < x + t_2; \\ u_2 + t_1 \equiv_m u_1 + t_2, & \text{if the form of } \alpha_i \text{ is } x + t_1 \equiv_m t_2, \end{cases}$$

where x does not occur in t_1, t_2 and $n > 0$. Then, we can show $\mathtt{P}_{+,\equiv} \vdash \exists x(\alpha_1 \wedge \dots \wedge \alpha_l) \leftrightarrow \exists x(\hat{\alpha}_1 \wedge \dots \wedge \hat{\alpha}_l)$. Since x does not occur in t_1, t_2, u_1, and u_2, we have $\mathtt{P}_{+,\equiv} \vdash \exists x(\hat{\alpha}_1 \wedge \dots \wedge \hat{\alpha}_l) \leftrightarrow \hat{\alpha}_1 \wedge \dots \wedge \hat{\alpha}_l$. Thus, $\mathtt{P}_{+,\equiv} \vdash \exists x(\alpha_1 \wedge \dots \wedge \alpha_l) \leftrightarrow \hat{\alpha}_1 \wedge \dots \wedge \hat{\alpha}_l$.

Assume the form of each atomic formula α_i is not $x + t_1 = t_2$. Then, the form of each α_i is

$$x + t_1 < t_2, \quad t_1 < x + t_2, \text{ or } x + t_1 \equiv_m t_2,$$

where x does not occur in t_1, t_2 and $m > 0$.

Assume the form of each α_i is either $x + t_1 < t_2$ or $t_1 < x + t_2$, where x does not occur in t_1, t_2. Then, the form of $\exists x(\alpha_1 \wedge \dots \wedge \alpha_l)$ is

$$\exists x \left(\left(\bigwedge_{i=1}^{l'} t_{1,i} < x + t_{2,i} \right) \wedge \left(\bigwedge_{j=l'+1}^{l} x + t_{1,j} < t_{2,j} \right) \right).$$

Then, we can show

$$\begin{aligned} \mathtt{P}_{+,\equiv} \vdash \exists x \left(\left(\bigwedge_{i=1}^{l'} t_{1,i} < x + t_{2,i} \right) \wedge \left(\bigwedge_{j=l'+1}^{l} x + t_{1,j} < t_{2,j} \right) \right) \leftrightarrow \\ \exists x \left(\left(\bigwedge_{i=1}^{l'} u_1^{(i)} < x + t_{2,1} + \dots + t_{2,l'} + t_{1,l'+1} + \dots + t_{1,l} \right) \wedge \right. \\ \left. \left(\bigwedge_{j=l'}^{l} x + t_{2,1} + \dots + t_{2,l'} + t_{1,l'+1} + \dots + t_{1,l} < u_2^{(j)} \right) \right), \end{aligned}$$

where $u_1^{(i)} \equiv t_{2,1} + \dots + t_{2,i-1} + t_{1,i} + t_{2,i+1} + \dots + t_{2,l'} + t_{1,l'+1} + \dots + t_{1,l}$ and $u_2^{(j)} \equiv t_{2,1} + \dots + t_{2,l'} + t_{1,l'+1} + \dots + t_{1,j-1} + t_{2,j} + t_{1,j+1} + \dots + t_{1,l}$. Then, we can show

$$\mathtt{P}_{+,\equiv} \vdash \exists x \left(\left(\bigwedge_{i=1}^{l'} t_{1,i} < x + t_{2,i} \right) \wedge \left(\bigwedge_{j=l'+1}^{l} x + t_{1,j} < t_{2,j} \right) \right) \leftrightarrow \exists x \left(\bigwedge_{\substack{i \in \{1,\dots,l'\} \\ j \in \{l'+1,\dots,l\}}} \mathrm{s}u_1^{(i)} < u_2^{(j)} \right).$$

Since x does not occur in $u_1^{(i)}$ and $u_2^{(j)}$, we have

$$\mathtt{P}_{+,\equiv} \vdash \exists x \left(\left(\bigwedge_{i=1}^{l'} t_{1,i} < x + t_{2,i} \right) \wedge \left(\bigwedge_{j=l'+1}^{l} x + t_{1,j} < t_{2,j} \right) \right) \leftrightarrow \bigwedge_{\substack{i \in \{1,\dots,l'\} \\ j \in \{l'+1,\dots,l\}}} \mathrm{s}u_1^{(i)} < u_2^{(j)}.$$

Assume there exists the form of $x + t_1 \equiv_m t_2$ among α_i's. Then, the form of $\exists x(\alpha_1 \wedge \cdots \wedge \alpha_{l+1})$ is

$$\exists x\left(\left(\bigwedge_{i=1}^{l_1} t_{1,i} < x + t_{2,i}\right) \wedge \left(\bigwedge_{j=l_1+1}^{l_2} x + t_{1,j} < t_{2,j}\right) \wedge \left(\bigwedge_{k=l_2+1}^{l} x + t_{1,k} \equiv_{m_k} t_{2,k}\right)\right).$$

Let M be the least common multiple of $m_{l_2+1}, \ldots, m_l$. Intuitively, the solution of $t_{1,i} < x + t_{2,i} \wedge \left(\bigwedge_{k=l_2+1}^{l} x + t_{1,k} \equiv_{m_k} t_{2,k}\right)$ is among $\mathrm{s}t_{1,q} - t_{2,i} + \overline{1}$, $\mathrm{s}t_{1,q} - t_{2,i} + \overline{2}$, $\ldots$, $\mathrm{s}t_{1,q} - t_{2,i} + \overline{M}$. Indeed, we can show

$$\begin{aligned}\mathtt{P}_{+,\equiv} \vdash \exists x\left(\left(\bigwedge_{i=1}^{l_1} t_{1,i} < x + t_{2,i}\right) \wedge \left(\bigwedge_{j=l_1+1}^{l_2} x + t_{1,j} < t_{2,j}\right) \wedge \left(\bigwedge_{k=l_2+1}^{l} x + t_{1,k} \equiv_{m_k} t_{2,k}\right)\right) \leftrightarrow \\ \exists x\left(\bigvee_{q=1}^{l_1}\bigvee_{p=1}^{M}\left(\bigwedge_{i=1}^{l_1} t_{1,i} + t_{2,q} < \mathrm{s}t_{1,q} + \overline{p} + t_{2,i}\right) \wedge \left(\bigwedge_{j=l_1+1}^{l_2} \mathrm{s}t_{1,q} + \overline{p} + t_{1,j} < t_{2,j} + t_{2,q}\right) \wedge \right. \\ \left.\left(\bigwedge_{k=l_2+1}^{l} \mathrm{s}t_{1,q} + \overline{p} + t_{1,k} \equiv_{m_k} t_{2,k} + t_{2,q}\right)\right).\end{aligned}$$

Since x does not occur in the right-hand side, we have

$$\begin{aligned}\mathtt{P}_{+,\equiv} \vdash \exists x\left(\left(\bigwedge_{i=1}^{l_1} t_{1,i} < x + t_{2,i}\right) \wedge \left(\bigwedge_{j=l_1+1}^{l_2} x + t_{1,j} < t_{2,j}\right) \wedge \left(\bigwedge_{k=l_2+1}^{l} x + t_{1,k} \equiv_{m_k} t_{2,k}\right)\right) \leftrightarrow \\ \left(\bigvee_{q=1}^{l_1}\bigvee_{p=1}^{M}\left(\bigwedge_{i=1}^{l_1} t_{1,i} + t_{2,q} < \mathrm{s}t_{1,q} + \overline{p} + t_{2,i}\right) \wedge \left(\bigwedge_{j=l_1+1}^{l_2} \mathrm{s}t_{1,q} + \overline{p} + t_{1,j} < t_{2,j} + t_{2,q}\right) \wedge \right. \\ \left.\left(\bigwedge_{k=l_2+1}^{l} \mathrm{s}t_{1,q} + \overline{p} + t_{1,k} \equiv_{m_k} t_{2,k} + t_{2,q}\right)\right).\end{aligned}$$

□

5.2.3 Completeness

In this section, we show the completeness of Presburger Arithmetic. To show the completeness, we show that the completeness of Presburger Arithmetic extended with modulo.

Lemma 5.8. *The following statements hold:*

(1) *For any variable-free $\mathcal{L}_{\mathtt{P}_{+,\equiv}}$ term, there exists $n \in \mathbb{N}$ such that $\mathtt{P}_{+,\equiv} \vdash t = \overline{n}$.*

(2) *For any variable-free atomic formula α, either $\mathtt{P}_{+,\equiv} \vdash \alpha$ or $\mathtt{P}_{+,\equiv} \vdash \neg\alpha$.*

(3) *For any quantifier-free variable-free formula ϑ, either $\mathtt{P}_{+,\equiv} \vdash \vartheta$ or $\mathtt{P}_{+,\equiv} \vdash \neg\vartheta$.*

Sketch of proof. (1) We show the statement by induction on the construction of t.

If t is 0, then we have the statement.

Assume t is of the form $\mathrm{s}t'$ with a term t'. By the induction hypothesis, $\mathtt{P}_{+,\equiv} \vdash t' = \overline{n}$ for $n \in \mathbb{N}$. Then, we have $\mathtt{P}_{+,\equiv} \vdash t = \overline{n+1}$.

Assume t is of the form st' with a term $t_1 + t_2$. By the induction hypothesis, $\mathtt{P}_{+,\equiv} \vdash t_1 = \overline{n}$ and $\mathtt{P}_{+,\equiv} \vdash t_2 = \overline{m}$ for n, $m \in \mathbb{N}$. Then, we have $\mathtt{P}_{+,\equiv} \vdash t_1 + t_2 = \overline{n} + \overline{m}$. Since we can show $\mathtt{P}_{+,\equiv} \vdash \overline{n} + \overline{m} = \overline{n+m}$, we see that $\mathtt{P}_{+,\equiv} \vdash t_1 + t_2 = \overline{n+m}$.

(2) By (1), without loss of generality, we can assume that the form of α is $\overline{n} = \overline{m}$, $\overline{n} < \overline{m}$, or $\overline{n} \equiv_l \overline{m}$. We can show the following statements:

- If $n = m$, we have $\mathtt{P}_{+,\equiv} \vdash \overline{n} = \overline{m}$; otherwise we have $\mathtt{P}_{+,\equiv} \vdash \neg(\overline{n} = \overline{m})$.
- If $n < m$, we have $\mathtt{P}_{+,\equiv} \vdash \overline{n} < \overline{m}$; otherwise we have $\mathtt{P}_{+,\equiv} \vdash \neg(\overline{n} < \overline{m})$.
- If there exists an integer k such that $n - m = l \cdot k$, we have $\mathtt{P}_{+,\equiv} \vdash \overline{n} \equiv_l \overline{m}$; otherwise we have $\mathtt{P}_{+,\equiv} \vdash \neg(\overline{n} \equiv_l \overline{m})$.

Then, we have the statement.

(3) We show the statement by induction on the construction of ϑ.

In the case where ϑ is an atomic formula, we have the statement by (2).

Assume ϑ is of the form $\neg\vartheta_1$. Assume $\mathtt{P}_{+,\equiv} \nvdash \neg\vartheta_1$. By the induction hypothesis, we have $\mathtt{P}_{+,\equiv} \vdash \vartheta_1$. Because of $\mathtt{P}_{+,\equiv} \vdash \vartheta_1 \leftrightarrow \neg\neg\vartheta_1$, we have $\mathtt{P}_{+,\equiv} \vdash \neg\vartheta$.

Assume ϑ is of the form $\vartheta_1 \wedge \vartheta_2$. Assume $\mathtt{P}_{+,\equiv} \nvdash \vartheta_1 \wedge \vartheta_2$. Then, we have $\mathtt{P}_{+,\equiv} \nvdash \vartheta_1$ or $\mathtt{P}_{+,\equiv} \nvdash \vartheta_2$. We consider the case $\mathtt{P}_{+,\equiv} \nvdash \vartheta_1$. By the induction hypothesis, $\mathtt{P}_{+,\equiv} \vdash \neg\vartheta_1$. Hence, we have $\mathtt{P}_{+,\equiv} \vdash \neg\vartheta_1 \vee \neg\vartheta_2$. Because of $\mathtt{P}_{+,\equiv} \vdash \neg\vartheta_1 \vee \neg\vartheta_2 \leftrightarrow \neg(\vartheta_1 \wedge \vartheta_2)$, we see that $\mathtt{P}_{+,\equiv} \vdash \neg(\vartheta_1 \wedge \vartheta_2)$. In the case $\mathtt{P}_{+,\equiv} \nvdash \vartheta_2$, we can show $\mathtt{P}_{+,\equiv} \vdash \neg(\vartheta_1 \wedge \vartheta_2)$ in the similar way.

Assume ϑ is of the form $\vartheta_1 \vee \vartheta_2$. Assume $\mathtt{P}_{+,\equiv} \nvdash \vartheta_1 \vee \vartheta_2$. Then, we have $\mathtt{P}_{+,\equiv} \nvdash \vartheta_1$ and $\mathtt{P}_{+,\equiv} \nvdash \vartheta_2$. By the induction hypothesis, $\mathtt{P}_{+,\equiv} \vdash \neg\vartheta_1$ and $\mathtt{P}_{+,\equiv} \vdash \neg\vartheta_2$. Hence, we have $\mathtt{P}_{+,\equiv} \vdash \neg\vartheta_1 \wedge \neg\vartheta_2$. Because of $\mathtt{P}_{+,\equiv} \vdash \neg\vartheta_1 \wedge \neg\vartheta_2 \leftrightarrow \neg(\vartheta_1 \vee \vartheta_2)$, we see that $\mathtt{P}_{+,\equiv} \vdash \neg(\vartheta_1 \vee \vartheta_2)$.

Assume ϑ is of the form $\vartheta_1 \to \vartheta_2$. Assume $\mathtt{P}_{+,\equiv} \nvdash \vartheta_1 \to \vartheta_2$. Because of $\mathtt{P}_{+,\equiv} \nvdash (\vartheta_1 \to \vartheta_2) \leftrightarrow (\neg\vartheta_1 \vee \vartheta_2)$, we have $\mathtt{P}_{+,\equiv} \nvdash \neg\vartheta_1 \vee \vartheta_2$. By the induction hypothesis, $\mathtt{P}_{+,\equiv} \vdash \vartheta_1$ and $\mathtt{P}_{+,\equiv} \vdash \neg\vartheta_2$. Hence we have $\mathtt{P}_{+,\equiv} \vdash \vartheta_1 \wedge \neg\vartheta_2$. Because of $\mathtt{P}_{+,\equiv} \vdash \vartheta_1 \wedge \neg\vartheta_2 \leftrightarrow \neg(\vartheta_1 \to \vartheta_2)$, we see that $\mathtt{P}_{+,\equiv} \vdash \neg(\vartheta_1 \to \vartheta_2)$. □

Proposition 5.9. *$\mathtt{P}_{+,\equiv}$ is complete i.e. for any closed formula φ, either $\mathtt{P}_{+,\equiv} \vdash \varphi$ or $\mathtt{P}_{+,\equiv} \vdash \neg\varphi$.*

Proof. Let φ be a closed formula. By Lemma 5.7, there exists a quantifier-free formula ψ such that $\mathtt{P}_{+,\equiv} \vdash \varphi \leftrightarrow \psi$. Let $x_0, \ldots, x_n$ be all free variables in ψ. Then, we have $\mathtt{P}_{+,\equiv} \vdash \varphi[x_0 := 0, \ldots, x_n := 0] \leftrightarrow \psi[x_0 := 0, \ldots, x_n := 0]$. Since φ is a closed formula, we have $\mathtt{P}_{+,\equiv} \vdash \varphi \leftrightarrow \psi[x_0 := 0, \ldots, x_n := 0]$. Let $\vartheta \equiv \psi[x_0 := 0, \ldots, x_n := 0]$. Then, ϑ is a quantifier-free variable-free formula where $\mathtt{P}_{+,\equiv} \vdash \varphi \leftrightarrow \vartheta$ and $\mathtt{P}_{+,\equiv} \vdash \neg\varphi \leftrightarrow \neg\vartheta$. By Lemma 5.8 (3), either $\mathtt{P}_{+,\equiv} \vdash \vartheta$ or $\mathtt{P}_{+,\equiv} \vdash \neg\vartheta$. Thus, we have either $\mathtt{P}_{+,\equiv} \vdash \varphi$ or $\mathtt{P}_{+,\equiv} \vdash \neg\varphi$. □

Now, we show the completeness of Presburger Arithmetic, Theorem 5.1.

Proof of Theorem 5.1. Assume $\mathtt{P}_{+} \nvdash \varphi$. Since $\mathtt{P}_{+,\equiv}$ is a conservative extension of $\mathtt{P}_{+}$, we have $\mathtt{P}_{+,\equiv} \nvdash \varphi$. By Proposition 5.9, we have $\mathtt{P}_{+,\equiv} \vdash \neg\varphi$. Since $\mathtt{P}_{+,\equiv}$ is a conservative extension of $\mathtt{P}_{+}$, we have $\mathtt{P}_{+} \vdash \neg\varphi$. □

5.3 Infinitary Presburger Arithmetic

In this section, we define an infinitary proof system for Presburger Arithmetic $\mathtt{P}_{+}^{\omega}$, called *Infinitary Presburger Arithmetic*, and show its soundness. This proof system is inspired by [18].

$$\frac{}{\Gamma, t < 0 \Rightarrow \Delta}\ (<_1) \qquad \frac{\Gamma, t < u \Rightarrow \Delta \qquad \Gamma, t = u \Rightarrow \Delta \qquad \Gamma, u < t \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (<_2)$$

$$\frac{\Gamma, t < \mathrm{s}t \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (<_3) \qquad (x \text{ is fresh})\ \frac{\Gamma, t = \mathrm{s}x \Rightarrow \Delta}{\Gamma, 0 < t \Rightarrow \Delta}\ (<_4)$$

$$\frac{\Gamma, t_1 = t_2 \Rightarrow \Delta}{\Gamma, \mathrm{s}t_1 = \mathrm{s}t_2 \Rightarrow \Delta}\ (\mathrm{s}_=) \qquad \frac{\Gamma, t_1 < t_2 \Rightarrow \Delta}{\Gamma, \mathrm{s}t_1 < \mathrm{s}t_2 \Rightarrow \Delta}\ (\mathrm{s}_<)$$

$$\frac{\Gamma, t + 0 = t \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (+_1) \qquad \frac{\Gamma, t + \mathrm{s}u = \mathrm{s}t + u \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (+_2)$$

Figure 5.2 Rules for Infinitary Presburger Arithmetic and Cyclic Presburger Arithmetic

We define a $\mathtt{P}^\omega_+$*-pre-proof* as a possibly infinite derivation tree with rules in Figures 2.1 and 5.2 and without buds. A $\mathtt{P}^\omega_+$*-pre-proof* whose root is assigned to $(\Gamma \Rightarrow \Delta, (\mathrm{R}))$ called a $\mathtt{P}^\omega_+$*-pre-proof* of $\Gamma \Rightarrow \Delta$.

To define *the global trace condition* for $\mathtt{P}^\omega_+$, we give the following concepts.

Definition 5.10 (Precursor, trace, progress)**.** Let $\{\Gamma_i \vdash \Delta_i\}_{i \geq 0}$ be an infinite path through a $\mathtt{P}^\omega_+$-pre-proof. For terms t, t', a *precursor* of t at i is defined as t' satisfying the following conditions:

(1) $\Gamma_i \vdash \Delta_i$ is the conclusion of (Sub) with a substitution θ, and t is $\theta(t')$.

(2) $\Gamma_i \vdash \Delta_i$ is the conclusion of $(=\mathrm{L})$ with the principal formula $u_1 = u_2$, and there exists a term u such that t is $u[x := u_2, y := u_1]$ and t' is $u[x := u_1, y := u_2]$ for some variables x, y.

(3) $\Gamma_i \vdash \Delta_i$ is the conclusion of other rules, and t' is t.

We say that a term t *occurs* in a sequent $\Gamma \vdash \Delta$ if it appears within some formula in Γ, Δ (possibly as a subterm of another term). A *trace* along $\{\Gamma_i \vdash \Delta_i\}_{i \geq 0}$ is a sequence $\{t_i\}_{i \geq 0}$ such that each term t_i occurs in $\Gamma_i \vdash \Delta_i$ and one of the following conditions holds:

(1) Either t_{i+1} is a precursor of t_i at i, or

(2) there exists $(t_{i+1} < t) \in \Gamma_{i+1}$ such that t is a precursor of t_i at i.

When the latter case holds, we say that the trace *progresses* at $i+1$. We call a trace $\{t_i\}_{i \geq 0}$ that progresses at infinitely many i *an infinitely progressing trace.*

Definition 5.11 (Global trace condition)**.** For a derivation tree, if, for every infinite path $\{\Gamma_i \Rightarrow \Delta_i\}_{i \geq 0}$ in the derivation tree, there exists an infinitely progressing trace following a tail of the path $\{\Gamma_i \Rightarrow \Delta_i\}_{i \geq k}$ with some $k \geq 0$, we say the derivation tree satisfies *the global trace condition.*

Definition 5.12 ($\mathtt{P}^\omega_+$-proof)**.** A $\mathtt{P}^\omega_+$*-proof* is a $\mathtt{P}^\omega_+$-pre-proof that satisfies the global trace condition.

We show the soundness of $\mathtt{P}^\omega_+$ for the standard interpretation in the set of natural numbers. It is proved in the same way as proving the soundness for ∞-proofs in [18].

If there is a $\mathtt{P}^\omega_+$-proof of $\Gamma \Rightarrow \Delta$, we write $\mathtt{P}^\omega_+ \vdash \Gamma \Rightarrow \Delta$. For simplicity, for a formula φ, we write $\mathtt{P}^\omega_+ \vdash \varphi$ for $\mathtt{P}^\omega_+ \vdash\ \Rightarrow \varphi$. We write $\mathbb{N} \models^\rho \varphi$ to say that formula φ is true in the standard interpretation in the set of natural numbers under a *valuation* ρ, which is a function mapping each free variable of φ to a natural number. We write $\mathbb{N} \models^{\rho} \Gamma \Rightarrow \Delta$ if $\mathbb{N} \models^{\rho} \varphi$ for all $\varphi \in \Gamma$ implies that there exists $\psi \in \Delta$ such that $\mathbb{N} \models^{\rho} \psi$. We write $\mathbb{N} \models \Gamma \Rightarrow \Delta$ if $\mathbb{N} \models^{\rho} \Gamma \Rightarrow \Delta$ for all valuations ρ holds.

Theorem 5.13 (Soundness for $\mathtt{P}^{\omega}_{+}$-proofs). *If there exists a $\mathtt{P}^{\omega}_{+}$ -proof of $\Gamma \Rightarrow \Delta$, then we have $\mathbb{N} \models \Gamma \Rightarrow \Delta$.*

Proof. We consider a $\mathtt{P}^{\omega}_{+}$-proof of $\Gamma \Rightarrow \Delta$. Assume $\mathbb{N} \not\models^{\rho} \Gamma \Rightarrow \Delta$ with a valuation ρ, for contradiction.

We inductively construct an infinite path $\{\Gamma_i \Rightarrow \Delta_i\}_{i \geq 0}$ from $\Gamma \Rightarrow \Delta$ in the $\mathtt{P}^{\omega}_{+}$-proof and an associated sequence $\{\rho_i\}_{i \geq 0}$ of valuations such that $\mathbb{N} \not\models^{\rho_i} \Gamma_i \Rightarrow \Delta_i$ on $i \geq 0$.

In the case $i = 0$, we define $\Gamma_0 \Rightarrow \Delta_0$ to be $\Gamma \Rightarrow \Delta$, and ρ_0 to be ρ.

We consider the case $i > 0$. By the induction hypothesis, $\mathbb{N} \not\models^{\rho_{i-1}} \Gamma_{i-1} \Rightarrow \Delta_{i-1}$ for all $i \geq 0$. Hence, the sequent $\Gamma_{i-1} \Rightarrow \Delta_{i-1}$ is not the conclusion of an inference rule with no assumption. If $\Gamma_{i-1} \Rightarrow \Delta_{i-1}$ is the conclusion of (Sub) with a substitution θ, then we define ρ_i to be $\rho_{i-1} \circ \theta$. Otherwise define ρ_i to be ρ_{i-1}. By the soundness of inference rules, there exists a sequent $\Gamma' \Rightarrow \Delta'$ which is an assumption of the rule such that $\mathbb{N} \not\models^{\rho_i} \Gamma' \Rightarrow \Delta'$ holds. We define $\Gamma_i \Rightarrow \Delta_i$ to be $\Gamma' \Rightarrow \Delta'$.

By the global trace condition, there is a infinitely progressing trace $\{t_i\}_{i \geq N}$ following a tail of the infinite path $\{\Gamma_i \Rightarrow \Delta_i\}_{i \geq k}$. Consider the sequence of numbers $\{t_i^{\rho_i}\}_{i \geq N}$. Since $\mathbb{N} \not\models^{\rho_i} \Gamma_i \Rightarrow \Delta_i$, we see that $\mathbb{N} \models^{\rho_i} \varphi$ for every $\varphi \in \Gamma_i$. By the definitions of precursor and of ρ_{i+1}, we see the following statements:

(1) $t_{i+1}^{\rho_{i+1}}$ is $t_i^{\rho_i}$ if t_{i+1} is a precursor of t_i.

(2) $t_{i+1}^{\rho_{i+1}}$ is less than $t_i^{\rho_i}$ if the trace progress at $i+1$ i.e. there exists a term t such that $(t_{i+1} < t) \in \Gamma_{i+1}$ and t is a precursor of t_i.

By the global trace condition, the second case applies infinitely often. Thus, $\{t_i^{\rho_i}\}_{i \geq N}$ is an infinite descending sequence of natural numbers. This is a contradiction. □

5.4 Cyclic Presburger Arithmetic is equivalent to Presburger Arithmetic

In this section, we define a cyclic proof system for Presburger Arithmetic $\mathtt{CP}^{\omega}_{+}$, called *Cyclic Presburger Arithmetic*, and show the equivalence between Presburger Arithmetic and Cyclic Presburger Arithmetic.

We define a $\mathtt{CP}^{\omega}_{+}$*-pre-proof* as a $\mathtt{P}^{\omega}_{+}$-pre-proof whose underlying tree is regular. A $\mathtt{CP}^{\omega}_{+}$*-proof* is a $\mathtt{CP}^{\omega}_{+}$-pre-proof that satisfies the global trace condition.

If there is a $\mathtt{CP}^{\omega}_{+}$-proof of $\Gamma \Rightarrow \Delta$, we write $\mathtt{CP}^{\omega}_{+} \vdash \Gamma \Rightarrow \Delta$. For simplicity, for a formula φ, we write $\mathtt{CP}^{\omega}_{+} \vdash \varphi$ for $\mathtt{CP}^{\omega}_{+} \vdash \Rightarrow \varphi$.

To show the equivalence between $\mathtt{P}_{+}$ and $\mathtt{CP}^{\omega}_{+}$, we show the following lemma.

Lemma 5.14. *The following rules are derivable without (Cut) in $\mathtt{CP}^{\omega}_{+}$:*

$$\frac{t = 0, \Gamma \Rightarrow \Delta \qquad 0 < t, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (<_{2'}) \quad .$$

Proof. It is simulated by the proof below without (Cut).

$$\frac{\dfrac{}{t < 0, \Gamma \Rightarrow \Delta}\ (<_1) \qquad t = 0, \Gamma \Rightarrow \Delta \qquad 0 < t, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (<_2)$$

□

We show the following theorem, which states the equivalence between $\mathtt{P}_+$ and $\mathtt{CP}_+^\omega$.

Theorem 5.15. *For any formula φ, the following statements are equivalent:*

(1) $\mathtt{P}_+ \vdash \varphi$,

(2) $\mathtt{CP}_+^\omega \vdash \varphi$,

(3) $\mathtt{P}_+^\omega \vdash \varphi$,

(4) $\mathbb{N} \models \varphi$.

Proof. (1) ⇒(2) It is sufficient to show that (Ai) for $i = 1, 2, 3, 4, 7, 8, 9$ are provable in $\mathtt{CP}_+^\omega$.
(A1)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{y<0\Rightarrow}\,(<_1)}{\mathrm{s}y=0, y<\mathrm{s}y\Rightarrow}\,(=\mathrm{L})}{\mathrm{s}y=0\Rightarrow}\,(<_3)}{\Rightarrow\neg(\mathrm{s}y=0)}\,(\neg\ \mathrm{R})}{\Rightarrow\forall x\neg(\mathrm{s}x=0)}\,(\forall\ \mathrm{R})$$

(A2)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{y_1=y_2\Rightarrow y_1=y_2}\,(\textsc{Axiom})}{\mathrm{s}y_1=\mathrm{s}y_2\Rightarrow y_1=y_2}\,(\mathrm{s}_=)}{\Rightarrow \mathrm{s}y_1=\mathrm{s}y_2\to y_1=y_2}\,(\to\ \mathrm{R})}{\Rightarrow\forall x_2(\mathrm{s}y_1=\mathrm{s}x_2\to y_1=x_2)}\,(\forall\ \mathrm{R})}{\Rightarrow\forall x_1\forall x_2(\mathrm{s}x_1=\mathrm{s}x_2\to x_1=x_2)}\,(\forall\ \mathrm{R})$$

(A3)

$$\dfrac{\dfrac{\dfrac{}{y+0=y\Rightarrow y+0=y}\,(\mathrm{Ax})}{\Rightarrow y+0=y}\,(+_1)}{\Rightarrow\forall x(x+0=x)}\,(\forall\ \mathrm{R})$$

(A4)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{y_1+\mathrm{s}y_2=\mathrm{s}y_1+y_2\Rightarrow y_1+\mathrm{s}y_2=\mathrm{s}y_1+y_2}\,(\mathrm{Ax})}{\Rightarrow y_1+\mathrm{s}y_2=\mathrm{s}y_1+y_2}\,(+_2)}{\Rightarrow\forall x_2(y_1+\mathrm{s}x_2=\mathrm{s}y_1+x_2)}\,(\forall\ \mathrm{R})}{\Rightarrow\forall x_1\forall x_2(x_1+\mathrm{s}x_2=\mathrm{s}x_1+x_2)}\,(\forall\ \mathrm{R})$$

(A7)

$$\dfrac{\dfrac{\dfrac{}{y<0\Rightarrow}\,(<_1)}{\Rightarrow\neg(y<0)}\,(\neg\ \mathrm{R})}{\Rightarrow\forall x\neg(x<0)}\,(\forall\ \mathrm{R})$$

(A8)

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\text{(A8)-1.1}}{y_1 < \mathrm{s}y_2 \Rightarrow y_1 < y_2 \lor y_1 = y_2}}{\Rightarrow y_1 < \mathrm{s}y_2 \to (y_1 < y_2 \lor y_1 = y_2)}\,(\to \mathrm{R}) \qquad \dfrac{\overset{\text{(A8)-2.1}}{y_1 < y_2 \lor y_1 = y_2 \Rightarrow y_1 < \mathrm{s}y_2}}{\Rightarrow (y_1 < y_2 \lor y_1 = y_2) \to y_1 < \mathrm{s}y_2}\,(\to \mathrm{R})}{\Rightarrow y_1 < \mathrm{s}y_2 \leftrightarrow (y_1 < y_2 \lor y_1 = y_2)}\,(\land \mathrm{R})}{\Rightarrow \forall x_2(y_1 < \mathrm{s}x_2 \leftrightarrow (y_1 < x_2 \lor y_1 = x_2))}\,(\forall \mathrm{R})}{\Rightarrow \forall x_1 \forall x_2(x_1 < \mathrm{s}x_2 \leftrightarrow (x_1 < x_2 \lor x_1 = x_2))}\,(\forall \mathrm{R})$$

(A8)-1.1

$$\dfrac{\dfrac{\dfrac{}{y_1 < \mathrm{s}y_2,\ y_1 < y_2 \Rightarrow y_1 < y_2,\ y_1 = y_2}\,(\textsc{Axiom}) \qquad \dfrac{}{y_1 < \mathrm{s}y_2,\ y_1 = y_2 \Rightarrow y_1 < y_2,\ y_1 = y_2}\,(\textsc{Axiom}) \qquad \overset{\text{(A8)-1.2}}{y_1 < \mathrm{s}y_2,\ y_2 < y_1 \Rightarrow y_1 < y_2,\ y_1 = y_2}}{y_1 < \mathrm{s}y_2 \Rightarrow y_1 < y_2, x_2 = y_2}\,(<_2)}{y_1 < \mathrm{s}y_2 \Rightarrow y_1 < y_2 \lor y_1 = y_2}\,(\lor \mathrm{R})$$

(A8)-1.2

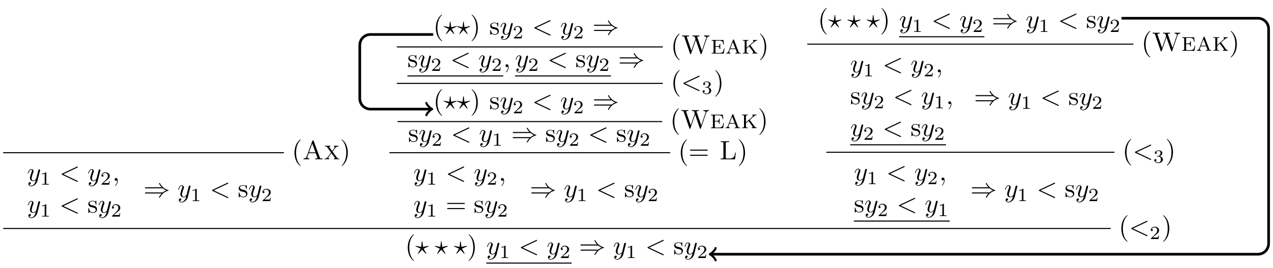
$$\dfrac{\dfrac{\dfrac{\dfrac{}{0 < \mathrm{s}y_2, y_2 < 0 \Rightarrow}\,(<_1)}{y_1 = 0, y_1 < \mathrm{s}y_2, y_2 < y_1 \Rightarrow}\,(=\mathrm{L}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(\star)\ y_1 < \mathrm{s}y_2, y_2 < y_1 \Rightarrow}{z_0 < y_2, y_2 < \mathrm{s}z_0 \Rightarrow}\,(\textsc{Sub})}{\mathrm{s}z_0 < \mathrm{s}y_2, y_2 < \mathrm{s}z_0 \Rightarrow}\,(\mathrm{s}_<)}{\underline{z_0 < \mathrm{s}z_0}, \mathrm{s}z_0 < \mathrm{s}y_2, y_2 < \mathrm{s}z_0 \Rightarrow}\,(\textsc{Weak})}{y_1 = \mathrm{s}z_0, z_0 < \mathrm{s}z_0, y_1 < \mathrm{s}y_2, y_2 < y_1 \Rightarrow}\,(=\mathrm{L})}{y_1 = \mathrm{s}z_0, y_1 < \mathrm{s}y_2, y_2 < y_1 \Rightarrow}\,(<_3)}{0 < y_1, y_1 < \mathrm{s}y_2, y_2 < y_1 \Rightarrow}\,(<_4)}{(\star)\ y_1 < \mathrm{s}y_2, y_2 < y_1 \Rightarrow}\,(<_{2'})}{y_1 < \mathrm{s}y_2, y_2 < y_1 \Rightarrow y_1 < y_2, y_1 = y_2}\,(\textsc{Weak})$$

We identify $(\star)$ nodes. The infinitely progressing trace along a tail of each infinite path that the $(\star)$ nodes occur infinitely many times is $\{y_1, y_1, y_1, y_1, z_0, z_0, z_0\}^\omega$, and the progress point is underlined.

(A8)-2.1

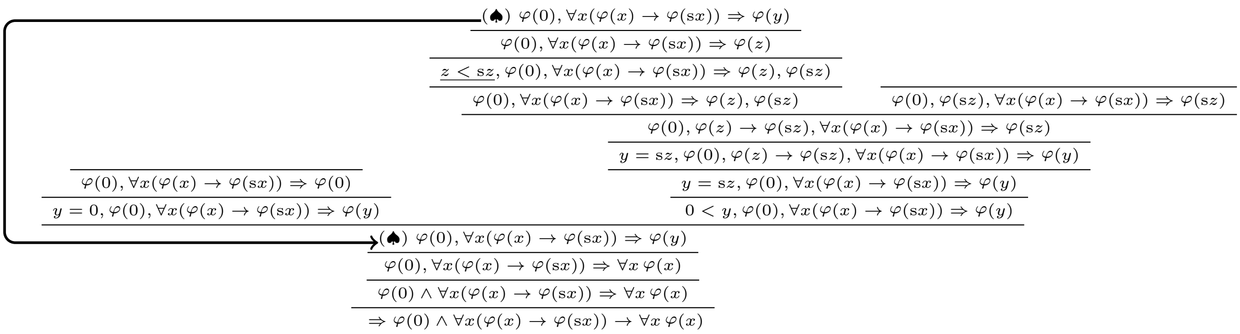
$$\dfrac{\dfrac{\dfrac{\dfrac{}{y_1 < \mathrm{s}y_2 \Rightarrow y_1 < \mathrm{s}y_2}\,(\mathrm{Ax})}{\Rightarrow y_1 < \mathrm{s}y_1}\,(<_3)}{y_1 = y_2 \Rightarrow y_1 < \mathrm{s}y_2}\,(=\mathrm{L}) \qquad \overset{\text{(A8)-2.2}}{y_1 < y_2 \Rightarrow y_1 < \mathrm{s}y_2}}{y_1 < y_2 \lor y_1 = y_2 \Rightarrow y_1 < \mathrm{s}y_2}\,(\lor \mathrm{L})$$

(A8)-2.2



We identify $(\star\star)$ and $(\star\star\star)$, respectively. The infinitely progressing trace along a tail of each infinite path that the $(\star\star)$ nodes occur infinitely many times is $\{y_2, \mathrm{s}y_2, \mathrm{s}y_2, y_2\}^\omega$, and the progress point is underlined. The infinitely progressing trace along a tail of each infinite path that the $(\star\star\star)$ nodes occur infinitely many times is $\{y_1, \mathrm{s}y_2, y_2\}^\omega$, and the progress point is underlined.

(A9)



We identify (♠). Labels of rules are omitted for limited space. The infinitely progressing trace along a tail of each infinite path that the (♠) nodes occur infinitely many times is $(y, y, y, y, y, \mathrm{s}z, \mathrm{s}z, z, z)^\omega$, and the progress point is underlined.

(2) ⇒(3) Obvious.

(3) ⇒(4) By Theorem 5.13.

(4) ⇒(1) We show the statement by contrapositive. Assume $\mathtt{P}_+ \nvdash \varphi$. Let $\overline{\varphi}$ be the universal closure of φ. Then, $\mathtt{P}_+ \nvdash \overline{\varphi}$. By Theorem 5.1, we have $\mathtt{P}_+ \vdash \neg\overline{\varphi}$. Since $\mathbb{N}$ is a model of $\mathtt{P}_+$, we have $\mathbb{N} \models \neg\overline{\varphi}$. Hence, $\mathbb{N} \not\models \overline{\varphi}$. Thus, $\mathbb{N} \not\models \varphi$. □

5.5 Discussion

This section discusses the equivalence between ordinary and cyclic proof systems and the cut-elimination property of Cyclic Presburger Arithmetic.

5.5.1 The equivalence between ordinary and cyclic proof systems

What is the condition for equivalence between a proof system with induction and the corresponding cyclic proof system? To show the equivalence between Presburger Arithmetic and Cyclic Presburger Arithmetic, the completeness of Presburger Arithmetic seems to be essential. Indeed, we can show the equivalence between a proof system for the theory of successor

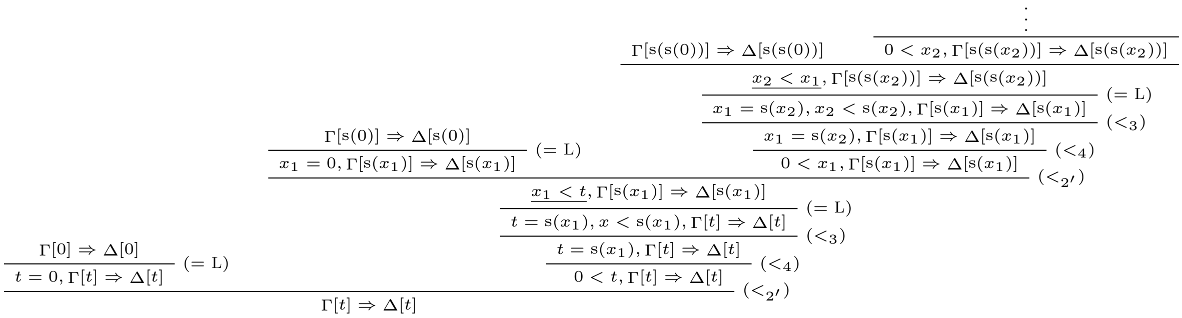


Figure 5.3 A simulation of ω-rule in Infinitary Presburger Arithmetic

and order, obtained by removing addition from Presburger Arithmetic, and the corresponding cyclic proof system obtained by removing addition from Cyclic Presburger Arithmetic in the same way as this thesis since the theory is complete [11, 19].

However, Peano Arithmetic is not complete, but Cyclic Arithmetic, the corresponding cyclic system for Peano Arithmetic, is equivalent to Peano Arithmetic [18]. Moreover, Berardi and Tatsuta [2] showed that the system obtained by adding Peano Arithmetic to $\mathtt{CLKID}^\omega$ is equivalent to that obtained by adding Peano Arithmetic to $\mathtt{LKID}$.

On the other hand, Berardi and Tatsuta [3] gave a sequent provable in $\mathtt{CLKID}^\omega$ but not in $\mathtt{LKID}$. Das [10] showed that the cyclic proof system for $\mathtt{I\Sigma_n}$ is more powerful than $\mathtt{I\Sigma_n}$.

There are some equivalence cases for theories not including Arithmetic. Afshari and Leigh [1] gave cyclic proof systems and a cut-free ordinary proof system for μ-calculus. They also showed that these proof systems are equivalent by transforming from each proof in one to a proof in others. Shamkov [17] showed the equivalence between an ordinary proof system and a cyclic proof system for Gödel-Löb provability logic by transforming each proof in the cyclic proof system into a proof in the ordinary system.

Someone may consider the issue of equivalence between ordinary and cyclic proof systems to depend on how to formalise the cyclic proof system. However, we do not imagine a more natural cyclic proof system for first-order logic with inductive definitions than $\mathtt{CLKID}^\omega$. For this reason, we cannot entirely agree with the idea

5.5.2 The cut-elimination property of Cyclic Presburger Arithmetic

The sequent calculus for first-order logic with the ω-rule

$$\frac{\Gamma[x := 0] \Rightarrow \Delta[x := 0] \quad \Gamma[x := s(0)] \Rightarrow \Delta[x := s(0)] \quad \Gamma[x := s(s(0))] \Rightarrow \Delta[x := s(s(0))] \quad \cdots\cdots}{\Gamma[t] \Rightarrow \Delta[t]} \ (\omega)$$

can eliminate the cut-rule [22].

The ω-rule is derivable in Infinitary Presburger Arithmetic. It is simulated as in Figure 5.3. We note that the infinitely progressing trace along the infinite path is $\{t, t, t, t, x_1, x_1, x_1, x_1, x_2, x_2, x_2, x_2, \ldots\}$. Therefore, the cut-elimination property of Infinitary Presburger Arithmetic holds.

However, it does not implies the cut-elimination property of Cyclic Presburger Arithmetic. In general, the cut-elimination property of an infinitary proof system does not imply that of the corresponding cyclic proof system, as we have seen in Chapters 3 and 4.

We conjecture that the cut-elimination property of Cyclic Presburger Arithmetic holds because of the equivalence between Cyclic Presburger Arithmetic and Infinitary Presburger Arithmetic. Shamkov [17] showed the equivalence between a cut-free infinitary proof system

$\mathtt{GL}_\infty$ for Gödel-Löb provability logic and the corresponding cut-free cyclic proof system $\mathtt{GL}_{\mathtt{circ}}$ by giving the transformation of each proof in $\mathtt{GL}_\infty$ to a proof in $\mathtt{GL}_{\mathtt{circ}}$. We conjecture that there is such a transformation of each cut-free proof in Infinitary Presburger Arithmetic to a cut-free proof in Cyclic Presburger Arithmetic.

6 Conclusions

This section concludes this thesis.

Section 6.1 summarise our contributions. Section 6.2 provides ideas for future work.

6.1 Summary of our contributions

In this thesis, we address three contributions.

Firstly, we have given a counterexample to cut-elimination in $\mathtt{CLKID}^{\omega}$, a cyclic proof system for first-order logic with inductive definitions. In other words, we have shown that the cut-elimination property of $\mathtt{CLKID}^{\omega}$ does not hold, which has been an open problem since Brotherston provided Conjecture 5.2.4. of [4].

Secondly, we have given a simpler counterexample to cut-elimination in $\mathtt{CLKID}^{\omega}$ with only unary predicates. Therefore, we have shown that the cut-elimination property in $\mathtt{CLKID}^{\omega}$ does not hold even if we restrict predicates in the language to unary predicates and $=$.

Thirdly, we have shown the equivalence between Presburger Arithmetic and Cyclic Presburger Arithmetic, a cyclic proof system for Presburger Arithmetic.

6.2 Future Work

There are five ideas for future work.

Firstly, we will examine the cases of “?” ’s in Table 4.1. As mentioned in Section 4.2, we conjectured that the cut-elimination property of $\mathtt{CLKID}^{\omega}$ holds with “$\underline{?}$” in Table 4.1 but does not hold with “?” in Table 4.1. Through this thesis, we have discussed how to show that the cut-elimination property of a cyclic proof system does not hold. Now, we discuss how to show the cut-elimination property of a cyclic proof system. To show the cut-elimination in first-order logic could be considered only around the cut-rule, but the global trace condition does not allow it in the cyclic proof system. Moreover, we cannot use a famous cut-elimination procedure for first-order logic since we cannot assure ourselves that the procedure in the cyclic proof system terminates and that the transformed proof figure is a regular tree. The cut-elimination property in cyclic proof systems is probably shown by either transforming from the corresponding cut-free infinitary proof into a cyclic proof, such as [17], or going through semantics, such as the proof of $\mathtt{LKID}^{\omega}$.

Secondly, we will examine the cut-elimination property of Cyclic Presburger Arithmetic. As mentioned inSection 5.5, the cut-elimination property of Infinitary Presburger Arithmetic holds, but it cannot imply that of Cyclic Presburger Arithmetic. We conjecture that the cut-elimination property of Cyclic Presburger Arithmetic holds, and it may be shown, as mentioned in the previous paragraph.

Thirdly, we will examine how to restrict cut formulas in $\mathtt{CLKID}^{\omega}$ without changing provability. Saotome, Nakazawa, and Kimura [15] showed that we could not restrict the cut formulas to formulas presumable from the goal sequent in the cyclic proof system for symbolic-heaps, a fragment of separation logic. The cut formulas in Figures 3.3 and 4.2 are presumable. Can we restrict the cut formulas to presumable formulas?

Fourthly, we will examine the efficiency of an algorithm to determine, for each given sequent, whether the sequent is provable in Presburger Arithmetic by using Cyclic Presburger Arithmetic.

Lastly, we will examine a subsystem of $\mathtt{LKID}^\omega$, which includes $\mathtt{CLKID}^\omega$ and satisfies the cut-elimination property.

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