Quantum infra-red effects in de Sitter space

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Abstract

In cosmic inflation at the early universe and dark energy at the present universe, our universe is exponentially expanding with the respective cosmological constants. To investigate the quantum effects on these universes, we need to understand the quantum field theory in de Sitter space. Exploring the quantum infra-red effects specific to de Sitter space, we may better understand inflation and dark energy. In investigating them, we divide the momentum scale into the two regions, inside the cosmological horizon and outside the cosmological horizon. The quantum effects inside the cosmological horizon respect the de Sitter symmetry, while the quantum effects outside the cosmological horizon break it. So the contributions to physical quantities are vastly different between these two regions. In this thesis, I summarize the quantum infra-red effects due to the degrees of freedom at the two regions.

1

Contents

3

Introduction

Concerning inflation in the early universe and dark energy of the present universe, the past and current exponential expansions of the universe are likely to be driven by the effective cosmological constant of the order of GUT and neutrino mass scales respectively. We have not understood why the huge disparity exists between their energy scales, and in addition, why they are so small compared with the Planck scale. Phenomenologically it appears that the cosmological constant has evolved with time. Although we may parametrize it by adopting a suitable potential, a microscopic perspective is totally lacking.

The quantum field theory in de Sitter (dS) space is necessary to investigate the above problem from a microscopic viewpoint. However our understanding of it is so sparse. There is still plenty of room which should be explored.

In investigating interacting field theories on a time dependent background like dS space, the standard Feynman-Dyson formalism breaks down. To investigate them, we need to employ the Schwinger-Keldysh formalism [1, 2]. The Feynman-Dyson formalism is the backbone not only in relativistic field theories but also in statistical mechanics for equilibrium systems. So it indicates that the quantum field theory in dS space belongs to nonequilibrium physics. A. M. Polyakov has proposed that we can evaluate the particle creation effects by using the Boltzmann equation, which is a standard tool in nonequilibrium physics [3].

There is a long history of studying Boltzmann equations in Schwinger-Keldysh formalism starting from Kadanoff-Baym [4, 2, 5]. In these studies, Boltzmann equations in Minkowski space have been investigated. Well inside the cosmological horizon where a particle description is valid, we have derived a Boltzmann equation in dS space from a Schwinger-Dyson equation [6]. The derivation of the Boltzmann equation in curved space-time has been studied to the leading order of the derivative expansion of the Moyal product in the Wigner representation [7]. However only the energy conserving process has been identified in such a limit. We go beyond the leading order of the expansion to investigate the particle creation effects due to energy non-conservation in dS space. As a result, we have found that the apparent time dependences of the physical quantities probed by the Boltzmann equation disappear after expressed by the physical scales.

We should note that the constant shift of the cosmic time: $t \to t + c$ can be compensated by rescaling the spatial coordinate: $\mathbf{x} \to e^{-Hc}\mathbf{x}$ to leave the metric of dS space invariant. So in investigating time dependences of physical quantities, the important issue is whether there is a mechanism to break this dS symmetry. The local physics probed by the Boltzmann equation respects the dS symmetry since the degrees of freedom inside the cosmological horizon are time independent.

On the other hand, the degrees of freedom outside the cosmological horizon increase with cosmic evolution. This increase gives rise to a growing time dependence to the propagator of a massless and minimally coupled scalar field and gravitational field [8, 9, 10]. It is a direct consequence of their scale invariant fluctuation spectrum. In some field theoretic models on dS space, the dS symmetry is dynamically broken and physical quantities acquire time dependences through such an quantum infra-red (IR) effect. In particular, R. P. Woodard and N. C. Tsamis have pointed out that this IR effect may be relevant to resolve the cosmological

constant problem [11].

In the Schwinger-Keldysh perturbation theory, the IR effects at each loop level manifest as polynomials in the logarithm of the scale factor of the universe $\log a(t)$, $a(t) = e^{Ht}$ [12]. At late times, the leading IR effect comes from the leading logarithm at each loop level. For example in $\lambda\varphi^4$ theory, the leading IR effect to the potential is the $2n$-th power of the logarithm at the n-th order of the coupling constant λ [13]. Their growing time dependences mean that the perturbation theory eventually breaks down after a large enough cosmic expansion. In order to understand such a situation, we have to investigate the IR effect nonperturbatively.

Remarkably in the models with interaction potentials, the leading IR effects can be evaluated nonperturbatively by the stochastic approach [14, 15]. Furthermore it has been found that the equilibrium solution in the stochastic approach can be rederived in an Euclidean field theory on S_4 [16]. However in a general model with derivative interactions, we still don't know how to evaluate the nonperturbative IR effects. Especially such a tool is required to understand the quantum IR effects of gravity. It is because the gravitational field contains massless and minimally coupled modes with derivative interactions.

As a simple model with derivative interactions, we have investigated the non-linear sigma model in [17, 18]. The global symmetry guarantees that it contains massless minimally coupled scalar fields. In addition, we can perform some nonperturbative investigations as it is exactly solvable in the large N limit on an N-sphere. Another point is that there is some similarity to the Einstein action as it consists of the derivative interactions of the metric tensor field. Here we have investigated the contribution to the cosmological constant by evaluating the expectation value of the energy-momentum tensor.

From the perturbative investigation, we have found that the coupling constant of the non-linear sigma model becomes time dependent at the one loop level in agreement with power counting of the IR logarithms. In contrast, the leading IR effects to the cosmological constant are canceled at the two loop level beyond the power counting [17]. In the further studies [18], we have shown that the cancellation of the leading IR effects works to all orders on an arbitrary target space. In fact even if we consider the full IR effects, the effective cosmological constant is time independent in the large N limit on an N-sphere. Although the sub-leading IR effects could arise at the three loop level in a generic non-linear sigma model, we have shown that there is a renormalization scheme to cancel it.

This thesis is divided into the following three parts. In Part I, we review a scalar field theory in dS space. Specifically we introduce propagators in dS space and the formalism to deal with the interacting field theories in a time dependent background. We investigate the quantum effects inside the cosmological horizon in Part II. In this region, the characteristic property in dS space can be investigated perturbatively from that in Minkowski space. Here we explain how to derive a Boltzmann equation in dS space from a Schwinger-Dyson equation and describe the local physics probed by this Boltzmann equation in φ^3, φ^4 theories. In Part III, we investigate the quantum effects from degrees of freedom outside the cosmological horizon. Unlike inside the cosmological horizon, the quantum IR effect in dS space breaks the dS symmetry. Firstly, we review the perturbative and nonperturbative investigation of the dS breaking effects in the models with interaction potentials. Secondly, we evaluate the dS breaking effects in the non-linear sigma model as a model with derivative interactions.

Part I

Scalar field theory in de Sitter space

1   Propagator in de Sitter space

In dealing with the quantum field theory on a certain background, we need to know the propagator on it. Here we introduce the propagator in de Sitter (dS) space.

In the Poincaré coordinate, the metric in dS space is

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2, \quad a(t) = e^{Ht}, \tag{1.1}$$

where the dimension of dS space is taken as $D = 4$ and H is the Hubble constant. In the conformally flat coordinate,

$$g_{\mu\nu} = a^2(\tau)\eta_{\mu\nu}, \quad a(\tau) = -\frac{1}{H\tau}. \tag{1.2}$$

Here the conformal time $\tau(-\infty < \tau < 0)$ is related to the cosmic time t as $\tau \equiv -\frac{1}{H}e^{-Ht}$.

In a general case, the quadratic action for a scalar field is written as

$$S_2 = \frac{1}{2} \int \sqrt{-g}d^4x \, [-g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - m^2\varphi^2 - \xi R_g\varphi^2]. \tag{1.3}$$

Here m^2 is the mass square and R_g is the Ricci scalar of the space-time. From this, the equation of motion is

$$\Big\{ -\frac{\partial^2}{\partial\tau^2} + \frac{2}{\tau}\frac{\partial}{\partial\tau} + \frac{\partial^2}{\partial\mathbf{x}^2} - \frac{m^2/H^2 + 12\xi}{\tau^2} \Big\}\varphi(x) = 0, \tag{1.4}$$

The corresponding wave function for the Bunch-Davies vacuum is

$$\phi_{\mathbf{p}}(x) = \frac{\sqrt{\pi}}{2}H(-\tau)^{\frac{3}{2}}H_\nu^{(1)}(-p\tau) \, e^{i\mathbf{p}\cdot\mathbf{x}}, \tag{1.5}$$

$$\nu = \sqrt{\Big(\frac{3}{2}\Big)^2 - \frac{m^2}{H^2} - 12\xi},$$

where $H_\nu^{(1)}(z)$ is the first kind of the Hankel function, $\mathbf{p}$ is the comoving momentum and $p = |\mathbf{p}|$. The normalization factor $\sqrt{\pi}H/2$ has been decided to satisfy

$$\sqrt{-g}\nabla^2\langle T\varphi(x)\varphi(x')\rangle = i\delta^{(4)}(x - x'), \tag{1.6}$$

where $\nabla^2 = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)$ and T denotes the time ordering.

The physical momentum P is defined as

$$P \equiv p/a(\tau). \tag{1.7}$$

At the large P limit, (1.5) approaches to the wave function in Minkowski space except for the scale factor

$$\phi_{\mathbf{p}}(x) \sim H\tau \times \frac{1}{\sqrt{2p}} e^{-ip\tau + i\mathbf{p}\cdot\mathbf{x}}. \tag{1.8}$$

We expand the scalar filed as

$$\varphi(x) = \int \frac{d^3p}{(2\pi)^3} \left(a_{\mathbf{p}}\phi_{\mathbf{p}}(x) + a_{\mathbf{p}}^\dagger \phi_{\mathbf{p}}^*(x) \right). \tag{1.9}$$

If we consider the Bunch-Davies vacuum $|0\rangle$ which is annihilated by all the annihilation operators $\forall a_{\mathbf{p}}|0\rangle = 0$, the propagator for such a vacuum is

$$\langle \varphi(x)\varphi(x')\rangle = \int \frac{d^3p}{(2\pi)^3} \phi_{\mathbf{p}}(x)\phi_{\mathbf{p}}^*(x'). \tag{1.10}$$

By performing the momentum integration, the propagator is written as

$$\langle \varphi(x)\varphi(x')\rangle = \frac{H^2}{16\pi^2}\Gamma(\frac{3}{2}+\nu)\Gamma(\frac{3}{2}-\nu) \, _2F_1(\frac{3}{2}+\nu, \frac{3}{2}-\nu; 2; 1-\frac{y}{4}), \tag{1.11}$$

where $_2F_1$ is the hypergeometric function and y is defined as

$$y \equiv \frac{-(\tau-\tau')^2 + (\mathbf{x}-\mathbf{x}')^2}{\tau\tau'}. \tag{1.12}$$

We call it the dS invariant distance since it has the following ten symmetries which leave the metric of dS space invariant:

$$\tau' = C\tau, \quad x'^i = Cx^i, \tag{1.13}$$

$$\tau' = \frac{\tau}{1 - 2\theta^j x^j + \theta^j \theta^j x_\mu x^\mu}, \quad x'^i = \frac{x^i - \theta^i x_\mu x^\mu}{1 - 2\theta^j x^j + \theta^j \theta^j x_\mu x^\mu}, \tag{1.14}$$

$$x'^i = x^i + b^i, \tag{1.15}$$

$$x'^i = R^i_{\ j}x^j, \quad R_k^{\ i}R^k_{\ j} = I, \tag{1.16}$$

where i is the spacial index.

In this thesis, we mainly investigate the massless and minimally coupled case: $m^2 = 0$, $\xi = 0$. In the case, the wave function is

$$\phi_{\mathbf{p}}(x) = \frac{H\tau}{\sqrt{2p}}(1 - i\frac{1}{p\tau}) \, e^{-ip\tau + i\mathbf{p}\cdot\mathbf{x}}, \tag{1.17}$$

and the propagator is

$$\langle \varphi(x_1)\varphi(x_2)\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{H\tau_1\tau_2}{2p}(1 - i\frac{1}{p\tau_1})(1 + i\frac{1}{p\tau_2})\ e^{-ip(\tau_1-\tau_2)+i\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_2)}. \qquad (1.18)$$

From (1.11) or (1.18), it is found that the propagator for a massless and minimally coupled field has an IR divergence. To investigate how the IR divergence contributes to physical quantities, we focus on the quantum effects outside the cosmological horizon $P \ll H$ in Part III. Besides the IR divergence, we investigate the quantum effects well inside the cosmological horizon $P \gg H$ in Part II.

2   Schwinger-Keldysh formalism (in-in formalism)

We evaluate the contributions from the interactions in Part II and III. Here we introduce the Schwinger-Keldysh formalism, which is necessary to deal with the interacting field theories in a time dependent background like a dS space.

Let us represent the vacuum at $t \to -\infty$ as $|in\rangle$, and $t \to +\infty$ as $|out\rangle$. In the Feynman-Dyson formalism, the vacuum expectation value (vev) is essentially given by the transition amplitude between $|in\rangle$ and $|out\rangle$

$$\langle \mathcal{O}_H(x)\rangle = \langle out|T[U(+\infty, -\infty)\mathcal{O}_I(x)]|in\rangle, \qquad (2.1)$$

where $\mathcal{O}_H$ and $\mathcal{O}_I$ denote the operators in the Heisenberg and the interaction pictures respectively. $U(t_1, t_2)$ is the time translation operator in the interaction picture

$$U(t_1, t_2) = \exp\Big\{i \int_{t_2}^{t_1} \sqrt{-g}dtd^3x\ \Delta\mathcal{L}_I(x)\Big\}. \qquad (2.2)$$

It is because $|in\rangle$ is equal to $|out\rangle$ up to a phase due to the time translation invariance.

On the other hand, there is no time translation symmetry in dS space, and so we can't prefix $|out\rangle$. In this case, we can evaluate the vev only with respect to $|in\rangle$

$$\langle \mathcal{O}_H(x)\rangle = \langle in|T_C[U(-\infty, \infty)U(\infty, -\infty)\mathcal{O}_I(x)]|in\rangle. \qquad (2.3)$$

Here we have adopted the operator ordering T_C specified by the following path instead of the time ordering T

$$\int_C dt = \int_{-\infty}^{\infty} dt_+ - \int_{-\infty}^{\infty} dt_-. \qquad (2.4)$$

8

Because there are two time indices $(+, -)$, the propagator has 4 components

$$\check{G}(x, x') \equiv \begin{pmatrix} G^{++}(x, x') & G^{+-}(x, x') \\ G^{-+}(x, x') & G^{--}(x, x') \end{pmatrix} \tag{2.5}$$

$$= \begin{pmatrix} \langle T\varphi(x)\varphi(x') \rangle & \langle \varphi(x')\varphi(x) \rangle \\ \langle \varphi(x)\varphi(x') \rangle & \langle \tilde{T}\varphi(x)\varphi(x') \rangle \end{pmatrix}.$$

Here $\tilde{T}$ denotes the anti time-ordering.

After performing the momentum integration, each propagator in (2.5) is distinguished by specifying the distance y as follows

$$y_{ij} = H^2 a(\tau) a(\tau') \Delta x_{ij}^2, \quad i = +, -, \tag{2.6}$$

$$\Delta x_{++}^2 \equiv -(|\tau - \tau'| - ie)^2 + (\mathbf{x} - \mathbf{x}')^2, \tag{2.7}$$
$$\Delta x_{-+}^2 \equiv -(\tau - \tau' - ie)^2 + (\mathbf{x} - \mathbf{x}')^2,$$
$$\Delta x_{+-}^2 \equiv -(\tau - \tau' + ie)^2 + (\mathbf{x} - \mathbf{x}')^2,$$
$$\Delta x_{--}^2 \equiv -(|\tau - \tau'| + ie)^2 + (\mathbf{x} - \mathbf{x}')^2,$$

where e is an infinitesimal constant.

For example, in investigating the effects of the interaction to the two point function, the Schwinger-Dyson equation is written as

$$\check{G}(x_1, x_2) = \check{G}_0(x_1, x_2) \tag{2.8}$$
$$+ \int \sqrt{-g_3} d^4 x_3 \sqrt{-g_4} d^4 x_4 \; \check{G}_0(x_1, x_3) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\times \check{\Sigma}(x_3, x_4) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \check{G}(x_4, x_2),$$

where G_0 is the free propagator, G is the full propagator, and Σ is the particle's self energy.

Especially, we focus on the $(-+)$ component of the propagator

$$G^{-+}(x_1, x_2) = G_0^{-+}(x_1, x_2) \tag{2.9}$$
$$+ \int \sqrt{-g_3} d^4 x_3 \sqrt{-g_4} d^4 x_4 \; G_0^{-+}(x_1, x_3) \Sigma^{++}(x_3, x_4) G^{++}(x_4, x_2)$$
$$- \int \sqrt{-g_3} d^4 x_3 \sqrt{-g_4} d^4 x_4 \; G_0^{-+}(x_1, x_3) \Sigma^{+-}(x_3, x_4) G^{-+}(x_4, x_2)$$
$$- \int \sqrt{-g_3} d^4 x_3 \sqrt{-g_4} d^4 x_4 \; G_0^{--}(x_1, x_3) \Sigma^{-+}(x_3, x_4) G^{++}(x_4, x_2)$$
$$+ \int \sqrt{-g_3} d^4 x_3 \sqrt{-g_4} d^4 x_4 \; G_0^{--}(x_1, x_3) \Sigma^{--}(x_3, x_4) G^{-+}(x_4, x_2)$$
$$= G_0^{-+}(x_1, x_2)$$

9

$$+ \int \sqrt{-g_3} d^4 x_3 \sqrt{-g_4} d^4 x_4 \; G_0^R(x_1, x_3) \Sigma^R(x_3, x_4) G^{-+}(x_4, x_2)$$

$$+ \int \sqrt{-g_3} d^4 x_3 \sqrt{-g_4} d^4 x_4 \; G_0^R(x_1, x_3) \Sigma^{-+}(x_3, x_4) G^A(x_4, x_2)$$

$$+ \int \sqrt{-g_3} d^4 x_3 \sqrt{-g_4} d^4 x_4 \; G_0^{-+}(x_1, x_3) \Sigma^A(x_3, x_4) G^A(x_4, x_2).$$

Here we have introduced the retarded and the advanced propagators as follows

$$G^R(x_1, x_2) \equiv \theta(t_1 - t_2)[G^{-+}(x_1, x_2) - G^{+-}(x_1, x_2)], \tag{2.10}$$
$$G^A(x_1, x_2) \equiv -\theta(t_2 - t_1)[G^{-+}(x_1, x_2) - G^{+-}(x_1, x_2)].$$

In the same way, the following identity also holds

$$G^{-+}(x_1, x_2) = G_0^{-+}(x_1, x_2) \tag{2.11}$$

$$+ \int \sqrt{-g_3} d^4 x_3 \sqrt{-g_4} d^4 x_4 \; G^R(x_1, x_3) \Sigma^R(x_3, x_4) G_0^{-+}(x_4, x_2)$$

$$+ \int \sqrt{-g_3} d^4 x_3 \sqrt{-g_4} d^4 x_4 \; G^R(x_1, x_3) \Sigma^{-+}(x_3, x_4) G_0^A(x_4, x_2)$$

$$+ \int \sqrt{-g_3} d^4 x_3 \sqrt{-g_4} d^4 x_4 \; G^{-+}(x_1, x_3) \Sigma^A(x_3, x_4) G_0^A(x_4, x_2).$$

In (2.9) and (2.11), we observe that a retarded or advanced propagator exists at each vertex. It is because of the causality. That is, the integrands are zero outside the past light corn.

In this formalism, the integrations over time are manifestly finite due to the causality. This formalism is called the Schwinger-Keldysh formalism. In order to understand the effects of the interaction, we derive a Boltzmann equation on the dS background from a Schwinger-Dyson equation in Part II.

Part II

Quantum effects from inside the cosmological horizon

3 Boltzmann equations from Schwinger-Dyson equations

In Part II, we investigate the quantum effects well inside the cosmological horizon. Since the particle description is valid in this region, we can evaluate how the particle creation effects in dS space emerge to physical quantities.

Here we redefine the scalar field as $\varphi \to H\tau\varphi$ for a convenience. We can simply scale it back to find the original scalar field. In terms of the rescaled field, the quadratic action for a massless and minimally coupled field becomes

$$S_2 = \frac{1}{2} \int d^4x \; \varphi \left(-\partial_\tau^2 + \partial_{\mathbf{x}}^2 + \frac{2}{\tau^2} \right) \varphi, \tag{3.1}$$

and the wave function is

$$\phi_{\mathbf{p}}(x) = \frac{1}{\sqrt{2p}} (1 - i\frac{1}{p\tau}) \; e^{-ip\tau + i\mathbf{p}\cdot\mathbf{x}}. \tag{3.2}$$

In a time dependent background, we need to consider excited states in general. For such a state, the expectation value of the number operator $\langle a^\dagger a \rangle$ is non-vanishing. We introduce a distribution function f for scalar particles as follows

$$\langle a_{\mathbf{p}}^\dagger a_{\mathbf{q}} \rangle \equiv f(\mathbf{p}) \times (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}). \tag{3.3}$$

One of our main objectives in this section is to understand the time dependence of the distribution function $f(\mathbf{p})$ due to the interaction. We utilize a Boltzmann equation for this purpose. Boltzmann equations govern the time evolution of the distribution functions. They are widely used to study non-equilibrium physics. In fact there is a long history of the microscopic derivation of Boltzmann equations in non-equilibrium physics using Schwinger-Keldysh formalism [4, 2, 5]. In this section, we systematically investigate the propagator in dS space from a Schwinger-Dyson equation.

We assume that the full propagator in dS space has the following form

$$\begin{aligned}
G^{-+}(x_1, x_2) = & \int \frac{d^3p}{(2\pi)^3} \left[(1 + f(p, \tau_c)) Z(p, \tau_c) \phi_{\mathbf{p}}(x_1) \phi_{\mathbf{p}}^*(x_2) \right. \tag{3.4} \\
& \left. + f(p, \tau_c) Z^*(p, \tau_c) \phi_{\mathbf{p}}^*(x_1) \phi_{\mathbf{p}}(x_2) \right] \\
& + \int_{\varepsilon > 0} \frac{d\varepsilon \, d^3p}{(2\pi)^4} \frac{1}{2\varepsilon} [F_+(\varepsilon, p, \tau_c) \; e^{-i\varepsilon(\tau_1 - \tau_2) + i\mathbf{p}\cdot(\mathbf{x}_1 - \mathbf{x}_2)} \\
& + F_-(\varepsilon, p, \tau_c) \; e^{+i\varepsilon(\tau_1 - \tau_2) - i\mathbf{p}\cdot(\mathbf{x}_1 - \mathbf{x}_2)}].
\end{aligned}$$

The propagator depends on the average and the relative time:

$$\tau_c \equiv \frac{\tau_1 + \tau_2}{2}, \qquad \bar{\tau} \equiv \tau_1 - \tau_2. \tag{3.5}$$

It consists of the on-shell part and the off-shell part. In the on-shell part, we have introduced the wave function renormalization factor $Z(p, \tau_c)$. The off-shell part depends on the spectral function $F_\pm(\varepsilon, p, \tau_c)$. We assume that $f, Z, F_\pm$ evolve with the average time τ_c. We investigate the propagator in the region:

$$|\tau_c| \gg |\bar{\tau}|, \qquad |\tau_c| \gg 1/p. \tag{3.6}$$

The second assumption implies that we investigate the propagator well inside the cosmological horizon.

From (2.9) and (2.11), we can derive the following identity

$$G_0^{-1}|_1 G^{-+}(x_1, x_2) - G_0^{-1}|_2 G^{-+}(x_1, x_2) \tag{3.7}$$

$$= + \sqrt{-g_1} \int \sqrt{-g_3} d^4 x_3 \; \Sigma^R(x_1, x_3) G^{-+}(x_3, x_2)$$

$$+ \sqrt{-g_1} \int \sqrt{-g_3} d^4 x_3 \; \Sigma^{-+}(x_1, x_3) G^A(x_3, x_2)$$

$$- \sqrt{-g_2} \int \sqrt{-g_3} d^4 x_3 \; G^R(x_1, x_3) \Sigma^{-+}(x_3, x_2)$$

$$- \sqrt{-g_2} \int \sqrt{-g_3} d^4 x_3 \; G^{-+}(x_1, x_3) \Sigma^A(x_3, x_2).$$

By substituting the expression for the full propagator (3.4) into the left-hand side of the Schwinger-Dyson equation (3.7), we obtain

$$G_0^{-1}|_1 G^{-+}(x_1, x_2) - G_0^{-1}|_2 G^{-+}(x_1, x_2) \tag{3.8}$$

$$\sim \int \frac{d^3 p}{(2\pi)^3} \left[\left(\frac{\partial}{\partial \tau_c} + \frac{i}{p} \frac{\partial^2}{\partial \bar{\tau} \partial \tau_c} \right) \{ (1 + f(p, \tau_c)) Z(p, \tau_c) \} \times e^{-ip\bar{\tau} + i\mathbf{p} \cdot \bar{\mathbf{x}}} \right.$$

$$\left. - \left(\frac{\partial}{\partial \tau_c} - \frac{i}{p} \frac{\partial^2}{\partial \bar{\tau} \partial \tau_c} \right) \{ f(p, \tau_c) Z^*(p, \tau_c) \} \times e^{+ip\bar{\tau} - i\mathbf{p} \cdot \bar{\mathbf{x}}} \right]$$

$$+ \int_{\varepsilon > 0} \frac{d\varepsilon d^3 p}{(2\pi)^4} \left[\left(\frac{\partial}{\partial \tau_c} + \frac{i}{\varepsilon} \frac{\partial^2}{\partial \bar{\tau} \partial \tau_c} \right) F_+(\varepsilon, p, \tau_c) \times e^{-i\varepsilon\bar{\tau} + i\mathbf{p} \cdot \bar{\mathbf{x}}} \right.$$

$$\left. - \left(\frac{\partial}{\partial \tau_c} - \frac{i}{\varepsilon} \frac{\partial^2}{\partial \bar{\tau} \partial \tau_c} \right) F_-(\varepsilon, p, \tau_c) \times e^{+i\varepsilon\bar{\tau} - i\mathbf{p} \cdot \bar{\mathbf{x}}} \right].$$

Here we recall the following definitions

$$G_0^{-1} \equiv i(\partial_\tau^2 - \partial_{\mathbf{x}}^2 - \frac{2}{\tau^2}), \tag{3.9}$$

$$G_0^{-1}|_1 G^R(x_1, x_2) = \delta^{(4)}(x_1 - x_2),$$

$$G_0^{-1}|_2 G^A(x_1, x_2) = \delta^{(4)}(x_1 - x_2).$$

In (3.8) we have shown the leading terms in the power series expansion of $1/p\tau_c$.

The right-hand side of Eq.(3.7) corresponds to the collision term $C[f]$. In this section, we investigate the effects of the interaction in $g\varphi^3$ theory at the one loop level. We subsequently find that this theory captures the essential features of more generic field theories such as $\lambda\varphi^4$ theory. The self-energy is

$$\tag{3.10}$$

$$\Sigma^{ij}(x_3, x_4) = \frac{(-ig)^2}{2} G^{ij}(x_3, x_4) G^{ij}(x_3, x_4), \quad i, j = +, -.$$

To the leading order in perturbation theory, we can approximate that $f(p, \tau_c) = f(p)$, $Z(p, \tau_c) = 1$, $F_\pm(\varepsilon, p, \tau_c) = 0$ in the collision term. We also expand the collision term

by the power series in $1/|p\tau_c|$ type factors which can be justified well inside the cosmological horizon. It is a kind of the derivative expansion of the Moyal product in the Wigner representation. We indeed find the particle production effects due to the non-conservation of the energy in this expansion.

In this investigation, we need to perform the following integrations at the interaction vertices.

$$\int_{-\infty}^{\tau_i} d\tau_3 \, \frac{1}{\tau_3^n} e^{i(\varepsilon \pm p)\tau_3} \qquad n \in \mathbf{N}, \quad i = 1, 2, \tag{3.11}$$

where $\varepsilon = \pm p_1 \pm p_2$. We evaluate these integrations in the assumption $|(\varepsilon \pm p)\tau_i| \gg 1$. For our purpose, it suffices to evaluate them to the next leading order

$$\int_{-\infty}^{\tau_i} d\tau_3 \, \frac{1}{\tau_3^n} e^{i(\varepsilon \pm p)\tau_3} \sim e^{i(\varepsilon \pm p)\tau_i} \times \left[\frac{1}{i(\varepsilon \pm p)\tau_i^n} + \frac{-n}{(\varepsilon \pm p)^2 \tau_i^{n+1}} \right]. \tag{3.12}$$

By using these approximations, we derive a Boltzmann equation in dS space. In what follows, we investigate the collision terms and their properties in detail.

We henceforth suppress the following integration factor in the propagator

$$\int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\bar{\mathbf{x}}}. \tag{3.13}$$

In other words we work in the momentum space by performing the Fourier transformation with respect to the spacial coordinate $\bar{\mathbf{x}}$.

3.1 The structure of the collision term

From the Schwinger-Dyson equation (3.7), we observe that the collision term has the on-shell part and the off-sell part. Firstly, the on-shell part comes from the following contributions

$$C_{\mathrm{on}}[f] = + \sqrt{-g_1} \int \sqrt{-g_3} d^4x_3 \, \Sigma^R(x_1, x_3) G^{-+}(x_3, x_2) \tag{3.14}$$

$$- \sqrt{-g_2} \int \sqrt{-g_3} d^4x_3 \, G^{-+}(x_1, x_3) \Sigma^A(x_3, x_2)$$

$$\propto e^{\mp i p \bar{\tau}}.$$

We evaluate the on-shell part to the leading non-trivial order $\mathcal{O}(1/\tau_c^3)$ as

$$C_{\mathrm{on}}[f] = - (1 + f(p)) e^{-ip\bar{\tau}} \frac{g^2}{16\pi p^2 H^2} \times \tag{3.15}$$

$$\left[\int_p^\infty \frac{d\varepsilon}{2\pi} \left\{ \left(\frac{1}{\varepsilon - p} + \frac{1}{\varepsilon + p} \right) \frac{i\bar{\tau}}{\tau_c^3} + \left(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2} \right) \frac{-1}{\tau_c^3} \right\} \right.$$

$$\left. \times \int_{\frac{\varepsilon-p}{2}}^{\frac{\varepsilon+p}{2}} dp_1 \left\{ (1 + f(p_1))(1 + f(\varepsilon - p_1)) - f(p_1)f(\varepsilon - p_1) \right\} \right.$$

13

$$+ 2 \int_0^p \frac{d\varepsilon}{2\pi} \Big\{ \Big(\frac{1}{\varepsilon - p} + \frac{1}{\varepsilon + p} \Big) \frac{i\bar{\tau}}{\tau_c^3} + \Big(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2} \Big) \frac{-1}{\tau_c^3} \Big\}$$

$$\times \int_{\frac{\varepsilon + p}{2}}^\infty dp_1 \big\{ (1 + f(p_1)) f(p_1 - \varepsilon) - f(p_1)(1 + f(p_1 - \varepsilon)) \big\} \quad \Big]$$

$$+ f(p) \, e^{+ip\bar{\tau}} \frac{g^2}{16\pi p^2 H^2} \times$$

$$\Big[\int_p^\infty \frac{d\varepsilon}{2\pi} \Big\{ \Big(\frac{1}{\varepsilon - p} + \frac{1}{\varepsilon + p} \Big) \frac{-i\bar{\tau}}{\tau_c^3} + \Big(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2} \Big) \frac{-1}{\tau_c^3} \Big\}$$

$$\times \int_{\frac{\varepsilon - p}{2}}^{\frac{\varepsilon + p}{2}} dp_1 \big\{ (1 + f(p_1))(1 + f(\varepsilon - p_1)) - f(p_1) f(\varepsilon - p_1) \big\}$$

$$+ 2 \int_0^p \frac{d\varepsilon}{2\pi} \Big\{ \Big(\frac{1}{\varepsilon - p} + \frac{1}{\varepsilon + p} \Big) \frac{-i\bar{\tau}}{\tau_c^3} + \Big(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2} \Big) \frac{-1}{\tau_c^3} \Big\}$$

$$\times \int_{\frac{\varepsilon + p}{2}}^\infty dp_1 \big\{ (1 + f(p_1)) f(p_1 - \varepsilon) - f(p_1)(1 + f(p_1 - \varepsilon)) \big\} \quad \Big].$$

See Appendix A for the details of the calculation.

Secondly, the off-shell part originates from the following contribution

$$C_{\text{off}}[f] = + \sqrt{-g_1} \int \sqrt{-g_3} d^4 x_3 \, \Sigma^{-+}(x_1, x_3) G^A(x_3, x_2) \qquad (3.16)$$

$$- \sqrt{-g_2} \int \sqrt{-g_3} d^4 x_3 \, G^R(x_1, x_3) \Sigma^{-+}(x_3, x_2)$$

$$\propto \int_0^\infty dp_1 \int_{|p_1 - p|}^{p_1 + p} dp_2 \, e^{-i(\pm p_1 \pm p_2)\bar{\tau}}.$$

The off-shell part is also calculated to $\mathcal{O}(1/\tau_c^3)$ as

$$C_{\text{off}}[f] = + \frac{g^2}{16\pi p^2 H^2} \times \qquad (3.17)$$

$$\Big[\int_p^\infty \frac{d\varepsilon}{2\pi} e^{-i\varepsilon\bar{\tau}} \Big(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2} \Big) \frac{-1}{\tau_c^3} \int_{\frac{\varepsilon - p}{2}}^{\frac{\varepsilon + p}{2}} dp_1 \, (1 + f(p_1))(1 + f(\varepsilon - p_1))$$

$$+ 2 \int_0^p \frac{d\varepsilon}{2\pi} e^{-i\varepsilon\bar{\tau}} \Big(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2} \Big) \frac{-1}{\tau_c^3} \int_{\frac{\varepsilon + p}{2}}^\infty dp_1 \, (1 + f(p_1)) f(p_1 - \varepsilon) \quad \Big]$$

$$- \frac{g^2}{16\pi p^2 H^2} \times$$

$$\Big[\int_p^\infty \frac{d\varepsilon}{2\pi} e^{+i\varepsilon\bar{\tau}} \Big(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2} \Big) \frac{-1}{\tau_c^3} \int_{\frac{\varepsilon - p}{2}}^{\frac{\varepsilon + p}{2}} dp_1 \, f(p_1) f(\varepsilon - p_1)$$

$$+ 2 \int_0^p \frac{d\varepsilon}{2\pi} e^{+i\varepsilon\bar{\tau}} \Big(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2} \Big) \frac{-1}{\tau_c^3} \int_{\frac{\varepsilon + p}{2}}^\infty dp_1 \, f(p_1)(1 + f(p_1 - \varepsilon)) \Big].$$

We note that the both on-shell (3.15) and off-shell (3.17) collision terms have infra-red divergences at $\varepsilon = p$. There is a standard procedure to deal with this problem in massless

field theory and we find that it also works here. First of all, we need to recall that any experiment has a finite energy resolution $\Delta\varepsilon$. So we need to add the on-shell and off-shell collision terms within the energy resolution $\Delta\varepsilon$. We first divide the integration range of $C_{\text{off}}[f]$ as follows

$$\int_p^\infty = \int_{p+\Delta\varepsilon}^\infty + \int_p^{p+\Delta\varepsilon} \quad , \quad \int_0^p = \int_0^{p-\Delta\varepsilon} + \int_{p-\Delta\varepsilon}^p . \tag{3.18}$$

We then redefine the on-shell term $C'_{\text{on}}[f]$ and the off-shell term $C'_{\text{off}}[f]$ by transferring the contribution of $C_{\text{off}}[f]$ within the energy resolution $p - \Delta\varepsilon \leq \varepsilon \leq p + \Delta\varepsilon$ to $C_{\text{on}}[f]$. The explicit expressions are shown in Appendix A.

When $f(p) = 0$, we find that infra-red divergences cancel out in this procedure. In the next subsection, we investigate the case when f is a thermal distribution. For a generic distribution, the cancellation does not take place and we seem to face linear IR divergences. However there is no real infra-red divergence in our problem since the time integration range in (3.12) is bounded by τ_c. We thus argue that the linear divergence should be cut-off at $|p - \varepsilon| \sim 1/|\tau_c|$.

Before investigating the thermal distribution case, we point out the difference between Minkowski space and dS space with respect to the collision term. In Minkowski space, the collision term does not have the off-shell term due to the time translation symmetry

$$C_{\text{off}}[f] \propto \int \frac{d\varepsilon}{2\pi} \, 2\pi\delta(\varepsilon - p)e^{\mp i\varepsilon\bar{\tau}} = \ e^{\mp ip\bar{\tau}} \ \implies \ C'_{\text{off}}[f] = 0. \tag{3.19}$$

On the other hand, as we observe in (3.17), the collision term in dS space has the off-shell term due to the absence of the time translation symmetry. This is why we have introduced the spectral function $F_\pm(\varepsilon, p, \tau_c)$ in the full propagator (3.4).

3.2 Thermal distribution case

We focus on the case that the initial distribution function is thermal in this subsection

$$f(p) = \frac{1}{e^{\beta p} - 1}, \tag{3.20}$$

where we introduce an inverse temperature β as a free parameter. In Minkowski space the thermal distribution is obtained as the solution of the Boltzmann equation. On the other hand, we find that the collision term in dS space is non-vanishing even for the thermal distribution.

The off-shell collision term can be evaluated as follows

$$C'_{\text{off}}[f] = + \frac{g^2}{16\pi p H^2} \times \tag{3.21}$$
$$\left[\int_{p+\Delta\varepsilon}^\infty \frac{d\varepsilon}{2\pi} \, (1 + f(\varepsilon))e^{-i\varepsilon\bar{\tau}} \left(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2} \right) \frac{-1}{\tau_c^3} (1 + G(\varepsilon, p, \beta)) \right.$$

$$+ \int_0^{p-\Delta\varepsilon} \frac{d\varepsilon}{2\pi} \, (1 + f(\varepsilon))e^{-i\varepsilon\bar{\tau}}\Big(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2}\Big)\frac{-1}{\tau_c^3}G(\varepsilon, p, \beta) \quad \Big]$$

$$- \frac{g^2}{16\pi p H^2} \times$$

$$\Big[\int_{p+\Delta\varepsilon}^{\infty} \frac{d\varepsilon}{2\pi} \, f(\varepsilon)e^{+i\varepsilon\bar{\tau}}\Big(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2}\Big)\frac{-1}{\tau_c^3}(1 + G(\varepsilon, p, \beta))$$

$$+ \int_0^{p-\Delta\varepsilon} \frac{d\varepsilon}{2\pi} \, f(\varepsilon)e^{+i\varepsilon\bar{\tau}}\Big(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2}\Big)\frac{-1}{\tau_c^3}G(\varepsilon, p, \beta) \quad \Big],$$

where

$$G(\varepsilon, p, \beta) \equiv \frac{2}{\beta p} \log \left(\frac{1 - e^{-\beta\frac{\varepsilon + p}{2}}}{1 - e^{-\beta\frac{|\varepsilon - p|}{2}}} \right). \tag{3.22}$$

We note that the above expression is of the following form

$$C'_{\text{off}}[f] = \int_{\varepsilon > 0} \frac{d\varepsilon}{2\pi} \left((1 + f(\varepsilon))A(\varepsilon, p, \tau_c)e^{-i\varepsilon\bar{\tau}} - f(\varepsilon)A^*(\varepsilon, p, \tau_c)e^{i\varepsilon\bar{\tau}} \right). \tag{3.23}$$

It is consistent with our ansatz for the full propagator (3.4).

Finally the on-shell collision term is evaluated as follows

$$C'_{\text{on}}[f] = -\frac{g^2}{16\pi p H^2}(1 + f(p))e^{-ip\bar{\tau}} \times \tag{3.24}$$

$$\Big[\int_{p+\Delta\varepsilon}^{\infty} \frac{d\varepsilon}{2\pi} \Big\{ \Big(\frac{1}{\varepsilon - p} + \frac{1}{\varepsilon + p}\Big)\frac{i\bar{\tau}}{\tau_c^3} + \Big(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2}\Big)\frac{-1}{\tau_c^3} \Big\}(1 + G(\varepsilon, p, \beta))$$

$$+ \int_0^{p-\Delta\varepsilon} \frac{d\varepsilon}{2\pi} \Big\{ \Big(\frac{1}{\varepsilon - p} + \frac{1}{\varepsilon + p}\Big)\frac{i\bar{\tau}}{\tau_c^3} + \Big(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2}\Big)\frac{-1}{\tau_c^3} \Big\}G(\varepsilon, p, \beta) \quad \Big]$$

$$+ \frac{g^2}{16\pi p H^2} \, f(p)e^{+ip\bar{\tau}} \times$$

$$\Big[\int_{p+\Delta\varepsilon}^{\infty} \frac{d\varepsilon}{2\pi} \Big\{ \Big(\frac{1}{\varepsilon - p} + \frac{1}{\varepsilon + p}\Big)\frac{-i\bar{\tau}}{\tau_c^3} + \Big(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2}\Big)\frac{-1}{\tau_c^3} \Big\}(1 + G(\varepsilon, p, \beta))$$

$$+ \int_0^{p-\Delta\varepsilon} \frac{d\varepsilon}{2\pi} \Big\{ \Big(\frac{1}{\varepsilon - p} + \frac{1}{\varepsilon + p}\Big)\frac{-i\bar{\tau}}{\tau_c^3} + \Big(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2}\Big)\frac{-1}{\tau_c^3} \Big\}G(\varepsilon, p, \beta) \quad \Big]$$

$$+ \frac{g^2}{32\pi^2 p H^2}\frac{-1}{\tau_c^3} \log|\Delta\varepsilon\tau_c| \, f'(p)e^{-ip\bar{\tau}} - \frac{g^2}{32\pi^2 p H^2}\frac{-1}{\tau_c^3} \log|\Delta\varepsilon\tau_c| \, f'(p)e^{+ip\bar{\tau}}.$$

The details of its derivation can be found in Appendix A.

Here we have cut-off the IR log divergences when $|\varepsilon - p| \sim 1/|\tau_c|$ because our time integration (3.12) does not diverge even when $\varepsilon = p$. From the on-shell collision term (3.24), we observe that it is necessary to introduce the wave function renormalization factor $Z(p, \tau_c)$. In the

16

last line, we find that the remaining logarithmic IR contribution leads to the modification of the thermal distribution function $\delta f(p, \tau_c)$.

So far, we have focused on the IR singularities due to the interaction. Of course, there are also the ultra-violet (UV) divergences in the collision term. The off-shell part (3.17) does not have the UV divergences because of the exponentially oscillating factor. We also assume that a generic distribution function vanishes exponentially at the UV region like the Bose distribution

$$f(p_i) \approx \frac{1}{e^{\beta p_i} - 1} \; \to \; 0. \tag{3.25}$$

From these facts, the UV divergences in the collision term is estimated as follows

$$C[f]_{\mathrm{UV}} = C'_{\mathrm{on}}[f] \tag{3.26}$$

$$\approx - \frac{g^2}{16\pi p H^2}(1 + f(p))e^{-ip\bar{\tau}} \int_{p+\Delta\varepsilon}^{\Lambda_{\mathrm{UV}}e^{Ht}} \frac{d\varepsilon}{2\pi} \, (\frac{1}{\varepsilon - p} + \frac{1}{\varepsilon + p})\frac{i\bar{\tau}}{\tau_c^3}$$

$$+ \frac{g^2}{16\pi p H^2} \, f(p)e^{+ip\bar{\tau}} \int_{p+\Delta\varepsilon}^{\Lambda_{\mathrm{UV}}e^{Ht}} \frac{d\varepsilon}{2\pi} \, (\frac{1}{\varepsilon - p} + \frac{1}{\varepsilon + p})\frac{-i\bar{\tau}}{\tau_c^3}$$

$$= - i\frac{g^2}{16\pi^2} \frac{2\bar{\tau}}{H^2\tau_c^3} \log \frac{\Lambda_{\mathrm{UV}}e^{Ht}}{q} \times (1 + f(p)) \, \frac{1}{2p}e^{-ip\bar{\tau}}$$

$$+ i\frac{g^2}{16\pi^2} \frac{2\bar{\tau}}{H^2\tau_c^3} \log \frac{\Lambda_{\mathrm{UV}}e^{Ht}}{q} \times f(p) \, \frac{1}{2p}e^{+ip\bar{\tau}}.$$

Since the integral is logarithmically divergent, we need to introduce a UV cut-off. We argue that we need to cut-off the integral at a fixed physical energy scale Λ_{UV}. As the physical energy is $\varepsilon H|\tau|$, this prescription leads to a time dependent UV cut-off $\Lambda_{\mathrm{UV}}/H|\tau| = \Lambda_{\mathrm{UV}}e^{Ht}$ in the above expression. We believe that this is a physically sensible prescription which is consistent with general covariance. In this prescription, the degrees of freedom inside the cosmological horizon remain the same with respect to time. The IR cut-off is provided by our energy resolution $\Delta\varepsilon$ in (3.26) as the IR singularity is canceled by the off-shell contribution. The final expression logarithmically depends on the virtuality $q^2 \equiv (p + \Delta\varepsilon)^2 - p^2$.

This UV divergence is renormalized by introducing a mass counter term in the action which leads to the following collision term

$$C[f]_{\delta m^2} = + i\frac{2\bar{\tau}}{H^2\tau_c^3}\delta m^2 \times (1 + f(p))\frac{1}{2p}e^{-ip\bar{\tau}} \tag{3.27}$$

$$- i\frac{2\bar{\tau}}{H^2\tau_c^3}\delta m^2 \times f(p)\frac{1}{2p}e^{+ip\bar{\tau}},$$

$$\delta m^2 = \frac{g^2}{16\pi^2} \log \frac{\Lambda_{\mathrm{UV}}e^{Ht}}{\mu},$$

where μ is the renormalization scale. After the renormalization, we obtain the following effective mass

$$m_{\mathrm{eff}}^2 = \frac{g^2}{16\pi^2} \left(\log \frac{q}{\mu} - \frac{1}{\beta p} \int_0^\infty d\varepsilon \{ \frac{1}{\varepsilon - p} + \frac{1}{\varepsilon + p} \} \log \left(\frac{1 - e^{-\beta(\varepsilon+p)/2}}{1 - e^{-\beta|\varepsilon-p|/2}} \right) \right), \tag{3.28}$$

17

including the finite temperature correction. In the zero temperature limit, it agrees with the renormalized mass in the flat space.

The IR logarithm in the collision term (3.24) leads to the change of the distribution function as we solve the Boltzmann equation

$$\delta f(p, \tau_c) = \frac{g^2}{64\pi^2 p} \frac{1}{H^2 \tau_c^2} \log |\Delta \varepsilon \tau_c| f'(p) \tag{3.29}$$

$$= -\frac{\lambda^2}{64\pi^2 p} \frac{1}{H^2 \tau_c^2} \log |\Delta \varepsilon \tau_c| \frac{\beta}{e^{\beta p} - 1} \frac{e^{\beta p}}{e^{\beta p} - 1}.$$

The wave function renormalization factor is determined as

$$\delta Z(p, \tau_c) = -\frac{g^2}{32\pi p H^2} \times \tag{3.30}$$

$$\Bigg[\int_{p+\Delta\varepsilon}^{\infty} \frac{d\varepsilon}{2\pi} \Big(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2} \Big) \frac{1}{\tau_c^2} (1 + G(\varepsilon, p, \beta))$$

$$+ \int_{0}^{p-\Delta\varepsilon} \frac{d\varepsilon}{2\pi} \Big(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2} \Big) \frac{1}{\tau_c^2} G(\varepsilon, p, \beta) \Bigg].$$

The off-shell part of the propagator is determined in terms $F_\pm$ as

$$F_+(\varepsilon, p, \tau_c) = + \frac{g^2}{32\pi p H^2} (1 + f(\varepsilon)) \times \tag{3.31}$$

$$\Bigg[\theta(\varepsilon - p) \Big(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2} \Big) \frac{1}{\tau_c^2} (1 + G(\varepsilon, p, \beta))$$

$$+ \theta(p - \varepsilon) \Big(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2} \Big) \frac{1}{\tau_c^2} G(\varepsilon, p, \beta) \Bigg],$$

$$F_-(\varepsilon, p, \tau_c) = + \frac{g^2}{32\pi p H^2} f(\varepsilon) \times \tag{3.32}$$

$$\Bigg[\theta(\varepsilon - p) \Big(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2} \Big) \frac{1}{\tau_c^2} (1 + G(\varepsilon, p, \beta))$$

$$+ \theta(p - \varepsilon) \Big(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2} \Big) \frac{1}{\tau_c^2} G(\varepsilon, p, \beta) \Bigg].$$

We observe that the on-shell weight represented by the wave function renormalization factor Z is reduced from the unity in a consistent way with the off-shell spectral weight. In this sense unitarity is respected by the interaction.

We have thus determined the full propagator inside the cosmological horizon to the leading order of the perturbation theory. We have found that the full propagator which is characterized by (3.29), (3.30), (3.31), (3.32) depends on τ_c. At first sight, it appears to change with cosmic evolution. More and more off-shell states are created with a lapse of time as on-shell states are correspondingly reduced. However we may represent (3.29), (3.30), (3.31), (3.32) by the physical quantities,

$$X \equiv \frac{x}{H|\tau}, \quad P \equiv H|\tau|p, \quad \Delta E \equiv H|\tau|\Delta\varepsilon, \quad T \equiv H|\tau|\frac{1}{\beta}, \quad M \equiv H|\tau|\mu. \tag{3.33}$$

In terms of the physical quantities, the full propagator of the original scalar field at the equal time $\bar{\tau} = 0$ is

$$G^{-+}(x_1, x_2) = \int \frac{d^3P}{(2\pi)^3 2P} \left(1 + 2(f + \delta f)\right)(1 + \delta Z)\left\{1 + (1 - \frac{m_{\text{eff}}^2}{2H^2})\frac{H^2}{P^2}\right\}e^{iP\cdot\bar{X}} \qquad (3.34)$$

$$+ \int_{E>0} \frac{dE d^3P}{(2\pi)^4 2E} \left(F_+ + F_-\right)e^{iP\cdot\bar{X}},$$

$$m_{\text{eff}}^2 = \frac{g^2}{32\pi^2} \left(\log\frac{Q^2}{M^2} - \frac{2T}{P}\int_0^\infty dE\{\frac{1}{E-P} + \frac{1}{E+P}\}\log\left(\frac{1 - e^{-(E+P)/2T}}{1 - e^{-|E-P|/2T}}\right)\right), \qquad (3.35)$$

$$\delta f = \frac{g^2}{64\pi^2 P}\frac{\partial f(P,T)}{\partial P}\log\frac{\Delta E}{H}, \qquad (3.36)$$

$$\delta Z = -\frac{g^2}{32\pi P} \times \qquad (3.37)$$

$$\left[\frac{1}{2\pi}(\frac{1}{\Delta E} - \frac{1}{2P}) + \int_{P+\Delta E}^\infty \frac{dE}{2\pi}(\frac{1}{(E-P)^2} - \frac{1}{(E+P)^2})\frac{2T}{P}\log\left(\frac{1 - e^{-\frac{E+P}{2T}}}{1 - e^{-\frac{E-P}{2T}}}\right)\right.$$

$$\left. + \int_0^{P-\Delta E} \frac{dE}{2\pi}(\frac{1}{(E-P)^2} - \frac{1}{(E+P)^2})\frac{2T}{P}\log\left(\frac{1 - e^{-\frac{E+P}{2T}}}{1 - e^{-\frac{P-E}{2T}}}\right)\right],$$

$$F_+ = \frac{g^2}{32\pi P}(1 + f(E,T)) \times \qquad (3.38)$$

$$\left[\theta(E-P)(\frac{1}{(E-P)^2} - \frac{1}{(E+P)^2})(1 + \frac{2T}{P}\log\left(\frac{1 - e^{-\frac{E+P}{2T}}}{1 - e^{-\frac{E-P}{2T}}}\right))\right.$$

$$\left. + \theta(P-E)(\frac{1}{(E-P)^2} - \frac{1}{(E+P)^2})\frac{2T}{P}\log\left(\frac{1 - e^{-\frac{E+P}{2T}}}{1 - e^{-\frac{P-E}{2T}}}\right)\right],$$

$$F_- = \frac{g^2}{32\pi P}f(E,T) \times \qquad (3.39)$$

$$\left[\theta(E-P)(\frac{1}{(E-P)^2} - \frac{1}{(E+P)^2})(1 + \frac{2T}{P}\log\left(\frac{1 - e^{-\frac{E+P}{2T}}}{1 - e^{-\frac{E-P}{2T}}}\right))\right.$$

$$\left. + \theta(P-E)(\frac{1}{(E-P)^2} - \frac{1}{(E+P)^2})\frac{2T}{P}\log\left(\frac{1 - e^{-\frac{E+P}{2T}}}{1 - e^{-\frac{P-E}{2T}}}\right)\right].$$

We find that the explicit τ_c dependence disappears in these expressions. If we focus on the physics at the fixed physical energy scale E, it remains the same with cosmic evolution. It is a very sensible conclusion as we do not expect physics such as particle mass to change with

19

cosmic evolution. For a fixed ε, the physical energy E decreases with time evolution. So the cosmic evolution is identical to the evolution under the renormalization group. We recall here that the radial coordinate in AdS space corresponds to the energy scale in AdS/CFT correspondence. Since the radial coordinate in AdS space is related to the time coordinate in dS space by analytic continuation, dS space seems to be related to AdS space in this respect. The only physical time dependence appears through the temperature T as it cools down linearly with τ_c for a fixed β.

We do find a non-trivial modification of the distribution function from the Bose distribution due to a large IR effect. The effect of the interaction on the distribution function (3.29) is such that it reduces the particle density in comparison to the Bose distribution. This effect can be understood as follows. A single particle can turn into two particles due to the cubic interaction. So such off-shell two particle states are created while the on-shell state weight is reduced by the same amount due to unitarity. The off-shell states cost more energy and so are less numerous due to the Bose distribution function. The net effect is the further reduction of the particle density.

In this section, we have investigated the effects of the interaction on the propagator well inside the cosmological horizon. The spectral weight of the off-shell states increases with time while the weight of the on-shell states decreases due to the interaction. The modification of the Bose distribution is analogous to QCD where the logarithmic divergence requires the scale dependent modification of the parton distribution function. In term of the physical energy and momentum variables, explicit time dependence disappears and the time evolution may be identified with the renormalization group evolution. So we find that the effects of the interaction in dS space parallel to those in flat space. As it is explained in Appendix B, these features are also shared by $\lambda\varphi^4$ theory. We thus expect they are the universal features of the interacting field theories in dS space.

Nevertheless we should keep in mind that we have investigated the propagator near flat space and the expansion in terms of $1/p\tau_c$ breaks down near the cosmological horizon. To fully understand the behavior of the two point function in dS space, we have to extend our work to the region $|p\tau_c| \sim 1$ and $|p\tau_c| \ll 1$. We investigate the quantum effects outside the cosmological horizon in Part III. In this region, the dS symmetry can be broken and the time dependence which is not absorbed by physical quantities can be induced.

Before concluding this section, we briefly investigate the non-thermal distribution case. The modification of the distribution function $\delta f(p, \tau_c)$ by the cubic interaction is roughly of the following magnitude

$$\frac{\partial f(p, \tau_1, \tau_2)}{\partial \tau_c} \sim \frac{g^2}{p} \frac{1}{H^2 \tau_c^3} \int_{p+|1/\tau_c|} d\varepsilon \frac{1}{(\varepsilon - p)^2} \tag{3.40}$$

$$\sim \frac{g^2}{p} \frac{1}{H^2 \tau_c^2},$$

$$\delta f(p, \tau_1, \tau_2) \sim \frac{g^2}{p} \frac{1}{H^2 \tau_c}.$$

So it is $O(1/p|\tau_c|)$ in a generic case instead of $O(1/(p\tau_c)^2)$ for the thermal case. While it is much larger than the change of the thermal distribution when $p|\tau_c| \gg 1$, it becomes only important near the cosmological horizon. Although the thermalization may take place when

the coupling is strong enough $g > H$, it could only occur near the cosmological horizon. At the higher loop level, the thermalization could also take place through the effective n point couplings. We find this is a very interesting problem which requires further investigations.

Part III

Quantum effects from outside the cosmological horizon

4   Infra-red divergence of propagator

As mentioned in Section 1, the propagator for a massless and minimally coupled field has an IR divergence. The origin of the IR divergence is the scale invariant fluctuation spectrum. It is dominant outside the cosmological horizon $P \ll H$:

$$\phi_{\mathbf{p}}(x) \sim \frac{H}{\sqrt{2p^3}}. \tag{4.1}$$

In this section, we explain how to regularize the IR divergence.

For simplicity, let us estimate the magnitude of the quantum fluctuation by taking the coincident limit of the propagator. It consists of the contributions from inside and outside the cosmological horizon as follows

$$\langle \varphi(x)\varphi(x) \rangle \sim \int_{P>H} \frac{d^3P}{(2\pi)^3} \frac{1}{2P} + H^2 \int_{P<H} \frac{d^3P}{(2\pi)^3} \frac{1}{2P^3}. \tag{4.2}$$

The UV contribution ($P > H$) is quadratically divergent just like in Minkowski space. It can be regularized and renormalized in an identical way. The logarithmic IR divergence due to the contributions from outside the cosmological horizon ($P < H$) is specific to dS space.

To regularize this IR divergence, we introduce an IR cut-off ε_0 which fixes the minimum value of the comoving momentum as in [19]:

$$\int_{\varepsilon_0 a^{-1}(\tau)}^{H} dP. \tag{4.3}$$

Note that the equation of motion is not satisfied if we fix the minimum value of the Physical momentum. With this prescription, more degrees of freedom go out of the cosmological horizon at $P = H$ with cosmic evolution. On the other hand, the UV cut-off Λ_{UV} fixes the maximum value of the physical momentum:

$$\int_{H}^{\Lambda_{UV}} dP. \tag{4.4}$$

$$+ \int \sqrt{-g_3}d^4x_3\sqrt{-g_4}d^4x_4 \ G_0^R(x_1, x_3)\Sigma^R(x_3, x_4)G^{-+}(x_4, x_2)$$

$$+ \int \sqrt{-g_3}d^4x_3\sqrt{-g_4}d^4x_4 \ G_0^R(x_1, x_3)\Sigma^{-+}(x_3, x_4)G^A(x_4, x_2)$$

$$+ \int \sqrt{-g_3}d^4x_3\sqrt{-g_4}d^4x_4 \ G_0^{-+}(x_1, x_3)\Sigma^A(x_3, x_4)G^A(x_4, x_2).$$

Here we have introduced the retarded and the advanced propagators as follows

$$G^R(x_1, x_2) \equiv \theta(t_1 - t_2)[G^{-+}(x_1, x_2) - G^{+-}(x_1, x_2)], \qquad (2.10)$$
$$G^A(x_1, x_2) \equiv -\theta(t_2 - t_1)[G^{-+}(x_1, x_2) - G^{+-}(x_1, x_2)].$$

In the same way, the following identity also holds

$$G^{-+}(x_1, x_2) = G_0^{-+}(x_1, x_2) \qquad (2.11)$$

$$+ \int \sqrt{-g_3}d^4x_3\sqrt{-g_4}d^4x_4 \ G^R(x_1, x_3)\Sigma^R(x_3, x_4)G_0^{-+}(x_4, x_2)$$

$$+ \int \sqrt{-g_3}d^4x_3\sqrt{-g_4}d^4x_4 \ G^R(x_1, x_3)\Sigma^{-+}(x_3, x_4)G_0^A(x_4, x_2)$$

$$+ \int \sqrt{-g_3}d^4x_3\sqrt{-g_4}d^4x_4 \ G^{-+}(x_1, x_3)\Sigma^A(x_3, x_4)G_0^A(x_4, x_2).$$

In (2.9) and (2.11), we observe that a retarded or advanced propagator exists at each vertex. It is because of the causality. That is, the integrands are zero outside the past light corn.

In this formalism, the integrations over time are manifestly finite due to the causality. This formalism is called the Schwinger-Keldysh formalism. In order to understand the effects of the interaction, we derive a Boltzmann equation on the dS background from a Schwinger-Dyson equation in Part II.

Part II

Quantum effects from inside the cosmological horizon

3   Boltzmann equations from Schwinger-Dyson equations

In Part II, we investigate the quantum effects well inside the cosmological horizon. Since the particle description is valid in this region, we can evaluate how the particle creation effects in dS space emerge to physical quantities.

Here we redefine the scalar field as $\varphi \to H\tau\varphi$ for a convenience. We can simply scale it back to find the original scalar field. In terms of the rescaled field, the quadratic action for a massless and minimally coupled field becomes

$$S_2 = \frac{1}{2} \int d^4x \ \varphi \left(-\partial_\tau^2 + \partial_{\mathbf{x}}^2 + \frac{2}{\tau^2} \right) \varphi, \tag{3.1}$$

and the wave function is

$$\phi_{\mathbf{p}}(x) = \frac{1}{\sqrt{2p}} (1 - i\frac{1}{p\tau}) \ e^{-ip\tau + i\mathbf{p}\cdot\mathbf{x}}. \tag{3.2}$$

In a time dependent background, we need to consider excited states in general. For such a state, the expectation value of the number operator $\langle a^\dagger a \rangle$ is non-vanishing. We introduce a distribution function f for scalar particles as follows

$$\langle a_{\mathbf{p}}^\dagger a_{\mathbf{q}} \rangle \equiv f(\mathbf{p}) \times (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}). \tag{3.3}$$

One of our main objectives in this section is to understand the time dependence of the distribution function $f(\mathbf{p})$ due to the interaction. We utilize a Boltzmann equation for this purpose. Boltzmann equations govern the time evolution of the distribution functions. They are widely used to study non-equilibrium physics. In fact there is a long history of the microscopic derivation of Boltzmann equations in non-equilibrium physics using Schwinger-Keldysh formalism [4, 2, 5]. In this section, we systematically investigate the propagator in dS space from a Schwinger-Dyson equation.

We assume that the full propagator in dS space has the following form

$$
\begin{aligned}
G^{-+}(x_1, x_2) = \ & \int \frac{d^3p}{(2\pi)^3} \left[(1 + f(p, \tau_c)) Z(p, \tau_c) \phi_{\mathbf{p}}(x_1) \phi_{\mathbf{p}}^*(x_2) \right. \\
& \left. + f(p, \tau_c) Z^*(p, \tau_c) \phi_{\mathbf{p}}^*(x_1) \phi_{\mathbf{p}}(x_2) \right] \\
& + \int_{\varepsilon>0} \frac{d\varepsilon d^3p}{(2\pi)^4} \frac{1}{2\varepsilon} [F_+(\varepsilon, p, \tau_c) \ e^{-i\varepsilon(\tau_1-\tau_2)+i\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_2)} \\
& + F_-(\varepsilon, p, \tau_c) \ e^{+i\varepsilon(\tau_1-\tau_2)-i\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_2)}].
\end{aligned}
\tag{3.4}
$$

The propagator depends on the average and the relative time:

$$\tau_c \equiv \frac{\tau_1 + \tau_2}{2}, \qquad \bar{\tau} \equiv \tau_1 - \tau_2. \tag{3.5}$$

It consists of the on-shell part and the off-shell part. In the on-shell part, we have introduced the wave function renormalization factor $Z(p, \tau_c)$. The off-shell part depends on the spectral function $F_\pm(\varepsilon, p, \tau_c)$. We assume that $f, Z, F_\pm$ evolve with the average time τ_c. We investigate the propagator in the region:

$$|\tau_c| \gg |\bar{\tau}|, \qquad |\tau_c| \gg 1/p. \tag{3.6}$$

The second assumption implies that we investigate the propagator well inside the cosmological horizon.

From (2.9) and (2.11), we can derive the following identity

$$G_0^{-1}|_1 G^{-+}(x_1, x_2) - G_0^{-1}|_2 G^{-+}(x_1, x_2) \tag{3.7}$$

$$= + \sqrt{-g_1} \int \sqrt{-g_3} d^4 x_3 \ \Sigma^R(x_1, x_3) G^{-+}(x_3, x_2)$$

$$+ \sqrt{-g_1} \int \sqrt{-g_3} d^4 x_3 \ \Sigma^{-+}(x_1, x_3) G^A(x_3, x_2)$$

$$- \sqrt{-g_2} \int \sqrt{-g_3} d^4 x_3 \ G^R(x_1, x_3) \Sigma^{-+}(x_3, x_2)$$

$$- \sqrt{-g_2} \int \sqrt{-g_3} d^4 x_3 \ G^{-+}(x_1, x_3) \Sigma^A(x_3, x_2).$$

By substituting the expression for the full propagator (3.4) into the left-hand side of the Schwinger-Dyson equation (3.7), we obtain

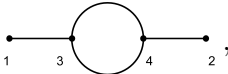$$G_0^{-1}|_1 G^{-+}(x_1, x_2) - G_0^{-1}|_2 G^{-+}(x_1, x_2) \tag{3.8}$$

$$\sim \int \frac{d^3 p}{(2\pi)^3} \left[\left(\frac{\partial}{\partial \tau_c} + \frac{i}{p} \frac{\partial^2}{\partial \bar{\tau} \partial \tau_c} \right) \{ (1 + f(p, \tau_c)) Z(p, \tau_c) \} \times e^{-ip\bar{\tau} + i\mathbf{p} \cdot \bar{\mathbf{x}}} \right.$$

$$\left. - \left(\frac{\partial}{\partial \tau_c} - \frac{i}{p} \frac{\partial^2}{\partial \bar{\tau} \partial \tau_c} \right) \{ f(p, \tau_c) Z^*(p, \tau_c) \} \times e^{+ip\bar{\tau} - i\mathbf{p} \cdot \bar{\mathbf{x}}} \right]$$

$$+ \int_{\varepsilon > 0} \frac{d\varepsilon d^3 p}{(2\pi)^4} \left[\left(\frac{\partial}{\partial \tau_c} + \frac{i}{\varepsilon} \frac{\partial^2}{\partial \bar{\tau} \partial \tau_c} \right) F_+(\varepsilon, p, \tau_c) \times e^{-i\varepsilon\bar{\tau} + i\mathbf{p} \cdot \bar{\mathbf{x}}} \right.$$

$$\left. - \left(\frac{\partial}{\partial \tau_c} - \frac{i}{\varepsilon} \frac{\partial^2}{\partial \bar{\tau} \partial \tau_c} \right) F_-(\varepsilon, p, \tau_c) \times e^{+i\varepsilon\bar{\tau} - i\mathbf{p} \cdot \bar{\mathbf{x}}} \right].$$

Here we recall the following definitions

$$G_0^{-1} \equiv i(\partial_\tau^2 - \partial_{\mathbf{x}}^2 - \frac{2}{\tau^2}), \tag{3.9}$$

$$G_0^{-1}|_1 G^R(x_1, x_2) = \delta^{(4)}(x_1 - x_2),$$

$$G_0^{-1}|_2 G^A(x_1, x_2) = \delta^{(4)}(x_1 - x_2).$$

In (3.8) we have shown the leading terms in the power series expansion of $1/p\tau_c$.

The right-hand side of Eq.(3.7) corresponds to the collision term $C[f]$. In this section, we investigate the effects of the interaction in $g\varphi^3$ theory at the one loop level. We subsequently find that this theory captures the essential features of more generic field theories such as $\lambda\varphi^4$ theory. The self-energy is

$$\tag{3.10}$$

$$\Sigma^{ij}(x_3, x_4) = \frac{(-ig)^2}{2} G^{ij}(x_3, x_4) G^{ij}(x_3, x_4), \quad i, j = +, -.$$

To the leading order in perturbation theory, we can approximate that $f(p, \tau_c) = f(p)$, $Z(p, \tau_c) = 1$, $F_\pm(\varepsilon, p, \tau_c) = 0$ in the collision term. We also expand the collision term

by the power series in $1/|p\tau_c|$ type factors which can be justified well inside the cosmological horizon. It is a kind of the derivative expansion of the Moyal product in the Wigner representation. We indeed find the particle production effects due to the non-conservation of the energy in this expansion.

In this investigation, we need to perform the following integrations at the interaction vertices.

$$\int_{-\infty}^{\tau_i} d\tau_3 \, \frac{1}{\tau_3^n} e^{i(\varepsilon \pm p)\tau_3} \qquad n \in \mathbf{N}, \quad i = 1, 2, \tag{3.11}$$

where $\varepsilon = \pm p_1 \pm p_2$. We evaluate these integrations in the assumption $|(\varepsilon \pm p)\tau_i| \gg 1$. For our purpose, it suffices to evaluate them to the next leading order

$$\int_{-\infty}^{\tau_i} d\tau_3 \, \frac{1}{\tau_3^n} e^{i(\varepsilon \pm p)\tau_3} \sim e^{i(\varepsilon \pm p)\tau_i} \times \left[\frac{1}{i(\varepsilon \pm p)\tau_i^n} + \frac{-n}{(\varepsilon \pm p)^2 \tau_i^{n+1}} \right]. \tag{3.12}$$

By using these approximations, we derive a Boltzmann equation in dS space. In what follows, we investigate the collision terms and their properties in detail.

We henceforth suppress the following integration factor in the propagator

$$\int \frac{d^3 p}{(2\pi)^3} e^{i\mathbf{p}\cdot\bar{\mathbf{x}}}. \tag{3.13}$$

In other words we work in the momentum space by performing the Fourier transformation with respect to the spacial coordinate $\bar{\mathbf{x}}$.

3.1 The structure of the collision term

From the Schwinger-Dyson equation (3.7), we observe that the collision term has the on-shell part and the off-sell part. Firstly, the on-shell part comes from the following contributions

$$C_{\text{on}}[f] = + \sqrt{-g_1} \int \sqrt{-g_3} d^4 x_3 \, \Sigma^R(x_1, x_3) G^{-+}(x_3, x_2) \tag{3.14}$$

$$- \sqrt{-g_2} \int \sqrt{-g_3} d^4 x_3 \, G^{-+}(x_1, x_3) \Sigma^A(x_3, x_2)$$

$$\propto e^{\mp i p \bar{\tau}}.$$

We evaluate the on-shell part to the leading non-trivial order $\mathcal{O}(1/\tau_c^3)$ as

$$C_{\text{on}}[f] = -(1 + f(p)) e^{-i p \bar{\tau}} \frac{g^2}{16\pi p^2 H^2} \times \tag{3.15}$$

$$\left[\int_p^\infty \frac{d\varepsilon}{2\pi} \left\{ \left(\frac{1}{\varepsilon - p} + \frac{1}{\varepsilon + p} \right) \frac{i\bar{\tau}}{\tau_c^3} + \left(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2} \right) \frac{-1}{\tau_c^3} \right\} \right.$$

$$\left. \times \int_{\frac{\varepsilon - p}{2}}^{\frac{\varepsilon + p}{2}} dp_1 \left\{ (1 + f(p_1))(1 + f(\varepsilon - p_1)) - f(p_1) f(\varepsilon - p_1) \right\} \right.$$

13

$$+ 2 \int_0^p \frac{d\varepsilon}{2\pi} \Big\{ \Big(\frac{1}{\varepsilon - p} + \frac{1}{\varepsilon + p} \Big) \frac{i\bar{\tau}}{\tau_c^3} + \Big(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2} \Big) \frac{-1}{\tau_c^3} \Big\}$$

$$\times \int_{\frac{\varepsilon + p}{2}}^{\infty} dp_1 \big\{ (1 + f(p_1)) f(p_1 - \varepsilon) - f(p_1)(1 + f(p_1 - \varepsilon)) \big\} \quad \Big]$$

$$+ f(p) \, e^{+ip\bar{\tau}} \frac{g^2}{16\pi p^2 H^2} \times$$

$$\Big[\int_p^{\infty} \frac{d\varepsilon}{2\pi} \Big\{ \Big(\frac{1}{\varepsilon - p} + \frac{1}{\varepsilon + p} \Big) \frac{-i\bar{\tau}}{\tau_c^3} + \Big(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2} \Big) \frac{-1}{\tau_c^3} \Big\}$$

$$\times \int_{\frac{\varepsilon - p}{2}}^{\frac{\varepsilon + p}{2}} dp_1 \big\{ (1 + f(p_1))(1 + f(\varepsilon - p_1)) - f(p_1) f(\varepsilon - p_1) \big\}$$

$$+ 2 \int_0^p \frac{d\varepsilon}{2\pi} \Big\{ \Big(\frac{1}{\varepsilon - p} + \frac{1}{\varepsilon + p} \Big) \frac{-i\bar{\tau}}{\tau_c^3} + \Big(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2} \Big) \frac{-1}{\tau_c^3} \Big\}$$

$$\times \int_{\frac{\varepsilon + p}{2}}^{\infty} dp_1 \big\{ (1 + f(p_1)) f(p_1 - \varepsilon) - f(p_1)(1 + f(p_1 - \varepsilon)) \big\} \quad \Big].$$

See Appendix A for the details of the calculation.

Secondly, the off-shell part originates from the following contribution

$$C_{\text{off}}[f] = + \sqrt{-g_1} \int \sqrt{-g_3} d^4 x_3 \, \Sigma^{-+}(x_1, x_3) G^A(x_3, x_2) \tag{3.16}$$

$$- \sqrt{-g_2} \int \sqrt{-g_3} d^4 x_3 \, G^R(x_1, x_3) \Sigma^{-+}(x_3, x_2)$$

$$\propto \int_0^{\infty} dp_1 \int_{|p_1 - p|}^{p_1 + p} dp_2 \, e^{-i(\pm p_1 \pm p_2)\bar{\tau}}.$$

The off-shell part is also calculated to $\mathcal{O}(1/\tau_c^3)$ as

$$C_{\text{off}}[f] = + \frac{g^2}{16\pi p^2 H^2} \times \tag{3.17}$$

$$\Big[\int_p^{\infty} \frac{d\varepsilon}{2\pi} e^{-i\varepsilon\bar{\tau}} \Big(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2} \Big) \frac{-1}{\tau_c^3} \int_{\frac{\varepsilon - p}{2}}^{\frac{\varepsilon + p}{2}} dp_1 \, (1 + f(p_1))(1 + f(\varepsilon - p_1))$$

$$+ 2 \int_0^p \frac{d\varepsilon}{2\pi} e^{-i\varepsilon\bar{\tau}} \Big(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2} \Big) \frac{-1}{\tau_c^3} \int_{\frac{\varepsilon + p}{2}}^{\infty} dp_1 \, (1 + f(p_1)) f(p_1 - \varepsilon) \quad \Big]$$

$$- \frac{g^2}{16\pi p^2 H^2} \times$$

$$\Big[\int_p^{\infty} \frac{d\varepsilon}{2\pi} e^{+i\varepsilon\bar{\tau}} \Big(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2} \Big) \frac{-1}{\tau_c^3} \int_{\frac{\varepsilon - p}{2}}^{\frac{\varepsilon + p}{2}} dp_1 \, f(p_1) f(\varepsilon - p_1)$$

$$+ 2 \int_0^p \frac{d\varepsilon}{2\pi} e^{+i\varepsilon\bar{\tau}} \Big(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2} \Big) \frac{-1}{\tau_c^3} \int_{\frac{\varepsilon + p}{2}}^{\infty} dp_1 \, f(p_1)(1 + f(p_1 - \varepsilon)) \Big].$$

We note that the both on-shell (3.15) and off-shell (3.17) collision terms have infra-red divergences at $\varepsilon = p$. There is a standard procedure to deal with this problem in massless

14

field theory and we find that it also works here. First of all, we need to recall that any experiment has a finite energy resolution $\Delta\varepsilon$. So we need to add the on-shell and off-shell collision terms within the energy resolution $\Delta\varepsilon$. We first divide the integration range of $C_{\text{off}}[f]$ as follows

$$\int_p^\infty = \int_{p+\Delta\varepsilon}^\infty + \int_p^{p+\Delta\varepsilon} \ , \quad \int_0^p = \int_0^{p-\Delta\varepsilon} + \int_{p-\Delta\varepsilon}^p \ . \tag{3.18}$$

We then redefine the on-shell term $C'_{\text{on}}[f]$ and the off-shell term $C'_{\text{off}}[f]$ by transferring the contribution of $C_{\text{off}}[f]$ within the energy resolution $p - \Delta\varepsilon \leq \varepsilon \leq p + \Delta\varepsilon$ to $C_{\text{on}}[f]$. The explicit expressions are shown in Appendix A.

When $f(p) = 0$, we find that infra-red divergences cancel out in this procedure. In the next subsection, we investigate the case when f is a thermal distribution. For a generic distribution, the cancellation does not take place and we seem to face linear IR divergences. However there is no real infra-red divergence in our problem since the time integration range in (3.12) is bounded by τ_c. We thus argue that the linear divergence should be cut-off at $|p - \varepsilon| \sim 1/|\tau_c|$.

Before investigating the thermal distribution case, we point out the difference between Minkowski space and dS space with respect to the collision term. In Minkowski space, the collision term does not have the off-shell term due to the time translation symmetry

$$C_{\text{off}}[f] \propto \int \frac{d\varepsilon}{2\pi} \, 2\pi\delta(\varepsilon - p)e^{\mp i\varepsilon\bar\tau} = \ e^{\mp ip\bar\tau} \ \implies \ C'_{\text{off}}[f] = 0. \tag{3.19}$$

On the other hand, as we observe in (3.17), the collision term in dS space has the off-shell term due to the absence of the time translation symmetry. This is why we have introduced the spectral function $F_\pm(\varepsilon, p, \tau_c)$ in the full propagator (3.4).

3.2 Thermal distribution case

We focus on the case that the initial distribution function is thermal in this subsection

$$f(p) = \frac{1}{e^{\beta p} - 1}, \tag{3.20}$$

where we introduce an inverse temperature β as a free parameter. In Minkowski space the thermal distribution is obtained as the solution of the Boltzmann equation. On the other hand, we find that the collision term in dS space is non-vanishing even for the thermal distribution.

The off-shell collision term can be evaluated as follows

$$C'_{\text{off}}[f] = + \frac{g^2}{16\pi p H^2} \times \tag{3.21}$$

$$\left[\int_{p+\Delta\varepsilon}^\infty \frac{d\varepsilon}{2\pi} \, (1 + f(\varepsilon))e^{-i\varepsilon\bar\tau} \left(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2} \right) \frac{-1}{\tau_c^3} (1 + G(\varepsilon, p, \beta)) \right.$$

15

$$+ \int_0^{p-\Delta\varepsilon} \frac{d\varepsilon}{2\pi} \ (1+f(\varepsilon))e^{-i\varepsilon\bar{\tau}} (\frac{1}{(\varepsilon-p)^2} - \frac{1}{(\varepsilon+p)^2}) \frac{-1}{\tau_c^3} G(\varepsilon,p,\beta) \quad \Big]$$

$$- \frac{g^2}{16\pi pH^2} \times$$

$$\Big[\int_{p+\Delta\varepsilon}^{\infty} \frac{d\varepsilon}{2\pi} \ f(\varepsilon)e^{+i\varepsilon\bar{\tau}} (\frac{1}{(\varepsilon-p)^2} - \frac{1}{(\varepsilon+p)^2}) \frac{-1}{\tau_c^3} (1+G(\varepsilon,p,\beta))$$

$$+ \int_0^{p-\Delta\varepsilon} \frac{d\varepsilon}{2\pi} \ f(\varepsilon)e^{+i\varepsilon\bar{\tau}} (\frac{1}{(\varepsilon-p)^2} - \frac{1}{(\varepsilon+p)^2}) \frac{-1}{\tau_c^3} G(\varepsilon,p,\beta) \quad \Big],$$

where

$$G(\varepsilon,p,\beta) \equiv \frac{2}{\beta p} \log \left(\frac{1-e^{-\beta\frac{\varepsilon+p}{2}}}{1-e^{-\beta\frac{|\varepsilon-p|}{2}}} \right). \tag{3.22}$$

We note that the above expression is of the following form

$$C'_{\text{off}}[f] = \int_{\varepsilon>0} \frac{d\varepsilon}{2\pi} \left((1+f(\varepsilon))A(\varepsilon,p,\tau_c)e^{-i\varepsilon\bar{\tau}} - f(\varepsilon)A^*(\varepsilon,p,\tau_c)e^{i\varepsilon\bar{\tau}} \right). \tag{3.23}$$

It is consistent with our ansatz for the full propagator (3.4).

Finally the on-shell collision term is evaluated as follows

$$C'_{\text{on}}[f] = - \frac{g^2}{16\pi pH^2}(1+f(p))e^{-ip\bar{\tau}} \times \tag{3.24}$$

$$\Big[\int_{p+\Delta\varepsilon}^{\infty} \frac{d\varepsilon}{2\pi} \{ (\frac{1}{\varepsilon-p} + \frac{1}{\varepsilon+p})\frac{i\bar{\tau}}{\tau_c^3} + (\frac{1}{(\varepsilon-p)^2} - \frac{1}{(\varepsilon+p)^2})\frac{-1}{\tau_c^3} \}(1+G(\varepsilon,p,\beta))$$

$$+ \int_0^{p-\Delta\varepsilon} \frac{d\varepsilon}{2\pi} \{ (\frac{1}{\varepsilon-p} + \frac{1}{\varepsilon+p})\frac{i\bar{\tau}}{\tau_c^3} + (\frac{1}{(\varepsilon-p)^2} - \frac{1}{(\varepsilon+p)^2})\frac{-1}{\tau_c^3} \}G(\varepsilon,p,\beta) \ \Big]$$

$$+ \frac{g^2}{16\pi pH^2} \ f(p)e^{+ip\bar{\tau}} \times$$

$$\Big[\int_{p+\Delta\varepsilon}^{\infty} \frac{d\varepsilon}{2\pi} \{ (\frac{1}{\varepsilon-p} + \frac{1}{\varepsilon+p})\frac{-i\bar{\tau}}{\tau_c^3} + (\frac{1}{(\varepsilon-p)^2} - \frac{1}{(\varepsilon+p)^2})\frac{-1}{\tau_c^3} \}(1+G(\varepsilon,p,\beta))$$

$$+ \int_0^{p-\Delta\varepsilon} \frac{d\varepsilon}{2\pi} \{ (\frac{1}{\varepsilon-p} + \frac{1}{\varepsilon+p})\frac{-i\bar{\tau}}{\tau_c^3} + (\frac{1}{(\varepsilon-p)^2} - \frac{1}{(\varepsilon+p)^2})\frac{-1}{\tau_c^3} \}G(\varepsilon,p,\beta) \ \Big]$$

$$+ \frac{g^2}{32\pi^2 pH^2} \frac{-1}{\tau_c^3} \log|\Delta\varepsilon\tau_c| \ f'(p)e^{-ip\bar{\tau}} - \frac{g^2}{32\pi^2 pH^2} \frac{-1}{\tau_c^3} \log|\Delta\varepsilon\tau_c| \ f'(p)e^{+ip\bar{\tau}}.$$

The details of its derivation can be found in Appendix A.

Here we have cut-off the IR log divergences when $|\varepsilon-p| \sim 1/|\tau_c|$ because our time integration (3.12) does not diverge even when $\varepsilon = p$. From the on-shell collision term (3.24), we observe that it is necessary to introduce the wave function renormalization factor $Z(p,\tau_c)$. In the

16

last line, we find that the remaining logarithmic IR contribution leads to the modification of the thermal distribution function $\delta f(p, \tau_c)$.

So far, we have focused on the IR singularities due to the interaction. Of course, there are also the ultra-violet (UV) divergences in the collision term. The off-shell part (3.17) does not have the UV divergences because of the exponentially oscillating factor. We also assume that a generic distribution function vanishes exponentially at the UV region like the Bose distribution

$$f(p_i) \approx \frac{1}{e^{\beta p_i} - 1} \ \to \ 0. \tag{3.25}$$

From these facts, the UV divergences in the collision term is estimated as follows

$$C[f]_{\mathrm{UV}} = C'_{\mathrm{on}}[f] \tag{3.26}$$

$$\approx - \frac{g^2}{16\pi p H^2}(1 + f(p))e^{-ip\bar{\tau}} \int_{p+\Delta\varepsilon}^{\Lambda_{\mathrm{UV}} e^{Ht}} \frac{d\varepsilon}{2\pi} \, (\frac{1}{\varepsilon - p} + \frac{1}{\varepsilon + p})\frac{i\bar{\tau}}{\tau_c^3}$$

$$+ \frac{g^2}{16\pi p H^2} \, f(p)e^{+ip\bar{\tau}} \int_{p+\Delta\varepsilon}^{\Lambda_{\mathrm{UV}} e^{Ht}} \frac{d\varepsilon}{2\pi} \, (\frac{1}{\varepsilon - p} + \frac{1}{\varepsilon + p})\frac{-i\bar{\tau}}{\tau_c^3}$$

$$= - i\frac{g^2}{16\pi^2} \frac{2\bar{\tau}}{H^2\tau_c^3} \log \frac{\Lambda_{\mathrm{UV}} e^{Ht}}{q} \times (1 + f(p)) \, \frac{1}{2p}e^{-ip\bar{\tau}}$$

$$+ i\frac{g^2}{16\pi^2} \frac{2\bar{\tau}}{H^2\tau_c^3} \log \frac{\Lambda_{\mathrm{UV}} e^{Ht}}{q} \times f(p) \, \frac{1}{2p}e^{+ip\bar{\tau}}.$$

Since the integral is logarithmically divergent, we need to introduce a UV cut-off. We argue that we need to cut-off the integral at a fixed physical energy scale Λ_{UV}. As the physical energy is $\varepsilon H|\tau|$, this prescription leads to a time dependent UV cut-off $\Lambda_{\mathrm{UV}}/H|\tau| = \Lambda_{\mathrm{UV}} e^{Ht}$ in the above expression. We believe that this is a physically sensible prescription which is consistent with general covariance. In this prescription, the degrees of freedom inside the cosmological horizon remain the same with respect to time. The IR cut-off is provided by our energy resolution $\Delta\varepsilon$ in (3.26) as the IR singularity is canceled by the off-shell contribution. The final expression logarithmically depends on the virtuality $q^2 \equiv (p + \Delta\varepsilon)^2 - p^2$.

This UV divergence is renormalized by introducing a mass counter term in the action which leads to the following collision term

$$C[f]_{\delta m^2} = + i\frac{2\bar{\tau}}{H^2\tau_c^3}\delta m^2 \times (1 + f(p))\frac{1}{2p}e^{-ip\bar{\tau}} \tag{3.27}$$

$$- i\frac{2\bar{\tau}}{H^2\tau_c^3}\delta m^2 \times f(p)\frac{1}{2p}e^{+ip\bar{\tau}},$$

$$\delta m^2 = \frac{g^2}{16\pi^2} \log \frac{\Lambda_{\mathrm{UV}} e^{Ht}}{\mu},$$

where μ is the renormalization scale. After the renormalization, we obtain the following effective mass

$$m_{\mathrm{eff}}^2 = \frac{g^2}{16\pi^2} \left(\log \frac{q}{\mu} - \frac{1}{\beta p} \int_0^\infty d\varepsilon \{\frac{1}{\varepsilon - p} + \frac{1}{\varepsilon + p}\} \log \left(\frac{1 - e^{-\beta(\varepsilon + p)/2}}{1 - e^{-\beta|\varepsilon - p|/2}} \right) \right), \tag{3.28}$$

17

including the finite temperature correction. In the zero temperature limit, it agrees with the renormalized mass in the flat space.

The IR logarithm in the collision term (3.24) leads to the change of the distribution function as we solve the Boltzmann equation

$$\delta f(p, \tau_c) = \frac{g^2}{64\pi^2 p} \frac{1}{H^2 \tau_c^2} \log |\Delta \varepsilon \tau_c| f'(p) \tag{3.29}$$

$$= -\frac{\lambda^2}{64\pi^2 p} \frac{1}{H^2 \tau_c^2} \log |\Delta \varepsilon \tau_c| \frac{\beta}{e^{\beta p} - 1} \frac{e^{\beta p}}{e^{\beta p} - 1}.$$

The wave function renormalization factor is determined as

$$\delta Z(p, \tau_c) = -\frac{g^2}{32\pi p H^2} \times \tag{3.30}$$

$$\left[\int_{p+\Delta\varepsilon}^{\infty} \frac{d\varepsilon}{2\pi} \left(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2} \right) \frac{1}{\tau_c^2} (1 + G(\varepsilon, p, \beta)) \right.$$

$$\left. + \int_0^{p-\Delta\varepsilon} \frac{d\varepsilon}{2\pi} \left(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2} \right) \frac{1}{\tau_c^2} G(\varepsilon, p, \beta) \right].$$

The off-shell part of the propagator is determined in terms $F_{\pm}$ as

$$F_+(\varepsilon, p, \tau_c) = +\frac{g^2}{32\pi p H^2} (1 + f(\varepsilon)) \times \tag{3.31}$$

$$\left[\theta(\varepsilon - p) \left(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2} \right) \frac{1}{\tau_c^2} (1 + G(\varepsilon, p, \beta)) \right.$$

$$\left. + \theta(p - \varepsilon) \left(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2} \right) \frac{1}{\tau_c^2} G(\varepsilon, p, \beta) \right],$$

$$F_-(\varepsilon, p, \tau_c) = +\frac{g^2}{32\pi p H^2} f(\varepsilon) \times \tag{3.32}$$

$$\left[\theta(\varepsilon - p) \left(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2} \right) \frac{1}{\tau_c^2} (1 + G(\varepsilon, p, \beta)) \right.$$

$$\left. + \theta(p - \varepsilon) \left(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2} \right) \frac{1}{\tau_c^2} G(\varepsilon, p, \beta) \right].$$

We observe that the on-shell weight represented by the wave function renormalization factor Z is reduced from the unity in a consistent way with the off-shell spectral weight. In this sense unitarity is respected by the interaction.

We have thus determined the full propagator inside the cosmological horizon to the leading order of the perturbation theory. We have found that the full propagator which is characterized by (3.29), (3.30), (3.31), (3.32) depends on τ_c. At first sight, it appears to change with cosmic evolution. More and more off-shell states are created with a lapse of time as on-shell states are correspondingly reduced. However we may represent (3.29), (3.30), (3.31), (3.32) by the physical quantities,

$$X \equiv \frac{x}{H|\tau}, \quad P \equiv H|\tau|p, \quad \Delta E \equiv H|\tau|\Delta\varepsilon, \quad T \equiv H|\tau|\frac{1}{\beta}, \quad M \equiv H|\tau|\mu. \tag{3.33}$$

18

In terms of the physical quantities, the full propagator of the original scalar field at the equal time $\bar{\tau} = 0$ is

$$G^{-+}(x_1, x_2) = \int \frac{d^3 P}{(2\pi)^3 2P} \left(1 + 2(f + \delta f)\right)(1 + \delta Z)\left\{1 + (1 - \frac{m_{\text{eff}}^2}{2H^2})\frac{H^2}{P^2}\right\}e^{iP \cdot \bar{X}} \qquad (3.34)$$

$$+ \int_{E>0} \frac{dE d^3 P}{(2\pi)^4 2E} \, (F_+ + F_-)e^{iP \cdot \bar{X}},$$

$$m_{\text{eff}}^2 = \frac{g^2}{32\pi^2} \left(\log \frac{Q^2}{M^2} - \frac{2T}{P} \int_0^\infty dE \{\frac{1}{E - P} + \frac{1}{E + P}\} \log\left(\frac{1 - e^{-(E+P)/2T}}{1 - e^{-|E-P|/2T}}\right)\right), \qquad (3.35)$$

$$\delta f = \frac{g^2}{64\pi^2 P} \frac{\partial f(P, T)}{\partial P} \log \frac{\Delta E}{H}, \qquad (3.36)$$

$$\delta Z = - \frac{g^2}{32\pi P} \times \qquad (3.37)$$

$$\left[\frac{1}{2\pi}(\frac{1}{\Delta E} - \frac{1}{2P}) + \int_{P+\Delta E}^\infty \frac{dE}{2\pi}(\frac{1}{(E - P)^2} - \frac{1}{(E + P)^2})\frac{2T}{P} \log\left(\frac{1 - e^{-\frac{E+P}{2T}}}{1 - e^{-\frac{E-P}{2T}}}\right) \right.$$

$$\left. + \int_0^{P-\Delta E} \frac{dE}{2\pi}(\frac{1}{(E - P)^2} - \frac{1}{(E + P)^2})\frac{2T}{P} \log\left(\frac{1 - e^{-\frac{E+P}{2T}}}{1 - e^{-\frac{P-E}{2T}}}\right) \right],$$

$$F_+ = \frac{g^2}{32\pi P}(1 + f(E, T)) \times \qquad (3.38)$$

$$\left[\theta(E - P)(\frac{1}{(E - P)^2} - \frac{1}{(E + P)^2})(1 + \frac{2T}{P} \log\left(\frac{1 - e^{-\frac{E+P}{2T}}}{1 - e^{-\frac{E-P}{2T}}}\right)) \right.$$

$$\left. + \theta(P - E)(\frac{1}{(E - P)^2} - \frac{1}{(E + P)^2})\frac{2T}{P} \log\left(\frac{1 - e^{-\frac{E+P}{2T}}}{1 - e^{-\frac{P-E}{2T}}}\right) \right],$$

$$F_- = \frac{g^2}{32\pi P} f(E, T) \times \qquad (3.39)$$

$$\left[\theta(E - P)(\frac{1}{(E - P)^2} - \frac{1}{(E + P)^2})(1 + \frac{2T}{P} \log\left(\frac{1 - e^{-\frac{E+P}{2T}}}{1 - e^{-\frac{E-P}{2T}}}\right)) \right.$$

$$\left. + \theta(P - E)(\frac{1}{(E - P)^2} - \frac{1}{(E + P)^2})\frac{2T}{P} \log\left(\frac{1 - e^{-\frac{E+P}{2T}}}{1 - e^{-\frac{P-E}{2T}}}\right) \right].$$

We find that the explicit τ_c dependence disappears in these expressions. If we focus on the physics at the fixed physical energy scale E, it remains the same with cosmic evolution. It is a very sensible conclusion as we do not expect physics such as particle mass to change with

cosmic evolution. For a fixed ε, the physical energy E decreases with time evolution. So the cosmic evolution is identical to the evolution under the renormalization group. We recall here that the radial coordinate in AdS space corresponds to the energy scale in AdS/CFT correspondence. Since the radial coordinate in AdS space is related to the time coordinate in dS space by analytic continuation, dS space seems to be related to AdS space in this respect. The only physical time dependence appears through the temperature T as it cools down linearly with τ_c for a fixed β.

We do find a non-trivial modification of the distribution function from the Bose distribution due to a large IR effect. The effect of the interaction on the distribution function (3.29) is such that it reduces the particle density in comparison to the Bose distribution. This effect can be understood as follows. A single particle can turn into two particles due to the cubic interaction. So such off-shell two particle states are created while the on-shell state weight is reduced by the same amount due to unitarity. The off-shell states cost more energy and so are less numerous due to the Bose distribution function. The net effect is the further reduction of the particle density.

In this section, we have investigated the effects of the interaction on the propagator well inside the cosmological horizon. The spectral weight of the off-shell states increases with time while the weight of the on-shell states decreases due to the interaction. The modification of the Bose distribution is analogous to QCD where the logarithmic divergence requires the scale dependent modification of the parton distribution function. In term of the physical energy and momentum variables, explicit time dependence disappears and the time evolution may be identified with the renormalization group evolution. So we find that the effects of the interaction in dS space parallel to those in flat space. As it is explained in Appendix B, these features are also shared by $\lambda\varphi^4$ theory. We thus expect they are the universal features of the interacting field theories in dS space.

Nevertheless we should keep in mind that we have investigated the propagator near flat space and the expansion in terms of $1/p\tau_c$ breaks down near the cosmological horizon. To fully understand the behavior of the two point function in dS space, we have to extend our work to the region $|p\tau_c| \sim 1$ and $|p\tau_c| \ll 1$. We investigate the quantum effects outside the cosmological horizon in Part III. In this region, the dS symmetry can be broken and the time dependence which is not absorbed by physical quantities can be induced.

Before concluding this section, we briefly investigate the non-thermal distribution case. The modification of the distribution function $\delta f(p, \tau_c)$ by the cubic interaction is roughly of the following magnitude

$$\frac{\partial f(p, \tau_1, \tau_2)}{\partial \tau_c} \sim \frac{g^2}{p} \frac{1}{H^2 \tau_c^3} \int_{p+|1/\tau_c|} d\varepsilon \frac{1}{(\varepsilon - p)^2} \qquad (3.40)$$

$$\sim \frac{g^2}{p} \frac{1}{H^2 \tau_c^2},$$

$$\delta f(p, \tau_1, \tau_2) \sim \frac{g^2}{p} \frac{1}{H^2 \tau_c}.$$

So it is $O(1/p|\tau_c|)$ in a generic case instead of $O(1/(p\tau_c)^2)$ for the thermal case. While it is much larger than the change of the thermal distribution when $p|\tau_c| \gg 1$, it becomes only important near the cosmological horizon. Although the thermalization may take place when

the coupling is strong enough $g > H$, it could only occur near the cosmological horizon. At the higher loop level, the thermalization could also take place through the effective n point couplings. We find this is a very interesting problem which requires further investigations.

Part III

Quantum effects from outside the cosmological horizon

4 Infra-red divergence of propagator

As mentioned in Section 1, the propagator for a massless and minimally coupled field has an IR divergence. The origin of the IR divergence is the scale invariant fluctuation spectrum. It is dominant outside the cosmological horizon $P \ll H$:

$$\phi_{\mathbf{p}}(x) \sim \frac{H}{\sqrt{2p^3}}. \tag{4.1}$$

In this section, we explain how to regularize the IR divergence.

For simplicity, let us estimate the magnitude of the quantum fluctuation by taking the coincident limit of the propagator. It consists of the contributions from inside and outside the cosmological horizon as follows

$$\langle \varphi(x)\varphi(x) \rangle \sim \int_{P>H} \frac{d^3P}{(2\pi)^3} \frac{1}{2P} + H^2 \int_{P<H} \frac{d^3P}{(2\pi)^3} \frac{1}{2P^3}. \tag{4.2}$$

The UV contribution ($P > H$) is quadratically divergent just like in Minkowski space. It can be regularized and renormalized in an identical way. The logarithmic IR divergence due to the contributions from outside the cosmological horizon ($P < H$) is specific to dS space.

To regularize this IR divergence, we introduce an IR cut-off ε_0 which fixes the minimum value of the comoving momentum as in [19]:

$$\int_{\varepsilon_0 a^{-1}(\tau)}^{H} dP. \tag{4.3}$$

Note that the equation of motion is not satisfied if we fix the minimum value of the Physical momentum. With this prescription, more degrees of freedom go out of the cosmological horizon at $P = H$ with cosmic evolution. On the other hand, the UV cut-off Λ_{UV} fixes the maximum value of the physical momentum:

$$\int_{H}^{\Lambda_{UV}} dP. \tag{4.4}$$

the coupling is strong enough $g > H$, it could only occur near the cosmological horizon. At the higher loop level, the thermalization could also take place through the effective n point couplings. We find this is a very interesting problem which requires further investigations.

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$$\int_{H}^{\Lambda_{UV}} dP. \tag{4.4}$$

If not, the UV divergence is associated with time dependence and not renormalizable by the counter term. Thus the degrees of freedom inside the cosmological horizon remains constant, while the degrees of freedom outside the cosmological horizon increases as time goes on. The contribution from outside the cosmological horizon gives a growing time dependence to the propagator

$$
\langle \varphi(x)\varphi(x) \rangle = (\text{UV const}) + \frac{H^2}{4\pi^2} \int_{\varepsilon_0 a^{-1}(\tau)}^{H} \frac{dP}{P} \tag{4.5}
$$
$$
= (\text{UV const}) + \frac{H^2}{4\pi^2} \log \left(\frac{H}{\varepsilon_0} a(\tau) \right).
$$

Physically speaking, we consider a situation that a universe with a finite spatial extension or a finite region of space starts dS expansion at an initial time t_i. The IR cut-off ε_0 is identified with the initial time t_i as

$$
\log \left(\frac{H}{\varepsilon_0} a(\tau) \right) = \log e^{H(t-t_i)}, \quad t_i \equiv \frac{1}{H} \log \frac{\varepsilon_0}{H}. \tag{4.6}
$$

Henceforth we adopt the following setting for simplicity

$$
\varepsilon_0 = H \Leftrightarrow t_i = 0. \tag{4.7}
$$

In this setting, the propagator at the coincident point is

$$
\langle \varphi(x)\varphi(x) \rangle = (\text{UV const}) + \frac{H^2}{4\pi^2} \log a(\tau). \tag{4.8}
$$

This time dependence breaks the dS invariance as has been pointed in [8, 9, 10].

In order to evaluate the quantum loop effects, more details about the propagator in dS space are necessary. That is, we need to know the explicit form of the propagator at the separated points and how to regularize the UV divergences of the loop amplitudes. Here we adopt the dimensional regularization. In $D = 4 - \varepsilon$, the wave function in (1.5) is generalized as

$$
\phi_{\mathbf{p}}(x) = \frac{\sqrt{\pi}}{2} H^{\frac{D-2}{2}} (-\tau)^{\frac{D-1}{2}} H_\nu^{(1)}(-p\tau) \, e^{i\mathbf{p}\cdot\mathbf{x}}, \tag{4.9}
$$
$$
\nu = \sqrt{\left(\frac{D-1}{2} \right)^2 - \frac{m^2}{H^2} - D(D-1)\xi}.
$$

The corresponding propagator is

$$
\langle \varphi(x)\varphi(x') \rangle = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(\frac{D-1}{2}+\nu)\Gamma(\frac{D-1}{2}-\nu)}{\Gamma(\frac{D}{2})} \, {}_2F_1\left(\frac{D-1}{2}+\nu, \frac{D-1}{2}-\nu; \frac{D}{2}; 1-\frac{y}{4} \right). \tag{4.10}
$$

Note that the propagator for a massless and minimally coupled field has an IR divergence

in any dimension. For convenient analyses, we expand (4.10) by y:

$$
\langle\varphi(x)\varphi(x')\rangle = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\Big\{\Gamma(\frac{D}{2}-1)(\frac{4}{y})^{\frac{D}{2}-1}\,{}_2F_1(\frac{1}{2}+\nu,\frac{1}{2}-\nu;2-\frac{D}{2};\frac{y}{4}) \tag{4.11}
$$

$$
+\frac{\Gamma(\frac{D-1}{2}-\nu)\Gamma(\frac{D-1}{2}+\nu)\Gamma(1-\frac{D}{2})}{\Gamma(\frac{1}{2}+\nu)\Gamma(\frac{1}{2}-\nu)}\,{}_2F_1(\frac{D-1}{2}+\nu,\frac{D-1}{2}-\nu;\frac{D}{2};\frac{y}{4})
$$

$$
=\frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\Big\{\Gamma(\frac{D}{2}-1)(\frac{4}{y})^{\frac{D}{2}-1}-\frac{(\frac{1}{2}+\nu)(\frac{1}{2}-\nu)\Gamma(\frac{D}{2})}{(1-\frac{D}{2})(2-\frac{D}{2})}(\frac{y}{4})^{2-\frac{D}{2}}
$$

$$
+\frac{\Gamma(1-\frac{D}{2})\Gamma(\frac{D-1}{2}+\nu)\Gamma(\frac{D-1}{2}-\nu)}{\Gamma(\frac{1}{2}+\nu)\Gamma(\frac{1}{2}-\nu)}
$$

$$
+\frac{\Gamma(\frac{D}{2})\Gamma(1-\frac{D}{2})}{\Gamma(\frac{1}{2}+\nu)\Gamma(\frac{1}{2}-\nu)}\sum_{n=1}^{\infty}\Big[\frac{\Gamma(\frac{D-1}{2}+\nu+n)\Gamma(\frac{D-1}{2}-\nu+n)}{\Gamma(\frac{D}{2}+n)n!}(\frac{y}{4})^n
$$

$$
-\frac{\Gamma(\frac{3}{2}+\nu+n)\Gamma(\frac{3}{2}-\nu+n)}{\Gamma(3-\frac{D}{2}+n)(n+1)!}(\frac{y}{4})^{n+2-\frac{D}{2}}\Big]\Big\}.
$$

Introducing the IR cut-off is equivalent to subtract the following term from (4.10)

$$
\int_{p<\varepsilon_0}\frac{d^{D-1}p}{(2\pi)^{D-1}}\,\frac{\pi}{4}H^{D-2}(-\tau)^{D-1}H_\nu^{(1)}(-p\tau)H_\nu^{(1)*}(-p\tau')e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')} \tag{4.12}
$$

$$
\simeq -\frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(\nu)\Gamma(2\nu)}{\Gamma(\frac{D-1}{2})\Gamma(\nu+\frac{1}{2})}\frac{\big(a(\tau)a(\tau')\big)^{\nu-\frac{D-1}{2}}}{\nu-\frac{D-1}{2}}.
$$

In the second line, we consider the case $(D-1)/2-\nu\ll 1$ and set $\varepsilon_0=H$ after neglecting the terms at $\mathcal{O}(\varepsilon_0)$. By taking the massless and minimally coupled limit after the subtraction, the propagator is written as follows [20, 21]

$$
\langle\varphi(x)\varphi(x')\rangle = A(y) + B\log(a(\tau)a(\tau')), \tag{4.13}
$$

where $A(y),B,\delta$ are defined as follows:

$$
A(y)=\frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\Big\{\Gamma(\frac{D}{2}-1)(\frac{4}{y})^{\frac{D}{2}-1}-\frac{\Gamma(\frac{D}{2}+1)}{2-\frac{D}{2}}(\frac{y}{4})^{2-\frac{D}{2}}+\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\delta \tag{4.14}
$$

$$
+\sum_{n=1}^{\infty}\Big[\frac{\Gamma(D-1+n)}{n\Gamma(\frac{D}{2}+n)}(\frac{y}{4})^n-\frac{\Gamma(\frac{D}{2}+1+n)}{(2-\frac{D}{2}+n)(n+1)!}(\frac{y}{4})^{n+2-\frac{D}{2}}\Big]\Big\},
$$

$$
B=\frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})},
$$

$$
\delta=-\psi(1-\frac{D}{2})+\psi(\frac{D-1}{2})+\psi(D-1)+\psi(1),\quad \psi(z)\equiv\Gamma'(z)/\Gamma(z).
$$

Since (4.10) is a function of the dS invariant distance y, the dimensional regularization for the UV divergences doesn't break the dS symmetry (1.13)-(1.16). In contrast, the IR cut-off

breaks the scale invariance (1.13) and the spatial special conformal symmetry (1.14) and induces the growing time dependent term $\log(a(\tau)a(\tau'))$.

We should note that (4.14) has infinite power series of y at $D = 4 - \varepsilon$, while it reduces to finite power series in the limit $\varepsilon \to 0$. The propagator for a massless and minimally coupled field at $D = 4$ is written as

$$G(x, x') = \frac{H^2}{4\pi^2}\Big\{\frac{1}{y} - \frac{1}{2}\log y + \frac{1}{2}\log a(\tau)a(\tau') + 1 - \gamma\Big\}, \qquad (4.15)$$

where γ is the Euler's constant. At the finite loop level, the higher series are necessary only when they are multiplied by the UV divergent terms like $1/\varepsilon$.

We also refer to the case that a field has a mass $m^2/H^2 \ll 1$. In this case, the propagator (4.11) behaves as follows at the IR region:

$$\langle\varphi(x)\varphi(x')\rangle \simeq \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D)}{\Gamma(\frac{D}{2})}\frac{H^2}{m^2}. \qquad (4.16)$$

So the IR cut-off is not necessary in terms of avoiding the IR divergence. Nevertheless, the IR cut-off is necessary since there is no smooth massless limit. The problem is clear in considering the vev of the mass term which emerges in the energy-momentum tensor

$$m^2\langle\varphi^2(x)\rangle \simeq \frac{H^D}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D)}{\Gamma(\frac{D}{2})}. \qquad (4.17)$$

Of course, this term is zero in the massless case, while is not zero in the massless limit of (4.17). We need the IR cut-off to resolve the difference. From (4.11) and (4.12), the propagator is written as

$$\langle\varphi(x)\varphi(x')\rangle \simeq \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D)}{\Gamma(\frac{D}{2})}\frac{H^2}{m^2}\Big\{1 - \exp(-\frac{1}{D-1}\frac{m^2}{H^2}\log a(\tau)a(\tau'))\Big\}. \qquad (4.18)$$

By substituting this, the mass term is

$$m^2\langle\varphi^2(x)\rangle \simeq \frac{H^D}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D)}{\Gamma(\frac{D}{2})}\Big\{1 - \exp(-\frac{2}{D-1}\frac{m^2}{H^2}\log a(\tau))\Big\}. \qquad (4.19)$$

The mass term approaches zero in the massless limit by introducing the IR cut-off. If a field has a finite mass, the value (4.17) is defined as a saturation value at $t \to \infty$.

Thus if a nearly massless and minimally coupled field exists, the propagator is time dependent beyond the dS invariance and physical quantities can acquire time dependences through the quantum loop corrections. In the subsequent sections, we investigate how the dS symmetry breaking contributes to physical quantities.

5   Energy-momentum tensor

We investigate the expectation value of the energy-momentum tensor as we are interested in how the IR logarithms contribute to the cosmological constant. The energy-momentum

24

tensor appears on the right-hand side of the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \quad \kappa = 8\pi G, \tag{5.1}$$

$$T_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}},$$

where Λ is the cosmological constant and G is the Newton's constant. As far as the dS symmetry is preserved, the vev of the energy-momentum tensor is proportional to $g_{\mu\nu}$ with a constant coefficient

$$\langle T_{\mu\nu}\rangle = g_{\mu\nu}T. \tag{5.2}$$

On the other hand, if the dS symmetry is broken down to the spatial translation and rotation symmetries, the coefficient of $g_{\mu\nu}$ becomes time dependent. We should note that the energy-momentum tensor is covariantly conserved as far as the equation of motion is satisfied:

$$0 = \int d^4x \, \frac{\delta S_{matter}}{\delta\varphi}\mathcal{L}_\xi\varphi = \frac{-1}{2}\int \sqrt{-g}d^4x \, T^{\mu\nu}\mathcal{L}_\xi g_{\mu\nu} \tag{5.3}$$

$$= \int \sqrt{-g}d^4x \, D_\mu T^{\mu\nu}\xi_\nu$$

$$\Rightarrow \ D_\mu T^\mu{}_\nu = 0,$$

where $\mathcal{L}_\xi$ denotes the Lie derivative. So in this case, the term which is proportional to $\delta_\mu{}^0\delta_\nu{}^0$ emerges to preserve the covariant conservation law of the energy-momentum tensor

$$\langle T_{\mu\nu}\rangle = g_{\mu\nu}T(\tau) + a^2(\tau)\delta_\mu{}^0\delta_\nu{}^0 U(\tau), \tag{5.4}$$

$$U(\tau) = \tau^3 \int d\tau \ \tau^{-3}\frac{d}{d\tau}T(\tau).$$

Since the time dependence is caused by the IR logarithms, T is logarithmically larger than U. It is in this sense that the matter quantum IR effect could induce the time dependent effective cosmological constant

$$\Lambda_{\text{eff}} = \Lambda - \kappa T(\tau). \tag{5.5}$$

In the free field theory, the vev of the energy-momentum tensor is

$$\langle T_{\mu\nu}\rangle = (\delta_\mu{}^\rho\delta_\nu{}^\sigma - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma})\langle\partial_\rho\varphi\partial_\sigma\varphi\rangle. \tag{5.6}$$

From (4.13) and (4.14), we find that the two differential operators cancel the IR logarithm

$$\partial_\rho\partial'_\sigma \log(a(\tau)a(\tau')) = 0. \tag{5.7}$$

Of the dS invariant part, only the linear y term contributes and its contribution is UV finite

$$\lim_{x'\to x}\partial_\rho\partial'_\sigma y = -2H^2 g_{\rho\sigma}. \tag{5.8}$$

25

Thus the contribution from the free field is the following time independent one

$$\langle T_{\mu\nu}\rangle = \frac{3H^4}{32\pi^2}g_{\mu\nu}, \quad \Lambda_{\text{eff}} = \Lambda - \kappa\frac{3H^4}{32\pi^2}. \tag{5.9}$$

In this thesis, we work with the Poincaré coordinate. The propagator and the energy-momentum tensor for a free field is investigated by using the global coordinate in [22, 23]. The result is a little different from (5.9). However the difference rapidly vanishes at late times with the spatial expansion. So we believe the Poincaré coordinate is sufficient to investigate the IR effects which grow with time. It is necessary that there exist interaction terms which contain undifferentiated scalar fields to identify the IR logarithms. In the subsequent sections, we investigate the models which satisfy this necessary condition.

Before investigating the interaction effects, we refer to the conformal anomaly. The conformal anomaly also contributes to the vev of the energy-momentum tensor [24]. In the case of the minimally coupled scalar field in dS space, it leads to the following energy-momentum tensor in addition

$$\langle T_{\mu\nu}\rangle = \frac{29H^4}{15\cdot64\pi^2}g_{\mu\nu}. \tag{5.10}$$

This contribution has no time dependence because the conformal anomaly is the UV effect. In the subsequent discussion, we focus on the time dependence of the effective cosmological constant induced by quantum IR effects.

6 Field theory with an interaction potential

In this section, we consider a scalar field theory with an interaction potential

$$S_{matter} = \int \sqrt{-g}d^4x \left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)\right]. \tag{6.1}$$

Here the energy-momentum tensor is given as follows

$$T_{\mu\nu} = (\delta_\mu{}^\rho\delta_\nu{}^\sigma - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma})\partial_\rho\varphi\partial_\sigma\varphi - g_{\mu\nu}V(\varphi). \tag{6.2}$$

As we have recalled in the previous section, the propagator of a massless minimally coupled scalar field contains the time dependent term $\log(a(\tau)a(\tau'))$. So the vev of the potential becomes time dependent. $\log(a(\tau)) = Ht$ factor grows with cosmic expansion which eventually gives rise to a large IR quantum effect.

6.1 Perturbative IR effects

By using the Schwinger-Keldysh perturbation theory, we can evaluate the quantum IR effects at each order. In the case of the polynomial interaction, each propagator could produce a single IR log factor. Although the retarded propagator resulting from the commutator in the

Schwinger-Keldysh perturbation theory does not contain any IR log factor, the associated time integration produces a single IR factor nevertheless. Thus, the maximum power of the IR logarithms can be estimated by counting the number of the propagators in a diagram.

In this way, the leading IR contributions coming from the potential are estimated as

$$-g_{\mu\nu}\langle V(\varphi)\rangle = -g_{\mu\nu}V(\tau) \simeq -g_{\mu\nu}\sum_{n=1}V_n\lambda^n\log^{mn}a(\tau), \quad m \in \mathbf{N}. \tag{6.3}$$

Here $\simeq$ means that we extract the leading logarithms at each order. They are dominant at late times when $\log a(\tau) \gg 1$. At the n-th order of the coupling constant in $\lambda\varphi^{2m}$ theory, we retain the term of order $(\lambda\log^m(a))^n$ where the power of the IR logarithm is maximum. Even if λ is small, higher order terms cannot be neglected as $\lambda\log^m(a)$ approaches $\mathcal{O}(\lambda^0)$. Thus we need to sum these leading logarithms first in order to understand the IR effects non-perturbatively. Sub-leading terms are suppressed by powers of λ just like the resummation of IR logarithms in QCD. Since the contribution from the potential is proportional to $g_{\mu\nu}$, it gives rise to a time dependent effective cosmological constant.

Next, we refer to the contribution from the kinetic term. $\langle\partial_\rho\varphi\partial_\sigma\varphi\rangle$ possesses the following structure

$$\langle\partial_\rho\varphi\partial_\sigma\varphi\rangle = -g_{\rho\sigma}\frac{3H^4}{32\pi^4} + a^2(\tau)\delta_\rho{}^0\delta_\sigma{}^0 K^1(\tau) + g_{\rho\sigma}K^2(\tau), \tag{6.4}$$

$$K^1(\tau) \simeq \sum_{n=1}K_n^1\lambda^n\log^{mn-1}a(\tau),$$

$$K^2(\tau) \simeq \sum_{n=1}K_n^2\lambda^n\log^{mn-1}a(\tau).$$

Here the first term is the the free field theory contribution at the one loop level. Note that the kinetic term is sub-dominant in comparison to the potential term except at the one loop level. It is because taking two derivative operations weakens the IR effect. So in the energy-momentum tensor, we can neglect the contribution $g_{\mu\nu}\{\frac{1}{2}K^1(\tau) - K^2(\tau)\}$. That is, we estimate the vev of the energy-momentum tensor as follows

$$\langle T_{\mu\nu}\rangle \simeq g_{\mu\nu}\frac{3H^4}{32\pi^4} + a^2(\tau)\delta_\mu{}^0\delta_\nu{}^0 K^1(\tau) - g_{\mu\nu}V(\tau). \tag{6.5}$$

In this approximation, From (5.4), the following identity is satisfied

$$K^1(\tau) = \frac{\tau}{3}\frac{d}{d\tau}V(\tau) \Leftrightarrow K_n^1(\tau) = -\frac{mn}{3}V_n(\tau). \tag{6.6}$$

We summarize the IR effects to the expectation value of the energy-momentum tensor here. At the one loop level, the contribution is identical to that of a free field and there is no IR effects. At $\mathcal{O}(\lambda^n)$ ($n \geq 1$) when the effect of the interaction becomes important, the potential term becomes dominant at late times in dS expansion. The dominant term is proportional to $g_{\mu\nu}$ and so contributes to the effective cosmological constant

$$\Lambda_{\text{eff}} = \Lambda - \kappa\frac{3H^4}{32\pi^4} + \kappa V(\tau). \tag{6.7}$$

The kinetic term at $\mathcal{O}(\lambda^n)$ ($n \geq 1$) is sub-dominant and contains the term which is proportional to $\delta_\mu{}^0\delta_\nu{}^0$. Such a term is related to the effective cosmological constant due to the conservation law.

6.1.1 Perturbative IR effects in φ^4 theory

Before starting the next subsection, we review the perturbative IR effects in φ^4 theory as a concrete example [13]. To focus on the IR effects from φ^4 interaction, we set the bare mass of the scalar field zero.

$$V(\varphi) = \frac{\lambda}{4!} \varphi^4. \tag{6.8}$$

The simplest IR effects in this model is the effective mass at the one loop level

$$m^2 + \delta m^2 = \frac{\lambda}{2} G^{++}(x, x), \tag{6.9}$$

where δm^2 is the counter term for the mass square. From (4.13) and (4.14), the propagator at the coincident point is UV divergent

$$G^{++}(x, x) = \langle \varphi^2(x) \rangle = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} (2 \log a(\tau) + \delta). \tag{6.10}$$

To renormalize the UV divergence, we set the counter term as follows

$$\delta m^2 = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \delta. \tag{6.11}$$

The effective mass square at the one loop level is

$$m^2 = \frac{\lambda H^2}{8\pi^2} \log a(\tau). \tag{6.12}$$

Next we evaluate the potential term of the energy-momentum tensor at the two loop level. From (6.10) and (6.11),

$$- g_{\mu\nu} \{ \langle V(\varphi) \rangle + \delta m^2 \langle \varphi^2 \rangle \} \tag{6.13}$$
$$= - g_{\mu\nu} \{ \frac{\lambda}{8} G^{++}(x, x) G^{++}(x, x) + \delta m^2 G^{++}(x, x) \}$$
$$= - g_{\mu\nu} \{ \frac{\lambda H^4}{2^7 \pi^4} \log^2 a(\tau) + \frac{\lambda}{2} \frac{H^{2D-4}}{(4\pi)^D} \frac{\Gamma^2(D-1)}{\Gamma^2(\frac{D}{2})} \delta^2 \}.$$

Note that the cross term between the IR logarithm and the UV divergence is canceled by the the counter term for the mass square. The remaining UV divergence is renormalizable by the counter term for the cosmological constant

$$\frac{\delta\Lambda}{\kappa} = -\frac{\lambda}{2} \frac{H^{2D-4}}{(4\pi)^D} \frac{\Gamma^2(D-1)}{\Gamma^2(\frac{D}{2})} \delta^2. \tag{6.14}$$

28

As a result, the contribution from the potential term is

$$-g_{\mu\nu}\frac{\lambda H^4}{2^7\pi^4}\log^2 a(\tau). \tag{6.15}$$

The evaluation for the kinetic term is not so simple. The kinetic term at the two loop level is written as follows in the Schwinger-Keldysh formalism

$$\langle\partial_\rho\varphi\partial_\sigma\varphi\rangle|_\lambda = -i\frac{\lambda}{2}\int\sqrt{-g'}d^Dx'\ \{G^{++}(x',x')+\delta m^2\} \tag{6.16}$$
$$\times\left[\partial_\rho G^{++}(x,x')\partial_\sigma G^{++}(x,x')-\partial_\rho G^{+-}(x,x')\partial_\sigma G^{+-}(x,x')\right].$$

From (4.13) and (4.14), we find

$$\partial_\rho G(x,x')\partial_\sigma G(x,x') = \frac{4^{D-2}H^{2D-2}}{(4\pi)^D}\Gamma^2(\frac{D}{2})a^2(\tau) \tag{6.17}$$
$$\times\left[4H^2a^2(\tau')\frac{\Delta x_\rho\Delta x_\sigma}{y^{4-\varepsilon}}+(4-\varepsilon)H^2a^2(\tau')\frac{\Delta x_\rho\Delta x_\sigma}{y^{3-\varepsilon}}\right.$$
$$+H^2a^2(\tau')\frac{\Delta x_\rho\Delta x_\sigma}{y^2}+2Ha(\tau')\frac{\Delta x_\rho\delta_\sigma{}^0+\Delta x_\sigma\delta_\rho{}^0}{y^{3-\varepsilon}}$$
$$\left.+Ha(\tau')\frac{\Delta x_\rho\delta_\sigma{}^0+\Delta x_\sigma\delta_\rho{}^0}{y^2}+\frac{\delta_\rho{}^0\delta_\sigma{}^0}{y^{2-\varepsilon}}\right],$$

where we abbreviate the indexes $++, +-$ because the above identities work out in both cases. Note that we have only to evaluate (6.16) up to $\mathcal{O}(\varepsilon^0)$. By substituting (6.10), (6.11) and (6.17) to (6.16),

$$\langle\partial_\rho\varphi\partial_\sigma\varphi\rangle|_\lambda = -i\frac{4^{D-2}H^{3D-4}}{(4\pi)^{\frac{3D}{2}}}\Gamma(D-1)\Gamma(\frac{D}{2})a^2(\tau) \tag{6.18}$$
$$\times\int d^{4-\varepsilon}x'\ a^{4-\varepsilon}(\tau')\log a(\tau')\sum_{m=1}^{6}H_{\rho\sigma}^m,$$

where the integrands are as follows

$$H_{\rho\sigma}^1 \equiv 4H^2a^2(\tau')[\frac{\Delta x_\rho\Delta x_\sigma}{y_{++}^{4-\varepsilon}}-\frac{\Delta x_\rho\Delta x_\sigma}{y_{++}^{4-\varepsilon}}], \tag{6.19}$$

$$H_{\rho\sigma}^2 \equiv (4-\varepsilon)H^2a^2(\tau')[\frac{\Delta x_\rho\Delta x_\sigma}{y_{++}^{3-\varepsilon}}-\frac{\Delta x_\rho\Delta x_\sigma}{y_{+-}^{3-\varepsilon}}], \tag{6.20}$$

$$H_{\rho\sigma}^3 \equiv H^2a^2(\tau')[\frac{\Delta x_\rho\Delta x_\sigma}{y_{++}^2}-\frac{\Delta x_\rho\Delta x_\sigma}{y_{+-}^2}], \tag{6.21}$$

$$H_{\rho\sigma}^4 \equiv 2Ha(\tau')[\frac{\Delta x_\rho\delta_\sigma{}^0+\Delta x_\sigma\delta_\rho{}^0}{y_{++}^{3-\varepsilon}}-\frac{\Delta x_\rho\delta_\sigma{}^0+\Delta x_\sigma\delta_\rho{}^0}{y_{+-}^{3-\varepsilon}}], \tag{6.22}$$

$$H^5_{\rho\sigma} \equiv Ha(\tau')[\frac{\Delta x_\rho \delta_\sigma{}^0 + \Delta x_\sigma \delta_\rho{}^0}{y^2_{++}} - \frac{\Delta x_\rho \delta_\sigma{}^0 + \Delta x_\sigma \delta_\rho{}^0}{y^2_{+-}}], \tag{6.23}$$

$$H^6_{\rho\sigma} \equiv [\frac{\delta_\rho{}^0 \delta_\sigma{}^0}{y^{2-\varepsilon}_{++}} - \frac{\delta_\rho{}^0 \delta_\sigma{}^0}{y^{2-\varepsilon}_{+-}}]. \tag{6.24}$$

We explain how to calculate these integrals containing $H^m_{\rho\sigma}$ in Appendix D. Here we simply list the results:

$$\int d^{4-\varepsilon}x' \; a^{4-\varepsilon}(\tau') \log a(\tau') H^1_{\rho\sigma} \tag{6.25}$$

$$\simeq \eta_{\rho\sigma} \times 4i\pi^2 H^{-4} \Big\{ \frac{1}{4} \big(\frac{\pi^{-\frac{\varepsilon}{2}} \mu^{-\varepsilon} H^{2\varepsilon}}{\Gamma(1-\frac{\varepsilon}{2})\varepsilon} + \log \frac{2\mu}{H} \big) \log a(\tau) + \frac{1}{8} \log a(\tau) \Big\}$$

$$+ \delta_\rho{}^0 \delta_\sigma{}^0 \times 4i\pi^2 H^{-4} \Big\{ -\frac{1}{2} \big(\frac{\pi^{-\frac{\varepsilon}{2}} \mu^{-\varepsilon} H^{2\varepsilon}}{\Gamma(1-\frac{\varepsilon}{2})\varepsilon} + \log \frac{2\mu}{H} \big) \log a(\tau) \Big\},$$

$$\int d^{4-\varepsilon}x' \; a^{4-\varepsilon}(\tau') \log a(\tau') H^2_{\rho\sigma} \tag{6.26}$$

$$\simeq \eta_{\rho\sigma} \times 4i\pi^2 H^{-4} \Big\{ -\frac{1}{2} \big(\frac{\pi^{-\frac{\varepsilon}{2}} \mu^{-\varepsilon} H^{2\varepsilon}}{\Gamma(1-\frac{\varepsilon}{2})\varepsilon} + \log \frac{2\mu}{H} \big) \log a(\tau) + \frac{1}{8} \log a(\tau) \Big\}$$

$$+ \delta_\rho{}^0 \delta_\sigma{}^0 \times 4i\pi^2 H^{-4} \Big\{ -\frac{1}{2} \log a(\tau) \Big\},$$

$$\int d^4 x' \; a^4(\tau') \log a(\tau') H^3_{\rho\sigma} \tag{6.27}$$

$$\simeq \eta_{\rho\sigma} \times 4i\pi^2 H^{-4} \Big\{ -\frac{1}{12} \log a(\tau) \Big\} + \delta_\rho{}^0 \delta_\sigma{}^0 \times 4i\pi^2 H^{-4} \Big\{ -\frac{1}{6} \log a(\tau) \Big\},$$

$$\int d^{4-\varepsilon}x' \; a^{4-\varepsilon}(\tau') \log a(\tau') H^4_{\rho\sigma} \tag{6.28}$$

$$\simeq \delta_\rho{}^0 \delta_\sigma{}^0 \times 4i\pi^2 H^{-4} \Big\{ \big(\frac{\pi^{-\frac{\varepsilon}{2}} \mu^{-\varepsilon} H^{2\varepsilon}}{\Gamma(1-\frac{\varepsilon}{2})\varepsilon} + \log \frac{2\mu}{H} \big) \log a(\tau) \Big\},$$

$$\int d^4 x' \; a^4(\tau') \log a(\tau') H^5_{\rho\sigma} \simeq \delta_\rho{}^0 \delta_\sigma{}^0 \times 4i\pi^2 H^{-4} \Big\{ \frac{1}{2} \log a(\tau) \Big\}, \tag{6.29}$$

$$\int d^{4-\varepsilon}x' \; a^{4-\varepsilon}(\tau') \log a(\tau') H^6_{\rho\sigma} \tag{6.30}$$

$$\simeq \eta_{\rho\sigma} \times 4i\pi^2 H^{-4} \Big\{ \frac{1}{4} \big(\frac{\pi^{-\frac{\varepsilon}{2}} \mu^{-\varepsilon} H^{2\varepsilon}}{\Gamma(1-\frac{\varepsilon}{2})\varepsilon} + \log \frac{2\mu}{H} \big) \log a(\tau) + \frac{1}{8} \log a(\tau) \Big\}$$

$$+ \delta_\rho{}^0 \delta_\sigma{}^0 \times 4i\pi^2 H^{-4} \Big\{ -\frac{1}{2} \big(\frac{\pi^{-\frac{\varepsilon}{2}} \mu^{-\varepsilon} H^{2\varepsilon}}{\Gamma(1-\frac{\varepsilon}{2})\varepsilon} + \log \frac{2\mu}{H} \big) \log a(\tau) \Big\}.$$

where μ is the mass parameter and we extract the contributions which contain the IR logarithm. The total of (6.25)-(6.30) is

$$\int d^{4-\varepsilon} x' \, a^{4-\varepsilon}(\tau') \log a(\tau') \sum_{m=1}^{6} H_{\rho\sigma}^{m} \qquad (6.31)$$

$$\simeq \eta_{\rho\sigma} \times 4i\pi^2 H^{-4} \Big\{ -\frac{1}{4} \Big(\frac{\pi^{-\frac{\varepsilon}{2}} \mu^{-\varepsilon} H^{2\varepsilon}}{\Gamma(1-\frac{\varepsilon}{2})\varepsilon} + \log \frac{2\mu}{H} \Big) \log a(\tau) + \frac{1}{6} \log a(\tau) \Big\}$$

$$+ \delta_\rho{}^0 \delta_\sigma{}^0 \times 4i\pi^2 H^{-4} \Big\{ -\frac{1}{6} \log a(\tau) \Big\}.$$

By substituting (6.31) to (6.18),

$$\langle \partial_\rho \varphi \partial_\sigma \varphi \rangle|_\lambda \simeq -g_{\rho\sigma} \frac{4^{D-2} \pi^2 \lambda H^{3D-8}}{(4\pi)^{\frac{3D}{2}}} \Gamma(D-1)\Gamma\Big(\frac{D}{2}\Big) \qquad (6.32)$$

$$\times \Big\{ \Big(\frac{\pi^{-\frac{\varepsilon}{2}} \mu^{-\varepsilon} H^{2\varepsilon}}{\Gamma(1-\frac{\varepsilon}{2})\varepsilon} + \log \frac{2\mu}{H} \Big) \log a(\tau) - \frac{2}{3} \log a(\tau) \Big\} - a^2(\tau) \delta_\rho{}^0 \delta_\sigma{}^0 \frac{\lambda H^4}{2^6 \cdot 3\pi^4} \log a(\tau).$$

Including the prefactor,

$$\Big(\delta_\mu{}^\rho \delta_\nu{}^\sigma - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \Big) \langle \partial_\rho \varphi \partial_\sigma \varphi \rangle|_\lambda \qquad (6.33)$$

$$\simeq g_{\mu\nu} \frac{4^{D-2} \pi^2 \lambda H^{3D-8}}{(4\pi)^{\frac{3D}{2}}} \Big(\frac{D}{2} - 1 \Big) \Gamma(D-1)\Gamma\Big(\frac{D}{2}\Big) \Big\{ \Big(\frac{\pi^{-\frac{\varepsilon}{2}} \mu^{-\varepsilon} H^{2\varepsilon}}{\Gamma(1-\frac{\varepsilon}{2})\varepsilon} + \log \frac{2\mu}{H} \Big) \log a(\tau) - \log a(\tau) \Big\}$$

$$- a^2(\tau) \delta_\mu{}^0 \delta_\nu{}^0 \frac{\lambda H^4}{2^6 \cdot 3\pi^4} \log a(\tau).$$

We should note that the kinetic term contains the IR logarithm with the UV divergent coefficient at the two loop level. The UV divergence is not renormalizable by the counter term for the cosmological constant $\delta\Lambda$. To renormalize it, we introduce the following counter term

$$\delta_C \mathcal{L} = \delta C (R_g - D(D-1)H^2) \varphi^2. \qquad (6.34)$$

Since $R_g = D(D-1)H^2$ on the dS background, the contribution from this counter term is non-zero only when it is differentiated with respect to $g_{\mu\nu}$. That is, it contributes only to the energy-momentum tensor:

$$\delta_C \langle T_{\mu\nu} \rangle = -2\delta C \Big\{ g_{\mu\nu} ((D-1)H^2 \langle \varphi^2 \rangle + \nabla^2 \langle \varphi^2 \rangle) - \nabla_\mu \nabla_\nu \langle \varphi^2 \rangle \Big\} \qquad (6.35)$$

$$\simeq -g_{\mu\nu} \delta C \frac{4H^D}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D)}{\Gamma(\frac{D}{2})} \log a(\tau).$$

To renormalize the UV divergence of (6.33), we set δC as follows

$$\delta C = \frac{4^{D-3} \pi^2 \lambda H^{2D-8}}{(4\pi)^D} \frac{\frac{D}{2}-1}{D-1} \Gamma^2\Big(\frac{D}{2}\Big) \Big\{ \Big(\frac{\pi^{-\frac{\varepsilon}{2}} \mu^{-\varepsilon} H^{2\varepsilon}}{\Gamma(1-\frac{\varepsilon}{2})\varepsilon} + \log \frac{2\mu}{H} \Big) - 1 \Big\}. \qquad (6.36)$$

31

As a result, the contribution from the kinetic term is

$$-a^2(\tau)\delta_\mu{}^0\delta_\nu{}^0\frac{\lambda H^4}{2^6\cdot 3\pi^4}\log a(\tau). \tag{6.37}$$

From (6.15) and (6.37), we can explicitly check (6.6) and (6.7) in φ^4 theory. The effective cosmological constant increases as time goes on

$$\Lambda_{\text{eff}} = \Lambda - \kappa\frac{3H^4}{32\pi^4} + \kappa\frac{\lambda H^4}{2^7\pi^4}\log^2 a(\tau). \tag{6.38}$$

Nevertheless the energy-momentum tensor is covariantly conserved

$$D_\mu\langle T^\mu{}_\nu\rangle = \delta_\nu{}^0\Big\{-\frac{3}{\tau}\frac{\lambda H^4}{2^6\cdot 3\pi^4}\log a(\tau) - \frac{d}{d\tau}\frac{\lambda H^4}{2^7\pi^4}\log^2 a(\tau)\Big\} = 0. \tag{6.39}$$

Here the sub-leading IR effects are canceled by the counter terms (6.11) and (6.34).

6.2   Stochastic approach

Perturbation theory eventually breaks down when $\lambda\log^m a(\tau) \sim H^{-2m+4}$. So we need a tool to investigate the non-perturbative effect in such a regime. There is a stochastic approach for investigating such a non-perturbative effect [14, 15]. It can be regarded as a resummation procedure of the leading IR logarithms due to an interaction potential. Here we briefly recall this prescription.

In a minimally coupled scalar field theory with a potential, the equation of motion is

$$\ddot{\varphi} + 3H\dot{\varphi} - \frac{1}{a^2}\partial_i^2\varphi + V'(\varphi) = 0, \tag{6.40}$$

where $\dot{\varphi} \equiv \frac{\partial}{\partial t}\varphi$. The equation of motion can be integrated as

$$\varphi(x) = \varphi_0(x) - i\int_0^t dt'a^3(t')\int d^3x'\ G^R(x,x')V'(\varphi(x')), \tag{6.41}$$

where $\varphi_0(x)$ denotes a free field and $G^R(x,x')$ is the retarded propagator

$$G^R(x,x') = \theta(t-t')[\langle\varphi_0(x)\varphi_0(x')\rangle - \langle\varphi_0(x')\varphi_0(x)\rangle]. \tag{6.42}$$

As we are interested in the dominant IR effect at late times, we extract the contribution from outside the cosmological horizon

$$\varphi_0(x) \simeq \int\frac{d^3p}{(2\pi)^3}\ \theta(Ha(t)-p)\big(a_{\mathbf{p}}\frac{H}{\sqrt{2p^3}}e^{+i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger\frac{H}{\sqrt{2p^3}}e^{-i\mathbf{p}\cdot\mathbf{x}}\big). \tag{6.43}$$

For the same reason, we extract the leading IR contribution of the propagator

$$G^R(x, x') \simeq \theta(t - t') \int \frac{d^3 p}{(2\pi)^3} \frac{-i}{3H} \left(\frac{1}{a^3(t')} - \frac{1}{a^3(t)} \right) e^{+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')} \tag{6.44}$$

$$= \frac{-i}{3H} \left(\frac{1}{a^3(t')} - \frac{1}{a^3(t)} \right) \theta(t - t') \delta^{(3)}(\mathbf{x} - \mathbf{x}'). \tag{6.45}$$

By substituting (6.43) and (6.44) to Eq.(6.41),

$$\varphi(x) = \varphi_0(x) - \frac{1}{3H} \int_0^t dt' \, V'(\varphi(t', \mathbf{x})), \tag{6.46}$$

where we have neglected the term : $a^{-3}(t) \int_0^t dt' a^3(t') V(\varphi(t', \mathbf{x})$ because it is sub-dominant.

By differentiating Eq.(6.46) with respect to t, we obtain the Langevin equation with the white noise

$$\dot{\varphi}(x) = \dot{\varphi}_0(x) - \frac{1}{3H} V'(\varphi(x)), \quad \langle \dot{\varphi}_0(x) \dot{\varphi}_0(x') \rangle = \frac{H^3}{4\pi^2} \delta(t - t'). \tag{6.47}$$

It describes a random walk in the field space. Since the fractal dimension of the random walk is two, the propagator grows linearly with the cosmic time $Ht = \log a(t)$ at the initial stage. The Langevin equation is equivalent to the Fokker-Planck equation

$$\dot{\rho}(t, \varphi) = \frac{1}{3H} \frac{\partial}{\partial \varphi} [V'(\varphi) \rho(t, \varphi)] + \frac{H^3}{8\pi^2} \frac{\partial^2}{\partial \varphi^2} \rho(t, \varphi), \tag{6.48}$$

where $\rho(t, \varphi)$ is the probability density. The vevs of the operators are given by

$$\langle F(\varphi(x)) \rangle = \int_{-\infty}^{\infty} d\omega \, F(\omega) \rho(t, \omega), \tag{6.49}$$

where F is a function of $\varphi(x)$.

We can reproduce the leading log terms in the perturbative expansion in this approach. Furthermore it allowed us to determine the non-perturbative effect at $t \to \infty$ when we assume that an equilibrium state is established at $t \to \infty$: $\rho(t, \varphi) \to \rho_\infty(\varphi)$. In this assumption, the Fokker-Planck equation is

$$0 = \frac{1}{3H} V'(\varphi) \rho_\infty(\varphi) + \frac{H^3}{8\pi^2} \frac{\partial}{\partial \varphi} \rho_\infty(\varphi). \tag{6.50}$$

The solution is

$$\rho_\infty(\varphi) = N \exp\left(-\frac{8\pi^2}{3H^4} V(\varphi) \right), \tag{6.51}$$

where N is the normalization factor : $\int_{-\infty}^{\infty} d\omega \, \rho_\infty(\omega) = 1$. From (6.49) and (6.51), we can evaluate the vevs of the operators at $t \to \infty$, especially the vev of the potential.

For example, in φ^4 theory, the probability density is

$$\rho_\infty(\varphi) = \frac{2}{\Gamma(\frac{1}{4})} \left(\frac{\pi^2 \lambda}{9H^4}\right)^{\frac{1}{4}} \exp\left(-\frac{\pi^2}{9}\lambda \frac{\varphi^4}{H^4}\right). \tag{6.52}$$

The vev of the potential is

$$\langle V(\varphi) \rangle = \frac{3H^4}{32\pi^2}. \tag{6.53}$$

As we observe in (6.15), the 2-loop effect increases the vev of the potential as time goes on. It is because the magnitude of φ field grows due to a random walk in a stochastic approach. Eventually the drift force due to the potential becomes important and reaches an equilibrium. Thus the following consistent picture emerges, namely the effective cosmological constant increases at the initial stage and the growth is eventually saturated at a constant value.

To evaluate the vev of the energy-momentum tensor, we also need to consider the kinetic term. Note that we retain only the leading IR effect in the stochastic approach. The kinetic term is sub-dominant in comparison to the potential term except at the one loop level. So we can't calculate the kinetic term directly in the stochastic approach. Nevertheless the structure of the kinetic term is constrained by the conservation law. As the potential term approaches a constant at $t \to \infty$, the dS symmetry breaking contribution from the kinetic term also vanishes. That is, the term which is proportional to $\delta_\mu^{\,0}\delta_\nu^{\,0}$ approaches to 0. Of course, it is possible that the sub-leading terms give a finite contribution to the cosmological constant. However these contributions are $\mathcal{O}(\lambda^{\frac{1}{m}})$ at most. We can neglect them if $\lambda \ll H^{-2m+4}$.

From (6.53), the effective cosmological constant at $t \to \infty$ is as follow in φ^4 theory

$$\Lambda_{\text{eff}} = \Lambda - \kappa \frac{3H^4}{32\pi^2} + \kappa \frac{3H^4}{32\pi^2} = \Lambda. \tag{6.54}$$

The contribution from the kinetic term at the one loop level and the non-perturbative contribution from the potential term cancel out each other. It is an accident in φ^4 theory. In φ^{2m} ($m \neq 2$) theory, there remains a finite contribution to the cosmological constant:

$$\Lambda_{\text{eff}} = \Lambda - \kappa \frac{3H^4}{32\pi^2} + \kappa \frac{4}{2m}\frac{3H^4}{32\pi^2}. \tag{6.55}$$

This stochastic approach has been applied to investigate non-perturbative IR effects in Yukawa theory [25] and Scalar QED [26, 27]. These models reduce to a scalar field theory with a potential (6.1) after integrating out the conformally coupled scalar fields, Dirac fields or vector fields.

6.3 In the large N limit

Another non-perturbative approach to investigate IR effects is to consider the large N limit where N counts the number of scalar field. In such a limit, we can solve the model by a

saddle point approximation. The action for φ^4 theory with $O(N)$ symmetry can be expressed as follows

$$S_{matter} = \int \sqrt{-g} d^4 x \ [-\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi^i \partial_\nu \varphi^i - \frac{\chi}{2} (\varphi^i)^2 + \frac{N}{2\lambda} \chi^2], \tag{6.56}$$

where $i = 1 \cdots N$ and χ is an auxiliary field. By differentiating the action with respect to χ, we find that χ represents a composite operator

$$(\varphi^i)^2 = \frac{2N}{\lambda} \chi. \tag{6.57}$$

In the large N limit, we can neglect the fluctuation of χ. So the action (6.56) reduces to a free massive scalar field theory plus the constant term $N\chi^2/2\lambda$.

Here χ acts as the mass of scalar fields $m^2 = \chi$. From (4.18) and (6.57), m^2 is self-consistently determined as

$$m^2 \simeq \frac{\lambda}{2} \times \frac{3H^4}{8\pi^2} \{1 - \exp(-\frac{2m^2}{3H^2} \log a(\tau))\}, \tag{6.58}$$

where we have assumed that m^2/H^2 is small. At the initial stage, m^2 grows with time:

$$m^2 = \chi \simeq \lambda \frac{H^2}{8\pi^2} \log a(\tau). \tag{6.59}$$

It is consistent with (6.12) up to $\mathcal{O}(N)$. From this, the contribution from the potential term is

$$g_{\mu\nu}(-\frac{m^2}{2} \langle \varphi^2(x) \rangle + \frac{N}{2\lambda} \chi^2) \simeq -g_{\mu\nu} N \frac{\lambda H^4}{2^7 \pi^4} \log a(\tau). \tag{6.60}$$

It is consistent with (6.15) up to $\mathcal{O}(N)$.

Furthermore, we find that (6.58) eventually approaches the following identity

$$m^2 \simeq \lambda \frac{3H^4}{16\pi^2 m^2}, \tag{6.61}$$

From this,

$$m^2 = \chi \simeq \frac{\sqrt{3\lambda} H^2}{4\pi}. \tag{6.62}$$

Thus the assumption $m^2/H^2 \ll 1$ is valid as far as $\lambda \ll 1$. Recall that the propagator of a massive scalar field eventually becomes the dS invariant. Therefore the vev of the energy-momentum tensor is written as follows

$$\langle T_{\mu\nu} \rangle = \frac{g_{\mu\nu}}{4} \langle T_\rho{}^\rho \rangle. \tag{6.63}$$

The trace of the energy-momentum tensor is

$$\langle T_\mu{}^\mu \rangle = \langle -g^{\mu\nu} \partial_\mu \varphi^i \partial_\nu \varphi^i - 2m^2 (\varphi^i)^2 + \frac{2N}{\lambda} \chi^2 \rangle \tag{6.64}$$

$$= \langle -\frac{1}{2} \nabla^2 (\varphi^i)^2 + \varphi^i \nabla^2 \varphi^i - 2m^2 (\varphi^i)^2 + \frac{2N}{\lambda} \chi^2 \rangle$$

$$= -\frac{1}{2} \nabla^2 \langle (\varphi^i)^2 \rangle - m^2 \langle (\varphi^i)^2 \rangle + \frac{2N}{\lambda} \chi^2.$$

In the third line, we have used the equation of motion

$$\nabla^2 \varphi^i - m^2 \varphi^i = 0. \tag{6.65}$$

By substituting (6.57) and (6.62) to (6.64),

$$\langle T_\mu{}^\mu \rangle = 0, \quad \Lambda_{\text{eff}} = \Lambda. \tag{6.66}$$

This result is consistent with that in the stochastic approach (6.54). However this is an exact non-perturbative result in the large N limit beyond the leading logarithmic approximation. Note that we extract the term which is proportional to H^2/m^2 in the right hand side of (6.61). Subsequent terms give $\mathcal{O}(\lambda^{\frac{1}{2}})$ shift to the cosmological constant.

Of course, we can confirm the consistent result with (6.55) in $\lambda \varphi^{2m}$ theory

$$\Lambda_{\text{eff}} = \Lambda - \kappa N \frac{3H^4}{32\pi^2} + \kappa N \frac{4}{2m} \frac{3H^4}{32\pi^2}. \tag{6.67}$$

6.4   Association with Euclidean field theory on S_4

At the last in this section, we refer to the association with the Euclidean field theory. dS space is wick rotated to a sphere S_4. Since the Euclidean field theory can't describe non-equilibrium physics, it doesn't deal with all physics which are described in the Lorentzian field theory. However the field theory on S_4 can describe an equilibrium state in dS_4 [16].

In the Euclidean field theory on S_4, the quadratic action for a massless scalar field which is minimally coupled to the background is

$$S_2 = \frac{1}{2} \int \sqrt{g} d^4x \ g^{ij} \partial_i \varphi \partial_j \varphi, \quad i = 1 \cdots 4. \tag{6.68}$$

In the following discussion, we adopt the path integral method. When we expand the field by the spherical harmonics $Y_{\mathbf{L}}(x)$,

$$\varphi(x) = \sum_{\mathbf{L}} \phi_{\mathbf{L}} Y_{\mathbf{L}}(x), \tag{6.69}$$

where $\mathbf{L}$ is the angular momentum: $\mathbf{L} = (L, L_1, L_2, L_3)$, $L \geq L_1 \geq L_2 \geq L_3 \geq 0$ and we normalize the basis as follows

$$\int \sqrt{g} d^4x \ Y_{\mathbf{L}}(x) Y_{\mathbf{L}'}^*(x) = \delta_{\mathbf{L}, \mathbf{L}'}. \tag{6.70}$$

By using the expansion (6.69), (6.70), the quadratic action is written as

$$S_2 = \frac{1}{2} \sum_{\mathbf{L}} L(L+3) H^2 \phi_{\mathbf{L}} \phi_{\mathbf{L}}^*. \tag{6.71}$$

Here we have used the following identity

$$\nabla^2 Y_{\mathbf{L}}(x) = -L(L+3)H^2 Y_{\mathbf{L}}(x). \tag{6.72}$$

From the quadratic action (6.71), the propagator is

$$\langle \varphi(x)\varphi(x')\rangle = \frac{\int \mathcal{D}\varphi \; \varphi(x)\varphi(x')e^{-S_2}}{\int \mathcal{D}\varphi \; e^{-S_2}} \tag{6.73}$$

$$= \sum_{\mathbf{L}} \frac{1}{L(L+3)H^2} Y_{\mathbf{L}}(x)Y_{\mathbf{L}}^*(x').$$

This propagator has an IR divergence at the zero mode $\mathbf{L} = \mathbf{0}$. Note that if a field is massive, the corresponding propagator is not IR divergent because its denominator is $L(L+3)H^2+m^2$.

The IR divergence in (6.73) means the breakdown of perturbation theories. If we adopt an interaction potential, the action is written as

$$S_{\text{matter}} = \frac{1}{2}\sum_{\mathbf{L}} L(L+3)H^2 \phi_{\mathbf{L}}\phi_{\mathbf{L}}^* + \int \sqrt{g}d^4x \; V(\varphi). \tag{6.74}$$

The perturbation theory is applicable as far as the linear term is dominant compared with the non-linear terms. Here we assume that the coupling constant λ is much smaller than 1. In this setting, the linear term is dominant compared with the non-linear terms except for the zero mode. At the zero mode, the linear term is zero and so the non-linear term is dominant. Considering the above, we have to treat the non-linear terms nonperturbatively at the zero mode. Focusing on the zero mode, the action is

$$S_{\text{matter}} \simeq \int \sqrt{g}d^4x \; V(\phi_{\mathbf{0}}Y_{\mathbf{0}}) \tag{6.75}$$

$$= \frac{8\pi^2}{3H^4}V(\phi_{\mathbf{0}}Y_{\mathbf{0}}).$$

Here we have used the fact that $Y_{\mathbf{0}}$ is constant and so the integral over the space is

$$\int \sqrt{g}d^4x = (\text{Area of } S_4) = \frac{8\pi^2}{3H^4}. \tag{6.76}$$

From (6.75), the vevs of operator's functions $F(\varphi)$ are

$$\langle F(\varphi(x))\rangle \simeq \frac{\int \mathcal{D}(\phi_{\mathbf{0}}Y_{\mathbf{0}}) \; F(\phi_{\mathbf{0}}Y_{\mathbf{0}}) \exp\left(-\frac{8\pi^2}{3H^4}V(\phi_{\mathbf{0}}Y_{\mathbf{0}})\right)}{\int \mathcal{D}(\phi_{\mathbf{0}}Y_{\mathbf{0}}) \; \exp\left(-\frac{8\pi^2}{3H^4}V(\phi_{\mathbf{0}}Y_{\mathbf{0}})\right)}. \tag{6.77}$$

It corresponds with the saturation value in the stochastic approach.

We may reflect on the result as follows. In a time dependent background like dS space, the Schwinger-Keldysh formalism is necessary to evaluate the perturbative effects [3]. Our problem belongs to nonequilibrium physics in this sense. However if an equilibrium state is eventually established, it may be described by an Euclidean field theory on S_4. From this

reason, the correspondence between the saturation value in the stochastic approach and the Euclidean evaluation is reasonable.

It should be noted that we consider only the zero mode to obtain the result (6.76) and it corresponds with the saturation value in the leading logarithm approximation. It is a natural question whether the corresponding is true up to the sub-leading IR effect. For example, in φ^4 theory on S_4, the vev of the potential up to the sub-leading IR effect is

$$\langle V(\varphi(x)) \rangle \tag{6.78}$$
$$\simeq \frac{\int \mathcal{D}(\phi_{\mathbf{0}} Y_{\mathbf{0}}) \, \frac{\lambda}{4!} (\phi_{\mathbf{0}} Y_{\mathbf{0}})^4 \exp\left(-\frac{\pi^2 \lambda}{9H^4}(\phi_{\mathbf{0}} Y_{\mathbf{0}})^4\right)}{\int \mathcal{D}(\phi_{\mathbf{0}} Y_{\mathbf{0}}) \, \exp\left(-\frac{\pi^2 \lambda}{9H^4}(\phi_{\mathbf{0}} Y_{\mathbf{0}})^4\right)}$$
$$+ \frac{\int \mathcal{D}(\phi_{\mathbf{0}} Y_{\mathbf{0}}) \prod_{\mathbf{L} \neq \mathbf{0}} \mathcal{D}(\phi_{\mathbf{L}} Y_{\mathbf{L}}) \, \frac{\lambda}{4}(\phi_{\mathbf{0}} Y_{\mathbf{0}})^2 (\sum_{\mathbf{L} \neq \mathbf{0}} \phi_{\mathbf{L}} Y_{\mathbf{L}})^2 \exp\left(-S_2 - \frac{\pi^2 \lambda}{9H^4}(\phi_{\mathbf{0}} Y_{\mathbf{0}})^4\right)}{\int \mathcal{D}(\phi_{\mathbf{0}} Y_{\mathbf{0}}) \prod_{\mathbf{L} \neq \mathbf{0}} \mathcal{D}(\phi_{\mathbf{L}} Y_{\mathbf{L}}) \, \exp\left(-S_2 - \frac{\pi^2 \lambda}{9H^4}(\phi_{\mathbf{0}} Y_{\mathbf{0}})^4\right)}$$
$$\simeq \frac{3H^4}{32\pi^2} + \frac{3\lambda^{\frac{1}{2}} H^2}{4\pi} \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \sum_{\mathbf{L} \neq \mathbf{0}} \frac{1}{L(L+3)H^2} Y_{\mathbf{L}}(x) Y_{\mathbf{L}}^*(x).$$

The sub-leading IR effect is proportional to $\lambda^{\frac{1}{2}}$. It is consistent with the stochastic approach where the sub-leading IR effect approaches $\mathcal{O}(\lambda^{\frac{1}{2}})$: $\lambda^n \log^{2n-1} a(\tau) \sim \lambda^{\frac{1}{2}}$.

Unlike the scalar field theory with an interaction potential, we don't know how to evaluate the non-perturbative IR effect in a generic model with derivative interactions. Non-linear sigma model is such an example while quantum gravity is another. It is very important to investigate IR effects in these models. With this motivation, we consider the non-linear sigma model in the next section. We can investigate some non-perturbative effects also since it is exactly solvable in the large N limit.

7   Non-linear sigma model

In this section, we investigate the IR effects of the non-linear sigma model in dS space. There are two reasons why we are interested in the non-linear sigma model. Firstly the non-linear sigma model contains massless and minimally coupled scalar fields due to the reparameterization invariance of the target space. Secondly we can investigate nonperturbative effects as it becomes exactly solvable in the large N limit.

The action of the non-linear sigma model is

$$S_{\text{matter}} = \frac{1}{2g^2} \int \sqrt{-g} d^4 x \, G_{ij}(\varphi)(-g^{\mu\nu} \partial_\mu \varphi^i \partial_\nu \varphi^j), \tag{7.1}$$

where $g_{\mu\nu}$ is the metric of the dS space, g^2 is the coupling constant and $G_{ij}(i = 1 \cdots N)$ is the metric of the target space. The reparameterization invariance of the target space is the important symmetry of the non-linear sigma model as it follows from the consistency as a quantum theory. The dimensional regularization respects this important symmetry. We

adopt the background field method which is manifestly covariant. The action is expanded as follows [28]

$$S_{\text{matter}} = \frac{-1}{2g^2} \int \sqrt{-g} d^4x \left[G_{ij}(\bar{\varphi}) g^{\mu\nu} \partial_\mu \bar{\varphi}^i \partial_\nu \bar{\varphi}^j - R_{cidj}(\bar{\varphi}) \xi^c \xi^d g^{\mu\nu} \partial_\mu \bar{\varphi}^i \partial_\nu \bar{\varphi}^j \right. \tag{7.2}$$

$$+ (-\frac{1}{12} D_e D_f R_{cidj}(\bar{\varphi}) + \frac{1}{3} R^g{}_{cad} R_{gebf}(\bar{\varphi})) \xi^c \xi^d \xi^e \xi^f g^{\mu\nu} \partial_\mu \bar{\varphi}^i \partial_\nu \bar{\varphi}^j$$

$$- \frac{4}{3} R_{cidb}(\bar{\varphi}) \xi^c \xi^d g^{\mu\nu} (D_\mu \xi)^b \partial_\nu \bar{\varphi}^i$$

$$- \frac{1}{2} D_e R_{cidb}(\bar{\varphi}) \xi^c \xi^d \xi^e g^{\mu\nu} (D_\mu \xi)^b \partial_\nu \bar{\varphi}^i$$

$$+ g^{\mu\nu} (D_\mu \xi)^a (D_\nu \xi)^a - \frac{1}{3} R_{cadb}(\bar{\varphi}) \xi^c \xi^d g^{\mu\nu} (D_\mu \xi)^a (D_\nu \xi)^b$$

$$- \frac{1}{6} D_e R_{cadb}(\bar{\varphi}) \xi^c \xi^d \xi^e g^{\mu\nu} (D_\mu \xi)^a (D_\nu \xi)^b$$

$$+ (-\frac{1}{20} D_e D_f R_{cadb}(\bar{\varphi}) + \frac{2}{45} R^g{}_{cad} R_{gebf}(\bar{\varphi})) \xi^c \xi^d \xi^e \xi^f g^{\mu\nu} (D_\mu \xi)^a (D_\nu \xi)^b + \cdots \left. \right],$$

where $\bar{\varphi}^i$ are the background fields, ξ^i are the quantum fluctuations. Here R_{ikjl} is the Riemann tensor [*] and the covariant derivative are

$$D_\mu \xi^i = \partial_\mu \xi^i + \Gamma^i{}_{jk} \partial_\mu \bar{\varphi}^j \xi^k. \tag{7.3}$$

By using the vielbein $e_i{}^a$, we can work in the flat tangential space E_N instead of the target space

$$\xi^a = e_i{}^a \xi^i, \quad (D_\mu \xi)^a = \partial_\mu \xi^a + \omega_i{}^{ab} \partial_\mu \bar{\varphi}^i \xi^b, \tag{7.4}$$

where $\omega_i{}^{ab}$ is the spin connection. Henceforth we rescale the quantum fluctuations $\xi^a/g \to \xi^a$ for convenience.

Since we are interested in the contribution to the cosmological constant, we can set the background fields $\bar{\varphi}^i$ zero. The vev of the energy-momentum tensor is

$$\langle T_{\mu\nu} \rangle = (\delta_\mu{}^\rho \delta_\nu{}^\sigma - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma}) \times \tag{7.5}$$

$$\langle \partial_\rho \xi^a \partial_\sigma \xi^a - \frac{g^2}{3} R_{cadb} \xi^c \xi^d \partial_\rho \xi^a \partial_\sigma \xi^b - \frac{g^3}{6} D_e R_{cadb} \xi^c \xi^d \xi^e \partial_\rho \xi^a \partial_\sigma \xi^b$$

$$+ (-\frac{g^4}{20} D_e D_f R_{cadb} + \frac{2g^4}{45} R^g{}_{cad} R_{gebf}) \xi^c \xi^d \xi^e \xi^f \partial_\rho \xi^a \partial_\sigma \xi^b + \cdots \rangle.$$

A propagator left intact by differential operators $\langle \xi(x) \xi(x') \rangle$ can induce a single IR logarithm. The power counting procedure for the leading IR logarithms in the expectation value of the energy-momentum tensor is explained in Appendix C. The conclusion is that the leading IR effect of the energy-momentum tensor at the n-th loop level is $\log^{n-1} a$. It also predicts the $\log^{n-2} a(\tau)$ factor as the sub-leading effect. We investigate the leading IR effect in Subsection 7.1, 7.2 and the sub-leading IR effect in Subsection 7.3, 7.5.

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[*]Our convention is $R^i{}_{jkl} = \partial_k \Gamma^i{}_{jl} - \partial_l \Gamma^i{}_{jk} + \cdots$ and $R_{ij} = R^k{}_{ikj}$.

39

Before investigating the effective cosmological constant, we refer to the effective cosmological constant. The quadratic part of the action $g^{\mu\nu}\partial_\mu\bar{\varphi}^i\partial_\nu\bar{\varphi}^j$ acquires the following quantum correction at the one loop level

$$-\frac{1}{2g^2}\big\{G_{ij}(\bar{\varphi}) - g^2 R_{ij}(\bar{\varphi})G^{++}(x,x)\big\}g^{\mu\nu}\partial_\mu\bar{\varphi}^i\partial_\nu\bar{\varphi}^j. \tag{7.6}$$

As seen in (6.10), the propagator at the coincident point has the UV divergence. To renormalize it, we introduce the following counter term:

$$-\frac{\delta\beta}{2g^2}R_{ij}(\varphi)g^{\mu\nu}\partial_\mu\varphi^i\partial_\nu\varphi^j, \quad \delta\beta = g^2\frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\delta. \tag{7.7}$$

Considering this, (7.6) is evaluated as

$$-\frac{1}{2g^2}\big\{G_{ij}(\bar{\varphi}) - g^2 R_{ij}(\bar{\varphi})\frac{H^2}{4\pi^2}\log a(\tau)\big\}g^{\mu\nu}\partial_\mu\bar{\varphi}^i\partial_\nu\bar{\varphi}^j. \tag{7.8}$$

In the case that the Ricci tensor R_{ij} is proportional to G_{ij} just as the maximally symmetric space, the effective coupling constant is found as follows

$$\frac{1}{g_{\text{eff}}^2} = \frac{1}{g^2} - \frac{R}{N}\frac{H^2}{4\pi^2}\log a(\tau). \tag{7.9}$$

The effective coupling constant increases with the cosmic evolution in the non-linear sigma model on S_N. On the other hand, the effective coupling constant decreases with cosmic evolution on a hyperboloid H_N.

As is well known, the non-linear sigma model on S_N is asymptotically free in 2-dimensional Minkowski space. The propagator at the coincident point is

$$\langle\xi(x)\xi(x)\rangle = \frac{1}{4\pi}\big\{\frac{2}{\varepsilon} - \log\mu^2 - \gamma + \log 4\pi\big\}. \tag{7.10}$$

We find

$$\frac{1}{g_{\text{eff}}^2} = \frac{1}{g^2} + \frac{R}{N}\frac{1}{2\pi}\log\mu. \tag{7.11}$$

The effective coupling constant increases as the mass scale μ is decreased in an analogous fashion. Although there are similarities between the non-linear sigma models in 4 dimensional dS space and in 2 dimensional Minkowski space, there are important differences. Namely the coupling constant in the non-linear sigma model in 4d dS space changes with time while that in 2d Minkowski space remains the constant. Its evolution takes place under the renormalization group not under the time evolution. If the dS invariance is maintained, the time evolution in a comoving coordinate can be related to the scale transformation and thus the renormalization group. However the dS invariance is broken by the IR quantum effects.

7.1 Leading IR effects at the two loop level

At the beginning of this section, we have found that the coupling constant of the non-linear sigma model becomes time dependent at the one loop level in agreement with power

counting of the IR logarithms. In this subsection, we investigate the leading IR effects to the cosmological constant at the two loop level.

The contributions to the energy-momentum tensor consist of the two terms. One is the vev of the twice differentiated propagator, the "propagator" term. The other is the vev of the non-linear term, the "vertex" term. In the model with an interaction potential, the "vertex" term is equal to the vev of the potential. In the non-linear sigma model, the "propagator" and "vertex" terms at the two loop level are written as follows in the Schwinger-Keldysh formalism:

$$\langle \partial_\rho \xi^a \partial_\sigma \xi^a \rangle|_{g^2} \tag{7.12}$$

$$= \int \sqrt{-g'} d^D x' \left\{ i \frac{g^2}{3} R G^{++}(x', x') - i(\delta\beta + 2\delta\gamma) R \right\}$$

$$\times g^{\alpha\beta}(\tau') \left[\partial_\rho \partial'_\alpha G^{++}(x, x') \partial_\sigma \partial'_\beta G^{++}(x, x') - \partial_\rho \partial'_\alpha G^{+-}(x, x') \partial_\sigma \partial'_\beta G^{+-}(x, x') \right]$$

$$+ \int \sqrt{-g'} d^D x' \ i \frac{g^2}{3} R \lim_{x'' \to x'} \partial'_\alpha \partial''_\beta G^{++}(x', x'')$$

$$\times g^{\alpha\beta}(\tau') \left[\partial_\rho G^{++}(x, x') \partial_\sigma G^{++}(x, x') - \partial_\rho G^{+-}(x, x') \partial_\sigma G^{+-}(x, x') \right]$$

$$- \int \sqrt{-g'} d^D x' \ i \frac{g^2}{6} R \partial'_\alpha G^{++}(x', x')$$

$$\times g^{\alpha\beta}(\tau') \partial'_\beta \left[\partial_\rho G^{++}(x, x') \partial_\sigma G^{++}(x, x') - \partial_\rho G^{+-}(x, x') \partial_\sigma G^{+-}(x, x') \right],$$

$$- \frac{g^2}{3} R_{cadb} \langle \xi^c \xi^d \partial_\rho \xi^a \partial_\sigma \xi^b \rangle|_{g^0} + (\delta\beta + 2\delta\gamma) R_{ab} \langle \partial_\rho \xi^a \partial_\sigma \xi^b \rangle|_{g^0} \tag{7.13}$$

$$= - \left\{ \frac{g^2}{3} R G^{++}(x, x) - (\delta\beta + 2\delta\gamma) R \right\} \lim_{x' \to x} \partial_\rho \partial'_\sigma G^{++}(x, x')$$

$$+ \frac{g^2}{12} R \partial_\rho G^{++}(x, x) \partial_\sigma G^{++}(x, x),$$

where we have introduced the counter term (7.7) and also renormalized the quantum fluctuations

$$\xi^i \to \xi^i + \delta\gamma R^i{}_j(\bar\varphi) \xi^j. \tag{7.14}$$

Here we set $\delta\gamma$ as follows to renormalize the UV divergence of (7.12):

$$\delta\beta + 2\delta\gamma = \frac{g^2}{3} \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \delta. \tag{7.15}$$

In each contribution in (7.12) and (7.13), there exists a propagator which is not affected by the derivatives. So the leading IR effects in the energy-momentum tensor are proportional to $g_{\mu\nu} g^2 \log a(\tau)$ at the two loop level. To evaluate them, we have only to calculate the terms which are proportional to $g_{\rho\sigma} g^2 \log a(\tau)$ in (7.12) and (7.13). As we have explained in Section 5, there could be no $\delta_\mu{}^0 \delta_\nu{}^0 g^2 \log a(\tau)$ type term in the energy-momentum tensor due to the

conservation law. From (7.5), it indicates that the $\delta_\rho{}^0\delta_\sigma{}^0 g^2 \log a(\tau)$ type terms are canceled between (7.12) and (7.13). The leading IR logarithms come from the following terms:

$$\langle\partial_\rho\xi^a\partial_\sigma\xi^a\rangle|_{g^2} \tag{7.16}$$
$$\simeq \int \sqrt{-g'}d^D x' \left\{i\frac{g^2}{3}RG^{++}(x',x') - i(\delta\beta + 2\delta\gamma)R\right\}$$
$$\times g^{\alpha\beta}(\tau)\left[\partial_\rho\partial'_\alpha G^{++}(x,x')\partial_\sigma\partial'_\beta G^{++}(x,x') - \partial_\rho\partial'_\alpha G^{+-}(x,x')\partial_\sigma\partial'_\beta G^{+-}(x,x')\right],$$

$$-\frac{g^2}{3}R_{cadb}\langle\xi^c\xi^d\partial_\rho\xi^a\partial_\sigma\xi^b\rangle|_{g^0} + (\delta\beta + 2\delta\gamma)R\langle\partial_\rho\xi^a\partial_\sigma\xi^a\rangle|_{g^0} \tag{7.17}$$
$$\simeq -\left\{\frac{g^2}{3}RG^{++}(x,x) - (\delta\beta + 2\delta\gamma)R\right\}\lim_{x'\to x}\partial_\rho\partial'_\sigma G^{++}(x,x').$$

From (4.13) and (4.14), we find

$$\lim_{x'\to x}\partial_\rho\partial'_\sigma G^{++}(x,x') = -\frac{H^D}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D)}{D\Gamma(\frac{D}{2})}g_{\rho\sigma}(\tau), \tag{7.18}$$

$$g^{\alpha\beta}(\tau')\partial_\rho\partial'_\alpha G(x,x')\partial_\sigma\partial'_\beta G(x,x') \tag{7.19}$$
$$\to \frac{4^{D-2}H^{2D}}{(4\pi)^D}\Gamma^2(\frac{D}{2})a^2(\tau)$$
$$\times \left\{\eta_{\rho\sigma}\left[\frac{4}{y^{4-\varepsilon}} + \frac{4(1-\frac{\varepsilon}{4})}{y^{3-\varepsilon}} + \frac{(1-\frac{\varepsilon}{4})^2}{y^{2-\varepsilon}} + \left(\frac{\Gamma(4-\frac{\varepsilon}{2})}{4\Gamma(2-\frac{\varepsilon}{2})}\frac{1}{y^{2-\varepsilon}} - \frac{\Gamma(4-\varepsilon)}{2\Gamma(3-\frac{\varepsilon}{2})\Gamma(2-\frac{\varepsilon}{2})}\frac{4^{\frac{\varepsilon}{2}}}{y^{2-\frac{\varepsilon}{2}}}\right)\right]$$
$$+ a^2(\tau')H^2\Delta x_\rho\Delta x_\sigma\left[\frac{32(1-\frac{3}{4}\varepsilon)}{y^{5-\varepsilon}} + \frac{4(1-\frac{7}{2}\varepsilon)}{y^{4-\varepsilon}} + \frac{-\frac{9}{2}\varepsilon}{y^{3-\varepsilon}}\right.$$
$$\left.- (4-\varepsilon)\left(\frac{\Gamma(4-\frac{\varepsilon}{2})}{4\Gamma(2-\frac{\varepsilon}{2})}\frac{1}{y^{3-\varepsilon}} - \frac{\Gamma(4-\varepsilon)}{2\Gamma(3-\frac{\varepsilon}{2})\Gamma(2-\frac{\varepsilon}{2})}\frac{4^{\frac{\varepsilon}{2}}}{y^{3-\frac{\varepsilon}{2}}}\right)\right]\right\}.$$

We have neglected the terms which are proportional to $\delta_\rho{}^0$ or $\delta_\sigma{}^0$ in (7.19) since they are not necessary to evaluate the leading IR effects. Henceforth we assign the indexes $++, +-$ in (7.19). To be exact, we should assign these indexes before the differential operators are acted. Only when we consider the twice differentiated time-ordered propagator $\partial_\rho\partial'_\alpha G^{++}(x,x')$, the difference emerges as

$$\frac{-i\delta^D(x-x')}{\sqrt{-g}g^{00}}\delta_\rho{}^0\delta_\alpha{}^0. \tag{7.20}$$

We don't consider it for the same reason above.

From (6.10), (7.15), (7.18) and (7.19), (7.17) is evaluated as

$$-\frac{g^2}{3}R_{cadb}\langle\xi^c\xi^d\partial_\rho\xi^a\partial_\sigma\xi^b\rangle|_{g^0} + (\delta\beta + 2\delta\gamma)R\langle\partial_\rho\xi^a\partial_\sigma\xi^a\rangle|_{g^0} \simeq +g_{\rho\sigma}\frac{g^2RH^6}{2^7\pi^4}\log a(\tau), \tag{7.21}$$

and (7.16) is written as

$$\langle \partial_\rho \xi^a \partial_\sigma \xi^a \rangle|_{g^2} \simeq i \frac{g^2 R}{3} \frac{2^{2D-3} H^{3D-2}}{(4\pi)^{\frac{3D}{2}}} \Gamma(D-1)\Gamma(\frac{D}{2}) a^2(\tau) \tag{7.22}$$

$$\times \int d^{4-\varepsilon}x' \, a^{4-\varepsilon}(\tau') \log a(\tau') \sum_{m=1}^{8} I^m_{\rho\sigma}.$$

Here the eight tensors $I^m_{\rho\sigma}$ are written as follows:

$$I^1_{\rho\sigma} \equiv \eta_{\rho\sigma} \Big[\frac{4}{y_{++}^{4-\varepsilon}} - \frac{4}{y_{+-}^{4-\varepsilon}} \Big], \tag{7.23}$$

$$I^2_{\rho\sigma} \equiv \eta_{\rho\sigma} \Big[\frac{4(1-\frac{\varepsilon}{4})}{y_{++}^{3-\varepsilon}} - \frac{4(1-\frac{\varepsilon}{4})}{y_{+-}^{3-\varepsilon}} \Big], \tag{7.24}$$

$$I^3_{\rho\sigma} \equiv \eta_{\rho\sigma} \Big[\frac{(1-\frac{\varepsilon}{4})^2}{y_{++}^{2-\varepsilon}} - \frac{(1-\frac{\varepsilon}{4})^2}{y_{+-}^{2-\varepsilon}} \Big], \tag{7.25}$$

$$I^4_{\rho\sigma} \equiv \eta_{\rho\sigma} \left(\frac{\Gamma(4-\frac{\varepsilon}{2})}{4\Gamma(2-\frac{\varepsilon}{2})} \Big[\frac{1}{y_{++}^{2-\varepsilon}} - \frac{1}{y_{+-}^{2-\varepsilon}} \Big] - \frac{\Gamma(4-\varepsilon)}{2\Gamma(3-\frac{\varepsilon}{2})\Gamma(2-\frac{\varepsilon}{2})} \Big[\frac{4^{\frac{\varepsilon}{2}}}{y_{++}^{2-\frac{\varepsilon}{2}}} - \frac{4^{\frac{\varepsilon}{2}}}{y_{+-}^{2-\frac{\varepsilon}{2}}} \Big] \right), \tag{7.26}$$

$$I^5_{\rho\sigma} \equiv a^2(\tau')H^2\Delta x_\rho \Delta x_\sigma \cdot 32(1-\frac{3}{4}\varepsilon)\Big[\frac{1}{y_{++}^{5-\varepsilon}} - \frac{1}{y_{+-}^{5-\varepsilon}} \Big], \tag{7.27}$$

$$I^6_{\rho\sigma} \equiv a^2(\tau')H^2\Delta x_\rho \Delta x_\sigma \cdot 4(1-\frac{7}{2}\varepsilon)\Big[\frac{1}{y_{++}^{4-\varepsilon}} - \frac{1}{y_{+-}^{4-\varepsilon}} \Big], \tag{7.28}$$

$$I^7_{\rho\sigma} \equiv a^2(\tau')H^2\Delta x_\rho \Delta x_\sigma \cdot -\frac{9}{2}\varepsilon\Big[\frac{1}{y_{++}^{3-\varepsilon}} - \frac{1}{y_{+-}^{3-\varepsilon}} \Big], \tag{7.29}$$

$$I^8_{\rho\sigma} \equiv a^2(\tau')H^2\Delta x_\rho \Delta x_\sigma \tag{7.30}$$

$$\times -(4-\varepsilon)\left(\frac{\Gamma(4-\frac{\varepsilon}{2})}{4\Gamma(2-\frac{\varepsilon}{2})} \Big[\frac{1}{y_{++}^{3-\varepsilon}} - \frac{1}{y_{+-}^{3-\varepsilon}} \Big] - \frac{\Gamma(4-\varepsilon)}{2\Gamma(3-\frac{\varepsilon}{2})\Gamma(2-\frac{\varepsilon}{2})} \Big[\frac{4^{\frac{\varepsilon}{2}}}{y_{++}^{3-\frac{\varepsilon}{2}}} - \frac{4^{\frac{\varepsilon}{2}}}{y_{+-}^{3-\frac{\varepsilon}{2}}} \Big] \right).$$

The integrals containing $I^m_{\rho\sigma}$ except with $m = 4, 8$ can be performed by the process which is introduced at the former part in Appendix D. We list the results:

$$\int d^{4-\varepsilon}x' a^{4-\varepsilon}(\tau') \log a(\tau') I^1_{\rho\sigma} \simeq 4i\pi^2 H^{-4} \log a(\tau) \, \eta_{\rho\sigma} \cdot 0, \tag{7.31}$$

$$\int d^{4-\varepsilon}x' a^{4-\varepsilon}(\tau') \log a(\tau') I^2_{\rho\sigma} \tag{7.32}$$

$$\simeq 4i\pi^2 H^{-4} \log a(\tau) \, \eta_{\rho\sigma} \cdot \left\{ \frac{1}{2}\Big(\frac{\pi^{-\frac{\varepsilon}{2}} \mu^{-\varepsilon} H^{2\varepsilon}}{\Gamma(1-\frac{\varepsilon}{2})\varepsilon} + \log\frac{2\mu}{H} \Big) + \frac{1}{8} \right\},$$

43

$$\int d^{4-\varepsilon}x' a^{4-\varepsilon}(\tau') \log a(\tau') I_{\rho\sigma}^3 \tag{7.33}$$

$$\simeq 4i\pi^2 H^{-4} \log a(\tau)\, \eta_{\rho\sigma} \cdot \left\{ -\frac{1}{2}\Big(\frac{\pi^{-\frac{\varepsilon}{2}}\mu^{-\varepsilon}H^{2\varepsilon}}{\Gamma(1-\frac{\varepsilon}{2})\varepsilon} + \log\frac{2\mu}{H}\Big) + \frac{1}{4} \right\},$$

$$\int d^{4-\varepsilon}x' a^{4-\varepsilon}(\tau') \log a(\tau') I_{\rho\sigma}^5 \tag{7.34}$$

$$\simeq 4i\pi^2 H^{-4} \log a(\tau)\, \eta_{\rho\sigma} \cdot \left\{ -\frac{1}{4}\Big(\frac{\pi^{-\frac{\varepsilon}{2}}\mu^{-\varepsilon}H^{2\varepsilon}}{\Gamma(1-\frac{\varepsilon}{2})\varepsilon} + \log\frac{2\mu}{H}\Big) + 0 \right\},$$

$$\int d^{4-\varepsilon}x' a^{4-\varepsilon}(\tau') \log a(\tau') I_{\rho\sigma}^6 \tag{7.35}$$

$$\simeq 4i\pi^2 H^{-4} \log a(\tau)\, \eta_{\rho\sigma} \cdot \left\{ \frac{1}{4}\Big(\frac{\pi^{-\frac{\varepsilon}{2}}\mu^{-\varepsilon}H^{2\varepsilon}}{\Gamma(1-\frac{\varepsilon}{2})\varepsilon} + \log\frac{2\mu}{H}\Big) - \frac{3}{4} \right\},$$

$$\int d^{4-\varepsilon}x' a^{4-\varepsilon}(\tau') \log a(\tau') I_{\rho\sigma}^7 \simeq 4i\pi^2 H^{-4} \log a(\tau)\, \eta_{\rho\sigma} \cdot \frac{9}{16}. \tag{7.36}$$

The evaluation of the integral containing $I_{\rho\sigma}^4$ and $I_{\rho\sigma}^8$ is a little different from others. We explain it in Appendix D.2. Here we simply show the result

$$\int d^{4-\varepsilon}x' a^{4-\varepsilon}(\tau') \log a(\tau')(I_{\rho\sigma}^4 + I_{\rho\sigma}^8) \simeq 4i\pi^2 H^{-4} \log a(\tau)\, \eta_{\rho\sigma} \cdot \frac{3}{16}. \tag{7.37}$$

The total of these eight contributions is

$$\int d^{4-\varepsilon}x' a^{4-\varepsilon}(\tau') \log a(\tau') \sum_{m=1}^{8} I_{\rho\sigma}^m \simeq 4i\pi^2 H^{-4} \log a(\tau)\, \eta_{\rho\sigma} \cdot \frac{3}{8}. \tag{7.38}$$

In this way, the quantum expectation value of the quadratic kinetic term is found as follows up to the two loop level

$$\langle \partial_\rho \xi^a \partial_\sigma \xi^a \rangle \simeq -g_{\rho\sigma} N \frac{3H^4}{32\pi^2} - g_{\rho\sigma} \frac{g^2 R H^6}{2^7 \pi^4} \log a(\tau). \tag{7.39}$$

Note that unlike in the scalar field theory with an interaction potential, the "propagator" term is of the same order with the "vertex" term. It is because the differential operators act on not only the "propagator" term but the "vertex" term.

By combining (7.21) and (7.39), we find that there is no time dependence of the vev of the energy-momentum tensor up to the two loop level.

$$\langle T_{\mu\nu} \rangle \simeq N \frac{3H^4}{32\pi^2} g_{\mu\nu}. \tag{7.40}$$

Although there are time dependent IR logarithms in each contribution in agreement with the power counting arguments, they cancel out each other. The contribution to the cosmological constant is identical to that in the free field theory

$$\Lambda_{eff} \simeq \Lambda - \kappa N \frac{3H^4}{32\pi^2}. \tag{7.41}$$

Note that we have neglected the sub-leading IR effects which is time independent at the two loop level.

7.2 Cancellation of the leading IR effects to the cosmological constant

In the previous subsection, we have confirmed that the leading IR effects to the cosmological constant cancel out each other at the two loop level from the explicit calculation. It is natural question whether the cancellation takes place at the higher loop level. However the explicit calculation becomes more hard as the loop level increases. In fact, the cancellation at the two loop level can be confirmed by a partial integration method. Furthermore by using this method, we can prove that the cancellation of the leading IR effects to the cosmological constant takes place to all orders. Here we explain how to prove it.

As pointed out in [15, 27], the partial integration is very useful to evaluate the time dependent contributions in the diagrams with derivative interactions. First, we reconfirm the cancellation at two loop level by using the partial integration. By using the partial integration, the "propagator" term (7.16) is written as

$$\langle \partial_\rho \xi^a \partial_\sigma \xi^a \rangle|_{g^2} \tag{7.42}$$

$$\simeq \int d^D x' \, \partial'_\alpha \big\{ i \frac{g^2}{3} R G^{++}(x', x') - i(\delta\beta + 2\delta\gamma)R \big\}$$

$$\times \sqrt{-g'} g^{\alpha\beta}(\tau') \big[\partial_\rho G^{++}(x, x') \partial_\sigma \partial'_\beta G^{++}(x, x') - \partial_\rho G^{+-}(x, x') \partial_\sigma \partial'_\beta G^{+-}(x, x') \big]$$

$$- \int d^D x' \, \big\{ i \frac{g^2}{3} R G^{++}(x', x') - i(\delta\beta + 2\delta\gamma)R \big\}$$

$$\times \big[\partial_\rho G^{++}(x, x') \partial_\sigma \sqrt{-g'} \nabla'^2 G^{++}(x, x') - \partial_\rho G^{+-}(x, x') \partial_\sigma \sqrt{-g'} \nabla'^2 G^{+-}(x, x') \big].$$

Note that the surface term is zero because $\tau \to 0$ is outside the past light corn and $\log a(\tau) = 0$ at $\tau = -\frac{1}{H}$. The first term doesn't induce a single logarithm and so we neglect it. By using the following identities

$$\sqrt{-g'} \nabla'^2 G^{++}(x, x') = i\delta^{(D)}(x - x'), \quad \sqrt{-g'} \nabla'^2 G^{+-}(x, x') = 0, \tag{7.43}$$

the "propagator" term is

$$\langle \partial_\rho \xi^a \partial_\sigma \xi^a \rangle|_{g^2} \tag{7.44}$$

$$\simeq \int d^D x' \, \big\{ \frac{g^2}{3} R G^{++}(x', x') - (\delta\beta + 2\delta\gamma)R \big\} \times \partial_\rho G^{++}(x, x') \partial_\sigma \delta^{(D)}(x - x')$$

$$= \big\{ \frac{g^2}{3} R G^{++}(x, x) - (\delta\beta + 2\delta\gamma)R \big\} \lim_{x' \to x} \partial_\rho \partial'_\sigma G^{++}(x, x')$$

$$+ \int d^D x' \, \partial'_\sigma \big\{ \frac{g^2}{3} R G^{++}(x', x') - (\delta\beta + 2\delta\gamma)R \big\} \partial_\rho G^{++}(x, x')$$

$$\simeq \big\{ \frac{g^2}{3} R G^{++}(x, x) - (\delta\beta + 2\delta\gamma)R \big\} \lim_{x' \to x} \partial_\rho \partial'_\sigma G^{++}(x, x').$$

Here we have used the partial integration and neglected the term which doesn't induce a single logarithm. From (7.17) and (7.44), the contributions from the "vertex" term and the "propagator" term cancel out each other up to the leading IR effect

$$\langle \partial_\rho \xi^a \partial_\sigma \xi^a \rangle|_{g^2} - \frac{g^2}{3} R_{cadb} \langle \xi^c \xi^d \partial_\rho \xi^a \partial_\sigma \xi^b \rangle|_{g^0} + (\delta\beta + 2\delta\gamma) R_{ab} \langle \partial_\rho \xi^a \partial_\sigma \xi^b \rangle|_{g^0} \simeq 0. \tag{7.45}$$

The above prescription is easily understood by using the Feynman diagrams. The leading IR contributions from the "propagator" term and the "vertex" term are represented by the following diagrams

$$\langle\partial_\rho\xi^a\partial_\sigma\xi^a\rangle|_{g^2} \simeq \quad , \tag{7.46}$$

$$-\frac{g^2}{3}R_{cadb}\langle\xi^c\xi^d\partial_\rho\xi^a\partial_\sigma\xi^b\rangle|_{g^0} + (\delta\beta + 2\delta\gamma)R\langle\partial_\rho\xi^a\partial_\sigma\xi^a\rangle|_{g^0} \simeq \quad , \tag{7.47}$$

where the dot denotes the location of the energy-momentum tensor x. The short line segments on the propagator denote the differential operators. By using the partial integration, the "propagator" term is

$$= 2 \quad - \quad . \tag{7.48}$$

We neglect the first diagram because it doesn't induce a single logarithm

$$\simeq - \quad . \tag{7.49}$$

Here the double line segments denote $\sqrt{-g'}\nabla'^2$. By using (7.43) and the partial integration,

$$= -2 \quad - \quad \simeq - \quad . \tag{7.50}$$

In the last process, we neglected the first diagram since it doesn't induce a IR logarithm. As a result, the "propagator" term cancels out the "vertex" term up to the leading IR effect.

The diagramatic investigation is useful beyond the two loop level. We can indeed confirm that the leading IR effects cancel between the "propagator" terms and the "vertex" terms. Let us recall that the interaction terms in the non-linear sigma model contain two derivatives. Each diagram with the leading IR logarithms contains a closed loop of the twice differentiated propagators which runs through the vertex located at the external point x. The other diagrams are obtained if we remove any of the differential operators from the closed loop and let them act on the other propagators outside the loop. We can show that such diagrams always have reduced powers of the IR logarithms. We explain the details of the IR power counting in non-linear sigma models in Appendix C.

Therefore in the "vertex" terms, the diagrams with the leading IR logarithms contain the following structure:

$$(\text{The "vertex" terms}) \simeq \quad + \quad . \tag{7.51}$$

46

To evaluate the leading IR effects of the "propagator" terms, we have only to consider the diagrams where $\partial_\rho \xi \partial_\sigma \xi$ is inserted to one of the propagators of such a loop:

$$\text{(The "propagator" terms)} \simeq \quad \text{} \quad + \quad \text{} \quad . \tag{7.52}$$

When the closed loop consists of a single propagator, we obtain

$$\text{} \quad \simeq - \quad \text{} \quad . \tag{7.53}$$

The important point is that there are equal number of the propagators and the vertices in a close loop. The "vertex" terms count the vertices while the "propagator" terms count the propagators. The "propagator" terms cancel the corresponding "vertex" terms. To prove the cancellation in general, we focus on a pair of the corresponding terms:

$$\text{} \quad = F \int \sqrt{-g'} d^D x' \; g^{\alpha\beta}(\tau') \sum_{i=\pm} sgn(+, i) \tag{7.54}$$

$$\times \cdots \partial_\varepsilon'''' \partial_\rho G^{l+}(x'''', x) \partial_\sigma \partial_\alpha' G^{+i}(x, x') \partial_\beta' \partial_\zeta''' G^{ik}(x', x''') \cdots ,$$

$$\text{} \quad = - iF \int \sqrt{-g''} d^D x'' \int \sqrt{-g'} d^D x' \; g^{\gamma\delta}(\tau'') g^{\alpha\beta}(\tau') \sum_{i,j=\pm} sgn(j, +) sgn(+, i)$$

$$\times \cdots \partial_\varepsilon'''' \partial_\gamma'' G^{lj}(x'''', x'') \partial_\delta'' \partial_\rho G^{j+}(x'', x) \partial_\sigma \partial_\alpha' G^{+i}(x, x') \partial_\beta' \partial_\zeta''' G^{ik}(x', x''') \cdots , \tag{7.55}$$

where F is a common coefficient between the "propagator" term and the "vertex" term which is a function of covariant tensors such as R_{cadb} and $sgn(i, j)$ is defined as

$$sgn(i, j) \equiv \begin{cases} +1 & \text{for } (i, j) = (+, +), (-, -), \\ -1 & \text{for } (i, j) = (+, -), (-, +). \end{cases} \tag{7.56}$$

Note that (7.55) has the extra prefactor $-i$ compared with (7.54). It is because the "propagator" terms have one more vertex than the "vertex" terms. By using the partial integration, (7.55) is

$$\text{} \quad \simeq + iF \int d^D x'' \int \sqrt{-g'} d^D x' \; g^{\alpha\beta}(\tau') \sum_{i,j=\pm} sgn(j, +) sgn(+, i) \tag{7.57}$$

$$\times \cdots \partial_\varepsilon'''' G^{lj}(x'''', x'') \partial_\rho \sqrt{-g''} \nabla''^2 G^{j+}(x'', x) \partial_\sigma \partial_\alpha' G^{+i}(x, x') \partial_\beta' \partial_\zeta''' G^{ik}(x', x''') \cdots ,$$

where we neglected the diagrams which don't induce the leading IR effects. By using (7.43) and the partial integration,

$$\cdots \simeq - F \int \sqrt{-g'}d^D x' \; g^{\alpha\beta}(\tau') \sum_{i=\pm} sgn(+,i) \tag{7.58}$$

$$\times \cdots \partial_\varepsilon^{''''}\partial_\rho G^{l+}(x'''',x)\partial_\sigma \partial_\alpha' G^{+i}(x,x')\partial_\beta' \partial_\zeta''' G^{ik}(x',x''') \cdots .$$

Here we neglected the diagrams which don't induce the leading IR effects again. From (7.54) and (7.58), we obtain

$$\cdots \simeq - \cdots . \tag{7.59}$$

This concludes the proof that the leading IR logarithms cancel in non-linear sigma models to all orders.

7.3 Sub-leading IR effects at the two loop level

In this section, we investigate the sub-leading IR effects to the cosmological constant at the two loop level.

To perform the calculation efficiently, we note that the dS invariance is preserved up to the two loop level. It is because the leading IR effect: $\log a(\tau)$ is absent. So the vev of the energy-momentum tensor is written as

$$\langle T_{\mu\nu} \rangle = \frac{g_{\mu\nu}}{D} \langle T_\rho{}^\rho \rangle. \tag{7.60}$$

We have only to evaluate the trace of the energy-momentum tensor.

In the non-linear sigma model, the trace of the energy-momentum tensor is

$$\langle T_\mu{}^\mu \rangle = (\frac{D}{2}-1)\langle -\{1+(\delta\beta+2\delta\gamma)R\}g^{\mu\nu}\partial_\mu\xi^a\partial_\nu\xi^a + \frac{g^2}{3}R_{cadb}\xi^c\xi^d g^{\mu\nu}\partial_\mu\xi^a\partial_\nu\xi^b\rangle \tag{7.61}$$

$$= (\frac{D}{2}-1)\langle -\{1+(\delta\beta+2\delta\gamma)R\}\frac{1}{2}\nabla^2(\xi^a\xi^a) + \{1+(\delta\beta+2\delta\gamma)R\}\xi^a\nabla^2\xi^a$$

$$+ \frac{g^2}{3}R_{cadb}\xi^c\xi^d g^{\mu\nu}\partial_\mu\xi^a\partial_\nu\xi^b\rangle$$

$$= (\frac{D}{2}-1)\langle -\{1+(\delta\beta+2\delta\gamma)R\}\frac{1}{2}\nabla^2(\xi^a\xi^a)$$

$$+ \frac{g^2}{6}(R_{cadb}+R_{cbda})\xi^a\frac{1}{\sqrt{-g}}\partial_\mu(\xi^c\xi^d\sqrt{-g}g^{\mu\nu}\partial_\nu\xi^b)\rangle.$$

48

In the third line of (7.61), we have used the equation of motion

$$\{1 + (\delta\beta + 2\delta\gamma)R\}\nabla^2\xi^a - \frac{g^2}{6}(R_{cadb} + R_{cbda})\frac{1}{\sqrt{-g}}\partial_\mu(\xi^c\xi^d\sqrt{-g}g^{\mu\nu}\partial_\nu\xi^b) \qquad (7.62)$$

$$+\frac{g^2}{6}(R_{cadb} + R_{cbda})\xi^b g^{\mu\nu}\partial_\mu\xi^c\partial_\nu\xi^d = 0.$$

Up to the two loop level,

$$\langle T_\mu{}^\mu \rangle = -(\frac{D}{2} - 1)\frac{1}{2}\{1 + (\delta\beta + 2\delta\gamma)R\}\nabla^2\langle\xi^a\xi^a\rangle \qquad (7.63)$$

$$+(\frac{D}{2} - 1)\frac{g^2}{6}(R_{cadb} + R_{cbda})\langle\xi^a\partial_\mu\xi^c\xi^d g^{\mu\nu}\partial_\nu\xi^b + \xi^a\xi^c\partial_\mu\xi^d g^{\mu\nu}\partial_\nu\xi^b\rangle.$$

$$= -(\frac{D}{2} - 1)\frac{1}{2}\nabla^2\langle\xi^a\xi^a\rangle + (\frac{D}{2} - 1)\frac{2g^2R}{3}\frac{H^{2D-2}}{(4\pi)^D}\frac{\Gamma^2(D-1)}{\Gamma^2(\frac{D}{2})}(D-1)\delta$$

$$+\frac{g^2RH^6}{2^5\pi^4}\log a(\tau) - \frac{g^2RH^6}{2^6\cdot 3\pi^4}.$$

In the third line of (7.63), we have used (7.15). To evaluate the sub-leading IR effects, we have to calculate the two point function up to $g^2\log a(\tau)$. In Appendix D.3, it is evaluated as:

$$\langle\xi^a\xi^a\rangle|_{g^2} \simeq \frac{g^2RH^4}{2^5\cdot 3\pi^4}\{-\log^2 a(\tau) + 6(-2 + \log 2 + \gamma)\log a(\tau)\} \qquad (7.64)$$

$$+\frac{2g^2R}{3}\frac{H^{2D-4}}{(4\pi)^D}\frac{\Gamma^2(D-1)}{\Gamma^2(\frac{D}{2})}\delta\log a(\tau).$$

From (7.63) and (7.64), the trace of the energy-momentum tensor up to the two loop level is

$$\langle T_\mu{}^\mu \rangle = N\frac{3H^4}{8\pi^2} + (\frac{D}{2} - 1)\frac{g^2RH^{2D-2}}{(4\pi)^D}\frac{\Gamma^2(D-1)}{\Gamma^2(\frac{D}{2})}(D-1)\delta \qquad (7.65)$$

$$-\frac{g^2RH^6}{2^6\pi^4}(13 - 6\log 2 - 6\gamma).$$

At the two loop level, we have confirmed that the matter contribution to the cosmological constant is time independent. To obtain the time dependence of the effective cosmological constant, we have to investigate the sub-leading IR effects beyond the two loop level. In Subsection 7.5, we investigate the sub-leading IR effects at the three loop level on an arbitrary target space. Before investigating it, we consider the non-linear sigma model on an S_N in the large N limit in the next section.

7.4 Non-linear sigma model on S_N in the large N limit

In the case that the target space is an S_N, by introducing the auxiliary field χ, the action of the non-linear sigma model is written as

$$S_{\text{matter}} = \int \sqrt{-g}d^4x \left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi^i\partial_\nu\varphi^i - \frac{\chi}{2}((\varphi^i)^2 - \frac{1}{g^2})\right], \qquad (7.66)$$

49

where $i = 1 \cdots N + 1$. The field χ imposes the following constraint

$$(\varphi^i)^2 = \frac{1}{g^2}. \tag{7.67}$$

In the large N limit, we can neglect the fluctuation of χ. So the action reduces to a free massive scalar field theory plus the constant term χ/g^2. Here the auxiliary field is identified as the mass term: $\chi = m^2$.

In order to satisfy the constraint (7.67), we have to introduce the classical expectation value $(\varphi^i_{cl}(x))^2$ in addition to the quantum one $\langle (\tilde{\varphi}^i(x))^2 \rangle$:

$$(\varphi^i)^2 = (\varphi^i_{cl}(x))^2 + \langle (\tilde{\varphi}^i(x))^2 \rangle = \frac{1}{g^2}. \tag{7.68}$$

It is because $1/g^2$ is a constant and even if a scalar field is massive, its propagator is time dependent until $t \sim 3H/2m^2$ [8, 9, 10]. From (4.10) and (4.12), the propagator for a massive field at the coincident point is written as

$$\langle (\tilde{\varphi}^i(x))^2 \rangle = (N+1) \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(1 - \frac{D}{2})\Gamma(\frac{D-1}{2} + \nu)\Gamma(\frac{D-1}{2} - \nu)}{\Gamma(\frac{1}{2} + \nu)\Gamma(\frac{1}{2} - \nu)} \tag{7.69}$$
$$+ (N+1) \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(\nu)\Gamma(2\nu)}{\Gamma(\frac{D-1}{2})\Gamma(\frac{1}{2} + \nu)} \frac{(a(\tau))^{2\nu - (D-1)}}{\nu - \frac{D-1}{2}},$$

where we have adopted the assumption: $m^2/H^2 \ll 1$.

The classical expectation value $(\varphi^i_{cl}(x))^2$ is identified with the effective coupling constant:

$$(\varphi^i_{cl}(x))^2 \equiv \frac{1}{g^2_{\text{eff}}}. \tag{7.70}$$

From (7.68) and (7.69), the effective coupling constant up to the one loop level is

$$\frac{1}{g^2_{\text{eff}}} = \frac{1}{g^2} - (N+1) \frac{H^2}{4\pi^2} \log a(\tau). \tag{7.71}$$

Here we have renormalized the UV divergence in (7.69) up to the one loop level by the coupling constant renormalization:

$$-\frac{\delta g^2}{g^4} = (N+1) \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \delta. \tag{7.72}$$

The effective coupling constant increases with time. It agrees with the one loop result (7.9) up to $\mathcal{O}(N)$.

From (5.4), the $g_{\mu\nu}$ term is always dominant in the energy-momentum tensor irrespective of whether the dS invariance is respected or broken

$$\langle T_{\mu\nu} \rangle \simeq \frac{g_{\mu\nu}}{D} \langle T_\rho{}^\rho \rangle. \tag{7.73}$$

The trace of the energy-momentum tensor is

$$\langle T_\mu{}^\mu \rangle = \langle -(\frac{D}{2} - 1)g^{\mu\nu}\partial_\mu\varphi^i\partial_\nu\varphi^i - \frac{D}{2}m^2\big((\varphi^i)^2 - (\frac{1}{g^2} - \frac{\delta g^2}{g^4})\big)\rangle \tag{7.74}$$

$$= (\frac{D}{2} - 1)\langle -\frac{1}{2}\nabla^2(\varphi^i)^2 + m^2(\varphi^i)^2\rangle$$

$$= (\frac{D}{2} - 1)m^2(\frac{1}{g^2} - \frac{\delta g^2}{g^4}).$$

Here we have used the constraint (7.67) and the equation of motion

$$\nabla^2\varphi^i - m^2\varphi^i = 0. \tag{7.75}$$

First, we confirm the result (7.65) in the leading order of N. To do so, we expand (7.69) up to $\mathcal{O}(m^2/H^2)$

$$\langle(\tilde{\varphi}^i(x))^2\rangle = (N+1)\frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}(2\log a(\tau) + \delta) \tag{7.76}$$

$$+ (N+1)\frac{m^2}{H^2}\Big[-\frac{H^2}{12\pi^2}\big\{ \log^2 a(\tau) + 2(2 - \log 2 - \gamma)\log a(\tau)\big\} + X\Big].$$

Here X denotes the UV divergent constant at $\mathcal{O}(m^2/H^2)$. To evaluate the two loop effect, we don't need to know its value. To renormalize the UV divergence up to the two loop level, we choose the counter term as

$$-\frac{\delta g^2}{g^4} = (N+1)\frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\delta + (N+1)\frac{m^2}{H^2}X, \tag{7.77}$$

$$(\varphi_{cl}^i(x))^2 = \frac{1}{g^2} - (N+1)\frac{2H^{D-2}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\log a(\tau) \tag{7.78}$$

$$+ (N+1)\frac{m^2}{2^2 \cdot 3\pi^2}\big\{ \log^2 a(\tau) + 2(2 - \log 2 - \gamma)\log a(\tau)\big\}.$$

By substituting (7.78) in the equation of motion

$$\nabla^2\varphi_{cl}^i - m^2\varphi_{cl}^i = 0, \tag{7.79}$$

we evaluate the mass term

$$m^2 = (N+1)g^2\frac{H^D}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D)}{\Gamma(\frac{D}{2})} - \frac{(N+1)^2g^4H^6}{2^6\pi^4}(13 - 6\log 2 - 6\gamma). \tag{7.80}$$

The value at $\mathcal{O}(g^2)$ is consistent with the result in [29]. Note that the assumption $m^2/H^2 \ll 1$ is consistent if $Ng^2H^2 \ll 1$. From (7.74), (7.77) and (7.80),

$$\langle T_\mu{}^\mu \rangle = (N+1)\frac{3H^4}{8\pi^2} + g^2(N+1)^2(\frac{D}{2} - 1)\frac{H^{2D-2}}{(4\pi)^D}\frac{\Gamma^2(D-1)}{\Gamma^2(\frac{D}{2})}(D-1)\delta \tag{7.81}$$

$$- \frac{g^2(N+1)^2H^6}{2^6\pi^4}(13 - 6\log 2 - 6\gamma).$$

51

As we recall $R = N(N - 1)$ on an S_N, the result coincides with (7.65) in the leading order of N.

Our interest is whether the effective cosmological constant becomes time dependent if we consider the higher loop effects. From (7.74) we find that the effective cosmological constant is time independent as long as the effective mass is time independent. If the effective mass becomes time dependent, the energy-momentum tensor has the UV divergent term whose coefficient is time dependent. The counter terms are highly restricted in the non-linear sigma model on an S_N in the large N limit. Since $\varphi^i \varphi^i$ is constrained to be a constant, possible scalar field dependent counter terms must contain $g^{\mu\nu} \partial_\mu \varphi^i \partial_\nu \varphi^i$. In the large N limit they must be bilinear in φ^i with the indices i contracted. Time dependent UV-divergences cannot be renormalized by the cosmological constant or possible other counter terms such as $R_g g^{\mu\nu} \partial_\mu \varphi^i \partial_\nu \varphi^i$ where R_g is the scalar curvature of dS space. The significance of this kind of counter term will be explained in the next section. On the other hand, we expect the renormalizability to hold if we allow all possible counter terms. Therefore we argue that the effective cosmological constant is time independent on an S_N in the large N limit even if we consider the full IR effects.

7.5   IR effects at the three loop level

Following the result in the previous section, it is natural to ask whether the effective cosmological constant has time dependence on a generic target space. As we have shown the cancellation of the leading IR logarithms to all orders, there is no $\log^2 a(\tau)$ type term at the three loop level. However there could still exist a sub-leading $\log a(\tau)$ type term in a generic non-linear sigma model. In this section, we investigate such IR effects on a generic target space.
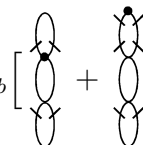
From (7.5), the vev of the energy-momentum tensor up to the three loop level is

$$\langle T_{\mu\nu} \rangle = (\delta_\mu{}^\rho \delta_\nu{}^\sigma - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma}) \times \tag{7.82}$$

$$\langle \partial_\rho \xi^a \partial_\sigma \xi^a - \frac{g^2}{3} R_{cadb} \xi^c \xi^d \partial_\rho \xi^a \partial_\sigma \xi^b$$

$$+ (-\frac{g^4}{20} D_e D_f R_{cadb} + \frac{2g^4}{45} R^g{}_{cad} R_{gebf}) \xi^c \xi^d \xi^e \xi^f \partial_\rho \xi^a \partial_\sigma \xi^b \rangle.$$

The contribution at the three loop level consists of the three kinds of diagrams

$$\langle T_{\mu\nu} \rangle = (\delta_\mu{}^\rho \delta_\nu{}^\sigma - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma}) \times \Big[\text{(The chain diagrams)} + \text{(The circle diagrams)} \tag{7.83}$$

$$+ \text{(The clover diagrams)} \Big].$$

These diagrams are represented as

$$\text{(The chain diagrams)} = -i \frac{g^4}{9} R^{ab} R_{ab} \Big[\, \vcenter{\hbox{}} + \vcenter{\hbox{}} + \cdots \Big], \tag{7.84}$$

(The circle diagrams) $= -i\dfrac{g^4}{6} R^{cadb} R_{cadb} \Big[\;\text{(diagram)} + \text{(diagram)} + \cdots\Big],$

(The clover diagrams) $= \Big(\dfrac{2g^4}{45} R^{ab} R_{ab} + \dfrac{g^4}{15} R^{cadb} R_{cadb} - \dfrac{g^4}{10} D^2 R\Big)\Big[\;\text{(diagram)} + \text{(diagram)} + \cdots\Big].$

Unlike in Subsection 7.2, we explicitly factor out the coefficients which are combinations of $R^{ab}R_{ab}$, $R^{cadb}R_{cadb}$, D^2R.

First, we reconfirm the cancellation of the leading IR effects of $\mathcal{O}(\log^2 a(\tau))$. By using the partial integration, we find

$$\text{(diagram)} + \text{(diagram)} = -2\,\text{(diagram)} - 2\,\text{(diagram)} = \mathcal{O}(\log a(\tau)), \tag{7.85}$$

$$\text{(diagram)} + \text{(diagram)} = -\text{(diagram)} - \text{(diagram)} - \text{(diagram)} - \text{(diagram)} = \mathcal{O}(\log a(\tau)),$$

$$\text{(diagram)} + \text{(diagram)} = -2\,\text{(diagram)} - 2\,\text{(diagram)} = \mathcal{O}(\log a(\tau)),$$

$$\text{(diagram)} + \text{(diagram)} = -2\,\text{(diagram)} - 2\,\text{(diagram)} = \mathcal{O}(\log a(\tau)),$$

$$\text{(diagram)} + \text{(diagram)} = -2\,\text{(diagram)} - 2\,\text{(diagram)} = \mathcal{O}(\log a(\tau)), \tag{7.86}$$

$$\text{(diagram)} + \text{(diagram)} = -4\,\text{(diagram)} - 4\,\text{(diagram)} = \mathcal{O}(\log a(\tau)). \tag{7.87}$$

From (7.85), (7.86) and (7.87), we can show that the total of the diagrams in (7.83) doesn't have the leading IR effect. Note that the leading IR effects cancel pairwise between a "propagator" term and a "vertex" term in accord with our proof in Subsection 7.2.

Next, we investigate the sub-leading IR effect. In Subsection 7.4, we have shown that the vev of the energy-momentum tensor has no time dependence on an S_N in the large N limit

53

where

$$R^{ab}R_{ab} = N(N-1)^2 = \mathcal{O}(N^3), \quad R^{cadb}R_{cadb} = 2N(N-1) = \mathcal{O}(N^2), \quad D^2R = 0. \quad (7.88)$$

Therefore, the result in the large N limit implies the cancellation of the time dependence between the following diagrams

$$-i\frac{g^4}{9}R^{ab}R_{ab}\Big[\; + \; + \cdots\Big] + \frac{2g^4}{45}R^{ab}R_{ab}\Big[\; + \; + \cdots\Big] = \text{const.} \qquad (7.89)$$

In order to investigate the sub-leading IR effect, we only need to consider the remaining diagrams. By using (7.86) and (7.87), the remaining diagrams are written as follows

$$-i\frac{g^4}{6}R^{cadb}R_{cadb}\Big[-4\; + \; + \; - 3 \qquad\qquad (7.90)$$

$$-\; - 2\; + 2\;\Big]$$

$$+\Big(\frac{g^4}{15}R^{cadb}R_{cadb} - \frac{g^4}{10}D^2R\Big)\Big[-5\; + 2\; - \; - 6\;\Big].$$

By using the partial integration, we find

$$\text{\Large X} = -\frac{1}{2}\text{\Large X} - \text{\Large X}. \qquad (7.91)$$

From this identity, the clover diagrams of (7.90) are written as follows

$$\Big(\frac{g^4}{3}R^{cadb}R_{cadb} - \frac{g^4}{2}D^2R\Big)\Big[\; - \; + \;\Big]. \qquad (7.92)$$

The third diagram in the right hand side does not induce an IR logarithm:

$$\text{\Large X} = \text{const.} \qquad (7.93)$$

We can confirm its time independence without an detailed calculation as explained in Appendix C. Thus the clover diagrams are estimated as

$$\Big(\frac{g^4}{3}R^{cadb}R_{cadb} - \frac{g^4}{2}D^2R\Big)\Big[\; - \;\Big]. \qquad (7.94)$$

In a similar way, we investigate the circle diagrams of (7.90). By using the partial integration, we find

$$\text{(diagram)} = - \text{(diagram)} - \text{(diagram)} - i\,\text{(diagram)}. \tag{7.95}$$

From this identity, the circle diagrams are evaluated as

$$-i\frac{g^4}{6}R^{cadb}R_{cadb}\Big[-4\,\text{(diagram)} + \text{(diagram)} + \text{(diagram)} + 3i\,\text{(diagram)}$$

$$-\,\text{(diagram)} + \text{(diagram)} + 5\,\text{(diagram)}\Big]. \tag{7.96}$$

In addition, we find the following identities by using the partial integration

$$\text{(diagram)} = -\frac{1}{2}\,\text{(diagram)} - i\frac{1}{2}\,\text{(diagram)}, \tag{7.97}$$

$$\text{(diagram)} = -\text{(diagram)} - \text{(diagram)} - i\,\text{(diagram)} - i\,\text{(diagram)} - i\,\text{(diagram)}. \tag{7.98}$$

From the above relations and (7.91), (7.96) is

$$-i\frac{g^4}{6}R^{cadb}R_{cadb}\Big[2i\,\text{(diagram)} - 2i\,\text{(diagram)} - 3i\,\text{(diagram)}$$

$$+\,3\,\text{(diagram)} - 2\,\text{(diagram)} + 5\,\text{(diagram)}\Big]. \tag{7.99}$$

By using the power counting in Appendix C like in (7.93), we can confirm the time independence of the following diagrams

$$\text{(diagram)} = \text{(diagram)} = \text{(diagram)} = \text{const.} \tag{7.100}$$

So the circle diagrams are estimated as

$$-\frac{g^4}{3}R^{cadb}R_{cadb}\Big[\text{(diagram)} - \text{(diagram)}\Big]. \tag{7.101}$$

From (7.90), (7.94) and (7.101), we conclude that the vev of the energy-momentum tensor at the three loop level is

$$\langle T_{\mu\nu}\rangle|_{g^4} \simeq (\delta_\mu{}^\rho \delta_\nu{}^\sigma - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}) \times -\frac{g^4}{2}D^2 R\Big[\vphantom{X} \quad - \quad \Big]. \qquad (7.102)$$

Here $\simeq$ denotes the equality with respect to the time dependent terms. The sub-leading IR effects which are proportional to $R^{cadb}R_{cadb}$ cancel out each other. Unlike the leading IR effects, this cancellation takes place between the different kinds of diagrams, between the clover diagrams and the circle diagrams. On the other hand, only the clover diagrams have the coefficient $D^2 R$. That is why the sub-leading IR logarithm is proportional to $D^2 R$. Note that $D^2 R$ vanishes on symmetric spaces such as an S_N. Therefore the time independence of the cosmological constant on an S_N also holds with finite N at the three loop level. Furthermore, we point out that the identity (7.89) can be confirmed also by using the above diagramatic investigation.

From (6.10), (6.17) and (7.18), the contribution from the second diagram in (7.102) is evaluated as

$$- \vphantom{X} = -\frac{1}{4}G^{++}(x,x)\partial_\rho G^{++}(x,x)\partial_\sigma G^{++}(x,x) \simeq -a^2(\tau)\delta_\rho{}^0\delta_\sigma{}^0\frac{H^8}{2^8\pi^6}\log a(\tau), \qquad (7.103)$$

and the contribution from the first diagram is written as

$$\vphantom{X} = -i\int \sqrt{-g'}d^D x'\ G^{++}(x',x')g^{\alpha\beta}(\tau')\lim_{x''\to x'}\partial'_\alpha\partial''_\beta G^{++}(x',x'') \qquad (7.104)$$

$$\times \left[\partial_\rho G^{++}(x,x')\partial_\sigma G^{++}(x,x') - \partial_\rho G^{+-}(x,x')\partial_\sigma G^{+-}(x,x')\right]$$

$$\simeq i\frac{2^{2D-3}H^{4D-4}}{(4\pi)^{2D}}(D-1)\Gamma^2(D-1)a^2(\tau)\int d^{4-\varepsilon}x'\ a^{4-\varepsilon}(\tau')\log a(\tau')\sum_{m=1}^{6}H^m_{\rho\sigma}.$$

Since $g^{\alpha\beta}(\tau')\lim_{x''\to x'}\partial'_\alpha\partial''_\beta G^{++}(x',x'') = $ const, the contribution from the second term is equal to (6.16) up to an overall coefficient. From (6.31),

$$\vphantom{X} \simeq g_{\rho\sigma}\frac{2^{2D-3}\pi^2 H^{4D-8}}{(4\pi)^{2D}}(D-1)\Gamma^2(D-1) \qquad (7.105)$$

$$\times \left\{\left(\frac{\pi^{-\frac{\varepsilon}{2}}\mu^{-\varepsilon}H^{2\varepsilon}}{\Gamma(1-\frac{\varepsilon}{2})\varepsilon} + \log\frac{2\mu}{H}\right)\log a(\tau) - \frac{2}{3}\log a(\tau)\right\} + a^2(\tau)\delta_\rho{}^0\delta_\sigma{}^0\frac{H^8}{2^8\pi^6}\log a(\tau).$$

From (7.103) and (7.105),

$$\vphantom{X} - \vphantom{X} \simeq g_{\rho\sigma}\frac{2^{2D-3}\pi^2 H^{4D-8}}{(4\pi)^{2D}}(D-1)\Gamma^2(D-1) \qquad (7.106)$$

$$\times \left\{\left(\frac{\pi^{-\frac{\varepsilon}{2}}\mu^{-\varepsilon}H^{2\varepsilon}}{\Gamma(1-\frac{\varepsilon}{2})\varepsilon} + \log\frac{2\mu}{H}\right)\log a(\tau) - \frac{2}{3}\log a(\tau)\right\}.$$

As a result, the contribution from the two diagrams is

$$\langle T_{\mu\nu}\rangle|_{g^4} \simeq (\delta_\mu{}^\rho \delta_\nu{}^\sigma - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}) \times -\frac{g^4}{2}D^2 R\left[\;\text{}\; - \;\text{}\;\right] \tag{7.107}$$

$$\simeq g_{\mu\nu}g^4 D^2 R \frac{2^{2D-4}\pi^2 H^{4D-8}}{(4\pi)^{2D}}(D-1)\Gamma^2(D-1)$$

$$\times \left\{ \left(\frac{\pi^{-\frac{\varepsilon}{2}}\mu^{-\varepsilon}H^{2\varepsilon}}{\Gamma(1-\frac{\varepsilon}{2})\varepsilon} + \log\frac{2\mu}{H}\right)\log a(\tau) - \frac{7}{6}\log a(\tau)\right\}$$

$$= g_{\mu\nu}g^4 D^2 R \frac{(D-1)(D-2)}{2}\frac{H^{3D-4}}{(4\pi)^{\frac{3D}{2}}}\frac{\Gamma^2(D-1)}{\Gamma(\frac{D}{2})}\left\{\frac{1}{\varepsilon}\log a(\tau) - \frac{7}{6}\log a(\tau)\right\}.$$

Note that the coefficient of $\log a(\tau)$ is UV divergent and it is not renormalizable by the existing counter terms (7.7), (7.14). The time dependent diagrams arising from (7.7) and (7.14) are

$$i\frac{g^2}{3}(\delta\beta + 2\delta\gamma)R^{ab}R_{ab}\left[2\;\text{}\; + \;\text{}\; + \;\text{}\; + \;\text{}\; + \;\text{}\;\right] \tag{7.108}$$

$$+ g^2\delta\beta(\frac{1}{2}D^2 R - \frac{1}{3}R^{ab}R_{ab})\left[\;\text{}\; + \;\text{}\;\right],$$

where a small dot denotes the counter term insertion. By using the partial integration, we find

$$\text{} \simeq -\text{},\quad \text{} \simeq \text{} \simeq -\text{},\quad \text{} \simeq -\text{}. \tag{7.109}$$

From these identities, the total contribution from (7.7) and (7.14) is time independent

$$\delta_{\beta,\gamma}\langle T_{\mu\nu}\rangle|_{g^4} \simeq 0. \tag{7.110}$$

It is why (7.107) is not renormalizable by (7.7) and (7.14).

This time dependent UV divergence can be renormalized by introducing the following counter term

$$\delta_\alpha \mathcal{L} = \frac{\delta\alpha}{g^2}(R_g - D(D-1)H^2)R_{ij}(\varphi)g^{\mu\nu}\partial_\mu\varphi^i\partial_\nu\varphi^j, \tag{7.111}$$

where R_g denotes the Ricci scalar of space-time. As seen in (6.34), the necessity of this kind of counter term in $\lambda\varphi^4$ theory has been pointed out in [13]. The only effect of the counter term is to modify the energy-momentum tensor as:

$$\delta_\alpha \langle T_{\mu\nu}\rangle = -2\delta\alpha\{g_{\mu\nu}((D-1)H^2 K + \nabla^2 K) - \nabla_\mu\nabla_\nu K\}, \tag{7.112}$$

$$K = \langle R_{ab}g^{\mu\nu}\partial_\mu\xi^a\partial_\nu\xi^b + (\frac{g^2}{2}D_cD_dR_{ab} - \frac{g^2}{3}R^e{}_{cad}R_{eb})\xi^c\xi^d g^{\mu\nu}\partial_\mu\xi^a\partial_\nu\xi^b\rangle. \qquad (7.113)$$

In a similar way to the leading IR effect at the two loop level, we find that the following part of (7.113) has no time dependence

$$R_{ab}\langle g^{\mu\nu}\partial_\mu\xi^a\partial_\nu\xi^b\rangle|_{g^2} - \frac{g^2}{3}R^e{}_{cad}R_{eb}\langle\xi^c\xi^d g^{\mu\nu}\partial_\mu\xi^a\partial_\nu\xi^b\rangle|_{g^0} \simeq 0. \qquad (7.114)$$

We fix $\delta\alpha$ to renormalize the two loop matter contribution to the cosmological constant in (7.65):

$$-2\delta\alpha(D-1)H^2 R_{ab}\langle g^{\mu\nu}\partial_\mu\xi^a\partial_\nu\xi^b\rangle|_{g^0} = -(\frac{D}{2}-1)\frac{g^2 RH^{2D-2}}{(4\pi)^D}\frac{\Gamma^2(D-1)}{\Gamma^2(\frac{D}{2})}\frac{D-1}{D}\delta \qquad (7.115)$$
$$+ \frac{g^2 RH^6}{2^8\pi^4}(13 - 6\log 2 - 6\gamma) + \frac{g^2 RH^6}{2^8\pi^4}C,$$

where we have used $\nabla_\mu\langle g^{\rho\sigma}\partial_\rho\xi^a\partial_\sigma\xi^b\rangle|_{g^0} = 0$. Note that there is a finite ambiguity C when we renormalize the UV divergence. In particular the two loop effect is completely canceled by the counter term up to $\mathcal{O}(\varepsilon^0)$ by setting $C = 0$. The result is

$$\delta\alpha = -\frac{D-2}{4D(D-1)}\frac{g^2 H^{D-4}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\delta + \frac{g^2}{2^6\cdot 3^2\pi^2}(13 - 6\log 2 - 6\gamma) + \frac{g^2}{2^6\cdot 3^2\pi^2}C. \quad (7.116)$$

At the three loop level, this counter term gives rise to the the following time dependent term

$$-2\delta\alpha g_{\mu\nu}(D-1)H^2 \times \frac{g^2}{2}D_cD_dR_{ab}\langle\xi^c\xi^d g^{\rho\sigma}\partial_\rho\xi^a\partial_\sigma\xi^b\rangle|_{g^0} \qquad (7.117)$$
$$\simeq -g_{\mu\nu}g^4 D^2 R\frac{(D-2)(D-1)}{2D}\frac{H^{3D-4}}{(4\pi)^{\frac{3D}{2}}}\frac{\Gamma^3(D-1)}{\Gamma^3(\frac{D}{2})}\delta\log a(\tau)$$
$$+ g_{\mu\nu}g^4 D^2 R\frac{H^8}{2^{11}\pi^6}(13 - 6\log 2 - 6\gamma)\log a(\tau) + g_{\mu\nu}g^4 D^2 R\frac{CH^8}{2^{11}\pi^6}\log a(\tau),$$

where we have used the fact that $\nabla_\mu\langle\xi^c\xi^d g^{\rho\sigma}\partial_\rho\xi^a\partial_\sigma\xi^b\rangle|_{g^0}$ is constant. From (7.107) and (7.117), we find

$$\langle T^{\text{total}}_{\mu\nu}\rangle|_{g^4} \simeq g_{\mu\nu}g^4 D^2 R\frac{CH^8}{2^{11}\pi^6}\log a(\tau). \qquad (7.118)$$

We have thus shown that the energy-momentum tensor can be renormalized up to the three loop level with the counter terms we have identified. The resultant time dependence of the cosmological constant is proportional to $D^2 R$. However it is also proportional to a finite subtraction ambiguity C. Therefore there exists a renormalization scheme with $C = 0$ in generic non-linear models which preserves the dS symmetry up to the three loop level.

8 IR effects of a higher derivative interaction

In the previous section, we have shown there exists a cancellation mechanism among IR logarithms beyond the power counting estimates in non-linear models on generic manifolds.

The leading cancellation occurs between the "propagator" and "vertex" terms as there are one to one correspondences between them. This feature is specific to the interaction terms with two derivatives. Therefore such a cancellation does not take place if we consider the higher derivative interaction terms. In this section we investigate a model with a higher derivative interaction term where the leading IR effects to the cosmological constant doesn't cancel out each other. We adopt the following model as a specific example:

$$S_{\text{matter}} = \int \sqrt{-g} d^4 x \left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi^i \partial_\nu \varphi^i - \frac{\lambda}{16N^2} (\varphi^i)^2 (g^{\mu\nu} \partial_\mu \varphi^j \partial_\nu \varphi^j)^2 \right], \qquad (8.1)$$

where $i = 1 \cdots N$. Note that we have also introduced the scalar field left intact by differential operators in the higher derivative interaction term. In addition, we impose $O(N)$ symmetry on the action because it becomes exactly solvable in the large N limit. The energy-momentum tensor is written as

$$\langle T_{\mu\nu} \rangle = (\delta_\mu{}^\rho \delta_\nu{}^\sigma - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma}) \langle \partial_\rho \varphi^i \partial_\sigma \varphi^i \rangle \qquad (8.2)$$

$$+ (\delta_\mu{}^\rho \delta_\nu{}^\sigma - \frac{1}{4} g_{\mu\nu} g^{\rho\sigma}) \langle \frac{\lambda}{4N^2} (\varphi^i)^2 \partial_\rho \varphi^j \partial_\sigma \varphi^j g^{\alpha\beta} \partial_\alpha \varphi^k \partial_\beta \varphi^k \rangle.$$

Note that the $g_{\mu\nu}$ dependences of the "propagator" term and the "vertex" term are different from those in the two derivative interaction models.

The quantum corrections arise at the three loop level. The leading IR effects from the "vertex" term and the "propagator" term are

$$\frac{\lambda}{4N^2} \langle (\varphi^i)^2 \partial_\rho \varphi^j \partial_\sigma \varphi^j g^{\alpha\beta} \partial_\alpha \varphi^k \partial_\beta \varphi^k \rangle |_{\lambda^0} \qquad (8.3)$$

$$\simeq N \frac{\lambda}{4} G^{++}(x,x) \lim_{x' \to x} \partial_\rho \partial'_\sigma G^{++}(x,x') g^{\alpha\beta} \partial_\alpha \partial'_\beta G^{++}(x,x')$$

$$+ \frac{\lambda}{2} G^{++}(x,x) \lim_{x' \to x} \partial_\rho \partial'_\beta G^{++}(x,x') g^{\alpha\beta} \partial_\alpha \partial'_\sigma G^{++}(x,x')$$

$$\simeq + g_{\rho\sigma} (N + \frac{1}{2}) \frac{3^2 \lambda H^{10}}{2^{12} \pi^6} \log a(\tau),$$

$$\langle \partial_\rho \varphi^i \partial_\sigma \varphi^i \rangle |_\lambda \qquad (8.4)$$

$$\simeq -iN \frac{\lambda}{4} \int \sqrt{-g'} d^D x' \, G^{++}(x',x') \lim_{x'' \to x'} \partial'_\alpha \partial''_\beta G^{++}(x',x'')$$

$$\times g^{\alpha\beta}(\tau') g^{\gamma\delta}(\tau') \left[\partial_\rho \partial'_\gamma G^{++}(x,x') \partial_\sigma \partial'_\delta G^{++}(x,x') - \partial_\rho \partial'_\gamma G^{+-}(x,x') \partial_\sigma \partial'_\delta G^{+-}(x,x') \right]$$

$$-i \frac{\lambda}{2} \int \sqrt{-g'} d^D x' \, G^{++}(x',x') \lim_{x'' \to x'} \partial'_\alpha \partial''_\delta G^{++}(x',x'')$$

$$\times g^{\alpha\beta}(\tau') g^{\gamma\delta}(\tau') \left[\partial_\rho \partial'_\gamma G^{++}(x,x') \partial_\sigma \partial'_\beta G^{++}(x,x') - \partial_\rho \partial'_\gamma G^{+-}(x,x') \partial_\sigma \partial'_\beta G^{+-}(x,x') \right].$$

By using the partial integration and extracting the leading IR effects, the "propagator" term

is

$$\langle \partial_\rho \varphi^i \partial_\sigma \varphi^i \rangle|_\lambda \tag{8.5}$$

$$\simeq + iN\frac{\lambda}{4} \int d^D x' \, G^{++}(x', x') \lim_{x'' \to x'} \partial'_\alpha \partial''_\beta G^{++}(x', x'')$$

$$\times g^{\alpha\beta}(\tau') \big[\partial_\rho G^{++}(x, x') \partial_\sigma \sqrt{-g'} \nabla'^2 G^{++}(x, x') - \partial_\rho G^{+-}(x, x') \partial_\sigma \sqrt{-g'} \nabla'^2 G^{+-}(x, x') \big]$$

$$+ i\frac{\lambda}{8} \int d^D x' \, G^{++}(x', x') \lim_{x'' \to x'} \partial'_\alpha \partial''_\beta G^{++}(x', x'')$$

$$\times g^{\alpha\beta}(\tau') \big[\partial_\rho G^{++}(x, x') \partial_\sigma \sqrt{-g'} \nabla'^2 G^{++}(x, x') - \partial_\rho G^{+-}(x, x') \partial_\sigma \sqrt{-g'} \nabla'^2 G^{+-}(x, x') \big].$$

Here we have used the fact : $\lim_{x'' \to x'} \partial'_\alpha \partial''_\beta G^{++}(x', x'') = g_{\alpha\beta}(\tau') \times \text{const}$, and

$$g^{\alpha\delta}(\tau') g_{\alpha\beta}(\tau') g^{\gamma\delta}(\tau') \tag{8.6}$$

$$\times \big[\partial_\rho \partial'_\gamma G^{++}(x, x') \partial_\sigma \partial'_\beta G^{++}(x, x') - \partial_\rho \partial'_\gamma G^{+-}(x, x') \partial_\sigma \partial'_\beta G^{+-}(x, x') \big]$$

$$= \frac{1}{D} g^{\alpha\beta}(\tau') g_{\alpha\beta}(\tau') g^{\gamma\delta}(\tau')$$

$$\times \big[\partial_\rho \partial'_\gamma G^{++}(x, x') \partial_\sigma \partial'_\delta G^{++}(x, x') - \partial_\rho \partial'_\gamma G^{+-}(x, x') \partial_\sigma \partial'_\delta G^{+-}(x, x') \big].$$

By using (7.43) and the partial integration,

$$\langle \partial_\rho \varphi^i \partial_\sigma \varphi^i \rangle|_\lambda \simeq - (N + \frac{1}{2}) \frac{\lambda}{4} G^{++}(x, x) \lim_{x' \to x} \partial_\rho \partial'_\sigma G^{++}(x, x') g^{\alpha\beta} \partial_\alpha \partial'_\beta G^{++}(x, x') \tag{8.7}$$

$$= - g_{\rho\sigma}(N + \frac{1}{2}) \frac{3^2 \lambda H^{10}}{2^{12} \pi^6} \log a(\tau).$$

By substituting (8.3) and (8.7) in (8.2),

$$\langle T_{\mu\nu} \rangle \simeq g_{\mu\nu} N \frac{3H^4}{32\pi^2} + g_{\mu\nu}(N + \frac{1}{2}) \frac{3^2 \lambda H^{10}}{2^{12} \pi^6} \log a(\tau) + a^2(\tau) \delta_\mu{}^0 \delta_\nu{}^0 (N + \frac{1}{2}) \frac{3 \lambda H^{10}}{2^{12} \pi^6}. \tag{8.8}$$

Here we have evaluated the coefficient of the $\delta_\mu{}^0 \delta_\nu{}^0$ term by the conservation law. Unlike the non-linear sigma model, the leading IR effect of the energy-momentum tensor is nonvanishing in this model. The effective cosmological constant decreases with cosmic evolution

$$\Lambda_{\text{eff}} \simeq \Lambda - \kappa N \frac{3H^4}{32\pi^2} - \kappa(N + \frac{1}{2}) \frac{3^2 \lambda H^{10}}{2^{12} \pi^6} \log a(\tau). \tag{8.9}$$

The perturbation theory breaks down when $\lambda H^6 \log a(\tau) \sim 1$. In such a situation we need to sum up all leading IR logarithms. We can evaluate such a nonperturbative IR effect in the large N limit. By using the auxiliary fields α, β, the action is written as

$$S_{\text{matter}} = \int \sqrt{-g} d^4 x \, \big[-\frac{1}{2}(1 + \alpha\beta) g^{\mu\nu} \partial_\mu \varphi^i \partial_\nu \varphi^i - \frac{1}{2} \beta^2 (\varphi^i)^2 + N \sqrt{\frac{2}{\lambda}} \alpha\beta^2 \big]. \tag{8.10}$$

By differentiating the action with respect to α, β,

$$\alpha = \frac{1}{N} \sqrt{\frac{\lambda}{2}} (\varphi^i)^2, \quad \beta = \frac{1}{2N} \sqrt{\frac{\lambda}{2}} g^{\mu\nu} \partial_\mu \varphi^i \partial_\nu \varphi^i. \tag{8.11}$$

60

In the large N limit, we can neglect the fluctuation of the auxiliary fields. So the action reduces to a free massive field theory plus the constant term $N\sqrt{2/\lambda}\alpha\beta^2$. We can evaluate the saturation value of the following vevs

$$\langle(\varphi^i)^2\rangle \simeq N\frac{3H^4}{8\pi^2\beta^2}, \tag{8.12}$$

$$\langle g^{\mu\nu}\partial_\mu\varphi^i\partial_\nu\varphi^i\rangle = \frac{1}{2}\nabla^2\langle(\varphi^i)^2\rangle - \langle\varphi^i\nabla^2\varphi^i\rangle$$

$$= -\frac{\beta^2}{1+\alpha\beta}\langle(\varphi^i)^2\rangle$$

$$\simeq \frac{-1}{1+\alpha\beta}N\frac{3H^4}{8\pi^2}.$$

Here we have adopted the assumption: $\beta^2/H^2 \ll 1$ and used the equation of motion

$$(1+\alpha\beta)\nabla^2\varphi^i - \beta^2\varphi^i = 0. \tag{8.13}$$

From (8.12), (8.11) is written as

$$\alpha \simeq \frac{1}{N}\sqrt{\frac{\lambda}{2}}\cdot N\frac{3H^4}{8\pi^2\beta^2}, \quad \beta = \frac{1}{2N}\sqrt{\frac{\lambda}{2}}\cdot\frac{-1}{1+\alpha\beta}N\frac{3H^4}{8\pi^2}. \tag{8.14}$$

By solving (8.14),

$$\alpha = \frac{4}{9}\sqrt{\frac{2}{\lambda}}\cdot\frac{8\pi^2}{3H^4}, \quad \beta = -\frac{3}{2}\sqrt{\frac{\lambda}{2}}\cdot\frac{3H^4}{8\pi^2}. \tag{8.15}$$

Furthermore, the trace of the energy-momentum tensor is written as

$$\langle T_\mu{}^\mu\rangle = \langle -(1+\alpha\beta)g^{\mu\nu}\partial_\mu\varphi^i\partial_\nu\varphi^i - 2\beta^2(\varphi^i)^2 + 4N\sqrt{\frac{2}{\lambda}}\alpha\beta^2\rangle \tag{8.16}$$

$$= \langle -(1+\alpha\beta)\frac{1}{2}\nabla^2(\varphi^i)^2 + (1+\alpha\beta)\varphi^i\nabla^2\varphi^i - 2\beta^2(\varphi^i)^2 + 4N\sqrt{\frac{2}{\lambda}}\alpha\beta^2\rangle$$

$$= \langle -\beta^2(\varphi^i)^2 + 4N\sqrt{\frac{2}{\lambda}}\alpha\beta^2\rangle.$$

In the third line, we have used the equation of motion (8.13). From (8.12), (8.15) and (8.16),

$$\langle T_\mu{}^\mu\rangle \simeq 3N\frac{3H^4}{8\pi^2}. \tag{8.17}$$

The vev of the energy-momentum tensor is

$$\langle T_{\mu\nu}\rangle \simeq g_{\mu\nu}N\frac{3H^4}{32\pi^2} + g_{\mu\nu}N\frac{3H^4}{16\pi^2}. \tag{8.18}$$

Note that the difference from the free field value is not suppressed by the coupling constant. It is the result of the resummation of the leading IR logarithms to all orders. The effective cosmological constant decreases with cosmic evolution at the initial stage, while it is eventually saturated at the value

$$\Lambda_{\text{eff}} = \Lambda - \kappa N\frac{3H^4}{32\pi^2} - \kappa N\frac{3H^4}{16\pi^2}. \tag{8.19}$$

61

9 Conclusion

In this thesis, I have summarized the quantum IR effects which are specific to dS space. In performing it, I have divided the momentum scale into the two regions, that is inside the cosmological horizon and outside the cosmological horizon.

In Part II, well inside the cosmological horizon, we have derived a Boltzmann equation in dS space from the Schwinger-Dyson equation. Here in order to investigate the particle creation effects, we have considered the collision term up to the order that the energy non-conservation processes emerge in.

From this Boltzmann equation, we have found that the total integral of the spectral weight remains to be unity as the particle creation effects are accompanied by the reduction of the on-shell states. In this sense, unitarity is respected by the interaction. At finite temperature, while the leading IR effects are canceled between the real and virtual processes, the remaining IR contribution leads to the modification of the particle distribution function. This effect doesn't emerge at zero temperature and decreases as the temperature is cooling down. We have confirmed these features both in φ^3 and φ^4 theories and expect that they are the universal features of the interacting field theories in dS space.

Although the above effects seem to be time dependent, their time dependences disappear after expressed by the physical scales. It is relevant that the degrees of freedom inside the cosmological horizon are time independent. We thus conclude that the local physics inside the cosmological horizon preserves the dS symmetry. In order for the physical quantities to obtain time dependences, the dS symmetry needs to be broken.

In Part III, we have investigated the contribution from outside the cosmological horizon. Unlike inside the cosmological horizon, the degrees of freedom outside the cosmological horizon increase with cosmic evolution. In addition, the propagator for a massless and minimally coupled field doesn't have the dS symmetry due to an IR divergence. So the existence of a massless and minimally coupled field indicates that the dS symmetry might be broken due to the increase. In some field theoretic models with this light field, the physical quantities acquire time dependences and their growing time dependences at each loop level eventually break the validity of perturbation theory.

We have reviewed how the dS symmetry breaking contributes to the physical quantities in the models with interaction potentials. Here the IR effect from the potential term is dominant compared with that of the kinetic term. In the perturbative investigation, the IR effect from the potential term makes the effective cosmological constant time dependent, while the IR effect in the kinetic term makes the energy-momentum tensor consistent with the covariant conservation law. Furthermore we can evaluate the saturation value of the contribution to the cosmological constant nonperturbatively by extracting the leading IR logarithm at each loop level. The saturation value is not suppressed by the coupling constant. We can rederive the same value in an Euclidean field theory on S_4.

In a general model with derivative interactions, we still don't know how to evaluate the nonperturbative IR effects. Ultimately, it is desirable that the quantum IR effects from gravity can be investigated by using such a tool. It is because the gravitational field contains

massless and minimally coupled modes. As a simple model with derivative interactions, we have investigated the non-linear sigma model.

In the perturbative investigation, the effective coupling constant of the non-linear sigma model is time dependent in agreement with power counting of the IR logarithms. Unlike in the models with interaction potentials, the contribution from the "propagator" term to the cosmological constant is of the same order with that from the "vertex" term in the models with derivative interactions. Especially in the non-linear sigma model, the leading IR effects to the cosmological constant cancel out each other between the "propagator" term and the "vertex" term. The cancellation of the leading IR effects takes place to all orders on an arbitrary target space. Furthermore the investigation in the large N limit on an N-sphere indicates that the effective cosmological constant is time independent even if we consider the full IR effects.

The above two nonperturbative considerations don't constrain the sub-leading IR effect on an arbitrary target space. We have investigated IR effects up to the three loop level where the sub-leading IR effect could induce time dependence. We have found that the sub-leading IR effect to the cosmological constant remains if $D^2 R \neq 0$ but its coefficient is UV divergent. We have identified a counter term which can cancel such a divergence. Furthermore a natural counter term can cancel the IR logarithm completely. Therefore there is a renormalization scheme in a generic non-linear sigma model which preserves dS symmetry up to the three loop level.

We may reflect these results as follows. If an equilibrium state is eventually established also in the non-linear sigma model, the correspondence between the stochastic approach and the Euclidean approach may work. Considering this conjecture, we may retain the zero mode in $G_{ij}(\varphi)$ and the nonzero modes in $g^{\mu\nu}\partial_\mu\varphi^i\partial_\nu\varphi^j$ to obtain the leading IR effects. In this approximation, the action is equal to the free field action because $G_{ij}(\varphi)$ has no coordinate dependence and can be put to identity by rescaling the nonzero modes. This argument may explain why the leading IR effects to the cosmological constant cancel out each other. Furthermore, the action on an S_N does not contain fields left intact by differential operators due to the constraint $(\varphi^i)^2 = 1/g^2$. So the effective cosmological constant is time independent because there is no contribution from the zero mode.

It should be noted that the above cancellations hold in the non-linear sigma model with two derivative interactions. In a general model with higher derivative interactions, the IR effects to the cosmological constant do not necessary cancel out each other. In fact, we have found that the cancellation of the leading IR effects does not take place in a field theory with higher derivative interactions. They could eventually sum up to the quantity as large as the one loop effect just like in the large N limit.

To understand the eventual IR effects in the physical quantities, we have to evaluate the IR effects nonperturbatively. The large N limit is available for some cases as is demonstrated in this thesis. However we still don't know how to evaluate the nonperturbative IR effects in a general model with derivative interactions. Our results may be relevant to investigate possible dS symmetry breaking due to IR effects in quantum gravity. It is because the gravitational field contain massless and minimally coupled modes [11]. When we consider the IR effects of gravity, an important question is to ask whether the IR effects emerge in the physical quantities or not [30, 31, 32, 33, 34, 35, 36]. Additionally, considering the

association with the Euclidean quantum gravity, the existence of an equilibrium state is also questionable.

As another project, it is an interesting question how the quantum IR effects emerge in the observables on the cosmic micro wave background. Also in approximate dS spaces like a slow-roll model, there may be strong quantum IR effects. That is, the quantum loop corrections may grow up to order one compared with the tree level if the e-folding time is long enough. So in each model of inflation, it is important to evaluate the quantum IR effects to the scalar spectral index, the tensor to scalar ratio and the non-gaussianity. Of course in these evaluations, the above test of the gauge invariance and development of the nonperturbative approach are necessary [32, 35, 36].

A   Collision term evaluation

In this appendix we explain the details of our calculation for the collision term.

In the first step, using our integration formula (3.12), the on-shell collision term (3.14) is evaluated as

$$C_{\mathrm{on}}[f] = + (1 + f(p))e^{-ip\bar{\tau}} \times \frac{1}{H^2} \frac{(-ig)^2}{2} \frac{1}{32\pi^2 p^2} \int_0^\infty dp_1 \int_{|p_1 - p|}^{p_1 + p} dp_2 \qquad (\mathrm{A.1})$$

$$\times \left[+ \left\{ \frac{1}{i(p_1 + p_2 - p)} \frac{-2\bar{\tau}}{\tau_c^3} + \frac{-1}{(p_1 + p_2 - p)^2} \frac{2}{\tau_c^3} \right\} \right.$$
$$\times \left\{ (1 + f(p_1))(1 + f(p_2)) - f(p_1)f(p_2) \right\}$$
$$+ \left\{ \frac{1}{i(p_1 - p_2 - p)} \frac{-2\bar{\tau}}{\tau_c^3} + \frac{-1}{(p_1 - p_2 - p)^2} \frac{2}{\tau_c^3} \right\}$$
$$\times \left\{ (1 + f(p_1))f(p_2) - f(p_1)(1 + f(p_2)) \right\}$$
$$+ \left\{ \frac{1}{i(-p_1 + p_2 - p)} \frac{-2\bar{\tau}}{\tau_c^3} + \frac{-1}{(-p_1 + p_2 - p)^2} \frac{2}{\tau_c^3} \right\}$$
$$\times \left\{ f(p_1)(1 + f(p_2)) - (1 + f(p_1))f(p_2) \right\}$$
$$+ \left\{ \frac{1}{i(-p_1 - p_2 - p)} \frac{-2\bar{\tau}}{\tau_c^3} + \frac{-1}{(-p_1 - p_2 - p)^2} \frac{2}{\tau_c^3} \right\}$$
$$\left. \times \left\{ f(p_1)f(p_2) - (1 + f(p_1))(1 + f(p_2)) \right\} \right]$$

$$- f(p) \, e^{+ip\bar{\tau}} \times \frac{1}{H^2} \frac{(-ig)^2}{2} \frac{1}{32\pi^2 p^2} \int_0^\infty dp_1 \int_{|p_1 - p|}^{p_1 + p} dp_2$$

$$\times \left[+ \left\{ \frac{1}{i(p_1 + p_2 - p)} \frac{+2\bar{\tau}}{\tau_c^3} + \frac{-1}{(p_1 + p_2 - p)^2} \frac{2}{\tau_c^3} \right\} \right.$$
$$\times \left\{ (1 + f(p_1))(1 + f(p_2)) - f(p_1)f(p_2) \right\}$$
$$+ \left\{ \frac{1}{i(p_1 - p_2 - p)} \frac{+2\bar{\tau}}{\tau_c^3} + \frac{-1}{(p_1 - p_2 - p)^2} \frac{2}{\tau_c^3} \right\}$$
$$\times \left\{ (1 + f(p_1))f(p_2) - f(p_1)(1 + f(p_2)) \right\}$$

64

$$+ \left\{ \frac{1}{i(-p_1 + p_2 - p)} \frac{+2\bar{\tau}}{\tau_c^3} + \frac{-1}{(-p_1 + p_2 - p)^2} \frac{2}{\tau_c^3} \right\}$$
$$\times \left\{ f(p_1)(1 + f(p_2)) - (1 + f(p_1))f(p_2) \right\}$$
$$+ \left\{ \frac{1}{i(-p_1 - p_2 - p)} \frac{+2\bar{\tau}}{\tau_c^3} + \frac{-1}{(-p_1 - p_2 - p)^2} \frac{2}{\tau_c^3} \right\}$$
$$\times \left\{ f(p_1)f(p_2) - (1 + f(p_1))(1 + f(p_2)) \right\} \Bigg].$$

Here we have used the following relation.

$$\frac{1}{2p} \int \frac{d^3 p_1}{(2\pi)^3 2p_1} \frac{d^3 p_2}{(2\pi)^3 2p_2} (2\pi)^3 \delta^{(3)}(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}) \tag{A.2}$$
$$= \frac{1}{32\pi^2 p^2} \int_0^\infty dp_1 \int_{|p_1 - p|}^{p_1 + p} dp_2.$$

For the comparison with the off-sell part, we insert the identity factor as

$$\int \frac{d\varepsilon}{2\pi} (2\pi)\delta(\varepsilon - (\pm p_1 \pm p_2)). \tag{A.3}$$

In this way, we obtain the expression (3.15) in the main text. The off-shell part is calculated just like the on-shell part.

In the main text, we have introduced the collision terms with a finite energy resolution $\Delta\varepsilon$ following a standard procedure in massless field theories. They are given explicitly as follows

$$C'_{\mathrm{on}}[f] \equiv C_{\mathrm{on}}[f] \tag{A.4}$$
$$+ \frac{g^2}{16\pi p^2 H^2} \times$$
$$\Bigg[\int_p^{p+\Delta\varepsilon} \frac{d\varepsilon}{2\pi} e^{-i\varepsilon\bar{\tau}} \left(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2} \right) \frac{-1}{\tau_c^3} \int_{\frac{\varepsilon - p}{2}}^{\frac{\varepsilon + p}{2}} dp_1 \, (1 + f(p_1))(1 + f(\varepsilon - p_1))$$
$$+ 2 \int_{p-\Delta\varepsilon}^p \frac{d\varepsilon}{2\pi} e^{-i\varepsilon\bar{\tau}} \left(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2} \right) \frac{-1}{\tau_c^3} \int_{\frac{\varepsilon + p}{2}}^\infty dp_1 \, (1 + f(p_1))f(p_1 - \varepsilon) \quad \Bigg]$$
$$- \frac{g^2}{16\pi p^2 H^2} \times$$
$$\Bigg[\int_p^{p+\Delta\varepsilon} \frac{d\varepsilon}{2\pi} e^{+i\varepsilon\bar{\tau}} \left(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2} \right) \frac{-1}{\tau_c^3} \int_{\frac{\varepsilon - p}{2}}^{\frac{\varepsilon + p}{2}} dp_1 \, f(p_1)f(\varepsilon - p_1)$$
$$+ 2 \int_{p-\Delta\varepsilon}^p \frac{d\varepsilon}{2\pi} e^{+i\varepsilon\bar{\tau}} \left(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2} \right) \frac{-1}{\tau_c^3} \int_{\frac{\varepsilon + p}{2}}^\infty dp_1 \, f(p_1)(1 + f(p_1 - \varepsilon)) \Bigg],$$

$$C'_{\mathrm{off}}[f] = + \frac{g^2}{16\pi p^2 H^2} \times \tag{A.5}$$
$$\Bigg[\int_{p+\Delta\varepsilon}^\infty \frac{d\varepsilon}{2\pi} e^{-i\varepsilon\bar{\tau}} \left(\frac{1}{(\varepsilon - p)^2} - \frac{1}{(\varepsilon + p)^2} \right) \frac{-1}{\tau_c^3} \int_{\frac{\varepsilon - p}{2}}^{\frac{\varepsilon + p}{2}} dp_1 \, (1 + f(p_1))(1 + f(\varepsilon - p_1))$$

$$+ 2 \int_0^{p-\Delta\varepsilon} \frac{d\varepsilon}{2\pi} e^{-i\varepsilon\bar{\tau}} \Big(\frac{1}{(\varepsilon-p)^2} - \frac{1}{(\varepsilon+p)^2}\Big)\frac{-1}{\tau_c^3} \int_{\frac{\varepsilon+p}{2}}^{\infty} dp_1 \ (1+f(p_1))f(p_1-\varepsilon) \ \Big]$$

$$- \frac{g^2}{16\pi p^2 H^2} \times$$

$$\Big[\int_{p+\Delta\varepsilon}^{\infty} \frac{d\varepsilon}{2\pi} e^{+i\varepsilon\bar{\tau}} \Big(\frac{1}{(\varepsilon-p)^2} - \frac{1}{(\varepsilon+p)^2}\Big)\frac{-1}{\tau_c^3} \int_{\frac{\varepsilon-p}{2}}^{\frac{\varepsilon+p}{2}} dp_1 \ f(p_1)f(\varepsilon-p_1)$$

$$+ 2 \int_0^{p-\Delta\varepsilon} \frac{d\varepsilon}{2\pi} e^{+i\varepsilon\bar{\tau}} \Big(\frac{1}{(\varepsilon-p)^2} - \frac{1}{(\varepsilon+p)^2}\Big)\frac{-1}{\tau_c^3} \int_{\frac{\varepsilon+p}{2}}^{\infty} dp_1 \ f(p_1)(1+f(p_1-\varepsilon)) \ \Big].$$

In the case of the thermal distribution function, the on-shell collision term (A.4) is evaluated as

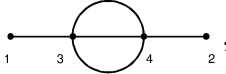$$C'_{\mathrm{on}}[f] \tag{A.6}$$

$$= - \frac{g^2}{16\pi p H^2}(1+f(p))e^{-ip\bar{\tau}} \times$$

$$\Big[\Big(\int_{p+\Delta\varepsilon}^{\infty} + \int_{p}^{p+\Delta\varepsilon}\Big)\frac{d\varepsilon}{2\pi}\Big\{\Big(\frac{1}{\varepsilon-p} + \frac{1}{\varepsilon+p}\Big)\frac{i\bar{\tau}}{\tau_c^3} + \Big(\frac{1}{(\varepsilon-p)^2} - \frac{1}{(\varepsilon+p)^2}\Big)\frac{-1}{\tau_c^3}\Big\}(1+G(\varepsilon,p,\beta))$$

$$+ \Big(\int_{p-\Delta\varepsilon}^{p} + \int_{0}^{p-\Delta\varepsilon}\Big)\frac{d\varepsilon}{2\pi}\Big\{\Big(\frac{1}{\varepsilon-p} + \frac{1}{\varepsilon+p}\Big)\frac{i\bar{\tau}}{\tau_c^3} + \Big(\frac{1}{(\varepsilon-p)^2} - \frac{1}{(\varepsilon+p)^2}\Big)\frac{-1}{\tau_c^3}\Big\}G(\varepsilon,p,\beta)\Big]$$

$$+ \frac{g^2}{16\pi p H^2} \ f(p)e^{+ip\bar{\tau}} \times$$

$$\Big[\Big(\int_{p+\Delta\varepsilon}^{\infty} + \int_{p}^{p+\Delta\varepsilon}\Big)\frac{d\varepsilon}{2\pi}\Big\{\Big(\frac{1}{\varepsilon-p} + \frac{1}{\varepsilon+p}\Big)\frac{-i\bar{\tau}}{\tau_c^3} + \Big(\frac{1}{(\varepsilon-p)^2} - \frac{1}{(\varepsilon+p)^2}\Big)\frac{-1}{\tau_c^3}\Big\}(1+G(\varepsilon,p,\beta))$$

$$+ \Big(\int_{p-\Delta\varepsilon}^{p} + \int_{0}^{p-\Delta\varepsilon}\Big)\frac{d\varepsilon}{2\pi}\Big\{\Big(\frac{1}{\varepsilon-p} + \frac{1}{\varepsilon+p}\Big)\frac{-i\bar{\tau}}{\tau_c^3} + \Big(\frac{1}{(\varepsilon-p)^2} - \frac{1}{(\varepsilon+p)^2}\Big)\frac{-1}{\tau_c^3}\Big\}G(\varepsilon,p,\beta)\Big]$$

$$+ \frac{g^2}{16\pi p H^2} \times$$

$$\Big[\int_{p}^{p+\Delta\varepsilon} \frac{d\varepsilon}{2\pi} (1+f(\varepsilon))e^{-i\varepsilon\bar{\tau}}\Big(\frac{1}{(\varepsilon-p)^2} - \frac{1}{(\varepsilon+p)^2}\Big)\frac{-1}{\tau_c^3}(1+G(\varepsilon,p,\beta))$$

$$+ \int_{p-\Delta\varepsilon}^{p} \frac{d\varepsilon}{2\pi} (1+f(\varepsilon))e^{-i\varepsilon\bar{\tau}}\Big(\frac{1}{(\varepsilon-p)^2} - \frac{1}{(\varepsilon+p)^2}\Big)\frac{-1}{\tau_c^3}G(\varepsilon,p,\beta) \ \Big]$$

$$- \frac{g^2}{16\pi p H^2} \times$$

$$\Big[\int_{p}^{p+\Delta\varepsilon} \frac{d\varepsilon}{2\pi} f(\varepsilon)e^{+i\varepsilon\bar{\tau}}\Big(\frac{1}{(\varepsilon-p)^2} - \frac{1}{(\varepsilon+p)^2}\Big)\frac{-1}{\tau_c^3}(1+G(\varepsilon,p,\beta))$$

$$+ \int_{p-\Delta\varepsilon}^{p} \frac{d\varepsilon}{2\pi} f(\varepsilon)e^{+i\varepsilon\bar{\tau}}\Big(\frac{1}{(\varepsilon-p)^2} - \frac{1}{(\varepsilon+p)^2}\Big)\frac{-1}{\tau_c^3}G(\varepsilon,p,\beta) \ \Big].$$

$G(\varepsilon, p, \beta)$ is defined in (3.22). We find that the linear infra-red divergences at $\varepsilon = p$ are canceled, but the apparent logarithmic divergences remain. The situation here is analogous

to QCD where the logarithmic divergences require the scale dependent modification of the parton distribution function. In our case, the IR singularity also leads to the modification of the particle distribution function as the final expression is shown in the main text (3.24).

B Boltzmann equation in $\lambda\varphi^4$ theory

In this appendix, we consider the Boltzmann equation in $\lambda\varphi^4$ theory. Since this theory is classically stable, it is a good example for investigating quantum effects on the dS background. Here the self-energy is

$$\Sigma^{ij}(x_3, x_4) = \frac{(-i\lambda)^2}{6} G^{ij}(x_3, x_4) G^{ij}(x_3, x_4) G^{ij}(x_3, x_4), \quad i, j = +, -. \tag{B.1}$$

As in the main text, we evaluate the time integrations with the assumption $|(\varepsilon \pm p)\tau_i| \gg 1$. In $\lambda\varphi^4$ theory, we need to retain higher order terms than (3.12) to investigate the particle production effects in dS space

$$\int_{-\infty}^{\tau_i} d\tau_3 \, \frac{1}{\tau_3^n} e^{i(\varepsilon\pm p)\tau_3} \sim \ e^{i(\varepsilon\pm p)\tau_i} \times \left[\frac{1}{i(\varepsilon \pm p)\tau_i^n} + \frac{-n}{(\varepsilon \pm p)^2 \tau_i^{n+1}} + \frac{-n(n+1)}{i(\varepsilon \pm p)^3 \tau_i^{n+2}} \right], \tag{B.2}$$
$$n = 1, 2, \cdots.$$

We should note that (B.2) can be evaluated exactly when $n = 0$

$$\int_{-\infty}^{\tau_i} d\tau_3 \, e^{i(\varepsilon\pm p)\tau_3} = \ e^{i(\varepsilon\pm p)\tau_i} \times \frac{1}{i(\varepsilon \pm p - ie)} \tag{B.3}$$
$$= \ e^{i(\varepsilon\pm p)\tau_i} \times \left(\frac{P}{i(\varepsilon \pm p)} + \pi\delta(\varepsilon \pm p) \right).$$

The $-ie$ prescription is necessary for the convergence at $\tau_3 = -\infty$.

In this appendix, we focus on the IR effects of the collision term at $\varepsilon - p = 0$. Therefore we consider only 2 bodies $\to$ 2 bodies processes. In these processes, the on-shell part of the collision term is as follows

$$C_{\rm on}[f] = + (1 + f(p)) \, e^{-ip\bar{\tau}} \times \frac{(-i\lambda)^2}{6} \frac{1}{2p} \int \prod_{i=1}^{3} \frac{d^3 p_i}{(2\pi)^3 2p_i} \, (2\pi)^3 \delta^{(3)}(\mathbf{p} + \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3) \tag{B.4}$$
$$\times \left[+ 3\{(1 + f(p_1))(1 + f(p_2))f(p_3) - f(p_1)f(p_2)(1 + f(p_3))\} \right.$$
$$\times \{ + 2\pi\delta(p_1 + p_2 - p_3 - p)$$
$$+ \frac{1}{i(p_1 + p_2 - p_3 - p)} \times (\frac{1}{p_1^2} + \frac{1}{p_2^2} + \frac{1}{p_3^2}) \frac{-2\bar{\tau}}{\tau_c^3}$$

$$+ \frac{1}{i(p_1 + p_2 - p_3 - p)^2} \times \left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} - \frac{1}{p} \right) \frac{-2\bar{\tau}}{\tau_c^3}$$

$$+ \frac{1}{(p_1 + p_2 - p_3 - p)^2} \times \left(\frac{1}{p_1^2} + \frac{1}{p_2^2} + \frac{1}{p_3^2} + \frac{1}{p^2} \right) \frac{-2}{\tau_c^3}$$

$$\left. \left. + \frac{1}{(p_1 + p_2 - p_3 - p)^3} \times \left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} - \frac{1}{p} \right) \frac{-4}{\tau_c^3} \quad \right\} \right]$$

$$- f(p) \, e^{+ip\bar{\tau}} \times \frac{(-i\lambda)^2}{6} \frac{1}{2p} \int \prod_{i=1}^{3} \frac{d^3 p_i}{(2\pi)^3 2p_i} \, (2\pi)^3 \delta^{(3)}(\mathbf{p} + \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3)$$

$$\times \left[+ 3 \{ (1 + f(p_1))(1 + f(p_2))f(p_3) - f(p_1)f(p_2)(1 + f(p_3)) \} \right.$$

$$\times \left\{ + 2\pi \delta(p_1 + p_2 - p_3 - p) \right.$$

$$- \frac{1}{i(p_1 + p_2 - p_3 - p)} \times \left(\frac{1}{p_1^2} + \frac{1}{p_2^2} + \frac{1}{p_3^2} \right) \frac{-2\bar{\tau}}{\tau_c^3}$$

$$- \frac{1}{i(p_1 + p_2 - p_3 - p)^2} \times \left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} - \frac{1}{p} \right) \frac{-2\bar{\tau}}{\tau_c^3}$$

$$+ \frac{1}{(p_1 + p_2 - p_3 - p)^2} \times \left(\frac{1}{p_1^2} + \frac{1}{p_2^2} + \frac{1}{p_3^2} + \frac{1}{p^2} \right) \frac{-2}{\tau_c^3}$$

$$\left. \left. + \frac{1}{(p_1 + p_2 - p_3 - p)^3} \times \left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} - \frac{1}{p} \right) \frac{-4}{\tau_c^3} \quad \right\} \right].$$

The off-shell part of collision term is as follows

$$C_{\text{off}}[f] = - \frac{(-i\lambda)^2}{6} \frac{1}{2p} \int \prod_{i=1}^{3} \frac{d^3 p_i}{(2\pi)^3 2p_i} \, (2\pi)^3 \delta^{(3)}(\mathbf{p} + \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3) \tag{B.5}$$

$$\times \left[+ 3(1 + f(p_1))(1 + f(p_2))f(p_3) \, e^{-i(p_1 + p_2 - p_3)\bar{\tau}} \right.$$

$$\times \left\{ + 2\pi \delta(p_1 + p_2 - p_3 - p) \right.$$

$$- \frac{1}{i(p_1 + p_2 - p_3 - p)} \times \frac{1}{p^2} \frac{-2\bar{\tau}}{\tau_c^3}$$

$$- \frac{1}{i(p_1 + p_2 - p_3 - p)^2} \times \left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} - \frac{1}{p} \right) \frac{-2\bar{\tau}}{\tau_c^3}$$

$$+ \frac{1}{(p_1 + p_2 - p_3 - p)^2} \times \left(\frac{1}{p_1^2} + \frac{1}{p_2^2} + \frac{1}{p_3^2} + \frac{1}{p^2} \right) \frac{-2}{\tau_c^3}$$

$$\left. \left. + \frac{1}{(p_1 + p_2 - p_3 - p)^3} \times \left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} - \frac{1}{p} \right) \frac{-4}{\tau_c^3} \quad \right\} \right]$$

$$+ \frac{(-i\lambda)^2}{6} \frac{1}{2p} \int \prod_{i=1}^{3} \frac{d^3 p_i}{(2\pi)^3 2p_i} \, (2\pi)^3 \delta^{(3)}(\mathbf{p} + \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3)$$

$$\times \left[+ 3 f(p_1)f(p_2)(1 + f(p_3)) \, e^{+i(p_1 + p_2 - p_3)\bar{\tau}} \right.$$

$$\times \left\{ + 2\pi \delta(p_1 + p_2 - p_3 - p) \right.$$

$$+ \frac{1}{i(p_1 + p_2 - p_3 - p)} \times \frac{1}{p^2} \frac{-2\bar{\tau}}{\tau_c^3}$$

$$+ \frac{1}{i(p_1 + p_2 - p_3 - p)^2} \times (\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} - \frac{1}{p}) \frac{-2\bar{\tau}}{\tau_c^3}$$
$$+ \frac{1}{(p_1 + p_2 - p_3 - p)^2} \times (\frac{1}{p_1^2} + \frac{1}{p_2^2} + \frac{1}{p_3^2} + \frac{1}{p^2}) \frac{-2}{\tau_c^3}$$
$$+ \frac{1}{(p_1 + p_2 - p_3 - p)^3} \times (\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} - \frac{1}{p}) \frac{-4}{\tau_c^3} \quad \} \bigg].$$

In (B.4) and (B.5), only the leading term in $1/p|\tau_c|$ expansion is shown for the energy conserving part containing $\delta(p_1 + p_2 - p_3 - p)$.

We note that the leading term in $1/p|\tau_c|$ expansion is the same with the collision term in Minkowski space

$$C[f]_{\text{leading}} \tag{B.6}$$

$$= \frac{\lambda^2}{2} \frac{1}{2p} \int \prod_{i=1}^{3} \frac{d^3 p_i}{(2\pi)^3 2p_i} \; (2\pi)^4 \delta^{(3)}(\mathbf{p} + \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3) \delta(p_1 + p_2 - p_3 - p)$$

$$\times \bigg[+ \{f(p_1)f(p_2)(1 + f(p_3))(1 + f(p)) - (1 + f(p_1))(1 + f(p_2))f(p_3)f(p)\} \; e^{-ip\bar{\tau}}$$

$$- \{f(p_1)f(p_2)(1 + f(p_3))(1 + f(p)) - (1 + f(p_1))(1 + f(p_2))f(p_3)f(p)\} \; e^{+ip\bar{\tau}} \bigg].$$

This is because the leading term is conformally invariant. We thus obtain the identical result with [7] to the leading order in $1/p|\tau_c|$ expansion.

In addition to the leading effect, we investigate the particle production effects due to energy non-conservation. Let us focus on the case that the initial distribution function is thermal. It solves the Boltzmann equation to the leading order as the following identity holds

$$(1 + f(p_1))(1 + f(p_2))f(p_3)f(p_1 + p_2 - p_3) \tag{B.7}$$
$$= f(p_1)f(p_2)(1 + f(p_3))(1 + f(p_1 + p_2 - p_3)).$$

Therefore the off-shell part is written as follows

$$C_{\text{off}}[f]_{\text{next leading}} \tag{B.8}$$

$$= -\frac{(-i\lambda)^2}{6} \frac{1}{2p} \int \prod_{i=1}^{3} \frac{d^3 p_i}{(2\pi)^3 2p_i} \; (2\pi)^3 \delta^{(3)}(\mathbf{p} + \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3)$$

$$\times \bigg[+ 3\{(1 + f(p_1))(1 + f(p_2))f(p_3) - f(p_1)f(p_2)(1 + f(p_3))\}$$

$$\times (1 + f(p_1 + p_2 - p_3)) \; e^{-i(p_1 + p_2 - p_3)\bar{\tau}}$$

$$\times \{ - \frac{1}{i(p_1 + p_2 - p_3 - p)} \times \frac{1}{p^2} \frac{-2\bar{\tau}}{\tau_c^3}$$

$$- \frac{1}{i(p_1 + p_2 - p_3 - p)^2} \times (\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} - \frac{1}{p}) \frac{-2\bar{\tau}}{\tau_c^3}$$

$$+ \frac{1}{(p_1 + p_2 - p_3 - p)^2} \times (\frac{1}{p_1^2} + \frac{1}{p_2^2} + \frac{1}{p_3^2} + \frac{1}{p^2}) \frac{-2}{\tau_c^3}$$

$$+ \frac{1}{(p_1 + p_2 - p_3 - p)^3} \times (\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} - \frac{1}{p})\frac{-4}{\tau_c^3} \quad \} \Big]$$

$$+ \frac{(-i\lambda)^2}{6} \frac{1}{2p} \int \prod_{i=1}^3 \frac{d^3 p_i}{(2\pi)^3 2p_i} (2\pi)^3 \delta^{(3)}(\mathbf{p} + \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3)$$

$$\times \Big[+ 3\{(1 + f(p_1))(1 + f(p_2))f(p_3) - f(p_1)f(p_2)(1 + f(p_3))\}$$

$$\times f(p_1 + p_2 - p_3)\, e^{+i(p_1 + p_2 - p_3)\bar{\tau}}$$

$$\times \{ + \frac{1}{i(p_1 + p_2 - p_3 - p)} \times \frac{1}{p^2}\frac{-2\bar{\tau}}{\tau_c^3}$$

$$+ \frac{1}{i(p_1 + p_2 - p_3 - p)^2} \times (\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} - \frac{1}{p})\frac{-2\bar{\tau}}{\tau_c^3}$$

$$+ \frac{1}{(p_1 + p_2 - p_3 - p)^2} \times (\frac{1}{p_1^2} + \frac{1}{p_2^2} + \frac{1}{p_3^2} + \frac{1}{p^2})\frac{-2}{\tau_c^3}$$

$$+ \frac{1}{(p_1 + p_2 - p_3 - p)^3} \times (\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} - \frac{1}{p})\frac{-4}{\tau_c^3} \quad \} \Big].$$

Most of the IR divergences at $p_1 + p_2 - p_3 - p = 0$ cancel out between $C_{\text{on}}[f]$ and $C_{\text{off}}[f]$. This is because the total spectral weight is conserved due to unitarity. The remaining IR divergence comes from momentum dependence of the distribution function

$$f(p_1 + p_2 - p_3) = f(p) + f'(p)(p_1 + p_2 - p_3 - p) + \cdots . \tag{B.9}$$

As explained in the main text, this IR divergence leads to the change of the distribution function

$$\delta f \sim f'(p)\, \frac{\lambda^2}{2} \frac{1}{2p} \int \prod_{i=1}^3 \frac{d^3 p_i}{(2\pi)^3 2p_i} (2\pi)^3 \delta^{(3)}(\mathbf{p} + \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3) \tag{B.10}$$

$$\times \Big[\{(1 + f(p_1))(1 + f(p_2))f(p_3) - f(p_1)f(p_2)(1 + f(p_3))\}$$

$$\times \frac{1}{(p_1 + p_2 - p_3 - p)^2}(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} - \frac{1}{p})\frac{2}{\tau_c^2} \Big].$$

Here again we may adopt the IR cut-off : $|p_1 + p_2 - p_3 - p| \sim 1/|\tau_c|$. τ_c dependence can be entirely absorbed into physical quantities $P_i = p_i H |\tau_c|$, $T = \beta H |\tau_c|$.

We may draw the following conclusion in this appendix. The leading order collision term is identical to that in Minkowski space. If we consider the higher order terms in $1/p|\tau_c|$ expansion, the off-shell part is generated due to the particle production while the total spectral weight is preserved due to unitarity. We further find the non-trivial change of the distribution function due to IR divergences. These features in $\lambda\varphi^4$ theory are qualitatively identical to those in $g\varphi^3$ theory.

C Power counting of $\log a(\tau)$

We can estimate the power of the IR logarithms induced by a diagram without a detailed calculation. Here we explain how to do it.

First of all, we recall that the interaction vertices are located in the past light-cone of the energy-momentum tensor. Since we are interested in logarithmically large contributions, we can assume that the conformal time of the interaction vertices τ_i are hierarchically separated $|\tau_1| \ll |\tau_2| \ll |\tau_3| \ll \cdots$. In such a configuration the separations of the interaction vertices are almost always time-like $|\tau_i - \tau_j| > |\mathbf{x}_i - \mathbf{x}_j|$.

For the power counting, we have only to focus on the following behavior of the constituents in the amplitude. The space-time metric and the propagator at the coincident point show the following time dependence:

$$g_{\alpha\beta}(\tau') \sim \frac{1}{\tau'^2}, \quad \sqrt{-g'}g^{\alpha\beta}(\tau') \sim \frac{1}{\tau'^2}, \quad G^{++}(x', x') \sim \log|\tau'|. \tag{C.1}$$

Concerning the retarded propagator $G^R(x, x')$ and the symmetric propagators $\bar{G}(x, x')$ between the separated points, we focus on the following behavior:

$$G^R(x, x') \sim \theta(\tau - \tau')\theta\big((\tau - \tau')^2 - |\mathbf{x} - \mathbf{x}'|^2\big), \tag{C.2}$$
$$\bar{G}(x, x') \sim \log\big((\tau - \tau')^2 - |\mathbf{x} - \mathbf{x}'|^2\big).$$

Note that they are functions of Δx^2 except for the factor $\theta(\tau - \tau')$. The behavior of the differentiated propagators follow from (C.1) and (C.2) except for the twice differentiated propagator at the coincident point:

$$\partial'_\alpha G^{++}(x', x') \sim \frac{1}{\tau'}, \tag{C.3}$$

$$\partial_\alpha G^R(x, x') = -\partial'_\alpha G^R(x, x') \sim \theta(\tau - \tau')\partial_\alpha\theta(-\Delta x^2), \tag{C.4}$$
$$\partial_\alpha\partial'_\beta G^R(x, x') \sim \theta(\tau - \tau')\partial_\alpha\partial'_\beta\theta(-\Delta x^2),$$
$$\partial_\alpha \bar{G}(x, x') = -\partial'_\alpha \bar{G}(x, x') \sim \frac{1}{\Delta x},$$
$$\partial_\alpha\partial'_\beta \bar{G}(x, x') \sim \frac{1}{\Delta x^2}.$$

We estimate the twice differentiated propagator at the coincident point as follows:

$$\lim_{x'' \to x'} \partial'_\alpha\partial''_\beta G^{++}(x', x'') \sim \frac{1}{\tau'^2}. \tag{C.5}$$

If we expand (C.2) and (C.4) in the power series of $|\mathbf{x} - \mathbf{x}'|/\tau - \tau'$ considering $\tau - \tau' > |\mathbf{x} - \mathbf{x}'|$,

the spatial integration doesn't induce a logarithm. We thus obtain

$$\int d^3x'\, G^R(x,x') \sim \theta(\tau-\tau') \times (\tau-\tau')^3, \tag{C.6}$$

$$\int d^3x'\, \partial_\alpha G^R(x,x') = -\int d^3x'\, \partial'_\alpha G^R(x,x') \sim \theta(\tau-\tau') \times (\tau-\tau')^2,$$

$$\int d^3x'\, \partial_\alpha \partial'_\beta G^R(x,x') \sim \theta(\tau-\tau') \times (\tau-\tau'),$$

$$\bar{G}(x,x') \sim \log(\tau-\tau').$$

$$\partial_\alpha \bar{G}(x,x') = -\partial'_\alpha \bar{G}(x,x') \sim \frac{1}{\tau-\tau'},$$

$$\partial_\alpha \partial'_\beta \bar{G}(x,x') \sim \frac{1}{(\tau-\tau')^2}.$$

In the above estimates, we have focued on the logarithm part of the propagator:

$$G(x,x') \sim \log(\Delta x^2). \tag{C.7}$$

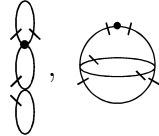To be more precise, the propagator has the inverse square part in addition:

$$G(x,x') \sim \frac{\tau\tau'}{\Delta x^2} - \frac{1}{2}\log(\Delta x^2). \tag{C.8}$$

If we take the zeroth order of the expansion by $|\mathbf{x}-\mathbf{x}'|/\tau-\tau'$ and the differentiations with respect to time, the twice differentiated propagators have different asymptotic behavior with respect to τ and τ' in comparison with (C.6):

$$\int d^3x'\, \partial_\alpha \partial'_\beta G^R(x,x') \sim \theta(\tau-\tau') \times \frac{\tau\tau'}{\tau-\tau'}, \tag{C.9}$$

$$\partial_\alpha \partial'_\beta \bar{G}(x,x') \sim \frac{\tau\tau'}{(\tau-\tau')^4}.$$

It seems that the estimation (C.6) is not entirely valid. Nevertheless it can be justified as we consider contributions from beyond the zeroth order expansion by $|\mathbf{x}-\mathbf{x}'|/\tau-\tau'$.

As a concrete example, let us perform the power counting of the IR logarithms induced by the following two diagrams

$$\tag{C.10}$$

The first diagram is written as

 $$\sim \int \sqrt{-g'} g^{\alpha\beta}(\tau') d^4x' \lim_{x'\to x} \partial_\rho \partial'_\sigma G^{++}(x,x') \tag{C.11}$$

$$\times \left[\partial'_\alpha G^R(x,x')\bar{G}(x,x') + \partial'_\alpha \bar{G}(x,x') G^R(x,x') \right] \partial'_\beta G^{++}(x',x').$$

By using (C.1), (C.3), (C.5) and (C.6), each integral is estimated as

$$\int \sqrt{-g'}g^{\alpha\beta}(\tau')d^4x' \lim_{x'\to x} \partial_\rho\partial'_\sigma G^{++}(x,x')\partial'_\alpha G^R(x,x')\bar{G}(x,x')\partial'_\beta G^{++}(x',x') \tag{C.12}$$

$$\sim \frac{1}{\tau^2}\int^\tau \frac{d\tau'}{\tau'^2}\frac{(\tau-\tau')^2}{\tau'}\log(\tau-\tau')$$

$$\sim \frac{1}{\tau^2}\int^\tau \frac{d\tau'}{\tau'}\left\{\log|\tau'|\sum_{n=0}A_n\left(\frac{\tau}{\tau'}\right)^n + \sum_{n=1}B_n\left(\frac{\tau}{\tau'}\right)^n\right\} \sim a^2(\tau)\log^2 a(\tau),$$

$$\int \sqrt{-g'}g^{\alpha\beta}(\tau')d^4x' \lim_{x'\to x} \partial_\rho\partial'_\sigma G^{++}(x,x')\partial'_\alpha \bar{G}(x,x')G^R(x,x')\partial'_\beta G^{++}(x',x') \tag{C.13}$$

$$\sim \frac{1}{\tau^2}\int^\tau \frac{d\tau'}{\tau'^2}\frac{(\tau-\tau')^2}{\tau'}$$

$$\sim \frac{1}{\tau^2}\int^\tau \frac{d\tau'}{\tau'}\sum_{n=0}C_n\left(\frac{\tau}{\tau'}\right)^n \sim a^2(\tau)\log a(\tau).$$

In the above expressions, we have expanded the integrands considering $|\tau| < |\tau'|$ where A_n, B_n, C_n are constant coefficients. For the power counting of the IR logarithms, we have only to retain the zeroth order $n = 0$. From (C.12) and (C.13),

$$\text{} \sim a^2(\tau)\log^2 a(\tau). \tag{C.14}$$

The second diagram is written as

$$\text{} \sim \int \sqrt{-g'}g^{\alpha\beta}(\tau')d^4x'\sqrt{-g''}g^{\gamma\delta}(\tau'')d^4x'' \tag{C.15}$$

$$\times 2\partial_\rho G^R(x,x')\big[\partial'_\alpha G^R(x',x'')\partial''_\gamma \bar{G}(x',x'')\partial'_\beta\partial''_\delta\bar{G}(x',x'')$$

$$+ \partial'_\alpha\bar{G}(x',x'')\partial''_\gamma G^R(x',x'')\partial'_\beta\partial''_\delta\bar{G}(x',x'')$$

$$+ \partial'_\alpha\bar{G}(x',x'')\partial''_\gamma\bar{G}(x',x'')\partial'_\beta\partial''_\delta G^R(x',x'')\big]\partial_\sigma\bar{G}(x,x'')$$

$$+ \int \sqrt{-g'}g^{\alpha\beta}(\tau')d^4x'\sqrt{-g''}g^{\gamma\delta}(\tau'')d^4x''$$

$$\times \partial_\rho G^R(x,x')\partial'_\alpha\bar{G}(x',x'')\partial''_\gamma\bar{G}(x',x'')\partial'_\beta\partial''_\delta\bar{G}(x',x'')\partial_\sigma G^R(x,x'').$$

By using (C.6), each integral is estimated as

$$\int \sqrt{-g'}g^{\alpha\beta}(\tau')d^4x'\sqrt{-g''}g^{\gamma\delta}(\tau'')d^4x'' \tag{C.16}$$

$$\times 2\partial_\rho G^R(x,x')\big[\partial'_\alpha G^R(x',x'')\partial''_\gamma\bar{G}(x',x'')\partial'_\beta\partial''_\delta\bar{G}(x',x'')$$

$$+ \partial'_\alpha\bar{G}(x',x'')\partial''_\gamma G^R(x',x'')\partial'_\beta\partial''_\delta\bar{G}(x',x'')$$

$$+ \partial'_\alpha \bar{G}(x', x'') \partial''_\gamma \bar{G}(x', x'') \partial'_\beta \partial''_\delta G^R(x', x'')] \partial_\sigma \bar{G}(x, x'')$$

$$\sim \int^\tau \frac{d\tau'}{\tau'^2} \int^{\tau'} \frac{d\tau''}{\tau''^2} \, (\tau - \tau')^2 \frac{1}{(\tau' - \tau'')} \frac{1}{(\tau - \tau'')}$$

$$\sim \int^\tau d\tau' \int^{\tau'} \frac{d\tau''}{\tau''^4} \sum_{p,q,r=0} D_{pqr} \left(\frac{\tau}{\tau'}\right)^p \left(\frac{\tau'}{\tau''}\right)^q \left(\frac{\tau}{\tau''}\right)^r \sim a^2(\tau),$$

$$\int \sqrt{-g'} g^{\alpha\beta}(\tau') d^4 x' \sqrt{-g''} g^{\gamma\delta}(\tau'') d^4 x'' \qquad\qquad\qquad (C.17)$$

$$\times \, \partial_\rho G^R(x, x') \partial'_\alpha \bar{G}(x', x'') \partial''_\gamma \bar{G}(x', x'') \partial'_\beta \partial''_\delta \bar{G}(x', x'') \partial_\sigma G^R(x, x'')$$

$$\sim \int^\tau \frac{d\tau'}{\tau'^2} \int^{\tau'} \frac{d\tau''}{\tau''^2} \, (\tau - \tau')^2 \frac{1}{(\tau' - \tau'')^4} (\tau - \tau'')^2$$

$$\sim \int^\tau d\tau' \int^{\tau'} \frac{d\tau''}{\tau''^4} \sum_{p,q,r=0} E_{pqr} \left(\frac{\tau}{\tau'}\right)^p \left(\frac{\tau'}{\tau''}\right)^q \left(\frac{\tau}{\tau''}\right)^r \sim a^2(\tau).$$

In the second line of (C.17), we have performed the integrals in the order $|\tau| < |\tau'| < |\tau''|$. In the third line, we have expanded the integrands respecting this ordering where D_{pqr}, E_{pqr} are constant coefficients. Just like the first diagram, we have only to retain the zeroth order $p = q = r = 0$ for the power counting of the IR logarithms. From (C.16) and (C.17), we conclude the second diagram has no IR logarithms

$$\sim a^2(\tau). \qquad\qquad\qquad (C.18)$$

The power counting procedure of the IR logarithms is summarized as follows. In the first step, we estimate the relevant behavior of the constituents of a diagram by using (C.1), (C.3), (C.5) and (C.6). In the second step, we time order the integrations over the interaction points. In the third step, we expand the integrand in the power series of the ratios of the conformal time respecting the time ordering. For the power counting of the leading IR logarithms of a diagram, we have only to integrate the zeroth order of the expansion.

In order to prove this statement, we first estimate the IR logarithms when there are no differential operators involved at the interaction point. The integral over the location of an interaction vertex induces some power of IR logarithms as:

$$\int \sqrt{-g'} d^4 x' \, A^R(x, x') \bar{B}(x', x'') \sim \log^{m-1} |\tau''| \log^{n+1} |\tau''|, \qquad (C.19)$$

where $|\tau| < |\tau''| < |\tau'|$. $A^R(x, x')$ consists of one retarded propagator $G^R(x, x')$ and $(m-1)$ symmetric propagators $\bar{G}(x, x')$. $\bar{B}(x, x')$ consists of n symmetric propagators $\bar{G}(x, x')$. We next estimate the effect of the minimal derivative coupling on the above estimate: The $g^{\alpha\beta}(\tau')$ at the interaction vertex induces τ'^2 behavior and there are at least two derivatives involved. At the zeroth order,

$$g^{\alpha\beta}(\tau')(\partial, \partial', \partial'')^p \sim \frac{1}{\tau'^{p-2}}, \quad p \geq 2. \qquad (C.20)$$

74

In the case $p = 2$, the integral over time induces a single logarithm. However the differentiations on the symmetric propagators reduce the previous estimate of the power of the IR logarithms (C.19). In the case $p > 2$, the integral over time doesn't induce the IR logarithm and the power of the IR logarithms is less than (C.19). From (C.19) and (C.20), we find that each integral doesn't induce the positive power of the scale setting conformal time τ''. For the power counting of the leading IR logarithms of a diagram, we have only to integrate the leading order of the expansion. We can iteratively continue this argument to cover the whole amplitude.

Finally we prove that each diagram with the leading IR logarithms contains a closed loop of the twice differentiated propagators which runs through the vertex located at the external point x. Each vertex integral of the closed loop corresponds to the $p = 4$ case in (C.20). If the closed loop has n vertices, the leading IR logarithms comes from the case that the closed loop has n retarded propagators and one symmetric propagator:

$$\int \sqrt{-g(\tau_1)}g^{\alpha_1\beta_1}(\tau_1)d^4x_1 \cdots \int \sqrt{-g(\tau_n)}g^{\alpha_n\beta_n}(\tau_n)d^4x_n \tag{C.21}$$
$$\times \partial_\rho\partial_{\alpha_1}G^R(x,x_1)\cdots\partial_{\beta_{n-1}}\partial_{\alpha_n}G^R(x_{n-1},x_n)\partial_{\beta_n}\partial_\sigma\bar{G}(x_n,x) \times L(x,x_1,\cdots,x_n),$$

where $L(x,x_1,\cdots,x_n)$ is some powers of the IR logarithm induced outside the closed loop. To be exact, the closed loop contains other permutations of propagators. The investigation of them can be performed in a similar way. We have only to estimate the integrand of (C.21) at the zeroth order

$$\int \sqrt{-g(\tau_1)}g^{\alpha_1\beta_1}(\tau_1)d^4x_1 \cdots \int \sqrt{-g(\tau_n)}g^{\alpha_n\beta_n}(\tau_n)d^4x_n \sim \int \frac{d\tau_1}{\tau_1^2}\cdots\int \frac{d\tau_n}{\tau_n^2}, \tag{C.22}$$

$$\partial_\rho\partial_{\alpha_1}G^R(x,x_1)\cdots\partial_{\beta_{n-1}}\partial_{\alpha_n}G^R(x_{n-1},x_n)\partial_{\beta_n}\partial_\sigma\bar{G}(x_n,x) \tag{C.23}$$
$$\sim \theta(\tau-\tau_1)\cdots\theta(\tau_{n-1}-\tau_n) \times \tau_1\cdots\tau_n \times \frac{1}{\tau_n^2},$$

$$L(x,x_1,\cdots,x_n) \sim \log^p|\tau|\log^{p_1}|\tau_1|\cdots\log^{p_n}|\tau_n|, \tag{C.24}$$

where $p, p_1, \cdots, p_n \geq 0$. So the integration (C.21) is estimated as

$$\int \sqrt{-g(\tau_1)}g^{\alpha_1\beta_1}(\tau_1)d^4x_1 \cdots \int \sqrt{-g(\tau_n)}g^{\alpha_n\beta_n}(\tau_n)d^4x_n \tag{C.25}$$
$$\times \partial_\rho\partial_{\alpha_1}G^R(x,x_1)\cdots\partial_{\beta_{n-1}}\partial_{\alpha_n}G^R(x_{n-1},x_n)\partial_{\beta_n}\partial_\sigma\bar{G}(x_n,x) \times L(x,x_1,\cdots,x_n)$$
$$\sim a^2(\tau)\big(\log a(\tau)\big)^{p+p_1+\cdots+p_n}.$$

Here the IR logarithms are induced by $L(x,x_1,\cdots,x_n)$ and the closed loop doesn't induce the IR logarithms.

The other diagrams are obtained if we remove any of the differential operators from the closed loop. As an example, we consider the diagram with the closed loop where one differential operator is removed. Such a differential operator acts on the IR logarithms outside the closed

loop $L(x, x_1, \cdots, x_n)$. On the other hand, the closed loop doesn't induce the IR logarithms. Therefore the power of this diagram is one less than (C.25):

$$\int \sqrt{-g(\tau_1)} g^{\alpha_1 \beta_1}(\tau_1) d^4 x_1 \cdots \int \sqrt{-g(\tau_n)} g^{\alpha_n \beta_n}(\tau_n) d^4 x_n \tag{C.26}$$
$$\times \partial_\rho G^R(x, x_1) \cdots \partial_{\beta_{n-1}} \partial_{\alpha_n} G^R(x_{n-1}, x_n) \partial_{\beta_n} \partial_\sigma \bar{G}(x_n, x) \times \partial_{\alpha_1} L(x, x_1, \cdots, x_n)$$
$$\sim a^2(\tau) \left(\log a(\tau) \right)^{p+p_1+\cdots+p_n-1}.$$

In the case where we remove any other differential operator from the closed loop, the power of the IR logarithms induced by the corresponding diagram is also one less than (C.25).

If we remove two differential operators from the closed loop of the twice differentiated propagators, it is possible that the closed loop induces a single IR logarithm more than otherwise. However in this case, the part outside the closed loop induces two less power of the IR logarithm. Therefore, also in this case, the power of the IR logarithm is one less than (C.25).

Even if we remove more than two differential operators from the closed loop, we can similarly conclude that the power of the IR logarithm induced by the corresponding diagram is less than (C.25).

D Evaluation of $\int d^{4-\varepsilon} x' \, G(a(\tau'))[F(\Delta x_{++}^2) - F(\Delta x_{+-}^2)]$

In this Appendix, we explain how to calculate the following integral:

$$\int d^{4-\varepsilon} x' \, G(a(\tau'))[F(\Delta x_{++}^2) - F(\Delta x_{+-}^2)], \tag{D.1}$$

where G is a arbitrary function of $a(\tau')$ and one example of F is

$$\frac{1}{\Delta x^{p-q\varepsilon}}. \tag{D.2}$$

Here p is a non-negative even number and q is a positive integer. In addition to (D.2), the method introduced here cover the following integrands with the Lorentz indexes:

$$\frac{\Delta x_\rho}{\Delta x^{p+2-q\varepsilon}}, \quad \frac{\Delta x_\rho \Delta x_\sigma}{\Delta x^{p+4-q\varepsilon}}, \quad \cdots . \tag{D.3}$$

Most integrals of them can be evaluated by applying the procedure developed in [13]. However special considerations are required in the case: $p \geq 4, q = 1$. Before discussing about the special cases, we review the procedure for other cases: $p \geq 4, q \geq 2$ or $p \leq 2$

D.1 $p \geq 4, q \geq 2$ or $p \leq 2$ case

Here we perform the integral where the integrand satisfy $p \geq 4, q \geq 2$ or $p \leq 2$. First, we show that the integrals containing the tensors (D.3) reduce to the integral of (D.2). The

reduction is performed by differential operators. For example,

$$\frac{\Delta x_\rho}{\Delta x^{p+2-q\varepsilon}} = -\frac{1}{p-q\varepsilon}\partial_\rho \frac{1}{\Delta x^{p-q\varepsilon}}, \tag{D.4}$$

$$\frac{\Delta x_\rho \Delta x_\sigma}{\Delta x^{p+4-q\varepsilon}} = \frac{1}{(p+2-q\varepsilon)(p-q\varepsilon)}\Big\{\partial_\rho\partial_\sigma + \frac{\eta_{\rho\sigma}\partial^2}{p-2-(q-1)\varepsilon}\Big\}\frac{1}{\Delta x^{p-q\varepsilon}}, \tag{D.5}$$

where we abbreviate the indexes $++, +-$ because the above identities work out in both cases. Note that (D.5) does not work out in the $p \geq 4, q = 1$ case. Two differential operators acted on $1/\Delta x^{p-\varepsilon}$ induces a delta function. We can put these differential operators outside the integral since $G(a(\tau'))$ is independent of τ. Considering these identity and the spatial translation and rotation symmetries, the following identity works

$$\int d^{4-\varepsilon}x' \ G(a(\tau'))\partial_\rho = \delta_\rho{}^0 \partial_0 \int d^{4-\varepsilon}x' \ G(a(\tau')). \tag{D.6}$$

So we have only to calculate the integral of (D.2).

Next, we explain how to perform the integral of (D.4) in the case: $p \geq 4,\ q \geq 2$. By the following iterate processes, (D.2) is written as

$$\frac{1}{\Delta x^{p-q\varepsilon}} = \frac{1}{(p-2-q\varepsilon)\{p-4-(q-1)\varepsilon\}}\partial^2 \frac{1}{\Delta x^{p-2-q\varepsilon}} \tag{D.7}$$

$$= \frac{1}{(p-2-q\varepsilon)(p-4-q\varepsilon)\cdots(2-q\varepsilon)}$$

$$\times \frac{1}{\{p-4-(q-1)\varepsilon\}\{p-6-(q-1)\varepsilon\}\cdots\{0-(q-1)\varepsilon\}}$$

$$\times \partial^{p-2}\frac{1}{\Delta x^{2-q\varepsilon}}.$$

To extract the UV divergence, we note the following part of (D.7)

$$\frac{\partial^2}{\varepsilon}\frac{1}{\Delta x^{2-q\varepsilon}}.$$

By using the identities

$$\partial^2 \frac{1}{\Delta x_{++}^{2-\varepsilon}} = \frac{2ie(2-\varepsilon)\delta(\tau-\tau')}{\big((\mathbf{x}-\mathbf{x}')^2+e^2\big)^{2-\frac{\varepsilon}{2}}} \to \frac{4i\pi^{2-\frac{\varepsilon}{2}}}{\Gamma(1-\frac{\varepsilon}{2})}\delta^{(D)}(x-x'), \tag{D.8}$$

$$\partial^2 \frac{1}{\Delta x_{+-}^{2-\varepsilon}} = 0,$$

(D.8) is evaluated as

$$\frac{\partial^2}{\varepsilon}\frac{1}{\Delta x_{++}^{2-q\varepsilon}} = \frac{\partial^2}{\varepsilon}\Big\{\frac{1}{\Delta x_{++}^{2-q\varepsilon}} - \frac{\mu^{-(q-1)\varepsilon}}{\Delta x_{++}^{2-\varepsilon}}\Big\} + \frac{4i\pi^{2-\frac{\varepsilon}{2}}\mu^{-(q-1)\varepsilon}}{\varepsilon\Gamma(1-\frac{\varepsilon}{2})}\delta^{(D)}(x-x') \tag{D.9}$$

$$= (q-1)\frac{\partial^2}{2}\Big\{\frac{\log(\mu^2\Delta x_{++}^2)}{\Delta x_{++}^2}\Big\} + \frac{4i\pi^{2-\frac{\varepsilon}{2}}\mu^{-(q-1)\varepsilon}}{\varepsilon\Gamma(1-\frac{\varepsilon}{2})}\delta^{(D)}(x-x'),$$

$$\frac{\partial^2}{\varepsilon}\frac{1}{\Delta x_{+-}^{2-q\varepsilon}} = \frac{\partial^2}{\varepsilon}\Big\{\frac{1}{\Delta x_{+-}^{2-q\varepsilon}} - \frac{\mu^{-(q-1)\varepsilon}}{\Delta x_{+-}^{2-\varepsilon}}\Big\}$$

$$= (q-1)\frac{\partial^2}{2}\Big\{\frac{\log(\mu^2\Delta x_{+-}^2)}{\Delta x_{+-}^2}\Big\},$$

where we introduce the mass parameter μ to correct the dimension. It should be noted that we set $\varepsilon = 0$ except at the coefficient of the delta function after this process.

By substituting (D.7) and (D.9), the integral of (D.2) is

$$\int d^{4-\varepsilon}x'\, G(a(\tau'))[\frac{1}{\Delta x_{++}^{p-q\varepsilon}} - \frac{1}{\Delta x_{+-}^{p-q\varepsilon}}] \tag{D.10}$$

$$= \frac{-1}{(p-2-q\varepsilon)!!\{p-4-(q-1)\varepsilon\}!!}(-\partial_0^2)^{\frac{p-4}{2}}\int dx^{4-\varepsilon}\, G(a(\tau'))$$

$$\times\Big\{\frac{4i\pi^{2-\frac{\varepsilon}{2}}\mu^{-\varepsilon}}{(q-1)\varepsilon\Gamma(1-\frac{\varepsilon}{2})}\delta^{(D)}(x-x') + \frac{\partial^2}{2}\Big[\frac{\log(\mu^2\Delta x_{++}^2)}{\Delta x_{++}^2} - \frac{\log(\mu^2\Delta x_{+-}^2)}{\Delta x_{+-}^2}\Big]\Big\},$$

where !! means the double factorial. To evaluate the UV finite part of (D.10), we use the following identities

$$\frac{\log(\mu^2\Delta x^2)}{\Delta x^2} = \frac{1}{8}\partial^2\big\{\log^2(\mu^2\Delta x^2) - 2\log(\mu^2\Delta x^2)\big\}, \tag{D.11}$$

$$\log(\mu^2\Delta x_{++}^2) = \log(\mu^2|\Delta\tau^2 - r^2|) + i\pi\theta(\Delta\tau^2 - r^2), \tag{D.12}$$
$$\log(\mu^2\Delta x_{+-}^2) = \log(\mu^2|\Delta\tau^2 - r^2|) - i\pi\theta(\Delta\tau^2 - r^2)\{\theta(\Delta\tau) - \theta(-\Delta\tau)\},$$

where $\Delta\tau = \tau - \tau'$, $r \equiv |\mathbf{x} - \mathbf{x}'|$. From these identities,

$$\int d^4x'\, G(a(\tau'))\frac{\partial^2}{2}\Big[\frac{\log(\mu^2\Delta x_{++}^2)}{\Delta x_{++}^2} - \frac{\log(\mu^2\Delta x_{+-}^2)}{\Delta x_{+-}^2}\Big] \tag{D.13}$$

$$= i\pi^2\partial_0^4\int_{-\frac{1}{H}}^{\tau} d\tau'\, G(a(\tau'))\int_0^{\Delta\tau} r^2 dr\,\big\{\log\big(\mu^2(\Delta\tau^2 - r^2)\big) - 1\big\}$$

$$= i\pi^2\partial_0^4\int_{-\frac{1}{H}}^{\tau} d\tau'\, G(a(\tau'))\Delta\tau^3\big\{\frac{2}{3}\log(2\mu\Delta\tau) - \frac{11}{9}\big\}$$

$$= 4i\pi^2\partial_0\int_{-\frac{1}{H}}^{\tau} d\tau'\, G(a(\tau'))\log(2\mu\Delta\tau)$$

$$= 4i\pi^2\Big\{G(a(\tau))\big(\log\frac{2\mu}{H} - \log a(\tau)\big) - a^2(\tau)\frac{\partial}{\partial a(\tau)}\int_1^{a(\tau)} da(\tau')\, G(a(\tau'))\sum_{n=1}^{\infty}\frac{a^{n-2}(\tau')}{na^n(\tau)}\Big\}.$$

By substituting (D.13) to (D.10),

$$\int d^{4-\varepsilon}x' \, G(a(\tau'))[\frac{1}{\Delta x_{++}^{p-q\varepsilon}} - \frac{1}{\Delta x_{+-}^{p-q\varepsilon}}] \tag{D.14}$$

$$= \frac{-4i\pi^2(-1)^{\frac{p-4}{2}}H^{p-4}}{(p-2-q\varepsilon)!!\{p-4-(q-1)\varepsilon\}!!}$$

$$\times \left(a^2(\tau)\frac{\partial}{\partial a(\tau)}\right)^{p-4}\left\{G(a(\tau))\left(\frac{\pi^{-\frac{\varepsilon}{2}}\mu^{-\varepsilon}}{(q-1)\varepsilon\Gamma(1-\frac{\varepsilon}{2})} + \log\frac{2\mu}{H} - \log a(\tau)\right)\right.$$

$$\left. - a^2(\tau)\frac{\partial}{\partial a(\tau)}\int_1^{a(\tau)} da(\tau') \, G(a(\tau'))\sum_{n=1}^{\infty}\frac{a^{n-2}(\tau')}{na^n(\tau)}\right\}.$$

In addition, the integrals of (D.4) and (D.5) are

$$\int d^{4-\varepsilon}x' \, G(a(\tau'))[\frac{\Delta x_\rho}{\Delta x_{++}^{p+2-q\varepsilon}} - \frac{\Delta x_\sigma}{\Delta x_{+-}^{p+2-q\varepsilon}}] \tag{D.15}$$

$$= \frac{4i\pi^2(-1)^{\frac{p-4}{2}}H^{p-3}}{(p-q\varepsilon)!!\{p-4-(q-1)\varepsilon\}!!}\delta_\rho^{\ 0}$$

$$\times \left(a^2(\tau)\frac{\partial}{\partial a(\tau)}\right)^{p-3}\left\{G(a(\tau))\left(\frac{\pi^{-\frac{\varepsilon}{2}}\mu^{-\varepsilon}}{(q-1)\varepsilon\Gamma(1-\frac{\varepsilon}{2})} + \log\frac{2\mu}{H} - \log a(\tau)\right)\right.$$

$$\left. - a^2(\tau)\frac{\partial}{\partial a(\tau)}\int_1^{a(\tau)} da(\tau') \, G(a(\tau'))\sum_{n=1}^{\infty}\frac{a^{n-2}(\tau')}{na^n(\tau)}\right\},$$

$$\int d^{4-\varepsilon}x' \, G(a(\tau'))[\frac{\Delta x_\rho\Delta x_\sigma}{\Delta x_{++}^{p+4-q\varepsilon}} - \frac{\Delta x_\rho\Delta x_\sigma}{\Delta x_{+-}^{p+4-q\varepsilon}}] \tag{D.16}$$

$$= \frac{-4i\pi^2(-1)^{\frac{p-2}{2}}H^{p-2}}{(p+2-q\varepsilon)!!\{p-4-(q-1)\varepsilon\}!!}\left\{\frac{\eta_{\rho\sigma}}{p-2-(q-1)\varepsilon} - \delta_\rho^{\ 0}\delta_\sigma^{\ 0}\right\}$$

$$\times \left(a^2(\tau)\frac{\partial}{\partial a(\tau)}\right)^{p-2}\left\{G(a(\tau))\left(\frac{\pi^{-\frac{\varepsilon}{2}}\mu^{-\varepsilon}}{(q-1)\varepsilon\Gamma(1-\frac{\varepsilon}{2})} + \log\frac{2\mu}{H} - \log a(\tau)\right)\right.$$

$$\left. - a^2(\tau)\frac{\partial}{\partial a(\tau)}\int_1^{a(\tau)} da(\tau') \, G(a(\tau'))\sum_{n=1}^{\infty}\frac{a^{n-2}(\tau')}{na^n(\tau)}\right\}.$$

The calculation of the remaining time integral depends on the explicit form of $G(a(\tau'))$.

We also refer to the case: $p \leq 2$ in (D.2). Note that the integral (D.1) has no UV divergence in this case and so we set $\varepsilon = 0$. The possible integrands are as follows

$$\log^r(H^2\Delta x^2), \quad \frac{1}{\Delta x^2}\log^r(H^2\Delta x^2). \tag{D.17}$$

where r is a non-negative integer. The latter integrand can be represented as the derivative of a polynomial in logarithms

$$\frac{1}{\Delta x^2}\log^r(H^2\Delta x^2) = \frac{1}{4}\partial^2\sum_{m=0}^{r}(-1)^m\frac{r!}{(r+1-m)!}\log^{r+1-m}(H^2\Delta x^2). \tag{D.18}$$

(D.11) is an instance of this identity at $r = 1$. In a similar way to the case: $p \geq 4$, $q \geq 2$, we can perform the integrals of these logarithms by using (D.12).

As a concrete example, we evaluate the integrals containing $H^1_{\rho\sigma}$ in (6.19). This integral can be rewritten in the form (D.1):

$$\int d^{4-\varepsilon}x' \, a^{4-\varepsilon}(\tau') \log a(\tau') H^1_{\rho\sigma} \tag{D.19}$$

$$= 4H^{-6+2\varepsilon}a^{-4+\varepsilon}(\tau) \int d^{4-\varepsilon}x' \, a^2(\tau') \log a(\tau') [\frac{\Delta x_\rho \Delta x_\sigma}{\Delta x_{++}^{8-2\varepsilon}} - \frac{\Delta x_\rho \Delta x_\sigma}{\Delta x_{+-}^{8-2\varepsilon}}].$$

From (D.16),

$$\int d^{4-\varepsilon}x' \, a^{4-\varepsilon}(\tau') \log a(\tau') H^1_{\rho\sigma} \tag{D.20}$$

$$= \frac{4i\pi^2 \cdot 4H^{-4+2\varepsilon}a^{-4+\varepsilon}}{(6-2\varepsilon)(4-2\varepsilon)(2-2\varepsilon)} \{ \frac{\eta_{\rho\sigma}}{2-\varepsilon} - \delta_\rho{}^0 \delta_\sigma{}^0 \}$$

$$\times \left(a^2(\tau) \frac{\partial}{\partial a(\tau)} \right)^2 \left\{ a^2(\tau) \log a(\tau) \left(\frac{\pi^{-\frac{\varepsilon}{2}} \mu^{-\varepsilon}}{\varepsilon \Gamma(1-\frac{\varepsilon}{2})} + \log \frac{2\mu}{H} - \log a(\tau) \right) \right.$$

$$\left. - a^2(\tau) \frac{\partial}{\partial a(\tau)} \int_1^{a(\tau)} da(\tau') \, \log a(\tau') \sum_{n=1}^{\infty} \frac{a^n(\tau')}{na^n(\tau)} \right\}$$

$$\simeq \frac{4i\pi^2 \cdot 4H^{-4+2\varepsilon}a^{-4+\varepsilon}}{(6-2\varepsilon)(4-2\varepsilon)(2-2\varepsilon)} \{ \frac{\eta_{\rho\sigma}}{2-\varepsilon} - \delta_\rho{}^0 \delta_\sigma{}^0 \}$$

$$\times \left\{ 6a^4(\tau) \log a(\tau) \left(\frac{\pi^{-\frac{\varepsilon}{2}} \mu^{-\varepsilon}}{\varepsilon \Gamma(1-\frac{\varepsilon}{2})} + \log \frac{2\mu}{H} - \log a(\tau) \right) \right.$$

$$\left. + 5a^4(\tau) \frac{\pi^{-\frac{\varepsilon}{2}} \mu^{-\varepsilon}}{\varepsilon \Gamma(1-\frac{\varepsilon}{2})} - 16a^4(\tau) \log a(\tau) \right\}$$

$$\simeq \eta_{\rho\sigma} \times 4i\pi^2 H^{-4} \left\{ \frac{1}{4} \left(\frac{\pi^{-\frac{\varepsilon}{2}} \mu^{-\varepsilon} H^{2\varepsilon}}{\varepsilon \Gamma(1-\frac{\varepsilon}{2})} + \log \frac{2\mu}{H} \right) \log a(\tau) + \frac{1}{8} \log a(\tau) \right\}$$

$$+ \delta_\rho{}^0 \delta_\sigma{}^0 \times 4i\pi^2 H^{-4} \left\{ -\frac{1}{2} \left(\frac{\pi^{-\frac{\varepsilon}{2}} \mu^{-\varepsilon} H^{2\varepsilon}}{\varepsilon \Gamma(1-\frac{\varepsilon}{2})} + \log \frac{2\mu}{H} \right) \log a(\tau) \right\}.$$

Here we extract the terms which are proportional to $\log a(\tau)$. The integrals containing $H^m_{\rho\sigma}, m = 2, \cdots, 6$ in (6.18) and $I^m_{\rho\sigma}$ except with $m = 4, 8$ in (7.22) are calculated analogously.

D.2 $\quad p \geq 4$, $q = 1$ case: Integrals containing (7.26) and (7.30)

We need a special consideration in the $p \geq 4$, $q = 1$ case of (D.2). As a example, we evaluate the integrals (7.26) and (7.30). These integrals consist of the two parts, one part containing $1/\Delta x^{p-2\varepsilon}$ and the other part containing $1/\Delta x^{p-\varepsilon}$. Specifically the integral containing $I^4_{\rho\sigma} + I^8_{\rho\sigma}$

is written as follows

$$\int d^{4-\varepsilon}x' a^{4-\varepsilon}(\tau') \log a(\tau')(I_{\rho\sigma}^4 + I_{\rho\sigma}^8) \tag{D.21}$$

$$= \frac{\Gamma(4 - \frac{\varepsilon}{2})}{4\Gamma(2 - \frac{\varepsilon}{2})} H^{-4+2\varepsilon} a^{-2+\varepsilon}(\tau) \int d^{4-\varepsilon}x' a^2(\tau') \log a(\tau') \left\{ \frac{\eta_{\rho\sigma}}{\Delta x^{4-2\varepsilon}} - (4 - \varepsilon) \frac{a(\tau')}{a(\tau)} \frac{\Delta x_\rho \Delta x_\sigma}{\Delta x^{6-2\varepsilon}} \right\}$$

$$- \frac{\Gamma(4 - \varepsilon)}{2\Gamma(3 - \frac{\varepsilon}{2})\Gamma(2 - \frac{\varepsilon}{2})} 4^{\frac{\varepsilon}{2}} H^{-4+\varepsilon} a^{-2+\frac{\varepsilon}{2}}(\tau) \int d^{4-\varepsilon}x' a^{2-\frac{\varepsilon}{2}}(\tau') \log a(\tau') \left\{ \frac{\eta_{\rho\sigma}}{\Delta x^{4-\varepsilon}} \right.$$

$$\left. - (4 - \varepsilon) \frac{a(\tau')}{a(\tau)} \frac{\Delta x_\rho \Delta x_\sigma}{\Delta x^{6-\varepsilon}} \right\}.$$

We can evaluate the part containing $1/\Delta x^{p-2\varepsilon}$ by the procedure which is introduced above

$$\frac{\Gamma(4 - \frac{\varepsilon}{2})}{4\Gamma(2 - \frac{\varepsilon}{2})} H^{-4+2\varepsilon} a^{-2+\varepsilon}(\tau) \int d^{4-\varepsilon}x' a^2(\tau') \log a(\tau') \left\{ \frac{\eta_{\rho\sigma}}{\Delta x^{4-2\varepsilon}} - (4 - \varepsilon) \frac{a(\tau')}{a(\tau)} \frac{\Delta x_\rho \Delta x_\sigma}{\Delta x^{6-2\varepsilon}} \right\}$$

$$\simeq 4i\pi^2 H^{-4} \log a(\tau) \eta_{\rho\sigma} \cdot \frac{-3}{16}. \tag{D.22}$$

When we calculate the part containing $1/\Delta x^{p-\varepsilon}$, we should note that the UV divergences at $\Delta x \sim 0$ are not regularized in the following integrals even if $\varepsilon > 0$

$$\int d^{4-\varepsilon}x' \frac{1}{\Delta x^{p-\varepsilon}}, \quad p \geq 4. \tag{D.23}$$

So we have to combine the terms in the integral so that these ill-defined terms don't appear. Herein the part containing $1/\Delta x^{p-\varepsilon}$ is rewritten as follows

$$- \frac{\Gamma(4 - \varepsilon)}{2\Gamma(3 - \frac{\varepsilon}{2})\Gamma(2 - \frac{\varepsilon}{2})} 4^{\frac{\varepsilon}{2}} H^{-4+\varepsilon} a^{-2+\frac{\varepsilon}{2}}(\tau) \int d^{4-\varepsilon}x' a^{2-\frac{\varepsilon}{2}}(\tau') \log a(\tau') \left\{ \frac{\eta_{\rho\sigma}}{\Delta x^{4-\varepsilon}} \right.$$

$$\left. - (4 - \varepsilon) \frac{a(\tau')}{a(\tau)} \frac{\Delta x_\rho \Delta x_\sigma}{\Delta x^{6-\varepsilon}} \right\}$$

$$= - \frac{\Gamma(4 - \varepsilon)}{2\Gamma(3 - \frac{\varepsilon}{2})\Gamma(2 - \frac{\varepsilon}{2})} 4^{\frac{\varepsilon}{2}} H^{-4+\varepsilon} a^{-2+\frac{\varepsilon}{2}}(\tau) \int d^{4-\varepsilon}x' a^{2-\frac{\varepsilon}{2}}(\tau') \log a(\tau') \left\{ \frac{\eta_{\rho\sigma}}{\Delta x^{4-\varepsilon}} \right.$$

$$\left. - (4 - \varepsilon) \frac{\Delta x_\rho \Delta x_\sigma}{\Delta x^{6-\varepsilon}} + (4 - \varepsilon) a(\tau') \frac{H \Delta \tau \Delta x_\rho \Delta x_\sigma}{\Delta x^{6-\varepsilon}} \right\}. \tag{D.24}$$

By the power counting, it is found that the term containing $\Delta \tau \Delta x_\rho \Delta x_\sigma / \Delta x^{6-\varepsilon}$ is not divergent. To evaluate this term, we use the following identity

$$\frac{\Delta \tau \Delta x_\rho \Delta x_\sigma}{\Delta x^{6-\varepsilon}} = \frac{-1}{(4 - \varepsilon)(2 - \varepsilon)} \left\{ -\eta_{\rho\sigma} \partial_0 \frac{1}{\Delta x^{2-\varepsilon}} + \partial_0 \partial_\sigma \frac{\Delta x_\rho}{\Delta x^{2-\varepsilon}} \right.$$

$$\left. + (\delta_\rho{}^0 \partial_\sigma + \delta_\sigma{}^0 \partial_\rho) \frac{1}{\Delta x^{2-\varepsilon}} \right\}. \tag{D.25}$$

It is found that the residual term has no UV divergence from the following identities

$$\frac{\eta_{\rho\sigma}}{\Delta x_{++}^{4-\varepsilon}} - (4 - \varepsilon) \frac{\Delta x_\rho \Delta x_\sigma}{\Delta x_{++}^{6-\varepsilon}} = \frac{-1}{2 - \varepsilon} \left\{ \partial_\rho \partial_\sigma \frac{1}{\Delta x_{++}^{2-\varepsilon}} + \frac{2(2 - \varepsilon) i e \delta(\Delta \tau) \delta_\rho{}^0 \delta_\sigma{}^0}{\Delta x_{++}^{4-\varepsilon}} \right\},$$

$$\frac{\eta_{\rho\sigma}}{\Delta x_{+-}^{4-\varepsilon}} - (4 - \varepsilon) \frac{\Delta x_\rho \Delta x_\sigma}{\Delta x_{+-}^{6-\varepsilon}} = \frac{-1}{2 - \varepsilon} \partial_\rho \partial_\sigma \frac{1}{\Delta x_{+-}^{2-\varepsilon}}. \tag{D.26}$$

By substituting (D.25) and (D.26) to (D.24) and extract the terms which are proportional to $\eta_{\rho\sigma} \log a(\tau)$,

$$
-\frac{\Gamma(4-\varepsilon)}{2\Gamma(3-\frac{\varepsilon}{2})\Gamma(2-\frac{\varepsilon}{2})} 4^{\frac{\varepsilon}{2}} H^{-4+\varepsilon} a^{-2+\frac{\varepsilon}{2}}(\tau) \int d^{4-\varepsilon} x' a^{2-\frac{\varepsilon}{2}}(\tau') \log a(\tau') \left\{ \frac{\eta_{\rho\sigma}}{\Delta x^{4-\varepsilon}} \right.
$$
$$
\left. - (4-\varepsilon)\frac{\Delta x_\rho \Delta x_\sigma}{\Delta x^{6-\varepsilon}} + (4-\varepsilon)a(\tau')\frac{H\Delta\tau\Delta x_\rho \Delta x_\sigma}{\Delta x^{6-\varepsilon}} \right\}
$$
$$
\rightarrow -\eta_{\rho\sigma}\frac{3}{4} H^{-3} a^{-2}(\tau)\partial_0 \int d^4 x' a^3(\tau') \log a(\tau') \times \left[\frac{1}{\Delta x_{++}^2} - \frac{1}{\Delta x_{+-}^2} \right]
$$
$$
\simeq 4i\pi^2 H^{-4} \log a(\tau)\, \eta_{\rho\sigma} \cdot \frac{3}{8}. \tag{D.27}
$$

From (D.21), (D.22) and (D.27), we obtain

$$
\int d^{4-\varepsilon} x' a^{4-\varepsilon}(\tau') \log a(\tau')(I_{\rho\sigma}^4 + I_{\rho\sigma}^8)
$$
$$
\simeq 4i\pi^2 H^{-4} \log a(\tau)\, \eta_{\rho\sigma} \cdot \frac{3}{16}. \tag{D.28}
$$

D.3 Two point function at the two loop level in the non-linear sigma model

Here we explain how to calculate the two point function up to $g^2 \log a(\tau)$. The two point function at the two loop level is written as

$$
\langle \xi^a \xi^a \rangle|_{g^2} = \int \sqrt{-g'} d^D x'\ i\frac{g^2}{3} R \lim_{x''\to x'} \partial'_\alpha \partial''_\beta G^{++}(x', x'') \tag{D.29}
$$
$$
\times g^{\alpha\beta}(\tau')\left[G^{++}(x, x')G^{++}(x, x') - G^{+-}(x, x')G^{+-}(x, x') \right]
$$
$$
- \int \sqrt{-g'} d^D x'\ i\frac{g^2}{6} R \partial'_\alpha G^{++}(x', x')
$$
$$
\times g^{\alpha\beta}(\tau')\partial'_\beta\left[G^{++}(x, x')G^{++}(x, x') - G^{+-}(x, x')G^{+-}(x, x') \right]
$$
$$
+ \int \sqrt{-g'} d^D x'\ \left\{ i\frac{g^2}{3} R G^{++}(x', x') - i(\delta\beta + 2\delta\gamma)R \right\}
$$
$$
\times g^{\alpha\beta}(\tau')\left[\partial'_\alpha G^{++}(x, x')\partial'_\beta G^{++}(x, x') - \partial'_\alpha G^{+-}(x, x')\partial'_\beta G^{+-}(x, x') \right].
$$

By using the partial integration,

$$
\langle \xi^a \xi^a \rangle|_{g^2} = i\frac{g^2}{3} R \int \sqrt{-g'} d^D x'\ \lim_{x''\to x'} \partial'_\alpha \partial''_\beta G^{++}(x', x'') \tag{D.30}
$$
$$
\times g^{\alpha\beta}(\tau')\left[G^{++}(x, x')G^{++}(x, x') - G^{+-}(x, x')G^{+-}(x, x') \right]
$$
$$
- i\frac{g^2}{3} R \int \sqrt{-g'} d^D x'\ \partial'_\alpha G^{++}(x', x')
$$
$$
\times g^{\alpha\beta}(\tau')\partial'_\beta\left[G^{++}(x, x')G^{++}(x, x') - G^{+-}(x, x')G^{+-}(x, x') \right]
$$
$$
+ \frac{g^2 R H^4}{2^4 \cdot 3\pi^4} \log^2 a(\tau) + \frac{2g^2 R}{3} \frac{H^{2D-4}}{(4\pi)^D} \frac{\Gamma^2(D-1)}{\Gamma^2(\frac{D}{2})} \delta \log a(\tau).
$$

To evaluate the two point function, we have to calculate the remaining integrals up to $g^2 \log a(\tau)$. To perform it, we have only to extract the following part from the propagator at the separated point

$$G(x, x')G(x, x') \simeq \left(\frac{H^2}{4\pi^2}\right)^2 \Big\{ -\frac{1}{y} \log(H^2 \Delta x^2) + \frac{1}{4} \log^2(H^2 \Delta x^2) \qquad \text{(D.31)}$$
$$- (1 - \gamma) \log(H^2 \Delta x^2) \Big\}.$$

It applies to the case: $p \leq 2$ and so we set $D = 4$. From (6.10), (7.15), (7.18) and (D.31), the integrals are

$$i\frac{g^2}{3} R \int \sqrt{-g'} d^D x' \lim_{x'' \to x'} \partial'_\alpha \partial''_\beta G^{++}(x', x'') \qquad \text{(D.32)}$$
$$\times g^{\alpha\beta}(\tau') \big[G^{++}(x, x')G^{++}(x, x') - G^{+-}(x, x')G^{+-}(x, x') \big]$$
$$= -i\frac{g^2 R H^8}{2^7 \pi^6} \int d^4 x' \, a^4(\tau')$$
$$\times \Big[\{ -\frac{1}{y_{++}} \log(H^2 \Delta x^2_{++}) + \frac{1}{4} \log^2(H^2 \Delta x^2_{++}) - (1 - \gamma) \log(H^2 \Delta x^2_{++}) \}$$
$$- \{ -\frac{1}{y_{+-}} \log(H^2 \Delta x^2_{+-}) + \frac{1}{4} \log^2(H^2 \Delta x^2_{+-}) - (1 - \gamma) \log(H^2 \Delta x^2_{+-}) \} \Big],$$

$$- i\frac{g^2}{3} R \int \sqrt{-g'} d^D x' \, \partial'_\alpha G^{++}(x', x') \qquad \text{(D.33)}$$
$$\times g^{\alpha\beta}(\tau') \partial'_\beta \big[G^{++}(x, x')G^{++}(x, x') - G^{+-}(x, x')G^{+-}(x, x') \big]$$
$$= +i\frac{g^2 R H^7}{2^6 \cdot 3\pi^6} \int d^4 x' \, a^3(\tau')$$
$$\times \partial'_0 \Big[\{ -\frac{1}{y_{++}} \log(H^2 \Delta x^2_{++}) + \frac{1}{4} \log^2(H^2 \Delta x^2_{++}) - (1 - \gamma) \log(H^2 \Delta x^2_{++}) \}$$
$$- \{ -\frac{1}{y_{+-}} \log(H^2 \Delta x^2_{+-}) + \frac{1}{4} \log^2(H^2 \Delta x^2_{+-}) - (1 - \gamma) \log(H^2 \Delta x^2_{+-}) \} \Big].$$

By using (D.12) and (D.18), each integral is evaluated as:

$$\int d^4 x' \, a^4(\tau') \Big[-\frac{1}{y_{++}} \log(H^2 \Delta x^2_{++}) + \frac{1}{y_{+-}} \log(H^2 \Delta x^2_{+-}) \Big] \qquad \text{(D.34)}$$
$$\simeq -\frac{4i\pi^2}{H^4} \log a(\tau),$$

$$\int d^4 x' \, a^4(\tau') \Big[\frac{1}{4} \log^2(H^2 \Delta x^2_{++}) - \frac{1}{4} \log(H^2 \Delta x^2_{+-}) \Big] \qquad \text{(D.35)}$$
$$\simeq \frac{4i\pi^2}{3H^4} \big\{ -\log^2 a(\tau) + (2\log 2 + 1) \log a(\tau) \big\},$$

$$\int d^4x' \, a^4(\tau') \big[-(1-\gamma)\log(H^2\Delta x_{++}^2) + (1-\gamma)\log(H^2\Delta x_{+-}^2) \big] \tag{D.36}$$

$$\simeq -(1-\gamma)\frac{8i\pi^2}{3H^4}\log a(\tau),$$

$$\int d^4x' \, a^3(\tau')\partial_0' \Big[-\frac{1}{y_{++}}\log(H^2\Delta x_{++}^2) + \frac{1}{y_{+-}}\log(H^2\Delta x_{+-}^2) \Big] \tag{D.37}$$

$$\simeq \frac{12i\pi^2}{H^3}\log a(\tau),$$

$$\int d^4x' \, a^3(\tau')\partial_0' \Big[\frac{1}{4}\log^2(H^2\Delta x_{++}^2) - \frac{1}{4}\log(H^2\Delta x_{+-}^2) \Big] \tag{D.38}$$

$$\simeq \frac{4i\pi^2}{H^3}\big\{ \log^2 a(\tau) - (2\log 2 + 1)\log a(\tau) \big\},$$

$$\int d^4x' \, a^3(\tau')\partial_0' \big[-(1-\gamma)\log(H^2\Delta x_{++}^2) + (1-\gamma)\log(H^2\Delta x_{+-}^2) \big] \tag{D.39}$$

$$\simeq (1-\gamma)\frac{8i\pi^2}{H^3}\log a(\tau).$$

From (D.32), (D.33) and (D.34)-(D.39),

$$i\frac{g^2}{3}R \int \sqrt{-g'}d^Dx' \lim_{x''\to x'} \partial_\alpha'\partial_\beta''G^{++}(x',x'') \tag{D.40}$$
$$\times g^{\alpha\beta}(\tau') \big[G^{++}(x,x')G^{++}(x,x') - G^{+-}(x,x')G^{+-}(x,x') \big]$$
$$\simeq \frac{g^2RH^4}{2^5\cdot 3\pi^4}\big\{ -\log^2 a(\tau) + 2(\log 2 - 2 + \gamma)\log a(\tau) \big\},$$

$$-i\frac{g^2}{3}R \int \sqrt{-g'}d^Dx' \, \partial_\alpha'G^{++}(x',x') \tag{D.41}$$
$$\times g^{\alpha\beta}(\tau')\partial_\beta' \big[G^{++}(x,x')G^{++}(x,x') - G^{+-}(x,x')G^{+-}(x,x') \big]$$
$$\simeq \frac{g^2RH^4}{2^4\cdot 3\pi^4}\big\{ -\log^2 a(\tau) + 2(\log 2 - 2 + \gamma)\log a(\tau) \big\}.$$

By substituting (D.40) and (D.41) in (D.30),

$$\langle \xi^a\xi^a \rangle|_{g^2} \simeq \frac{g^2RH^4}{2^5\cdot 3\pi^4}\big\{ -\log^2 a(\tau) + 6(-2 + \log 2 + \gamma)\log a(\tau) \big\} \tag{D.42}$$
$$+ \frac{2g^2R}{3}\frac{H^{2D-4}}{(4\pi)^D}\frac{\Gamma^2(D-1)}{\Gamma^2(\frac{D}{2})}\delta\log a(\tau).$$

References

[1] J. S. Schwinger, J. Math. Phys. **2**, 407 (1961).

[2] L. V. Keldysh, Zh. Eksp. Teor. Fiz. **47**, 1515 (1964) [Sov. Phys. JETP **20**, 1018 (1965)].

[3] A.M. Polyakov, Nucl. Phys. B **797**, 199 (2008) [arXiv:0709.2899 [hep-th]].
A.M. Polyakov, Nucl. Phys. B **834**, 316 (2010) [arXiv:0912.5503 [hep-th]].

[4] L.P. Kadanoff, G. Baym, *Quantum Statistical Mechanics* (Benjamin, New York, 1962).

[5] T. Kita, J. Phys. Soc. Japan **75**, 114005 (2006).

[6] H. Kitamoto and Y. Kitazawa, Nucl. Phys. B **839**, 552 (2010) [arXiv:1004.2451 [hep-th]].

[7] A. Hohenegger, A. Kartavtsev and M. Lindner, Phys. Rev. D **78**, 085027 (2008) [arXiv:0807.4551 [hep-ph]].

[8] A. Vilenkin and L. H. Ford, Phys. Rev. D **26**, 1231 (1982).

[9] A. D. Linde, Phys. Lett. B **116**, 335 (1982).

[10] A. A. Starobinsky, Phys. Lett. B**117**, 175 (1982).

[11] N. C. Tsamis and R. P. Woodard, Nucl. Phys. B **474**, 235 (1996) [arXiv:hep-ph/9602315].
N. C. Tsamis and R. P. Woodard, Annals Phys. **253**, 1 (1997) [arXiv:hep-ph/9602316].
N. C. Tsamis and R. P. Woodard, Phys. Rev. D **54**, 2621 (1996) [arXiv:hep-ph/9602317].

[12] S. Weinberg, Phys. Rev. D**72**, 043514 (2005) [arXiv:hep-th/0506236].
S. Weinberg, Phys. Rev. D**74**, 023508 (2006) [arXiv:hep-th/0605244].

[13] V. K. Onemli and R. P. Woodard, Class. Quant. Grav. **19**, 4607 (2002) [arXiv:gr-qc/0204065].
V. K. Onemli, R. P. Woodard, Phys. Rev. **D70**, 107301 (2004) [gr-qc/0406098].

[14] A. A. Starobinsky, Lect. Notes Phys. **246**, 107 (1986).
A. A. Starobinsky and J. Yokoyama, Phys. Rev. D **50**, 6357 (1994) [arXiv:astro-ph/9407016].

[15] N. C. Tsamis and R. P. Woodard, Nucl. Phys. B **724**, 295 (2005) [arXiv:gr-qc/0505115].

[16] A. Rajaraman, Phys. Rev. D **82**, 123522 (2010) [arXiv:1008.1271 [hep-th]].

[17] H. Kitamoto and Y. Kitazawa, Phys. Rev. D **83**, 104043 (2011) [arXiv:1012.5930 [hep-th]].

[18] H. Kitamoto and Y. Kitazawa, arXiv:1109.4892 [hep-th].

[19] D. H. Lyth, JCAP **0712**, 016 (2007) [arXiv:0707.0361 [astro-ph]].

[20] T. M. Janssen, S. P. Miao, T. Prokopec, and R. P. Woodard, Class. Quant. Grav. **25**, 245013 (2008) [arXiv:0808.2449 [gr-qc]].

[21] S. P. Miao, N. C. Tsamis, R. P. Woodard, J. Math. Phys. **51**, 072503 (2010) [arXiv:1002.4037 [gr-qc]].

[22] B. Allen, A. Folacci, Phys. Rev. **D35**, 3771 (1987).

[23] A. Folacci, J. Math. Phys. **32**, 2828-2838 (1991).

[24] N. D. Birrell and P. C. W. Davies, Quantum fields in curved space (Cambridge University Press, Cambridge, 1982).

[25] S. P. Miao and R. P. Woodard, Phys. Rev. D **74**, 044019 (2006) [arXiv:gr-qc/0602110].

[26] E. O. Kahya and R. P. Woodard, Phys. Rev. D**74**, 084012 (2006) [arXiv:gr-qc/0608049]. T. Prokopec, N. C. Tsamis and R. P. Woodard, Annals Phys. **323**, 1324 (2008) [arXiv:0707.0847 [gr-qc]].

[27] T. Prokopec, N. C. Tsamis and R. P. Woodard, Class. Quant. Grav. **24**, 201 (2007) [arXiv:gr-qc/0607094].

[28] L. Alvarez-Gaume, D. Z. Freedman and S. Mukhi, Annals Phys. **134**, 85 (1981).

[29] R. L. Davis, Phys. Rev. D **45**, 2155 (1992).

[30] J. Garriga and T. Tanaka, Phys. Rev. D **77**, 024021 (2008) [arXiv:0706.0295 [hep-th]].

[31] N. C. Tsamis and R. P. Woodard, Phys. Rev. D **78**, 028501 (2008) [arXiv:0708.2004 [hep-th]].

[32] Y. Urakawa and T. Tanaka, arXiv:1007.0468 [hep-th]. Y. Urakawa and T. Tanaka, arXiv:1009.2947 [hep-th].

[33] A. Higuchi, D. Marolf and I. A. Morrison, arXiv:1107.2712 [hep-th].

[34] S. P. Miao, N. C. Tsamis and R. P. Woodard, arXiv:1107.4733 [gr-qc].

[35] S. B. Giddings and M. S. Sloth, JCAP **1101**, 023 (2011) [arXiv:1005.1056 [hep-th]].

[36] D. Chialva and A. Mazumdar, arXiv:1103.1312 [hep-th].