

Quantum Fluctuations and Space-time Horizons

Sen Zhang

DOCTOR OF PHILOSOPHY

*Department of Particle and Nuclear Physics
School of High Energy Accelerator Science
The Graduate University for Advanced Studies*

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Abstract

In this Thesis I discuss quantum fluctuations in physics of horizons. There are two main topics. The first topic is the fluctuations in Unruh effect. When a particle uniformly accelerated in Minkowski vacuum, it will experience the vacuum as a thermal bath. Using a stochastic approach, we investigated the fluctuations of this particle and proved the equipartition theorem for the transverse fluctuations. We also obtained the relaxation time of the fluctuations and the radiation due to the fluctuations(the Unruh radiation [12]). These result are also related to the experiment for detecting the Unruh effect by using high intensity laser(ELI) which is under construction in Europe now. The second topic is to apply fluctuation theorem to black holes. There is an analogue between black hole physics and thermodynamics. This analogue was well established in equilibrium region. We investigated the fluctuations of the black holes. We considered a system with a black hole coupled with matter fields and derive a non-equilibrium relation for the black holes. This relation corresponds to the non-equilibrium fluctuation theorem of Crooks and Jarzynski. As a result, we can also obtain the generalized second law from this relation. In our derivation, the second law holds only after taking a thermodynamic average, and it should be violated as individual process in a way to satisfy the Jarzynski equality. The thermodynamic features of the horizons should be closed related to a more fundamental structure of space-time. So it will be important to investigate the fluctuations of the horizons.

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Chapter 1

Introduction

One of the most interesting problem for theoretical physics must be the unification of quantum theory with general relativity. It is expected that the answer to this problem will tell us about the structure of spacetime. There are various approaches to this problem, it is widely believed that the thermodynamical behaviors of black holes and the Hawking effect will play a key role.

A black hole is a region in spacetime where the gravitational field is so strong that even light can not escape from there to infinity. The black hole itself is just a solution of Einstein equation which is a hyperbolic second order partial differential equation. However, people noticed that there is an analogy between black hole physics and thermodynamics [1]. After that, Hawking showed that due to the quantum effects, there is a thermal radiation with a black body spectrum from the black hole [2]. This means that the black hole thermodynamics is not just an analogue, it should has some physical meaning. The Hawking radiation also gives many implications about microscopic structure of spacetime itself. For example, if the thermodynamical quantities of black hole are physical, then how do we explain them from statistical mechanics? How to count the number of states and obtain the black hole entropy? The theory of quantum gravity should answer these questions.

Indeed there are varieties of works to explain the black hole entropy from microscopic point of view, for example see [3] [4].

In the thermodynamics of black holes, the event horizon plays an important role. Unruh found that the Minkowski vacuum will appear as a thermal state for a uniformly accelerated observer [5], this is known as the Unruh effect. Just as the black hole has event horizon, there is an event horizon for the uniformly accelerated observer. Indeed, the Unruh effect is related to Hawking radiation via equivalence theorem. A derivation of the Hawking radiation by using quantum anomaly can clearly show that the existence of event horizon is essential for the Hawking radiation [6]. Furthermore, Ted Jacobson showed that one can derive Einstein equation by assuming the thermodynamics of horizons [7].

One sees that the thermodynamics of event horizon is a key to the quantum aspect of gravity. However, most of the discussions were done in equilibrium region. We would like to investigate the fluctuations related to the event horizons. We expect that this will be important to understand the structure of spacetime. In this thesis, I am going to show two approaches. The first one is a stochastic approach to the Unruh effect and the second one is to show a fluctuation theorem for black holes.

When a particle is uniformly accelerated in Minkowski vacuum, it will experience the vacuum as a thermal bath. Due to the interactions with this thermal bath, the motion of the particle will be stochastic. Using the stochastic approach, we investigated the fluctuations of this particle and proved the equipartition theorem for the transverse fluctuations. We also obtained the relaxation time of the fluctuations and the radiation due to the fluctuations (the Unruh radiation [12]). These results are also useful in experiments which are under planning to detect the Unruh radiation by using ultrahigh intensity lasers [13, 14].

For black holes. We applied the recent developments in non-equilibrium statistical physics to area changing processes of a black hole interacting with

external matter. We derived the non-equilibrium fluctuation theorems of Crooks and Jarzynski for the black holes. And this procedure also gives another derivation of the generalized second law of black hole thermodynamics. In our derivation, the second law holds only after taking a thermodynamic average, and it should be violated as individual process in a way to satisfy the Jarzynski equality. This is a first step to understand the non-equilibrium nature of the black hole horizons.

The plan of this thesis is following. In chapter 2, I am going to review the fundamental facts of the event horizons. In chapter 3, I am going to show the stochastic approach and the results for the Unruh effect. In chapter 4, I am going to show our fluctuation theorem of the black holes.

Chapter 2

Physics of Horizons

The organization of this chapter is following. First I am going to review the Unruh effect. Then is the black hole physics, the thermodynamics, the Hawking radiation, the argument of Ted Jacobson. Finally I would like to review the Membrane paradigm, which is a very interesting approach to the horizon.

2.1 Unruh Effect

An uniformly accelerated observer sees the Minkowski vacuum as thermally excited, this is called Unruh effect. Unruh effect is very fundamental and important since it means that in field theory the content of particle is observer dependent. The existence of an event horizon is essential for Unruh effect, and Unruh effect is also related to Hawking radiation by equivalence principle.

2.1.1 Scalar Fields in curved space

The Unruh effect is in flat spacetime. But it is useful to briefly review the framework of quantum field theory in curved spacetime. This framework will also be used in derivation of Hawking radiation. Here, we only consider a

scalar field. The other case can add some technical details (for example to concern the spin components), but the essence is same.

Consider the action

$$S = \int d^4x \frac{1}{2} \sqrt{-g} (g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - m^2 \phi^2), \quad (2.1)$$

the field equation is given by

$$\left(\frac{1}{\sqrt{-g}} \partial_\mu g^{\mu\nu} \sqrt{-g} \partial_\nu + m^2 \right) \phi = 0. \quad (2.2)$$

Using the complete sets of the solutions of this equation of motion, one can expand operator $\phi(x)$ as:

$$\phi(x) = \sum_k (a_k u_k(x) + a_k^\dagger u_k^*(x)). \quad (2.3)$$

Define an inner product as

$$(f, g) = (g, f)^* = i \int_\Sigma \left(f^*(x) n^\mu \nabla_\mu g(x) - (n^\mu \nabla_\mu f^*(x)) g(x) \right) \sqrt{h} d^3x. \quad (2.4)$$

If $f(x)$ and $g(x)$ are both solutions of the equation of motion, then the value of their inner product will be independent of the choice of hypersurface Σ .

The ortho-normality condition is given by:

$$(u_k, u_{k'}) = \delta(k, k'), \quad (u_k^*, u_{k'}^*) = -\delta(k, k'), \quad (u_k, u_{k'}^*) = 0. \quad (2.5)$$

Note that we took $u_k(x)$ to be always positive norm and u_k^* to be negative norm. One can written a_k and a_k^\dagger by the inner product

$$a_k = (u_k, \phi), \quad a_k^\dagger = -(u_k^*, \phi). \quad (2.6)$$

Now, consider to different complete sets $u_{M,k}$ and $u_{R,k}$. Then there are tow different expansions of $\phi(x)$

$$\begin{aligned} \phi(x) &= \sum_k (a_{M,k} u_{M,k}(x) + a_{M,k}^\dagger u_{M,k}^*(x)) \\ &= \sum_k (a_{R,k} u_{R,k}(x) + a_{R,k}^\dagger u_{R,k}^*(x)). \end{aligned} \quad (2.7)$$

Generally, operators $(a_{M,k}, a_{M,k}^\dagger)$ and $(a_{R,k}, a_{R,k}^\dagger)$ are different, they are related by

$$\begin{aligned} a_{R,\tilde{k}} &= \sum_k \left(\alpha^*(\tilde{k}, k) a_{M,k} + \beta^*(\tilde{k}, k) a_{M,k}^\dagger \right) \\ a_{R,\tilde{k}}^\dagger &= \sum_k \left(\alpha(\tilde{k}, k) a_{M,k}^\dagger + \beta(\tilde{k}, k) a_{M,k} \right). \end{aligned} \quad (2.8)$$

The coefficients $\alpha(\tilde{k}, k)$ and $\beta(\tilde{k}, k)$ are given by:

$$\alpha^*(\tilde{k}, k) = (u_{R,\tilde{k}}, u_{M,k}), \quad \beta^*(\tilde{k}, k) = (u_{R,\tilde{k}}, u_{M,k}^*). \quad (2.9)$$

This transformation is known as Bogolubov transformation. The coefficients $\alpha(\tilde{k}, k)$ and $\beta(\tilde{k}, k)$ are called Bogolubov coefficients. The Bogolubov coefficients possess the following properties

$$\begin{aligned} \sum_k \left(\alpha(i, k) \alpha^*(j, k) - \beta(i, k) \beta^*(j, k) \right) &= \delta(i, j), \\ \sum_k \left(\alpha(i, k) \beta(j, k) - \beta(i, k) \alpha(j, k) \right) &= 0, \end{aligned} \quad (2.10)$$

which retains the commutation relations,

$$[a_k, a_{k'}^\dagger] = \delta(k, k'), \quad [a_k, a_{k'}] = 0, \quad [a_k^\dagger, a_{k'}^\dagger] = 0, \quad (2.11)$$

for both $a_{M,k}$ and $a_{R,k}$.

With this two sets of annihilation and creation operators $a_{M,k}$ and $a_{R,k}$, there are also two definitions the vacuum: $|0\rangle_M$ defined by $a_{Mk}|0\rangle_M = 0$, or $|0\rangle_R$ defined by $a_{Rk}|0\rangle_R = 0$. Generally this two vacuums are not identical. For example, the expectation value of the number operator defined by u_R is zero for $|0\rangle_R$, but is generally nonzero at state $|0\rangle_M$

$$\begin{aligned} \langle N_{R,k} \rangle_R &= {}_R\langle 0 | a_{R,k}^\dagger a_{R,k} | 0 \rangle_R = 0 \\ {}_M\langle N_{R,k} \rangle_M &= {}_M\langle 0 | a_{R,k}^\dagger a_{R,k} | 0 \rangle_M \\ &= {}_M\langle 0 | \sum_{k'} \beta(k, k') \beta^*(k, k') a_{M,k'} a_{M,k'}^\dagger | 0 \rangle_M \\ &= \sum_{k'} \beta(k, k') \beta^*(k, k'). \end{aligned} \quad (2.12)$$

Finally, the exact relation between $|0\rangle_M$ and $|0\rangle_R$ can be obtained by inserting a set of complete states

$$|0\rangle_M = \sum_{\{n_\omega\}} \langle n_\omega | 0 \rangle_M |n_\omega\rangle_R. \quad (2.13)$$

Where $|n_\omega\rangle_R = \prod_{\omega} \frac{1}{\sqrt{n_\omega!}} (a_{R\omega}^\dagger)^{n_\omega} |0\rangle_R$. The two vacuum $|0\rangle_M$ and $|0\rangle_R$ are equal if and only if $\beta(\tilde{k}, k) = 0$ for all \tilde{k} and k .

2.1.2 Rindler Space

Now we are going to show that the vacuum for the inertial observer is looks like a thermal state for the uniformly accelerated observer. Here we consider the scalar field in flat space. The coordinates correspond the uniformly accelerated observers are

$$\begin{aligned} t &= \rho \sinh \tau \\ x &= \rho \cosh \tau. \end{aligned} \quad (2.14)$$

Then the metric takes the form:

$$ds^2 = \rho^2 d\tau^2 - d\rho^2 - dy^2 - dz^2. \quad (2.15)$$

Where $\rho \geq 0$ and this Rindler coordinates (τ, ρ, x^2, x^3) covers only the region $z \geq |t|$. It is easy to check that $\rho = \rho_0$ describe a world line with constant proper acceleration Fig. 2.1.

It is convenient to define the coordinates (u, v) and (U, V)

$$\begin{aligned} u &= \tau - \log \rho, & v &= \tau + \log \rho \\ U &= t - x = -e^{-u}, & V &= t + x = e^v. \end{aligned} \quad (2.16)$$

Now the metric takes the form $ds^2 = \rho^2 dudv - dy^2 - dz^2 = dUdV - dy^2 - dz^2$. The Rindler coordinates covers $V \geq 0, U \leq 0$. The future horizon is given

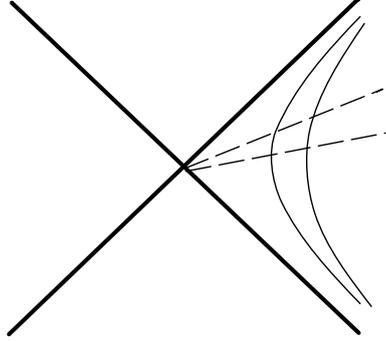


Figure 2.1: Rindler Coordinates.

by $U = 0$, corresponds to $v = v_0$ and $u \rightarrow \infty$. The past horizon is given by $V = 0$, corresponds to $v \rightarrow -\infty$ and $u = u_0$.

Next we solve the wave equation in the two coordinates and calculate $\beta(\tilde{\omega}, \omega)$ explicitly. The solutions of the equation of motion can be written in the form

$$\begin{aligned}\phi_{M, \omega \vec{k}} &= \frac{e^{-i\omega t - i\vec{k} \cdot \vec{x}}}{\sqrt{(2\pi)^3 |\omega|}} \\ \phi_{R, \tilde{\omega} \vec{k}} &= \frac{e^{-i\tilde{\omega} \tau - i(k_1 x + k_2 y)}}{\sqrt{(2\pi)^3 |\tilde{\omega}|}} g_{\tilde{\omega} \vec{k}}(\rho),\end{aligned}\quad (2.17)$$

for the Minkowski observers and the Rindler observers respectively. Here $\omega = (\mu^2 + \vec{k} \cdot \vec{k})^{\frac{1}{2}}$ and $\tilde{\omega}$ is a free parameter. $g_{\tilde{\omega} \vec{k}}(\rho)$ satisfies the equation:

$$\tilde{\omega}^2 + \rho \partial_\rho \rho \partial_\rho - \rho^2 (k_1^2 + k_2^2 + m^2) g_{\tilde{\omega} \vec{k}}(\rho) = 0. \quad (2.18)$$

Near horizon ($\rho \rightarrow 0$) the mass term and the transverse momentum is negligible, the equation behaves like $\{(\frac{\partial}{\partial \log \rho})^2 + \tilde{\omega}^2\} g_{\tilde{\omega} \vec{k}}(\rho) = 0$. Then $\phi_{R, \tilde{\omega} \vec{k}}$ is solved as

$$\phi_{R, \tilde{\omega} \vec{k}} = \frac{(e^{-i\tilde{\omega} u} + \alpha e^{-i\tilde{\omega} v}) e^{-i(k_1 x + k_2 y)}}{\sqrt{(2\pi)^3 |\tilde{\omega}|}}. \quad (2.19)$$

Here α is a complex number satisfies $|\alpha| = 1$. Near the past horizon ($V = 0$ and $v \rightarrow \infty$)

$$\phi_{R,\tilde{\omega}\vec{k}} \rightarrow \frac{e^{-i\tilde{\omega}u - ik_1x - ik_2y}}{\sqrt{(2\pi)^3|2\tilde{\omega}|}} = \frac{|\frac{1}{2}U|^{i2\tilde{\omega}} e^{-ik_1x - ik_2y}}{\sqrt{(2\pi)^3|2\tilde{\omega}|}}. \quad (2.20)$$

Since we are considering the null surface, the normalization of the wave function is need to be specified. Here we the normalization as

$$\begin{aligned} & i \int_{-\infty}^{\infty} du \int d^2\vec{x} (\phi_{R,\tilde{\omega}\vec{k}}^* \partial_u \phi_{R,\tilde{\omega}'\vec{k}'} - \partial_u (\phi_{R,\tilde{\omega}\vec{k}}^*) \phi_{R,\tilde{\omega}'\vec{k}'}) \\ = & i \int_{-\infty}^0 dU \int d^2\vec{k} (\phi_{R,\tilde{\omega}\vec{k}}^* \partial_U \phi_{R,\tilde{\omega}'\vec{k}'} - \partial_U (\phi_{R,\tilde{\omega}\vec{k}}^*) \phi_{R,\tilde{\omega}'\vec{k}'}) \\ = & \delta(\tilde{\omega} - \tilde{\omega}') \delta^2(\vec{k} - \vec{k}') \end{aligned} \quad (2.21)$$

With the same normalization $\phi_{M,\omega\vec{k}}$ is

$$\phi_{M,\omega\vec{k}} = \frac{e^{-i(\omega+k_3)\frac{U}{2}} e^{-ik_1x - ik_2y}}{\sqrt{(2\pi)^3|\bar{\omega}|}}. \quad (2.22)$$

Where $\bar{\omega} = \omega + k_3$. With these preparation we can calculate $\beta(\omega, \tilde{\omega})$ and obtain the expectation values of N_ω

$$\begin{aligned} \beta(\omega, \tilde{\omega}) &= - \int_{-\infty}^0 dU \frac{\omega}{2\pi \sqrt{|\tilde{\omega}\omega|}} |U|^{i\tilde{\omega}} e^{-i\omega U} \\ &= - \int_0^{\infty} \frac{d\tau}{2\pi} \sqrt{\frac{|\omega|}{|\tilde{\omega}|}} e^{2\tilde{\omega}i \log \tau} e^{i\omega\tau} \\ &= - \int_0^{\infty} \frac{ds}{2\pi} \frac{1}{|\omega\tilde{\omega}|} e^{2\tilde{\omega}i \log s + is} e^{-2\tilde{\omega}i \log \omega} \\ &= - \frac{1}{2\pi} e^{-\tilde{\omega}\pi - 2\tilde{\omega}i \log \omega} \Gamma(1 + 2\tilde{\omega}i) \frac{1}{|\tilde{\omega}\omega|}. \end{aligned} \quad (2.23)$$

The last line can be carried out by complex path integral. N_ω is given by

$$\begin{aligned} \int \beta(\tilde{\omega}, \omega) \beta^*(\tilde{\omega}', \omega) d\omega &= e^{-2\tilde{\omega}\pi} \int e^{-2i \log \omega (\tilde{\omega} - \tilde{\omega}')} \frac{\Gamma(1 + 2\tilde{\omega}i) \Gamma(1 - 2\tilde{\omega}')}{|\tilde{\omega}\tilde{\omega}'|} \frac{d\omega}{4\pi\omega} \\ &= \frac{e^{-2\tilde{\omega}\pi}}{2 \sinh(2\tilde{\omega}\pi)} \delta(\tilde{\omega} - \tilde{\omega}'). \end{aligned} \quad (2.24)$$

Here we used equation $\frac{1}{\Gamma(s)\Gamma(1-s)} = \frac{\sin(\pi s)}{\pi}$ in the last line. Now we obtain the thermal distribution $N_\omega = \frac{1}{1-e^{4\omega\pi}}$. Note that energy of the wave function $\phi_{R\omega}$ is given by $\frac{\omega}{2}$ (at past horizon $U = T + Z = 2T$), the temperature of this thermal distribution is $T_R = \frac{1}{\beta_R} = \frac{1}{2\pi}$.

To make the meaning of Unruh effect concrete, the Unruh detector is proposed

(a) The detector will react to states which have positive frequency with respect to the detectors proper time, not with respect to any universal time.

(b) The process of detection of a field quanta by a detector, defined as the exciting of the detector by the field, may correspond to either the absorption or the emission of a field quanta when the detector is an accelerated one.

With these assumptions one can show that a uniformly detector will record finite temperature while a inertial detector record nothing. We will come back to this model in Chapter 3. As we have already seen, the essential reason for this result is that the detector measures frequencies with respect to its own proper time. For an accelerated observer, this definition of positive frequency is not equivalent to that of a nonaccelerated observer. In Minkowski spacetime, positive frequency defined with respect to any geodesic detectors are all equivalent. However in noneflat spacetime, two equally valid geodesic detectors may disagree on whether there are field quanta present. Hence the concept of particles is observer dependent. It is natural to ask that how can we detect the Unruh effect. I will discuss this point in Chapter 3.

2.2 Black Hole Physics

Black hole gives many implications to the microscopic structure of spacetime. Here I am going to review the thermodynamics and the Hawking radiation of the black holes.

2.2.1 Black Hole Thermodynamics

Consider that a black hole swallows a hot body possessing a certain amount of entropy. Then the observer outside it finds that the total entropy in the part of the world accessible to his observation has decreased. This disappearance of entropy could be avoided in a purely formal way if we simply assigned the entropy of the ingested body to the inner region of the black hole. But this solution is unsatisfactory because the outside observer can not determine the amount of entropy absorbed by the black hole. Quite soon after the absorption, the black hole becomes stationary and completely forgets the information of the ingest body and its entropy.

If we are not inclined to forgo the law of non-decreasing entropy because a black hole has formed somewhere in the Universe, we have to conclude that any black hole by itself possesses a certain amount of entropy. A hot body falling into it not only transfers its mass, angular momentum and electric charge to the black hole, but also transfers its entropy S . As a result, the entropy of the black hole will increase when something falling to it. Bekenstein noticed that the properties of the black hole horizon area, A , resemble those of entropy. Indeed with the following correspondences

$$T = \frac{\hbar\kappa}{2\pi kc}, \quad S = \frac{A}{4G\hbar}, \quad E = Mc^2, \quad (2.25)$$

there is an analogy between thermodynamics and black hole physics. People formulated the four laws of black hole physics, which are similar to the four laws of thermodynamics.

Zeroth law: The surface gravity of a stationary black hole is constant everywhere on the surface of the event horizon.

The zeroth law corresponds to that thermodynamics does not permit equilibrium when different parts of a system are at different temperatures. The existence of a state of thermodynamic equilibrium and temperature is postulated by the zeroth law of thermodynamics. This zeroth law of black hole physics plays a similar role. This proposition was proved under the

assumption of the energy dominance condition

$$T_{\alpha\beta}T_{\gamma\theta}g^{\alpha\gamma}u^\beta u^\gamma \geq 0, \quad (2.26)$$

which is that $T_{\alpha\beta}u^\beta$ is a non-spacelike vector. Where $T_{\alpha\beta}$ is the energy momentum tensor and u^μ is an arbitrary timelike vector field.

First law: When the system incorporating a black hole switches from one stationary state to another, its mass changes by

$$dM = TdS + \Omega dJ + \mu dQ + \delta q, \quad (2.27)$$

where dJ and dQ are the changes in the total angular momentum and electric charge of the black hole, respectively, and δq is the contribution to the change in the total mass due to the change in the stationary matter distribution outside the black hole.

The first law is known as a mass formula of the black holes. And this can be generalized to the higher derivative gravity, known as Wald formula [8]. The entropy of black hole is just the Noether charge.

Second law: In any classical process, the area of a black hole, A , and hence its entropy S , do not decrease:

$$\Delta S \geq 0. \quad (2.28)$$

The second law is known as the Hawking's area theorem, which is proved with the weak energy condition

$$T_{\alpha\beta}u^\alpha u^\beta \geq 0. \quad (2.29)$$

In both cases of thermodynamics and black hole physics, the second law signals the irreversibility in the system. As in thermodynamics, the entropy stems from the impossibility of extracting any information about the structure of the system, the structure of the black hole. Note that this second law is classical, the quantum effects can violate Hawking's area theorem. Hawking radiation will reduce the black hole area. On the other hand, the

radiation itself is thermal and it will rise the entropy outside the black hole. So people expect the generalized second law which says that the sum of the black hole entropy and the entropy of the radiation or matter outside the black hole will not decrease. I will derive this generalized second law at chapter 4, in context of the fluctuation theorem for black hole.

Third law: It is impossible by any procedure, no matter how idealized, to reduce the black hole temperature to zero by a finite sequence of operations.

The impossibility of transforming a black hole into an extremal one is closely related to the impossibility of realizing a state with $M^2 < a^2 + Q^2$ in which a naked singularity would appear. Israel [9] proposed and proved the following version of the third law: A non-extremal black hole cannot become extremal at a finite advanced time in any continuous process in which the stress-energy tensor of accreted matter stays bounded and satisfies the weak energy condition in a neighborhood of the outer apparent horizon. It must be emphasized that unlike the thermodynamics, the entropy of black hole generally will not vanishes at zero temperature. The horizon area A remains finite as $\kappa \rightarrow 0$.

In this section, we only reviewed the analogues between the thermodynamics and the black hole physics. The physical meanings of this black hole thermodynamics, especially for the black hole temperature will be clear in Hawking radiation.

2.2.2 Hawking Radiation

Here we consider the Schwarzschild black holes for simplicity. The essence does not change for general black holes.

The Schwarzschild black holes are described by the metric

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1}dr^2 - r^2d\Omega^2. \quad (2.30)$$

Here M is the mass of the black hole. This metric will goes to flat at infinity, $r \rightarrow \infty$, so (r, t) is the coordinates of the observer who stays at the infinity,

the asymptotic observer. This coordinates is singular at $r = 2M$, this corresponds to the horizon of the black hole. However this singularity is just a coordinate singularity, and physical quantities will not diverge here. For example, a coordinate which is regular at the horizon can be given by

$$\begin{aligned} U &= \left(\frac{r}{2GM} - 1\right)^{\frac{1}{2}} e^{\frac{r+t}{4GM}} \\ V &= -\left(\frac{r}{2GM} - 1\right)^{\frac{1}{2}} e^{\frac{r-t}{4GM}}. \end{aligned} \quad (2.31)$$

Then the metric are

$$ds^2 = \frac{32G^3M^3}{r} e^{-r/2GM} (dU dV) - r^2 d\Omega^2, \quad (2.32)$$

which is regular at $r = 2M$, and this coordinates correspond to coordinates of free falling observers.

Hawking radiation is related to Unruh effect via equivalence principle. Indeed, Rindler space will emerge in the near horizon limit of the black holes. For example, consider the near horizon limit ($r \rightarrow 2M$) of the Schwarzschild metric

$$\begin{aligned} ds^2 &= \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 \\ &\rightarrow (r - 2M) \frac{dt^2}{2M} - \frac{2M dr^2}{r - 2M} - r^2 d\Omega^2 \\ &= \rho^2 \left(\frac{dt}{4M}\right)^2 - d\rho^2 - r^2 d\Omega^2. \end{aligned} \quad (2.33)$$

Here $d\rho = \frac{\sqrt{2M}}{\sqrt{r-2M}} dr$ and $\rho = 2\sqrt{2M(r-2M)} > 0$. The black hole horizon ($r = 2M$) corresponds $\rho = 0$ in the Rindler coordinates.

Denote $|0\rangle_U$ for Unruh vacuum which defined as the vacuum for the free falling observer, and denote $|0\rangle_S$ for Schwarzschild vacuum which defined as the vacuum for the observer at spacetime infinity. Here $|0\rangle_U$ corresponds to $|0\rangle_R$ and $|0\rangle_S$ corresponds to $|0\rangle_M$ at the previous section.

One can evaluate the Bogolubov transformation at the horizon. Then the geometry will becomes the Rindler space. Unruh vacuum is vacuum for free

falling observers near horizon, and Schwarzschild is vacuum for observers at infinity.

However the temperature $T_R = \frac{1}{2\pi}$ obtained before is defined by Rindler time and is different from the temperature observed at infinity. One should take account the redshift from Rindler time to Schwarzschild time. For example, as we showed before (2.33), the near horizon Schwarzschild geometries can be written by:

$$ds^2 = -\rho^2 \left(\frac{dt}{4M} \right)^2 + d\rho^2 + r^2 d\Omega^2. \quad (2.34)$$

Where $\frac{t}{4M}$ is the Rindler time, and t is the Schwarzschild time. So the radiation will experience a redshift from the Rindler coordinates to the infinity observers. Hence the temperature of Hawking radiation is given by

$$T = \frac{1}{8\pi M}. \quad (2.35)$$

The Unruh effect just tells us that $|0\rangle_R$ is different from $|0\rangle_M$, and one can find the explicit relation between $|0\rangle_M$ and $|0\rangle$. There is no problem for that which state is more natural to be vacuum. However Hawking radiation occurs for vacuum $|0\rangle_U$. If one chooses vacuum $|0\rangle$ there will be no Hawking radiation.

There are a number of reasons why people favor the Unruh vacuum over the Schwarzschild vacuum:

(1) As we mentioned before, Unruh vacuum is the vacuum for free falling observers. It seems more natural to take the vacuum that free falling observer observes nothing than the vacuum that free observers observes finite temperature.

(2) Most physical properties of the scalar field (charge density, energy, etc.) are regular for Unruh vacuum and will become singular for Schwarzschild vacuum.

(3) Unruh vacuum preserves more symmetris than Schwarzschild vacuum. For the case of flat spacetime, it is the property that $|0\rangle_M$ is invariant under the full Poincaré group while $|0\rangle_R$ does not.

(4) Consider the situation that a star shell collapse to black hole, for example, the Vaidya metric, then it will turn out that Unruh vacuum is preferred.

2.2.3 Information Puzzle

The Hawking radiation from a black hole is thermal. This result is obtained in the semi-classical approximation, which is supposed to be valid until the black hole has shrunk to nearly the Planck mass. Then, if the black hole disappears completely, what is left is just thermal radiation. This also occurs even in the case when we start from some pure state collapse into a black hole. This means whatever initial state one start from, once it collapse to a black hole, the final state will be the same which is just thermal radiation. The information of the initial state are lost in this process. Or in another way to say, a pure state seems to go to a mixed state in this process. This can not happen in a quantum system with unitary time evolution. This problem is called the information loss puzzle, or information paradox.

There are many discussions on the information puzzle. However, none of them are satisfactory. The information paradox appears more like a hint or motivation than a problem, which helps people to investigate the microscopic structure of spacetime.

2.3 The Argument of Ted Jacobson

Though the black holes are just solutions of the Einstein equation, but the thermodynamics do hold for black holes. This fact is very surprisingly since the Einstein equation is just a hyperbolic second order partial differential equation. This fact may implies some connections between gravity and thermodynamics. Some people also expects that the gravity may just be effective descriptions, like thermodynamics. Ted Jacobson [7] push this idea further.

He reversed the argument by assuming the thermodynamics of horizons first and then derived the Einstein equation.

The basic idea is following. First assume that in quantum spacetime there is a universal entropy density α per unit horizon area

$$S = \alpha A, \quad (2.36)$$

here A is the horizon area. Next consider the energy flux flows through this horizon as the heat, dQ . Then determine the temperature from the Unruh effect. And finally obtain the Einstein equation from the relation

$$\delta S = \frac{\delta Q}{T}. \quad (2.37)$$

The local causal horizon at a point p is defined like this: choose a spacelike 2-surface patch B including p and consider the past boundary of B . Near p , this boundary is a congruence of null geodesics orthogonal to B . These comprise the horizon. Here B is chosen as such that the expansion θ and shear σ_{ab} of this congruence are vanishing at p . This means the equilibrium state at p .

To define the heat flux and temperature, employ an approximate boost Killing vector field χ^a that vanishes at p , so its flow leaves the tangent plane B_p to B at p invariant. The direction of χ^a is chosen to pointing on future of the causal horizon. The normalization of χ^a is chosen so that $\chi_{a;b}\chi^{a;b} = -2$, just like the usual boost Killing vector $x\partial_t + t\partial_x$ in Minkowski spacetime. One can obtain χ^a by solving Killing's equation $\chi_{a;b} + \chi_{b;a} = 0$ in Riemann normal coordinates, y^μ , order by order. However, there may be no solutions at $\mathcal{O}(y^3)$, since a general curved spacetime may not to have a Killing vector.

Killing time along the horizon is given by the parameter v such that $\chi^\mu \nabla_\mu v = 1$. This Killing time is related to the affine parameter along the horizon generators by $\lambda = -e^{-v}$, the point p is located at infinite Killing time and $\lambda = 0$. It will be convenient to work in terms of horizon tangent vector k^μ which is related to χ^a by $\chi^a = -\lambda k^a$.

Now consider the heat as the energy current of matter across the horizon

$$\delta Q = \int T_{\mu\nu}^M \chi^\mu d\Sigma^\nu, \quad (2.38)$$

where $T_{\mu\nu}^M$ is the stress tensor of matter. The integral is taken over a short segment of a thin pencil of horizon generators centered on the one that terminates at p . The temperature is given by $T = \frac{\hbar}{2\pi}$ which is the Unruh temperature of the Rindler space. Thus one have

$$\frac{\delta Q}{T} = \frac{2\pi}{\hbar} \int T_{\mu\nu}^M k^\mu k^\nu (-\lambda) d\lambda d^2 A. \quad (2.39)$$

The entropy change $\delta S = \alpha \delta A$ is given by

$$\delta A = \int \theta d\lambda d^2 A, \quad (2.40)$$

where $\theta = d(\ln d^2 A)/d\lambda$ is the expansion of the congruence of null geodesics generation the horizon. Using the Raychaudhuri equation

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} - R_{\mu\nu}k^\mu k^\nu, \quad (2.41)$$

one can expand θ around p where $\lambda = 0$, then

$$\delta S = \alpha \int \left[\theta - \lambda \left(\frac{1}{2}\theta^2 + \sigma_{\mu\nu}\sigma^{\mu\nu} + R_{\mu\nu}k^\mu k^\nu \right) \right]_{|_p} d\lambda d^2 A. \quad (2.42)$$

Note that all quantities in the integrand are evaluated at p . Using the assumption of the equilibrium, $\theta = \sigma^{ab} = 0$ at p

$$\delta S = \alpha \int R_{ab}k^a k^b (-\lambda) d\lambda d^2 A. \quad (2.43)$$

Now if one require that $\delta S = \delta Q/T$ holds for all local Rindler horizons through all points p . Then the integrand of (2.39) and (2.43) should be same at every point

$$R_{ab}k^a k^b = \frac{2\pi}{\alpha \hbar} T_{ab}^M k^a k^b. \quad (2.44)$$

This should hold for all null vectors k^a . So

$$R_{ab} + \Lambda g_{ab} = \frac{2\pi}{\hbar\alpha} T_{ab}^M, \quad (2.45)$$

here the λ is a undetermined constant since $g_{ab}k^ak^b = 0$. Finally, with the entropy density α

$$\alpha = \frac{1}{4\hbar G}, \quad (2.46)$$

the Einstein equation with cosmological constant is derived.

2.4 Membrane Paradigm

Another interesting issue closely related to the horizons is the Membrane Paradigm. It has been shown that observer who remains outside a black hole will see the horizon to behave according to equations that describe a fluid bubble with electrical conductivity as well as shear and bulk viscosities. The membrane paradigm is a point of view that consider a black hole as a dynamical time-like surface, the membrane. It was first proposed by Kip S. Thorn, R. H. Price and D. A. Macdonald [10]. Here I am going to review an approach given by M. K. Parikh and F. Wilczek [11].

The basic idea is like this, since (Classically) nothing can emerge from a blackhole, an observer who remains outside a blackhole cannot be affected by the dynamics inside the hole. Hence we should be able to obtain the equations of motion for external region from varying the action restricted to the external universe.

But the boundary of external region is not the boundary of spacetime. Then some surface terms will be left when we varying the action to obtain the equation of motion. So we have to add some surface terms which cancel the residual surface terms and enable us to get the complete equations of motion. Interpreting these added surface terms as sources residing on the horizon will give the picture of membrane.

Here what we actually do is just dividing the space-time into two regions (the boundary is not necessary to be horizon) and rewriting the total action as

$$S_{total} = (S_{out} + S_{surf}) + (S_{in} - S_{surf}), \quad (2.47)$$

requiring $\delta S_{out} + \delta S_{surf} = 0$. Then S_{surf} corresponds to sources on the boundary.

Here instead of the true horizon we consider the stretched horizon, which is a time-like surface just outside the true horizon. We parameterize it's location by α in the way that $\alpha \rightarrow 0$ corresponds the limit that the stretched horizon coincides with the true horizon. Many intermediate quantities will diverge at the true horizon, α also plays the role of a regulator.

This stretched horizon has several advantages over the true horizon. Unlike the true horizon the stretched horizon is a time-like (rather than null) surface. So the metric is nondegenerate, and this enable us to write down a conventional action of the stretched horizon. We can also say that the stretched horizon is more fundamental by this sense: an external observer can make a measurement and report it at the stretched horizon which can not be done at the true horizon. By the way a one-to-one correspondence between points on the true horizon and the stretched horizon is always possible.

2.4.1 The Electromagnetic Membrane

For example, we consider the electromagnetic case here. The action is given by:

$$S = \int d^4x \sqrt{-g} \left(-\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} + J^\mu A_\mu \right). \quad (2.48)$$

Restrict the integral region to external:

$$\begin{aligned}
\delta S_{out} &= \int d^4x \sqrt{-g} \left(-\frac{1}{4\pi} F^{\mu\nu} \nabla_\mu \delta A_\nu + J^\nu \delta A_\nu \right) \\
&= \int d^4x \sqrt{-g} \left\{ -\frac{1}{4\pi} \nabla_\mu (F^{\mu\nu} \delta A_\nu) + \delta A_\nu \left(\frac{\nabla_\mu F^{\mu\nu}}{4\pi} + J^\nu \right) \right\} \\
&= \int dS \sqrt{-h} \frac{1}{4\pi} n_\mu F^{\mu\nu} \delta A_\nu + \int d^4x \sqrt{-g} \delta A_\nu \left(\frac{1}{4\pi} \nabla_\mu F^{\mu\nu} + J^\nu \right).
\end{aligned} \tag{2.49}$$

Put the second term to zero gives the Maxwell equations: $\nabla_\mu F^{\mu\nu} = -4\pi J^\nu$. The first term is left because we can not set δA_μ to zero at the boundary of external region. To cancel this term, adding S_{surf} :

$$S_{surf} = \int d^3x \sqrt{-h} j_s^\mu A_\mu. \tag{2.50}$$

Here the equations of motion at the boundary are given by:

$$j_s^\mu = \frac{1}{4\pi} F^{\mu\nu} n_\nu. \tag{2.51}$$

From the antisymmetry of $F^{\mu\nu}$ this current is parallel to the horizon ($j_s^\mu n_\mu = 0$). By component, it is given by:

$$\begin{aligned}
j_s^0 &= -\frac{1}{4\pi} F^{0b} n_b = E_\perp \\
j_s^A &= \frac{1}{4\pi} F^{Ab} n_b = \frac{1}{4\pi} (\vec{n} \times \vec{B})^A.
\end{aligned} \tag{2.52}$$

This is just the relation between the surface charge and surface current and the discontinuity of electromagnetic field at the surface. So it is natural to interpret j_s^μ as the electromagnetic current at the surface. One can also show that j_s satisfies the continuity equation by using the equations of motion,

$$\begin{aligned}
\nabla_\mu j_s^\mu &= j_{s|\mu}^\mu = \frac{1}{4\pi} \nabla_\mu F^{\mu\nu} n_\nu \\
&= -J^\nu n_\nu.
\end{aligned} \tag{2.53}$$

This means that any charges that fall into the in-region can be regarded as remaining on the surface. We did not choose the surface to be the horizon

until now. So the results obtained here do not depend on which surface we choose.

It can be show that the membrane behaves like a conductor with certain conductivity if we choose the surface to be the horizon. We impose the regularity condition that the electromagnetic fields measured by free falling observer (FFO) should not diverge at horizon. The fiducial observer (denoted by FIDO, the observer who just remaining outside the horizon) is related to the free falling observer by some infinite Lorentz boosts ($\beta \rightarrow 1$). For example, E_θ^{FIDO} and B_ϕ^{FIDO} are given by:

$$\begin{aligned} E_\theta^{FIDO} &= F_{FIDO}^{0\theta} = \gamma F_{FFO}^{0\theta} - \beta \gamma F_{FFO}^{r\theta} \\ &= \gamma (E_\theta^{FFO} - B_\phi^{FFO}) \\ B_\phi^{FIDO} &= F_{FIDO}^{r\theta} = -\gamma \beta F_{FFO}^{0\theta} + \gamma F_{FFO}^{r\theta} \\ &= \gamma (B_\phi^{FFO} - E_\theta^{FFO}). \end{aligned} \quad (2.54)$$

We obtain relation: $E_\theta^{FIDO} = -B_\phi^{FIDO}$. Similarly we can also obtain $E_\phi^{FIDO} = B_\theta^{FIDO}$. In summary:

$$\vec{E}_{\parallel}^{FIDO} = \vec{n} \times \vec{B}_{\parallel}^{FIDO} = 4\pi \vec{j}_s. \quad (2.55)$$

This is the Ohm's law for the conductor with resistivity $\rho = 4\pi \approx 377\Omega$. Here the regularity condition is equivalent to the statement that classically all radiation in the normal direction is ingoing (a black hole acts as a perfect absorber). Indeed the Poynting flux can be calculated,

$$\vec{S} = \frac{1}{4\pi} \vec{E} \times \vec{B} = -j_s^2 \rho \vec{n}. \quad (2.56)$$

It is always inward. This equation also describes the Joule heating.

2.4.2 The Gravitational Membrane

In order to explore more about the membrane, we fix our convention here. We denote U^a to be the tangential vectors of the world lines of the fiducial

observers($U^a = (\frac{d}{d\tau})^a$, where τ is the proper time of the fiducial observer). The normal vector of the hypersurface is denoted by n^a . Also we choose the normal vector congruence to be along the geodesics. We can parameterize the location of the stretched horizon in the way that $\alpha U^a \rightarrow l^a$ and $\alpha n^a \rightarrow l^a$; in other words αU^a and αn^a are equal in the true horizon limit, and the null vector l^a is both normal and tangential to the horizon.

The hypersurface can be regarded as a $d + 1$ space-time splitting, and n^a is the normal vector. We can also consider the $(d - 1) + 1$ splitting of the hypersurface, take U^a to be the normal vector, and the $d - 1$ part can be regarded as the space-like section of the hypersurface. We denote the metric on the hypersurface and on the space-like section by h^{ab} and γ^{AB} .

If some vector fields lie in the hypersurface, then we can define a covariant derivative on the hypersurface, denote $|_a$ to be the d -covariant derivative (the derivative on the stretched horizon) and denote $||_A$ to be the $(d - 1)$ -covariant derivative (the derivative on the space-like section of the stretched horizon). The covariant derivatives are related by $h_d^c \nabla_c w^a = w_{|d}^a - K_d^c w_c n^a$, where $K_b^a = h_b^c \nabla_c n^a$ is the extrinsic curvature of the stretched horizon. In summary

$$U^a = (\frac{d}{d\tau})^a, \quad U^2 = -1, \quad \lim_{\alpha \rightarrow 0} \alpha U^a = l^a \quad (2.57)$$

$$n^2 = 1, \quad a^c = n^a \nabla_a n^c = 0, \quad \lim_{\alpha \rightarrow 0} \alpha n^a = l^a \quad (2.58)$$

$$l^2 = 0 \quad (2.59)$$

$$h_b^a = g_b^a - n^a n_b,$$

$$\gamma_b^a = h_b^a + U^a U_b = g_b^a - n^a n_b + U^a U_b \quad (2.60)$$

For w^a lies in the surface $h_d^c \nabla_c w^a = w_{|d}^a - K_d^c w_c n^a$

$$\nabla_c w^c = w_{|c}^c. \quad (2.61)$$

Action

Now we turn to the gravitational membrane. First we write down the action for the membrane, second we show that this membrane behaves like a fluid

which satisfies the “Navier-Stokes equation”.

The way to obtain the action of membrane is same as the electromagnetic case. But the calculation is more complicated and for gravitational case the surface has to be the stretched horizon in order to write down the action.

The Einstein-Hilbert action is given by:

$$S[g^{ab}] = \frac{1}{16\pi} \int d^4x \sqrt{-g} R + \frac{1}{8\pi} \oint d^3x \sqrt{\pm h} K + S_{matter}, \quad (2.62)$$

the integral of the second term is only over the outer boundary of the space-time. It is necessary to obtain the Einstein equations since the Ricci scalar contains second derivatives of g_{ab} . The equations of motion well known

$$R_{ab} - \frac{1}{2} g_{ab} R = 8\pi T_{ab}. \quad (2.63)$$

The surface term only comes from $g^{ab} \delta R_{ab}$

$$g^{ab} \delta R_{ab} = \nabla^a [\nabla^b (\delta g_{ab}) - g^{cd} \nabla_a (\delta g_{cd})]. \quad (2.64)$$

Note that the normal vector n^a is taken to point outward, and Gauss’s theorem gives

$$\int d^4x \sqrt{-g} (g^{ab} \delta R_{ab}) = - \int d^3x \sqrt{-h} n^a g^{bc} [\nabla_c (\delta g_{ab} - \nabla_a (\delta g_{bc}))]. \quad (2.65)$$

Equation $n^a n^b n^c [\nabla_c (\delta g_{ab} - \nabla_a (\delta g_{bc}))] = 0$ and equation $g^{ab} = h^{ab} - n^a n^b$ enable us to replace $g^{bc} \delta R_{bc}$ to $h^{bc} \delta R_{bc}$:

$$\int d^4x \sqrt{-g} (g^{ab} \delta R_{ab}) = - \int d^3x \sqrt{-h} n^a h^{bc} [\nabla_c (\delta g_{ab} - \nabla_a (\delta g_{bc}))]. \quad (2.66)$$

Since the variation of membrane action takes the form $\delta S_{surf} = \int d^3x \sqrt{-h} t_{sab} \delta h^{ab}$, we have to rewrite this term in order to obtain the membrane action,

$$\begin{aligned} \int d^4x \sqrt{-g} (g^{ab} \delta R_{ab}) = \int d^3x \sqrt{-h} h^{bc} [& \nabla_a (n^a \delta g_{bc}) - \delta g_{bc} \nabla_a (n^a) \\ & - \nabla_c (n^a \delta g_{ab}) + \delta g_{ab} \nabla_c (n^a)]. \end{aligned} \quad (2.67)$$

Terms involving derivative of δg^{ab} will vanish in the limit that the stretched horizon approaches the true horizon:

$$\int d^3x \sqrt{-h} h^{bc} [\nabla_a (n^a \delta g_{bc}) - \nabla_c (n^a \delta g_{ab})] = 0. \quad (2.68)$$

With $K^{ab} = h^{bc} \nabla_c n^a$ the variation of the external action is

$$\delta S_{out}[g^{ab}] = \frac{1}{16\pi} \int d^3x \sqrt{-h} (K h_{ab} - K_{ab}) \delta h^{ab}. \quad (2.69)$$

We have to integrate equation (2.69) to obtain the membrane action.

$$S_{surf}[h^{ab}] = \int d^3x \sqrt{-h} (B_{ab} h^{ab} - b) \quad (2.70)$$

is a solution, with source terms to be

$$\begin{aligned} B_{ab} &= \frac{1}{16\pi} K_{ab} \\ b &= -\frac{1}{16\pi} K. \end{aligned} \quad (2.71)$$

If we write the variation of the membrane action by the form

$$\delta S_{surf}[h^{ab}] = -\frac{1}{2} \int d^3x \sqrt{-h} t_{sab} \delta h^{ab}, \quad (2.72)$$

then the membrane stress tensor is given by

$$t_s^{ab} = \frac{1}{8\pi} (K h^{ab} - K^{ab}). \quad (2.73)$$

Just as a surface charge produces a discontinuity in the normal component of the electric field, a surface stress tensor creates a discontinuity in the extrinsic curvature. The membrane stress tensor is consistent with the Israel junction condition:

$$t_s^{ab} = \frac{1}{8\pi} ([K] h^{ab} - [K]^{ab}), \quad (2.74)$$

where $[K] = K_+ - K_-$ is the discontinuity of the extrinsic curvature through the surface. One can also show that t_s^{ab} satisfies the continuity equation by using the equations of motion:

$$\begin{aligned} t_s^{ab}{}_{|b} &= \frac{1}{8\pi} [(K h^{ab})_{|b} - K^{ab}{}_{|b}] \\ &= -T^{ma} \\ &= -h_c^a T^{cd} n_d. \end{aligned} \quad (2.75)$$

The “Navier-Stokes equation”

The continuity equation and the fact that t_s^{ab} consistent with Israel junction condition imply that stretched horizon can be considered as a fluid membrane. Indeed, it can be shown that membrane obeys the Navier-Stokes equation.

In the limit that the stretched horizon approaches the true horizon both αU^a and αn^a approach l^a :

$$U^c \nabla_c n^a \rightarrow \frac{1}{\alpha^2} l^c \nabla_c l^a = \frac{g_H}{\alpha^2} l^a. \quad (2.76)$$

Using this fact we can write down all the components of K_b^a . For K_U^U :

$$\begin{aligned} K_{ab} U^a U^b &= U^a U^b \nabla_a n_b \\ &\rightarrow \alpha^{-2} U^a l^b \nabla_a l_b = \alpha^{-2} U^b l_b g_H \\ &\rightarrow \alpha^{-1} U^a U_a g_H = -\alpha^{-1} g_H = -g. \end{aligned} \quad (2.77)$$

For K_A^U :

$$K_A^U = \gamma_A^a U^b K_{ab} \rightarrow g \gamma_A^a U_a = 0.$$

If we define the extrinsic curvature of the space-like section of stretched horizon as: $k_B^A = \gamma_A^d l_{B|d} = \frac{1}{2} \mathcal{L}_{l^a} \gamma_{AB}$, then K_B^A is given by:

$$\begin{aligned} K_A^B &= \gamma_A^a K_a^b \gamma_b^B = \gamma_A^a \gamma_b^B \nabla_a n^b \\ &\rightarrow \frac{1}{\alpha} \gamma_A^a \gamma_b^B \nabla_a l^b = \frac{1}{\alpha} k_A^B. \end{aligned} \quad (2.78)$$

We can decompose k_B^A into traceless part and trace part: $K_{AB} = \sigma_{AB} + \frac{1}{2} \gamma_{AB} \theta$. Where θ is the traceless part, correspond to shear of the membrane. And θ is the trace part, correspond to expansion of the membrane. Then we can write down all the components of t_{sab} explicitly ($K = \frac{g_H + \theta}{\alpha}$ here):

$$\begin{aligned} \pi_A &= t_{sa}^b \gamma_A^a U_b = \frac{1}{8\pi} (K h^{ab} \gamma_{aA} U_b - K_A^U) = 0 \\ \Sigma &= t_{sa}^b U_b U^a = -\frac{K}{8\pi} + \frac{K_U^U}{8\pi} = -\frac{\theta}{8\pi\alpha} \\ t_s^{AB} &= \frac{1}{8\pi\alpha} [\gamma^{AB} (g_H + \frac{1}{2}\theta) - \sigma^{AB}]. \end{aligned} \quad (2.79)$$

The ordinary fluid stress tensor is given by:

$$\Pi_{ik} = p\delta_{ik} + \rho v_i v_k - \eta(\partial_k v_i + \partial_i v_k - \frac{2}{3}\delta_{ik}\partial_l v_l) - \zeta\delta_{ik}\partial_l v_l, \quad (2.80)$$

where v_l is velocity of the fluid and p, η, ζ is pressure, shear viscosity and bulk viscosity of the fluid. If we identify U^a with v_l , then the membrane pressure ($p = \frac{g_H}{8\pi\alpha}$), shear viscosity ($\eta = \frac{1}{16\pi\alpha}$) and bulk viscosity ($\zeta = -\frac{1}{16\pi\alpha}$) are obtained.

Now we can write down the Navier-Stokes equation for membrane. Define the A-momentum density by: $\pi_A = t_{sa}^b \gamma_A^a U_b$. Using the continuity equation:

$$\begin{aligned} -\gamma_A^a T_a^c n_c &= -T_A^n = \gamma_A^a (t_{sa}^b) \\ &= (\gamma_A^a t_a^b)|_b - t_a^b (\gamma_A^a|_b) \\ &= -(\pi_A U^b)|_b + (t_{ab} \gamma_A^a \gamma^{bB})_{||B} - t_a^b (U_{A|b}) \\ &= (t_A^B)_{||B} - U^b \pi_{A|b} - \pi^B U_{A|B} + \pi_A U_{|b}^b + \Sigma U^b U_{A|b} \\ &= (t_A^B)_{||B} - \gamma_A^B \mathcal{L}_U \pi_B \end{aligned} \quad (2.81)$$

We have used equation $\pi_A = 0$ to set $\pi U_b^b = 0$ and used equation $U^b U_{A|b} = U^b \nabla_b U_A \rightarrow \frac{1}{\alpha^2} l^b \nabla_b l_A = \frac{g_H}{\alpha^2} l_A = 0$ to set $\Sigma U^b U_{A|b} = 0$. Substitute t_{sAB} , then:

$$-\gamma_A^B \mathcal{L}_U \pi_B = -\nabla_A p + \zeta \nabla_A \theta + 2\eta \sigma_{A||B}^B - T_A^n. \quad (2.82)$$

This equation is quit similar to the Navier-Stokes equation of the ordinary fluid in 4-dimension (except the source term T_A^n):

$$\begin{aligned} \rho \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} \right] &= -\vec{\nabla} p + \eta \Delta \vec{v} + \left(\zeta + \frac{1}{3} \right) \vec{\nabla} (\vec{\nabla} \cdot \vec{v}) \\ &= -\vec{\nabla} p + 2\eta \vec{\nabla} \cdot \sigma + \zeta \vec{\nabla} (\vec{\nabla} \cdot \vec{v}) \end{aligned} \quad (2.83)$$

Where σ_{ij} is shear of the fluid in 4-dimension ($2\sigma_{ij} = \partial_i v_j + \partial_j v_i - \delta_{ij} \frac{2}{3} \partial_l v_l$).

The viscosity and the Joule heating mean there are dissipation on membrane. It seems mysterious that dissipation can appears, since the equations of motion of bulk fields are symmetric under time-reversal.

The breaking of time-reversal symmetry can come from the definition of the stretched horizon: given data on some suitable surface, for the exterior we can predict the future but cannot determine the entire past; however inside the blackhole we can “postdict” the past but cannot determine the entire future. Thus, our choice of the horizon contains some asymmetry between past and future. Beside this global properties, the local property that the normal vector to the horizon is also the tangential to the horizon can also cause time-reversal asymmetry. In our calculation, this local property manifests itself as the regularity conditions. Without regularity conditions we can also obtain surface terms, but no dissipation.

Thermodynamics

We can extend the analogy to fluid dynamics to thermodynamics by writing down the equation for $\Sigma = T^{ab}U_aU_b$. Using the continuity equation:

$$\begin{aligned} (U_a t_s^{ab})|_b &= -T_{ab}U^a n^b + t_s^{ab}U_{a|b} \\ &= -T_{ab}U^a n^b + t_s^{ab}U_a U_b U^c U^d U_{c|d} + t_s^{AB}U_{A|B} \\ &= -T_{ab}U^a n^b - g\Sigma + \frac{1}{\alpha}k_{AB}t_s^{AB}. \end{aligned} \quad (2.84)$$

$(U_a t_s^{ab})|_b$ itself can be written by:

$$\begin{aligned} (U_a t_s^{ab})|_b &= -\Sigma|_c U^c - \Sigma U|_b^b + (U_a t_s^{ab} \gamma_b^B)|_B \\ &= -\mathcal{L}_U \Sigma - \Sigma g - \frac{1}{\alpha} \Sigma \theta. \end{aligned} \quad (2.85)$$

Then we obtain:

$$\begin{aligned} \mathcal{L}_U \Sigma + \frac{1}{\alpha} \Sigma \theta &= \frac{1}{\alpha} K_{AB} t_s^{AB} + T_{ab} U^a n^b \\ &= -p\theta + \zeta\theta^2 + 2\eta\sigma_{AB}\sigma^{AB} + T_b^a n_a U^b. \end{aligned} \quad (2.86)$$

This is just the equation of energy conservation. From the analogy with fluids we may expect that this equation can be related to the heat transfer equation for a two-dimensional fluid. Indeed if we writing the expansion of

the fluid in terms of the area, $\theta = \Delta A$, we can indeed write the equation as the heat transfer equation:

$$T\left(\frac{d\Delta S}{d\tau} - \frac{1}{g} \frac{d^2\Delta S}{d\tau^2}\right) = (\zeta\theta^2 + 2\eta\sigma_{AB}\sigma^{AB} + T_b^a n_a U^b)\Delta A, \quad (2.87)$$

with the entropy S and temperature T given by:

$$S = F \frac{k_B}{\hbar} A, \quad T = \frac{\hbar}{8\pi k_B F} g. \quad (2.88)$$

F is some constant remains undetermined.

The temperature T and entropy S can be obtained by another way. Temperature can be obtained by performing an analytic continuation to imaginary time, $\tau = it$. For removing the conical singularity τ have to be periodic. Temperature can be read from the period:

$$\beta = \int d\tau = \frac{2\pi}{g_H}.$$

Entropy can be calculated using partition function. Since the dominant contribution to the path integral comes from the classical solution, we just evaluate the partition function in a stationary phase approximation:

$$\begin{aligned} Z &= \int Dg_E^{ab} \exp\left(-\frac{1}{\hbar}(S_{out}^E[g_E^{ab}] + S_{surf}^E[h_E^{ab}])\right) \\ &= \exp\left(-\frac{1}{\hbar}(S_{out}^E[g_E^{ab}] + S_{surf}^E[h_E^{ab}])\right). \end{aligned} \quad (2.89)$$

Where $S_{out} = S_{bulk} + S_{\infty}$, and for a blackhole alone in the universe, $S_{bulk} = 0$. A term proportional to the surface area at infinity can be included in S_{∞} without effect the Einstein equations. We fix S_{∞} by:

$$S_{\infty} = \frac{1}{8\pi} \int d^3x \sqrt{-h} [K], \quad (2.90)$$

where $[K]$ is the difference in the trace of the extrinsic curvature at the spacetime boundary for the metric g_{ab} and the flat-space metric η_{ab} ($[K] = K[g] - K[\eta]$).

For Schwarzschild blackhole, $\beta = 8\pi M$ and

$$\begin{aligned} K[g]_{ab} &= \Gamma_{ab}^r \frac{1}{\sqrt{1 - \frac{2M}{r}}} \\ \Rightarrow K[g] &= \frac{1}{r^2 \sqrt{1 - \frac{2M}{r}}} (2r - 3M). \end{aligned} \quad (2.91)$$

Here we take the surface of S_∞ to be a sphere. Then $K[\eta]$ is given by:

$$K[\eta]_{ab} = \Gamma_{ab}^r \Rightarrow K[\eta] = \frac{2}{r}. \quad (2.92)$$

Using equation $\int d^3x \sqrt{-h} = \int d\tau d\Omega r^2 \sqrt{1 - \frac{2M}{r}} = 4\pi\beta r^2 \sqrt{1 - \frac{2M}{r}}$ one can evaluate S_∞ as:

$$\begin{aligned} S_\infty &= \lim_{r \rightarrow \infty} \frac{4\pi}{8\pi} \beta (2r - 3M - 2r \sqrt{1 - \frac{2M}{r}}) \\ &= 4\pi M^2 \end{aligned} \quad (2.93)$$

Now we turn to the membrane action, S_{surf} . Using equation (2.70,2.71) we obtain:

$$S_{surf}[h_{cl}^{ab}] = \frac{1}{8\pi} \int d^3x \sqrt{-h_{cl}^{ab}} K_{cl} \quad (2.94)$$

For the stationary blackhole, the expansion is zero: $\theta = 0$. Then K is given by $K = g + \theta = g$. And using equation $\int \sqrt{-h} d^3x = \beta\alpha 4\pi r^2$, we obtain:

$$S_{surf} = \lim_{r \rightarrow r_H} \frac{1}{8\pi} 4\pi r_H^2 \alpha g = -\pi r_H^2 = -4\pi M^2. \quad (2.95)$$

Where $r_H = 2M$ and $g_H = \alpha g = \frac{1}{4M}$.

Here the membrane action exactly cancels the external action. Hence the entropy is zero. This is what makes the membrane paradigm attractive: to an external observer, there is no black hole but only a membrane and no a generalized entropy. The entropy of the outside is simply the logarithm of the number of quantum states.

We just using the action of outside and membrane to obtain the entropy of outside region. To recover the Bekenstein-Hawking entropy, we have to use the action of internal and membrane actions. Since $R = 0$ everywhere $S_{in} = \frac{1}{16\pi} \int \sqrt{-g} R = 0$. So we only have to evaluate S_{surf} . Remember we divided the action in the fashion: $S = (S_{in} - S_{surf}) + (S_{out} + S_{surf})$. The membrane action contribute in the different sign. Now we obtain the partition function:

$$Z_{BH} = \exp\left(-\frac{1}{\hbar}(4\pi M)\right). \quad (2.96)$$

Then the Bekenstein-Hawking entropy is given by:

$$S_{BH} = \beta\left(M + \frac{\log Z_{BH}}{\beta}\right) = \frac{A}{4}. \quad (2.97)$$

Chapter 3

Stochastic Approach to Unruh Radiation

The Unruh effect is closely related to the Hawking radiation and the horizon physics. It tells us that an accelerated particle will see the vacuum of Minkowski as thermally excited. This can be shown by the Bogolubov transformation, which is the transformation of the wave function. These discussions were done in the equilibrium region, and the interactions were ignored. Here we would like to investigate the fluctuations of the uniformly accelerated particle. Due to the interactions, the motion of the uniformly accelerated particle should become stochastic. We are going to analyze this stochastic motion and the radiation due to it. This study is also motivated from a point of view from experiment.

3.1 Detectability of Unruh Effect

The Unruh temperature is given by

$$T_U = \frac{\hbar a}{2\pi c k_B} = 4 \times 10^{-23} [K] \times \left(\frac{a}{1 \text{ cm/s}^2} \right). \quad (3.1)$$

Generally it is very small for ordinary acceleration. To detect Unruh effect, we need a extremely high acceleration, and the recent development of ultra-high intensity lasers makes it possible. In the electro-magnetic field of a laser with intensity $I[W/cm^2]$, an electron can be accelerated to

$$a = 2 \times 10^{12} [cm/s^2] \times \sqrt{I} \quad (3.2)$$

and the Unruh temperature is given by

$$T_U = 8 \times 10^{-11} [K] \times \sqrt{I}. \quad (3.3)$$

The ELI (Extreme Light Infrastructure) project [14] recently approved is planning to construct Peta Watt lasers with an intensity as high as $5 \times 10^{26} [W/cm^2]$. Then the expected Unruh temperature becomes more than $10^3 K$ which is much higher than the room temperature. Now, the question is that how can we experimentally observe such a high Unruh temperature of an accelerated electron in the laser field. One proposal was given by P.Chen and T.Tajima [12]. Their basic idea is the following.

Their basic idea is the following. An electron is accelerated in the oscillating electro-magnetic field of lasers. It is not a uniform acceleration, but they approximated the electron's motion around the turning points by a uniform acceleration. Since the electron feels the vacuum as thermally excited with the Unruh temperature, the motion of the electron will be thermalized and fluctuate in the transverse directions to the direction of the acceleration (Fig. 3.1). Because of this fluctuating motion of an electron, they conjectured that additional radiation, apart from the classical Larmor radiation, will be emanated. Using an intuitive argument, they estimated the additional radiation (which they called the Unruh radiation). Though the estimated amount of radiation is much smaller than the classical one ($\times 10^{-5}$), the angular dependence is different. Especially in the direction along the acceleration there is a blind spot for the Larmor radiation while the Unruh radiation is radiated more spherically. Hence they proposed to detect the additional radiation in this direction.

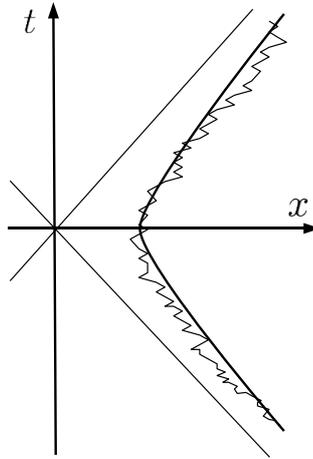


Figure 3.1: Stochastic trajectories induced by quantum field fluctuations.

The above heuristic argument sounds physically correct, but it has been known in a simpler situation that such a radiation is canceled by an interference effect between the radiation field emanated from the fluctuating motion and the quantum fluctuation of the radiation field itself [15, 16]. The cancellation was shown to occur for an internal detector in 1+1 dimensions. Hu and Lin [19] extended the same calculation to 3+1 dimensions and showed that the cancellations. The detail will be briefly reviewed in Section 3.2.

In this chapter, I am going to investigate a stochastic motion of a uniformly accelerated charged particle and to study whether there is additional radiation (the *Unruh radiation*) associated with the stochastic motion of the particle. The situation becomes much more complicated than the internal detector case because the equations of motion are highly nonlinear. When the particle's motion $z(\tau)$ is affected by the vacuum fluctuation, the Green function $G_R(x, z(\tau))$ is also changed accordingly unlike the internal detector case. Hence we need to approximate that fluctuations are small in the transverse directions. We first study a stochastic equation for a charged particle coupled with the scalar field [22]. This gives a simplified model of the real QED. A self-interaction with a scalar field created by the particle itself gives

a backreaction to the particle's motion, and it gives a radiation damping term of the Abraham-Lorentz-Dirac equation [20]. If we further regard the vacuum fluctuation as stochastic noise, the particle's motion obeys a generalized Langevin equation. The equation determines stochastic fluctuations of the particle's momenta. Here, I mainly focus on the scalar QED for simplicity. The generalization to real QED case is straightforward.

The organization of this chapter is following. In section 3.2 I review the Unruh detector, the model and the cancellations of Unruh radiation. In section 3.3, I summarize the basic framework and obtain a generalized Langevin equation for a charged particle coupled with a scalar field. In section 3.4, I consider small fluctuations in the transverse directions. Then the stochastic equation can be solved and one can prove the equipartition theorem for the transverse momenta, i.e. a stochastic average of a square of the momentum fluctuations in the transverse directions is shown to be proportional to the Unruh temperature. I also discuss the relaxation time of the thermalization process. In section 3.5, I show the radiation emitted by a charged particle in the scalar QED. The interference terms partially cancel the radiation coming from the contribution $\langle \phi_{inh}(x)\phi_{inh}(x') \rangle$, but unlike the internal case, they do not cancel exactly. In section 3.6, I obtain a similar stochastic equation for an accelerated charged particle in the real QED, and show the equipartition theorem for transverse momenta.

3.2 Unruh Detector

The Unruh detector is a toy model proposed to investigate Unruh effect. The model is composed by a quantum field and a detector. The detector is a box moving on some specified path. Inside the box there is a quantum state which will interact with the quantum field outside. 3.2 Here I review the cancellation of the Unruh radiation for Unruh detector at 1+1 dimensions and 3+1 dimensions.

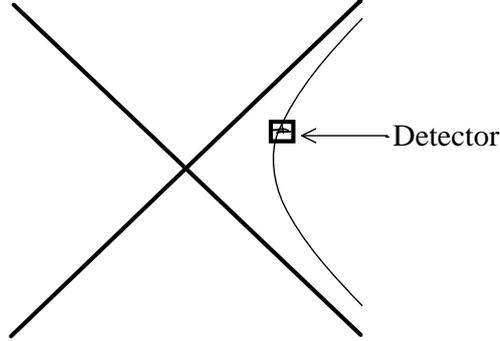


Figure 3.2: Unruh Detector

3.2.1 1+1 Dimensions

The action is given by [15, 16]

$$S = S(Q) + S(\phi) + e \int d\tau \frac{dQ}{d\tau} \phi(\mathbf{z}(\tau)) \quad (3.4)$$

where $\mathbf{z}(\tau) = (t(\tau), x(\tau))$ represents a classical trajectory of the internal detector. $S(Q)$ and $S(\phi)$ are quadratic actions of the internal detector (a harmonic oscillator) and the scalar field in 1+1 dimensions respectively;

$$S(Q) = \int d\tau \left(\frac{1}{2} \dot{Q}(\tau)^2 - \frac{\omega_0^2}{2} Q^2(\tau) \right) \quad (3.5)$$

$$S(\phi) = \int d^2x \frac{1}{2} (\partial\phi(x))^2. \quad (3.6)$$

Since the coupling term is linear both in Q and ϕ , the Heisenberg equations of motion can be exactly solved. The scalar field is written as a sum of the vacuum fluctuation $\phi_h(\mathbf{x})$ (a solution to the homogeneous equation in the absence of Q) and an inhomogeneous term $\phi_{inh}(\mathbf{x})$ as

$$\phi(\mathbf{x}) = \phi_h(\mathbf{x}) + \phi_{inh}(\mathbf{x}), \quad (3.7)$$

where ϕ_{inh} is given by

$$\phi_{inh}(\mathbf{x}) = \int d\tau G_R(\mathbf{x}, \mathbf{z}(\tau)) \frac{dQ}{d\tau} \quad (3.8)$$

and G_R is the retarded Green function of the scalar field. The equation of motion of the internal detector becomes

$$\ddot{Q} + \omega_0^2 Q = -e \frac{d\phi_h}{d\tau} - e \frac{d\phi_{inh}}{d\tau}. \quad (3.9)$$

The inhomogeneous part ϕ_{inh} is solved linearly in $Q(\tau)$ and accordingly the second term of the r.h.s. gives a dissipative term $\gamma \dot{Q}$ for the internal detector where $\gamma = e/2\pi$. It also renormalizes the frequency ω_0 . Hence $Q(\tau)$ can be solved as

$$\tilde{Q}(\omega) = h(\omega)\varphi(\omega) \quad (3.10)$$

where $\tilde{Q}(\omega)$ and $\varphi(\omega)$ are the Fourier transformations of $Q(\tau)$ and $\phi_h(z(\tau))$ with respect to τ , and $h(\omega) \sim i\omega/(\omega^2 - \omega_0^2 - i\omega\gamma)$. By inserting the solution to (3.8), the inhomogeneous solution ϕ_{inh} is written linearly in terms of the vacuum fluctuation $\phi_h(\mathbf{x})$. Then it is straightforward to calculate the energy flux. Since the energy flux is written in terms of the 2-point function, they calculated the 2-point function

$$G(\mathbf{x}, \mathbf{x}') = \langle \phi(\mathbf{x})\phi(\mathbf{x}') \rangle. \quad (3.11)$$

It is written as a sum of the following terms,

$$G(\mathbf{x}, \mathbf{x}') - G_0(\mathbf{x}, \mathbf{x}') = \langle \phi_{inh}(\mathbf{x})\phi_{inh}(\mathbf{x}') \rangle + \langle \phi_{inh}(\mathbf{x})\phi_h(\mathbf{x}') \rangle + \langle \phi_h(\mathbf{x})\phi_{inh}(\mathbf{x}') \rangle \quad (3.12)$$

where an uninteresting vacuum fluctuation $G_0(\mathbf{x}, \mathbf{x}') = \langle \phi_h(\mathbf{x})\phi_h(\mathbf{x}') \rangle$ is subtracted. Since ϕ_{inh} is induced by the internal detector, the first term $\langle \phi_{inh}(\mathbf{x})\phi_{inh}(\mathbf{x}') \rangle$ can be considered as an analog of the *Unruh radiation* proposed in [12]. It is actually nonzero because the internal detector is thermally excited from the classical ground state $Q = 0$. However, Sciamia et.al. [15] and Hu et.al. [16] have shown that the contributions from the interference terms $\langle \phi_{inh}(\mathbf{x})\phi_h(\mathbf{x}') \rangle + \langle \phi_h(\mathbf{x})\phi_{inh}(\mathbf{x}') \rangle$ exactly cancel the radiation $\langle \phi_{inh}(\mathbf{x})\phi_{inh}(\mathbf{x}') \rangle$ in (1+1)-dimensional case.

3.2.2 3+1 Dimensions

Now I am going to see how the *Unruh radiation* is canceled by the interference effects in (3+1) dimensions[19].

The action is given by

$$S = \int d\tau \frac{m}{2} ((\partial_\tau Q(\tau))^2 - \Omega_0^2 Q^2) + \int d^4x \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) + \lambda \int d^4x d\tau Q(\tau) \phi(x) \delta^4(x - z(\tau)), \quad (3.13)$$

where ∂_τ is used to denote a derivative with respect to the proper time τ . The equations of motion are given by

$$\partial^2 \phi(x) = \lambda \int d\tau Q(\tau) \delta^4(x - z(\tau)) \quad (3.14)$$

$$(\partial_\tau^2 + \Omega_0^2)Q(\tau) = \frac{\lambda}{m} \phi(z(\tau)). \quad (3.15)$$

Substituting the solution $\phi = \phi_h + \phi_{inh}$,

$$\phi_{inh}(x) = \lambda \int d\tau Q(\tau) G_R(x - z(\tau)), \quad (3.16)$$

to the equation of the internal detector, we get the following equation,

$$(\partial_\tau^2 + \Omega_0^2)Q(\tau) - \frac{\lambda^2}{m} \int d\tau' Q(\tau') G_R(z(\tau) - z(\tau')) = \frac{\lambda}{m} \phi_h(z(\tau)). \quad (3.17)$$

Here ϕ_h is the homogeneous solution representing the vacuum fluctuations. The inhomogeneous term is evaluated by expanding the Green function with respect to $(\tau - \tau')$ as we did in (3.63). Then after a renormalization of the mass term, we get the diffusive term of the radiation reaction,

$$\int d\tau' Q(\tau') G_R(z(\tau) - z(\tau')) \Rightarrow \frac{Q'(\tau)}{4\pi}. \quad (3.18)$$

The stochastic equation can be solved by the Fourier transformation on the path as

$$\tilde{Q}(\tau) = \lambda h(\omega) \varphi(\omega), \quad (3.19)$$

where

$$h(\omega)^{-1} = -m\omega^2 + m\Omega^2 - i\frac{\omega\lambda^2}{4\pi} \quad (3.20)$$

and the Fourier transformations are defined as

$$\tilde{Q}(\omega) = \int d\tau e^{i\omega\tau} Q(\tau), \quad (3.21)$$

$$\varphi(\omega) = \int d\tau e^{i\omega\tau} \phi_h(z(\tau)). \quad (3.22)$$

Note that $G_R(z(\tau) - z(\tau'))$ is a function of $(\tau - \tau')$ if the classical solution $z(\tau)$ represents the accelerated path (3.48). The 2-point correlation function is decomposed into

$$\langle \phi(x)\phi(y) \rangle = \langle \phi_h(x)\phi_h(y) \rangle + \langle \phi_{inh}(x)\phi_h(y) \rangle + \langle \phi_h(x)\phi_{inh}(y) \rangle + \langle \phi_{inh}(x)\phi_{inh}(y) \rangle \quad (3.23)$$

where

$$\begin{aligned} & \langle \phi_{inh}(x)\phi_h(y) \rangle + \langle \phi_h(x)\phi_{inh}(y) \rangle \\ &= \int d\tau \frac{d\omega}{2\pi} e^{-i\omega\tau} \lambda^2 h(\omega) (G_R(y - z(\tau)) \langle \phi_h(x)\varphi(\omega) \rangle + G_R(x - z(\tau)) \langle \varphi(\omega)\phi_h(y) \rangle) \end{aligned} \quad (3.24)$$

$$\begin{aligned} & \langle \phi_{inh}(x)\phi_{inh}(y) \rangle \\ &= \int d\tau d\tau' \frac{d\omega}{2\pi} \frac{d\omega'}{2\pi} e^{-i(\omega\tau + \omega'\tau')} \lambda^4 G_R(x - z(\tau)) G_R(y - z(\tau')) h(\omega) h(\omega') \langle \varphi(\omega)\varphi(\omega') \rangle. \end{aligned} \quad (3.25)$$

We first evaluate the interference term (3.24);

$$\begin{aligned} \langle \phi_h(x)\varphi(\omega) \rangle &= \int d\tau e^{i\omega\tau} \langle \phi_0(x)\phi_0(z(\tau)) \rangle \\ &= -\frac{1}{4\pi^2} \int d\tau \frac{e^{i\omega\tau}}{(x^0 - z^0(\tau) - i\epsilon)^2 - (x^1 - z^1(\tau))^2 - \rho^2} \\ &= -\frac{1}{4\pi^2} P(x, \omega). \end{aligned} \quad (3.26)$$

Poles of the denominator are given by solving the equation,

$$0 = \left(x^0 - \frac{\sinh a\tau}{a}\right)^2 - \left(x^1 - \frac{\cosh a\tau}{a}\right)^2 - \rho^2 \quad (3.27)$$

$$= -u \frac{e^{a\tau}}{a} + u \frac{e^{-a\tau}}{a} + x^2 - \frac{1}{a^2}. \quad (3.28)$$

The solutions of this equation are classified according to two different types of observers (See Fig.3.3);

\mathcal{O}_F (in future wedge) : $u > 0, v > 0$

$$\Rightarrow e^{a\tau^F} = \frac{a}{2u} \left(-L^2 + \sqrt{L^4 + \frac{4}{a^2}uv}\right) \quad (3.29)$$

$$-e^{a\tau_+^F} = \frac{a}{2u} \left(-L^2 - \sqrt{L^4 + \frac{4}{a^2}uv}\right) \quad (3.30)$$

\mathcal{O}_R (in right wedge) : $u < 0, x^0 + x^1 > 0$

$$\Rightarrow e^{a\tau^R} = \frac{a}{2|u|} \left(L^2 - \sqrt{L^4 - \frac{4}{a^2}|uv|}\right) \quad (3.31)$$

$$e^{a\tau_+^R} = \frac{a}{2|u|} \left(L^2 + \sqrt{L^4 - \frac{4}{a^2}|uv|}\right), \quad (3.32)$$

where, $L^2 = -x^2 + 1/a^2$. The poles at $\tau_-^{F,R}$ correspond to the proper times at the intersections of the particle's world line and the past light cone of the observer's position. Hence they are the physically acceptable poles. On the other hand, τ_+^F correspond to the proper time at a point on a "virtual path" in the left wedge. τ_+^R lies at an intersection of the world line and the future light cone of the observer. Both of them are classically unacceptable.

Summing these contributions to the integral, we obtain

$$P(x, \omega) = \frac{-\pi i}{\rho_0} \frac{1}{e^{2\pi\omega/a} - 1} (e^{i\omega\tau_-} - e^{i\omega\tau_+^x Z_x(\omega)}), \quad (3.33)$$

where

$$Z_x = e^{\pi\omega/a} \theta(u) + \theta(-u), \quad (3.34)$$

$$\rho_0 = \frac{a}{2} \sqrt{L^4 + \frac{4}{a^2}uv}. \quad (3.35)$$

Using the following relation,

$$\int d\tau G_R(x - z(\tau))f(\tau) = \frac{1}{4\pi\rho_0}f(\tau_-), \quad (3.36)$$

a part of the interference term depending on τ_-^R or τ_-^F can be written as

$$\langle \phi_h(x)\phi_{inh}(y) \rangle \rightarrow i\lambda^2 \int d\tau d\tau' \frac{d\omega}{2\pi} G_R(x - z(\tau))G_R(y - z(\tau')) e^{i\omega(\tau - \tau')} \frac{h(\omega)}{e^{2\pi\omega/a} - 1}. \quad (3.37)$$

Similarly, we have

$$\langle \phi_{inh}(x)\phi_h(y) \rangle \rightarrow i\lambda^2 \int d\tau d\tau' \frac{d\omega}{2\pi} G_R(x - z(\tau))G_R(y - z(\tau')) e^{-i\omega(\tau - \tau')} \frac{h(\omega)}{1 - e^{-2\pi\omega/a}}, \quad (3.38)$$

where we have used the identity

$$\langle \tilde{\varphi}(\omega)\phi_h(y) \rangle = (\langle \phi_h(y)\tilde{\varphi}(-\omega) \rangle)^*. \quad (3.39)$$

The correlation function of inhomogeneous terms is given by

$$\begin{aligned} \langle \phi_{inh}(x)\phi_{inh}(y) \rangle &= \lambda^4 \int d\tau d\tau' \frac{d\omega}{2\pi} \frac{d\omega'}{2\pi} e^{-i(\omega\tau + \omega'\tau')} G_R(x - z(\tau))G_R(y - z(\tau')) \\ &\quad \times h(\omega)h(\omega') \langle \tilde{\varphi}(\omega)\tilde{\varphi}(\omega') \rangle \\ &= \lambda^4 \int d\tau d\tau' \frac{d\omega}{2\pi} \frac{d\omega'}{2\pi} e^{-i\omega(\tau - \tau')} G_R(x - z(\tau))G_R(y - z(\tau')) h(\omega)h(-\omega) \\ &\quad \times \int (d\tau_a - \tau_b) 2\pi \delta(\omega + \omega') e^{i\omega(\tau_a - \tau_b)} \langle \phi_0(z(\tau_a))\phi_0(z(\tau_b)) \rangle \\ &= \lambda^4 \int d\tau d\tau' \frac{d\omega}{2\pi} e^{-i\omega(\tau - \tau')} G_R(x - z(\tau))G_R(y - z(\tau')) \frac{\omega}{2\pi} \frac{h(\omega)h(-\omega)}{1 - e^{-2\pi\omega/a}}. \end{aligned} \quad (3.40)$$

These three contributions (3.37), (3.38), (3.40) to the correlation function are shown to be canceled each other because of the relation

$$h(\omega) - h(-\omega) = \frac{i\omega\lambda^2}{2\pi} |h(\omega)|^2. \quad (3.41)$$

Therefore if we neglect the contributions from the classically unacceptable poles at τ^+ the 2-point function vanishes, and therefore there are no energy-momentum flux after the thermalization occurs.

The remaining term in the 2-point function is the contributions of the τ_+ dependent terms to the interference term, and written as

$$\int \frac{d\omega}{2\pi} \frac{-ia^2\lambda^2}{8\pi\rho_0(x)\rho_0(y)} \frac{1}{1 - e^{-2\pi\omega/a}} (h(\omega)e^{-i\omega(\tau_-(x)-\tau_+(y))} Z_y(-\omega) - h(-\omega)e^{-i\omega(\tau_+(x)-\tau_-(y))} Z_x(-\omega)). \quad (3.42)$$

It looks strange why we have such a (classically unacceptable) term in the final result.

3.3 Stochastic Equation of an Accelerated Particle

We consider the scalar QED. The model is analyzed in [22] and here we briefly review the settings and the derivation of the stochastic ALD equations. In [22], the authors have used the Feynman-Vernon formalism or the influence functional approach, but here we take a simplified method. The system composes of a relativistic particle $z^\mu(\tau)$ and the scalar field $\phi(x)$. The action is given by

$$S[z, \phi, h] = S[z, h] + S[\phi] + S[z, \phi], \quad (3.43)$$

with

$$\begin{aligned} S[z, h] &= -m \int d\tau \sqrt{\dot{z}^\mu \dot{z}_\mu}, \\ S[\phi] &= \int d^4x \frac{1}{2} (\partial_\mu \phi)^2, \\ S[z, \phi] &= \int d^4x j(x; z) \phi(x). \end{aligned} \quad (3.44)$$

The scalar current $j(x; z)$ is defined as

$$j(x; z) = e \int d\tau \sqrt{\dot{z}^\mu \dot{z}_\mu} \delta^4(x - z(\tau)), \quad (3.45)$$

where e is negative for an electron. We can parametrize the particle's path satisfying $\dot{z}^2 = 1$ by taking τ properly.

The equation of motion of the particle is given by

$$m\ddot{z}^\mu = F^\mu - \int d^4x \frac{\delta j(x; z)}{\delta z_\mu(\tau)} \phi(x) \quad (3.46)$$

where we have added the external force F^μ so as to accelerate the particle uniformly;

$$F^\mu = ma(\dot{z}^1, \dot{z}^0, 0, 0). \quad (3.47)$$

The external force can be considered as a gradient of an external potential $V(x)$, but it does not matter in the following discussions. Then a classical solution of the particle (in the absence of the coupling to ϕ) is given by

$$z_{\text{cl}}^\mu = \left(\frac{1}{a} \sinh a\tau, \frac{1}{a} \cosh a\tau, 0, 0 \right). \quad (3.48)$$

Note that the external force satisfies $F^\mu \dot{z}_\mu = 0$ and therefore the classical equation of motion preserves the gauge condition $\dot{z}^2 = 1$. From the definition of the current (3.45), it is easy to prove the identity,

$$\int d^4x \frac{\delta j(x; z)}{\delta z^\mu(\tau)} f(x) = e \vec{\omega}_\mu f(x)|_{x=z(\tau)} \quad (3.49)$$

where $\vec{\omega}^\mu$ is given by

$$\vec{\omega}_\mu = \dot{z}^\nu \dot{z}_{[\nu} \partial_{\mu]} - \ddot{z}_\mu. \quad (3.50)$$

Here we have used the gauge condition $\dot{z}^2 = 1$ and $\ddot{z} \cdot \dot{z} = 0$. Hence the equation of motion (3.46) becomes

$$m\ddot{z}^\mu = F^\mu - e \vec{\omega}^\mu \phi(z(\tau)) \quad (3.51)$$

Since the differential operator $\vec{\omega}_\mu$ satisfies $\vec{\omega}_\mu \dot{z}^\mu = 0$ for a classical path satisfying the gauge condition, the stochastic equation (3.51) continues to preserve the condition $\dot{z}^2 = 1$. The second term of (3.51) represents a self-interaction of the particle with the radiation emitted by the particle itself.

The equation of motion of the radiation field $\partial^\mu \partial_\mu \phi(x) = j(x)$ is solved by using the retarded Green function G_R as

$$\phi(x) = \phi_h(x) + \phi_{inh}, \quad \phi_{inh} = \int d^4x' G_R(x, x') j(x'; z) \quad (3.52)$$

where ϕ_h is the homogeneous solution of the equation of motion and represents the vacuum fluctuation. It is responsible for the particle's fluctuating motion under a uniform acceleration. The retarded Green function satisfies

$$\partial^\mu \partial_\mu G_R(x, x') = \delta^{(4)}(x - x') \quad (3.53)$$

and is given by

$$G_R(x, x') = i \langle [\phi(x), \phi(x')] \rangle \theta(t - t') = \frac{\theta(t - t') \delta((x - x')^2)}{2\pi} = \frac{\delta((t - t') - r)}{4\pi r} \quad (3.54)$$

where $r^2 = |\mathbf{x} - \mathbf{x}'|^2$. Inserting the solution (3.52) into (3.51), we have the following stochastic equation for the particle

$$m \ddot{z}^\mu(\tau) = F^\mu(z(\tau)) - e \vec{\omega}^\mu \left(\phi_h(z(\tau)) + e \int d\tau' G_R(z(\tau), z(\tau')) \right). \quad (3.55)$$

Here we have used the gauge condition $\dot{z}^2 = 1$. The operator $\vec{\omega}_\mu$ acts on $z(\tau)$. The homogeneous part $\phi_h(z(\tau))$ of the scalar field describes Gaussian fluctuations of the vacuum, and hence the first term in the parenthesis can be interpreted as random noise to the particle's motion. Expanding ϕ_h as

$$\phi_h(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} (a_k e^{-ik^\mu x_\mu} + a_k^\dagger e^{ik^\mu x_\mu}), \quad (3.56)$$

the vacuum fluctuation is given by

$$\langle \phi_h(x) \phi_h(y) \rangle = -\frac{1}{4\pi^2} \frac{1}{(t - t' - i\epsilon)^2 - r^2}. \quad (3.57)$$

It is essentially quantum mechanical, but if it is evaluated on a world line of a uniformly accelerated particle $x = z(\tau)$, $y = z(\tau')$, it behaves as the ordinary finite temperature noise.

The second term in the parenthesis of (3.55) is a functional of the total history of the particle's motion $z(\tau')$ for $\tau' \leq \tau$, but it can be reduced to the so called radiation damping term of a charged particle coupled with a radiation field. It is generally nonlocal, but since the Green function damps rapidly as a function of the distance r , the term is approximated by local derivative terms. First define $y^\mu(s) = z^\mu(\tau) - z^\mu(\tau')$ where $\tau' = \tau - s$ with τ kept fixed. Then it can be expanded as

$$y^\mu(s) = s\dot{z}^\mu(\tau) - \frac{s^2}{2}\ddot{z}^\mu(\tau) + \frac{s^3}{6}\dddot{z}^\mu(\tau) + \dots \quad (3.58)$$

A square of the space-time distance σ is given by

$$\sigma(s) \equiv y^\mu y_\mu = s^2(1 - \frac{s^2}{12}(\ddot{z})^2 + \dots) \quad (3.59)$$

and

$$\frac{d\sigma(s)}{ds} = 2y^\mu \dot{y}_\mu = 2s(1 - \frac{s^2}{6}(\ddot{z})^2 + \dots). \quad (3.60)$$

In deriving them, we have used the gauge condition $(\dot{z})^2 = 1$, $\dot{z} \cdot \ddot{z} = 0$ and $\dot{z} \cdot \ddot{z} = -(\ddot{z})^2$. The derivative ∂_μ appearing in the operator $\vec{\omega}_\mu$ can be written in terms of $\frac{d}{ds}$, when it acts on a function of σ , as

$$\begin{aligned} \partial_\mu &= \frac{\partial \sigma}{\partial z^\mu} \frac{d}{d\sigma} = 2y_\mu \left(\frac{d\sigma}{ds}\right)^{-1} \frac{d}{ds} = \frac{y_\mu}{y^\nu \dot{y}_\nu} \frac{d}{ds} \\ &= \left(\dot{z}_\mu - \frac{s}{2}\ddot{z}_\mu + \frac{s^2}{6}(\ddot{z}_\mu + \dot{z}_\mu(\ddot{z})^2) + \dots\right) \frac{d}{ds}. \end{aligned} \quad (3.61)$$

Hence the backreaction part of the stochastic equation can be simplified as

$$\begin{aligned} e^2 \vec{\omega}_\mu \int_{-\infty}^{\tau} G_R(z(\tau), z(\tau')) d\tau' &= ds \vec{\omega}_\mu G_R(s) \\ &= e^2 \int_0^\infty ds \{a_\mu(\tau) G^R(s) + a_\mu(\tau) \frac{s}{2} \frac{d}{ds} G^R(s) \\ &\quad + (\dot{z}_\mu \ddot{z}^2 + \ddot{z}_\mu) \frac{s^2}{6} \frac{d}{ds} G^R(s) + \mathcal{O}(s^3)\}. \end{aligned} \quad (3.62)$$

In the first equality, we have neglected the singular term proportional to $\delta(\sigma)$. The first two terms can be absorbed by a mass renormalization. The last one is the radiation reaction. Since the retarded Green function is given by (3.54), the mass renormalization is divergent. The radiation reaction term can be evaluated by using the identity

$$\int_0^\infty ds s^2 \frac{d}{ds} G^R(s) = \int_0^\infty ds s^2 \frac{d}{ds} \frac{\delta(s)}{4\pi s} = -\frac{1}{2\pi} \quad (3.63)$$

After the mass renormalization, we get the following generalized Langevin equation for the charged particle,

$$m\ddot{z}^\mu - F^\mu - \frac{e^2}{12\pi}(\dot{z}^\mu \dot{z}^\mu + \ddot{z}^\mu) = -e\vec{\omega}^\mu \phi_h(z). \quad (3.64)$$

This is an analog of the Abraham-Lorentz-Dirac equation for a charged particle interacting with the electromagnetic field. The dissipation term is induced by an effect of the backreaction of the particle's radiation to the particle's motion. Note that, if the noise term is absent, the classical solution (3.48) with a constant acceleration is still a solution to the equation (3.64).

3.4 Thermalization of Transverse Momentum Fluctuations

The stochastic equation (3.64) is nonlinear and difficult to solve. Here we consider small fluctuations around the classical trajectory induced by the vacuum fluctuation ϕ_h . Especially we consider fluctuations in the transverse directions. First we expand the particle's motion around the classical trajectory z_0^μ as

$$z^\mu(\tau) = z_0^\mu + \delta z^\mu. \quad (3.65)$$

The particle is accelerated along the x direction. In the following we consider small fluctuation in transverse directions. By expanding the stochastic equation (3.64), we can obtain a linearized stochastic equation for the transverse

velocity fluctuation $\delta v^i \equiv \delta \dot{z}^i$ as,

$$m\delta \dot{v}^i = e\partial_i\phi_h + \frac{e^2}{12\pi}(\delta \ddot{v}^i - a^2\delta v^i). \quad (3.66)$$

Performing the Fourier transformation with respect to the trajectory's parameter τ

$$\delta v^i(\tau) = \int \frac{d\omega}{2\pi} \delta \tilde{v}^i(\omega) e^{-i\omega\tau}, \quad \partial_i\phi_h(\tau) = \int \frac{d\omega}{2\pi} \partial_i\varphi(\omega) e^{-i\omega\tau} \quad (3.67)$$

the stochastic equation can be solved as

$$\delta \tilde{v}^i(\omega) = eh(\omega)\partial_i\varphi(\omega). \quad (3.68)$$

where

$$h(\omega) = \frac{1}{-im\omega + \frac{e^2(\omega^2+a^2)}{12\pi}}. \quad (3.69)$$

The vacuum 2-point function along the classical trajectory can be evaluated from (3.57) as

$$\begin{aligned} \langle \partial_i\phi_h(x)\partial_j\phi_h(x') \rangle|_{x=z(\tau),x'=z(\tau')} &= \frac{1}{2\pi^2} \frac{\delta_{ij}}{((t-t'-i\epsilon)^2 - r^2)^2} \\ &= \frac{a^4}{32\pi^2} \frac{\delta_{ij}}{\sinh^4\left(\frac{a(\tau-\tau'-i\epsilon)}{2}\right)}. \end{aligned} \quad (3.70)$$

It has originated from the quantum fluctuations of the vacuum, but it can be interpreted as finite temperature noise if it is evaluated on the accelerated particle's trajectory [5] *. Its Fourier transformation is evaluated as

$$\langle \partial_i\varphi(\omega)\partial_j\varphi(\omega') \rangle = 2\pi\delta(\omega + \omega')\delta_{ij}I(\omega) \quad (3.71)$$

*The finite temperature (Unruh) effect is caused by the appearance of the horizon for a uniformly accelerated observer in the Minkowski space-time and analogous to the Hawking radiation of the black hole, but you should not confuse the radiation we are discussing in this paper with the Hawking radiation. The accelerated observer sees the Minkowski vacuum as thermally excited, but it is excited from the Rindler vacuum (not from the Minkowski vacuum) and the energy momentum tensor remains zero as ever. The radiation discussed in the paper is, if exists, produced by an interaction with the vacuum and the accelerated charged particle.

where

$$I(\omega) = \frac{a^4}{32\pi^2} \int_{-\infty}^{\infty} d\tau_- \frac{e^{i\omega\tau_-}}{\sinh^4\left(\frac{a(\tau_- - i\epsilon)}{2}\right)} = \frac{1}{6\pi} \frac{\omega^3 + \omega a^2}{1 - e^{-2\pi\omega/a}}. \quad (3.72)$$

By symmetrizing it, i.e., $\langle \partial\phi(x)\partial\phi(x') \rangle_S = \langle \{\partial\phi(x), \partial\phi(x')\} \rangle / 2$, the Wightman Green function becomes

$$I_S(\omega) = \frac{1}{12\pi} \coth\left(\frac{\pi\omega}{a}\right) (\omega^3 + \omega a^2), \quad (3.73)$$

which is an even function of ω . The correlator $I(\omega)$ or $I_S(\omega)$ should be UV cut-off for large ω , or for short proper time difference, where quantum field theoretic effects of electron become important. Full QED treatment is necessary there.

For small ω , this is expanded as

$$I(\omega) = \frac{a}{12\pi^2} (a^2 + a\pi\omega + \dots). \quad (3.74)$$

The expansion corresponds to the derivative expansion

$$\langle \partial_i \phi_h(x) \partial_j \phi_h(x') \rangle |_{x=z(\tau), x'=z(\tau')} = \frac{a^3}{12\pi^2} \delta_{ij} \delta(\tau - \tau') - i \frac{a^2}{12\pi} \delta_{ij} \delta'(\tau - \tau') \quad (3.75)$$

We approximate the 2-point function by the first term, which corresponds to the white noise approximation. The coefficient determines the strength of the noise. We show that it is consistent with the fluctuation-dissipation theorem at the Unruh temperature.

The expectation value of the square of the transverse velocity fluctuation can be evaluated as

$$\begin{aligned} \langle \delta v^i(\tau) \delta v^j(\tau') \rangle_S &= e^2 \int \frac{d\omega d\omega'}{(2\pi)^2} \langle \partial_i \varphi(\omega) \partial_j \varphi(\omega') \rangle_S h(\omega) h(\omega') e^{-i(\omega\tau + \omega'\tau')} \\ &= e^2 \delta_{ij} \int \frac{d\omega}{2\pi} I_S(\omega) |h(\omega)|^2 e^{-i\omega(\tau - \tau')} \\ &\sim e^2 \delta_{ij} \int \frac{d\omega}{24\pi^3} \frac{a^3}{(m\omega)^2 + \left(\frac{e^2}{12\pi}\right) (\omega^2 + a^2)^2} e^{-i\omega(\tau - \tau')} \quad (3.76) \end{aligned}$$

The denominator has four poles at $\omega = \pm i\Omega_{\pm}$ where

$$\Omega_+ = \frac{12\pi m}{e^2}(1 + \mathcal{O}(a^2/m^2)), \quad \Omega_- = \frac{a^2 e^2}{12\pi m}(1 + \mathcal{O}(a^2/m^2)). \quad (3.77)$$

The acceleration of an electron in high-intensity laser fields in near future can be at most 0.1 eV and much smaller than the electron mass 0.5 MeV. Hence, the values of these poles satisfy the following inequalities,

$$\Omega_+ \gg a \gg \Omega_-. \quad (3.78)$$

Since the energy scale of the dynamics of the accelerated particle is much smaller than the electron mass, the poles at $\pm i\Omega_+$ should be considered spurious and we should not take the contributions of the residues at $\pm i\Omega_+$ [†]. By taking the residue at $\pm\omega = i\Omega_-$, we can evaluate the integral and get the following result,

$$\frac{m}{2} \langle \delta v^i(\tau) \delta v^j(\tau) \rangle = \frac{1}{2} \frac{a\hbar}{2\pi c} \delta_{ij} (1 + \mathcal{O}(a^2/m^2)). \quad (3.79)$$

Here we have recovered c and \hbar . This gives the equipartition relation for the transverse momentum fluctuations in the Unruh temperature $T_U = a\hbar/2\pi c$.

The thermalization process of the stochastic equation (3.66) can be also discussed. For simplicity, we approximate the stochastic equation by dropping the second derivative term. Then it is solved as

$$\delta v^i(\tau) = e^{-\Omega_-\tau} \delta v^i(0) + \frac{e}{m} \int_0^\tau d\tau' \partial_i \phi(z(\tau')) e^{-\Omega_-(\tau-\tau')}. \quad (3.80)$$

The relaxation time is given by $\tau_R = 1/\Omega_-$. The momentum square can be also calculated as

$$\begin{aligned} \langle \delta v^i(\tau) \delta v^j(\tau) \rangle &= e^{-2\Omega_-\tau} \delta v^i(0) \delta v^j(0) \\ &+ e^2 \int_0^\tau d\tau' \int_0^\tau d\tau'' e^{-\Omega_-(\tau-\tau')} e^{-\Omega_-(\tau-\tau'')} \langle \partial_i \phi(z(\tau')) \partial_j \phi(z(\tau'')) \rangle \\ &= e^{-2\Omega_-\tau} \delta v^i(0) \delta v^j(0) + \frac{a\delta_{ij}}{2\pi m} (1 - e^{-2\Omega_-\tau}). \end{aligned} \quad (3.81)$$

[†]Or we can simply approximate the denominator by dropping the ω^4 term. Then only the poles at $\pm i\Omega_-$ survive.

For $\tau \rightarrow \infty$, it approaches the thermalized average (3.79). The relaxation time in the proper time can be estimated, for $a = 0.1$ eV and $m = 0.5$ MeV, to be

$$\tau_R = \frac{12\pi m}{a^2 e^2} = 5 \times 10^{-5} \text{sec.} \quad (3.82)$$

This relaxation time should be compared with the laser frequency. The planned wavelength of the laser at ELI is around 10^3 nm and the oscillation period of the laser field is very short; 3×10^{-15} seconds. Hence the relaxation time is much longer and the charged particle does not thermalize during each oscillation. Hence an assumption of the uniform acceleration in the laser field is not good. Even in such a situation, if the electron is accelerated in the laser field for a long time, an electron may feel an averaged temperature.

The position of the particle in the transverse directions also fluctuates like the ordinary Brownian motion in a heat bath. The mean square of the transverse coordinate is given by

$$R^2(\tau) = \sum_{i=y,z} \langle (z^i(\tau) - z^i(0))^2 \rangle = 2D \left(\tau - \frac{3 - 4e^{-\Omega_- \tau} + e^{-2\Omega_- \tau}}{2\Omega_-} \right). \quad (3.83)$$

The diffusion constant D is given by

$$D = \frac{2T_U}{\Omega_- m} = \frac{12}{ae^2}, \quad (3.84)$$

which is estimated for the above parameters as $D \sim 6 \text{ m}^2/\text{s}$. In the Ballistic region where $\tau < \tau_R$, the mean square becomes

$$R^2(\tau) = \frac{2T_U}{m} \tau^2 \quad (3.85)$$

while in the diffusive region ($\tau > \tau_R$), it is proportional to the proper time as

$$R^2(\tau) = 2D\tau. \quad (3.86)$$

As the ordinary Brownian motion, the mean square of the particle's transverse position grows linearly with time. If it becomes possible to accelerate the particle for a sufficiently long period, it may be possible to detect such a Brownian motion in future laser experiments.

3.5 Quantum Radiation by Transverse Fluctuation

Once we obtain the stochastic motion of the accelerated particle, it is straightforward to calculate the energy flux of the radiation field emitted by this particle. In this section, we calculate the radiation induced by the fluctuation in the transverse direction. First we evaluate the two point function

$$G(x, x') - G_0(x, x') = \langle \phi_{inh}(x)\phi_{inh}(x') \rangle + \langle \phi_{inh}(x)\phi_h(x') \rangle + \langle \phi_h(x)\phi_{inh}(x') \rangle + \langle \phi_h(x)\phi_h(x') \rangle \quad (3.87)$$

The inhomogeneous part ϕ_{inh} is a direct consequence of the presence of an accelerated charged particle while the homogeneous part ϕ_h is the vacuum fluctuation of the quantum field ϕ . The Unruh radiation estimated in [12] corresponds to calculating the 2-point correlation function of the inhomogeneous terms $\langle \phi_{inh}(x)\phi_{inh}(y) \rangle$. (As we will see later, the same term also contains the classical Larmor radiation.) However, this is not the end of the story. As it has been discussed in [15], the interference terms $\langle \phi_{inh}\phi_h \rangle + \langle \phi_h\phi_{inh} \rangle$ cannot be neglected and may possibly cancel the Unruh radiation in $\langle \phi_{inh}\phi_{inh} \rangle$ after the thermalization occurs. This is shown for an internal detector in (1+1) dimensions, but it is not obvious whether the same cancellation occurs for the case of a charged particle we are considering.

The inhomogeneous solution of the scalar field is written as

$$\phi_{inh}(x) = e \int d\tau G_R(x - z(\tau)) = e \int d\tau \frac{\theta(t - z^0(\tau))\delta((x - z(\tau))^2)}{2\pi} = \frac{e}{4\pi\rho}. \quad (3.88)$$

where ρ is defined by

$$\rho = \dot{z}(\tau_-^x) \cdot (x - z(\tau_-^x)). \quad (3.89)$$

Because of the step and the delta functions in the integrand of (3.88), τ_-^x satisfies

$$(x - z(\tau_-^x))^2 = 0, \quad x^0 > z^0(\tau_-^x), \quad (3.90)$$

which is the proper time of the particle whose radiation travels to the space-time point x . Hence $z(\tau_-^x)$ lies on an intersection between the particle's world line and the light cone extending from the observer's position x (See Fig 2 below). We write the superscript x to make the x dependence of τ explicitly. The meaning of the subscript $(-)$ will be made clear later. By using the light cone condition, ρ can be also written as

$$\rho(x) = \frac{dz^0(\tau_-^x)}{d\tau} r(\tau_-^x) \left(1 - \frac{\mathbf{v} \cdot \mathbf{r}}{r}\right) \quad (3.91)$$

where $\mathbf{v} = \frac{d\mathbf{z}}{dz^0}$, $\mathbf{r}(\tau_-^x) = \mathbf{x} - \mathbf{z}(\tau_-^x)$ and $r = |\mathbf{r}|$. It is the spacial distance for the observer moving with the particle.

The particle's trajectory is fluctuating and expressed as $z = z_0 + \delta z + \delta^2 z + \dots$ where we have expanded the fluctuation with respect to the interaction with the radiation field (i.e. e). Then ρ is also expanded as $\rho = \rho_0 + \delta\rho + \delta^2\rho + \dots$ and (3.88) becomes

$$\phi_{inh} = \frac{e}{4\pi\rho_0} \left(1 - \frac{\delta\rho}{\rho_0} + \left(\frac{\delta\rho}{\rho_0}\right)^2 - \frac{\delta^2\rho}{\rho_0} + \dots\right). \quad (3.92)$$

The first term is the classical potential, but since the particle's trajectory deviates from the classical one, the potential also receives corrections. Here ρ_0 , $\delta\rho$ and $\delta^2\rho$ are given by

$$\rho_0 = \dot{z}_0(\tau_-^x) \cdot (x - z_0(\tau_-^x)) \quad (3.93)$$

$$\delta\rho = \delta\dot{z}(\tau_-^x) \cdot (x - z_0(\tau_-^x)) - \dot{z}_0(\tau_-^x) \cdot \delta z(\tau_-^x), \quad (3.94)$$

$$\delta^2\rho = \delta^2\dot{z}(\tau_-^x) \cdot (x - z_0(\tau_-^x)) - \delta\dot{z}(\tau_-^x) \cdot \delta z(\tau_-^x) - \dot{z}_0(\tau_-^x) \cdot \delta^2 z(\tau_-^x). \quad (3.95)$$

From now on we only consider the transverse fluctuations. Then $\dot{z}_0 \cdot \delta z = 0$ is satisfied. In the above, we neglected the change of τ_-^x since it corresponds to the longitudinal fluctuation. As seen from (3.91), ρ is proportional to the spacial distance from the particle to the observer. The variation of ρ becomes negligible for large distance r if we take a variation of $(x - z_0(\tau))$ in ρ . On

the contrary, if we take a variation of \dot{z}_0 , $\delta\rho$ or $\delta^2\rho$ is still proportional to the spacial distance r . Hence for large r , we can approximate the variations by

$$\delta\rho \sim \delta\dot{z}(\tau_-^x) \cdot (x - z_0(\tau_-^x)), \quad \delta^2\rho = \delta^2\dot{z}(\tau_-^x) \cdot (x - z_0(\tau_-^x)). \quad (3.96)$$

Note also $\langle\delta^2\dot{z}^i\rangle = 0$ since the velocity in the transverse directions fluctuates uniformly and its expectation value vanishes.

Let us calculate the 2-point function. If we take the classical part without the fluctuation of ρ , the 2-point function becomes

$$G(x, x') - G_0(x, x') \rightarrow \langle\phi_{inh}(x)\phi_{inh}(x')\rangle = \left(\frac{e}{4\pi}\right)^2 \frac{1}{\rho_0(x)\rho_0(y)}. \quad (3.97)$$

This gives the classical radiation corresponding to the Larmor radiation. The interference term vanishes because 1-point function vanishes identically $\langle\phi_h\rangle = 0$.

Corrections to the classical Larmor radiation are induced by the transverse fluctuating motion $\delta\rho$. First we consider 2-pt function between the inhomogeneous part up to the second order of the transverse fluctuations. Since $\langle\delta^2\rho\rangle = 0$, we have

$$\begin{aligned} \langle\phi_{inh}(x)\phi_{inh}(y)\rangle &= \left(\frac{e}{4\pi}\right)^2 \left\langle \frac{1}{\rho(x)\rho(y)} \right\rangle \\ &= \left(\frac{e}{4\pi}\right)^2 \frac{1}{\rho_0(x)\rho_0(y)} \left(1 + \frac{\langle\delta\rho(x)\delta\rho(y)\rangle}{\rho_0(x)\rho_0(y)} + \frac{\langle(\delta\rho(x))^2\rangle}{\rho_0^2(x)} + \frac{\langle(\delta\rho(y))^2\rangle}{\rho_0^2(y)} \right). \end{aligned} \quad (3.98)$$

Note that all the terms in the parenthesis behave constantly as the distance r between the observer and the particle becomes large. The first term gives the Larmor radiation mentioned above. The other terms correspond to the radiation induced by the fluctuations. The calculation of them is easy, because one can write $\langle\delta\rho\delta\rho\rangle$ in terms of $\langle\delta\dot{z}^i\delta\dot{z}^i\rangle = \langle\delta v^i\delta v^i\rangle$ which we have already

evaluated in the previous section. With the expression (3.76), it becomes

$$\begin{aligned} \langle \phi_{inh}(x)\phi_{inh}(y) \rangle &= \left(\frac{e}{4\pi}\right)^2 \frac{1}{\rho_0(x)\rho_0(y)} \left[1 + e^2 \int \frac{d\omega}{2\pi} |h(\omega)|^2 I(\omega) \right. \\ &\quad \left. \times \left(\frac{x^i y^i e^{-i\omega(\tau_-^x - \tau_-^y)}}{\rho_0(x)\rho_0(y)} + \frac{x^i x^i}{\rho_0(x)\rho_0(x)} + \frac{y^i y^i}{\rho_0(y)\rho_0(y)} \right) \right]. \end{aligned} \quad (3.99)$$

As before, since we are considering the fluctuating motion whose frequency is smaller than the acceleration, we may as well approximate $I(\omega)$ by $a^3/12\pi^2$. If we calculate the symmetrized correlation function between x and y , $I(\omega)$ is replaced by $I_S(\omega)$.

Next let us calculate the interference terms. They are rewritten as

$$\langle \phi_{inh}(x)\phi_h(y) \rangle + \langle \phi_h(x)\phi_{inh}(y) \rangle = -\frac{e}{4\pi} \left(\frac{\langle \delta\rho(x)\phi_h(y) \rangle}{\rho_0^2(x)} + \frac{\langle \phi_h(x)\delta\rho(y) \rangle}{\rho_0^2(y)} \right). \quad (3.100)$$

Calculation of the interference terms are more complicated since we need to evaluate the following correlation functions;

$$\begin{aligned} \langle \delta\rho(x)\phi_h(y) \rangle &= -x^i \langle \delta z^i(\tau_-^x)\phi_h(y) \rangle = -ex^i \int \frac{d\omega}{2\pi} e^{-i\omega\tau_-^x} h(\omega) \langle \partial_i \varphi(\omega)\phi_h(y) \rangle \\ \langle \phi_h(x)\delta\rho(y) \rangle &= -y^i \langle \phi_h(x)\delta z^i(\tau_-^y) \rangle = -ey^i \int \frac{d\omega}{2\pi} e^{-i\omega\tau_-^y} h(\omega) \langle \phi_h(x)\partial_i \varphi(\omega) \rangle. \end{aligned} \quad (3.101)$$

Since two terms $\langle \partial_i \varphi(\omega)\phi_h(y) \rangle$ and $\langle \phi_h(x)\partial_i \varphi(\omega) \rangle$ are related by

$$\langle \partial_i \varphi(\omega)\phi_h(y) \rangle = (\langle \phi_h(y)\partial_i \varphi(-\omega) \rangle)^*, \quad (3.102)$$

it is sufficient to calculate one of them. From the definition of φ in (3.67),

the interference term $\langle \phi_h(x) \partial_i \varphi(\omega) \rangle$ is written as

$$\begin{aligned}
 \langle \phi_h(x) \partial_i \varphi(\omega) \rangle &= \int d\tau e^{i\omega\tau} \left(\frac{\partial}{\partial y^i} \langle \phi_h(x) \phi_h(y) \rangle \right)_{y=z(\tau)} \\
 &= -\frac{1}{4\pi^2} \int d\tau e^{i\omega\tau} \left(\frac{\partial}{\partial y^i} \frac{1}{(x^0 - y^0 - i\epsilon)^2 - (\vec{x} - \vec{y})^2} \right)_{y=z(\tau)} \\
 &= \frac{1}{4\pi^2} \frac{\partial}{\partial x^i} \int d\tau \frac{e^{i\omega\tau}}{(x^0 - z^0(\tau) - i\epsilon)^2 - (x^1 - z^1(\tau))^2 - x_\perp^2},
 \end{aligned} \tag{3.103}$$

where $x_\perp^2 = (x^2)^2 + (x^3)^2$ is the transverse distance. We first evaluate the integral and then take the derivative. The integral

$$P(x, \omega) \equiv \int d\tau \frac{e^{i\omega\tau}}{(x^0 - z^0(\tau) - i\epsilon)^2 - (x^1 - z^1(\tau))^2 - x_\perp^2}, \tag{3.104}$$

can be evaluated by the contour integral in the complex τ plane. The positions of the pole are given by a series of points

$$\tau_\pm^n = T_\pm + \frac{2n\pi i}{a} - i\epsilon, \tag{3.105}$$

where n is an integer. T_\pm are complex numbers whose imaginary parts are 0 or π/a and satisfy

$$e^{aT_\pm} = \frac{a}{2u} \left(-L^2 \mp \sqrt{L^4 + \frac{4}{a^2} uv} \right). \tag{3.106}$$

Here we have defined

$$L^2 = -x^\mu x_\mu + \frac{1}{a^2}, \tag{3.107}$$

$$u = x^0 - x^1, \quad v = x^0 + x^1. \tag{3.108}$$

Note the relation $e^{aT_+} e^{aT_-} = -v/u$. The positions of the poles reflect the finite temperature property of the uniformly accelerated observer. In the following we will consider two different types of observers as shown in Fig.3.3. The first observer is to observe the radiation in the right wedge (O_R) while the

second one is in the future wedge (O_F). For both cases, $v > 0$ is satisfied and the radiation can travel causally from the particle to the observers. There are two different types of poles τ_{\pm} . A pole at $\tau_- = T_-$, which is real, is located at a classically acceptable point. Namely, τ_- is the proper time of the particle whose radiation travels to the observer in a causal way. The other pole at T_+ is more subtle. For $u < 0$ (in the right wedge), $T_+ = \tau_+^R$ is real and corresponds to the advanced causal proper time. For $u > 0$ (in the future wedge), $T_+ = \tau_+^F + i\pi/a$ has an imaginary part and one can interpret it as the proper time of a trajectory of a virtual particle in the left wedge, as in Fig. 3.3. In the following, we drop the superscript F or R . In the region where $v < 0$, ϕ_{inh} does not exist and no nontrivial correlation is observed there.

The residue of the pole at τ_{\pm}^n is given by $-e^{i\omega\tau_{\pm}^n}/2\rho(\tau_{\pm}^n)$ where

$$\rho(\tau_{\pm}^n) = \dot{z}(\tau_{\pm}^n) \cdot (x - z_0(\tau_{\pm}^n)) = \frac{1}{2}(ue^{a\tau_{\pm}^n} + ve^{-a\tau_{\pm}^n}). \quad (3.109)$$

Because of the periodicity, $\rho(\tau_{\pm}^n)$ is independent of n . The integral is now given by

$$P(x, \omega) = \frac{-\pi i}{\rho_0} \frac{1}{e^{2\pi\omega/a} - 1} (e^{i\omega\tau_-^x} - e^{i\omega\tau_+^x} Z_x(\omega)), \quad (3.110)$$

where

$$Z_x(\omega) = e^{\pi\omega/a}\theta(u) + \theta(-u) \quad (3.111)$$

$\rho_0 = \rho(\tau_{\pm}^n)$ can be rewritten in terms of L^2 as

$$\rho_0 = \frac{a}{2} \sqrt{L^4 + \frac{4}{a^2}uv}. \quad (3.112)$$

Note that the relation $\rho(\tau_+^n) = -\rho_0$ follows the identity $e^{aT_+}e^{aT_-} = -v/u$.

The second term of the parenthesis in (3.110) depends on τ^+ . With naive intuition based on classical causality, the term may be removed by hand, but the calculation of the interference terms is essentially quantum mechanical,

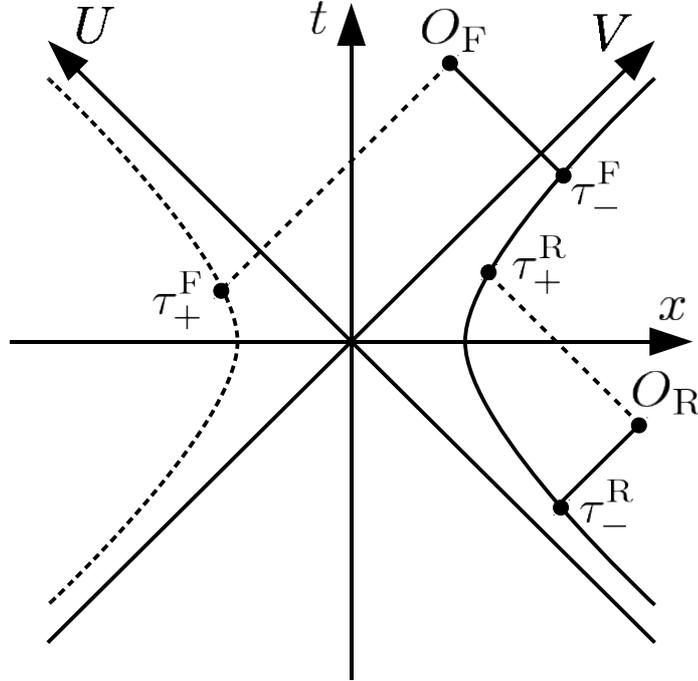


Figure 3.3: The hyperbolic line in the right wedge denotes the world line of the particle. The points O_F and O_R are observers in the future and right wedges, respectively. For an observer in the right wedge, the light-cone of the observer has two intersections with the world line, and the proper time of the intersections are given by τ_{\pm}^R . For an observer in the future wedge, there is only one intersection on the particle's real trajectory which corresponds to τ_-^F . The other solution $T_+^F = \tau_+^F + i\pi/a$ is complex. One may interpret this complex proper time as the intersection between the light-cone of the observer and the world line of a virtual particle with a real proper time τ_+^F in the left wedge. The superscript letters R or F are used to distinguish two different observers, but we do not use them in the body of the paper to leave the space for the observer's position x .

and it should not be neglected. It is puzzling how we can physically interpret such τ^+ dependence of the integral.

Taking a derivation of $P(x, \omega)$, we obtain $\langle \phi_h(x) \partial_i \varphi(\omega) \rangle$ as

$$\langle \phi_h(x) \partial_i \varphi(\omega) \rangle = \frac{iax^i}{4\pi\rho_0^2} \frac{1}{e^{2\pi\omega/a} - 1} \left(\left(\frac{aL^2}{2\rho_0} + \frac{i\omega}{a} \right) e^{i\omega\tau^-} + \left(-\frac{aL^2}{2\rho_0} + \frac{i\omega}{a} \right) e^{i\omega\tau^+} Z_x(\omega) \right). \quad (3.113)$$

Here we have used the following identities,

$$\frac{\partial \rho_0}{\partial x^i} = \frac{a^2 L^2}{2\rho_0} x^i, \quad \frac{\partial \tau_{\pm}^x}{\partial x^i} = \pm \frac{1}{\rho_0} x^i, \quad (3.114)$$

where i is the transverse direction. The second identity can be obtained by differentiating $(x - z(\tau_{\pm}^x))^2 = 0$ with respect to x^i . (See (3.123) below.)

The whole interference terms are now given by

$$\begin{aligned} & \langle \phi_h(x) \phi_{inh}(y) \rangle + \langle \phi_{inh}(x) \phi_h(y) \rangle \\ &= \frac{-iae^2 x^i y^i}{(4\pi)^2 \rho_0(x)^2 \rho(y)^2} \int \frac{d\omega}{2\pi} \frac{1}{1 - e^{-2\pi\omega/a}} \left[e^{-i\omega(\tau_-^x - \tau_-^y)} \left(h(-\omega) \left(\frac{aL_x^2}{2\rho_0(x)} - \frac{i\omega}{a} \right) - h(\omega) \left(\frac{aL_y^2}{2\rho_0(y)} + \frac{i\omega}{a} \right) \right) \right. \\ & \left. + e^{-i\omega(\tau_+^x - \tau_+^y)} h(-\omega) \left(-\frac{aL_x^2}{2\rho_0(x)} - \frac{i\omega}{a} \right) Z_x(-\omega) - e^{-i\omega(\tau_-^x - \tau_+^y)} h(\omega) \left(-\frac{aL_y^2}{2\rho_0(y)} + \frac{i\omega}{a} \right) Z_y(-\omega) \right]. \end{aligned} \quad (3.115)$$

In the following, in order to see whether there is a cancellation between the interference terms and the correlation function of the inhomogeneous terms, we study the first term in the parenthesis of (3.115) which depends only on τ_- . (Note that the correlation function of the inhomogeneous terms (3.99) depends only on τ_- .) Using the relation,

$$h(\omega) + h(-\omega) = \frac{e^2}{6\pi} (\omega^2 + a^2) |h(\omega)|^2, \quad (3.116)$$

one can show that a part of the interference terms

$$\begin{aligned}
 & \frac{ia e^2 x^i y^i}{(4\pi)^2 \rho_0(x)^2 \rho_0(y)^2} \int \frac{d\omega}{2\pi} \frac{1}{1 - e^{-2\pi\omega/a}} e^{-i\omega(\tau_x^+ - \tau_y^-)} \left(h(-\omega) \frac{i\omega}{a} + h(\omega) \frac{i\omega}{a} \right) \\
 &= - \left(\frac{e}{4\pi} \right)^2 \frac{x^i y^i}{\rho_0(x)^2 \rho_0(y)^2} \int \frac{d\omega}{2\pi} \frac{1}{1 - e^{-2\pi\omega/a}} e^{-i\omega(\tau_x^+ - \tau_y^-)} \frac{e^2}{6\pi} |h(\omega)|^2 (\omega^2 + a^2) \omega \\
 &= - \left(\frac{e}{4\pi} \right)^2 \frac{\langle \delta\rho(x) \delta\rho(y) \rangle}{\rho_0(x) \rho_0(y)} \tag{3.117}
 \end{aligned}$$

cancels the first correction term of the inhomogeneous part in (3.99). This term was obtained by taking a derivative of $e^{i\omega\tau_x^+}$ in $P(x, \omega)$.

Summing up both contributions, (3.99) and (3.115), we get the following result of the 2-point function;

$$\langle \phi(x) \phi(y) \rangle - \langle \phi_h(x) \phi_h(y) \rangle = \frac{e^2}{(4\pi)^2 \rho_0(x) \rho_0(y)} F(x, y) \tag{3.118}$$

where

$$\begin{aligned}
 F(x, y) &= 1 + e^2 \int \frac{d\omega}{2\pi} \frac{|h(\omega)|^2}{6\pi} I(\omega) \left(\left(\frac{x^i}{\rho_0(x)} \right)^2 + \left(\frac{y^i}{\rho_0(y)} \right)^2 \right) \\
 &- \frac{ia^2 x^i y^i}{\rho_0(x) \rho_0(y)} \int \frac{d\omega}{4\pi} \frac{1}{1 - e^{-2\pi\omega/a}} \left[e^{-i\omega(\tau_x^+ - \tau_y^-)} \left(h(-\omega) \frac{L_x^2}{\rho_0(x)} - h(\omega) \frac{L_y^2}{\rho_0(y)} \right) \right. \\
 &\left. - e^{-i\omega(\tau_x^+ - \tau_y^+)} h(-\omega) \left(\frac{L_x^2}{\rho_0(x)} + i \frac{2\omega}{a^2} \right) Z_x(-\omega) - e^{-i\omega(\tau_x^+ - \tau_y^+)} h(\omega) \left(-\frac{L_y^2}{\rho_0(y)} + i \frac{2\omega}{a^2} \right) Z_y(-\omega) \right]. \tag{3.119}
 \end{aligned}$$

The first term in F is the classical effect of radiation corresponding to the Larmor radiation. The second term comes from the inhomogeneous term $\langle (\delta\rho(x)/\rho_0(x))^2 \rangle + \langle (\delta\rho(y)/\rho_0(y))^2 \rangle$. The third term comes from the interference term, which is obtained by taking a derivative of $\rho(x)$ in $P(x, \omega)$. The fourth term is also an interference effect and depends on τ_+ .

Let us compare the above result with the calculation for an internal detector. In the case of an internal detector in (1+1) dimensions, there are no terms depending on τ_+ . All the contributions to radiation (derived from the 2-point correlation function) are canceled. In the case of an internal detector

in (3+1) dimensions, there are τ_+ dependent terms. But if we neglect these terms, it was shown [22] that the interference terms completely cancel the radiation. (The calculation is reviewed in the appendix.) On the contrary, in the case of a charged particle, since the position of the particle is fluctuating, only a part of terms are canceled even if we neglect the τ^+ dependent terms. Hence there is a possibility to detect additional radiation besides the classical Larmor radiation. In the following we neglect the τ_+ dependent terms because the oscillating function $e^{-i\omega\tau}$ remains even after setting $x = y$ and suppresses the ω integral.

In calculating the symmetrized 2-point function, $F(x, y)$ is replaced by $F_S(x, y)$

$$\begin{aligned}
 F_S(x, y) = & 1 + e^2 \int \frac{d\omega}{2\pi} \frac{|h(\omega)|^2}{6\pi} I_S(\omega) \left(\left(\frac{x^i}{\rho_0(x)} \right)^2 + \left(\frac{y^i}{\rho_0(y)} \right)^2 \right) \\
 & - \frac{ia^2 x^i y^i}{\rho_0(x) \rho_0(y)} \int \frac{d\omega}{8\pi} \coth\left(\frac{\pi\omega}{a}\right) e^{-i\omega(\tau_x^+ - \tau_y^+)} \left(h(-\omega) \frac{L_x^2}{\rho_0(x)} - h(\omega) \frac{L_y^2}{\rho_0(y)} \right) \\
 & + \tau_+\text{-dependent terms.}
 \end{aligned} \tag{3.120}$$

In the remainder of this section we consider the radiation emitted by the accelerated particle. The energy momentum tensor of the scalar field is given by

$$\langle T_{\mu\nu} \rangle = \langle : \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial^\alpha \phi \partial_\alpha \phi : \rangle_S. \tag{3.121}$$

Hence we can evaluate it by taking a derivative of the 2-point function (3.120).

The following relations are useful in taking derivatives;

$$\begin{aligned}
 \partial_\mu \rho_0 &= (\ddot{z}_0 \cdot (x - z_0) - 1) \partial_\mu \tau_- + \dot{z}_{0\mu} = -\frac{a^2 L^2}{2} \partial_\mu \tau_- + \dot{z}_{0\mu} \\
 \partial_\mu \tau_- &= \frac{x_\mu - z_{0\mu}}{\rho_0}
 \end{aligned} \tag{3.122}$$

In the last line of the first equation, we used the explicit form of the classical trajectory (3.48) and $\ddot{z}_0 \cdot x = -a^2 L^2 / 2$. The derivative $\partial_\mu \tau_-$ was obtained by

taking a variation of the light-cone condition $(x - z_0(\tau_-^x))^2 = 0$ as

$$2(x_\nu - z_{0\nu})(\delta x^\nu - \dot{z}_0^\nu \delta \tau_-^x) = 0 \longrightarrow \frac{\delta \tau_-^x}{\delta x^\nu} = \frac{x_\nu - z_{0\nu}}{\rho_0}. \quad (3.123)$$

In particular, u and v derivatives are given by

$$\begin{aligned} \partial_u \tau_- &= \frac{v - v_z}{2\rho_0}, & \partial_v \tau_- &= \frac{u - u_z}{2\rho_0} \\ \partial_u \rho_0 &= -\frac{a^2 L^2}{2} \partial_u \tau_- + \frac{av_z}{2}, & \partial_v \rho_0 &= -\frac{a^2 L^2}{2} \partial_v \tau_- + \frac{1}{2av_z} \end{aligned} \quad (3.124)$$

where $u_z = -e^{-a\tau_-}/a$, $v_z = e^{a\tau_-}/a$. From (3.122), we have $(\partial\rho_0)^2 = a^2 x^2$. Since $(x - z(\tau))^2 = 0$, $x^2 \sim \mathcal{O}(r)$ and $(\partial\rho_0)^2$ is approximately proportional to the spacial distance r , not r^2 . On the other hand, since $L^2 = -x_\mu^2 + 1/a^2$ is $\mathcal{O}(r)$, $\partial_\mu \rho_0$ itself is growing as $\mathcal{O}(r)$.

First we calculate the classical part of the energy momentum tensor. It becomes

$$T_{\text{cl},\mu\nu} = \frac{e^2(\partial_\mu \rho_0 \partial_\nu \rho_0 - \frac{g_{\mu\nu}}{2} \partial_\alpha \rho_0 \partial^\alpha \rho_0)}{(4\pi)^2 \rho_0^4} \sim \frac{e^2 \partial_\mu \rho_0 \partial_\nu \rho_0}{(4\pi)^2 \rho_0^4}. \quad (3.125)$$

Note that $\partial_\alpha \rho_0 \partial^\alpha \rho_0$ does not make a contribution here, since it is the order of ρ_0 at the infinity while $\partial_\mu \rho_0 \partial_\nu \rho_0$ is in general ρ_0^2 . This part of the energy momentum tensor corresponds to the classical Larmor radiation and behaves as $1/\rho_0^2 \sim 1/r^2$ at infinity. The term $\dot{z}_{0\mu}(\tau_-^x)$ in $\partial_\mu \rho_0$ seems to be negligible, since it is $\mathcal{O}(1)$ while $\partial_\mu \rho_0$ is $\mathcal{O}(r)$. However, a care should be taken of because $\dot{z}_{0\mu}(\tau_-^x) = (\cosh a\tau_-^x, \sinh a\tau_-^x, 0, 0)$ behaves singularly if the observer is located near the horizon.

Next we evaluate the other parts of the energy momentum tensor. We especially consider the (u, u) and (v, v) -components in the following. From (3.120), extra terms of the energy momentum tensor besides the classical

ones are given by

$$\begin{aligned}
 T_{\text{fluc},\mu\nu} = \frac{(x^i)^2}{\rho_0^2} & \left[\left(\frac{e^2}{\pi} I_m - \frac{6ma^2 I_1 L^2}{\rho_0} \right) T_{\text{cl},\mu\nu} - \frac{e^2 a^2 L^2}{(4\pi)^2 \rho_0^3} \left(m I_3 \partial_\mu \tau_-^x \partial_\nu \tau_-^x \right. \right. \\
 & + \frac{2m I_1}{\rho_0 L^2} (x_\mu \partial_\nu \rho_0 + x_\nu \partial_\mu \rho_0) + \frac{e^2 I_m}{12\pi L^2} (x_\mu \partial_\nu \tau_-^x + x_\nu \partial_\mu \tau_-^x) \\
 & \left. \left. - \frac{e^2 I_m}{24\pi \rho_0} (\partial_\mu \tau_-^x \partial_\nu \rho_0 + \partial_\nu \tau_-^x \partial_\mu \rho_0) \right) \right] \quad (3.126)
 \end{aligned}$$

where we have defined the following ω integrals

$$\begin{aligned}
 I_1 &= \int \frac{d\omega}{4\pi} |h(\omega)|^2 \coth\left(\frac{\pi\omega}{a}\right) \omega, \\
 I_3 &= \int \frac{d\omega}{4\pi} |h(\omega)|^2 \coth\left(\frac{\pi\omega}{a}\right) \omega^3, \\
 I_m &= \int \frac{d\omega}{4\pi} |h(\omega)|^2 \coth\left(\frac{\pi\omega}{a}\right) (\omega^3 + a^2\omega) = I_3 + a^2 I_1. \quad (3.127)
 \end{aligned}$$

These integrals can be similarly evaluated as in section 3, and we have

$$I_1 = \frac{3}{2mae^2}, \quad I_m \sim a^2 I_1. \quad (3.128)$$

Because of the inequality $\Omega_- \ll a$, terms containing I_3 are generally negligible compared to other terms; $I_3 \sim \Omega_-^2 I_1 \ll a^2 I_1$.

Near the past horizon, the $v \rightarrow 0$, the u -derivatives of ρ_0 and τ_-^x become very small and negligible. On the other hand, v -derivative of τ_- becomes potentially large. u -derivatives of them are approximately given by

$$\partial_v \tau_-^x \rightarrow \frac{1}{av}, \quad \partial_v \rho_0 \rightarrow -\frac{au}{2}. \quad (3.129)$$

A singular term of $\partial_v \rho_0$ near $v \sim 0$ is canceled and it remains finite near the past horizon. Hence the second term in (3.126) proportional to $(\partial \tau_-)^2$ may becomes large there. However, there are two reasons that the term cannot grow so large. One is a suppression by the ω integral, which is proportional to a very small coefficient I_3 . The other reason is the overall factor $(x^i)^2/\rho_0^2$. Since the observer is much further than the acceleration scale $1/a$ from the particle, L^2 is much larger than $1/a^2$. Then $\rho_0 = (a/2)\sqrt{L^4 + (4/a^2)uv}$ can

be approximated by $\rho_0 \sim (a/2)|x_\mu|^2$ and $(x^i)^2/\rho_0^2$ is also suppressed. Because of these two reasons, the singular behavior near the past horizon seems to be difficult to be observed experimentally.

3.6 Thermalization in Electromagnetic Field

In this section, we consider the thermalization of an accelerated charged particle in the realistic electromagnetic field (QED). Calculations of the energy momentum tensors are more involved and left for a future investigation. We study the thermalization of the transverse momenta of a uniformly accelerated particle in an electromagnetic field. The calculation is almost the same, but due to the presence of the polarization, several quantities become twice as large as those in the scalar case.

The action is given by

$$S_{EM} = -m \int d\tau \sqrt{\dot{z}^\mu \dot{z}_\mu} - \int d^4x j^\mu(x) A_\mu(x) - \frac{1}{4} \int d^4x F^{\mu\nu} F_{\mu\nu}, \quad (3.130)$$

where the current is defined as

$$j^\mu(x) = e \int d\tau \dot{z}^\mu(\tau) \delta^4(x - z(\tau)). \quad (3.131)$$

The equations of motion are

$$\begin{aligned} m\ddot{z}_\mu &= eF_{\mu\nu}\dot{z}^\nu \\ \partial_\mu F^{\mu\nu}(x) &= j^\nu. \end{aligned} \quad (3.132)$$

Using the gauge

$$\partial^\mu A_\mu = 0, \quad (3.133)$$

the equation of motion for A_μ becomes

$$\partial^\mu \partial_\mu A^\nu = j^\nu. \quad (3.134)$$

One can solve this equation as

$$\begin{aligned} A_\mu &= A_{h\mu} + \int d^4y G_R(x, y) j_\mu(y) \\ &= A_{h\mu} + e \int d\tau G_R(x, z(\tau)) \dot{z}_\mu(\tau), \end{aligned} \quad (3.135)$$

where $A_{h\mu}$ is the homogeneous part of the equation of motion which satisfies $\partial^2 A_h^\mu = 0$. $G_R(x - y)$ is the retarded Green function

$$G_R(x, y) = \theta(x^0 - y^0) \frac{\delta((x - y)^2)}{2\pi}, \quad \partial^2 G_R(x, y) = \delta^4(x - y). \quad (3.136)$$

Inserting the solution of $A_\mu(x)$ back to the equation of motion for z^μ , we obtain the following stochastic equation

$$\begin{aligned} m\ddot{z}_\mu(\tau) &= F_\mu + e\{\partial_\mu A_{h\nu}(z) - \partial_\nu A_{h\mu}(z)\}\dot{z}^\nu \\ &\quad + e^2 \int d\tau' \dot{z}^\nu(\tau') \{\dot{z}_\nu(\tau') \partial_\mu - \dot{z}_\mu(\tau') \partial_\nu\} G_R(z(\tau), z(\tau')) \end{aligned} \quad (3.137)$$

The second line is the radiation reaction which can be treated similarly to the scalar case. It becomes

$$\begin{aligned} &e^2 \int d\tau' \{\dot{z}^\nu \dot{z}_{[\nu} \partial_{\mu]} - \frac{s^2}{2} (\ddot{z}^2 \partial_\mu - \ddot{z}_\mu \dot{z}^\nu \partial_\nu)\} \frac{d}{ds} G_R(z(\tau), z(\tau')) \\ &= -e^2 \int_{-\infty}^{\infty} ds \frac{s^2}{3} \{\ddot{z}_\mu(\tau) + \dot{z}_\mu(\tau) \ddot{z}^2(\tau)\} \frac{d}{ds} \frac{\delta(s^2)}{2\pi} \\ &= \frac{e^2}{6\pi} \{\ddot{z}_\mu + \dot{z}_\mu \ddot{z}^2\}, \end{aligned} \quad (3.138)$$

which has exactly the same form as the scalar case, but the coefficient is twice as large since the gauge field have two different polarizations. This is the Abraham-Lorentz-Dirac self-force term.

For the transverse momentum fluctuations $\delta v^i \equiv \delta \dot{z}^i$, we can similarly simplify the stochastic equation and solve it, by the Fourier transformation, as a function of the homogeneous part of the gauge field as

$$\delta \tilde{v}^i(\omega) = -eh(\omega)(v_{0\alpha} \partial_i + \delta_\alpha^i (v_0 \cdot \partial)) A_h^\alpha, \quad (3.139)$$

where

$$h(\omega) = \frac{1}{-im\omega + \frac{e^2}{6\pi}(\omega^2 + a^2)}. \quad (3.140)$$

The relaxation time is also twice as large as the scalar case. The noise correlation of A_h^μ in the rhs of (3.139) can be evaluated as

$$(v_{0\alpha}\partial_i + \delta_\alpha^i(v_0 \cdot \partial))(v'_{0\beta}\partial'_j + \delta_\beta^j(v'_0 \cdot \partial'))\langle A_h^\alpha(z)A_h^\beta(z')\rangle = \frac{a^4}{16\pi^2} \frac{\delta_{ij}}{\sinh^4\left(\frac{a(\tau-\tau'-i\epsilon)}{2}\right)}. \quad (3.141)$$

which is also twice as large as the scalar case. Note that the quantity is gauge invariant

$$(\dot{z}_\alpha k_\mu - \eta_{\alpha\mu}(\dot{z} \cdot k))(\dot{z}'_\beta k'_\nu - \eta_{\beta\nu}(\dot{z}' \cdot k'))k^\alpha k^\beta = 0. \quad (3.142)$$

Hence following the same calculations in the scalar case, the fluctuations of the transverse momentum becomes

$$\frac{m}{2}\langle \delta v^i(\tau)\delta v^j(\tau)\rangle = \frac{1}{2} \frac{a\hbar}{2\pi c} \delta_{ij} \left(1 + \mathcal{O}\left(\frac{a^2}{m^2}\right) \right). \quad (3.143)$$

The relaxation time is twice as large as the scalar case.

3.7 Summary

In this chapter, we studied a stochastic motion of a uniformly accelerated charged particle in the scalar QED. The particle's motion fluctuates because of the thermal behavior of the uniformly accelerated observer (the Unruh effect). Because of this fluctuating motion, Chen and Tajima [12] conjectured that there is additional radiation besides the classical Larmor radiation. On the other hand, it was argued [15, 16] that interferences between the radiation field induced by the fluctuating motion and the quantum fluctuation of the vacuum may cancel the above additional radiation. The cancellation was shown in the case of an internal detector, but it was not yet settled whether

the same kind of cancellation occurs in the case of a fluctuating charged particle in QED.

In order to investigate the above issue systematically, we first formulated a motion of a uniformly accelerated particle in terms of the stochastic (Langevin) equation. By using this formalism, we showed that the momenta in the transverse directions actually get thermalized so as to satisfy the equipartition relation with the Unruh temperature. Then we calculated correlation functions and energy flux from the accelerated particle. Partial cancellation is actually shown to occur, but some terms still remain. Hence there is still a possibility that, besides the classical Larmor radiation, we can detect additional radiation associated with the fluctuating motion caused by the Unruh effect.

There are several issues to be clarified. First in calculating the energy flux at infinity there appeared classically unacceptable contributions (i.e. those depend on τ_+). If the observer is in the right wedge, the contribution to the energy flux come from the particle in the future of the observer. In the case of the observer in the future wedge, this contribution comes from the virtual particle in the left wedge. Both of them are classically unacceptable, and we do not yet have physical understanding why these contributions appear in the calculation.

Another issue is the calculation of longitudinal fluctuations. Since the particle is moving at a relativistic speed in the longitudinal direction, such small fluctuations caused by the Unruh effect seem to be difficult to be separated from the classical motion. Even the meaning of the thermalization is unclear because once the particle fluctuates in the longitudinal direction it is kinematically unstable.

Chapter 4

Fluctuation Theorem and Black Hole

As already reviewed in Chapter 2, the black hole physics are related to the thermodynamics. And the Hawking radiation causes the information problem. The information problem says that in the systems involve black hole evaporation, one may not be able to know the initial state only from the information of final state. People think this may be because that the black hole are just some thermodynamic description, which are the descriptions after some kind of coarse graining of some microscopic theory. We would consider this point from the view of thermodynamics.

From statistic mechanics, one can calculate various thermodynamic quantities from a microscopic theory. The microscopic theory we start from are time reversible but the thermodynamics are generally not time reversible (the process with increasing entropy). The information is lost here. One interesting explanation is that the coarse graining is responsible for this. However, this problem is still not solved completely. There is a recent development of non-equilibrium statistic physics called the Jarzynski equality. The Jarzynski equality itself is an equality which can be derived from quantum mechanics, however, from this equality one can obtain a in-

equality which have the same form to the second law of thermodynamics. So we expect that the Jarzynski equality may shed new lights on this problem. And here, we would like to apply the thought of this Jarzynski equality to the black hole physics.

The organization of this chapter is following. First I review the Jarzynski equality and fluctuation theorem briefly. Then after some preparation of the correction transition rates of the Hawking radiation, I will apply the fluctuation theorem to black hole and derive the generalized second law of black hole as a result.

4.1 Jarzynski Equality and Fluctuation Theorem

4.1.1 Jarzynski Equality

Consider a system with some parameter λ and change this parameter with time, denote by $\lambda(t)$. This is, for example corresponds to change the position of a cylinder, or to change some potential of the system. The operation does not have to be quasi-static. Now let the system starting from a thermal equilibrium state, and under the operation $\lambda(t)$ the system went to some final state which does not have to be thermal equilibrium. Then the Jarzynski equality says that the work done by the external force which caused the change of the parameter should obey

$$\langle e^{-\beta W} \rangle = e^{-\beta \Delta F}. \quad (4.1)$$

Here W is work done to the system during the whole process. Microscopically it is defined by $W = E_f - E_i$, with E_i (E_f) is the energy of the initial (final) state. And ΔF is the difference of the free energy between the initial and the final state, $\Delta F = F_i - F_f$, with F_i is the free energy of the initial state. Note that the thermal equilibrium state will not stay at thermal equi-

librium during general operation. And here the final state is not at thermal equilibrium. So generally, F_f can not be the free energy of the final state. Here F_f is defined by the free energy of the thermal equilibrium state at the temperature of the initial state, and with the parameter $\lambda(t_f)$. This equality was proved in various systems. The average in the left hand side are taken by the probability of the microscopic process, the exact definition are depend on the system one consider.

One can confirm equation (4.1) using quantum mechanics. Assuming that the system is controlled by a Hamiltonian $H_{\lambda(t)}$, which depends on the parameter $\lambda(t)$ and changing with time. Denote $\hat{\rho}_0$ for the density matrix of the initial state,

$$\hat{\rho}_0 = \frac{e^{-\beta\hat{H}_{\lambda(0)}}}{\text{Tr}(e^{-\beta\hat{H}_{\lambda(0)}})} \quad (4.2)$$

then the probability of the external work to be W are given by

$$P(W) \equiv \sum_{E_i, E_f} |\langle E_f | \hat{U}(t, 0) | E_i \rangle|^2 \langle E_i | \hat{\rho}_0 | E_i \rangle \delta(W - E_f + E_i). \quad (4.3)$$

Then $\langle e^{-\beta W} \rangle$ is

$$\begin{aligned} \langle e^{-\beta W} \rangle &\equiv \sum_W P(W) e^{-\beta W} \\ &= \sum_{E_i, E_f} |\langle E_f | \hat{U}(t, 0) | E_i \rangle|^2 \langle E_i | \frac{e^{-\beta\hat{H}_{\lambda(0)}}}{\text{Tr}(e^{-\beta\hat{H}_{\lambda(0)}})} | E_i \rangle e^{-\beta(E_f - E_i)} \\ &= e^{\beta F_i} \sum_{E_i, E_f} \langle E_f | \hat{U}(t, 0) | E_i \rangle \langle E_i | \hat{U}^\dagger(t, 0) | E_f \rangle e^{-\beta E_f} \\ &= e^{\beta F_i} \sum_{E_f} e^{-\beta E_f} \\ &= e^{-\beta \Delta F}. \end{aligned} \quad (4.4)$$

With equation (4.1), using the Jensen's inequality

$$\langle e^x \rangle \geq e^{\langle x \rangle} \quad (4.5)$$

one obtain

$$W - \Delta F \geq 0 \quad \rightarrow \quad \Delta S \geq 0 \quad (4.6)$$

which is just the second law of thermodynamics. I have to note a fact that what the second law of thermodynamics says is about two states both at thermal equilibrium. And here what we considered is starting from one thermal equilibrium state to another general state. To complete the settings one have to put the final state to some thermal bath and wait it to go to the thermal equilibrium. But this process generally may not be controlled by a Hamiltonian, since it involves another systems. So the second law of thermodynamics can not be proved only with this discussions, one need other assumptions to complete this proof. However I believe that this Jarzynski equality catches some important informations on our problem.

4.1.2 Fluctuation Theorem

The Jarzynski equality is closely related to the fluctuation theorem. The fluctuation theorem was discovered earlier, and one can derive the Jarzynski equality from the fluctuation theorem.

The non-equilibrium fluctuation theorem was first discovered in [24]. There are several variations of the theorem. Here I am going to introduce the Crooks fluctuation theorem [25].

The setting is same to the previous session. Assuming that the system is initially in thermal equilibrium at inverse temperature β with the external parameter $\lambda^F(0)$. Then changing the external parameter $\lambda^F(t)$ as a function of time from $t = 0$ to $t = T$. The procedure of changing the parameter corresponds, for example, to a process of moving a piston and it needs not to be quasi-static.

The process of changing the external parameter $\lambda^F(t)$ is called a forward protocol, this is what the index F refers for. We also consider a reversed protocol which defined as changing the external parameter in a reversed way

as $\lambda^R(t) \equiv \lambda^F(T-t)$ from $t = 0$ to $t = T$. In the reversed protocol, the system is assumed to be initially in thermal equilibrium at the same temperature, but with a different external parameter $\lambda^R(0) = \lambda^F(T)$.

In changing the external parameter, the system becomes out-of-equilibrium. For each microscopic state, one does some measurements on the system, and takes an ensemble average over the initial density matrix. For general microscopic states $|a_f\rangle$ and $|a_i\rangle$, define a function $K^F(a_f, a_i)$

$$K(a_f, a_i) = K^F(a_f, a_i) = \ln \frac{P^F[a_f, a_i]}{P^R[a_i, a_f]} \quad (4.7)$$

and $K^R(a_i, a_f)$

$$K^R(a_i, a_f) = \ln \frac{P^R[a_i, a_f]}{P^F[a_f, a_i]} = -K(a_f, a_i). \quad (4.8)$$

Where $P^F[a_f, a_i]$ and $P^R[a_i, a_f]$ are the probability of the transition from $|a_i\rangle$ to $|a_f\rangle$ in the forward protocol and from $|a_f\rangle$ to $|a_i\rangle$ in the reversed protocol

$$\begin{aligned} P^F[a_f, a_i] &= \sum_{a_i} |\langle a_f | \hat{U}(t, 0) | a_i \rangle|^2 \langle a_i | \hat{\rho}_0^F | a_i \rangle \\ P^R[a_i, a_f] &= \sum_{a_f} |\langle a_f | \hat{U}(t, 0) | a_i \rangle|^2 \langle a_f | \hat{\rho}_0^R | a_f \rangle \end{aligned} \quad (4.9)$$

As in (4.3), define $P^F(K)$ and $P^R(K)$ as the probability of measurement K in the forward and reversed protocol respectively

$$\begin{aligned} P^F(K) &\equiv \sum_{a_i, a_f} P^F[a_f, a_i] \delta(K - K(a_f, a_i)) \\ P^R(K) &\equiv \sum_{a_i, a_f} P^R[a_i, a_f] \delta(K - K^R(a_i, a_f)). \end{aligned} \quad (4.10)$$

Using equation (4.7) and (4.8)

$$P^R(K) = \sum_{a_i, a_f} e^{-K(a_i, a_f)} P^F(a_f, a_i) \delta(K + K(a_f, a_i)) = e^K P^F(-K). \quad (4.11)$$

Replace K to $-K$ one obtains the fluctuation theorem

$$\frac{P^F(K)}{P^R(-K)} = e^K. \quad (4.12)$$

To obtain the probability of the exerted work W under the change of parameter $\lambda^F(t)$. One only need to choose the state $|a_i\rangle$ and $|a_f\rangle$ to the energy eigen state $|E_i\rangle$ and $|E_f\rangle$. Then $K(E_f, E_i)$ becomes

$$K(E_f, E_i) = \ln \frac{\langle E_i | \hat{\rho}_0^F | E_f \rangle}{\langle E_f | \hat{\rho}_0^R | E_f \rangle} = \beta[(E_f - E_i) - (F_f - F_i)], \quad (4.13)$$

since F_f and F_i are state independent, $P(K) = P(W)$. Then

$$\frac{P^F(W)}{P^R(-W)} = e^{\beta(W - \Delta F)}, \quad (4.14)$$

this equation states that the ratio of these two probabilities is given in terms of the work and the difference of free energies $F(\lambda)$ between the two equilibrium states. The Jarzynski equality [26] can be obtained by summing over W

$$\langle e^{-\beta(W - \Delta F)} \rangle = \sum_W e^{-\beta(W - \Delta F)} P^F(W) = \sum_W P^R(-W) = 1 \quad (4.15)$$

can be obtained. Here, the angled bracket stands for the average with the probability $\rho^F(W)$. It is surprising since the average of exponentiated work in non-equilibrium processes in the left hand side is related to the difference of equilibrium quantities at the beginnings of the protocols. By using the Jensen's inequality, the Jarzynski relation is reduced to

$$\langle (W - \Delta F) \rangle \geq 0, \quad (4.16)$$

which implies the second law of thermodynamics. Note that since e^x are always smaller than 1 for negative x , though the average of $W - \Delta F$ is always non-negative, to satisfy the Jarzynski equality (4.15), there must be a nonzero probability for the quantity to take a negative value microscopically.

4.2 Application to Black Holes

Now we are going to apply the fluctuation theorem and Jarzynski equality to the black hole physics.

4.2.1 Transition Rates

We consider a coupled system of a black hole and matter. The external parameter $\lambda(t)$ characterizing the system Hamiltonian, which appeared in the fluctuation theorem, is, for example, height or shape of the potential for the matter field. If the whole system is controlled by a unitary time evolution with time-reversal symmetry, a transition probability $\mathcal{W}_{\lambda^F(t)}(\mathcal{C} \rightarrow \mathcal{C}')$ from one configuration \mathcal{C} to another \mathcal{C}' under a time evolution of the external parameter $\lambda^F(t)$ is equal to a probability $\mathcal{W}_{\lambda^R(t)}(\mathcal{C}' \rightarrow \mathcal{C})$ from \mathcal{C}' to \mathcal{C} under the reversed change of the parameter $\lambda^R(t)$. In the presence of a black hole, however, the time-reversal symmetry is violated by imposing the ingoing boundary condition at the horizon. In a black hole space-time, regular coordinates near the horizon, i.e. Kruskal coordinates (U, V) , are defined by $U = -\kappa^{-1}e^{-\kappa(t-r_*)}$ and $V = \kappa^{-1}e^{\kappa(t+r_*)}$. Here, t and r_* are the Schwarzschild-time and the tortoise coordinates. Quantum fields near the horizon are classified into two types of chiral fields (in a two-dimensional sense on (t, r_*) plane), one depending on U and the other on V . Fields depending on V are ingoing waves falling into the black hole while those depending on U are propagating nearly along the horizon and correspond to outgoing modes. The regularity at the horizon requires occupation of outgoing modes $\phi(\omega) \sim e^{-i\omega U}$ to vanish at the (future) horizon. Namely, we must impose the vacuum condition for the outgoing modes in the Kruskal coordinates. On the contrary, there is no constraint for the ingoing modes, and the conditions are asymmetric between U and V . The time-reversal transformation $t \rightarrow -t$ exchanges the coordinates U and V , and the presence of horizon violates the time-reversal symmetry of the quantum states. Therefore, the above transition probabilities are not necessary the same.

The ratio of the above transition probabilities was evaluated by Massar and Parentani [28] for Hawking radiation processes. They have shown that the transition rates for systems with a black hole horizon are governed by changes in the horizon area. In the present case, the ratio of transition

probabilities between a configuration \mathcal{C} with black hole area A and another one \mathcal{C}' with area A' under a fixed value of external parameter λ is given by

$$\frac{\mathcal{W}_\lambda(\mathcal{C}(A) \rightarrow \mathcal{C}'(A'))}{\mathcal{W}_\lambda(\mathcal{C}'(A') \rightarrow \mathcal{C}(A))} = \exp(\Delta A/4G\hbar), \quad (4.17)$$

where the change of area $\Delta A = A' - A$ is assumed to be small. In deriving this, they used the WKB approximation for the system wave function, and calculated the transition rates in the first Born approximation for the interaction between the detector and radiation field. A similar result was obtained in a different way in [29]. If we identify ΔA as the energy ΔE emanated from the black hole by $\Delta A/4G\hbar = -\Delta E/T_H$, it becomes the Boltzmann factor $\exp(-\Delta E/T_H)$ of the Hawking radiation. The ratio eq.(4.17) takes into account the back reaction of the radiation to the black hole area. If the detailed balance condition is satisfied in the processes, the ratio eq.(4.17) is identified as the ratio of a probability in the configuration $\mathcal{C}'(A')$ to that in $\mathcal{C}(A)$, and hence consistent with the entropy of the black hole $S_{BH} = A/4G\hbar$.

In proving (4.17), they have used an observation by Carlip and Teitelboim [30] that, if we consider a coupled system of a black hole and matter exchanging energy between them, one needs to add a boundary term to the bulk action, $S = S_{bulk} + A\Theta/8\pi G$. Here, $\Theta = \kappa t$ for on-shell and stationary metrics. Then in quantizing the system, the Wheeler-DeWitt equation $\hat{H}_{tot}\Psi = 0$ in the bulk must be supplemented by the boundary Schrödinger equation [30] $i\hbar\partial_\Theta\Psi = -(\hat{A}/8\pi G)\Psi$, and the total system's wave function evolves as $\exp(iA\Theta/8\pi G\hbar)$ in the WKB approximation.

The ratio (4.17) is valid also for processes including classical absorption of energy into the black hole, if we generalize the notion of transition probabilities in the following way. The matter system outside the horizon dissipates the energy by transferring it into the black hole. Furthermore, the matter system feels thermal noise due to the Hawking radiation. Hence, by including both effects of the heat transfer, in and out, at the horizon, the effective equation of motion for matter is controlled by a stochastic Langevin equa-

tion with dissipation and noise terms. In such a situation, one can define a probability distribution of the system to take some configuration. Time evolution of the probability distribution function is described by the Fokker-Planck equation. Clearly the time reversal symmetry is violated, and there is an asymmetry between the probabilities of the forward and the reversed processes. The ratio is evaluated in general Langevin processes in [27]. By applying it to our case, the energy transfer into the black hole can be rewritten as the area change of the black hole through the first law of black hole thermodynamics ($\Delta S_{BH} = \Delta E/T_{BH}$). Hence the probability ratio (4.17) is valid for wider situations including classical absorption of energy into the black hole.

4.2.2 Non-equilibrium Fluctuations of Horizons

We consider a sequence of configurations of a coupled system of a black hole and matter, and denote it as $\Gamma = \{\mathcal{C}_0(A_0), \mathcal{C}_1(A_1), \dots, \mathcal{C}_M(A_M)\}$. The configuration \mathcal{C}_k is realized at a discretized time $t = k\Delta t$. Each transition probability is given by $\mathcal{W}_{\lambda^F(t_k)}(\mathcal{C}_k(A_k) \rightarrow \mathcal{C}_{k+1}(A_{k+1}))$. If we assume the Markov process, the transition probability for the sequence of configurations Γ to be realized is given by a product of them,

$$P^F(\Gamma) = \prod_{k=0}^{M-1} \mathcal{W}_{\lambda^F(t_k)}(\mathcal{C}_k(A_k) \rightarrow \mathcal{C}_{k+1}(A_{k+1})). \quad (4.18)$$

The sequence represents a general process of absorbing and emitting matter through the black hole horizon. On the other hand, the probability for the reversed sequence of configurations $\Gamma^* = \{\mathcal{C}_M(A_M), \dots, \mathcal{C}_1(A_1), \mathcal{C}_0(A_0)\}$ with the reserved change of the external parameter is given by

$$\begin{aligned} P^R(\Gamma^*) &= \prod_{k=0}^{M-1} \mathcal{W}_{\lambda^R(t_k)}(\mathcal{C}_{M-k}(A_{M-k}) \rightarrow \mathcal{C}_{M-k+1}(A_{M-k+1})) \\ &= \prod_{k=0}^{M-1} e^{\frac{\Delta A}{4G\hbar}} \mathcal{W}_{\lambda^F(t_k)}(\mathcal{C}_{k+1}(A_{k+1}) \rightarrow \mathcal{C}_k(A_k)). \end{aligned} \quad (4.19)$$

Here we have used eq. (4.17). The ratio of these two probabilities is given by

$$\frac{P^F(\Gamma)}{P^R(\Gamma^*)} = \exp\left(\frac{A_M - A_0}{4G\hbar}\right) \equiv \exp(S_P(\Gamma)). \quad (4.20)$$

$S_P(\Gamma)$ is defined as the logarithm of the ratio, and proportional to the difference of area, which is not necessarily small.

We now derive a Crook's type fluctuation theorem in the black hole system. The matter is assumed to be in thermal equilibrium with Hawking temperature T_H with an external parameter $\lambda^F(0)$ at the beginning. First define the total dissipation $\Delta S(\Gamma)$ by

$$\exp(-\Delta S(\Gamma)) \equiv \frac{p_{\lambda^R(0)}(\mathcal{C}_M)}{p_{\lambda^F(0)}(\mathcal{C}_0)} \exp(-S_P(\Gamma)), \quad (4.21)$$

where $p_{\lambda^{(F, R)}(t_0)}$ is the initial probability distribution for matters under the forward or reversed protocols. We assume that these probability distributions are canonical distributions with the Hawking temperature. A relation $\Delta S(\Gamma^*) = -\Delta S(\Gamma)$ follows the identity $S_P(\Gamma^*) = -S_P(\Gamma)$. A transition probability to produce the total dissipation $\Delta S(\Gamma) = \Delta S$ under the forward protocol $\lambda^F(t)$ is now given by

$$\rho^F(\Delta S) = \sum_{\mathcal{C}_0, \Gamma} p_{\lambda^F(0)}(\mathcal{C}_0) P^F(\Gamma) \delta(\Delta S(\Gamma) - \Delta S). \quad (4.22)$$

Here, $\sum_{\mathcal{C}_0} p_{\lambda^F(0)}(\mathcal{C}_0)$ stands for a sum over all possible initial states weighted by the initial distribution. \sum_{Γ} is a path integral for all possible trajectories. By using (4.20) and (4.21), it is straightforward to show the Crook's type fluctuation theorem

$$\rho^F(\Delta S) e^{-\Delta S} = \rho^R(-\Delta S). \quad (4.23)$$

From the definition of $\Delta S(\Gamma)$ in (4.21), it is identified as a change of a sum of the black hole entropy and the entropy of matter;

$$\Delta S(\Gamma) = \frac{\Delta A}{4G\hbar} + \beta_H(\Delta E - \Delta F). \quad (4.24)$$

By integrating the equation (4.23) over ΔS , it gives a Jarzynski type equality

$$\langle e^{-\Delta S} \rangle = 1. \quad (4.25)$$

It can be expanded as

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \langle (\Delta S)^n \rangle = 0. \quad (4.26)$$

Since the black hole entropy is inversely proportional to \hbar , it is an equality relating different powers of $(1/\hbar^n)$. This means that the fluctuation of the horizon area $\langle (\Delta A)^n \rangle$ is a nontrivial function of \hbar , and contains information of the microstates of black holes.

The generating function to compute correlation functions of ΔS such as $\langle (\Delta S)^n \rangle$ is

$$Z^F(f) \equiv \int_{-\infty}^{\infty} d(\Delta S) \rho^F(\Delta S) e^{if\Delta S}. \quad (4.27)$$

This is the generating function of correlators under a variation of external parameter $\lambda^F(t)$. The fluctuation theorem (4.23) suggests a following general relation.

$$\begin{aligned} Z^F(f) &= \int_{-\infty}^{\infty} d(\Delta S) \rho^R(-\Delta S) e^{if\Delta S + \Delta S} \\ &= \int_{-\infty}^{\infty} dx \rho^R(x) e^{ix(i-f)} \\ &= Z^R(i-f). \end{aligned} \quad (4.28)$$

Non-equilibrium fluctuations of black hole horizons and matters under forward protocol and reversed protocol are related this way. Because correlation functions have different quantum corrections under forward or reversed protocol, the equation (4.28) relates complex quantum nature of non-equilibrium fluctuations.

By using the Jensen inequality $\langle \exp(x) \rangle \geq \exp(\langle x \rangle)$, the generalized second law of black hole thermodynamics [1]

$$\langle \Delta S \rangle \geq 0 \quad (4.29)$$

is derived as a corollary of the Jarzynski equality. An important point here is that it is satisfied only in an averaged sense, and in order to satisfy the Jarzynski equality (4.25), entropy decreasing processes ($\Delta S < 0$) must exist as individual processes (otherwise (4.25) can not be satisfied). The probabilities to microscopically violate the second law are arranged to satisfy the Jarzynski equality.

4.3 Future Work

Here, we have considered a special class of the fluctuation theorem where the matter system is initially in the thermal equilibrium with the Hawking temperature. If the matter field is interacting with another thermal bath with a different temperature from T_H , there is a constant flow of energy between the black hole and the matter. The fluctuation theorem is also applicable to such a situation. From the steady state fluctuation theorem, we can derive a fluctuation-dissipation relation and calculate various response functions. We may also be able to obtain Green-Kubo relations that relates the correlation of energy flow and a proportionality coefficient between area change of horizon and energy flow. Moreover, if we consider charged rotating black holes, we will obtain the Onsager reciprocal relations among proportionality coefficients of several currents.

Another important issue is to take into account the effect of back reaction of the radiation to the Hawking temperature. In the present letter, we have assumed that the temperature of the black hole is not affected and the matter system continues to be in thermal equilibrium with the black hole. In the real dynamical process of evaporation, the temperature is varying and it drastically changes the fluctuation of the horizon area, if the size of the black hole is small. In particular, the Schwarzschild black hole has a negative specific heat and it is thermodynamically unstable. The ordinary linear response theorem can not be valid for such an unstable system. The

fluctuation theorem is, however, applicable and we can in principle calculate fluctuations of horizon area for an evaporating black hole. Furthermore, the way a black hole reacts to radiation is dependent on the details of the microstates of the black hole, and so is the fluctuation theorem. Then we may reveal microscopic structures of a black hole by observing details of horizon area fluctuations against a change of external parameters of the black hole.

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Appendix A

Radiation Damping

Classically, an accelerating charged particle will emit radiation. Since those radiation carries energy and momentum, the conservation laws requires that the charged particle should receive some change due to the radiation. This is called the radiation damping. In this section, I am going to review the classical treatment of this back reaction on the charged particle due to the radiation. First I am going to review the Abraham-Lorentz-Dirac(ALD) force and its problems. Next I am going to see the approach by Landau and Lifshitz.

In classical electromagnetic dynamics, usually we only determine the field with the source specified or determine the classical motion of the charged particle with the external field specified. So generally we don't consider the problem of back reaction. One reason is that this kind of effect are negligible for most case under dealing. There is a simple way to briefly estimate that if or when the back reaction will be important. The radiation power of the accelerated charged particle can be obtained from the Lienard-Wiechert Potentials

$$E = -\frac{e^2}{6\pi m^2 c^3} \int dt \frac{dp^\mu}{d\tau} \frac{dp_\mu}{d\tau}, \quad (\text{A.1})$$

this is called the Larmor radiation. Then, consider a particle of charge e has an acceleration of typical magnitude a for a period of time T , the energy

radiated is of the order of

$$E_{\text{rad}} \sim \frac{e^2 a^2 T}{6\pi c^3}. \quad (\text{A.2})$$

On the other hand, the relevant energy E_0 of the problem can be estimated from the kinetic energy

$$E_0 \sim m(aT)^2. \quad (\text{A.3})$$

So the condition for the back reaction to be unimportant is

$$E_0 \gg E_{\text{rad}} \quad \longrightarrow \quad ma^2 T^2 \gg \frac{e^2 a^2 T}{6\pi c^3} \quad (\text{A.4})$$

or

$$T \gg \frac{e^2}{6\pi m c^3}. \quad (\text{A.5})$$

Here we find a characteristic time

$$\tau_0 = \frac{e^2}{6\pi m c^3}, \quad (\text{A.6})$$

for the phenomenon with time T much longer than τ_0 , the back reaction can be neglected. For electron, $\tau_0 \sim 10^{-24} \text{s}$, which is the time taken for light to travel 10^{-15}m (the Compton wavelength is 10^{-13}m).

A.1 Abraham-Lorentz-Dirac Force

Writing the equation of motion for the charged particle in the form

$$\frac{dp^\mu}{d\tau} = F_{\text{ext}}^\mu + f^\mu, \quad (\text{A.7})$$

here F_{ext}^μ denotes the external force which cause the acceleration of the charged particle, and f^μ denotes the back reaction force due to the radiation. Then we are going to determine the form of f^μ . It can be done from

the energy momentum conservation. The covariant form of the power for the Larmor radiation (A.1) can be written by

$$P_{\text{rad}}^{\mu} = -\frac{e^2}{6\pi c^5} \int d\tau \frac{dx^{\mu}}{d\tau} \frac{d^2 x^{\nu}}{d\tau^2} \frac{d^2 x_{\nu}}{d\tau^2}. \quad (\text{A.8})$$

With this expression, we can have

$$\int d\tau f^{\mu} = \frac{e^2}{6\pi c^5} \int d\tau \frac{dx^{\mu}}{d\tau} \frac{d^2 x^{\nu}}{d\tau^2} \frac{d^2 x_{\nu}}{d\tau^2}. \quad (\text{A.9})$$

Taking off the integral

$$\hat{f}^{\mu} = \frac{e^2}{6\pi c^5} \frac{dx^{\mu}}{d\tau} \frac{d^2 x^{\nu}}{d\tau^2} \frac{d^2 x_{\nu}}{d\tau^2}. \quad (\text{A.10})$$

However, this result suffers an ambiguity when we taking off the integral. This ambiguity can be fixed by the on-shell condition. Fix the parameter τ to be the proper time

$$\frac{dx^{\mu}}{d\tau} \frac{dx_{\mu}}{d\tau} = c^2, \quad (\text{A.11})$$

then

$$\frac{dx^{\mu}}{d\tau} \frac{dp^{\mu}}{d\tau} = 0 \quad \longrightarrow \quad f^{\mu} \frac{dx_{\mu}}{d\tau} = 0, \quad (\text{A.12})$$

to satisfy this condition, one have to add f^{μ} a total differential term

$$f^{\mu} = \frac{e^2}{6\pi c^3} \left(\frac{d^3 x^{\mu}}{d\tau^3} + \frac{1}{c^2} \frac{d^2 x^{\nu}}{d\tau^2} \frac{d^2 x_{\nu}}{d\tau^2} \frac{dx^{\mu}}{d\tau} \right), \quad (\text{A.13})$$

then f^{μ} satisfies the on-shell condition, $f^{\mu}(dx_{\mu}/d\tau) = 0$. This f^{μ} is called the Abraham-Lorentz-Dirac force.

The Abraham-Lorentz-Dirac force is very different from the ordinary force on the point particle because it contains third derivatives of the particle path. Corresponding this feature, there are several problems on the ALD force. To show the problems explicitly, it will be convenient to use the non-relativistic limit. Then the three-dimensional notation of ALD force can be written by

$$m(\dot{\vec{v}} - \tau_0 \ddot{\vec{v}}) = \vec{F}_{\text{ext}}. \quad (\text{A.14})$$

From this expression, one can see that in the absence of external force, there are two solutions. One is the constant velocities, $\vec{v} = \text{const}$, the other is the so called runaway solution,

$$\dot{\vec{v}} = \dot{\vec{v}}_0 e^{t/\tau_0}.$$

The runaway solution is unphysical, since it suggests that even there are no forces, the particle can be accelerated to speed of light!

One might consider that the problem will be resolved by just neglecting the unphysical runaway solutions and only taking the regular solutions. However this is not true, there is an acausal problem for the regular solutions. The regular solutions can be specified by insisting proper boundary conditions (in particular $\dot{\vec{v}} \rightarrow 0$ at $t \rightarrow \infty$ with \vec{F}_{ext} vanishes in this limit). With this condition, the solutions can be written by an integral form

$$m \dot{\vec{v}} = \int_0^\infty ds e^{-s} \vec{F}_{\text{ext}}(t + \tau_0 s). \quad (\text{A.15})$$

The runaway solutions are eliminated from this form, but another unpleasant feature arise. Consider that the external force are turned on to constant at some instance, so \vec{F}_{ext} takes value 0 for negative t and takes value of some constant, \vec{F}_0 for positive t , as in Fig. A.1. With this external force, the $\dot{\vec{v}}$

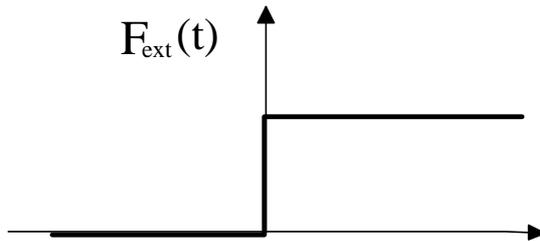


Figure A.1: External Force

can be solved as

$$m \dot{\vec{v}} = \begin{cases} \vec{F}_0 e^{-|t|/\tau_0} & \text{for } t < 0 \\ \vec{F}_0 & \text{for } t > 0, \end{cases} \quad (\text{A.16})$$

one can see that even when \vec{F}_{ext} is zero for negative t , the acceleration $\dot{\vec{v}}$ does not vanish but rather starts increasing at earlier times of order τ_0 ($\sim 10^{-24} \text{ s}$ for an electron), which is the time required for light to cross the electromagnetic radius, Fig. A.2. This means that the electron will know the

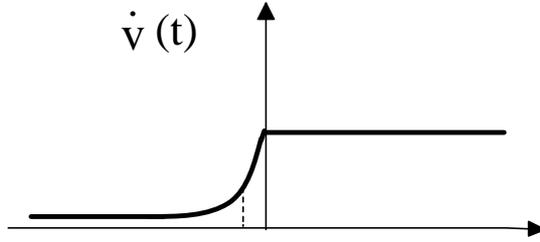


Figure A.2: Preacceleration of a classical charge

force and starting to accelerate before one switching on the force. However, this acausal effects are only occur at the time scale, τ_0 , which are much smaller than the Compton length. So one may hope that the acausal effects are unobservable because the quantum effect.

The acausal effects can also be seen in another case. We just considered is that to turn on an external force for infinity time. Now we consider that only turn on the external force for an finite time interval, $\delta t = T$. So the external force takes value \vec{F}_0 between $t = 0$ and $t = T$, as shown in Fig. A.3. In this case, the solution becomes

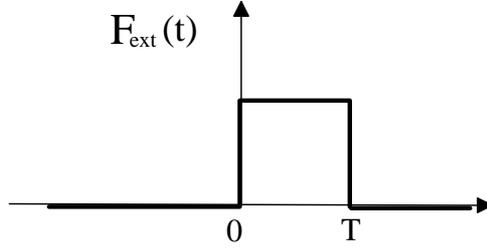


Figure A.3: External Force

$$\begin{aligned}
 m \dot{\vec{v}} &= \vec{F}_0 e^{-|t|/\tau_0} (1 - e^{-T/\tau_0}) & \text{for } t < 0 \\
 & \vec{F}_0 (1 - e^{(t-T)/\tau_0}) & \text{for } 0 < t < T \\
 & 0 & \text{for } t > T,
 \end{aligned} \tag{A.17}$$

here one can see different acausal effects. As in Fig. A.4, the electron starts to accelerate before the force acts on it. Beside that, the acceleration of the electron also starts to decrease to zero before the external force vanishes, again at earlier times of order τ_0 . And there also exist a screen effect due to the "finite time effect", the maximum acceleration of the charge will not be $|\vec{F}_0|$, but $|\vec{F}_0|(1 - e^{-T/\tau_0})$. The suppression e^{-T/τ_0} are negligible for $T \gg \tau_0$,

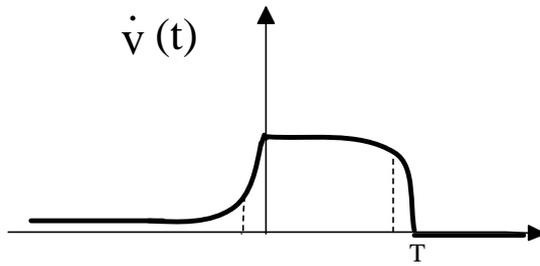


Figure A.4: Preacceleration of a classical charge

and will be important for $T \sim \tau_0$, which again in a region that the quantum effects should be essential.

A.2 Landau-Lifshitz Equation

To avoid the unpleasant feature of the ALD equation, Landau and Lifshitz have proposed another equation to describe the radiation damping. The basic idea is like this, since the radiation damping are generally small compare to the leading order of the motion for the classical charge, so the radiation damping term can be treated as perturbation. Writing the ALD equation in the form

$$\begin{aligned}\frac{du^\mu}{ds} &= \frac{e}{mc^2} F^{\mu\nu} u_\nu + \frac{e^2}{6\pi mc^2} g^\mu, \\ g^\mu &= \frac{d^2 u^\mu}{ds^2} - u^\mu u^\nu \frac{d^2 u_\nu}{ds^2},\end{aligned}\tag{A.18}$$

where $u^\mu = \frac{dx^\mu}{ds}$, note that we changed the variable from τ to s with $ds = cd\tau$, when $c = 1$ they are same. In this expression, g^μ describes the effects of radiation damping. Here we treat g^μ as the perturbation.

For the leading order, the equation is

$$\left(\frac{du^\mu}{ds}\right)_0 = \frac{e}{mc^2} F^{\mu\nu} u_\nu,\tag{A.19}$$

which is just the equation of motion without radiation damping. For the next order

$$\left(\frac{du^\mu}{ds}\right)_1 = \frac{e}{mc^2} F^{\mu\nu} u_\nu + \frac{e^2}{6\pi mc^2} g_0^\mu,\tag{A.20}$$

with g_0^μ given by

$$g_0^\mu = \left(\frac{d^2 u^\mu}{ds^2}\right)_0 - u^\mu u^\nu \left(\frac{d^2 u_\nu}{ds^2}\right)_0,\tag{A.21}$$

substituting (A.19) to $\left(\frac{d^2 u^\mu}{ds^2}\right)_0$

$$\begin{aligned} \left(\frac{d^2 u^\mu}{ds^2}\right)_0 &= \frac{d}{ds} \left(\frac{e}{mc^2} F^{\mu\nu} u_\nu\right) \\ &= \left(\frac{e}{mc^2}\right)^2 F^{\mu\nu} F_{\nu\rho} u^\rho + \frac{e}{mc^2} u^\rho u_\nu \partial_\rho F^{\mu\nu}. \end{aligned} \quad (\text{A.22})$$

Finally we have

$$g^\mu = \frac{e}{mc^2} u^\rho u_\nu \partial_\rho F^{\mu\nu} + \left(\frac{e}{mc^2}\right)^2 F^{\mu\nu} F_{\nu\rho} u^\rho - \left(\frac{e}{mc^2}\right)^2 (F_{\nu\rho} u^\rho)(F^{\nu\theta} u_\theta) u^\mu. \quad (\text{A.23})$$

The Landau-Lifshitz equation is a second order differential equation. Since the radiation damping term in this equation are all depending on the field strength or its derivative, it is free from the runaway solution and the acausal problem.

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