BRANE DYNAMICS IN M-THEORY

YOSHINORI HONMA

Institute of Particle and Nuclear Studies
High Energy Accelerator Research Organization (KEK)
and
Department of Particle and Nuclear Physics
The Graduate University for Advanced Studies (SOKENDAI)

Oho 1-1, Tsukuba, Ibaraki 305-0801, Japan

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School of High Energy Accelerator Science
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PREFACE

M-theory is an eleven dimentional theory and provides a useful framework to understand the nonperturbative physics of superstring theory. M-theory can be regarded as a strong coupling limit of Type IIA superstring theory and is related to other superstring theories through the S, T and U-dualities. M2-branes and M5-branes exist as BPS objects and these branes reduces to D-branes, NS5-branes, Kaluza-Klein monopoles and fundamental strings in superstring theory. Until recently, the low energy effective theory of multiple M-theory branes has not been known. However, triggered by the pioneer papers [1, 2, 3], fruitful developments about the multiple M2-branes have been achieved in the recent past.

One of the novelties in the developments is the appearance of Lie 3-algebra $[T^a, T^b, T^c] = f^{abc}{}_d T^d$ for the gauge symmetry, and the theory based on this algebra has appropriate symmetries as the effective theory of multiple M2-branes. This is called Bagger-Lambert-Gustavsson (BLG) theory. For the concrete expressions of Lie 3-algebra, it is known that the following theories with maximal supersymmetry can be derived from the original BLG theory: $\mathcal{A}_4$ BLG theory for two M2-branes [2], Lorentzian BLG theory for multiple D2-branes [4, 5, 6], extended Lorentzian BLG theory for multiple Dp-branes ($p > 2$) [7, 8], and Nambu-Poisson worldvolume theory for a single M5-brane [9, 10] or finite number of multiple M2-branes [11]. Another approach to construct the action of multiple M2-branes is given by [12], and this Aharony-Bergman-Jafferis-Maldacena (ABJM) theory describes an arbitrary number N of multiple M2-branes on an orbifold $\mathbb{C}^4/\mathbb{Z}_k$. This theory has $U(N) \times U(N)$ gauge symmetry and only in special cases it can have a maximal supersymmetry. In fact, ABJM theory in a certain scaling limit reproduces Lorentzian BLG theory [13], and the latter theory can be reduced to the 3-dim super Yang-Mills theory through the new kind of Higgs mechanism [14]. Therefore, the relation between M2-branes and D2-branes can be understood only in the viewpoint of Lagrangians [13, 15, 16] (see also [17, 18]). In addition, when we start from the extended Lorentzian BLG theory [7, 8] or the orbifolded ABJM theory [19, 20], we obtain Dp-branes whose worldvolume is a flat torus T^{p-2} bundle over the membrane worldvolume. In these cases, the moduli of torus compactification of M-theory is properly realized, and the U-duality transformation can be expressed in terms of Lie 3-algebra or the quiver of Lie groups.

On the other hand, there has been a long time mystery about M5-brane. It is known that the low energy dynamics of M5-brane is described by 6-dim $(2, 0)$ SCFT, and that the field contents are five scalars, a spinor and a self-dual 2-form field. However, the covariant description of the self-dual field is not easy, and thus only the covariant action of single M5-brane is known [21, 22, 23]. For the multiple M5-brane dynamics, it has not been known even in the level of the equations of motion. Recently, however, Lambert and Papageorgakis [24] proposed a set of equations of motion of the nonabelian $(2, 0)$ theory by using the Lie 3-algebra, which may shed light on the underlying cause of the mystery. Starting from the supersymmetry transformations of the multiple D4-branes theory, they conjectured those of the nonabelian

$(2,0)$ theory. Note that they introduce an auxiliary field which doesn't appear in the abelian case. Although this theory seems simply reduced to 5-dim super Yang-Mills theory and might be nothing more than the reformulation of D4-brane theory, this is the first step toward the covariant description of multiple M5-branes.

This thesis is organized as follows. In part I, we give a brief review about M-theory and its brane solution. According to the AdS/CFT correspondence, we can extract the expected properties about dual field theories. In part II, we take a quick look at the recent developments about multiple M2-branes. There are two types of Lie 3-algebras classified by the metric of generators, namely Euclidean and Lorentzian. We first explain the general reduction of the Lorentzian-BLG theory to D2-brane theory and confirm that the Lorentzian-BLG theory can be regarded as a reformulation of D2-brane theory. However, such a formulation of Lorentzian-BLG theory in terms of ordinary gauge theory enables us to connect this theory to the ABJM theory.

Then, in part III, we confirm that the 3-dim $\mathcal{N}=8$ BLG theory based on the Lorentzian type 3-algebra can be derived by taking a certain scaling limit of 3d $\mathcal{N}=6$ $U(N)_k \times U(N)_{-k}$ ABJM theory whose moduli space is $Sym^N(\mathbb{C}^4/\mathbb{Z}_k)$. The scaling limit which can be interpreted as the Inönü-Wigner contraction is to scale the trace part of the bifundamental fields and an axial combination of the two gauge fields. Simultaneously we scale the Chern-Simons level. In this scaling limit, M2-branes are located far from the origin of $\mathbb{C}^4/\mathbb{Z}_k$ compared to their fluctuations and $\mathbb{Z}_k$ identification becomes a circle identification. Furthermore, we show that the BLG theory with two pairs of negative norm generators is derived from the scaling limit of an orbifolded ABJM theory. The BLG theory with many Lorentzian pairs is known to be reduced to the Dp-brane theory via the Higgs mechanism. Therefore our scaling procedure can be used to derive Dp-branes from M2-branes. We also investigate the scaling limits of various quiver Chern-Simons theories obtained from different orbifoldings. Remarkably, in the case of $\mathcal{N}=2$ quiver CS theories, the resulting D3-brane action covers a larger region in the parameter space of the complex structure moduli than the $\mathcal{N}=4$ quiver CS theories. How the $SL(2,Z)$ duality transformation is realized in the resultant D3-brane theory is also discussed.

Moreover, we explain the recent progress on the application of Lie 3-algebra to M5-branes. For M5-branes, its nonabelian action has not been discovered due to the lack of understanding about consistent coupling between arbitrary number of tensor multiplets and Yang-Mills multiplets. Recently, however, it was suggested that the equations of motion of M5-branes can be constructed by using Lie 3-algebra. We describe its consistency with the known string dualities and confirm that the proposed system has to be modified to realize the dynamics of multiple M5-branes [25]. We also comment about type IIA/IIB NS5-brane and Kaluza-Klein monopoles by taking various compactification cycles. Because both longitudinal and transverse directions to the worldvolume can be compactified in the proposed model, we can realize these systems. This situation is entirely different from the case of BLG theory. Realization of the moduli parameters in the U-duality group is also discussed.

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Contents

Part I

Foundations of M-theory

Chapter 1

M-theory

1.1 11-dim supergravity

M-theory is an eleven-dimensional quantum theory whose low energy effective action is given by 11-dim $\mathcal{N} = 1$ supergravity

$$S = \frac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{-g} \left(R - \frac{1}{48} F_{mnkl} F^{mnkl} \right) - \frac{1}{2\kappa_{11}^2} \int \frac{1}{3!} C_3 \wedge F_4 \wedge F_4. \tag{1.1}$$

Here $F_4 = dC_3$ and κ_{11} is a 11-dim gravitational coupling constant which is related to 11-dim Newton's constant and Planck length as

$$2\kappa_{11}^2 = 16\pi G_{11} = \frac{1}{2\pi} (2\pi l_p)^9. \tag{1.2}$$

The field content of 11-dim supergravity is quite simple. It consists of the vierbein E_m^a, a Majorana spin 3/2 field (gravitino) ψ_m and a completely antisymmetric tensor C_{mnl} where $m, n, l = 1, \cdots, 11$ are spacetime indices and a is a tangent space index. The action (1.1) is invariant under the following supersymetry transformations

$$\begin{aligned}
\delta E_m^a &= \bar{\epsilon} \Gamma^a \psi_m, \\
\delta \psi_m &= \partial_m \epsilon + \frac{1}{4} \omega_{mab} \Gamma^{ab} \epsilon + \frac{1}{288} F_{nklp} \left(\Gamma_m \Gamma^{nklp} + 12 \Gamma^{nkl} \delta_m^p \right) \epsilon, \\
\delta C_{mnl} &= -3 \bar{\epsilon} \Gamma_{[mn} \psi_{l]}.
\end{aligned} \tag{1.3}$$

Note that the introduction of cosmological constant is not allowed by supersymmetry.

Now we consider the Kaluza-Klein reduction and reduce the 11-dim supergravity to 10-dim. We take the eleven-dimensional metric to be

$$ds^2 = e^{-2\Phi/3} g_{\mu\nu} dx^\mu dx^\nu + e^{4\Phi/3} (dx^{11} + A_\mu dx^\mu)^2 \tag{1.4}$$

to describe ten-dimentional metric along with a 1-form A_1, dilaton ϕ. The 3-form C_3 reduces to the R-R 3-form and the NS-NS 2-form through a proper rescaling. Finally we obtain the 10-dim Type IIA supergravity and its string coupling constant g_s is given by e^Φ. From (1.4), we find that $l_p = g_s^{1/3} l_s$. Through the KK-reduction on a circle of radius R_{11}, the Newton's constant in 11-dim and 10-dim are related as

$$G_{11} = 2\pi R_{11} G_{10}, \tag{1.5}$$

while the 10-dim Newton's constant is given by $16\pi G_{10} = (2\pi)^7 l_s^8 g_s^2$. Combining these with (1.2), we obtain the famous relation

$$R_{11} = g_s l_s. \tag{1.6}$$

This means that the strong coupling limit of Type IIA string theory is eleven dimensional. This is the M-theory.

1.2 M2-branes and M5-branes in 11-dim supergravity

Here we describe the brane solutions of 11-dim supergravity (1.1) obtained by solving the Killing spinor equation

$$\delta\psi_m = \partial_m \epsilon + \frac{1}{4}\omega_{mab}\Gamma^{ab}\epsilon + \frac{1}{288}F_{nklp}\left(\Gamma_m \Gamma^{nklp} + 12\Gamma^{nkl}\delta_m^p\right)\epsilon = 0 \tag{1.7}$$

We don't have to consider other SUSY variations because we take a bosonic background.

The flat coincident N M2-branes in 11-dim have $SO(1,2) \times SO(8)$ symmetry and the metric and 4-from field strength are given by

$$ds^2 = H(r)^{-2/3}\eta_{\mu\nu}dx^\mu dx^\nu + H(r)^{1/3}(dr^2 + r^2 d\Omega_7^2), \tag{1.8}$$

$$F_4 = dx^0 \wedge dx^1 \wedge dx^2 \wedge dH^{-1} \tag{1.9}$$

where $\mu, \nu = 0, 1, 2$ and $H(r)$ is the harmonic function on $\mathbb{R}_8$

$$H(r) = 1 + \frac{R^6}{r^6}. \tag{1.10}$$

Here $R = (32\pi^2 N)^{1/6} l_p$. Note that F_4 has nonzero time components and thus M2-branes are electrically coupled to the 4-form flux. In the near horizon limit this solution becomes $AdS_4 \times S^7$

$$\begin{aligned} ds^2 &= \left(\frac{r}{R}\right)^4 \eta_{\mu\nu}dx^\mu dx^\nu + \left(\frac{R}{r}\right)^2 dr^2 + R^2 d\Omega_7^2 \\ &= R^2\left[\frac{1}{4}ds^2_{AdS} + d\Omega_7^2\right], \end{aligned} \tag{1.11}$$

$$F_4 = \frac{3}{8}R^3 \epsilon_{AdS_4} \tag{1.12}$$

where we have rescaled the worldvolume coordinate of M2-branes and ϵ_{AdS_4} is a volume form of AdS_4 spacetime. According to the AdS/CFT correspondence, the dual field theory is expected to be a 3-dim $\mathcal{N} = 8$ SCFT with $SO(8)$ R-symmetry.

The flat coincident $\tilde{N}$ M5-branes in 11-dim have $SO(1,5)\times SO(5)$ symmetry and the metric and 4-from field strength are given by

$$ds^2 = H(r)^{-1/3}\eta_{\mu\nu}dx^\mu dx^\nu + H(r)^{2/3}(dr^2 + r^2 d\Omega_4^2), \tag{1.13}$$
$$F_4 = *(dx^0\wedge dx^1\wedge dx^2\wedge\cdots\wedge dx^5\wedge dH^{-1}) \tag{1.14}$$

where $\mu,\nu = 0,1,\cdots,5$ and $H(r)$ is the harmonic function on $\mathbb{R}_5$

$$H(r) = 1 + \frac{\tilde{R}^3}{r^3}. \tag{1.15}$$

Here $\tilde{R} = (\pi\tilde{N})^{1/3}l_p$. We can easily show that the near horizon geometry of this solution is $AdS_7\times S^4$ and we expect that dual CFT is 6-dim $\mathcal{N}=(2,0)$ SCFT with $SO(5)$ R-symmetry.

1.3 Exact vacua of M-theory

The on-shell 11-dimensional supergravity in superspace was formulated in [26]. There is a single superfield $W_{rstu}(x,\theta)$ whose local Lorentz indices are totally antisymmetric. All components of the supertorsion and supercurvatures can be expressed in terms of W_{rstu} and its first and second covariant derivatives. The first few components of this superfield are

$$W_{rstu}(x,\theta)|_{\theta=0} = \hat{F}_{rstu}(x), \tag{1.16}$$
$$(D_\alpha W_{rstu}(x,\theta))|_{\theta=0} = 6(\gamma_{[rs}\hat{D}_t\psi_{u]})(x), \tag{1.17}$$
$$\begin{aligned}(D_\alpha(\hat{D}_r\psi_{s]})_\beta)|_{\theta=0} = \Big(&\frac{1}{8}\hat{R}_{rsmn}\gamma^{mn} + \frac{1}{2}[T_r^{tuvw}, T_s^{xyzp}]\hat{F}_{tuvw}(x)\hat{F}_{xrzp}(x)\\ &+ T_{[s}^{tuvw}\hat{D}_{r]}\hat{F}_{tuvw}(x)\Big)_{\alpha\beta}\end{aligned} \tag{1.18}$$

where $\hat{F}_{rstu} = F_{rstu} - 3\bar{\psi}_r\gamma_{st}\psi_u$ is a (shifted) 4-form flux and $T^{rstuv} = (1/12^2)(\gamma^{rstuv} - 8\gamma^{[stu}\eta^{v]r})$. The equation of motion is

$$(\gamma^{rst}D)_\alpha W_{rstu}(x,\theta) = 0. \tag{1.19}$$

In a generic background we can write down corrections to the RHS of equation of motion involving superfields and derivatives of superfields. However, it was shown in [27] that there are no corrections to the $AdS_4\times S^7$ and $AdS_7\times S^4$ solutions in M-theory and thus they are exact.

The lowest component of the superfield W is given by 4-form flux. In the case of AdS_4 or AdS_7, 4-form flux is given by the volume form of AdS_4 or S^4 and these are covariantly constant. The next component of the superfield (1.17) is derivative of gravitino and this vanishes due to considering the bosonic background. From explicit computation or differentiating Killing spinor equation, we can verify the component (1.18) vanishes as well. The remaining higher components are given by some derivatives of the previous ones and thus all vanish. Therefore we see that W_{rstu} is supercovariantly constant.

Now we reconsider the correction to the equation of motion. Because W_{rstu} is supercovariantly constant, the possible corrections can depend only on W_{rstu} and other constant tensors like γ-matrices etc. The equation of motion is written in a form which have one free spinorial index and so do the corrections. Although it is impossible to construct the one spinorial index without using spinorial derivatives, however derivative terms are all nonzero. Therefore there is no possible correction we can write down. This means that the $AdS_4 \times S^7$ and $AdS_7 \times S^4$ spacetimes are exact vacua of M-theory.

1.3.1 M2-brane entropy from the gravity dual

For $n+1$ spacetime dimensions, the (Euclidean) gravitational action has two contributions

$$I_{bulk} + I_{surf} = -\frac{1}{16\pi G_N}\int_{\mathcal{M}} d^{n+1}x\ \sqrt{g}\left(R + \frac{2n(n-1)}{L^2}\right) - \frac{1}{8\pi G_N}\int_{\partial\mathcal{M}} d^n x\ \sqrt{h}K \tag{1.20}$$

where G_N is n-dimensional Newton's constant. The first term is the Einstein-Hilbert action with cosmological constant $\Lambda = -\frac{n(n-1)}{L^2}$. The second term is the Gibbons-Hawking term. Here K is the extrinsic curvature, h is the induced metric on the boundary. On the AdS background, both of these terms are divergent because of the noncompactness of the space. The modern approach to circumventing this problem is to perform a "counterterm subtraction" [28], namely a gravitational analogue of Minimal Subtraction scheme and the counterterm action is given by

$$I_{ct} = \frac{1}{8\pi G_N}\int_{\partial\mathcal{M}} d^n x\ \sqrt{h}\left[\frac{n-1}{L} + \frac{L}{2(n-2)}\mathcal{R} + \frac{L^3}{2(n-4)(n-2)^2}\left(\mathcal{R}_{ab}\mathcal{R}^{ab} - \frac{n}{4(n-1)}\mathcal{R}^2\right) + \cdots\right] \tag{1.21}$$

where $\mathcal{R}$ and $\mathcal{R}_{ab}$ are Ricciscalar and Ricci tensor for the induced metric h, respectively. These three terms are sufficient to cancel divergence for $n \leq 6$.

Now we explicitly compute on Euclidean AdS background which has a boundary $\mathbb{S}^n$. According to the AdS/CFT dictionary, it correspondes to the free energy of CFT on $\mathbb{S}^n$. As a metric of Euclidean AdS space, we choose

$$ds^2 = \frac{dr^2}{1+\frac{r^2}{L^2}} + r^2 d\Omega_n^2. \tag{1.22}$$

Then the bulk action is

$$I_{bulk} = \frac{n\mathrm{vol}(\mathbb{S}^n)}{8\pi G_N L}\int_0^r ds\ \frac{s^n}{\sqrt{L^2+s^2}} \tag{1.23}$$

where we computed with a cutoff at the boundary located at r. Finally we will take $r \to \infty$ limit. By using the useful relation

$$\sqrt{h}K = \mathcal{L}_n\sqrt{h} \tag{1.24}$$

and the expression of unit normal vector to the boundary as $n = \sqrt{1+r^2/L^2}\partial/\partial r$, we obtain

$$I_{surf} = -\frac{1}{8\pi G_N}\int_{\partial\mathcal{M}} d^n x\ \mathcal{L}_n\sqrt{h} = -\frac{nr^{n-1}}{8\pi G_N}\sqrt{1+\frac{r^2}{L^2}}\mathrm{vol}(\mathbb{S}^n). \tag{1.25}$$

Combining these terms with the first two counter terms, we obtain

$$I_{AdS_{n+1}} = I_{bulk} + I_{surf} + I_{ct} \tag{1.26}$$

$$= \frac{\mathrm{vol}(\mathbb{S}^n)}{8\pi G_N L}\left[\int_0^{r/L} dt\ \frac{t^n}{\sqrt{1+t^2}} - nr^{n-1}\sqrt{r^2+L^2} + r^n(n-1)\left(1+\frac{n}{2(n-2)}\frac{L^2}{r^2}\right)\right]. \tag{1.27}$$

Taking a limit $r \to \infty$, we find

$$I_{AdS_4} = \frac{\mathrm{vol}(\mathbb{S}^3)}{8\pi G_N L}\left(2L^3 + \mathcal{O}(1/r)\right) \approx \frac{\pi L^2}{2G_N}. \tag{1.28}$$

Let us rewrite this expression in terms of charge or number of M2-branes. As we will see later in part II of this thesis, the gravity dual of ABJM theory is known to be $AdS_4 \times \mathbb{S}^7/\mathbb{Z}_k$. The eleven dimensional metric and 4-from flux are given by

$$ds_{11}^2 = R^2\left(\frac{1}{4}ds^2_{AdS_4} + ds^2_{\mathbb{S}^7/\mathbb{Z}_k}\right), \tag{1.29}$$

$$F_4 = \frac{3}{8}R^3\epsilon_{AdS_4}. \tag{1.30}$$

The radius R is determined by the flux quantization condition

$$(2\pi l_p)^6 Q = \int_{\partial M_8} *F_4 = 6R^6 \mathrm{vol}(\mathbb{S}^7/\mathbb{Z}_k). \tag{1.31}$$

As expained in [29], the charge Q is related to the number of M2-branes as

$$Q = N - \frac{1}{24}\left(k - \frac{1}{k}\right). \tag{1.32}$$

The four dimensional Newton's constant is written as

$$\frac{1}{G_N} = \frac{2\sqrt{6}\pi^2 Q^{3/2}}{9\sqrt{\mathrm{vol}(\mathbb{S}^7/\mathbb{Z}_k)}}\frac{1}{R^2}. \tag{1.33}$$

Thus we finally obtain

$$I_{AdS_4} = \frac{\pi R^2}{2G_N} = Q^{3/2}\sqrt{\frac{2\pi^6}{27\mathrm{vol}(\mathbb{S}^7/\mathbb{Z}_k)}}. \tag{1.34}$$

In the large N limit, $Q \approx N$ and we find that the planar free energy

$$-F^{(0)}_{ABJM}(\mathbb{S}^3) = \sqrt{\frac{2\pi^6}{27\mathrm{vol}(\mathbb{S}^7/\mathbb{Z}_k)}}N^{3/2} = \frac{\sqrt{2}\pi}{3}k^2\lambda^{3/2}. \tag{1.35}$$

This is the famous strong coupling behaviour of the free energy of M2-branes.

1.4 Supergravity on $AdS_4\times$ Hopf fibrations

Here we consider the way to obtain the gravity duals of SCFTs with less than 16 supercharges. It is known that odd sphere can be considered to be a $U(1)$ fibration over $\mathbf{CP}^n$. Then the metric is given by

$$d\Omega_{2n+1} = d\Sigma_{2n}^2 + (dz + \mathcal{A})^2 \tag{1.36}$$

where the $d\Sigma_{2n}^2$ is the Fubini-Study metric of $\mathbf{CP}^n$ and 1-form potential $\mathcal{A}$ has a field strength given by $\mathcal{F} = 2J$ where J is the Kähler form of $\mathbf{CP}^n$. The coordinate z has a period 4π.

By taking S^7 to be a Hopf fibration over $\mathbf{CP}^3$, we obtain

$$ds_{10}^2 = ds^2(AdS_4) + d\Sigma_6^2 + (dz + \mathcal{A})^2. \tag{1.37}$$

Then we can Hopf reduce the $AdS_4 \times S^7$ over the $U(1)$ fiber and this gives the $AdS_4 \times CP^3$

$$ds_{10}^2 = ds^2(AdS_4) + d\Sigma_6^2 \tag{1.38}$$

which is a solution of 10-dim Type IIA supergravity. $SO(8)$ isometry of S^7 reduces to that of $\mathbf{CP}^3 \times U(1)$ which is $SU(4) \times U(1)$.

In the gauged supergravity on AdS_4 with $SO(8)$ gauge group, we have gravitino in 8_s representation and gauge fields in 28 representation. Decomposing these representations into $SU(4) \times U(1)$, we obtain

$$\begin{aligned} 8_s &\to 1_2 + 1_{-2} + 6_0, \\ 28 &\to 1_0 + 6_2 + 6_{-2} + 15_0. \end{aligned} \tag{1.39}$$

The $U(1)$ neutral subsets survive under the Hopf reduction and only the 6_0 representation remains for the gravitino. Therefore we conslude that bulk SUSY reduces from 4-dim $\mathcal{N} = 8$ to 4-dim $\mathcal{N} = 6$ and the dual field theory is 3-dim $\mathcal{N} = 6$ SCFT with $SU(4) \times U(1)$ R-symmetry.

Another way to obtain the nonmaximal supersymmetric gravity dual is to consider the supergravity on $AdS_4 \times S^7/\mathbb{Z}_k$. In this case we identify the coordinate of $U(1)$ fiber over $\mathbf{CP}^3$ with a period $1/k$ times than that of S^7. Then only a subset of the original states which have a $U(1)$ charge $q = kn/2$ remain in the massless spectrum on $AdS_4 \times S^7/\mathbb{Z}_k$.

For $k = 2$, charge projection condition becomes $q = n$ and all the gravitino are left. Thus the bulk theory is maximally supersymmetric and we expect the dual theory is 3-dim $\mathcal{N} = 8$ SCFT with $SU(4) \times SO(4)^2 \times U(1)$ R-symmetry. As we will see later in part II, this corresponds to $U(N)_2 \times U(N)_{-2}$ ABJM theory.

For $k = 3$, charge condition becomes $q = 3n/2$ and only the six gravitino 6_0 remain and bulk theory has $\mathcal{N} = 6$ SUSY. The corresponding field theory dual is thought to be 3-dim $\mathcal{N} = 6$ SCFT with $SU(4) \times U(1)$ R-symmetry. Generically bulk theory has $\mathcal{N} = 6$ SUSY in $k \geq 3$ and the dual CFT becomes $U(N)_k \times U(N)_{-k}$ ABJM theory.

Part II

Recent developments in M-theory branes

Chapter 2

Low energy effective theory of M2-branes

2.1 Bagger-Lambert-Gustavsson theory

We first briefly review the Bagger-Lambert-Gustavsson (BLG) theory and its symmetry properties. It is a (2+1)-dimensional nonabelian gauge theory with $\mathcal{N}=8$ supersymmetries. It contains 8 real scalar fields $X^I=\sum_a X^I_a T^a,\ I=3,...,10$, gauge fields $A^\mu=\sum_{ab} A^\mu_{ab} T^a\otimes T^b, \mu=0,1,2$ with two internal indices and 11-dimensional Majorana spinor fields $\Psi=\sum_a \Psi_a T^a$ with a chirality condition $\Gamma_{012}\Psi=\Psi$. The action of BLG theory is given by

$$\mathcal{L}=-\frac{1}{2}\mathrm{Tr}(D^\mu X^I, D_\mu X^I)+\frac{i}{2}\mathrm{Tr}(\bar{\Psi},\Gamma^\mu D_\mu \Psi)+\frac{i}{4}\mathrm{Tr}(\bar{\Psi},\Gamma_{IJ}[X^I,X^J,\Psi])-V(X)+\mathcal{L}_{CS}. \tag{2.1}$$

where D_μ is the covariant derivative defined by:

$$(D_\mu X^I)_a=\partial_\mu X^I_a - f^{cdb}{}_a A_{\mu cd}(x) X^I_b. \tag{2.2}$$

$V(X)$ is a sextic potential term

$$V(X)=\frac{1}{12}\mathrm{Tr}([X^I,X^J,X^K],[X^I,X^J,X^K]), \tag{2.3}$$

and the Chern-Simons term for the gauge potential is given by

$$\mathcal{L}_{CS}=\frac{1}{2}\epsilon^{\mu\nu\lambda}(f^{abcd}A_{\mu ab}\partial_\nu A_{\lambda cd}+\frac{2}{3}f^{cda}{}_g f^{efgb}A_{\mu ab}A_{\mu cd}A_{\lambda ef}). \tag{2.4}$$

This action is invariant under the SUSY transformation

$$\begin{aligned}\delta X^I_a &= i\bar{\epsilon}\Gamma^I\Psi_a,\\ \delta\Psi_a &= D_\mu X^I_a\Gamma^\mu\Gamma^I\epsilon-\frac{1}{6}X^I_b X^J_c X^K_d f^{bcd}{}_a\Gamma^{IJK}\epsilon,\\ \delta\tilde{A}_\mu{}^b{}_a &= i\bar{\epsilon}\Gamma_\mu\Gamma_I X^I_c\Psi_d f^{cdb}{}_a, \qquad \tilde{A}_\mu{}^b{}_a\equiv A_{\mu cd}f^{cdb}{}_a,\end{aligned} \tag{2.5}$$

and the gauge transformation

$$\begin{aligned}\delta X^I &= \Lambda_{ab}[T^a,T^b,X^I],\\ \delta\Psi &= \Lambda_{ab}[T^a,T^b,\Psi],\\ \delta\tilde{A}_\mu{}^b{}_a &= D_\mu\tilde{\Lambda}^b{}_a, \qquad \tilde{\Lambda}\,{}^b_a\equiv\Lambda_{cd}f^{cdb}{}_a,\end{aligned} \tag{2.6}$$

provided that the triple product $[X, Y, Z]$ has the fundamental identity and Tr satisfies the property discussed in the next subsection. The most peculiar property of the model is that the gauge transformation and the associated gauge fields have two internal indices. This must come from the volume preserving diffeomorphism of the membrane action [30, 31] but the concrete realization of the gauge symmetry from the supermembrane action is not yet clear.

2.2 A specific realization of Lie 3-algebra

BLG theory is based on the Lie 3-algebra

$$[T^a, T^b, T^c] = f^{abc}{}_d T^d. \tag{2.7}$$

where T^a is generator and f^{abcd} is structure constant of this algebra. In order to obtain the consistent gauge transformations, this algebra must satisfy the generalized Jacobi identity, so called fundamental identity

$$[T^a, T^b, [T^c, T^d, T^e]] = [[T^a, T^b, T^c], T^d, T^e] + [T^c, [T^a, T^b, T^d], T^e] + [T^c, T^d, [T^a, T^b, T^e]]. \tag{2.8}$$

If this identity holds, we can show that the gauge transformations generated by $T^a \otimes T^b$ form Lie algebra[1]. Namely, if we write $\tilde{T}^{ab}X = [T^a, T^b, X]$, a commutator closes among the generators $\tilde{T}^{ab}$;

$$\begin{aligned} [\tilde{T}^{ab}, \tilde{T}^{cd}]X &= [T^a, T^b, [T^c, T^d, X]] - [T^c, T^d, [T^a, T^b, X]] \\ &= [[T^a, T^b, T^c], T^d, X] + [T^c, [T^a, T^b, T^d], X] \\ &= (f^{abc}{}_e \tilde{T}^{ed} + f^{abd}{}_e \tilde{T}^{ce})X. \end{aligned} \tag{2.9}$$

A specific choice of the 3-algebra satisfying the fundamental identity is given by [4, 5, 6]. It contains an ordinary set of Lie algebra generators as well as two extra generators T^{-1} and T^0. The algebra is given by

$$\begin{aligned} &[T^{-1}, T^a, T^b] = 0, \\ &[T^0, T^i, T^j] = f^{ij}{}_k T^k, \\ &[T^i, T^j, T^k] = f^{ijk} T^{-1}, \end{aligned} \tag{2.10}$$

where $a, b = \{-1, 0, i\}$. T^i is a generator of the Lie algebra and $f^{ij}{}_k$ is its structure constants. Here T^{-1} is the central generator meaning that its triple product with any other generators vanishes. T^0 is also special since it is not generated by the 3-algebra and does not appear in the right hand side of the triple product. One can easily check that this triple product satisfies the

[1] Strictly speaking, $\tilde{T}^{ab}$ satisfies ordinary Lie algebras only when they act on X. If we write the commutation relations of $\tilde{T}^{ab}$ without acting on X, they are not necessarily associative and contain associativity-violating 3-cocycles.

fundamental identity. In order to construct a gauge invariant field theory Lagrangian, we need the trace operation with the identity

$$\mathrm{Tr}([T^a, T^b, T^c], T^d) + \mathrm{Tr}(T^c, [T^a, T^b, T^d]) = 0. \tag{2.11}$$

After a suitable redefinition of generators, such a trace can be given by

$$\begin{aligned} &\mathrm{Tr}(T^{-1}, T^{-1}) = \mathrm{Tr}(T^{-1}, T^i) = 0, \quad \mathrm{Tr}(T^{-1}, T^0) = -1, \\ &\mathrm{Tr}(T^0, T^i) = 0, \quad \mathrm{Tr}(T^0, T^0) = 0, \quad \mathrm{Tr}(T^i, T^j) = h^{ij}. \end{aligned} \tag{2.12}$$

If we define f^{abcd} as $f^{abcd} = f^{abc}{}_e h^{ed}$, f^{abcd} is totally antisymmetry.

The above construction of the 3-algebra contains the ordinary Lie algebra as a sub-algebra. The generators of the gauge transformation can be classified into 3 classes.

- $\mathcal{I}$=$\{T^{-1} \otimes T^a, a = 0, i\}$
- $\mathcal{A}$=$\{T^0 \otimes T^i\}$
- $\mathcal{B}$=$\{T^i \otimes T^j\}$

Then it is easy to show that

$$[\mathcal{I}, \mathcal{I}] = [\mathcal{I}, \mathcal{A}] = [\mathcal{I}, \mathcal{B}] = 0, \ [\mathcal{A}, \mathcal{A}] = \mathcal{A}, \ [\mathcal{A}, \mathcal{B}] = \mathcal{B}, \ [\mathcal{B}, \mathcal{B}] = \mathcal{I} \tag{2.13}$$

and hence the generators of $\mathcal{A}$ form a sub-algebra, which can be identified as the Lie algebra of N D2-branes.

2.3 BLG theory to D2 branes

Now we decompose the modes of the fields as

$$\begin{aligned} X^I &= X_0^I T^0 + X_{-1}^I T^{-1} + X_i^I T^i, \\ \Psi &= \Psi_0 T^0 + \Psi_{-1} T^{-1} + \Psi_i T^i, \\ A_\mu &= T^{-1} \otimes A_{\mu(-1)} - A_{\mu(-1)} \otimes T^{-1} \\ &\quad + A_{\mu 0 j} T^0 \otimes T^j - A_{\mu j 0} T^j \otimes T^0 + A_{\mu i j} T^i \otimes T^j. \end{aligned} \tag{2.14}$$

It will be convenient to define the following fields as in [6]

$$\begin{aligned} &\hat{X}^I = X_i^I T^i, \qquad \hat{\Psi} = \Psi_i T^i \\ &\hat{A}_\mu = 2A_{\mu 0 i} T^i, \quad B_\mu = f^{ij}{}_k A_{\mu i j} T^k. \end{aligned} \tag{2.15}$$

The gauge field $A_{\mu(-1)}$ is decoupled from the action and we drop it in the following discussions. The gauge field $\hat{A}_\mu$ is associated with the gauge transformation of the sub-algebra $\mathcal{A}$. Another gauge field B_μ will play a role of the B-field of the BF theory and can be integrated out. With these expression the BLG action can be written as

$$\begin{aligned}\mathcal{L} &= \mathrm{Tr}\Big(-\frac{1}{2}(\hat{D}_\mu \hat{X}^I - B_\mu X_0^I)^2 + \frac{i}{2}\bar{\hat{\Psi}}\Gamma^\mu \hat{D}_\mu \hat{\Psi} + i\bar{\Psi}_0\Gamma^\mu B_\mu \hat{\Psi} + \frac{1}{4}(X_0^K)^2([\hat{X}^I, \hat{X}^J])^2 \\ &\quad -\frac{1}{2}(X_0^I[\hat{X}^I, \hat{X}^J])^2 - \frac{1}{2}\bar{\Psi}_0 \hat{X}^I[\hat{X}^J, \Gamma_{IJ}\hat{\Psi}] + \frac{1}{2}\bar{\hat{\Psi}} X_0^I[\hat{X}^J, \Gamma_{IJ}\hat{\Psi}] + \frac{1}{2}\epsilon^{\mu\nu\lambda}\hat{F}_{\mu\nu}B_\lambda \\ &\quad -\partial_\mu X_0^I \; B_\mu \hat{X}^I\Big) + \mathcal{L}_{gh}, \end{aligned} \tag{2.16}$$

where the ghost term is

$$\mathcal{L}_{gh} = (\partial_\mu X_0^I)(\partial^\mu X_{-1}^I) - i\bar{\Psi}_{-1}\Gamma^\mu \partial_\mu \Psi_0. \tag{2.17}$$

The covariant derivative and the field strength

$$\hat{D}_\mu \equiv \partial_\mu \hat{X}^I + i[\hat{A}_\mu, \hat{X}^I], \quad \hat{D}_\mu \Psi \equiv \partial_\mu \hat{\Psi} + i[\hat{A}_\mu, \hat{\Psi}], \quad \hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + i[\hat{A}_\mu, \hat{A}_\nu] \tag{2.18}$$

are the ordinary covariant derivative and field strength for the sub-algebra $\mathcal{A}$. As emphasized in [4, 5, 6], a coupling constant can be always absorbed by the field redefinition and there is no tunable parameters in this model.

The supersymmetry transformations for each mode are given by

$$\begin{aligned} \delta X_0^I &= i\bar{\epsilon}\Gamma^I \Psi_0, \\ \delta X_{-1}^I &= i\bar{\epsilon}\Gamma^I \Psi_{-1}, \\ \delta \hat{X}^I &= i\bar{\epsilon}\Gamma^I \hat{\Psi}, \\ \delta \Psi_0 &= \partial_\mu X_0^I \Gamma^\mu \Gamma^I \epsilon, \\ \delta \Psi_{-1} &= \{\partial_\mu X_{-1}^I - \mathrm{Tr}(B_\mu, \hat{X}^I)\}\Gamma^\mu \Gamma^I \epsilon + \frac{i}{6}\mathrm{Tr}(\hat{X}^I, [\hat{X}^J, \hat{X}^K])\Gamma^{IJK}\epsilon, \\ \delta \hat{\Psi} &= \hat{D}_\mu \hat{X}^I \Gamma^\mu \Gamma^I \epsilon - B_\mu X_0^I \Gamma^\mu \Gamma^I \epsilon + \frac{i}{2} X_0^I [\hat{X}^J, \hat{X}^K]\Gamma^{IJK}\epsilon, \\ \delta \hat{A}_\mu &= i\bar{\epsilon}\Gamma_\mu \Gamma_I (X_0^I \hat{\Psi} - \hat{X}^I \Psi_0), \\ \delta B_\mu &= \bar{\epsilon}\Gamma_\mu \Gamma_I [\hat{X}^I, \hat{\Psi}]. \end{aligned} \tag{2.19}$$

Here note that X_{-1}^I and Ψ_{-1} appear only linearly in the Lagrangian and thus they are Lagrange multipliers. By integrating out these fields, we have the following constraints for the other problematic fields associated with T^0;

$$\partial^2 X_0^I = 0, \qquad \Gamma^\mu \partial_\mu \Psi_0 = 0. \tag{2.20}$$

This should be understood as a physical state condition $\partial^2 X_0^I |\mathrm{phys}\rangle = 0$. In the path integral formulation, these constraints appear as a delta function $\delta(\partial^2 X_0^I)$ and those fields are constrained to satisfy the massless wave equations. In order to fully quantize the theory, we need to sum all the solutions satisfying the constraints, but we here take a special solution to the constraint equations and see what kind of field theory can be obtained.

The simplest solution is given by

$$X_0^I = v\ \delta_{10}^I, \qquad \Psi_0 = 0, \tag{2.21}$$

where v is some constant. This solution was considered in [4, 5, 6] and preserves all the 16 supersymmetries, the gauge symmetry generated by the subalgebra $\mathcal{A}$, and $SO(7)$ R-symmetry rotating $X^A,\ A = 3, ..., 9$. Another interesting solution is given by

$$X_0^I = v(x^0 + x^1)\delta_{10}^I\ , \qquad \Psi_0 = 0 \tag{2.22}$$

where $v(x^0 + x^1)$ is an arbitrary function on the light cone coordinate. As we see the supersymmetry transformation for Ψ_0,

$$\delta\Psi_0 = \partial_\mu X_0^I \Gamma^\mu \Gamma^I \epsilon, \tag{2.23}$$

the solution $X_0^I = v(x^0 + x^1)\delta_{10}^I$ preserves half of the supersymmetries.

In both cases, if we fix the fields X_0^I and Ψ_0 as above, we can integrate over the gauge field B_μ and obtain the effective action for N D2 branes[2]

$$\mathcal{L} = \mathrm{Tr}\left[-\frac{1}{2}(\hat{D}_\mu \hat{X}^A)^2 + \frac{1}{4}v^2[\hat{X}^A, \hat{X}^B]^2 + \frac{i}{2}\bar{\hat{\Psi}}\Gamma^\mu \hat{D}_\mu \hat{\Psi} - \frac{1}{4v^2}\hat{F}_{\mu\nu}^2 + \frac{1}{2}v\bar{\hat{\Psi}}[\hat{X}^A, \Gamma_{10,A}\hat{\Psi}]\right], \tag{2.24}$$

where $A, B = 3, \cdots, 9$. The coupling v is given by the vev of X_0^{10} and it is either a constant or an arbitrary function on the light-cone $v(x^0 + x^1)$. This may be identified as the compactification radius of 11-th direction in M-theory, $v = 2\pi g_s l_s$. The supersymmetric YM theories with a space-time dependent coupling are known as Janus field theories and originally considered to be a dual of supergravity solutions with space-time dependent dilaton fields [32](see also [33, 34]). A salient feature is that the 10-th spacial fields X^{10} completely disappear from the Lagrangian by integrating out the redundant gauge field B_μ. It is interesting that Janus field theories are naturally obtained from BLG theory and we will discuss this point in the Appendix.

The $v \to 0$ limit cannot be taken after integrating the redundant gauge field B_μ. In the case of vanishing v, the Lagrangian is simply given by

$$\mathcal{L} = \mathrm{Tr}\left[-\frac{1}{2}(\hat{D}_\mu \hat{X}^I)^2 + \frac{i}{2}\bar{\hat{\Psi}}\Gamma^\mu \hat{D}_\mu \hat{\Psi}\right] \tag{2.25}$$

with a constraint $\hat{F}_{\mu\nu} = 0$. The action is of course invariant under the full $SO(8)$ R-symmetry.

2.4 Aharony-Bergman-Jafferis-Maldacena theory

The action of the ABJM theory is given by (we use the convention used in [35])

$$\mathcal{S} = \int d^3x\ \mathrm{tr}\,[-(D_\mu Z_A)^\dagger D^\mu Z^A - (D_\mu W^A)^\dagger D^\mu W_A + i\zeta_A^\dagger \Gamma^\mu D_\mu \zeta^A + i\omega^{\dagger A}\Gamma^\mu D_\mu \omega_A] \\ + \mathcal{S}_{CS} - \mathcal{S}_{V_f} - \mathcal{S}_{V_b}, \tag{2.1}$$

[2]The fermion here is a 32 component spinor satisfying $\Gamma_{012}\Psi = \Psi$. In order to recover the ordinary notation for D2 branes, we rearrange it as $\tilde{\Psi} = (1 + \Gamma_{10})\Psi$. Then it satisfies $\Gamma_{10}\tilde{\Psi} = \tilde{\Psi}$ and the action is written in the usual form (no Γ_{10} in the last term).

with $A = 1, 2$. This is an $\mathcal{N} = 6$ superconformal $U(N) \times U(N)$ Chern-Simons theory. Z is a bifundamental field under the gauge group and its covariant derivative is defined by

$$D_\mu X = \partial_\mu X + iA_\mu^{(L)} X - iXA_\mu^{(R)}. \tag{2.2}$$

The gauge transformations $U(N) \times U(N)$ act from the left and the right on this field as $Z \to UZV^\dagger$.

The level of the Chern-Simons gauge theories is $(k, -k)$ and the coefficients of the Chern-Simons terms for the two $U(N)$ gauge groups, $A_\mu^{(L)}$ and $A_\mu^{(R)}$, are opposite. Hence the action $\mathcal{S}_{CS}$ is given by

$$\mathcal{S}_{CS} = \int d^3x \ 2K\epsilon^{\mu\nu\lambda} \,\mathrm{tr}\,[A_\mu^{(L)}\partial_\nu A_\lambda^{(L)} + \frac{2i}{3}A_\mu^{(L)}A_\nu^{(L)}A_\lambda^{(L)} - A_\mu^{(R)}\partial_\nu A_\lambda^{(R)} - \frac{2i}{3}A_\mu^{(R)}A_\nu^{(R)}A_\lambda^{(R)}]. \tag{2.3}$$

The potential term for bosons is given by

$$\begin{aligned} S_{V_b} = -\frac{1}{48K^2}\int d^3x \ \mathrm{tr}\,[&Y^AY_A^\dagger Y^BY_B^\dagger Y^CY_C^\dagger + Y_A^\dagger Y^AY_B^\dagger Y^BY_C^\dagger Y^C \\ &+ 4Y^AY_B^\dagger Y^CY_A^\dagger Y^BY_C^\dagger - 6Y^AY_B^\dagger Y^BY_A^\dagger Y^CY_C^\dagger], \end{aligned} \tag{2.4}$$

and for fermions by

$$\begin{aligned} S_{V_f} = \frac{i}{4K}\int d^3x \,\mathrm{tr}\,[&Y_A^\dagger Y^A\psi^{B\dagger}\psi_B - Y^AY_A^\dagger\psi_B\psi^{B\dagger} + 2Y^AY_B^\dagger\psi_A\psi^{B\dagger} - 2Y_A^\dagger Y^B\psi^{A\dagger}\psi_B \\ &+ \epsilon^{ABCD}Y_A^\dagger\psi_BY_C^\dagger\psi_D - \epsilon_{ABCD}Y^A\psi^{B\dagger}Y^C\psi^{D\dagger}]. \end{aligned} \tag{2.5}$$

Y^A and ψ_A $(A = 1\cdots 4)$ are defined by

$$Y^C = \{Z^A, W^{\dagger A}\}, \qquad \psi_C = \{\epsilon_{AB}\zeta^Be^{i\pi/4}, \epsilon_{AB}\omega^{\dagger B}e^{-i\pi/4}\}, \tag{2.6}$$

where the index C runs from 1 to 4. The $SU(4)$ R-symmetry of the potential terms is manifest in terms of Y^A and ψ_A.

Part III

More details about M-theory branes

Chapter 3

Derivation of Lorentzian BLG theory from ABJM theory

3.1 Gauge structures and Inönü-Wigner contraction

We first look at the gauge structures of the Lorentzian BLG theory [4, 5, 6]. As we have seen, BLG theory [2, 3] has a gauge symmetry generated by $\tilde{T}^{ab}X = [T^a, T^b, X]$ and the Lorentzian Lie 3-algebra is defined by

$$[T^{-1}, T^a, T^b] = 0, \tag{3.1}$$

$$[T^0, T^i, T^j] = f^{ij}{}_k T^k, \tag{3.2}$$

$$[T^i, T^j, T^k] = f^{ijk} T^{-1}, \tag{3.3}$$

where $a, b = \{-1, 0, i\}$ and T^i are generators of the ordinary Lie algebra with the structure constant f^{ijk} as $[T^i, T^j] = if^{ij}{}_k T^k$. Moreover, the gauge generators of the Lorentzian BLG theory can be classified into 3 classes

- $\mathcal{I}$=$\{T^{-1} \otimes T^a, a = 0, i\}$
- $\mathcal{A}$=$\{T^0 \otimes T^i\}$
- $\mathcal{B}$=$\{T^i \otimes T^j\}$.

The generators in the class $\mathcal{I}$ vanish when they act on X, hence we set these generators to zero in the following. Since the generators in the class $\mathcal{B}$ always appear as a combination with the structure constant, we define generators $S^i \equiv f^i_{jk}\tilde{T}^{jk}$. Then they satisfy the algebra

$$[\tilde{T}^{0i}, \tilde{T}^{0j}] = if^{ij}_k \tilde{T}^{0k}, \quad [\tilde{T}^{0i}, S^j] = if^{ij}_k S^k, \quad [S^i, S^j] = 0. \tag{3.4}$$

The last commutator was originally proportional to the generators in the class $\mathcal{I}$. If we had kept these generators, the algebra would have become nonassociative. The algebra (3.4) is a semi direct sum of $SU(N)$ (or $U(N)$) and translations. In the case of $SU(2)$, it becomes the $ISO(3)$ gauge group, which is the gauge group of the 3-dimensional gravity. Lorentzian BLG theory has the above gauge symmetries and corresponding gauge fields $\hat{A}_\mu$ and B_μ as we will see in the next section.

On the other hand, the theory proposed by Aharony et.al. [12] is a Chern-Simons (CS) gauge theory with the gauge group $U(N)\times U(N)$. They act on the bifundamental fields (e.g. X^I) from the left and the right as $X\to UXV^\dagger$. If we write the generators as T^i_L and T^i_R, the combination $T^i=T^i_L+T^i_R$ and $S^i=T^i_L-T^i_R$ satisfy the algebra

$$[T^i,T^j]=if^{ij}_kT^k,\quad [T^i,S^j]=if^{ij}_kS^k,\quad [S^i,S^j]=if^{ij}_kT^k. \tag{3.5}$$

By taking the Inönü-Wigner contraction, i.e. scaling the generators as $S^i\to\lambda^{-1}S^i$ and taking $\lambda\to 0$ limit, the algebra (3.5) becomes the algebra (3.4) of the Lorentzian BL theory. Therefore it is tempting to think that the Lorentzian BL theory can be obtained by taking an appropriate scaling limit of the ABJM theory. We will see later that it is indeed the case. Interestingly, even the constraint equations in the BL theory (obtained by integrating the Lagrange multiplier fields) can be derived from this scaling procedure.

3.2 Lorentzian BLG theory and ABJM theory

We have shown that the Lorentzian BLG Lagrangian can be written as $\mathcal{L}=\mathcal{L}_0+\mathcal{L}_{gh}$ where

$$\begin{aligned}\mathcal{L}_0=\ \mathrm{tr}\ \Big[&-\frac{1}{2}(\hat{D}_\mu\hat{X}^I-B_\mu X^I_0)^2+\frac{1}{4}(X^K_0)^2([\hat{X}^I,\hat{X}^J])^2-\frac{1}{2}(X^I_0[\hat{X}^I,\hat{X}^J])^2\\&+\frac{i}{2}\bar{\hat{\Psi}}\Gamma^\mu\hat{D}_\mu\hat{\Psi}+i\bar{\Psi}_0\Gamma^\mu B_\mu\hat{\Psi}-\frac{1}{2}\bar{\Psi}_0\hat{X}^I[\hat{X}^J,\Gamma_{IJ}\hat{\Psi}]+\frac{1}{2}\bar{\hat{\Psi}}X^I_0[\hat{X}^J,\Gamma_{IJ}\hat{\Psi}]\\&+\frac{1}{2}\epsilon^{\mu\nu\lambda}\hat{F}_{\mu\nu}B_\lambda-\partial_\mu X^I_0\ B_\mu\hat{X}^I\Big],\end{aligned} \tag{3.6}$$

and

$$\mathcal{L}_{gh}=(\partial_\mu X^I_0)(\partial^\mu X^I_{-1})-i\bar{\Psi}_{-1}\Gamma^\mu\partial_\mu\Psi_0. \tag{3.7}$$

The ghosts X^I_{-1} and Ψ_{-1} appear only linearly and can be integrating out. Then we obtain the following constraints

$$\partial^2X^I_0=0,\qquad \Gamma^\mu\partial_\mu\Psi_0=0. \tag{3.8}$$

The constraint equations (3.8) and the Lagrangian $\mathcal{L}_0$ are what we want to obtain from the ABJM theory by taking a scaling limit.

ABJM theory is similar to the Lorentzian BLG theory, but different in the following points. First, the gauge group of ABJM theory is $U(N)\times U(N)$ while it is a semi direct product of $U(N)$ and translations in the Lorentzian BLG theory. Accordingly the matter fields are in the bifundamental representation in the ABJM theory. Furthermore the Lorentzian BLG theory contains an extra field X_0 and Ψ_0 associated with the generator T_0, and they are required to obey the constraint equations (3.8).

The bosonic potential terms in both theories are sextic, but the potential in the Lorentzian BLG theory contains two X_0^I fields and four adjoint matter fields $\hat{X}^I$ while the potential terms in the ABJM theory are written in the product of six bifundamental matter fields Y^A. Hence it is natural to think that the trace part of Y^A will play a role of X_0 in the Lorentzian BLG theory. We will see that, if we separate the matter field Y^A into a trace and a traceless part, the potential terms coincide in a certain scaling limit.

3.3 Scaling limit of ABJM theory

In order to take a scaling limit, we first recombine the gauge fields as

$$\hat{A}_\mu = \frac{A_\mu^{(L)} + A_\mu^{(R)}}{2}, \qquad B_\mu = \frac{A_\mu^{(L)} - A_\mu^{(R)}}{2}, \tag{3.9}$$

then the gauge transformations corresponding to $\hat{A}_\mu$ and B_μ are $Z \to e^{i\sigma_a T^a} Z e^{-i\sigma_b T^b}$ and $Z \to e^{i\sigma_a T^a} Z e^{i\sigma_b T^b}$ respectively. They are vectorial and axial gauge transformations. Matter fields are in the adjoint representation for the $\hat{A}_\mu$ gauge fields. Hence the $U(1)$ part of $\hat{A}_\mu$ decouples from the matter sector.

The covariant derivative can be written in terms of $\hat{A}_\mu$ and B_μ as

$$\begin{aligned} D_\mu Z &= \partial_\mu Z + i[\hat{A}_\mu, Z] + i\{B_\mu, Z\} \\ &= \hat{D}_\mu Z + i\{B_\mu, Z\}, \end{aligned} \tag{3.10}$$

where $\hat{D}_\mu$ is the covariant derivative with respect to the gauge field $\hat{A}_\mu$. S_{CS} can be written in terms of $\hat{A}_\mu$ and B_μ as

$$S_{CS} = \int d^3x \; 4K\epsilon^{\mu\nu\rho} \, \mathrm{tr}\, [B_\mu \hat{F}_{\nu\rho} + \frac{2}{3} B_\mu B_\nu B_\rho], \tag{3.11}$$

where $\hat{F}_{\mu\nu}$ is field strength of $\hat{A}_\mu$.
The gauge fields $\hat{A}_\mu$, B_μ are associated with the gauge transformations generated by T^i and S^i in (3.5). Hence in order to take the Inönü-Wigner contraction to obtain the gauge structure of the Lorentzian BL theory (3.4), we need to rescale the gauge field B_μ as $B^\mu \to \lambda B^\mu$ and take the $\lambda \to 0$ limit. Simultaneously we need to scale the coefficient K by $\lambda^{-1}K$. Since the coefficient K is proportional to the level of the Chern-Simons theory k as $K = k/8\pi$, the scaling limit corresponds to taking the large k limit. In this scaling limit, the cubic term of the B_μ fields vanishes and the Chern-Simons action coincides with the BF-type action in the Lorentzian BLG theory:

$$S_{CS} \to \int d^3x \; 4K\epsilon^{\mu\nu\rho} \, \mathrm{tr}\, B_\mu \hat{F}_{\nu\rho}. \tag{3.12}$$

In order to match the covariant derivatives in the Lorentzian BLG action (3.6) and in the ABJM theory (3.10), we separate the bifundamental fields into the trace and the traceless part, and scale them differently. We write the matter fields Y^A as

$$Y_{ij}^A = Y_0^A \delta_{ij} + \tilde{Y}_a^A T_{ij}^a, \tag{3.13}$$

where T^a is the generator of $SU(N)$.

Now we perform the following rescaling:

$$
\begin{aligned}
B_\mu &\to \lambda B_\mu, \\
Y_0^A &\to \lambda^{-1} Y_0^A, \\
\psi_{A0} &\to \lambda^{-1}\psi_{A0}, \\
K &\to \lambda^{-1} K,
\end{aligned}
\tag{3.14}
$$

where Y_0^A and ψ_{A0} is the trace part of Y^A and ψ_A. All the other fields are kept fixed. Then take the $\lambda \to 0$ limit. If we take the scaling limit, we can show that the covariant derivatives in both theories exactly match.

In the following we consider the ABJM theory with the $SU(N) \times SU(N)$ gauge group. In the presence of the $U(1) \times U(1)$ group, a little more care should be taken for the scaling of the $U(1)$ part of the B_μ gauge field.

In taking the above scaling limit, many terms vanish. The kinetic term of the ABJM action becomes

$$
\begin{aligned}
\mathrm{tr}\Big[&-\frac{1}{\lambda^2}\partial_\mu Y^\dagger_{0A}\partial^\mu Y_0^A + \frac{1}{\lambda^2}\psi^\dagger_{0A}\Gamma^\mu\partial_\mu\psi_0^A + 2(i\partial_\mu Y^\dagger_{0A}B^\mu Y^A + h.c.) \\
&-(\hat{D}_\mu\tilde{Y}_A + 2i\tilde{B}_\mu Y_{0A})^\dagger(\hat{D}^\mu\tilde{Y}^A + 2i\tilde{B}^\mu Y_0^A) + i\tilde{\psi}^\dagger_A\Gamma^\mu\hat{D}_\mu\tilde{\psi}^A - 2\tilde{\psi}^\dagger_A\Gamma^\mu\tilde{B}_\mu\psi_0^A - 2\psi^\dagger_{0A}\Gamma^\mu\tilde{B}_\mu\tilde{\psi}^A\Big].
\end{aligned}
\tag{3.15}
$$

The first and the second terms are divergent for small λ. In order to make the action finite, we need to impose that the trace part of the bifundamental fields must satisfy the constraint equations

$$
\partial^2 Y_0^I = 0, \quad \Gamma^\mu\partial_\mu\psi_{A0} = 0
$$

in the $\lambda \to 0$ limit. They are precisely the same constraint equations (3.8) in the Lorentzian BLG theory. In that case, the constraints are obtained by integrating out the Lagrange multiplier fields X_{-1} and Ψ_{-1}. Here they arise from a condition that the action should be finite in the scaling limit.

The other terms in (3.15) are finite in the scaling limit and it can be easily shown that they are precisely the same kinetic terms as that of the Lorentzian BLG theory (after a redefinition of the gauge field $2B_\mu \to B_\mu$ and setting $K = 1/2$). The trace part of the bifundamental fields is identified with the fields X_0 associated with one of the extra generators T^0 in the Lorentzian Bagger-Lambert theory. This is the reason why we have used the same convention with subscript 0 for both of the trace part of the bifundamental fields and the field associated with the generator T^0.

Now let us check the potential terms. The potential terms of the ABJM theory are invariant under the $SU(4)$ symmetries but not under full $SO(8)$. By decomposing the matter fields Y^A

into the trace part Y_0^A and the traceless part $\tilde{Y}^A$, the bosonic sextic potential becomes a sum of $V_B = \sum_{n=0}^{6} V_B^{(n)}$, where $V_B^{(n)}$ contains n Y_0 fields and $(6-n)$ $\tilde{Y}$ fields. Since the coefficient of the bosonic potential is proportional to K^{-2}, $V_B^{(n)}$ term scales as λ^{2-n}. It can be easily checked that the coefficients of $V_B^{(n)}$ vanishes for $n > 3$. On the other hand, the potential terms $V_B^{(n)}$ for $n < 2$ vanish in the scaling limit of $\lambda \to 0$. Hence the only remaining term in the scaling limit is $V_B^{(2)}$. This part of the potential has the full $SO(8)$ symmetry and becomes identical with the potential in the Lorentzian BL theory. In order to see that the BL potential is obtained, we assume that only the field Z^1 has the trace part for simplicity. Let us write the 4 complex scalar field Y^A by 8 real scalar fields as

$$\begin{aligned} Z^1 &= X_0^1 + iX_0^5 + i\tilde{X}_a^1 T^a - \tilde{X}_a^5 T^a, \\ Z^2 &= i\tilde{X}_a^2 T^a - \tilde{X}_a^6 T^a, \\ W_1^\dagger &= i\tilde{X}_a^3 T^a - \tilde{X}_a^7 T^a \\ W_2^\dagger &= i\tilde{X}_a^4 T^a - \tilde{X}_a^8 T^a. \end{aligned} \tag{3.16}$$

Substituting them into S_{V_b} and taking the scaling limit, we can obtain the following bosonic potential:

$$S_{V_b} = -\frac{1}{8K^2} \int d^3x \ \mathrm{tr} \left(((X_0^1)^2 + (X_0^5)^2)[P_I, P_J][P^I, P^J] \right). \tag{3.17}$$

P^I is defined by

$$\begin{aligned} P^I &\equiv (P^1, \tilde{X}^2, \tilde{X}^3, \tilde{X}^4, \tilde{X}^6, \tilde{X}^7, \tilde{X}^8), \\ &= \left(\frac{1}{2}(\tilde{Y}^A + \tilde{Y}_A^\dagger), \frac{1}{2i}(\tilde{Y}^B - \tilde{Y}_B^\dagger) \right), \\ \tilde{Y}^A &\equiv (P^1, Z^2, W_1^\dagger, W_2^\dagger), \\ P^1 &\equiv \frac{X_0^1 \tilde{X}^5 - X_0^5 \tilde{X}^1}{\sqrt{(X_0^1)^2 + (X_0^5)^2}}. \end{aligned} \tag{3.18}$$

We can rewrite it as,

$$S_{V_b} = -\frac{1}{8K^2} \int d^3x \, \mathrm{tr} \left[\frac{1}{4}(X_0^K)^2 \left([\tilde{X}^I, \tilde{X}^J] \right)^2 - \frac{1}{2} \left(X_0^I [\tilde{X}^I, \tilde{X}^J] \right)^2 \right], \tag{3.19}$$

where we have used $X_0^I = (X_0^1, 0, 0, 0, X_0^5, 0, 0, 0)$. This is the potentials for bosons in the Lorentzian BL theory (3.6). It is straightforward to see that the complete potential of the BL theory can be obtained by considering general X_0^I and the full $SO(8)$ invariance is restored.

It should be noted that the above potential term is written in terms of the commutators. This shows that, if we replace more than two bosons by their trace components, the potential vanishes. This assures that the would-be divergent terms $V_B^{(n)}$ for $n > 3$ vanish and the only remaining term in the scaling limit is given by the above potential.

Finally consider the fermion potential. We expand the potential as $V_f = \sum_{n=0}^{4} V_f^{(n)}$ where $V_f^{(n)}$ contains n trace parts and $(4-n)$ traceless parts. Since the coefficient of the fermion potential is proportional to $1/K$, $V_f^{(n)}$ scales as λ^{1-n}. $V_f^{(n)}$ for $n > 1$ diverges in the scaling limit and their coefficients must vanish. $V_f^{(0)}$ vanishes in the scaling limit $\lambda \to 0$. Hence the only remaining finite terms are $V_f^{(1)}$. In the following we look at the potential term with one of the bosons replaced by the trace part X_0^I. Such a term can be written as

$$\begin{aligned}
S_{V_f} = \frac{i}{2K} X_0^1 \,\mathrm{tr}\, \Big[& -\psi_1^\dagger[\tilde{X}^5, \psi_1] + \psi_2^\dagger[\tilde{X}^5, \psi_2] + \psi_3^\dagger[\tilde{X}^5, \psi_3] + \psi_4^\dagger[\tilde{X}^5, \psi_4] \\
& + \psi_1^\dagger[Y_2, \psi_2] + \psi_2^\dagger[Y_2^\dagger, \psi_1] + \psi_3^\dagger[Y_2, \psi_4^\dagger] + \psi_4[Y_2{}^\dagger, \psi_3] \\
& + \psi_1^\dagger[Y_3, \psi_3] + \psi_3^\dagger[Y_3^\dagger, \psi_1] + \psi_4^\dagger[Y_3, \psi_2^\dagger] + \psi_2[Y_3^\dagger, \psi_4] \\
& + \psi_1^\dagger[Y_4, \psi_4] + \psi_4^\dagger[Y_4^\dagger, \psi_1] + \psi_2^\dagger[Y_4, \psi_3^\dagger] + \psi_3[Y_4^\dagger, \psi_2] \Big] \\
+ \frac{i}{2K} X_0^5 \,\mathrm{tr}\, \Big[& +\psi_1^\dagger[\tilde{X}^1, \psi_1] - \psi_2^\dagger[\tilde{X}^1, \psi_2] - \psi_3^\dagger[\tilde{X}^1, \psi_3] - \psi_4^\dagger[\tilde{X}^1, \psi_4] \\
& - \psi_1^\dagger[iY_2, \psi_2] + \psi_2^\dagger[iY_2^\dagger, \psi_1] + \psi_3^\dagger[iY_2, \psi_4^\dagger] - \psi_4[iY_2^\dagger, \psi_3] \\
& - \psi_1^\dagger[iY_3, \psi_3] + \psi_3^\dagger[iY_3^\dagger, \psi_1] + \psi_4^\dagger[iY_3, \psi_2^\dagger] - \psi_2[iY_3^\dagger, \psi_4] \\
& - \psi_1^\dagger[iY_4, \psi_4] + \psi_4^\dagger[iY_4^\dagger, \psi_1] + \psi_2^\dagger[iY_4, \psi_3^\dagger] - \psi_3[iY_4^\dagger, \psi_2] \Big] . \qquad (3.20)
\end{aligned}$$

Here for simplicity we have assumed that the trace part of the boson X_0^I is nonvanishing for $I = 1, 5$. This can be done by using the original $SU(4)$ symmetry. Note again that these potential terms are written as a form of commutators.

To get the 3-dimensional Majorana fermion as the BL theory, we rewrite the $SU(4)$ complex fermion in terms of the real variables [1].

$$\begin{aligned}
&\psi_1 = i\chi_1 - \chi_5, \quad \psi_2 = i\chi_2 - \chi_6, \\
&\psi_3 = i\chi_3 - \chi_7, \quad \psi_4 = i\chi_4 - \chi_8, \qquad (3.21)
\end{aligned}$$

where χ_I are real 2-component spinors. We also expand the complex bosons as the real ones (3.16). Then the fermion potential (3.20) becomes by using the 8×8 Γ matrice as

$$\begin{aligned}
S_{V_f} &= -\frac{1}{2K} \,\mathrm{tr}\, \bar{\Psi} X_0^I [\tilde{X}^J, \Gamma_{IJ} \Psi], \\
\Psi &\equiv (\chi_1, \chi_2, \chi_3, \chi_4, \chi_5, \chi_6, \chi_7, \chi_8)\,, \qquad (3.22)
\end{aligned}$$

where the indices I, J run from 1 to 8 and $X_0^I = (X_0^1, 0, 0, 0, X_0^5, 0, 0, 0)$. The explicit forms of the Γ matrices are given in the Appendix G. This fermion potential has the same $SO(8)$ invariant form as that of the Lorentzian BLG action (3.6). In the same fashion as the bosonic potential, the full $SO(8)$ invariance can be seen easily by considering the general X_0^I.

[1] When we give a VEV to the X_0^4 part only, we will get 7 Γ matrices as in [36]. In our case we need 8 Γ matrices and their antisymmetrized-products because we give a VEV to a more general direction.

Chapter 4

Generalizing the scaling procedure

4.1 Generalization of the Lorentzian BLG theory

In [8] (see also [7, 37]), the BLG theory based on the Lorentzian Lie 3-algebra was generalized by adding d pairs of negative norm generators. Then, they showed that the worldvolume theory of Dp-branes ($p = d+2$) is produced. The proposed 3-algebra is

$$\begin{aligned}
&[u_0, u_a, u_b] = 0, \\
&[u_0, u_a, T^i_{\vec{m}}] = -im_a T^i_{\vec{m}}, \\
&[u_0, T^i_{\vec{m}}, T^j_{\vec{n}}] = im_a v^a \delta_{\vec{m}+\vec{n}} \delta^{ij} + f^{ij}{}_k T^k_{\vec{m}+\vec{n}}, \\
&[T^i_{\vec{l}}, T^j_{\vec{m}}, T^k_{\vec{n}}] = f^{ijk} \delta_{\vec{l}+\vec{m}+\vec{n}} v^0,
\end{aligned} \tag{4.1}$$

where $a, b = 1, \cdots, d$ and $\vec{l}, \vec{m}, \vec{n} \in \mathbb{Z}^d$. a and b correspond to the label of the compactified direction and $\vec{m}$ to the Kaluza-Klein momentum[1] along the T^d. f^{ijk} $(i, j, k = 1, \cdots, \dim \mathbf{g})$ is a structure constant of an arbitrary Lie algebra $\mathbf{g}$. This 3-algebra actually satisfies the fundamental identity. The nonvanishing part of the metric is

$$\mathrm{tr}(u_A, v^B) = -\delta^B_A, \quad \mathrm{tr}(T^i_{\vec{m}}, T^j_{\vec{n}}) = \delta^{ij} \delta_{\vec{m}+\vec{n}}. \quad (A = 0, 1, \cdots, d) \tag{4.2}$$

Following [8], we will rewrite the BLG action and derive the action of Dp-branes ($p = d+2$). The steps are summarized as follows. First, we derive 3d $\mathcal{N} = 8$ SYM through the Higgs mechanism [14]. The difference from the original L-BLG theory is that the resulting D2-brane action has a Kaluza-Klein tower. Then, we obtain the Dp-brane action with a rearrangement of fields corresponding to T-duality. The worldvolume of Dp-brane is given as a flat T^d bundle over the membrane worldvolume $\mathcal{M}$.

In the remainder of this subsection, we look at the above procedure more explicitly. For the

[1] Instead, we can consider $\vec{m}$ as the index describing open string modes that interpolate the mirror images of a point in $S^1 = \mathbb{R}/\mathbb{Z}$ in the spirit of Taylor's T-duality [38].

3-algebra (4.1), we expand the fields as

$$
\begin{aligned}
X^I &= X^I_{(i\vec{m})} T^i_{\vec{m}} + X^{IA} u_A + \underline{X}^I_A v^A, \\
\psi &= \psi_{(i\vec{m})} T^i_{\vec{m}} + \psi^A u_A + \underline{\psi}_A v^A, \\
A_\mu &= A_{\mu(i\vec{m})(j\vec{n})} T^i_{\vec{m}} \wedge T^j_{\vec{n}} + \frac{1}{2} A_{\mu(i\vec{m})} u_0 \wedge T^i_{\vec{m}} + \frac{1}{2} A^a_{\mu(i\vec{m})} u_a \wedge T^i_{\vec{m}} \\
&\quad + \frac{1}{2} A^a_\mu u_0 \wedge u_a + A^{ab}_\mu u_a \wedge u_b + (\text{terms including } v^A).
\end{aligned}
\tag{4.3}
$$

Each bosonic component has the following role:

- $X^I_{(i\vec{m})}$: These fields become scalar fields corresponding to the transverse coordinates of Dp-branes and gauge fields along the fiber direction.
- X^{IA} : Higgs fields whose VEVs determine the moduli of T^d and the circle radius in the M-direction.
- $\underline{X}^I_A$: Ghost fields that can be removed by Higgs mechanism.
- $A_{\mu(i\vec{m})}$: Gauge fields along $\mathcal{M}$.

The other bosonic terms do not show up in the following discussion.

Because the ghost fields $\underline{X}$ and $\underline{\psi}$ appear linearly in the action, these fields become Lagrange multipliers and can be integrated out. This gives constraint equations for X^{IA} and ψ^A:

$$
\partial^\mu \partial_\mu X^{IA} = 0, \quad \Gamma^\mu \partial_\mu \psi^A = 0. \tag{4.4}
$$

As a solution, we choose a constant vector $\vec{X}^A = \vec{\lambda}^A$ and it determines the (d+1)-dimensional subspace $\mathbb{R}^{d+1} \subset \mathbb{R}^8$. $\mathbb{R}^{d+1}$ is compactified on T^{d+1} and VEVs $\vec{\lambda}^{IA}$ give the moduli of the T^d compactification and the M-theory circle. We can represent the metric of torus T^d as

$$
G^{AB} = \vec{\lambda}^A \cdot \vec{\lambda}^B. \tag{4.5}
$$

The covariant derivative becomes

$$
(D_\mu X^I)_{(i\vec{m})} = (\hat{D}_\mu X^I)_{(i\vec{m})} - A'_{\mu(i\vec{m})} \lambda^{I0} - i m_a A_{\mu(i\vec{m})} \lambda^{Ia}, \tag{4.6}
$$

where

$$
\begin{aligned}
(\hat{D}_\mu X^I)_{(i\vec{m})} &= \partial_\mu X^I_{(i\vec{m})} - f^{jk}{}_i A_{\mu(k\vec{n})} X^I_{(j,\vec{m}-\vec{n})}, \\
A'_{\mu(i\vec{m})} &= -i m_a A^a_{\mu(i\vec{m})} + f^{jk}{}_i A_{\mu(j,\vec{m}-\vec{n})(k\vec{n})}.
\end{aligned}
\tag{4.7}
$$

The Chern-Simons term is written as

$$
L_{CS} = \frac{1}{2} A'_{(i\vec{m})} \wedge F_{(i,-\vec{m})} + (\text{total derivative}), \tag{4.8}
$$

where $F_{\mu\nu(i,\vec{m})} = \partial_\mu A_{\nu(i\vec{m})} - \partial_\nu A_{\mu(i\vec{m})} - f^{jk}{}_i A_{\mu(j\vec{n})} A_{\nu(k,\vec{m}-\vec{n})}$. Integrating $A'_{(i\vec{m})}$, Chern-Simons gauge fields obtain a degree of freedom and the usual F^2 term emerges.

The bosonic potential term is given by the square of a triple product

$$[X^I, X^J, X^K]_{(i\vec{m})} = -im_a \lambda^{[I0}\lambda^{Ja} X^{K]}_{(i\vec{m})} + f^{jk}{}_i \lambda^{[I0} X^J_{(j\vec{n})} X^{K]}_{(k,\vec{m}-\vec{n})}. \tag{4.9}$$

The square of this term gives

$$\begin{aligned} &6g^{ab} m_a m_b X^I_{\vec{m}} P^{IJ}_{\vec{m}} X^J_{-\vec{m}} - i\lambda^{[I0}\lambda^J_{\vec{m}} X^{K]}_{(i\vec{m})} f^{jk}{}_i \lambda^{[I0} X^J_{(j\vec{n})} X^{K]}_{(k,-\vec{m}-\vec{n})} \\ &- 3\Big[G^{00}\langle [X^J, X^K]^2\rangle - 2\langle [(\vec{\lambda}^0\cdot\vec{X}), X^I]^2\rangle\Big], \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} P^{IJ}_{\vec{m}} &\equiv \delta^{IJ} - \frac{|\vec{\lambda}^0|^2 \lambda^I_{\vec{m}}\lambda^J_{\vec{m}} + |\lambda_{\vec{m}}|^2 \lambda^{I0}\lambda^{J0} - (\vec{\lambda}^0\cdot\vec{\lambda}_{\vec{m}})(\lambda^{I0}\lambda^J_{\vec{m}} + \lambda^{J0}\lambda^I_{\vec{m}})}{|\vec{\lambda}^0|^2|\vec{\lambda}_{\vec{m}}|^2 - (\vec{\lambda}^0\cdot\vec{\lambda}_{\vec{m}})^2}, \\ \vec{\lambda}_{\vec{m}} &\equiv m_a \vec{\lambda}^a. \end{aligned} \tag{4.11}$$

By collecting all the results, we obtain the D2-brane action with Kaluza-Klein tower. Then, we decompose X^I as

$$X^I = P^{IJ}X^J + \frac{1}{G^{00}}\lambda^{I0}(\vec{\lambda}^0\cdot\vec{X}) + \left(-\frac{G^{0a}}{G^{00}}\lambda^{I0} + \lambda^{Ia}\right), \tag{4.12}$$

and regard the Kaluza-Klein masses m_a with the derivatives of fiber direction $-i\partial_a$, we obtain the kinetic term of the fiber direction and the interaction term in the language of the Dp-brane worldvolume.

As a result, we obtain the following standard Dp-brane action[2]

$$\begin{aligned} L_{Dp} &= L_A + L_{F\tilde{F}} + L_X + L_{pot}, \\ L_A &= -\frac{1}{4G^{00}}\int \frac{d^dy}{(2\pi)^d}\sqrt{g}\ (\tilde{F}^2_{\mu\nu} + 2g^{ab}\tilde{F}_{\mu a}\tilde{F}_{\mu b} + g^{ac}g^{bd}\tilde{F}_{ab}\tilde{F}_{cd}), \\ L_{F\tilde{F}} &= \frac{G^{0a}}{8G^{00}}\int \frac{d^dy}{(2\pi)^d}\sqrt{g}\ (4\epsilon^{\mu\nu\lambda}\tilde{F}_{\mu a}\tilde{F}_{\nu\lambda}), \\ L_X &= -\frac{1}{2}\int \frac{d^dy}{(2\pi)^d}\sqrt{g}\ (\hat{D}_\mu \tilde{X}^I P^{IJ}\hat{D}_\mu \tilde{X}^J + g^{ab}\hat{D}_a\tilde{X}^I P^{IJ}\hat{D}_b\tilde{X}^J), \\ L_{pot} &= \frac{G^{00}}{4}\int \frac{d^dy}{(2\pi)^d}\sqrt{g}[P^{IK}\tilde{X}^K, P^{JL}\tilde{X}^L]^2, \end{aligned} \tag{4.13}$$

whose worldvolume is $\mathcal{M}\times T^d$ with the metric

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + g_{ab}dy^a dy^b, \tag{4.14}$$

where $g_{ab} = (G^{00}G^{ab} - G^{a0}G^{b0})^{-1}$ is the metric of dual torus.

[2]The tilde indicates that the fields are (3+d)-dimensional: $\tilde{\Phi}(x,y) = \sum_{\vec{m}} \Phi_{\vec{m}}(x)e^{i\vec{m}\cdot\vec{y}}$. $P^{IJ} \equiv \delta^{IJ} - \lambda^{IA}\pi^J_A$ is a projector into the subspace orthogonal to all $\vec{\lambda}^A$, where $\vec{\pi}_A$ is a dual basis satisfying $\vec{\lambda}^A\cdot\vec{\pi}_B = \delta^A_B$.

4.2 Orbifolding the ABJM theory

The ABJM theory is a 3d $\mathcal{N} = 6$ $U(N) \times U(N)$ Chern-Simons matter theory. This theory is conjectured to describe the low energy physics of N M2-branes probing $\mathbb{C}^4/\mathbb{Z}_k$. The bosonic action of the ABJM theory is given by

$$\begin{aligned} S = \int d^3x \Big[&- \mathrm{tr}\{(D_\mu Z^A)^\dagger D^\mu Z^A + (D_\mu W^A)^\dagger D^\mu W^A\} - V(Z,W) \\ &+ \frac{k}{4\pi}\epsilon^{\mu\nu\lambda}\mathrm{tr}\Big(A^{(1)}_\mu \partial_\nu A^{(1)}_\lambda + \frac{2i}{3}A^{(1)}_\mu A^{(1)}_\nu A^{(1)}_\lambda \\ &\qquad - A^{(2)}_\mu \partial_\nu A^{(2)}_\lambda - \frac{2i}{3}A^{(2)}_\mu A^{(2)}_\nu A^{(2)}_\lambda\Big)\Big], \end{aligned} \tag{4.15}$$

where $A = 1, 2$. Z^A and $W^{\dagger A}$ are bifundamental matter fields and their covariant derivatives are defined by

$$\begin{aligned} D_\mu Z^A &= \partial_\mu Z^A + iA^{(1)}_\mu Z^A - iZ^A A^{(2)}_\mu, \\ D_\mu W^A &= \partial_\mu W^A + iA^{(2)}_\mu W^A - iW^A A^{(1)}_\mu. \end{aligned} \tag{4.16}$$

In [13], we explicitly show that the original L-BLG theory can be derived from the ABJM theory. Motivated by the agreement of the gauge structure of these two theories through the Inönü-Wigner contraction, we performed the following rescaling:

$$\begin{aligned} &Z^A_0 \to \lambda^{-1} Z^A_0, \\ &W^A_0 \to \lambda^{-1} W^A, \\ &B_\mu \equiv (A^{(1)}_\mu - A^{(2)}_\mu)/2 \to \lambda B_\mu, \\ &k \to \lambda^{-1} k, \end{aligned} \tag{4.17}$$

to the ABJM theory and took the $\lambda \to 0$ limit, where Z^A_0 and W^A_0 are the VEV of Z^A and W^A. Then, we obtained the action of the L-BLG theory. This scaling limit corresponds to locate the M2-branes very far from the origin of the $\mathbb{Z}_k$ orbifold so as not to feel the singularity and simultaneously take $k \to \infty$. Thus, this procedure is effectively the same as the ordinary S^1 compactification and that is why we obtain the L-BLG theory, which is almost D2-branes theory.

As explained in [8], the Extended Lorentzian 3-algebra (4.1) can be regarded as the original Lorentzian 3-algebra with a loop algebra. Thus, it is natural to presume that even the Extended L-BLG theory might be derived from an M2-brane theory in a certain scaling limit. So which M2-brane theory is appropriate? In [19], it was shown that the D3-brane action can be derived by orbifolding the ABJM theory and taking a limit. Because the Extended L-BLG theory with $d = 1$ also reduces to the D3-brane theory via the Higgs mechanism, these two theories might be connected directly. The main purpose of this paper is to clarify the relationship between the orbifolded ABJM theory and the Extended L-BLG theory.

In the remainder of this section, we review the orbifolded ABJM action. By applying the standard orbifolding technique [39] to the ABJM theory or alternatively using the brane construction, we can derive various quiver Chern-Simons matter theories. Here, we see a particular 3d $\mathcal{N}=4$ theory whose bosonic action is[3]

$$
\begin{aligned}
S = \int d^3x \Big[& - \mathrm{tr} \sum_{s=1}^{2n} \{ (D_\mu Z^{(s)})^\dagger D^\mu Z^{(s)} + (D_\mu W^{(s)})^\dagger D^\mu W^{(s)} \} - V_{bos} \\
& + \frac{k}{4\pi} \epsilon^{\mu\nu\lambda} \sum_{l=1}^{n} \mathrm{tr} \{ A_\mu^{(2l-1)} \partial_\nu A_\lambda^{(2l-1)} + \frac{2i}{3} A_\mu^{(2l-1)} A_\nu^{(2l-1)} A_\lambda^{(2l-1)} \\
& - A_\mu^{(2l)} \partial_\nu A_\lambda^{(2l)} - \frac{2i}{3} A_\mu^{(2l)} A_\nu^{(2l)} A_\lambda^{(2l)} \} \Big].
\end{aligned} \tag{4.18}
$$

The explicit forms of the covariant derivatives and bosonic potential are given by

$$
\begin{aligned}
D_\mu Z^{(2l-1)} &= \partial_\mu Z^{(2l-1)} + i A_\mu^{(2l-1)} Z^{(2l-1)} - i Z^{(2l-1)} A_\mu^{(2l)}, \\
D_\mu Z^{(2l)} &= \partial_\mu Z^{(2l)} + i A_\mu^{(2l+1)} Z^{(2l)} - i Z^{(2l)} A_\mu^{(2l)}, \\
D_\mu W^{(2l-1)} &= \partial_\mu W^{(2l-1)} + i A_\mu^{(2l)} W^{(2l-1)} - i W^{(2l-1)} A_\mu^{(2l-1)}, \\
D_\mu W^{(2l)} &= \partial_\mu W^{(2l)} + i A_\mu^{(2l)} W^{(2l)} - i W^{(2l)} A_\mu^{(2l+1)},
\end{aligned} \tag{4.19}
$$

$$
\begin{aligned}
V_{bos} = -\frac{4\pi^2}{3k^2} \sum_{l=1}^{n} \Big[& \mathrm{tr} Y_{2l}^A Y_{A,2l}^\dagger Y_{2l}^B Y_{B,2l}^\dagger Y_{2l}^C Y_{C,2l}^\dagger + 3\mathrm{tr} Y_{2l}^A Y_{A,2l}^\dagger Y_{2l}^B Y_{B,2l}^\dagger Y_{2l+1}^C Y_{C,2l+1}^\dagger \\
& + 3\mathrm{tr} Y_{2l}^A Y_{A,2l}^\dagger Y_{2l+1}^B Y_{B,2l+1}^\dagger Y_{2l+1}^C Y_{C,2l+1}^\dagger + \mathrm{tr} Y_{2l+1}^A Y_{A,2l+1}^\dagger Y_{2l+1}^B Y_{B,2l+1}^\dagger Y_{2l+1}^C Y_{C,2l+1}^\dagger \\
& + \mathrm{tr} Y_{A,2l-1}^\dagger Y_{2l-1}^A Y_{B,2l-1}^\dagger Y_{2l-1}^B Y_{C,2l-1}^\dagger Y_{2l-1}^C + 3\mathrm{tr} Y_{A,2l-1}^\dagger Y_{2l-1}^A Y_{B,2l-1}^\dagger Y_{2l-1}^B Y_{C,2l}^\dagger Y_{2l}^C \\
& + 3\mathrm{tr} Y_{A,2l-1}^\dagger Y_{2l-1}^A Y_{B,2l}^\dagger Y_{2l}^B Y_{C,2l}^\dagger Y_{2l}^C + \mathrm{tr} Y_{A,2l}^\dagger Y_{2l}^A Y_{B,2l}^\dagger Y_{2l}^B Y_{C,2l}^\dagger Y_{2l}^C \\
& + 4\mathrm{tr} Y_{2l-1}^A Y_{B,2l-1}^\dagger Y_{2l-1}^C Y_{A,2l-1}^\dagger Y_{2l-1}^B Y_{C,2l-1}^\dagger + 12\mathrm{tr} Y_{2l}^A Y_{B,2l}^\dagger Y_{2l+1}^C Y_{A,2l+2}^\dagger Y_{2l+2}^B Y_{C,2l+1}^\dagger \\
& + 12\mathrm{tr} Y_{2l+1}^A Y_{B,2l+1}^\dagger Y_{2l}^C Y_{A,2l-1}^\dagger Y_{2l-1}^B Y_{C,2l}^\dagger + 4\mathrm{tr} Y_{2l}^A Y_{B,2l}^\dagger Y_{2l}^C Y_{A,2l}^\dagger Y_{2l}^B Y_{C,2l}^\dagger \\
& - 6\mathrm{tr} Y_{2l-1}^A Y_{B,2l-1}^\dagger Y_{2l-1}^B Y_{A,2l-1}^\dagger Y_{2l-1}^C Y_{C,2l-1}^\dagger - 6\mathrm{tr} Y_{2l}^A Y_{B,2l}^\dagger Y_{2l}^B Y_{A,2l}^\dagger Y_{2l}^C Y_{C,2l}^\dagger \\
& - 6\mathrm{tr} Y_{2l+1}^A Y_{B,2l+1}^\dagger Y_{2l+1}^B Y_{A,2l+1}^\dagger Y_{2l}^C Y_{C,2l}^\dagger - 6\mathrm{tr} Y_{2l}^A Y_{B,2l}^\dagger Y_{2l}^B Y_{A,2l}^\dagger Y_{2l+1}^C Y_{C,2l+1}^\dagger \\
& - 6\mathrm{tr} Y_{2l-1}^A Y_{B,2l}^\dagger Y_{2l}^B Y_{A,2l-1}^\dagger Y_{2l-1}^C Y_{C,2l-1}^\dagger - 6\mathrm{tr} Y_{2l}^A Y_{B,2l-1}^\dagger Y_{2l-1}^B Y_{A,2l}^\dagger Y_{2l}^C Y_{C,2l}^\dagger \\
& - 6\mathrm{tr} Y_{2l+1}^A Y_{B,2l+2}^\dagger Y_{2l+2}^B Y_{A,2l+1}^\dagger Y_{2l}^C Y_{C,2l}^\dagger - 6\mathrm{tr} Y_{2l}^A Y_{B,2l-1}^\dagger Y_{2l-1}^B Y_{A,2l}^\dagger Y_{2l+1}^C Y_{C,2l+1}^\dagger \Big],
\end{aligned} \tag{4.20}
$$

where we used $SU(2)$ doublets

$$
Y_l^A = \{ Z^{(l)}, W^{(l)\dagger} \}, \quad Y_{A,l}^\dagger = \{ Z^{(l)\dagger}, W^{(l)} \}, \qquad (A = 1, 2) \tag{4.21}
$$

[3] This is the "non-chiral orbifold gauge theory" described in [35] and we use their notation. This theory can also be regarded as case II in [40] and the $n_A = n_B$ case in [41] with alternate NS5- and (k,1)5-branes. The "generalized ABJM model" described in [19] is obtained by interchanging our $Z^{(2l)}$ and $W^{(2l)}$ in (4.18).

for each link l. The quiver diagram of this theory is given in Figure 4.1.

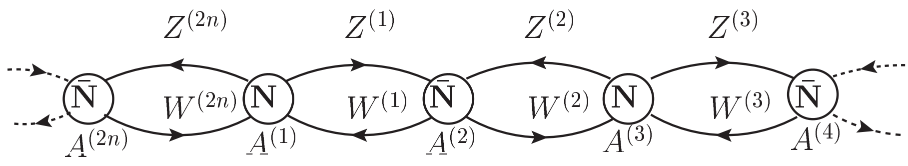

Figure 4.1: Quiver diagram for $\mathcal{N}=4$ quiver CS theory (4.18). This theory has global $SU(2)_o \times SU(2)_e$ symmetry and the $SU(2)_o$ part rotates the fields on the odd links and the $SU(2)_e$ part corresponds to the even links.

This theory has product gauge group $U(N)^{2n}$ and its moduli space is $Sym^N(\mathbb{C}^4/(\mathbb{Z}_{kn}\times\mathbb{Z}_n))$. $\mathbb{Z}_{nk}$ corresponds to the original ABJM orbifold action,

$$y^1 \to e^{2\pi i/nk}y^1, \quad y^2 \to e^{2\pi i/nk}y^2, \quad y^3 \to e^{2\pi i/nk}y^3, \quad y^4 \to e^{2\pi i/nk}y^4. \tag{4.22}$$

Note that in order to have a correct moduli space, as explained in [40], the levels of the Chern-Simons terms in (4.18) must be $\pm k$, not $\pm nk$. Another $\mathbb{Z}_n$ action is given by

$$y^1 \to e^{2\pi i/n}y^1, \quad y^2 \to y^2, \quad y^3 \to e^{2\pi i/n}y^3, \quad y^1 \to y^4. \tag{4.23}$$

This kind of further orbifolding is essential for deriving the Extended L-BLG theory from the ABJM theory. In [13], we obtained a circle by taking a limit of the original ABJM orbifold action and rescaling the fields. Therefore, in a similar fashion, the emergence of an additional circle is expected in a suitable limit of $\mathbb{Z}_n$ action. Naively, it seems that the more we orbifold the ABJM theory, the more we have additional circles. However, in this paper, we only consider the case for one additional circle, namely, T^2 compactification of M-theory. We show that a proper scaling limit leads to the Extended L-BLG theory with $d=1$.

4.3 Scaling limit of $\mathcal{N}=4$ quiver Chern-Simons theory

Here we explicitly show how the Extended L-BLG theory with $d=1$ is derived from a $\mathcal{N}=4$ quiver Chern-Simons theory (4.18). First, we take linear combinations for the gauge fields as

$$A_\mu^{(\pm)(2l-1)} = \frac{1}{2}(A_\mu^{(2l-1)} \pm A_\mu^{(2l+2s)}), \qquad (s \in \mathbb{Z}) \tag{4.24}$$

and decompose the bifundamental fields into trace and traceless parts as $Y = Y_0 \mathbf{1}_{N\times N} + \hat{Y}$. VEV Y_0 is interpreted as a classical position of the center of mass of the multiple M2-branes, and $\hat{Y} = \hat{Y}_a T^a$ is a fluctuation around it. T^a is the generator of $SU(N)$. Next, we rescale the fields as

$$Y^1_{0\ (2l-1)} \to \sqrt{\frac{n}{2}} Y^{(1)}_0, \quad Y^1_{0\ (2l)} \to \sqrt{\frac{n}{2}} Y^{(2)}_0, \quad Y^2_{0\ (2l-1)} \to \sqrt{\frac{n}{2}} Y^{(3)}_0, \quad Y^2_{0\ (2l)} \to \sqrt{\frac{n}{2}} Y^{(4)}_0,$$
$$\hat{Y}^1_{(2l-1)} \to \frac{q^{lm}}{\sqrt{n}} \frac{Y^{(1)}_{(m)}}{\sqrt{2}}, \quad \hat{Y}^1_{(2l)} \to \frac{q^{lm}}{\sqrt{n}} \frac{Y^{(2)}_{(m)}}{\sqrt{2}}, \quad \hat{Y}^2_{(2l-1)} \to \frac{q^{lm}}{\sqrt{n}} \frac{Y^{(3)}_{(m)}}{\sqrt{2}}, \quad \hat{Y}^2_{(2l)} \to \frac{q^{lm}}{\sqrt{n}} \frac{Y^{(4)}_{(m)}}{\sqrt{2}},$$
$$A^{(+)(2l-1)}_\mu \to q^{lm} A_{\mu(m)}, \quad A^{(-)(2l-1)}_\mu \to \frac{\pi}{n} q^{lm} A'_{\mu(m)} \tag{4.25}$$

and finally take $n \to \infty$. Here, $q \equiv e^{\frac{2\pi i}{n}}$ and multiplying q^{lm} corresponds to the Fourier transformation. The normalization is determined by $\sum_l q^{lm} = n\delta_{m,0}$. Recalling that this $\mathcal{N} = 4$ quiver CS theory describes multiple M2-branes at the singularity of an orbifold $\mathbb{C}^4/(\mathbb{Z}_{nk} \times \mathbb{Z}_n)$, this scaling limit corresponds to locating the M2-branes far from the origin of the orbifold and simultaneously making each $\mathbb{Z}_{nk}, \mathbb{Z}_n$ identifications into the independent circle identifications. This is effctively the same as the ordinary T^2 compactification. Therefore, we can expect that the Extended L-BLG theory with $d = 1$ emerges from this limit.

First, let us check the kinetic term. The covariant derivatives (4.19) are scaled as

$$D_\mu Z_{(2l-1)} \to \frac{q^{lm}}{\sqrt{n}} \cdot \frac{1}{\sqrt{2}} \Big[\partial_\mu Y^{(1)}_{(m)} + i[A_{\mu(n)}, Y^{(1)}_{(m-n)}] - 2\pi s m A_{\mu(m)} Y^{(1)}_0 + 2\pi i A'_{\mu(m)} Y^{(1)}_0 + \mathcal{O}(n^{-1})\Big],$$
$$D_\mu Z_{(2l)} \to \frac{q^{lm}}{\sqrt{n}} \cdot \frac{1}{\sqrt{2}} \Big[\partial_\mu Y^{(2)}_{(m)} + i[A_{\mu(n)}, Y^{(2)}_{(m-n)}] - 2\pi (s+1) m A_{\mu(m)} Y^{(2)}_0 + 2\pi i A'_{\mu(m)} Y^{(2)}_0$$
$$+ \mathcal{O}(n^{-1})\Big],$$
$$D_\mu W_{(2l-1)} \to \frac{1}{\sqrt{2}} \Big[\frac{q^{-lm}}{\sqrt{n}} \partial_\mu Y^{(3)\dagger}_{(m)} + i\frac{q^{lm}}{\sqrt{n}}[A_{\mu(n)}, Y^{(3)\dagger}_{(n-m)}] + \frac{2\pi s m}{\sqrt{n}} q^{lm} A_{\mu(m)} Y^{(3)\dagger}_0$$
$$- i\frac{2\pi}{\sqrt{n}} q^{lm} A'_{\mu(m)} Y^{(3)\dagger}_0 + \mathcal{O}(n^{-1})\Big],$$
$$D_\mu W_{(2l)} \to \frac{1}{\sqrt{2}} \Big[\frac{q^{-lm}}{\sqrt{n}} \partial_\mu Y^{(4)\dagger}_{(m)} + i\frac{q^{lm}}{\sqrt{n}}[A_{\mu(n)}, Y^{(4)\dagger}_{(n-m)}] + \frac{2\pi (s+1) m}{\sqrt{n}} q^{lm} A_{\mu(m)} Y^{(4)\dagger}_0$$
$$- i\frac{2\pi}{\sqrt{n}} q^{lm} A'_{\mu(m)} Y^{(4)\dagger}_0 + \mathcal{O}(n^{-1})\Big]. \tag{4.26}$$

The $\mathcal{O}(n^{-1})$ terms do not contribute to the action in the limit $n \to \infty$.

In our notation, complex scalar fields are decomposed to real fields as

$$Y^{(A)}_0 = X^A_0 + iX^{A+4}_0,$$
$$Y^{(A)}_{(m)} = i\hat{X}^A_{(m)} - \hat{X}^{A+4}_{(m)}. \tag{4.27}$$

We note that hermitian conjugation changes the sign of the label m such as

$$Y^{(A)\dagger}_{(m)} = -i\hat{X}^A_{(-m)} - \hat{X}^{A+4}_{(-m)}, \qquad A^\dagger_{\mu(m)} = A_{\mu(-m)}. \tag{4.28}$$

Combining (4.26),(4.27) and (4.28), we can write out a rescaled kinetic term using real fields. Let us compare this kinetic term with that of the Extended L-BLG theory given by

$$
\begin{aligned}
-\frac{1}{2}(D_\mu X^I_{(-m)})(D^\mu X^I_{(m)}) =& -\frac{1}{2}\partial_\mu X^I_{(-m)}\partial^\mu X^I_{(m)} - i\partial_\mu X^I_{(-m)}[A^\mu_{(n)}, X^I_{(m-n)}] \\
&- \frac{1}{2}[X^I_{(-m+n)}, A_{\mu(-n)}][A^\mu_{(k)}, X^I_{(m-k)}] + A'^\mu_{(m)}\lambda^{I0}\partial_\mu X^I_{(-m)} + imA^\mu_{(m)}\lambda^{I1}\partial_\mu X^I_{(-m)} \\
&- iA'^\mu_{(m)}\lambda^{I0}[X^1_{(-m+n)}, A_{\mu(-n)}] + mA^\mu_{(m)}\lambda^{I1}[X^I_{(-m+n)}, A_{\mu(-n)}] \\
&- \frac{1}{2}A'_{\mu(-m)}A'^\mu_{(m)}(\lambda^{I0})^2 - \frac{1}{2}m^2 A_{\mu(-m)}A^\mu_{(m)}(\lambda^{I1})^2 + imA_{\mu(-m)}A'^\mu_{(m)}\lambda^{I0}\lambda^{I1}.
\end{aligned}
\tag{4.29}
$$

Then, we see that if we identify

$$
\begin{aligned}
\lambda^{I0} &= -2\pi(X^1_0, X^2_0, X^3_0, X^4_0, X^5_0, X^6_0, X^7_0, X^8_0), \\
\lambda^{I1} &= -2\pi\Big(sX^1_0, (s+1)X^2_0, sX^3_0, (s+1)X^4_0, sX^5_0, (s+1)X^6_0, sX^7_0, (s+1)X^8_0\Big),
\end{aligned}
\tag{4.30}
$$

both kinetic terms completely agree.

For the Chern-Simons term, we can show the agreement easily:

$$
\begin{aligned}
&\frac{k}{4\pi}\epsilon^{\mu\nu\lambda}\Big[A^{(2l-1)}_\mu\partial_\nu A^{(2l-1)}_\lambda + \frac{2i}{3}A^{(2l-1)}_\mu A^{(2l-1)}_\nu A^{(2l-1)}_\lambda - A^{(2l)}_\mu\partial_\nu A^{(2l)}_\lambda - \frac{2i}{3}A^{(2l)}_\mu A^{(2l)}_\nu A^{(2l)}_\lambda\Big] \\
&= \frac{k}{2\pi}\epsilon^{\mu\nu\lambda}A^{(-)(2l-1)}_\mu F^{(2l-1)}_{\nu\lambda} + \frac{4i}{3}\epsilon^{\mu\nu\lambda}A^{(-)(2l-1)}_\mu A^{(-)(2l-1)}_\nu A^{(-)(2l-1)}_\lambda \\
&= \frac{k}{2}\epsilon^{\mu\nu\lambda}\frac{q^{l(m+n)}}{n}A'_{\mu(m)}F_{\nu\lambda(n)} + \frac{ik}{3\pi}\epsilon^{\mu\nu\lambda}\frac{q^{lm}}{n^3}A'_{\mu(n)}A'_{\nu(k)}A'_{\lambda(m-n-k)} \\
&\to \frac{k}{2}\epsilon^{\mu\nu\lambda}A'_{\mu(m)}F_{\nu\lambda(-m)},
\end{aligned}
\tag{4.31}
$$

where $F^{(2l-1)}_{\nu\lambda} = \partial_\nu A^{(+)(2l-1)}_\lambda - \partial_\lambda A^{(+)(2l-1)}_\nu + i[A^{(+)(2l-1)}_\nu, A^{(+)(2l-1)}_\lambda]$. Note that we have chosen $k = 1$ in the BLG side.

In the Extended L-BLG theory, VEVs λ^{IA} are related to the metric of two-torus as (4.5). By constructing the metric G^{AB} from (4.30), we see that the metric components are connected as

$$
G^{11} = -s(s+1)G^{00} + (2s+1)G^{01}. \tag{4.32}
$$

Thus, in the scaling limit of the $\mathcal{N} = 4$ quiver CS theory, only a specific class of the T^2 compactification is realizable. This is because we have chosen a particular $\mathbb{Z}_n$ orbifold. Owing to the constraint (4.32), the complexified coupling constant τ of the resultant D3-brane theory is limited to the one that depends on only one real variable. We will return to this point in Section 5.

Now, let us check the potential term. By decomposing the matter fields Y^A_l into the trace part Y^A_0 and the traceless part $\hat{Y}^A_l$, the bosonic sextic potential term becomes $V_{bos} = \sum_{s=0}^{6} V^{(s)}_{bos}$, where $V^{(s)}_{bos}$ contains s Y_0 fields and $(6-s)$ $\hat{Y}$ fields. It can be easily checked that $V^{(6)}_{bos}$ and $V^{(5)}_{bos}$

are indentically zero. Since $V_{bos}^{(s)}$ scales as $n^{\frac{s}{2}-\frac{6-s}{2}+1} = n^{s-2}$ in our limit (4.25), $V_{bos}^{(0)}$ and $V_{bos}^{(1)}$ vanish. Note that there is an additional factor n that comes from the relation $\sum_l q^{lm} = n\delta_{m,0}$. Therefore, the remaining terms are $V_{bos}^{(2)}$, $V_{bos}^{(3)}$, and $V_{bos}^{(4)}$.

First, we consider the scaling limit of $V_{bos}^{(2)}$. In this case, we can utilize the result in [13] and obtain the scaling limit easily. The key point is the fact that the relative difference of label l becomes $\mathcal{O}(n^{-\frac{3}{2}})$ under the expansion $q^{lm} = 1 + \frac{2\pi i lm}{n} + \mathcal{O}(n^{-2})$:

$$(\hat{Y}_{2l} - \hat{Y}_{2(l+k)}) \to \frac{q^{lm}}{\sqrt{n}}(Y_m - q^{km}Y_m) = \mathcal{O}(n^{-\frac{3}{2}}). \tag{4.33}$$

This means that in the scaling limit of $V_{bos}^{(2)}$, the relative difference between the labels of $\hat{Y}_{2l}$ (or $\hat{Y}_{2l-1}$ in the odd case) does not contribute to the result. To show this explicitly, let us consider the scaling limit of the following substraction:

$$Y_{0,2l}Y_{0\ 2l}^{\dagger}\hat{Y}_{2l}\hat{Y}_{2l}^{\dagger}(\hat{Y}_{2(l+k)} - \hat{Y}_{2l})\hat{Y}_{2l}^{\dagger} \to \mathcal{O}(n^{-1}) = 0. \tag{4.34}$$

Note that if the numbers of $Y_{0,l}$ and $\hat{Y}_l$ are different, the situation entirely changes. Indeed, for the scaling limit of $V_{bos}^{(3)}$ and $V_{bos}^{(4)}$, the relative difference between the labels of $\hat{Y}_l$ is essential. The relation like (4.34) holds in all the terms of (4.20). Therefore, even if we replace all the $Y_{2(l+k)-1}^{A}$ with Y_{2l-1}^{A} (and $Y_{2(l+k)}^{A}$ with Y_{2l}^{A}) in (4.20), the resultant potential gives the same scaling limit as long as we focus on the $Y_{0,l}$-squared term. We denote this new potential as V'

$$\begin{aligned}
V' = -\frac{4\pi^2}{3k^2}\Big[& \mathrm{tr}Y_{2l}^{A}Y_{A,2l}^{\dagger}Y_{2l}^{B}Y_{B,2l}^{\dagger}Y_{2l}^{C}Y_{C,2l}^{\dagger} + 3\mathrm{tr}Y_{2l}^{A}Y_{A,2l}^{\dagger}Y_{2l}^{B}Y_{B,2l}^{\dagger}Y_{2l-1}^{C}Y_{C,2l-1}^{\dagger} \\
& + 3\mathrm{tr}Y_{2l}^{A}Y_{A,2l}^{\dagger}Y_{2l-1}^{B}Y_{B,2l-1}^{\dagger}Y_{2l-1}^{C}Y_{C,2l-1}^{\dagger} + \mathrm{tr}Y_{2l-1}^{A}Y_{A,2l-1}^{\dagger}Y_{2l-1}^{B}Y_{B,2l-1}^{\dagger}Y_{2l-1}^{C}Y_{C,2l-1}^{\dagger} \\
& \\
& + \mathrm{tr}Y_{A,2l-1}^{\dagger}Y_{2l-1}^{A}Y_{B,2l-1}^{\dagger}Y_{2l-1}^{B}Y_{C,2l-1}^{\dagger}Y_{2l-1}^{C} + 3\mathrm{tr}Y_{A,2l-1}^{\dagger}Y_{2l-1}^{A}Y_{B,2l-1}^{\dagger}Y_{2l-1}^{B}Y_{C,2l}^{\dagger}Y_{2l}^{C} \\
& + 3\mathrm{tr}Y_{A,2l-1}^{\dagger}Y_{2l-1}^{A}Y_{B,2l}^{\dagger}Y_{2l}^{B}Y_{C,2l}^{\dagger}Y_{2l}^{C} + \mathrm{tr}Y_{A,2l}^{\dagger}Y_{2l}^{A}Y_{B,2l}^{\dagger}Y_{2l}^{B}Y_{C,2l}^{\dagger}Y_{2l}^{C} \\
& \\
& + 4\mathrm{tr}Y_{2l-1}^{A}Y_{B,2l-1}^{\dagger}Y_{2l-1}^{C}Y_{A,2l-1}^{\dagger}Y_{2l-1}^{B}Y_{C,2l-1}^{\dagger} + 12\mathrm{tr}Y_{2l}^{A}Y_{B,2l}^{\dagger}Y_{2l-1}^{C}Y_{A,2l}^{\dagger}Y_{2l}^{B}Y_{C,2l-1}^{\dagger} \\
& + 12\mathrm{tr}Y_{2l-1}^{A}Y_{B,2l-1}^{\dagger}Y_{2l}^{C}Y_{A,2l-1}^{\dagger}Y_{2l-1}^{B}Y_{C,2l}^{\dagger} + 4\mathrm{tr}Y_{2l}^{A}Y_{B,2l}^{\dagger}Y_{2l}^{C}Y_{A,2l}^{\dagger}Y_{2l}^{B}Y_{C,2l}^{\dagger} \\
& \\
& - 6\mathrm{tr}Y_{2l-1}^{A}Y_{B,2l-1}^{\dagger}Y_{2l-1}^{B}Y_{A,2l-1}^{\dagger}Y_{2l-1}^{C}Y_{C,2l-1}^{\dagger} - 6\mathrm{tr}Y_{2l}^{A}Y_{B,2l}^{\dagger}Y_{2l}^{B}Y_{A,2l}^{\dagger}Y_{2l}^{C}Y_{C,2l}^{\dagger} \\
& - 6\mathrm{tr}Y_{2l-1}^{A}Y_{B,2l-1}^{\dagger}Y_{2l-1}^{B}Y_{A,2l-1}^{\dagger}Y_{2l}^{C}Y_{C,2l}^{\dagger} - 6\mathrm{tr}Y_{2l}^{A}Y_{B,2l}^{\dagger}Y_{2l}^{B}Y_{A,2l}^{\dagger}Y_{2l-1}^{C}Y_{C,2l-1}^{\dagger} \\
& - 6\mathrm{tr}Y_{2l-1}^{A}Y_{B,2l}^{\dagger}Y_{2l}^{B}Y_{A,2l-1}^{\dagger}Y_{2l-1}^{C}Y_{C,2l-1}^{\dagger} - 6\mathrm{tr}Y_{2l}^{A}Y_{B,2l-1}^{\dagger}Y_{2l-1}^{B}Y_{A,2l}^{\dagger}Y_{2l}^{C}Y_{C,2l}^{\dagger} \\
& - 6\mathrm{tr}Y_{2l-1}^{A}Y_{B,2l}^{\dagger}Y_{2l}^{B}Y_{A,2l-1}^{\dagger}Y_{2l}^{C}Y_{C,2l}^{\dagger} - 6\mathrm{tr}Y_{2l}^{A}Y_{B,2l-1}^{\dagger}Y_{2l-1}^{B}Y_{A,2l}^{\dagger}Y_{2l-1}^{C}Y_{C,2l-1}^{\dagger}\Big].
\end{aligned} \tag{4.35}$$

V' is convenient because it can be simplified. If we rewrite each field as

$$Y_{2l-1}^{1} \to Y_l^1, \quad Y_{2l}^{1} \to Y_l^2, \quad Y_{2l-1}^{2} \to Y_l^3, \quad Y_{2l}^{2} \to Y_l^4, \tag{4.36}$$

V' becomes

$$-\frac{4\pi^2}{3k^2}\Big[Y_l^{A'}Y^\dagger_{A',l}Y_l^{B'}Y^\dagger_{B',l}Y_l^{C'}Y^\dagger_{C',l}+Y^\dagger_{A',l}Y_l^{A'}Y^\dagger_{B',l}Y_l^{B'}Y^\dagger_{C',l}Y_l^{C'} \\ +4Y_l^{A'}Y^\dagger_{B',l}Y_l^{C'}Y^\dagger_{A',l}Y_l^{B'}Y^\dagger_{C',l}-6Y_l^{A'}Y^\dagger_{B',l}Y_l^{B'}Y^\dagger_{A',l}Y_l^{C'}Y^\dagger_{C',l}\Big], \tag{4.37}$$

where $A', B', C' = 1, \cdots, 4$. This is just the original ABJM potential with an extra label l. The scaling limit of the original ABJM bosonic potential is already obtained in [13] and the result is

$$\mathrm{tr}(X_0^I)^2([P^{IK}X^K, P^{JL}X^L])^2. \tag{4.38}$$

Using this result, we can obtain the scaling limit of $V^{(2)}_{bos}$:

$$V^{(2)}_{bos} \to -\frac{\pi^2}{k^2}(X_0^I)^2[P^{IK}X^K_{(m)}, P^{JL}X^L_{(-m)}]. \tag{4.39}$$

This agrees with the last term of (4.10).

Next we consider the scaling limit of $V^{(4)}_{bos}$ and $V^{(3)}_{bos}$. As before, we can decompose V' as $V' = \sum_{s=0}^{6} V'^{(s)}$. Using the same argument, we see that only $V'^{(2)}$, $V'^{(3)}$, and $V'^{(4)}$ remain in the scaling limit.

In (4.38), more insertion of X_0^K to X^K gives zero. Therefore, $V'^{(3)}$ and $V'^{(4)}$ are zero. This means that the scaling limit of $V_{bos} - V'$ is the same as the scaling limit of $V^{(3)}_{bos} + V^{(4)}_{bos}$. It is convenient to consider $V_{bos} - V_0$ because it is much simpler than V_{bos} itself. The explicit form of $V_{bos} - V'$ is given by

$$V_{bos} - V' = V_1 + V_2, \tag{4.40}$$

where

$$V_1 = -\frac{4\pi^2}{3k^2}\mathrm{tr}\big[3Y^A_{2l-1}Y^\dagger_{A,2l-1}Y^B_{2l-1}Y^\dagger_{B,2l-1}(Y^C_{2l-2}Y^\dagger_{C,2l-2}-Y^C_{2l}Y^\dagger_{C,2l}) \\ +12Y^C_{2l}Y^\dagger_{A,2l-1}Y^B_{2l-1}Y^\dagger_{C,2l}(Y^A_{2l+1}Y^\dagger_{B,2l+1}-Y^A_{2l-1}Y^\dagger_{B,2l-1}) \\ -6Y^A_{2l-1}Y^\dagger_{B,2l-1}Y^B_{2l-1}Y^\dagger_{A,2l-1}(Y^C_{2l-2}Y^\dagger_{C,2l-2}-Y^C_{2l}Y^\dagger_{C,2l}) \\ -6Y^C_{2l}Y^\dagger_{A,2l-1}Y^A_{2l-1}Y^\dagger_{C,2l}(Y^B_{2l+1}Y^\dagger_{B,2l+1}-Y^B_{2l-1}Y^\dagger_{B,2l-1})\big], \tag{4.41}$$

and

$$V_2 = -\frac{4\pi^2}{3k^2}\mathrm{tr}\big[3Y^A_{2l}Y^\dagger_{A,2l}Y^B_{2l}Y^\dagger_{B,2l}(Y^C_{2l+1}Y^\dagger_{C,2l+1}-Y^C_{2l-1}Y^\dagger_{C,2l-1}) \\ +12Y^C_{2l-1}Y^\dagger_{A,2l}Y^B_{2l}Y^\dagger_{C,2l-1}(Y^A_{2l-2}Y^\dagger_{B,2l-2}-Y^A_{2l}Y^\dagger_{B,2l}) \\ -6Y^A_{2l}Y^\dagger_{B,2l}Y^B_{2l}Y^\dagger_{A,2l}(Y^C_{2l+1}Y^\dagger_{C,2l+1}-Y^C_{2l-1}Y^\dagger_{C,2l-1}) \\ -6Y^C_{2l-1}Y^\dagger_{A,2l}Y^A_{2l}Y^\dagger_{C,2l-1}(Y^B_{2l-2}Y^\dagger_{B,2l-2}-Y^B_{2l}Y^\dagger_{B,2l})\big]. \tag{4.42}$$

Note that V_1 and V_2 can be translated into each other by exchanging Y^A_{2l} for Y^A_{2l-1} and Y^A_{2l-2} for Y^A_{2l+1}. Since the rescaling rule (4.25) is written as

$$Y^A_{2l} \to \frac{q^{lm}}{\sqrt{n}}\frac{Y^{2A}_m}{\sqrt{2}}, \quad Y^A_{2l-1} \to \frac{q^{lm}}{\sqrt{n}}\frac{Y^{2A-1}_m}{\sqrt{2}}, \quad Y^A_{2l-2} \to q^{-m}\frac{q^{lm}}{\sqrt{n}}\frac{Y^{2A}_m}{\sqrt{2}}, \quad Y^A_{2l+1} \to q^{m}\frac{q^{lm}}{\sqrt{n}}\frac{Y^{2A-1}_m}{\sqrt{2}}, \tag{4.43}$$

the above translation corresponds to a translation between Y_m^{2A} and Y_m^{2A+1}.

Therefore, to obtain the scaling limit of V_1 and V_2, we only need to calculate one of them. The other one is obtained from the translation.

With the above simplifications, the scaling limit of $V_{bos}^{(4)}$ can be calculated more easily. The result is

$$\frac{m^2(16\pi^4)}{2}(X_0^{2C}X_0^{2C}X_0^{2A-1}X_0^{2A-1}\hat{X}_{(i,m)}^{2B-1}\hat{X}_{(i,-m)}^{2B-1} - X_0^{2C}X_0^{2C}X_0^{2A-1}X_0^{2B-1}\hat{X}_{(i,m)}^{2A-1}\hat{X}_{(i,-m)}^{2B-1}$$
$$+X_0^{2C-1}X_0^{2C-1}X_0^{2A}X_0^{2A}\hat{X}_{(i,m)}^{2B}\hat{X}_{(i,-m)}^{2B} - X_0^{2C-1}X_0^{2C-1}X_0^{2A}X_0^{2B}\hat{X}_{(i,m)}^{2A}\hat{X}_{(i,-m)}^{2B}). \quad (4.44)$$

This is just the first term of (4.10) with the assignment (4.30). To see how the above terms come from the Extended L-BLG potential, it is convenient to use an expression

$$m^2\lambda^{[I0}\lambda^{J1}X_{i,m}^{K]}\lambda^{[I0}\lambda^{J1}X_{i,-m}^{K]} \quad (4.45)$$

and substitute (4.30) into this term. Then, we obtain (4.44). Note that the result does not depend on s, because the s-dependent part of λ^{I1} is proportional to λ^{I0} and the indices I, J, and K are antisymmetrized so that s dependent terms are cancelled.

Similarly, the scaling limit of $V_{bos}^{(3)}$ is given by

$$(2\pi)^3\mathrm{tr}\{(2m+n)X_0^{2C}X_0^{2A-1}X_0^{2B-1}\hat{X}_m^{2A-1}[\hat{X}_n^{2C}, \hat{X}_{-m-n}^{2B-1}]$$
$$+mX_0^{2C}X_0^{2C}X_0^{2B-1}\hat{X}_m^{2A-1}[\hat{X}_n^{2B-1}, \hat{X}_{-m-n}^{2A-1}] - mX_0^{2C}X_0^{2B-1}X_0^{2B-1}\hat{X}_m^{2A-1}[\hat{X}_n^{2C}, \hat{X}_{-m-n}^{2A-1}]$$
$$-(2m+n)X_0^{2C-1}X_0^{2A}X_0^{2B}\hat{X}_m^{2A}[\hat{X}_n^{2C-1}, \hat{X}_{-m-n}^{2B}]$$
$$-mX_0^{2C-1}X_0^{2C-1}X_0^{2B}\hat{X}_m^{2A}[\hat{X}_n^{2B}, \hat{X}_{-m-n}^{2A}] + mX_0^{2C-1}X_0^{2B}X_0^{2B}\hat{X}_m^{2A}[\hat{X}_n^{2C-1}, \hat{X}_{-m-n}^{2A}]\}.$$
$$(4.46)$$

Note that the overall signs of $V_1^{(3)}$ and $V_2^{(3)}$ are opposite owing to the factors $q^{\pm m}$ in (4.43). (4.46) agrees with the second term of (4.10).

Fermionic sector We have seen the agreement of the bosonic sector. Here, we consider the fermionic sector of the $\mathcal{N} = 4$ quiver CS theory and confirm the emergence of the Extended L-BLG theory. The nontrivial part is the fermionic potential.

In the Extended L-BLG theory, the fermionic interaction term is given by

$$L_{int} = \frac{m_a}{4}\bar{\psi}_{(i-\vec{m})}(\Gamma_{IJ}\lambda^{I0}\lambda^{Ja})\psi_{(i,\vec{m})} + \frac{1}{4}\bar{\psi}_{(i\vec{m})}\lambda^{I0}[X^J, \Gamma_{IJ}\psi]_{(i,-\vec{m})}. \quad (4.47)$$

Substituting (4.12) into (4.47), we can indeed obtain the fermionic sector of the Dp-brane action.

On the other hand, the fermionic potential of the $\mathcal{N}=4$ quiver CS theory is given by

$$
\begin{aligned}
V_{ferm} = -\frac{iL}{4}\mathrm{tr}\Big[& Y^{\dagger}_{A,2l-1}Y^{A}_{2l-1}\Psi^{B\dagger}_{2l-1}\Psi_{B,2l-1} + Y^{\dagger}_{A,2l-1}Y^{A}_{2l-1}\Psi^{B\dagger}_{2l}\Psi_{B,2l} \\
& + Y^{\dagger}_{A,2l}Y^{A}_{2l}\Psi^{B\dagger}_{2l-1}\Psi_{B,2l-1} + Y^{\dagger}_{A,2l}Y^{A}_{2l}\Psi^{B\dagger}_{2l}\Psi_{B,2l} \\
& \\
& - Y^{A}_{2l-1}Y^{\dagger}_{A,2l-1}\Psi_{B,2l-1}\Psi^{B\dagger}_{2l-1} - Y^{A}_{2l+1}Y^{\dagger}_{A,2l+1}\Psi_{B,2l}\Psi^{B\dagger}_{2l} \\
& - Y^{A}_{2l}Y^{\dagger}_{A,2l}\Psi_{B,2l+1}\Psi^{B\dagger}_{2l+1} - Y^{A}_{2l}Y^{\dagger}_{A,2l}\Psi_{B,2l}\Psi^{B\dagger}_{2l} \\
& \\
& + 2Y^{A}_{2l-1}Y^{\dagger}_{B,2l}\Psi_{A,2l}\Psi^{B\dagger}_{2l-1} + 2Y^{A}_{2l}Y^{\dagger}_{B,2l-1}\Psi_{A,2l-1}\Psi^{B\dagger}_{2l} \\
& + 2Y^{A}_{2l}Y^{\dagger}_{B,2l}\Psi_{A,2l+1}\Psi^{B\dagger}_{2l+1} + 2Y^{A}_{2l+1}Y^{\dagger}_{B,2l+1}\Psi_{A,2l}\Psi^{B\dagger}_{2l} \\
& \\
& - 2Y^{\dagger}_{A,2l-1}Y^{B}_{2l-1}\Psi^{A\dagger}_{2l}\Psi_{B,2l} - 2Y^{\dagger}_{A,2l}Y^{B}_{2l}\Psi^{A\dagger}_{2l-1}\Psi_{B,2l-1} \\
& - 2Y^{\dagger}_{A,2l}Y^{B}_{2l+1}\Psi^{A\dagger}_{2l+1}\Psi_{B,2l} - 2Y^{\dagger}_{A,2l+1}Y^{B}_{2l}\Psi^{A\dagger}_{2l}\Psi_{B,2l+1} \\
& \\
& - \epsilon^{AB}\epsilon^{CD}Y^{\dagger}_{A,2l-1}\Psi_{C,2l-1}Y^{\dagger}_{B,2l-1}\Psi_{D,2l-1} - \epsilon^{AB}\epsilon^{CD}Y^{\dagger}_{A,2l}\Psi_{C,2l}Y^{\dagger}_{B,2l}\Psi_{D,2l} \\
& + 2\epsilon^{AB}\epsilon^{CD}Y^{\dagger}_{A,2l-1}\Psi_{C,2l-1}Y^{\dagger}_{D,2l}\Psi_{B,2l} + 2\epsilon^{AB}\epsilon^{CD}Y^{\dagger}_{A,2l+1}\Psi_{B,2l}Y^{\dagger}_{C,2l}\Psi_{D,2l+1} \\
& \\
& + \epsilon_{AB}\epsilon_{CD}Y^{A}_{2l-1}\Psi^{C\dagger}_{2l-1}Y^{B}_{2l-1}\Psi^{D\dagger}_{2l-1} + \epsilon_{AB}\epsilon_{CD}Y^{A}_{2l}\Psi^{C\dagger}_{2l}Y^{B}_{2l}\Psi^{D\dagger}_{2l} \\
& - 2\epsilon_{AB}\epsilon_{CD}Y^{A}_{2l-1}\Psi^{B\dagger}_{2l}Y^{C}_{2l}\Psi^{D\dagger}_{2l-1} - 2\epsilon_{AB}\epsilon_{CD}Y^{A}_{2l+1}\Psi^{C\dagger}_{2l+1}Y^{D}_{2l}\Psi^{B\dagger}_{2l}\Big],
\end{aligned} \tag{4.48}
$$

where $\epsilon^{12} = -\epsilon_{12} = 1$ and we used doublets

$$
Y_l{}^{A} = \{Z^{(l)}, W^{(l)\dagger}\} \quad , \quad \Psi_{A,l} = \{(-1)^{l-1}e^{-i\pi/4}\zeta^{(l)}, (-1)^{l}e^{i\pi/4}\omega^{(l)\dagger}\}. \qquad (A=1,2) \tag{4.49}
$$

The label l of $\zeta^{(l)}$ and $\omega^{(l)}$ was determined from the following orbifold projection of the $nN \times nN$ ABJM fermions:

$$
\zeta^{1} = \begin{pmatrix} 0 & \zeta^{(1)} & & & \\ & 0 & \zeta^{(3)} & & \\ & & 0 & \ddots & \\ & & & 0 & \zeta^{(2n-3)} \\ \zeta^{(2n-1)} & & & & 0 \end{pmatrix}, \quad \omega_{1} = \begin{pmatrix} 0 & & & & \omega^{(2n-1)} \\ \omega^{(1)} & 0 & & & \\ & \omega^{(3)} & 0 & & \\ & & \ddots & 0 & \\ & & & \omega^{(2n-3)} & 0 \end{pmatrix},
$$

$$
\zeta^{2} = \mathrm{diag}(\zeta^{(2n)}, \zeta^{(2)}, \cdots, \zeta^{(2n-2)}) \ , \qquad \omega_{2} = \mathrm{diag}(\omega^{(2n)}, \omega^{(2)}, \cdots, \omega^{(2n-2)}). \tag{4.50}
$$

Each $\zeta^{(l)}$ and $\omega^{(l)}$ $(l = 1, 2, \cdots, 2n)$ are $N \times N$ matrices.

Now, we investigate the scaling limit of (4.48). The appropriate rescalings of the fermions are given by

$$
\Psi^{1}_{(2l-1)} \to \frac{q^{lm}}{\sqrt{n}}\frac{\Psi^{(2)}_{(m)}}{2} \ , \ \Psi^{1}_{(2l)} \to \frac{q^{lm}}{\sqrt{n}}\frac{\Psi^{(1)}_{(m)}}{2} \ , \ \Psi^{2}_{(2l-1)} \to \frac{q^{(l-2)m}}{\sqrt{n}}\frac{\Psi^{(4)}_{(m)}}{2} \ , \ \Psi^{2}_{(2l)} \to \frac{q^{lm}}{\sqrt{n}}\frac{\Psi^{(3)}_{(m)}}{2}. \tag{4.51}
$$

In analogy with the bosonic potential, after the decomposition $Y^A_{(l)} = Y_0^A \mathbf{1}_{N\times N} + \hat{Y}^A_{(l)}$, the fermionic potential becomes $V_{ferm} = \sum_{s=0}^{2} V^{(s)}_{ferm}$, where $V^{(s)}_{ferm}$ contains s Y_0 fields and $(2-s)$ $\hat{Y}$ fields. Obviously, $V^{(0)}_{ferm}$ vanishes in the limit $n \to \infty$. Thus, the remaining terms are $V^{(1)}_{ferm}$ and $V^{(2)}_{ferm}$.

First, let us consider the $V^{(2)}_{ferm}$ term. For simplicity, we consider the case where only the $Y_0^{(1)}$ and $Y_0^{(2)}$ are nonzero. Then the surviving terms in the limit $n \to \infty$ are summarized as

$$\begin{aligned}\frac{4\pi^2 m}{k}\mathrm{tr}\Big[&2Y_0^{(2)\dagger}Y_0^{(1)}\Psi^{(2)\dagger}_{(m)}\Psi^{(1)}_{(m)} - 2Y_0^{(1)\dagger}Y_0^{(2)}\Psi^{(1)\dagger}_{(m)}\Psi^{(2)}_{(m)}\\ &- 2Y_0^{(1)\dagger}Y_0^{(2)\dagger}\Psi^{(3)}_{(-m)}\Psi^{(4)}_{(m)} + 2Y_0^{(1)}Y_0^{(2)}\Psi^{(4)\dagger}_{(m)}\Psi^{(3)\dagger}_{(-m)}\Big].\end{aligned} \tag{4.52}$$

After the decomposition of the fermions into the 2-component Majorana spinors as

$$\Psi_{A(m)} = i\chi_{A(m)} - \chi_{A+4(m)}, \tag{4.53}$$

we obtain various bilinear terms of $\chi_{1(m)}, \cdots, \chi_{8(m)}$. Using the appropriate Gamma matrices, the assignment (4.30), and the identification $\psi^T_{(m)} = (\chi^T_{1(m)}, \cdots, \chi^T_{8(m)})$, we can show that these bilinear terms agree with the first term of (4.47). The explicit forms of the Gamma matrices are written in the Appendix.

As for the $V^{(1)}_{ferm}$ term, the situation is the same as the $V^{(2)}_{bos}$ term. In the scaling limit, we just need to consider whether the index l of Y^A_l and Ψ^A_l is odd or even, namely, we can replace all the $Y^A_{l'}$ ($l' \in \mathbb{Z}$) with Y^A_{2l-1} or Y^A_{2l}. This denotes that the fermion potential of the original ABJM theory with the additional labels l

$$\begin{aligned}-\frac{2\pi i}{k}\mathrm{tr}\,[&Y^\dagger_{A,l}Y^A_l\Psi^{B\dagger}_l\Psi_{B,l} - Y^A_l Y^\dagger_{A,l}\Psi_{B,l}\Psi^{B\dagger}_l + 2Y^A_l Y^\dagger_{B,l}\Psi_{A,l}\psi^{B\dagger}_l - 2Y^\dagger_{A,l}Y^B_l\Psi^{A\dagger}_l\Psi_{B,l}\\ &+ \epsilon^{ABCD}Y^\dagger_{A,l}\Psi_{B,l}Y^\dagger_{C,l}\Psi_{D,l} - \epsilon_{ABCD}Y^A_l\Psi^{B\dagger}_l Y^C_l\Psi^{D\dagger}_l],\end{aligned} \tag{4.54}$$

and the $V^{(1)}_{ferm}$ term become coincident in the scaling limit. Therefore, using the result in [13] that the ABJM fermionic potential scales as

$$\bar{\psi}X^I_0[X^J, \Gamma_{IJ}\psi], \tag{4.55}$$

we can say that the scaling limit of the $V^{(1)}_{ferm}$ term is given by

$$-\frac{\pi}{2k}\bar{\psi}_{(m)}X^I_0[X^J, \Gamma_{IJ}\psi]_{(-m)}, \tag{4.56}$$

where $\psi^T_{(m)} = (\chi^T_{1(m)}, \cdots, \chi^T_{8(m)})$. This agrees with the second term of (4.47).

Therefore, we completely verify the emergence of the Extended L-BLG theory with two Lorentzian pairs from the scaling limit of the $\mathcal{N} = 4$ quiver CS theory. This means that we obtain a concrete prescription for gaining D3-brane theory from the ABJM theory, because the Extended L-BLG theory with $d = 1$ can be reduced to the D3-brane theory.

4.4 Applications to the other quiver Chern-Simons theories

Thus far, we have only discussed a particular $\mathcal{N}=4$ quiver CS theory (4.18). However, by orbifolding the ABJM theory, we can obtain infinitely many quiver CS theories. Thus, here, we apply our scaling limit to various quiver CS theories.

(I) $\mathbb{C}^2 \times \mathbb{C}^2/\mathbb{Z}_n$

The $\mathbb{Z}_n$ action (4.23) was of the $\mathbb{C}^2 \times \mathbb{C}^2/\mathbb{Z}_n$ type. As another example of this type, let us consider the following $\mathbb{Z}_n$ orbifolding action[4]:

$$y^1 \to e^{2\pi i/n} y^1, \quad y^2 \to e^{-2\pi i/n} y^2, \quad y^3 \to y^3, \quad y^4 \to y^4. \tag{4.57}$$

This preserves $\mathcal{N}=2$ supersymmetry and $SU(2)$ global symmetry. The covariant derivatives are

$$\begin{aligned} D_\mu Z^{(2l-1)} &= \partial_\mu Z^{(2l-1)} + iA_\mu^{(2l-1)} Z^{(2l-1)} - iZ^{(2l-1)} A_\mu^{(2l)}, \\ D_\mu Z^{(2l)} &= \partial_\mu Z^{(2l)} + iA_\mu^{(2l+1)} Z^{(2l)} - iZ^{(2l)} A_\mu^{(2l-2)}, \\ D_\mu W^{(2l-1)} &= \partial_\mu W^{(2l-1)} + iA_\mu^{(2l-2)} W^{(2l-1)} - iW^{(2l-1)} A_\mu^{(2l-1)}, \\ D_\mu W^{(2l)} &= \partial_\mu W^{(2l)} + iA_\mu^{(2l)} W^{(2l)} - iW^{(2l)} A_\mu^{(2l+1)}, \end{aligned} \tag{4.58}$$

where $l = 1, \cdots, n$. The $Z^{(2l)}, W^{(2l-1)}$ parts are changed from the $\mathcal{N}=4$ case (4.19). Figure 4.2 is the corresponding quiver diagram.

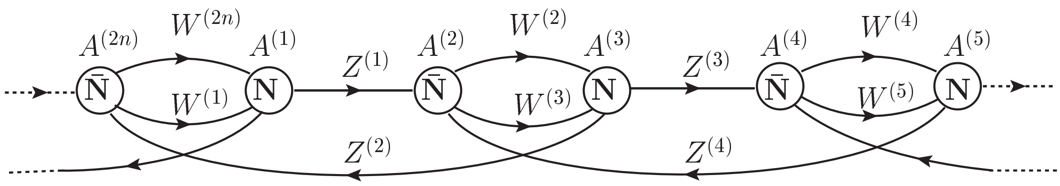

Figure 4.2: Quiver diagram for case (I).

In this theory, the Chern-Simons term is unchanged from the $\mathcal{N}=4$ case. Thus, its scaling limit is completely the same as that of (4.31). As for the kinetic term, the covariant derivatives are scaled as

$$\begin{aligned} D_\mu Z_{(2l)} \to\ & \frac{q^{lm}}{\sqrt{2n}} \partial_\mu Y^{(2)}_{(m)} + i\frac{q^{lm}}{\sqrt{2n}} [A_{\mu(n)}, Y^{(2)}_{(m-n)}] - \frac{2\pi(s+2)mq^{lm}}{\sqrt{2n}} A_{\mu(m)} Y^{(2)}_0 + i\frac{2\pi q^{lm}}{\sqrt{2n}} A'_{\mu(m)} Y^{(2)}_0, \\ D_\mu W_{(2l-1)} \to\ & \frac{q^{-lm}}{\sqrt{2n}} \partial_\mu Y^{(3)\dagger}_{(m)} + i\frac{q^{lm}}{\sqrt{2n}} [A_{\mu(n)}, Y^{(3)\dagger}_{(n-m)}] + \frac{2\pi(s+1)mq^{lm}}{\sqrt{2n}} A_{\mu(m)} Y^{(3)\dagger}_0 \\ & - i\frac{2\pi q^{lm}}{\sqrt{2n}} A'_{\mu(m)} Y^{(3)\dagger}_0. \end{aligned} \tag{4.59}$$

[4]This is the "chiral orbifold gauge theory" described in [35].

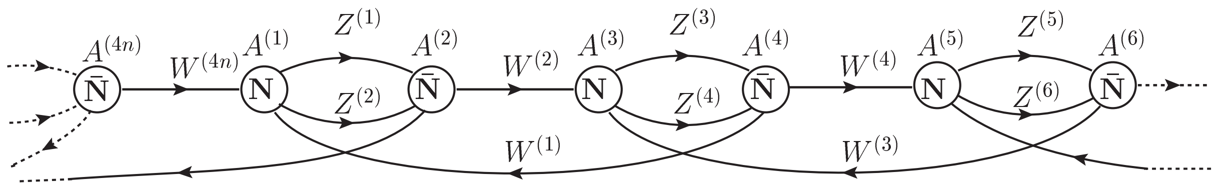

Figure 4.3: Quiver diagram for case (II)-(i).

Again, through the assignments

$$\begin{aligned}\lambda^{I0} &= -2\pi(X_0^1, X_0^2, X_0^3, X_0^4, X_0^5, X_0^6, X_0^7, X_0^8),\\ \lambda^{I1} &= -2\pi\Big(sX_0^1, (s+2)X_0^2, (s+1)X_0^3, (s+1)X_0^4, sX_0^5, (s+2)X_0^6, (s+1)X_0^7, (s+1)X_0^8\Big),\end{aligned} \tag{4.60}$$

we see that the kinetic term completely agrees with (4.29). The constraint for the metric of two-torus is calculated as

$$G^{11} = -s(s+1)G^{00} + (2s+1)G^{01} + 8\pi^2[(X_0^2)^2 + (X_0^6)^2]. \tag{4.61}$$

The difference from the previous case is an appearance of a term $(X_0^2)^2 + (X_0^6)^2$. This indicates that we can cover a larger parameter space of the coupling constant τ than the $\mathcal{N} = 4$ quiver CS theories, as we will see in Section 4.

(II) $\mathbb{C} \times \mathbb{C}^3/\mathbb{Z}_n$

(i) Now, we consider the $\mathbb{Z}_{2n}$ action given by

$$y^1 \to e^{2\pi i/2n} y^1, \quad y^2 \to e^{2\pi i/2n} y^2, \quad y^3 \to e^{2\pi i/n} y^3, \quad y^4 \to y^4. \tag{4.62}$$

The quiver CS theory based on this orbifolding also has $\mathcal{N} = 2$ SUSY and $SU(2)$ global symmetry. The quiver diagram of this theory is given in Figure 4.3. The covariant derivatives are given by

$$\begin{aligned}D_\mu Z^{(2l-1)} &= \partial_\mu Z^{(2l-1)} + iA_\mu^{(2l-1)} Z^{(2l-1)} - iZ^{(2l-1)} A_\mu^{(2l)},\\ D_\mu Z^{(2l)} &= \partial_\mu Z^{(2l)} + iA_\mu^{(2l-1)} Z^{(2l)} - iZ^{(2l)} A_\mu^{(2l)},\\ D_\mu W^{(2l-1)} &= \partial_\mu W^{(2l-1)} + iA_\mu^{(2l+2)} W^{(2l-1)} - iW^{(2l-1)} A_\mu^{(2l-1)},\\ D_\mu W^{(2l)} &= \partial_\mu W^{(2l)} + iA_\mu^{(2l)} W^{(2l)} - iW^{(2l)} A_\mu^{(2l+1)},\end{aligned} \tag{4.63}$$

where $l = 1, \cdots, 2n$. The $Z^{(2l)}, W^{(2l-1)}$ parts are changed from (4.19). The Chern-Simons term is unchanged from the one in (4.18) except that l runs 1 to $2n$.

In this case, we have to change the scaling limit (4.25) slightly. Because we took a $\mathbb{Z}_{2n}$ orbifolding, we must change n to $2n$ in (4.25) and redefine q as $q \equiv e^{\frac{2\pi i}{2n}}$. Under this limit, the

CS term of the Extended L-BLG theory is properly derived. The covariant derivatives are scaled as

$$
\begin{aligned}
D_\mu Z_{(2l)} \to & \frac{q^{lm}}{\sqrt{4n}}\partial_\mu Y^{(2)}_{(m)} + i\frac{q^{lm}}{\sqrt{4n}}[A_{\mu(n)}, Y^{(2)}_{(m-n)}] - \frac{2\pi s m q^{lm}}{\sqrt{4n}} A_{\mu(m)} Y^{(2)}_0 + i\frac{2\pi q^{lm}}{\sqrt{4n}} A'_{\mu(m)} Y^{(2)}_0, \\
D_\mu W_{(2l-1)} \to & \frac{q^{-lm}}{\sqrt{4n}}\partial_\mu Y^{(3)\dagger}_{(m)} + i\frac{q^{lm}}{\sqrt{4n}}[A_{\mu(n)}, Y^{(3)\dagger}_{(n-m)}] + \frac{2\pi(s-1) m q^{lm}}{\sqrt{4n}} A_{\mu(m)} Y^{(3)\dagger}_0 \\
& - i\frac{2\pi q^{lm}}{\sqrt{4n}} A'_{\mu(m)} Y^{(3)\dagger}_0.
\end{aligned}
\tag{4.64}
$$

Under the identifications

$$
\begin{aligned}
\lambda^{I0} &= -2\pi(X^1_0, X^2_0, X^3_0, X^4_0, X^5_0, X^6_0, X^7_0, X^8_0), \\
\lambda^{I1} &= -2\pi\Big(sX^1_0, sX^2_0, (s-1)X^3_0, (s+1)X^4_0, sX^5_0, sX^6_0, (s-1)X^7_0, (s+1)X^8_0\Big),
\end{aligned}
\tag{4.65}
$$

we can show the agreement of kinetic terms. The constraint to the T^2 metric is

$$
G^{11} = -s(s+1)G^{00} + (2s+1)G^{01} + 8\pi^2[(X^3_0)^2 + (X^7_0)^2]. \tag{4.66}
$$

Note that we have a degree of freedom that corresponds to tuning $[(X^3_0)^2 + (X^7_0)^2]$ as with the case (I).

(ii) Next, as another example of the $\mathbb{C} \times \mathbb{C}^3/\mathbb{Z}_n$ type, we consider the $\mathbb{Z}_{6n}$ action given by

$$
y^1 \to e^{2\pi i/6n} y^1, \quad y^2 \to e^{2\pi i/3n} y^2, \quad y^3 \to e^{2\pi i/2n} y^3, \quad y^4 \to y^4. \tag{4.67}
$$

This orbifold projection also preserves $\mathcal{N}=2$ SUSY, but the remaining global symmetry is less than before. The quiver CS theory obtained from this orbifold action has the following covariant derivatives,

$$
\begin{aligned}
D_\mu Z^{(2l-1)} &= \partial_\mu Z^{(2l-1)} + iA^{(2l-1)}_\mu Z^{(2l-1)} - iZ^{(2l-1)} A^{(2l)}_\mu, \\
D_\mu Z^{(2l)} &= \partial_\mu Z^{(2l)} + iA^{(2l-1)}_\mu Z^{(2l)} - iZ^{(2l)} A^{(2l+2)}_\mu, \\
D_\mu W^{(2l-1)} &= \partial_\mu W^{(2l-1)} + iA^{(2l+4)}_\mu W^{(2l-1)} - iW^{(2l-1)} A^{(2l-1)}_\mu, \\
D_\mu W^{(2l)} &= \partial_\mu W^{(2l)} + iA^{(2l)}_\mu W^{(2l)} - iW^{(2l)} A^{(2l+1)}_\mu,
\end{aligned}
\tag{4.68}
$$

where $l = 1, \cdots, 6n$. Again, the $Z^{(2l)}, W^{(2l-1)}$ parts are changed from (4.19). The corresponding quiver diagram is given in Figure 4.4.

For the Chern-Simons term, under the scaling limit (4.25) with n being replaced by $6n$, the agreement between both theories is easily shown as before. For the kinetic term, the covariant derivatives are scaled as

$$
\begin{aligned}
D_\mu Z_{(2l)} \to & \frac{q^{lm}}{\sqrt{12n}}\partial_\mu Y^{(2)}_{(m)} + i\frac{q^{lm}}{\sqrt{12n}}[A_{\mu(n)}, Y^{(2)}_{(m-n)}] - \frac{2\pi(s-1) m q^{lm}}{\sqrt{12n}} A_{\mu(m)} Y^{(2)}_0 + i\frac{2\pi q^{lm}}{\sqrt{12n}} A'_{\mu(m)} Y^{(2)}_0, \\
D_\mu W_{(2l-1)} \to & \frac{q^{-lm}}{\sqrt{12n}}\partial_\mu Y^{(3)\dagger}_{(m)} + i\frac{q^{lm}}{\sqrt{12n}}[A_{\mu(n)}, Y^{(3)\dagger}_{(n-m)}] + \frac{2\pi(s-2) m q^{lm}}{\sqrt{12n}} A_{\mu(m)} Y^{(3)\dagger}_0 \\
& - i\frac{2\pi q^{lm}}{\sqrt{12n}} A'_{\mu(m)} Y^{(3)\dagger}_0.
\end{aligned}
\tag{4.69}
$$

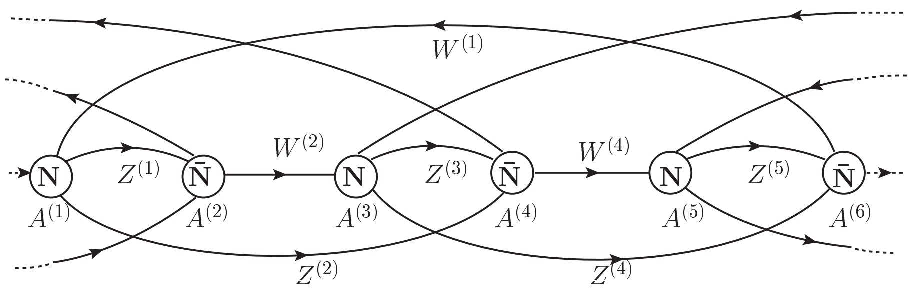

Figure 4.4: Quiver diagram for case (II)-(ii).

The agreement of kinetic terms is achieved using the assignment

$$\begin{aligned}
\lambda^{I0} &= -2\pi(X_0^1, X_0^2, X_0^3, X_0^4, X_0^5, X_0^6, X_0^7, X_0^8), \\
\lambda^{I1} &= -2\pi\Big(sX_0^1, (s-1)X_0^2, (s-2)X_0^3, (s+1)X_0^4, sX_0^5, (s-1)X_0^6, (s-2)X_0^7, (s+1)X_0^8\Big).
\end{aligned} \tag{4.70}$$

In this case, the metric of T^2 is constrained to satisfy

$$G^{11} = -s(s+1)G^{00} + (2s+1)G^{01} + 8\pi^2\{(X_0^2)^2 + (X_0^6)^2\} + 24\pi^2\{(X_0^3)^2 + (X_0^7)^2\}. \tag{4.71}$$

Once again, we have a degree of freedom that corresponds to the sum of VEV squared.

(III) $\mathbb{C}^4/\mathbb{Z}_n$

Finally, we consider the $\mathbb{C}^4/\mathbb{Z}_n$ type. When we consider the $\mathbb{Z}_n$ action given by

$$y^1 \to e^{2\pi i/n}y^1, \quad y^2 \to e^{2\pi i/n}y^2, \quad y^3 \to e^{-2\pi i/n}y^3, \quad y^4 \to e^{-2\pi i/n}y^4, \tag{4.72}$$

$\mathcal{N} = 4$ SUSY and $SU(2) \times SU(2)$ global symmetry are preserved. The covariant derivatives are given by

$$\begin{aligned}
D_\mu Z^{(2l-1)} &= \partial_\mu Z^{(2l-1)} + iA_\mu^{(2l-1)} Z^{(2l-1)} - iZ^{(2l-1)} A_\mu^{(2l)}, \\
D_\mu Z^{(2l)} &= \partial_\mu Z^{(2l)} + iA_\mu^{(2l-1)} Z^{(2l)} - iZ^{(2l)} A_\mu^{(2l)}, \\
D_\mu W^{(2l-1)} &= \partial_\mu W^{(2l-1)} + iA_\mu^{(2l-2)} W^{(2l-1)} - iW^{(2l-1)} A_\mu^{(2l+1)}, \\
D_\mu W^{(2l)} &= \partial_\mu W^{(2l)} + iA_\mu^{(2l-2)} W^{(2l)} - iW^{(2l)} A_\mu^{(2l+1)},
\end{aligned} \tag{4.73}$$

where $l = 1, \cdots, n$. In this case, only the $Z^{(2l-1)}$ part is unchanged from (4.19). The quiver diagram of this theory is given in Figure 4.5.

The CS term and its scaling behaviour are exactly the same as (4.18) and (4.31), respectively.

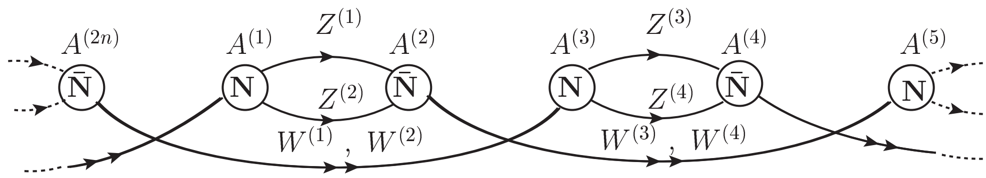

Figure 4.5: Quiver diagram for case (III).

On the other hand, the covariant derivatives are scaled as

$$
\begin{aligned}
D_\mu Z_{(2l)} &\to \frac{q^{lm}}{\sqrt{n}}\partial_\mu Y^{(2)}_{(m)} + i\frac{q^{lm}}{\sqrt{n}}[A_{\mu(n)}, Y^{(2)}_{(m-n)}] - \frac{2\pi s m q^{lm}}{\sqrt{n}}A_{\mu(m)}Y^{(2)}_0 + i\frac{2\pi q^{lm}}{\sqrt{n}}A'_{\mu(m)}Y^{(2)}_0, \\
D_\mu W_{(2l-1)} &\to \frac{q^{-lm}}{\sqrt{n}}\partial_\mu Y^{(3)\dagger}_{(m)} + i\frac{q^{lm}}{\sqrt{n}}[A_{\mu(n)}, Y^{(3)\dagger}_{(n-m)}] + \frac{2\pi(s+2)m}{\sqrt{n}}q^{lm}A_{\mu(m)}Y^{(3)\dagger}_0 - i\frac{2\pi q^{lm}}{\sqrt{n}}A'_{\mu(m)}Y^{(3)\dagger}_0, \\
D_\mu W_{(2l)} &\to \frac{q^{-lm}}{\sqrt{n}}\partial_\mu Y^{(4)\dagger}_{(m)} + i\frac{q^{lm}}{\sqrt{n}}[A_{\mu(n)}, Y^{(4)\dagger}_{(n-m)}] + \frac{2\pi(s+2)m q^{lm}}{\sqrt{n}}A_{\mu(m)}Y^{(4)\dagger}_0 - i\frac{2\pi q^{lm}}{\sqrt{n}}A'_{\mu(m)}Y^{(4)\dagger}_0.
\end{aligned}
\tag{4.74}
$$

Using the identifications

$$
\begin{aligned}
\lambda^{I0} &= -2\pi(X^1_0, X^2_0, X^3_0, X^4_0, X^5_0, X^6_0, X^7_0, X^8_0), \\
\lambda^{I1} &= -2\pi\Big(sX^1_0, sX^2_0, (s+2)X^3_0, (s+2)X^4_0, sX^5_0, sX^6_0, (s+2)X^7_0, (s+2)X^8_0\Big),
\end{aligned}
\tag{4.75}
$$

we can show that the kinetic term of the Extended L-BLG theory emerges precisely. Therefore, the T^2 metric is limited to satisfy

$$
G^{11} = -s(s+2)G^{00} + (2s+2)G^{01}. \tag{4.76}
$$

In this section, we checked the emergence of the Extended L-BLG theory from the various quiver CS theories for the kinetic and CS terms. Naively, whenever an additional circle exists, independently of how to realize it, the Extended L-BLG theory and D3-brane theory are expected to emerge. Therefore, it is just conceivable that independently of how the further $\mathbb{Z}_n$ orbifolding acts on $\mathbb{C}^4/\mathbb{Z}_k$, namely, regardless of the remaining SUSY and global symmetry, the orbifolded ABJM theories lead us to the Extended L-BLG theory from our scaling procedure. All the examples we have studied display positive signs for this expectation. Further research in this direction may be interesting.

4.5 T^2 compactification and $SL(2,Z)$ transformations

We have seen the emergence of the Extended Lorentzian BLG theory from the scaling limit of quiver Chern-Simons theories. Our procedure realizes ordinary T^2 compactification. However, starting from the orbifolded ABJM theory, the resultant metric of two-torus G^{AB} $(A, B = 0, 1)$ is constrained. This means that after the reduction to the D3-brane theory, the realizable parameter region of the complexified coupling constant τ is also limited. In this section, we focus on this constraint and a realization of $SL(2,Z)$ transformations.

In section 2, we have seen that the Extended L-BLG theory with $d=1$ is reduced to the D3-brane worldvolume theory through the Higgs mechanism. The gauge sector of the resultant D3-brane action is given by

$$
\begin{aligned}
L_A + L_{F\tilde{F}} &= -\frac{1}{4G^{00}}\int \frac{dy}{2\pi}\sqrt{g^{11}}F^2 + \frac{G^{01}}{8G^{00}}\int \frac{dy}{2\pi}F\tilde{F} \\
&\equiv -\frac{1}{8\pi}\int dy\left[\mathrm{Im}(\tau)F^2 + \frac{1}{2}\mathrm{Re}(\tau)F\tilde{F}\right],
\end{aligned}
\tag{4.77}
$$

where

$$
\begin{aligned}
F^2 &= \tilde{F}_{\mu\nu}^2 + 2g^{11}\tilde{F}_{\mu 1}\tilde{F}_{\mu 1}, \\
F\tilde{F} &= (4\sqrt{g^{11}}\epsilon^{\mu\nu\lambda})\tilde{F}_{\mu 1}\tilde{F}_{\nu\lambda}.
\end{aligned}
\tag{4.78}
$$

Thus, the complexified coupling constant τ is represented as

$$
\tau = -\frac{G^{01}}{G^{00}} + i\sqrt{\frac{G^{11}}{G^{00}} - \left(\frac{G^{01}}{G^{00}}\right)^2}. \tag{4.79}
$$

Note that we have chosen $k=1$.

In the previous section, we have seen that the T^2 metric G^{AB} is constrained to satisfy a certain relation. Now, we substitute these constraints into (4.79) and investigate the parameter space of τ and the $SL(2,Z)$ transformations.

(I) $\mathcal{N}=4$

First, we consider the $\mathcal{N}=4$ case. Substituting (4.30) into (4.79), we obtain

$$
\tau = -\frac{G^{01}}{G^{00}} + i\sqrt{-\left(\frac{G^{01}}{G^{00}} - s\right)\left[\frac{G^{01}}{G^{00}} - (s+1)\right]}, \tag{4.80}
$$

where

$$
\frac{G^{01}}{G^{00}} = s + \frac{(X_0^2)^2 + (X_0^4)^2 + (X_0^6)^2 + (X_0^8)^2}{(X_0^I)^2}. \tag{4.81}
$$

This denotes that in a fixed s, namely, in certain linear combinations of the gauge fields (4.24), the realizable parameter space of τ is limited to the one that depends on only one real parameter, the ratio of the VEVs G^{01}/G^{00}. Remarkably, s appears in τ only through the real part. When we shift s as $s \to s+a$ ($a \in \mathbb{Z}$), τ changes as $\tau \to \tau + a$. Therefore, the linear combinations of the gauge fields and the T-transformations have one-to-one correspondence. This is an extension of the work in [19]. This correspondence also works in all the other examples (I), (II), (III) in Section 4.

If we define $\tau \equiv x + iy$, the realizable region of the coupling τ is represented as

$$\left(x + \frac{2s+1}{2} \right)^2 + y^2 = \frac{1}{4}. \tag{4.82}$$

This is an upper part of a circle of radius 1/2 whose center depends on the combinations of gauge fields.

Similarly, if we consider the constraint (4.76), the realizable parameter space of τ is represented as

$$(x + s + 1)^2 + y^2 = 1. \tag{4.83}$$

Again, τ becomes a one parameter curve.

In both cases, even if we move all the values of VEVs X_0^I and indices s ($s \in \mathbb{Z}$), we cannot cover the full parameter space of the complex structure moduli τ.

(II) $\mathcal{N} = 2$

In the $\mathcal{N} = 2$ case, the situation slightly changes. Now, τ is represented as

$$\tau = -\frac{G^{01}}{G^{00}} + i\sqrt{-\left(\frac{G^{01}}{G^{00}} - s \right) \left[\frac{G^{01}}{G^{00}} - (s+1) \right] + A}\ , \tag{4.84}$$

where

$$A \equiv \begin{cases} 8\pi^2[(X_0^2)^2 + (X_0^6)^2]/G^{00} & \text{for (5.66)}, \\ 8\pi^2[(X_0^3)^2 + (X_0^7)^2]/G^{00} & \text{for (4.66)}, \\ [8\pi^2\{(X_0^2)^2 + (X_0^6)^2\} + 24\pi^2\{(X_0^3)^2 + (X_0^7)^2\}]/G^{00} & \text{for (4.71)}. \end{cases} \tag{4.85}$$

Now, owing to the existence of the term A, we can move a larger region of the complex structure τ than in the $\mathcal{N} = 4$ case. The realizable region of τ is represented as

$$\left(x + \frac{2s+1}{2} \right)^2 + y^2 = \frac{1}{4} + A. \tag{4.86}$$

Compared with case (I), we can change a radius of a circle by tuning A. Therefore, moving all the values of allowed x ($= -G^{01}/G^{00}$), s ($s \in \mathbb{Z}$), and A, we can realize the parameter space of τ more widely. Hence, it seems that the one parameter dependence of τ in the previous case is the reflection of the fact that 3d $\mathcal{N} = 4$ SUSY is very restricted.

Finally, we comment on the A term. Because A is bounded above, again the whole region of the complex structure moduli cannot be reproduced. Naively, even if we consider the $\mathbb{Z}_n$ action that preserves no supersymmetry, the situation seems to be unchanged. This is slightly mysterious and more work is required.

Chapter 5

Lie 3-algebra in six dimension

5.1 6-dim (2,0) theory with Lie 3-algebra

In this section we consider the 6-dim $(2,0)$ theory with Lie 3-algebra proposed in [24]. This model was proposed for the purpose of constructing multiple M5-branes. Here we check the consistency with various string dualities [25, 42]. The proposed set of equations of motion (EOM) is given by

$$\begin{aligned} D_\mu^2 X_a^I - \frac{i}{2}[C^\mu, \bar{\Psi}, \Gamma_\mu \Gamma^I \Psi]_a - [C^\mu, X^J, [C_\mu, X^J, X^I]]_a &= 0 \\ \Gamma^\mu D_\mu \Psi_a + \Gamma_\mu \Gamma^I [C^\mu, X^I, \Psi]_a &= 0 \\ D_{[\mu} H_{\nu\rho\sigma]a} + \frac{1}{4}\epsilon_{\mu\nu\rho\sigma\lambda\tau}[C^\lambda, X^I, D^\tau X^I]_a + \frac{i}{8}\epsilon_{\mu\nu\rho\sigma\lambda\tau}[C^\lambda, \bar{\Psi}, \Gamma^\tau \Psi]_a &= 0 \\ \tilde{F}_{\mu\nu}{}^b{}_a - C_c^\rho H_{\mu\nu\rho,d} f^{cdb}{}_a &= 0 \\ D_\mu C_a^\nu &= 0 \,, \end{aligned} \tag{5.1}$$

and constraints

$$C_c^\mu D_\mu X_d^I f^{cdb}{}_a = C_c^\mu D_\mu \Psi_d f^{cdb}{}_a = C_c^\mu D_\mu H_{\nu\rho\sigma,d} f^{cdb}{}_a = C_c^\mu C_d^\nu f^{cdb}{}_a = 0 \,. \tag{5.2}$$

This theory has 6-dim $\mathcal{N} = (2,0)$ supersymmetry and nontrivial gauge symmetry, so this formulation is expected to be a new approach to understand the multiple M5-brane dynamics. Here the indices $I = 6, \cdots, 10$ specify the transverse directions of M5-branes and $\mu, \nu = 0, \cdots, 5$ indicate the longitudinal directions. $a, b, \cdots$ denote the gauge indices.

The field contents are as follows: X_a^I are scalar fields, Ψ_a is a spinor field, $A_{\mu,ab}$ is a gauge field, and C_a^μ is a new auxiliary field. It is well known that the 6-dim $\mathcal{N} = (2,0)$ tensor multiplet contains the 2-form field $B_{\mu\nu,a}$ besides X_a^I and Ψ_a. In this theory, only its field strength $H_{\mu\nu\rho,a} = 3\partial_{[\mu} B_{\nu\rho]a}$ appears and it satisfies the self-dual condition

$$H_{\mu\nu\rho,a} = \frac{1}{3!}\epsilon_{\mu\nu\rho\sigma\lambda\tau} H^{\sigma\lambda\tau}{}_a \,. \tag{5.3}$$

The covariant derivative of the fields $\Phi = X^I$, Ψ, $H_{\mu\nu\rho}$, C^μ is defined by

$$(D_\mu \Phi)_a := \partial_\mu \Phi_a - i f^{cdb}{}_a A_{\mu,cd} \Phi_b \,, \tag{5.4}$$

where the notation is slightly different from the original one [24], so that the gauge field $A_{\mu,ab}$ becomes Hermitian.

Lie 3-algebra In general, Lie 3-algebra is defined with the totally antisymmetric 3-bracket and the inner product

$$[T^a, T^b, T^c] = f^{abc}{}_d T^d \,, \quad \langle T^a, T^b \rangle = h^{ab} \,, \tag{5.5}$$

where $f^{abc}{}_d$ is a structure constant and h^{ab} is a metric. For the closure of gauge transformation, the structure constant must satisfy the fundamental identity

$$f^{abc}{}_f f^{def}{}_g + f^{abd}{}_f f^{ecf}{}_g + f^{abe}{}_f f^{cdf}{}_g = f^{cde}{}_f f^{abf}{}_g \,. \tag{5.6}$$

Also, we impose the invariance of the inner product

$$f^{abc}{}_e h^{ed} = -f^{abd}{}_e h^{ec} \,, \tag{5.7}$$

which is required when one will write down the Lagrangian in the future. Unfortunately, Lagrangian of this nonabelian $(2,0)$ theory cannot be written down at this stage, since the self-dual 2-form field $B_{\mu\nu,a}$ cannot be properly defined. Although this is not the matter with our present discussion, this must be a very important subject of future research.

Symmetry transformation The nonabelian $(2,0)$ theory is invariant under the gauge symmetry transformation defined by

$$\begin{aligned} &\delta_\Lambda X^I_a = \tilde{\Lambda}^b{}_a X^I_b \,, \quad \delta_\Lambda \Psi_a = \tilde{\Lambda}^b{}_a \Psi_b \,, \quad \delta_\Lambda H_{\mu\nu\rho,a} = \tilde{\Lambda}^b{}_a H_{\mu\nu\rho,b} \,, \\ &\delta_\Lambda C^\mu_a = \tilde{\Lambda}^b{}_a C^\mu_b \,, \quad \delta_\Lambda \tilde{A}_\mu{}^b{}_a = D_\mu \tilde{\Lambda}^b{}_a \,, \end{aligned} \tag{5.8}$$

where $\tilde{A}_\mu{}^b{}_a := A_{\mu cd} f^{cdb}{}_a$ and $\tilde{\Lambda}^b{}_a := \Lambda_{cd} f^{cdb}{}_a$. And it is also invariant under the 6-dim $\mathcal{N} = (2,0)$ supersymmetry transformation

$$\begin{aligned} \delta_\epsilon X^I_a &= i\bar{\epsilon}\Gamma^I \Psi_a \\ \delta_\epsilon \Psi_a &= \Gamma^\mu \Gamma^I D_\mu X^I_a \epsilon + \frac{1}{12}\Gamma_{\mu\nu\rho} H^{\mu\nu\rho}_a \epsilon - \frac{1}{2}\Gamma_\mu \Gamma^{IJ} [C^\mu, X^I, X^J]_a \epsilon \\ \delta_\epsilon H_{\mu\nu\rho,a} &= 3i\bar{\epsilon}\Gamma_{[\mu\nu} D_{\rho]} \Psi_a + i\bar{\epsilon}\Gamma^I \Gamma_{\mu\nu\rho\sigma} [C^\sigma, X^I, \Psi]_a \\ \delta_\epsilon \tilde{A}_\mu{}^b{}_a &= i\bar{\epsilon}\Gamma_{\mu\nu} C^\nu_c \Psi_d f^{cdb}{}_a \\ \delta_\epsilon C^\mu_a &= 0 \,, \end{aligned} \tag{5.9}$$

where ϵ and Ψ are 32-component Majorana spinors under the chirality condition

$$\Gamma_{012345}\epsilon = +\epsilon \,, \quad \Gamma_{012345}\Psi = -\Psi \,. \tag{5.10}$$

Thus the nonabelian $(2,0)$ theory is equipped with the expected symmetries of multiple M5-branes. The main purpose of our work is to explore its properties through the reduction to branes in superstring theory and to clarify the availability of this formulation. In the next section, starting from this theory, we will show that this theory actually reproduce the multiple Dp-branes.

5.2 Dp-brane theory from nonabelian (2,0) theory

First we briefly review how the nonabelian $(2,0)$ theory reproduces D4-brane action [24]. In this case, we use the Lorentzian Lie 3-algebra $\{T^a, u_0, v^0\}$ defined by

$$[u_0, T^a, T^b] = if^{ab}{}_c T^c\,, \quad [T^a, T^b, T^c] = -if^{abc}v^0\,,$$
$$\langle T^a, T^b\rangle = h^{ab}\,, \quad \langle u_0, v^0\rangle = 1\,, \quad \text{otherwise} = 0\,, \tag{5.11}$$

where T^a are generators of the ordinary Lie algebra, so this algebra is a central extension of Lie algebra. Since $u_0 - \alpha v^0$ ($\alpha > 0$) is a negative norm generator, the u_0- and v^0-component fields become ghosts. Then we have to remove them in order to obtain a physical theory. It is well known that this can be performed by the new kind of Higgs mechanism [6, 14]. In this mechanism, we assign a VEV (vacuum expectation value) to the u_0-component field without breaking gauge and supersymmetry. When we set a VEV for the longitudinal field $C^\mu_{u_0}$, D4-brane worldvolume theory can be reproduced from the nonabelian $(2,0)$ theory. In BLG theory, on the other hand, we can obtain D2-brane worldvolume theory, when we set a VEV for the transverse scalar field $X^I_{u_0}$. In both cases, the direction specified by the VEV becomes compactified and then M-branes are reduced to D-branes in type IIA superstring theory. In fact, the VEV can be interpreted as the compactification radius of the M-theory direction.

In this section, we show that the nonabelian $(2,0)$ theory can also reproduce Dp-brane system ($p > 4$) on a torus T^{p-4}. We realize this by using the central extension of Lorentzian Lie 3-algebra, which is called the generalized loop algebra. The number of its centers corresponds to the dimension of compactified torus. It is already known that BLG theory with this algebra reproduces Dp-brane system ($p > 2$) on a torus T^{p-2} [7, 8]. Therefore, the following discussion is similar to BLG theory case.

Setup

Now we start with the generalized loop algebra $\{T^i_{\vec m}, u_A, v^A\}$ [7, 8] defined by

$$
\begin{aligned}
&[u_0, u_a, u_b] = 0\\
&[u_0, u_a, T^i_{\vec m}] = m_a T^i_{\vec m}\\
&[u_0, T^i_{\vec m}, T^j_{\vec n}] = m_a v^a \delta_{\vec m+\vec n}\delta^{ij} + if^{ij}{}_k T^k_{\vec m+\vec n}\\
&[T^i_{\vec m}, T^j_{\vec n}, T^k_{\vec l}] = -if^{ijk}v^0\delta_{\vec m+\vec n+\vec l}\\
&\langle T^i_{\vec m}, T^j_{\vec n}\rangle = h^{ij}\delta_{\vec m+\vec n}\,, \quad \langle u_A, v^B\rangle = \delta^B_A\,, \quad \text{otherwise} = 0\,,
\end{aligned}
\tag{5.12}
$$

where $\vec m, \vec n, \vec l \in \mathbb{Z}^d$, $A = 0, 1, \cdots, d$ and $a = 1, \cdots, d$. $f^{ij}{}_k$ ($i, j, k = 1, \cdots, \dim\mathfrak{g}$) is a structure constant of an arbitrary Lie algebra $\mathfrak{g}$ defined as

$$[T^i, T^j] = if^{ij}{}_k T^k. \tag{5.13}$$

It can be easily shown that this Lie 3-algebra satisfies the fundamental identity (5.6) and the invariant metric condition (5.7). This algebra is characteristic in that the generators u_A are not produced by any 3-brackets, *i.e.* $[\star,\star,\star]_{u_A} = 0$, and the generators v^A are the center of the algebra, *i.e.* $[v^A,\star,\star] = 0$. According to systematic discussion in [7], these conditions are necessary if we want to remove ghost fields by the Higgs mechanism.

Actually, this algebra can be regarded as the original Lorentzian Lie 3-algebra (5.11) with an infinite dimensional Lie algebra $\{T^i_{\vec{m}}, u_a, v^a\}$ given by

$$\begin{aligned}
&[u_a, u_b] = 0\,, \quad [u_a, T^i_{\vec{m}}] = m_a T^i_{\vec{m}}\,, \quad [T^i_{\vec{m}}, T^j_{\vec{n}}] = m_a v^a \delta_{\vec{m}+\vec{n}}\delta^{ij} + i f^{ij}{}_k T^k_{\vec{m}+\vec{n}}\,, \\
&\langle T^i_{\vec{m}}, T^j_{\vec{n}}\rangle = h^{ij}\delta_{\vec{m}+\vec{n}}\,, \quad \langle u_a, v^b\rangle = \delta^b_a\,, \quad \text{otherwise} = 0\,.
\end{aligned} \tag{5.14}$$

This is a higher loop generalization of the Kac-Moody algebra, and can be regarded as a Lie algebra on a torus T^d. As we mentioned, the nonabelian (2,0) theory with Lorentzian Lie 3-algebra reproduces D4-brane theory. In our case, in the following discussion, we define the higher dimensional fields by collecting the infinite $T^i_{\vec{m}}$-component fields and using Fourier transformation. In other words, we interpret the index $\vec{m} \in \mathbb{Z}^d$ as the Kaluza-Klein momentum along the torus T^d to recover the higher dimension. As a result, we will obtain the higher dimensional Dp-brane theory whose worldvolume is given by the flat torus T^d bundle over the original D4-brane worldvolume $\mathcal{M}_5$ (*i.e.* $p = 4 + d$).

Component Expansion

Then, we expand all the fields into their components of Lie 3-algebra as

$$\begin{aligned}
\Phi &= \Phi_{(i\vec{m})} T^i_{\vec{m}} + \Phi^A u_A + \underline{\Phi}_A v^A \\
A_\mu &= A_{\mu(i\vec{m})(j\vec{n})} T^i_{\vec{m}} \wedge T^j_{\vec{n}} + \frac{1}{2} A^A_{\mu(i\vec{m})} u_A \wedge T^i_{\vec{m}} + A^{AB}_\mu u_A \wedge u_B + \cdots\,,
\end{aligned} \tag{5.15}$$

where $\Phi = X^I$, Ψ, $H_{\mu\nu\rho}$, C^μ. For simplicity, we set $A^{AB}_\mu = 0$ in the following. The omitted terms in the expansion of A_μ are the terms including v^A which never appear in EOM's.

Each component of the covariant derivatives is written as

$$\begin{aligned}
(D_\mu\Phi)_{(i\vec{m})} &= (\hat{D}_\mu\Phi)_{(i\vec{m})} + A'_{\mu(i\vec{m})}\Phi^0 + i m_a A^0_{\mu(i\vec{m})}\Phi^a \\
(D_\mu\Phi)_{u_A} &= \partial_\mu \Phi^A \\
(D_\mu\Phi)_{v^0} &= \partial_\mu \underline{\Phi}_0 + i m_a (A^a_{\mu(i\vec{m})}\Phi_{(i,-\vec{m})} + A_{\mu(i\vec{m})(i,-\vec{m})}\Phi^a) \\
&\quad - f^{ijk} A_{\mu(i\vec{m})(j\vec{n})}\Phi_{(k,-\vec{m}-\vec{n})} \\
(D_\mu\Phi)_{v^a} &= \partial_\mu \underline{\Phi}_a - i m_a (A^0_{\mu(i\vec{m})}\Phi_{(i,-\vec{m})} + A_{\mu(i\vec{m})(i,-\vec{m})}\Phi^0)\,,
\end{aligned} \tag{5.16}$$

where

$$\begin{aligned}
(\hat{D}_\mu\Phi)_{(i\vec{m})} &= \partial_\mu \Phi_{(i\vec{m})} + f^{jk}{}_i A^0_{\mu(j,\vec{m}-\vec{n})}\Phi_{(k\vec{n})} \\
A'_{\mu(i\vec{m})} &= -i m_a A^a_{\mu(i\vec{m})} + f^{jk}{}_i A_{\mu(j,\vec{m}-\vec{n})(k\vec{n})}\,.
\end{aligned} \tag{5.17}$$

Solving the ghost sector

The generalized loop algebra (5.12) has $d+1$ negative norm generators $u_A - \alpha v^A$ ($\alpha > 0$), so the u_A and v^A-component fields become ghosts. Then one may wonder whether this theory is unitary. However, as we will see, it doesn't matter because these ghosts can be removed by the Higgs mechanism. The detailed procedure is as follows.

First, we consider u_A-component fields. Their EOM's are

$$\partial_\mu^2 X^{IA} = 0\,, \quad \Gamma^\mu \partial_\mu \Psi^A = 0\,, \quad \partial_{[\mu} H^A_{\nu\rho\sigma]} = 0\,, \quad \partial_\mu C^{\nu A} = 0\,. \tag{5.18}$$

The gauge transformation is given by

$$\delta_\Lambda X^{IA} = 0\,, \quad \delta_\Lambda \Psi^A = 0\,, \quad \delta_\Lambda H^A_{\mu\nu\rho} = 0\,, \quad \delta_\Lambda C^{\mu A} = 0\,, \tag{5.19}$$

and the supersymmetry transformation is

$$\begin{aligned} &\delta_\epsilon X^{IA} = i\bar{\epsilon}\Gamma^I \Psi^A\,, \quad \delta_\epsilon \Psi^A = \Gamma^\mu \Gamma^I \partial_\mu X^{IA} \epsilon\,, \quad \delta_\epsilon H^A_{\mu\nu\rho} = 3i\bar{\epsilon}\Gamma_{[\mu\nu}\partial_{\rho]}\Psi^A\,, \\ &\delta_\epsilon C^{\mu A} = 0\,. \end{aligned} \tag{5.20}$$

This means that we can insert the VEV's as

$$X^{IA} = \text{const.}\,, \quad \Psi^A = 0\,, \quad H^A_{\mu\nu\rho} = \text{arbitrary}\,, \quad C^{\mu A} = \text{arbitrary} \tag{5.21}$$

without breaking gauge symmetry and supersymmetry. Then, in the following, we consider

$$C^{\mu 0} = \lambda^0 \delta^\mu_5\,, \quad X^{Ia} = \lambda^{Ia}\,, \quad \text{otherwise} = 0\,, \tag{5.22}$$

where $\vec{\lambda}^a$ are constant vectors in $\mathbb{R}^5$ (the transverse directions of M5-branes), namely,

$$\vec{\lambda}^a \in \mathbb{R}^d \subset \mathbb{R}^5. \tag{5.23}$$

In the following, we use $\{\vec{\lambda}^a\}$ as the basis of $\mathbb{R}^d$. Therefore, it is useful for later discussion to define the dual basis $\vec{\pi}_a$ and the projection operator P^{IJ} as

$$\vec{\lambda}^a \cdot \vec{\pi}_b = \delta^a_b\,, \quad P^{IJ} = \delta^{IJ} - \sum_a \lambda^{Ia}\pi^J_a\,. \tag{5.24}$$

The operator P projects a vector onto subspace of $\mathbb{R}^5$ which is orthogonal to all $\vec{\lambda}^a$, and it satisfies the projector condition $P^2 = P$. In the next subsection, we will compactify this $\mathbb{R}^d$ space on a torus T^d, and identify it with the torus T^d defined by loop algebra (5.14).

Next, we look at v^A-component fields. For simplicity, we set $C^\mu_{(i\vec{m})} = 0$ only here. After setting VEV's (5.22), their EOM's become

$$\begin{aligned}
0 &= D_\mu^2 \underline{X}_0^I \\
&= D_\mu^2 \underline{X}_a^I + \frac{1}{2} m_a \lambda^0 \bar{\Psi}_{(i\vec{m})} \hat{\Gamma}^I \Psi_{(i,-\vec{m})} \\
&\quad -m_a^2 (\lambda^0)^2 \lambda^{[Ia} X_{(i\vec{m})}^{J]} X_{(i,-\vec{m})}^J - m_a (\lambda^0)^2 f^{ijk} X_{(i\vec{m})}^J X_{(j\vec{n})}^J X_{(k,-\vec{m}-\vec{n})}^I \\
0 &= \Gamma^\mu D_\mu \underline{\Psi}_0 = \Gamma^\mu D_\mu \underline{\Psi}_a - i m_a \lambda^0 X_{(i\vec{m})}^I \hat{\Gamma}^I \Psi_{(i,-\vec{m})} \\
0 &= D_{[\mu} \underline{H}_{\nu\rho\sigma],0} = D_{[\mu} \underline{H}_{\nu\rho\sigma],a} + \epsilon_{\mu\nu\rho\sigma 5\tau} m_a \lambda^0 \left(\frac{1}{4} X_{(i\vec{m})}^I D^\tau X_{(i,-\vec{m})}^I + \frac{i}{8} \bar{\Psi}_{(i\vec{m})} \Gamma^\tau \Psi_{(i,-\vec{m})} \right) \\
0 &= \underline{\tilde{F}}_{\mu\nu}{}^a{}_0 = \underline{\tilde{F}}_{\mu\nu}{}^{(i\vec{m})}{}_0 = \underline{\tilde{F}}_{\mu\nu}{}^0{}_a = \underline{\tilde{F}}_{\mu\nu}{}^{(i\vec{m})}{}_a - m_a \lambda^0 H_{\mu\nu 5(i\vec{m})} \\
0 &= D_\mu \underline{C}_A^\nu \,, \qquad (5.25)
\end{aligned}$$

where $\hat{\Gamma}^I := i\Gamma_5\Gamma^I$ and these satisfies $\frac{1}{2}\{\hat{\Gamma}^I, \hat{\Gamma}^J\} = \delta^{IJ}$.[1] Note that all the equations of v^0-component fields are free, while the equations of v^a-component fields are necessarily not. This doesn't matter as long as we consider the VEV's of u_A-component fields to be constants.

Derivation of Dp-brane action

Now we concentrate on the EOM's for $T^i_{\vec{m}}$-component fields. In order to obtain the Dp-brane action, we compactify the $\mathbb{R}^d$ space spanned by $\vec{\lambda}^a$ on a torus T^d and regard the index $\vec{m} \in \mathbb{Z}^d$ as the Kaluza-Klein momentum along the torus. Then we identify the infinite $T^i_{\vec{m}}$-component fields with the $(6+d)$-dim fields through the Fourier transformation on it:

$$\hat{\Phi}_i(x,y) := \sum_{\vec{m}} \Phi_{(i\vec{m})}(x) e^{-i\vec{m}\cdot\vec{y}} \,, \quad \hat{A}_{\mu i}(x,y) := \sum_{\vec{m}} A^0_{\mu(i\vec{m})}(x) e^{-i\vec{m}\cdot\vec{y}} \,, \qquad (5.26)$$

where x^μ are coordinates of M5-brane worldvolume, and $y^a \in [0, 2\pi]$ are coordinates of the d-dim torus T^d [7, 8]. We will also use the notation of field strength

$$\hat{F}_{\mu\nu,i}(x,y) := \sum_{\vec{m}} F^0_{\mu\nu(i\vec{m})}(x) e^{-i\vec{m}\cdot\vec{y}} \,, \qquad (5.27)$$

where $F^0_{\mu\nu(i\vec{m})} := \partial_\mu A^0_{\nu(i\vec{m})} - \partial_\nu A^0_{\mu(i\vec{m})} + f^{jk}{}_i A^0_{\mu(j,\vec{m}-\vec{n})} A^0_{\nu(k\vec{n})}$. In fact, this procedure corresponds to taking the field theoretical T-duality [38] for the directions of T^d, since it means that we make the brane worldvolume extended to these directions.

C^μ-field and constraints After inserting the VEV's (5.22), the EOM (5.1) for C^μ-field and the constraints (5.2) become

$$D_5 X^I_{(i\vec{m})} = D_5 \Psi_{(i\vec{m})} = D_5 H_{\mu\rho\sigma(i\vec{m})} = D_\mu C^\nu_{(i\vec{m})} = C^\alpha_{(i\vec{m})} = 0 \,, \qquad (5.28)$$

where $\alpha = 0, \cdots, 4$. Also, from eq. (5.8) and (5.9), we find that one can set a VEV as

$$C^5_{(i\vec{m})} = \text{const.} \qquad (5.29)$$

[1] In our notation, $\frac{1}{2}\{\Gamma_\mu, \Gamma_\nu\} = g_{\mu\nu} = \text{diag.}(- + \cdots +)$ and $\frac{1}{2}\{\Gamma^I, \Gamma^J\} = \delta^{IJ}$.

without breaking gauge symmetry and supersymmetry. However, as we will see, this field and its VEV has no influence on the EOM's in the final form.

Spinor field After inserting the VEV's (5.22), we obtain

$$[C^\mu, X^I, \Psi]_{(i\vec{m})} = \lambda^0 \delta_5^\mu \left(m_a \lambda^{Ia} \Psi_{(i\vec{m})} + i f^{jk}{}_i X^I_{(j\vec{n})} \Psi_{(k,\vec{m}-\vec{n})} \right) . \tag{5.30}$$

Then, using the projector (5.24), we define the field $A_{a(i\vec{m})}$ as

$$X^I_{(i\vec{m})} = P^{IJ} X^J_{(i\vec{m})} + \lambda^{Ia} (\vec{\pi}_a \cdot \vec{X})_{(i\vec{m})} =: P^{IJ} X^J_{(i\vec{m})} + \lambda^{Ia} A_{a(i\vec{m})} . \tag{5.31}$$

This field can be regarded as the gauge field along the fiber torus T^d. Therefore, by using these equations and eq. (5.28), the EOM (5.1) for spinor field becomes

$$\begin{aligned} 0 &= \Gamma^\alpha \hat{D}_\alpha \Psi_{(i\vec{m})} + \lambda^0 \lambda^{Ia} \Gamma_5 \Gamma^I (m_a \Psi_{(i\vec{m})} + i f^{jk}{}_i A_{a(j\vec{n})} \Psi_{(k,\vec{m}-\vec{n})}) \\ &\quad + \lambda^0 \Gamma_5 \Gamma^I [P^{IJ} X^J, \Psi]_{(i\vec{m})} . \end{aligned} \tag{5.32}$$

After the field redefinition (5.26), this can be represented as

$$0 = \Gamma^\alpha \hat{D}_\alpha \hat{\Psi} + \Gamma^a \hat{D}_a \hat{\Psi} + \lambda^0 \hat{\Gamma}^I [P^{IJ} \hat{X}^J, \hat{\Psi}] , \tag{5.33}$$

where the covariant derivative is defined as $\hat{D}_a \hat{\Phi}_i := \partial_a \hat{\Phi}_i - i[\hat{A}_a, \hat{\Phi}]$. The Γ-matrices $\Gamma^a := i\lambda^0 \lambda^{Ia} \Gamma_5 \Gamma^I$ satisfy $\frac{1}{2}\{\Gamma^a, \Gamma^b\} = g^{ab}$ which is the metric on the torus T^d given by

$$g^{ab} := |\vec{\lambda}^0|^2 \vec{\lambda}^a \cdot \vec{\lambda}^b . \tag{5.34}$$

Scalar fields Similarly, after inserting the VEV's, we obtain

$$[C^\mu, X^I, X^J]_{(i\vec{m})} = \lambda^0 \delta_5^\mu \left(m_a \lambda^{[Ia} X^{J]}_{(i\vec{m})} + i f^{jk}{}_i X^{[I}_{(j\vec{n})} X^{J]}_{(k,\vec{m}-\vec{n})} \right) . \tag{5.35}$$

Then, by using eq. (5.28) and (5.31), we obtain

$$(D_\alpha^2 X^I)_{(i\vec{m})} = P^{IJ} (\hat{D}_\alpha^2 X^J)_{(i\vec{m})} + \lambda^{Ia} (\hat{D}^\alpha F_{\alpha a})_{(i\vec{m})} , \tag{5.36}$$

where $(F_{\alpha a})_{(i\vec{m})} := \hat{D}_\alpha A_{a(i\vec{m})} + i m_a A^0_{\alpha(i\vec{m})}$.

After the field redefinition, the EOM's (5.1) for scalar fields become

$$\begin{aligned} 0 &= P^{IJ} \hat{D}_\alpha^2 \hat{X}^J + P^{IJ} \hat{D}_a^2 \hat{X}^J \\ &\quad + i(\lambda^0)^2 \lambda^{Ia} [P^{JL} \hat{X}^L, P^{JK} \hat{D}_a \hat{X}^K] - (\lambda^0)^2 [P^{JM} \hat{X}^M, [P^{JL} \hat{X}^L, P^{IK} \hat{X}^K]] \\ &\quad + \lambda^{Ib} (\hat{D}^\alpha \hat{F}_{\alpha b}) + \lambda^{Ib} (\hat{D}^a \hat{F}_{ab}) + \frac{i\lambda^0}{2} [\bar{\hat{\Psi}}, \hat{\Gamma}^I \hat{\Psi}] , \end{aligned} \tag{5.37}$$

where $\hat{D}^a = g^{ab} \hat{D}_b$ and $\hat{F}_{ab} := \partial_a \hat{A}_b - \partial_b \hat{A}_a - i[\hat{A}_a, \hat{A}_b]$.

Gauge field The EOM for gauge field becomes

$$\begin{aligned} 0 &= \tilde{F}_{\mu\nu}{}^{(j\vec{n})}{}_{(i\vec{m})} - i\lambda^0 f^{kj}{}_i H_{\mu\nu 5(k,\vec{m}-\vec{n})} \\ &= \tilde{F}_{\mu\nu}{}^0{}_{(i\vec{m})} - i f^{jk}{}_i C^5_{(j\vec{n})} H_{\mu\nu 5(k,\vec{m}-\vec{n})} \\ &= \tilde{F}_{\mu\nu}{}^a{}_{(i\vec{m})} + m_a \lambda^0 H_{\mu\nu 5(i\vec{m})} \,. \end{aligned} \tag{5.38}$$

In fact, we don't use the second equation in the following, since we now regard only $A^0_{\mu(i\vec{m})}$ as the gauge field, as we can see in eq. (5.17) or (5.26). This is a direct reason why $C^5_{(im)}$-field gives no effects on the EOM's in the final form.

2-form field Similarly, the EOM for self-dual 2-form field becomes

$$\begin{aligned} 0 &= \hat{D}_{[\mu} H_{\nu\rho\sigma](i\vec{m})} + \frac{\lambda^0}{4}\epsilon_{\mu\nu\rho\sigma 5\tau}[P^{IJ}X^J, P^{IK}\hat{D}^\tau X^K]_{(i\vec{m})} + \frac{\lambda^0\lambda^{Ia}}{4}\epsilon_{\mu\nu\rho\sigma 5\tau}P^{IJ}\hat{D}^\tau \hat{D}_a X^J_{(i\vec{m})} \\ &\quad + \frac{1}{\lambda^0}\epsilon_{\mu\nu\rho\sigma 5\tau}\hat{D}^a F_{a\tau(i\vec{m})} + \frac{i\lambda^0}{8}\epsilon_{\mu\nu\rho\sigma 5\tau}[\bar{\Psi}, \Gamma^\tau \Psi]_{(i\vec{m})} \,. \end{aligned} \tag{5.39}$$

Then, by using eq. (5.38), the self-duality of $H_{\mu\nu\rho}$ (5.3), and the field redefinition (5.27), this can be rewritten as

$$0 = \frac{1}{(\lambda^0)^2}\left(\hat{D}^\alpha \hat{F}_{\alpha\beta} + \hat{D}^a \hat{F}_{a\beta}\right) + i[P^{IJ}\hat{X}^J, P^{IK}D_\beta \hat{X}^K] - \frac{1}{2}[\hat{\bar{\Psi}}, \Gamma_\beta \hat{\Psi}] \,. \tag{5.40}$$

Summary First, we note that the Higgs mechanism removes the ghost sector completely without breaking gauge symmetry and supersymmetry. In fact, the ghost fields *never* appear in the EOM's for $T^i_{\vec{m}}$-component fields.

Then we can finally show that all the EOM's derived above, *i.e.* eq. (5.28), (5.33), (5.37), (5.38) and (5.40), are successfully reproduced from the $(5+d)$-dim super Yang-Mills action

$$\begin{aligned} S &= \lambda^0 \int d^5x \, \frac{d^dy}{(2\pi)^d} \sqrt{g}\, \mathcal{L} \,, \\ \mathcal{L} &= -\frac{1}{2}(\hat{D}_{\underline{\mu}}\hat{X}^I)P^{IJ}(\hat{D}^{\underline{\mu}}\hat{X}^J) + \frac{i}{2}\hat{\bar{\Psi}}\Gamma^{\underline{\mu}}\hat{D}_{\underline{\mu}}\hat{\Psi} - \frac{1}{4(\lambda^0)^2}\hat{F}^2_{\underline{\mu\nu}} \\ &\quad - \frac{(\lambda^0)^2}{4}[P^{IK}\hat{X}^K, P^{JL}\hat{X}^L]^2 + \frac{i\lambda^0}{2}\hat{\bar{\Psi}}\hat{\Gamma}^I[P^{IJ}\hat{X}^J, \hat{\Psi}] \,. \end{aligned} \tag{5.41}$$

where the spacetime indices are summarized as $\underline{\mu} = (\alpha, a)$, and $g := \det g^{ab}$. This is nothing but the low energy effective action of multiple Dp-branes ($p = 4 + d$) on $\mathcal{M}_5 \times T^d$. Therefore, we conclude that one can reproduce Dp-brane system from nonabelian $(2,0)$ theory.

5.3 NS5-brane theory from nonabelian (2,0) theory

In the previous section, we successfully derive Dp-brane system on a torus T^{p-4} from the non-abelian $(2,0)$ theory by using the Higgs mechanism (5.22) and the field redefinition (5.26). Let us see here the physical meaning of each step. From the discussion in Lorentzian BLG theory, it is well known that putting a VEV of u_A-component field corresponds to the compactification. Therefore, in eq. (5.22), we put a VEV $C^{\mu 0}$ to compactify one of the x^μ-directions which becomes M-theory direction, and then we also put VEV's X^{Ia} to compactify some of the x^I-directions. After the field redefinition (5.26) which is equivalent to the field theoretical T-duality for the latter compactified directions, we finally obtain Dp-brane system on a torus T^{p-4}.

In this section, we change the way of setting VEV's from the previous case. This should correspond to changing the directions of M-compactification and that of taking T-duality. Especially, we now consider the reduction to type IIA/IIB NS5-brane system, and investigate whether these branes can be reproduced from the nonabelian (2,0) theory.

Type IIA NS5-brane theory

In order to obtain type IIA NS5-branes from M5-branes, we change the M-direction, compared with D4-brane case. Therefore, here we use the original Lorentzian Lie 3-algebra $\{T^a, u_0, v^0\}$ defined by

$$
\begin{aligned}
&[u_0, T^a, T^b] = i f^{ab}{}_c T^c\,, \quad [T^a, T^b, T^c] = -i f^{abc} v^0\,,\\
&\langle T^a, T^b\rangle = h^{ab}\,, \quad \langle u_0, v^0\rangle = 1\,, \quad \text{otherwise} = 0\,.
\end{aligned}
\tag{5.42}
$$

In D4-brane case, we put a non-zero VEV into the longitudinal field $C^{\mu 0}$ in order to compactify one of x^μ-direction. Then in this case, we put a VEV into u_0-components as

$$
X^{I0} = \lambda \delta^I_{10}\,, \quad \text{otherwise} = 0\,, \tag{5.43}
$$

in order to compactify one of the transverse x^I-direction as M-theory direction.

On gauge field In this setup, the EOM for gauge field $\tilde{A}_\mu{}^b{}_a$ is

$$
\tilde{F}_{\mu\nu}{}^b{}_a = 0\,, \tag{5.44}
$$

and its supersymmetry transformation is

$$
\delta_\epsilon \tilde{A}_\mu{}^b{}_a = 0\,. \tag{5.45}
$$

This means that the gauge field $\tilde{A}_\mu{}^b{}_a$ have no physical degrees of freedom, and can be set to zero up to gauge transformation. Therefore, the covariant derivative $\hat{D}_\mu$ in eq. (5.17) is reduced to the partial derivative ∂_μ.

Equations of motion The remaining EOM's are

$$
\begin{aligned}
\partial_\mu^2 X_a^i - \lambda^2 [C^\mu, [C_\mu, X^i]]_a &= 0 \\
\partial_\mu^2 X_a^{10} &= 0 \\
\Gamma^\mu \partial_\mu \Psi_a - \lambda \Gamma_\mu \Gamma^{10} [C^\mu, \Psi]_a &= 0 \\
\partial_{[\mu} H_{\nu\rho\sigma]a} - \frac{\lambda}{4} \epsilon_{\mu\nu\rho\sigma\lambda\tau} [C^\lambda, (\partial^\tau X^{10} + \lambda \tilde{A}^{\tau 0})]_a &= 0 \\
\tilde{F}_{\mu\nu}{}^0{}_a - [C^\rho, H_{\mu\nu\rho}]_a &= 0 \\
\partial_\mu C_a^\nu &= 0
\end{aligned}
\tag{5.46}
$$

where $i = 6, \cdots, 9$, and we set $\partial^\mu \tilde{A}_\mu{}^0{}_a = 0$ using the gauge transformation.

For the multiple Dp-branes, the interaction terms like $[X, [X, X]]$ or $[X, \Psi]$ come from strings ending on different branes. In this case, however, C^μ-field has no dynamical degrees of freedom because they have no kinetic terms. Therefore, we naively guess that the terms including this field doesn't describe the interaction between different NS5-branes, and so the resultant EOM's (5.46) seem practically the simple copies of free theory of $\mathcal{N} = (2, 0)$ multiplet. In order to obtain the interaction terms, we need to go beyond the present construction of the nonabelian (2,0) theory.

Type IIB NS5-brane theory

In order to obtain type IIB NS5-branes from M5-branes, we interchange the direction of M-compactification and that of taking T-duality, compared with D5-brane case. Therefore, in this case, we use a generalized loop algebra $\{T_m^i, u_{0,1}, v^{0,1}\}$ defined by

$$
\begin{aligned}
&[u_0, u_1, T_m^i] = m T_m^i\,, \quad [u_0, T_m^i, T_n^j] = m v^1 \delta_{m+n} \delta^{ij} + i f^{ij}{}_k T_{m+n}^k\,, \\
&[T_m^i, T_n^j, T_l^k] = -i f^{ijk} v^0 \delta_{m+n+l}\,, \\
&\langle T_m^i, T_n^j \rangle = h^{ij} \delta_{m+n}\,, \quad \langle u_0, v^0 \rangle = \langle u_1, v^1 \rangle = 1\,, \quad \text{otherwise} = 0\,.
\end{aligned}
\tag{5.47}
$$

In D5-brane case, we put non-zero VEV's into $C^{\mu 0}$ and X^{I1} as eq. (5.22). Then, we now put VEV's into $u_{0,1}$-components as

$$
X^{I0} = \lambda^0 \delta_{10}^I\,, \quad C^{\mu 1} = \lambda^1 \delta_5^\mu\,, \quad \text{otherwise} = 0\,. \tag{5.48}
$$

We also redefine the fields in a similar but slightly different way from eq. (5.26) as

$$
\hat{\Phi}_i(x, y) = \sum_m \Phi_{(im)}(x) e^{-imy}\,, \quad \hat{A}_{\mu,i}(x, y) = \sum_m A^1_{\mu(im)}(x) e^{-imy}\,, \quad \cdots\,. \tag{5.49}
$$

Note that we now regard $A^1_{\mu(im)}$ field as the gauge field, while we use $A^0_{\mu(im)}$ field in D5-brane case (5.26).

C-field and constraints The EOM for C-field and the constraints become

$$D_5 X^I_{(im)} = D_5 \Psi_{(im)} = D_5 H_{\nu\rho\sigma(im)} = D_\mu C^\nu_{(im)} = C^\alpha_{(im)} = 0\,. \tag{5.50}$$

where $\mu = 0, \cdots, 5$ and $\alpha = 0, \cdots, 4$.

Gauge field The EOM for gauge field becomes

$$\begin{aligned} \tilde{F}_{\mu\nu}{}^0{}_{(im)} - m\lambda^1 H_{\mu\nu 5(im)} - i f^{jk}{}_i C^5_{(j,m-n)} H_{\mu\nu 5(kn)} &= 0 \\ \tilde{F}_{\mu\nu}{}^1{}_{(im)} = \tilde{F}_{\mu\nu}{}^{(jn)}{}_{(im)} &= 0\,, \end{aligned} \tag{5.51}$$

and the supersymmetry transformation becomes

$$\begin{aligned} \delta_\epsilon \tilde{A}_\mu{}^0{}_{(im)} &= i\bar{\epsilon}\Gamma_{\mu 5}\left(m\lambda^1 \Psi_{(im)} + i f^{jk}{}_i C^5_{(j,m-n)} \Psi_{(kn)}\right) \\ \delta_\epsilon \tilde{A}_\mu{}^1{}_{(im)} &= \delta_\epsilon \tilde{A}_\mu{}^{(jn)}{}_{(im)} = 0\,. \end{aligned} \tag{5.52}$$

Therefore, we can see that $\tilde{A}_\mu{}^1{}_{(im)}$ and $\tilde{A}_\mu{}^{(jn)}{}_{(im)}$ have no physical degrees of freedom, and can be set to zero up to gauge transformation. This means that the covariant derivative $\hat{D}_\alpha \Phi_{(im)} = \partial_\alpha \Phi_{(im)} - i\tilde{A}_\mu{}^{(jn)}{}_{(im)}\Phi_{(jn)}$ is reduced to the partial derivative. Moreover, $\tilde{F}_{\mu\nu}{}^0{}_{(im)}$ is also reduced to

$$\begin{aligned} \tilde{F}_{\mu\nu}{}^0{}_{(im)} &= \partial_\mu \tilde{A}_\nu{}^0{}_{(im)} - \partial_\nu \tilde{A}_\mu{}^0{}_{(im)} \\ &= m\big(\partial_\mu A^1_{\nu(im)} - \partial_\nu A^1_{\mu(im)}\big) + i f^{jk}{}_i\big(\partial_\mu A_{\nu(j,m-n)(kn)} - \partial_\nu A_{\mu(j,m-n)(kn)}\big)\,. \end{aligned} \tag{5.53}$$

Then from eq. (5.51), we obtain

$$\begin{aligned} F^1_{\mu\nu(im)} &:= \partial_\mu A^1_{\nu(im)} - \partial_\nu A^1_{\mu(im)} = \lambda^1 H_{\mu\nu 5(im)} \\ F_{\mu\nu(im)(jn)} &:= \partial_\mu A_{\nu(im)(jn)} - \partial_\nu A_{\mu(im)(jn)} = C^5_{(im)} H_{\mu\nu 5(jn)}\,. \end{aligned} \tag{5.54}$$

Here we define the field strength $F_{\mu\nu}$, but unfortunately, the interaction term like $f^{jk}{}_i A^1_{\mu(j,m-n)} A^1_{\nu(kn)}$ cannot appear in this setup.

Scalar and spinor fields Then, the EOM's for scalar fields and spinor fields are

$$\begin{aligned} \hat{D}^2_\alpha \hat{X}^i + \hat{D}^2_y \hat{X}^i &= 0 \\ \Gamma^\alpha \hat{D}_\alpha \hat{\Psi} + \Gamma^y \hat{D}_y \hat{\Psi} &= 0 \end{aligned} \tag{5.55}$$

where $i = 6, \cdots, 9$, and we define

$$\hat{D}_y \hat{\Phi} := \partial_y \hat{\Phi} - i[\hat{C}_y, \hat{\Phi}]\,, \quad \hat{C}_y := -\frac{1}{\lambda^1}\hat{C}_5\,, \quad \Gamma^y := i\lambda^0\lambda^1\Gamma_5\Gamma^{10}\,, \tag{5.56}$$

satisfying $\frac{1}{2}\{\Gamma^y, \Gamma^y\} = g^{yy} = (\lambda^0\lambda^1)^2$. Note that $\hat{C}_y$-field has no kinetic terms, so it is not a gauge field, although the theory in this setup is invariant under the transformation

$$\delta_\Lambda \Phi_{(im)} = i f^{jk}{}_i \Lambda_{(j,m-n)} \Phi_{(kn)}\,, \quad \delta_\Lambda C_{y(im)} = \hat{D}_y \Lambda_{(im)}\,. \tag{5.57}$$

This means that the covariant derivative $\hat{D}_y$ can be also reduced to the partial derivative if we gauge away the $\hat{C}_y$-field. Anyway, it is interesting that C^μ-field appears in EOM's, which is different from D5-brane case.

The remaining EOM for the scalar field is

$$\hat{D}^\alpha\big(\hat{D}_\alpha X^{10}_{(im)} + \lambda^0 A'_{\alpha(im)}\big) = 0\,, \tag{5.58}$$

where $A'_{\alpha(im)}$ is defined in eq. (5.17). Here, by using eq. (5.49) and (5.54), we can see that

$$\hat{D}_y \hat{A}_{\alpha,i} = \sum_m A'_{\alpha(im)} e^{-imy} \tag{5.59}$$

is satisfied. Therefore, if we redefine the field as

$$\hat{A}_y := -\frac{1}{\lambda^0}\hat{X}^{10}\,, \tag{5.60}$$

we can define the field strength $\hat{F}_{\alpha y}$ and show that

$$\hat{D}^\alpha \hat{F}_{\alpha y} \;:=\; \hat{D}^\alpha\big(\hat{D}_\alpha \hat{A}_y - \hat{D}_y \hat{A}_\alpha - i[\hat{A}_\alpha, \hat{A}_y]\big) = -i[\hat{D}^\alpha \hat{A}_\alpha, \hat{A}_y] = 0\,, \tag{5.61}$$

where we use eq. (5.58) at the second equality, and the last equality is satisfied up to gauge transformation.

2-form field Using the above results, the EOM for 2-form field

$$\hat{D}_{[\mu}\hat{H}_{\nu\rho\sigma]} - \frac{i\lambda^0\lambda^1}{4}\epsilon_{\mu\nu\rho\sigma 5\tau}\hat{D}_y(\hat{D}_\tau \hat{X}^{10} + \lambda^0 \hat{A}'_\tau) \;=\; 0 \tag{5.62}$$

can be rewritten, by using eq. (5.54) for the first term and eq. (5.58)–(5.60) for the second term, as

$$\hat{D}^\beta \hat{F}_{\alpha\beta} + \hat{D}^y \hat{F}_{\alpha y} \;=\; 0\,, \tag{5.63}$$

where we use $\hat{D}^y[\hat{A}_\alpha, \hat{A}_y] = 0$ up to gauge transformation, similarly to eq. (5.61).

Summary We have obtained all the EOM's (5.50), (5.54), (5.55), (5.61) and (5.63). Note that they are practically free part of the EOM's of 6-dim $\mathcal{N} = (1, 1)$ super Yang-Mills theory which is known as the low energy effective theory of type IIB NS5-branes. Therefore, we conclude that one can partially reproduce the type IIB NS5-brane theory on $\mathcal{M}_5 \times S^1$ from the nonabelian $(2, 0)$ theory. Further justification from the viewpoint of S-duality will be done in § 5.5.

Finally, let us look at the kinetic part of the theory. The EOM's of original nonabelian $(2, 0)$ theory can be reproduced from the Lagrangian

$$\mathcal{L} \;=\; -\frac{1}{2}(D_\mu X^I)^2 + \frac{i}{2}\bar{\Psi}\Gamma^\mu D_\mu \Psi - \frac{1}{12}H^2_{\mu\nu\rho} + \cdots\,. \tag{5.64}$$

Then, by using the field redefinition (5.49) and (5.54), this Lagrangian becomes

$$\mathcal{L} \;=\; -\frac{1}{2}(\hat{D}_{\underline{\mu}}\hat{X}^i)^2 + \frac{i}{2}\hat{\bar{\Psi}}\Gamma^{\underline{\mu}}\hat{D}_{\underline{\mu}}\hat{\Psi} - \frac{1}{4(\lambda^1)^2}\hat{F}^2_{\underline{\mu\nu}} + \cdots\,, \tag{5.65}$$

where $\underline{\mu} = (\alpha, y)$. This is nothing but the kinetic part of 6-dim $\mathcal{N} = (1,1)$ super Yang-Mills Lagrangian. However, we should remind that $\hat{D}_{\underline{\mu}}$ is *not* the covariant derivative, that is, it does *not* include the gauge field $\hat{A}_{\underline{\mu}}$: In fact, both $\hat{D}_\alpha$ and $\hat{D}_y$ are simply the partial derivatives up to gauge transformation. In order to make $\hat{D}_{\underline{\mu}}$ the covariant derivative and also to obtain all the interaction terms in super Yang-Mills Lagrangian, we must generalize the original nonabelian $(2,0)$ theory. This must be a very interesting subject, but we put off detailed discussion as a future work.

5.4 More comments on nonabelian $(2,0)$ theory

Generalization of setting VEV's and total derivative terms

In the previous sections, we chose the VEV's as eq. (5.22) for Dp-branes or as eq. (5.48) for type IIB NS5-branes. This means that we have seen only the case where the direction of M-compactification and that of taking T-duality are perpendicular to each other.

If we want to discuss more general cases where the directions are not perpendicular, we may turn on an additional VEV $C^{\mu a}$ or X^{I1} as

$$\begin{aligned}
&C^{\mu 0} = \lambda^0\delta^\mu_5\,, \quad C^{\mu a} = \tilde{\lambda}^a\delta^\mu_5\,, \quad X^{Ia} = \lambda^{Ia} && \text{for D}p\text{-branes}\\
&X^{I0} = \lambda^0\delta^I_{10}\,, \quad X^{I1} = \tilde{\lambda}^1\delta^I_{10}\,, \quad C^{\mu 1} = \lambda^1\delta^\mu_5 && \text{for type IIB NS5-branes}
\end{aligned} \tag{5.66}$$

since putting these VEV's can be regarded as the M-compactification for the direction of

$$\begin{aligned}
&\vec{\lambda}^0 = (\vec{0}, \lambda^0; 0,0,0,0,0,0) && \text{for D}p\text{-branes}\\
&\vec{\lambda}^0 = (\vec{0}, 0\,; 0,0,0,0,0,\lambda^0 l_p^3) && \text{for type IIB NS5-branes}
\end{aligned} \tag{5.67}$$

and taking T-duality for the direction of

$$\begin{aligned}
&\vec{\lambda}^a = (\vec{0}, \tilde{\lambda}^a; \lambda^{Ia} l_p^3) && \text{for D}p\text{-branes}\\
&\vec{\lambda}^1 = (\vec{0}, \lambda^1; 0,0,0,0,0,\tilde{\lambda}^1 l_p^3) && \text{for type IIB NS5-branes}
\end{aligned} \tag{5.68}$$

where $\vec{0}$ is the $(4+1)$-dim zero vector, and l_p is 11-dim Planck length. Note that we now recover the factors l_p^3 which were previously set to 1. They have to appear here, since the canonical mass dimension of C^μ (and $\vec{\lambda}^{0,a}$) is -1, while that of X^I is 2.

After a straightforward calculation, we can show that this generalization of setting VEV's (5.66) does *not* change any terms of the EOM's in all the cases. This means that this generalization affects at most only the terms which doesn't appear in EOM's, for example, total derivative terms in Lagrangian. In fact, it is well known that such a shift of T-duality directions corresponds to T-transformation which affects the Chern-Simons term in Dp-brane Lagrangian. To see this, therefore, we now try to discuss total derivative terms in Lagrangian of the nonabelian $(2,0)$ theory.

Since the nonabelian (2,0) theory must not have dimensionful parameters, we only consider the total derivative terms with mass dimension 6. Then one natural candidate is

$$\mathcal{L} \supset \epsilon^{\mu\nu\rho\sigma\lambda\tau} \tilde{F}_{\mu\nu}{}^{a}{}_{b} \tilde{F}_{\rho\sigma}{}^{b}{}_{c} \tilde{F}_{\lambda\tau}{}^{c}{}_{a} \,. \tag{5.69}$$

Let us now consider the Dp-brane ($p > 4$) case with VEV's (5.66). In this case, both $\vec{\lambda}^0$ and $\vec{\lambda}^a$ have nonzero elements for x^5-direction, so the projector (5.24) must be redefined as

$$P^{MN} = \delta^{MN} - \sum_{A} \lambda^{MA} \pi_A^N \,, \quad \vec{\lambda}^A \cdot \vec{\pi}^B = \delta_B^A \,, \tag{5.70}$$

where $M, N = 5, 6, \cdots, 10$ and $A = 0, 1, \cdots, d\,(= p-4)$. By using this, the gauge field $A_{a(i\vec{m})}$ can be defined like as eq. (5.31)

$$\begin{aligned} X^M_{(i\vec{m})} &= P^{MN} X^N_{(i\vec{m})} + \lambda^{MA} (\vec{\pi}_A \cdot \vec{X})_{(i\vec{m})} \\ &=: P^{MN} X^N_{(i\vec{m})} + \lambda^{M0} (\vec{\pi}_0 \cdot \vec{X})_{(i\vec{m})} + \lambda^{Ma} A_{a(i\vec{m})} \,, \end{aligned} \tag{5.71}$$

where we naturally define as

$$X^5_{(i\vec{m})} := \frac{1}{\lambda^0} A^0_{\mu=5,(i\vec{m})} \,, \quad X^5_{u_A} := C^{5A} \,. \tag{5.72}$$

Note that we set $l_p = 1$ again for readability. Therefore, the nontrivial factor in eq. (5.69) can be written as

$$\begin{aligned} F^0_{\mu 5,(i\vec{m})} &= \lambda^0 D_\mu X^5_{(i\vec{m})} - \partial_5 A^0_{\mu(i\vec{m})} \\ &= \lambda^0 \left[P^{5M} \hat{D}_\mu X^M_{(i\vec{m})} + \lambda^0 F_{\mu 0(i\vec{m})} + \sum_a \tilde{\lambda}^a F_{\mu a(i\vec{m})} \right] - \partial_5 A^0_{\mu(i\vec{m})} \,, \end{aligned} \tag{5.73}$$

where $F_{\mu 0(i\vec{m})} := \hat{D}_\mu (\vec{\pi}_0 \cdot \vec{X})_{(i\vec{m})} + A'_{\mu(i\vec{m})}$ and $F_{\mu a(i\vec{m})} := \hat{D}_\mu A_{a(i\vec{m})} + i m_a A^0_{\mu(i\vec{m})}$. The notation of other fields is defined around eq. (5.17) and (5.27). Then we obtain the total derivative terms in Dp-brane action which can be derived from the term (5.69) as

$$S \supset \int d^5x \, \frac{d^d y}{(2\pi)^d} \sqrt{g} \left[(\lambda^0)^2 \tilde{\lambda}^a \epsilon^{\mu\nu\rho\sigma\lambda 5} \hat{F}_{\mu\nu,i} \hat{F}_{\rho\sigma,j} \hat{F}_{\lambda a,k} f^{il}{}_m f^{jm}{}_n f^{kn}{}_l + \cdots \right] , \tag{5.74}$$

where '$\cdots$' are the total derivative terms which don't vanish in the $\tilde{\lambda}^a \to 0$ limit. We neglect them here, since it is known that the total derivative terms don't play any role, when M-compactification direction is perpendicular to T-duality direction, *i.e.* $\vec{\lambda}^0 \cdot \vec{\lambda}^a = 0$ or $\tilde{\lambda}^a = 0$. Note that the metric g^{ab} in this case is different from eq. (5.34) as

$$g^{ab} := |\vec{\lambda}^0|^2 (\vec{\lambda}^a \cdot \vec{\lambda}^b) - (\vec{\lambda}^0 \cdot \vec{\lambda}^a)(\vec{\lambda}^0 \cdot \vec{\lambda}^b) \,. \tag{5.75}$$

From the discussion above, we can conclude that the nonabelian $(2,0)$ theory can have an additional total derivative term of the form (5.69) in its Lagrangian, and that the $F \wedge F \wedge F$

term in Dp-brane Lagrangian can be derived from this term. Here we should remember again that Lagrangian of the nonabelian $(2,0)$ theory is not defined properly at this stage, but this discussion is still meaningful, since the problematic self-dual 2-form field $B_{\mu\nu}$ doesn't appear here at all. Further justification of this result from the viewpoint of T-transformation will be done in § 5.5.

Kaluza-Klein monopoles

For completeness of our discussion, we now comment on type IIA/IIB Kaluza-Klein monopoles reproduced from the nonabelian $(2,0)$ theory.

Type IIA KK monopoles

It is known that type IIA KK monopoles can be obtained from type IIB NS5-branes by taking T-duality for a transverse direction [43]. Therefore, in this case, we use a generalized loop algebra $\{T^i_{\vec{m}}, u_{0,1,2}, v^{0,1,2}\}$ defined by eq. (5.12). Then we put VEV's into $u_{0,1,2}$-component fields as

$$X^{I0} = \lambda^0 \delta^I_{10}\,, \quad C^{\mu 1} = \lambda^1 \delta^\mu_5\,, \quad X^{I2} = \lambda^2 \delta^I_9\,, \quad \text{otherwise} = 0\,. \tag{5.76}$$

This setup can be generalized into the case where these VEV's are not perpendicular to each other, but all the following results remain the same. Finally, we redefine the fields in a similar way to eq. (5.49) as

$$\hat{\Phi}_i(x, y_1, y_2) = \sum_{\vec{m}} \Phi_{(i\vec{m})}(x) e^{-i\vec{m}\cdot\vec{y}}\,, \quad \hat{A}_{\mu,i}(x, y_1, y_2) = \sum_{\vec{m}} A^1_{\mu(i\vec{m})}(x) e^{-i\vec{m}\cdot\vec{y}}\,, \quad \cdots\,. \tag{5.77}$$

As a result, we obtain the EOM's of the same form as type IIB NS5-brane case in § 5.3, except that of the scalar field $\hat{X}^9$

$$\hat{D}^2_\alpha \hat{X}^9 + \hat{D}^2_{y_1} \hat{X}^9 - (\lambda^0)^2 \lambda^1 \lambda^2 \hat{D}_{y_1} \partial_{y_2} \hat{C}^5 \;=\; 0\,, \tag{5.78}$$

which has an additional term with a y_2 derivative, compared with eq. (5.55). We should remember that a factor like $\partial_{y_2}\hat{C}^\mu$ never appear in the previous discussions. From the viewpoint of Lorentz invariance for the condition $\partial_\mu C^\nu_{(i\vec{m})} = 0$, it is natural here to impose $\partial_{y_2}\hat{C}^5 = 0$, or equivalently, $C^5_{(i\vec{m})}\big|_{m_2\neq 0} = 0$. This, of course, does not break gauge symmetry nor supersymmetry. After imposing this, the final result does not contain any y_2 derivatives, so this y_2 direction becomes isometry. In fact, it must correspond to Taub-NUT isometry direction. Therefore, we can integrate out the y_2 dependence from all the redefined fields (5.77), and then we obtain the 6-dim worldvolume fields in type IIA KK monopole theory which depend on only $x^{0,\cdots,4}$ and y_1 coordinates.

The field contents of this theory are three embedding scalars $\hat{X}^{6,7,8}$, a 1-form field $\hat{A}_{\underline{\mu}}$, a 0-form field $\hat{X}^9$ and a fermion $\hat{\Psi}$. Therefore, they are exactly reproduced from the nonabelian $(2,0)$ theory only by specializing the scalar field $\hat{X}^9$.

Type IIB KK monopoles

On the other hand, type IIB KK monopoles can be obtained from type IIA NS5-branes by taking T-duality for a transverse direction [43]. Therefore, in this case, we use a generalized loop algebra $\{T^i_m, u_{0,1}, v^{0,1}\}$ defined by eq. (5.12) or (5.47). Then we put VEV's into $u_{0,1}$-component fields as

$$X^{I0} = \lambda^0 \delta^I_{10}\,, \quad X^{I1} = \lambda^1 \delta^I_9\,, \quad \text{otherwise} = 0\,. \tag{5.79}$$

Similarly, even if we make these VEV's not perpendicular, the following results are unchanged. Finally, we redefine the field in a similar way to eq. (5.49) as

$$\hat{\Phi}_i(x,y) = \sum_m \Phi_{(im)}(x) e^{-imy}\,, \quad \cdots\,. \tag{5.80}$$

As a result, at this time, we obtain the EOM's of the same form as type IIA NS5-brane case (5.46), except that of the scalar field $\hat{X}^9$

$$\partial^2_\mu \hat{X}^9 - (\lambda^0)^2[\hat{C}^\mu, [\hat{C}_\mu, \hat{X}^9]] + i(\lambda^0)^2\lambda^1[\hat{C}_\mu, \partial_y \hat{C}^\mu] \;=\; 0\,, \tag{5.81}$$

which has an additional term with a y derivative. By similar discussion to type IIA KK monopole case, it is natural to impose $\partial_y \hat{C}^\mu = 0$ to eliminate the y derivative, and to regard the y direction as Taub-NUT isometry direction. Therefore, we can integrate out the y dependence from all the redefined fields (5.80), and then we obtain 6-dim worldvolume fields in type IIB KK monopole theory which depend on only $x^{0,\cdots,5}$ coordinates. The field contents of this theory are three embedding scalars $\hat{X}^{6,7,8}$, a self-dual 2-form field $\hat{B}_{\mu\nu}$, two 0-form fields $\hat{X}^{9,10}$ and a fermion $\hat{\Psi}$. Therefore, they are exactly reproduced from the nonabelian $(2,0)$ theory only by specializing the scalar fields $\hat{X}^{9,10}$.

It is also known that type IIB KK monopole theory must be invariant under S-duality transformation. In our setup, this transformation corresponds to the interchange of VEV's X^{I0} and X^{I1}, as we will see in § 5.5. Since C^μ-field has no dynamical degrees of freedom, we can regard the resultant theory as practically the simple copies of free theory, just as we discussed in § 5.3. Therefore, all the interaction terms are negligible, and then we can see that S-self-duality of type IIB KK monopole is trivially satisfied. If one wants to reproduce S-self-duality including the interaction terms, some generalization of the nonabelian $(2,0)$ theory must be needed.

Role of C^μ-field

Let us make short comments on C^μ-field here. This field is a nondynamical auxiliary field, since it never has the kinetic term. Moreover, it seems conveniently introduced instead of a dimensionful parameter in order to make interaction terms appear in the theory, since any dimensionful parameters cannot exist in M5-brane system in flat background.

However, let us now try to find some physical meanings of this field.

In fact, it seems related to the gauge fixing condition for the general coordinate transformation symmetry on the M5-brane worldvolume as

$$X^{\mu}(\sigma) = \sigma^{\mu}\mathbf{1} + C^{\mu}_{a}(\sigma)\, T^{a}\,, \tag{5.82}$$

under the condition $D_{\mu}C^{\nu}_{a} = 0$. Here σ^{μ} are worldvolume coordinates and $\mathbf{1}$ is a trivial element, satisfying $[\mathbf{1}, T^{a}, T^{b}] = 0$ and $\langle \mathbf{1}, \mathbf{1}\rangle = 1$. It corresponds to the center-of-mass mode in brane system which is decoupled from the theory. In the case of generalized loop algebra (5.12), for example, $\mathbf{1}$ is equivalent to $T^{0}_{\vec{0}}$.

This discussion suggests that we can regard $[C^{\mu}, \star, \star]$ as $[X^{\mu}, \star, \star]$. This identification must be natural: As we saw in § 5.2 and § 5.3, putting a VEV for u-component of C^{μ}-field means the compactification for one of x^{μ}-directions, while putting a VEV for u-component of X^{I}-field means the compactification for one of x^{I}-directions. Therefore, it seems very natural to expect that C^{μ}-field is related to X^{μ}.

Moreover, we consider in § 5.4 that gauge field $A_{\mu,ab}$ and C^{μ}_{a}-field play the complementary roles of X^{μ}. In fact, in eq. (5.72), we have treated the gauge field $A^{0}_{\mu(i\vec{m})}$ as $X^{\mu}_{(i\vec{m})}$, while the field $C^{\mu A}$ as $X^{\mu}_{u_A}$. The former is natural from the viewpoint of dimensional reduction where a higher dimensional gauge field is decomposed into a lower dimensional gauge field and transverse scalars. However, the latter seems unusual and very interesting. This makes us again suppose that C^{μ}-field is related to X^{μ}.

If the identification (5.82) is correct, the condition $D_{\mu}C^{\nu}_{a} = 0$ can be regarded as a gauge fixing for a part of general coordinate transformation symmetry, which assures that the factor $D_{\mu}X^{\nu}_{a}$ doesn't appear in Lagrangian. Therefore, in order to check our assumption, we need to write down DBI-like action for generalization of the nonabelian $(2,0)$ theory, since such factors should appear in it. We hope to discuss it in the future.

5.5 Discussion on U-duality

In § 5.2 and § 5.3, we show that the Dp-brane and NS5-brane theories can be obtained from the nonabelian $(2,0)$ theory. Strictly speaking, however, they are only (part of) super Yang-Mills theories, which are low energy effective theories of the brane systems. Then in this section, as a further justification of our discussion, we study whether our results reproduce the expected U-duality relation among M5-branes, Dp-branes and NS5-branes. This must be a highly nontrivial check for the nonabelian $(2,0)$ theory as a formulation of M5-brane system.

D5-branes on S^1

We start with the simplest case. This corresponds to the $d = 1$ case in § 5.2. The notation for VEV's $\vec{\lambda}^{A}$ is defined in eq. (5.67) and (5.68).

T-duality For simplicity, only in this and next paragraphs, let us assume $\vec{\lambda}^0 \perp \vec{\lambda}^1$. As we mentioned, putting the VEV $\vec{\lambda}^0$ means the compactification of M-theory direction with the radius

$$R_0 = |\vec{\lambda}^0| \,. \tag{5.83}$$

Similarly, putting a VEV $\vec{\lambda}^1$ must imply the compactification of another direction with the radius $R_1 = |\vec{\lambda}^1|$ before taking T-duality. Then we have D4-brane worldvolume theory with string coupling [24]

$$g_s = g_{YM}^2 l_s^{-1} = |\vec{\lambda}^0| l_s^{-1} \tag{5.84}$$

where l_s is the string length, satisfying $l_p^3 = g_s l_s^3$. In §5.2, D5-brane theory is obtained, since we take T-duality for the $\vec{\lambda}^1$ direction (by field redefinition). After taking T-duality, the compactification radius is

$$\tilde{R}_1 = \frac{l_s^2}{R_1} = \frac{l_p^3}{|\vec{\lambda}^0||\vec{\lambda}^1|} \,, \tag{5.85}$$

which is consistent with the metric component g^{11} on the torus S^1 (5.34). From the kinetic term for gauge field in Lagrangian (5.41), the string coupling in this theory can be read as

$$g'_s = g'^2_{YM} l_s^{-2} = \frac{|\vec{\lambda}^0|^2}{|\vec{\lambda}^0||\vec{\lambda}^1|} \frac{l_p^3}{R_0\, l_s^2} = \frac{|\vec{\lambda}^0|}{|\vec{\lambda}^1|} \,, \tag{5.86}$$

which is compatible with the expected result from string duality, namely $g'_s = g_s l_s / \tilde{R}_1 = R_0/R_1$. Therefore, we can conclude that T-duality relation is exactly reproduced.

S-duality We continuously assume $\vec{\lambda}^0 \perp \vec{\lambda}^1$ in this paragraph. In §5.3, we discuss the worldvolume theory on type IIB NS5-branes. From the kinetic term for gauge field in Lagrangian (5.65), we can read off the string coupling in this theory as

$$g''_s = g''^2_{YM} l_s^{-2} = \frac{|\vec{\lambda}^1|^2}{|\vec{\lambda}^0||\vec{\lambda}^1|} \frac{l_p^3}{R_0\, l_s^2} = \frac{|\vec{\lambda}^1|}{|\vec{\lambda}^0|} \,. \tag{5.87}$$

This is exactly the inverse of string coupling in D5-brane theory (5.86), so we can conclude that S-duality relation is successfully reproduced. Moreover, we can find that S-duality is realized as a part of $SL(2,\mathbb{Z})$ transformation of VEV's

$$\vec{\lambda}^0 \to -\vec{\lambda}^1 \,, \quad \vec{\lambda}^1 \to \vec{\lambda}^0 \,. \tag{5.88}$$

T-transformation We consider this transformation in § 5.4. By comparing the setting of VEV's after transformation (5.66) with the original one (5.22), we can find that this transformation is identified with another part of $SL(2,\mathbb{Z})$ transformation of VEV's

$$\vec{\lambda}^0 \to \vec{\lambda}^0 \,, \quad \vec{\lambda}^1 \to \vec{\lambda}^1 + n\vec{\lambda}^0 \,. \tag{5.89}$$

Interestingly enough, it is related to automorphism of Lie 3-algebra [8]

$$u_0 \to u_0 - n u_1\,, \quad u_1 \to u_1\,, \\ v^0 \to v^0\,, \quad v^1 \to v^1 + n v^0\,, \tag{5.90}$$

that is, this transformation changes neither structure constant nor metric of Lie 3-algebra. The relation between them can be understood as the redefinition of ghost fields

$$X^M = X^{M0} u_0 + X^{M1} u_1 + \cdots = X^{M0}(u_0 - n u_1) + (X^{M1} + n X^{M0})\, u_1 + \cdots\,, \tag{5.91}$$

where $M = (\mu, I)$ and $X^{\mu A} := C^{\mu A}$ as in eq. (5.72). Of course, there is no reason that the parameter n must be quantized at the classical level, but it is still interesting that part of the duality transformation comes from the automorphism of Lie 3-algebra.

It is well known that this transformation (5.89) causes the change of axion field $C_{(0)}$, which appears in D5-brane Lagrangian as a Chern-Simons term $C_{(0)} \wedge F_{(2)} \wedge F_{(2)} \wedge F_{(2)}$. Therefore, the value of $C_{(0)}$ field can be read from eq. (5.74) as

$$C_{(0)} = \frac{|\vec{\lambda}^0|(\vec{\lambda}^0 \cdot \vec{\lambda}^1)}{3!\, 2\pi l_p^3} = \frac{\tau_1}{3!\, 2\pi} \frac{|e|^3}{l_p^3}\,, \tag{5.92}$$

and the inverse of string coupling can be read from eq. (5.41) as

$$g_s^{-1} = \frac{|\vec{\lambda}^0| \sqrt{g^{11}}}{2\pi l_p^3} = \frac{\tau_2}{2\pi} \frac{|e|^3}{l_p^3}\,, \tag{5.93}$$

where we define the new basis $\{\vec{e}^{\,0}, \vec{e}^{\,1}\}$ as

$$\vec{\lambda}^0 = \vec{e}^{\,0}\,, \quad \vec{\lambda}^1 = \tau_1 \vec{e}^{\,0} + \tau_2 \vec{e}^{\,1}\,; \quad \vec{e}^{\,0} \cdot \vec{e}^{\,1} = 0\,, \quad |\vec{e}^{\,0}| = |\vec{e}^{\,1}| =: |e|\,. \tag{5.94}$$

In this basis, T-transformation is written as $\tau_1 \to \tau_1 + n$, $\tau_2 \to \tau_2$. Therefore, this result shows that T-transformation is also perfectly reproduced in our discussion.

Taylor's T-duality This transformation [38] interchanges D5- and D4-branes, and corresponds to the different identification of T^i_m-component fields in our discussion. To obtain D5-brane system, we constructed 6-dim field $\hat{X}^I(x, y)$ from the component fields $X^I_{(im)}(x)$ by Fourier transformation (5.26). On the other hand, one can interpret $X^I_{(im)}(x)$ as the 5-dim fields and the index $m \in \mathbb{Z}$ as open string modes which interpolate mirror images of a point in $T^1 = \mathbb{R}/\mathbb{Z}$. In this way, Taylor's T-duality transformation $\mathbb{Z}_2$ is reproduced.

Summary As we already mentioned, S-duality and T-transformation can be written as the $SL(2, \mathbb{Z})$ transformation of VEV's

$$\begin{pmatrix} \vec{\lambda}^1 \\ \vec{\lambda}^0 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \vec{\lambda}^1 \\ \vec{\lambda}^0 \end{pmatrix}, \tag{5.95}$$

which is equivalent to the transformation of the moduli parameter $\tau := \tau_1 + i\tau_2$

$$\tau \to \frac{a\tau + b}{c\tau + d} . \tag{5.96}$$

In fact, S-duality $\tau \to -1/\tau$ is given as $(a, b, c, d) = (0, 1, -1, 0)$, while T-transformation $\tau \to \tau + n$ is given as $(a, b, c, d) = (1, n, 0, 1)$. It is well known that any element of $SL(2, \mathbb{Z})$ transformation can be composed as combination of these two kinds of transformation.

As a result, together with Taylor's T-duality, it is finally shown that the whole of U-duality transformation in the case of D5-branes on S^1 (or M-theory on T^2)

$$SL(2, \mathbb{Z}) \bowtie \mathbb{Z}_2 \tag{5.97}$$

is completely reproduced in our discussion, where the first factor is described by the rotation of VEV's and the second factor is described by the different representation of the field theory. Here, the symbol $\bowtie$ denotes the product group defined by the two noncommuting subgroups.

Dp-branes on T^{p-4} $(p \geq 5)$

Finally, we discuss the U-duality in general $d \geq 1$ cases in § 5.2. In these cases, we consider M-theory compactified on T^{d+1} (where $d = p - 4$). This theory has U-duality group

$$E_{d+1}(\mathbb{Z}) = SL(d + 1, \mathbb{Z}) \bowtie SO(d, d; \mathbb{Z}) \tag{5.98}$$

and its moduli parameters take values in E_{d+1}/H_{d+1}, where H_{d+1} is the maximal compact subgroup of E_{d+1}. (See *e.g.* [44] for a review.)

Now let us read off the values of these moduli in Dp-brane case from our results. For readability, we set $l_p = 1$ again in the following. First, the metric on the torus T^d (5.75) is

$$g^{ab} = |\vec{\lambda}^0|^2 (\vec{\lambda}^a \cdot \vec{\lambda}^b) - (\vec{\lambda}^0 \cdot \vec{\lambda}^a)(\vec{\lambda}^0 \cdot \vec{\lambda}^b) , \tag{5.99}$$

where $a, b = 1, \cdots, d$. Secondly, the Yang-Mills coupling (5.41) is

$$g_{YM}^2 = \frac{(2\pi)^d |\vec{\lambda}^0|}{\sqrt{g}} , \tag{5.100}$$

where $g := \det g^{ab}$. Finally, we read off the value of R-R $(d-1)$-form field $C_{(d-1)}$. This field may appear in Dp-brane Lagrangian as a Chern-Simons term $C_{(d-1)} \wedge F_{(2)} \wedge F_{(2)} \wedge F_{(2)}$. Therefore, this can be read from eq. (5.74) as

$$C_{(d-1)} = \frac{|\lambda^0| (\vec{\lambda}^0 \cdot \vec{\lambda}^a)}{6 (2\pi)^d (d - 1)!} \frac{\sqrt{g}}{\sqrt{g^{aa}}} , \tag{5.101}$$

where no sum is taken on the index a. This represents the components of $C_{(d-1)}$ with the indices $1\,2 \cdots \hat{a} \cdots d$, *i.e.* except a.

Therefore, the number of moduli written by VEV's (5.99)–(5.101) is

$$\frac{1}{2}d(d+1)+1+d=\frac{1}{2}(d+1)(d+2)\,. \tag{5.102}$$

This coincides with the number of parameters in $G^{AB}:=\vec{\lambda}^A\cdot\vec{\lambda}^B$, which is transformed under $SL(2,\mathbb{Z})$ transformation

$$\vec{\lambda}^A\to\vec{\lambda}'^A:=\Lambda^A{}_B\vec{\lambda}^B\,;\quad \Lambda^A{}_B\in SL(d+1,\mathbb{Z})\,. \tag{5.103}$$

This means that our discussion correctly reproduces the $SL(d+1,\mathbb{Z})$ symmetry as the first factor of U-duality (5.98), and that $G^{AB}=G^{AB}(g^{ab},\,g_{YM}^2,\,C_{(d-1)})$ gives the moduli parameter which is transformed covariantly under the $SL(d+1,\mathbb{Z})$ transformation.

The second factor $SO(d,d;\mathbb{Z})$ of U-duality (5.98) can be also reproduced. It consists of the permutation of T-duality directions, Taylor's T-duality transformation, and the shift of the value of NS-NS 2-form field. The first one can be seen trivially in our setup, and the second one is reproduced in a similar way to the $d=1$ case. The third one is rather nontrivial. The NS-NS 2-form field B_{ab} can be introduced as the deformation of Lie 3-algebra [7]

$$[u_0,u_a,u_b]=B_{ab}T^0_{\vec{0}} \tag{5.104}$$

instead of ordinary generalized loop algebra (5.12), since it provides the noncommutativity on the torus T^d. It is interesting that some part of moduli (5.99)–(5.101) are described in terms of VEV's, while another part comes from the structure constant of Lie 3-algebra.

However, this is not the end of the story. The U-duality group is a product of these *non-commuting* subgroups, and so unfortunately, the whole moduli space of U-duality cannot be described by only the moduli parameters obtained above. In the following, we check the dimension of moduli space, and discuss what kinds of parameters are lacked in our setup. In fact, in the $d\geq 3$ cases, some missing parameters exist.

D5-branes ($d=1$) M-theory compactified on T^2 is considered. The moduli space in this case is $\big(SL(2)/U(1)\big)\times\mathbb{R}$ which gives 3 parameters. They correspond to g^{11}, ϕ and $C_{(0)}$.

D6-branes ($d=2$) M-theory compactified on T^3 is considered. The moduli space in this case is $\big(SL(3)/SO(3)\big)\times\big(SL(2)/U(1)\big)$ which gives 7 parameters. They correspond to g^{ab}, B_{ab}, ϕ and $C_{(1)}$ which transform in the $\mathbf{3}+\mathbf{1}+\mathbf{1}+\mathbf{2}$ representations of $SL(2)$.

D7-branes ($d=3$) M-theory compactified on T^4 is considered. The moduli space in this case is $SL(5)/SO(5)$ which gives 14 parameters. They correspond to g^{ab}, B_{ab}, ϕ, $C_{(2)}$ and $C_{(0)}$ which transform in the $\mathbf{6}+\mathbf{3}+\mathbf{1}+\mathbf{3}+\mathbf{1}$ representations of $SL(3)$.

R-R 0-form field $C_{(0)}$ is lacked in our discussion. This field causes the Chern-Simons interaction $C_{(0)}\wedge F_{(2)}\wedge F_{(2)}\wedge F_{(2)}\wedge F_{(2)}$ which cannot be derived in a similar way to § 5.4. Therefore, in order to include this parameter, we might need to consider the nontrivial backgrounds. For the missing parameters below, similar discussions would be made.

D8-branes $(d=4)$ M-theory compactified on T^5 is considered. The moduli space in this case is $SO(5,5)/\big(SO(5)\times SO(5)\big)$ which gives 25 parameters. They correspond to g^{ab}, B_{ab}, ϕ, $C_{(3)}$ and $C_{(1)}$ which transform in the $\mathbf{10}+\mathbf{6}+\mathbf{1}+\mathbf{4}+\mathbf{4}$ representation of $SL(4)$. R-R 1-form field $C_{(1)}$ is lacked in our discussion.

D9-branes $(d=5)$ M-theory compactified on T^6 is considered. The moduli space in this case is $E_6/USp(8)$ which gives 42 parameters. They correspond to g^{ab}, B_{ab}, ϕ, $C_{(4)}$, $C_{(2)}$ and $C_{(0)}$ which transform in the $\mathbf{15}+\mathbf{10}+\mathbf{1}+\mathbf{5}+\mathbf{10}+\mathbf{1}$ representations of $SL(5)$. R-R 2-form and 0-form field $C_{(2)}$, $C_{(0)}$ are lacked in our discussion.

Conclusions and discussions

In order to understand the nonperturbative aspects of superstring theory, it is essential to investigate the dynamics of M-theory. Although some aspects of M-theory has been clarified due to the develpoment such as Matrix model and AdS/CFT correspondence, further studies are needed to uncover the characteristics of M-theory and its branes.

Finally we would like to comment that there are still many important open problems related to M-theory branes. Some of them are listed below in random order.

M5-branes and anomaly Quite recently, 6-dim (1,0) SCFT with nonabelian gauge coupling between multiple tensor multiplets were proposed in [45]. This construction is based on a method originally considered in the context of gauged supergravity. This success may shed light to understand M5-branes. The proposed model consists of tensor multiplets and vector multiplets. To complete the field content to that of (2,0) theory, we have to include the hypermultiplets. However, in general, the anomaly-free condition heavily restricts the number of these multiplets and only a few gauge group is allowed. Therefore, it is indispensable to study the anomaly structure in order to construct the maximally supersymmetric M5-brane action in the future.

Lie 3-algebra in M5-branes Applicating Lie 3-algebra to M5-branes is a challenging problem. Although there was some recent progress in constructing M5-brane theory in terms of Lie 3-algebra, completely sufficient results has not been obtained. The gauging procedure used in [45] has been also applied to construct the multiple M2-branes and the relationship between structure constant of Lie 3-algebra and certain invariant tensor crucial for the gauging are clarified. It may be possible to utilize this results for rewriting (1,0) SCFT of [45] in terms of the Lie 3-algebra. Searching a connection to the construction of (2,0) theory in [24] is also interesting.

M2-brane entropy The crucial difference between M-branes and D-branes is a scaling property of the entropy. From AdS/CFT correspondence, it is known that the degrees of freedom on the worldvolume of N M2-branes is proportional to $N^{3/2}$, not N^2 like N D-branes. Although it has not been fully understood how and why such a phenomenon occurs, a remarkable progress about this issue was achieved in [46]. They observed exact results about free energy of M2-branes from ABJM matrix model obtained by the use of localization technique. In the strong coupling limit of t'Hooft parameter, they realized the expected anomalous scaling for the M2-brane theory.

In [47], it was shown that the partition function of ABJM theory reduced to a matrix model can be reformulated as an ideal Fermi gas with one-particle Hamiltonian. It is very important to explore the physical meaning of anomalous scaling of M2-brane entropy along this approach. Worldsheet and membrane instantons are responsible for the nonperturbative correction to the partition function of M2-branes and understanding thier effects leads to reveal unknown dynamics of M-theory.

M5-brane and 5-dim SYM It is well known that one dimensional reduction of M5-brane theory leads to 5-dim SYM. However, it has been not enough understood how M-theoretic information appears in 5-dim SYM in UV. The reason is that the ordinary Kaluza-Klein compactification is not allowed in this case. This is because the dimensional analysis of 5-dim SYM gauge coupling is inconsistent with the conformal symmetry of M5-brane theory. This is a peculiar problem of M5-brane and further research is required to extract M-theoretic properties behind it. There is a recent attempt to identify self-dual string soliton obtained from M2-M5 system as instantons of 5-dim SYM [48]. This means that the information of M-theory is already included as soliton solutions and this remarkable identification needs to be further investigated. Moreover, there is a possibility that the difference of the entropy of M5-branes and that of D-branes are due to the appearance of certain bound states and this is also an interesting topic. Meanwhile, caluculation technique of gauge and gravity amplitudes has seen dramatic growth within the recent past and its application to M-theory branes draws increasing attention.

5-dim supersymmetric Yang-Mills theory in the UV scale Revisiting the UV-completion of 5-dim SYM may be important in the viewpoint of M-theory. If KK-states coming from M5-brane on S^1 and instantons of 5-dim SYM are equivalent as considered in [48], this means that 5-dim SYM doesn't acquire extra degrees of freedom at all in the cut-off scale and, therefore, it may be UV-finite. Then we need to reconsider UV-completion mechanism beyond the standard Wilsonian approach.

On the other hand, a novel approach to UV-completion of a class of non-renormalizable theories was suggested in [49]. This idea is inspired by a black hole formation and they conjecture that a formation of classical objects in high energy scattering procceses induces inaccessibility of short distance. Although Nambu-Goldstone type scalar is given as an example, examination of its validity and further generalization is required.

Massive Gravity and Higher Spin Gauge Theories Non-linear theories of massive gravity generally suffer from ghost instability. However, recently proposed theories of massive gravity have been shown to be ghost-free. Inspired by these developments, a ghost-free bimetric theory of spin-2 fields were proposed in [50]. This is the first construction of a consistent theory of interacting multiple spin-2 fields.

This remarkable progress might be applied to several issues about M-theory. It is known that there are some no-go theorems prohibiting nonabelian deformation of self-dual antysymmetric gauge field on M5-brane. Searching potential loop-holes for nontrivial interacting theories of M5-branes using techniques of massive gravity is intriguing. The bimetric gravity is also attractive in the AdS/CFT point of view. Investigating its relationship to Fradkin-Vasiliev cubic vertices and Vasiliev's full higher spin equation of motion is need to be clarified.

As we have seen, investigating M-theory physics from the explicit models of its branes starts only recently. We expect further fruitful developments in this fascinating subjects.

Appendix A

Mass deformation and Janus solutions

A.1 Janus field theory with dynamical coupling

In the previous section, we discussed BLG theory with Lorentzian Lie 3-algebra. There we have fixed the solution of the constraint equations (2.20). But in the quantization of the Bagger-Lambert-Gustavsson theory, the solutions should be summed in the path integral. So we will consider more general solutions in this subsection. After integrating the modes associated with the T^{-1} generator, the partition function becomes

$$Z = \int \mathcal{D}X_0^I \mathcal{D}\Psi_o \mathcal{D}B_\mu \mathcal{D}\hat{X}^I \mathcal{D}\hat{\Psi} \mathcal{D}A_\mu \ \delta(\partial^2 X_0^I) \ \delta(\Gamma^\mu \partial_\mu \Psi_0) \ e^{iS(X_o^I, \Psi_0, B_\mu, \hat{X}^I, \hat{\Psi}, A_\mu)}. \tag{A.1}$$

The integrations over X_0^I and Ψ_0 are constrained to obey the massless wave equations and can be expanded as

$$X_0^I = \sum_n c_n^I f_n(x), \quad \Psi_0 = \sum_n b_n u_n(x) \tag{A.2}$$

where $f_n(x), u_n(x)$ are complete sets of functions satisfying the massless wave equations. Then the integration over X_0^I and Ψ_0 can be reduced to integrations over c_n^I and b_n.

Let us now choose a general solution $(X_0^I = v^I(x), \Psi_0)$ to the constraints and expand the action around it. In this case all the supersymmetries are generally broken if we fix v^I and Ψ_0. Inserting this general solution into the action, terms including the B_μ gauge field are given by

$$-\frac{1}{2}(\hat{D}_\mu \hat{X}^I - B_\mu X_0^I)^2 + i\bar{\Psi}_0 \Gamma^\mu B_\mu \hat{\Psi} + \frac{1}{2}\epsilon^{\mu\nu\lambda} \hat{F}_{\mu\nu} B_\lambda - \partial_\mu X_0^I B_\mu \hat{X}^I. \tag{A.3}$$

The integration over the B_μ gauge field can be similarly performed. It is convenient to introduce the locally defined projection operator

$$P_{IJ}(x) = \delta_{IJ} - \frac{v_I v_J}{v^2}, \tag{A.4}$$

This operator satisfies $P^2 = P$ and $P_{IJ}v^J = 0$. In the simplest case considered in the previous subsection, $v^I = v(t+x)\delta_{10}^I$, this projects out the 10-th direction if it acts on $\hat{X}^I$. Generally, the direction removed is dependent on the space-time position.

After integrating over the B_μ field, the Lagrangian becomes $\mathcal{L}_{Janus} = \mathcal{L}_0 + \mathcal{L}'$ where

$$\begin{aligned}\mathcal{L}_0 &= \mathrm{Tr}\Big[-\frac{1}{2}(\hat{D}_\mu Y^I)^2 + \frac{1}{4}v^2[Y^I,Y^J]^2 + \frac{i}{2}\bar{\hat{\Psi}}\Gamma^\mu \hat{D}_\mu\hat{\Psi} + \frac{1}{2}\bar{\hat{\Psi}}[Y^I,(v^J\Gamma_J)\Gamma_I\hat{\Psi}] \\ &\quad + \frac{1}{2(v^I)^2}\big(\frac{1}{2}\epsilon^{\mu\nu\lambda}\hat{F}_{\nu\lambda} + i\bar{\Psi}_0\Gamma^\mu\hat{\Psi} - 2Y_I\partial^\mu v^I\big)^2 - \frac{1}{2}\bar{\Psi}_0\Gamma_{IJ}\hat{\Psi}[Y^I,Y^J]\Big], \qquad (A.5)\\ \mathcal{L}' &= \frac{1}{v^2}\mathrm{Tr}\Big[\Big(\bar{\Psi}_0\Gamma_I(v^J\Gamma_J)[Y^I,\hat{\Psi}] - i\bar{\Psi}_0\Gamma_\mu\hat{D}_\mu\hat{\Psi}\Big)(v^K\hat{X}^K)\Big]. \qquad (A.6)\end{aligned}$$

Here $I,J = 3,\cdots,10$ and we have defined a new scalar field $Y^I = P_{IJ}\hat{X}^J$ with 7 degrees of freedom. In spite of it, the action has $SO(8)$ invariance if v^I and Ψ_0 also transform under it. Also note that Y^I is invariant under the gauge transformations associated with B_μ gauge fields. Is is also interesting to notice that the action will have a generalized conformal symmetry [51] even with the dimensionful coupling because it is a dynamical variable here. This may have its origin in the conformal symmetry of M2 branes. In this sense, the reduced action is not exactly the same as the ordinary D2 brane effective action with a fixed gauge coupling. This issue is now under investigations.

This is a Janus field theory whose coupling varies with space-time. The Lagrangian $\mathcal{L}_{YM}$ contains only the projected scalar field Y^I. On the other hand, in the presence of Ψ_0, the scalar field $(v^I\hat{X}^I)$ does not decouple from the Lagrangian $\mathcal{L}'$. If we can set $\Psi_0 = 0$, $\mathcal{L}'$ vanishes and the resultant Lagrangian is given by a similar form to the ordinary Super Yang-Mills Lagrangian, but the kinetic term of the gauge field $\hat{F}_{\mu\nu}$ is modified to $\hat{F}_{\mu\nu} + 2\epsilon_{\mu\nu\rho}Y_I\partial^\rho v^I$. All the supersymmetries are generally broken if we fix one solution to the constraint equations of $(X_0^I(x),\Psi_0)$ as above.

By using the above calculation, the partition function can be simply rewritten as

$$Z = \int\prod_n dc_n^I\, db_n\, W(v^I)\int \mathcal{D}\hat{X}^I\mathcal{D}\hat{\Psi}\mathcal{D}A_\mu\, e^{iS_{Janus}(\hat{X}^I,\hat{\Psi},A_\mu;v^I(x),\Psi_0)}. \qquad (A.7)$$

Here $W(v^I) \sim ((v^I)^2)^{-3/2}$ came from the integration over the B_μ field. It is a sum of Janus field theories. The coupling constant v^I is dynamical and varies with space-time coordinates. It is constrained to satisfy the massless equations. If we fix the "slow" variable v and perform the path integration over the other "fast" variables first, then we can get an effective action for the dynamical coupling v^I. This will determine the most stable configuration of $v^I(x)$, and accordingly one of the Janus gauge theory with the most stable coupling is determined. If the variable v^I fluctuates rapidly and cannot be considered as a slow variable, the theory becomes very different from the ordinary gauge theory with a fixed (either constant or varying) gauge coupling. This may be related to the dynamical determination of the compactification radius of 11-th direction in M-theory.

Finally we would like to comment on the unitarity of the Bagger-Lambert-Gustavsson theory. If we fix one solution to the constraints, each theory behaves regularly if the coupling constant does not vary drastically. The quantization of the coupling is very difficult, but since it is not a propagating mode, it will not violate the unitarity of the theory. However the unitarity should be more carefully analyzed.

A.2 Mass deformation of BLG theory

The BLG theory in the previous section gives a familiar effective action of N D2 branes with either a constant or a varying coupling. (For general solutions, the kinetic term of the gauge field contains a non-familiar term of $Y_I \partial^\mu v^I$.)

In this section we start from a mass deformed BLG action given by [52, 53] and show that supersymmetric Janus field theories with a Myers-term are obtained.

One parameter deformation of the Bagger-Lambert action preserving the full supersymmetries is given by adding the following mass and flux terms to the original Lagrangian. The mass term is given by

$$\mathcal{L}_{mass} = -\frac{1}{2}\mu^2 \mathrm{Tr}(X^I, X^I) + \frac{i}{2}\mu \mathrm{Tr}(\bar{\Psi}\Gamma_{3456}, \Psi), \tag{A.8}$$

and a flux term is

$$\mathcal{L}_{flux} = -\frac{1}{6}\mu\epsilon_{EFGH}\mathrm{Tr}([X^E, X^F, X^G], X^H) - \frac{1}{6}\mu\epsilon_{E'F'G'H'}\mathrm{Tr}([X^{E'}, X^{F'}, X^{G'}], X^{H'}). \tag{A.9}$$

Here $E, F, G, H = 3, 4, 5, 6$ and $E', F', G', H' = 7, 8, 9, 10$. This action is invariant under the original gauge transformation and the deformed SUSY transformation [1]

$$\begin{aligned} \delta X^I &= i\bar{\epsilon}\Gamma^I \Psi, \\ \delta\Psi &= (D_\mu X^I)\Gamma^\mu \Gamma_I \epsilon - \frac{1}{6}[X^I, X^J, X^K]\Gamma_{IJK}\epsilon - \mu\Gamma_{3456}\Gamma^I X^I \epsilon, \\ \delta\tilde{A}_{\mu}{}^{b}{}_{a} &= i\bar{\epsilon}\Gamma_\mu \Gamma_I X^I_c \Psi_d f^{cdb}{}_a. \end{aligned} \tag{A.10}$$

This deformed theory breaks the original $SO(8)$ R-symmetry down to $SO(4)\times SO(4)$. By setting $\mu \to 0$ both the action and SUSY transformation reduce to the original BLG action. In addition there is another supersymmetry transformation

$$\begin{aligned} &\delta X^I_a = 0, \qquad \delta\tilde{A}_{\mu}{}^{b}{}_{a} = 0, \\ &\delta\Psi = \exp\left(-\frac{\mu}{3}\Gamma_{3456}\Gamma_\mu x^\mu\right) T^{-1}\eta, \end{aligned} \tag{A.11}$$

where x^μ is the coordinates of the world volume. In the massless limit of $\mu \to 0$, this becomes a constant shift of the fermion $\delta\Psi = T^{-1}\eta$. These inhomogeneous supersymmetries correspond to the spontaneously broken supersymmetries in $d = 11$ by the presence of M2 branes. As in the case of D-brane effective theories, they will play an important role in the full $d = 11$ superalgebras with 32 supercharges.

[1] To give a rigorous proof of the closure of the supersymmetry, we should check the Jacobi identity of $[Q, \{Q, Q\}]$ (appendix E of [54]) because there are non-central terms, i.e. $SO(4)\times SO(4)$ rotation term, in the algebra $\{Q, Q\}$. We thank Dr. Hai Lin for informing us of the paper [54]

A.3 Mass deformed BLG to Janus

This model can be similarly investigated by expanding the fields into modes with internal indices $a = (-1, 0, i)$. The mode expansions of the mass and the flux terms become

$$\mathcal{L}_{mass} = \mu^2 X^I_{-1} X^I_0 - \frac{\mu^2}{2}\mathrm{Tr}(\hat{X}^I, \hat{X}^I) - i\mu\bar{\Psi}_{-1}\Gamma_{3456}\Psi_0 + \frac{i}{2}\mu\mathrm{Tr}(\bar{\hat{\Psi}}\Gamma_{3456}, \hat{\Psi}), \tag{A.12}$$

and

$$\mathcal{L}_{flux} = \frac{2i}{3}\mu\epsilon_{EFGH}X^E_0\mathrm{Tr}(\hat{X}^F, [\hat{X}^G, \hat{X}^H]) + \frac{2i}{3}\mu\epsilon_{E'F'G'H'}X^{E'}_0\mathrm{Tr}(\hat{X}^{F'}, [\hat{X}^{G'}, \hat{X}^{H'}]). \tag{A.13}$$

Now X^I_{-1} and Ψ_{-1} again appear linearly in the action, and they are Lagrange multipliers. Because of the mass terms, the constraint equations are modified to

$$(\partial^2 - \mu^2)X^I_0 = 0, \qquad (\Gamma^\mu\partial_\mu + \mu\Gamma_{3456})\Psi_0 = 0. \tag{A.14}$$

Namely the fields with the T^0 component are constrained to obey the massive wave equations. Since X^I are real fields, instead of the plane waves $\exp(ik_\mu x^\mu)$ with a time-like vector k_μ, we take the following solution to the constraint equation;

$$X^I_0 = fe^{p_\mu x^\mu}\delta^I_{10} = v(x)\delta^I_{10}, \qquad \Psi_0 = 0, \tag{A.15}$$

where f is an arbitrary constant and p_μ is a spacelike vector satisfying $p^2 = \mu^2$. Without loss of generality, we can take $p_\mu = (0, \mu, 0)$. This configuration preserves half of the 16 supersymmetries, since Ψ_0 transforms as:

$$\delta\Psi_0 = v(x)\mu(\Gamma^1 - \Gamma_{3456})\Gamma^{10}\epsilon. \tag{A.16}$$

Hence around the above configuration, we will get Janus gauge field theories with 8 supersymmetries. (For general solutions, more supersymmetries are broken.)

Inserting this configuration to the action, one can again integrate the redundant gauge field B_μ. Terms involving B_μ are given by:

$$\mathrm{Tr}\left[-\frac{1}{2}(\hat{D}_\mu\hat{X}^{10} - vB_\mu)^2 + \frac{1}{2}\epsilon^{\mu\nu\lambda}\hat{F}_{\mu\nu}B_\lambda - p^\mu vB_\mu\hat{X}^{10}\right]. \tag{A.17}$$

Integrating B_μ gives

$$\begin{aligned}&\mathrm{Tr}\left[\frac{1}{2v}\epsilon^{\mu\nu\lambda}\hat{F}_{\mu\nu}p_\lambda\hat{X}^{10} + \frac{1}{8v^2}(\epsilon^{\mu\nu\lambda}\hat{F}_{\mu\nu} - 2v\hat{X}^{10}p^\lambda)^2\right]\\ &= -\frac{1}{4v^2}\mathrm{Tr}\hat{F}^2_{\mu\nu} + \frac{\mu^2}{2}\mathrm{Tr}(\hat{X}^{10}, \hat{X}^{10}).\end{aligned} \tag{A.18}$$

Interestingly the second term is canceled by the mass term of $\hat{X}^{10}$ and all the terms involving $\hat{X}^{10}$ have disappeared. To summarize, the resultant effective Lagrangian is given by:

$$\begin{aligned}\mathcal{L} \;=\; &-\frac{1}{2}\mathrm{Tr}(\hat{D}_\mu\hat{X}^A)^2 - \frac{\mu^2}{2}\mathrm{Tr}(\hat{X}^A, \hat{X}^A) + \frac{1}{4}v^2[\hat{X}^A, \hat{X}^B]^2\\ &+\frac{i}{2}\mathrm{Tr}\left(\bar{\hat{\Psi}}\Gamma^\mu\hat{D}_\mu\hat{\Psi}\right) + \frac{i}{2}\mu\mathrm{Tr}(\bar{\hat{\Psi}}\Gamma_{3456}, \hat{\Psi}) + \frac{1}{2}v\mathrm{Tr}\left(\bar{\hat{\Psi}}[\hat{X}^A, \Gamma_{10,A}\hat{\Psi}]\right) - \frac{1}{4v^2}\mathrm{Tr}\hat{F}^2_{\mu\nu}\\ &-\frac{2i}{3}v\mu\epsilon^{A'B'C'10}\mathrm{Tr}(\hat{X}^{A'}, [\hat{X}^{B'}, \hat{X}^{C'}]).\end{aligned} \tag{A.19}$$

This is a Janus field theory whose coupling constant is given by $v = f\ \exp(\mu x^1)$. The Lagrangian is invariant under the following 8 supersymmetries

$$
\begin{aligned}
\delta\hat{X}^A &= i\bar{\epsilon}\Gamma^A\hat{\Psi}, \\
\delta\hat{\Psi} &= \hat{D}_\mu\hat{X}^A\Gamma^\mu\Gamma^A\epsilon - \frac{1}{2v}\epsilon_{\mu\nu\lambda}\hat{F}^{\nu\lambda}\Gamma^\mu\Gamma^{10}\epsilon + \frac{i}{2}v[\hat{X}^A,\hat{X}^B]\Gamma^{AB}\Gamma^{10}\epsilon - \mu\Gamma_{3456}\Gamma^A\hat{X}^A\epsilon, \\
\delta\hat{A}_\mu &= iv\bar{\epsilon}\Gamma_\mu\Gamma^{10}\hat{\Psi},
\end{aligned}
\tag{A.20}
$$

Finally if v vanishes, i.e. for $X_0^I = 0$ and $\Psi_0 = 0$, the Lagrangian becomes

$$
\mathcal{L} = -\frac{1}{2}\mathrm{Tr}(\hat{D}_\mu\hat{X}^I)^2 + \frac{i}{2}\mathrm{Tr}\left(\bar{\hat{\Psi}}\Gamma^\mu\hat{D}_\mu\hat{\Psi}\right) - \frac{\mu^2}{2}\mathrm{Tr}(\hat{X}^I,\hat{X}^I) + \frac{i}{2}\mu\mathrm{Tr}(\bar{\hat{\Psi}}\Gamma_{3456},\hat{\Psi}), \tag{A.21}
$$

with a constraint $\hat{F}_{\mu\nu} = 0$. The supersymmetry transformation is given by

$$
\begin{aligned}
\delta\hat{X}^I &= i\bar{\epsilon}\Gamma^I\hat{\Psi}, \\
\delta\hat{\Psi} &= \hat{D}_\mu\hat{X}^I\Gamma^\mu\Gamma^I\epsilon - \mu\Gamma_{3456}\Gamma^I\hat{X}^I\epsilon, \\
\delta\hat{A}_\mu &= 0
\end{aligned}
\tag{A.22}
$$

and the Lagrangian has the $SO(4)\times SO(4)$ R-symmetry.

Appendix B

Conformal Symmetry of ABJM and L-BLG

B.1 Conformal invariance of ABJM

As shown in [55], the ABJM theory is invariant under the superconformal transformations. Here we study the invariance of the ABJM theory under the conformal transformations, in particular the special conformal transformations.

ABJM theory is a $U(N) \times U(N)$ or $SU(N) \times SU(N)$ gauge theory. The other choices of gauge groups are possible but here we consider these two types. The actions of the gauge fields are given by the Chern-Simons action with coefficients k and $-k$. Matter fields Y^A and ψ^A are in the bifundamental representation and the covariant derivative is defined by

$$D_\mu Y = \partial_\mu Y + iA_\mu^{(L)} Y - iY A_\mu^{(R)}. \tag{B.1}$$

The action is invariant under $\mathcal{N} = 6$ superconformal transformations. In the following we check the explicit invariance under the conformal transformations.

First it is obvious that the action is invariant under the dilatation. Dilatation is defined by $x \to e^\epsilon x$ and simultaneously we transform each field by multiplying $e^{-n\epsilon}$ where n is the conformal weight. The scalars Y^A, fermions ψ^A and the gauge fields A_μ have weights $1/2, 1, 1$ respectively.

A little more nontrivial transformation is a special conformal transformation. It is given by

$$\delta x^\mu = 2\epsilon \cdot x x^\mu - \epsilon^\mu x^2. \tag{B.2}$$

If we write the infinitesimal transformation for each field $Y(x)$ as $\delta Y(x) = Y'(x') - Y(x)$, they are given by

$$\begin{aligned} \delta Y^A(x) &= -\epsilon \cdot x Y^A(x), \\ \delta A_\mu^{(L,R)}(x) &= -2\epsilon \cdot x A_\mu^{(L,R)}(x) - 2(x \cdot A^{(L,R)} \epsilon_\mu - \epsilon \cdot A^{(L,R)} x_\mu), \\ \delta\psi^A(x) &= -2\epsilon \cdot x \psi^A(x) - \epsilon_{\mu\nu\lambda}\epsilon^\nu x^\lambda \Gamma^\mu \psi^A(x). \end{aligned} \tag{B.3}$$

These transformations can be understood as follows. They look like the general coordinate transformations, but are different since the theory is restricted to live in the flat space-time with a fixed metric and the change of the metric under the general coordinate transformations must be compensated by the transformations of the fields. The first terms in each transformation reflect the conformal weight of each field. The second term in the transformation of the fermion is the local Lorentz transformation which pulls back the flat local Lorentz frame (where we use $\Gamma^{012}\psi = \psi$). The transformation for the gauge field A_μ is nothing but the general coordinate transformation with the transformation parameter (B.2).

The action is invariant under the above special conformal transformations. In order to see it, the following transformation rules are useful:

$$\begin{aligned} d^3x &\to e^{6\epsilon\cdot x} d^3x, \\ \partial_\mu &\to e^{-2\epsilon\cdot x}[\partial_\mu - 2(\epsilon_\mu x^\nu \partial_\nu - x_\mu \epsilon^\nu \partial_\nu)], \\ D_\mu Y &\to e^{-3\epsilon\cdot x}\Big[D_\mu Y - \{Y + 2x^\nu\partial_\nu Y + 2i(x\cdot A^{(L)}Y - Yx\cdot A^{(R)})\}\epsilon_\mu \\ &\qquad + \{2\epsilon^\nu\partial_\nu Y + 2i(\epsilon\cdot A^{(L)}Y - Y\epsilon\cdot A^{(R)})\}x_\mu\Big], \\ F_{\mu\nu} &\to e^{-4\epsilon\cdot x}\left[F_{\mu\nu} - 2(\epsilon_\nu x^\rho F_{\mu\rho} - \epsilon_\mu x^\rho F_{\nu\rho}) + 2(x_\nu\epsilon^\rho F_{\mu\rho} - x_\mu \epsilon^\rho F_{\nu\rho})\right]. \end{aligned} \tag{B.4}$$

Though ϵ is an infinitesimal parameter, we write the overall factors as $e^{-2n\epsilon\cdot x}$ for convenience. They are cancelled in the action because n is the conformal weight of each field and coordinates.

Here let us check the invariance of the Chern-Simons term as an example. First the derivative part transforms as

$$\begin{aligned} &\epsilon^{\mu\nu\lambda}\mathrm{tr} F_{\mu\nu}A_\sigma \\ &\to \epsilon^{\mu\nu\lambda}e^{-6\epsilon\cdot x}\mathrm{tr}[F_{\mu\nu}A_\lambda + 4(\epsilon_\mu x^\rho - x_\mu\epsilon^\rho)A_\lambda F_{\nu\rho} - 2F_{\mu\nu}(x\cdot A\epsilon_\lambda - \epsilon\cdot A x_\lambda)]. \end{aligned} \tag{B.5}$$

The pre-factor $e^{-6\epsilon\cdot x}$ is cancelled with the transformation of d^3x in (B.4). The rest vanishes because

$$\begin{aligned} &\epsilon^{\mu\nu\lambda}\mathrm{tr}[2(\epsilon_\mu x^\rho - x_\mu\epsilon^\rho)A_\lambda F_{\nu\rho} - F_{\mu\nu}(x\cdot A\epsilon_\lambda - \epsilon\cdot A x_\lambda)] \\ &= \epsilon^{\mu\nu\lambda}\mathrm{tr}[2\epsilon_\mu{}^{\rho\alpha} f_\alpha F_{\nu\rho}A_\lambda - \epsilon_\lambda{}^{\rho\alpha} f_\alpha F_{\mu\nu}A_\rho] = 0. \end{aligned} \tag{B.6}$$

In the second line we have defined $f^\alpha = \epsilon^{\mu\nu\alpha}x_\mu\epsilon_\nu$. Similarly the invariance of the term $\epsilon^{\mu\nu\lambda}A_\mu A_\nu A_\lambda$ can be shown by noting that the gauge field transforms as

$$A_\mu \to e^{-2\epsilon\cdot x}(A_\mu + 2\epsilon_{\mu\alpha\beta}f^\alpha A^\beta). \tag{B.7}$$

Hence the Chern-Simons terms are invariant under the special conformal transformation. Though we have checked it explicitly, the invariance can be naturally understood because the Chern-Simons term is independent of the metric if it is defined in a curved background space-time.

The other terms in the action are also straightforwardly shown to be invariant under the special conformal transformations.

B.2 ABJM to L-BLG

As shown in [13], the L-BLG theory is obtained by taking a scaling limit of the ABJM theory with a gauge group $SU(N)\times SU(N)$. In the gauge theory with $U(N)\times U(N)$ there is a subtlety in the scaling of the $U(1)$ part. We will discuss the issue in the Appendix D and here restrict the discussions to the $SU(N)\times SU(N)$ case.

The scaling is given as follows:

$$\begin{aligned} B_\mu &\to \lambda B_\mu, \\ X_0^I &\to \lambda^{-1} X_0^I, \\ \psi_{A0} &\to \lambda^{-1}\psi_{A0}, \\ k &\to \lambda^{-1} k \end{aligned} \tag{B.8}$$

where

$$Y^A = X_0^{2A-1} + iX_0^{2A} - \hat{X}^{2A} + i\hat{X}^{2A-1}, \qquad B_\mu = \frac{1}{2}(A_\mu^{(L)} - A_\mu^{(R)}) \tag{B.9}$$

and X_0^I and ψ_{0A} are trace components of the bifundamental matter fields, and $I = 1,\cdots,8$. When we take $\lambda \to 0$ limit and keep the other fields fixed, the action of the ABJM theory is reduced to the action of the L-BLG theory. Since the $k\to\infty$ limit is taken before taking the large N, our scaling corresponds to a vanishing 't Hooft coupling $N/k \to 0$. Besides the action, the same constraint equations as those in the L-BLG theory can be obtained from the ABJM theory:

$$\partial^2 X_0^I = 0, \quad \Gamma^\mu \partial_\mu \Psi_0 = 0, \tag{B.10}$$

by requiring *finiteness of the action* in the $\lambda \to 0$ limit.

In the above scaling limit we arrive at the L-BLG theory:

$$\begin{aligned} \mathcal{L}_0 = \mathrm{Tr}\Big[&-\frac{1}{2}(\hat{D}_\mu \hat{X}^I - B_\mu X_0^I)^2 + \frac{1}{4}(X_0^K)^2([\hat{X}^I,\hat{X}^J])^2 - \frac{1}{2}(X_0^I[\hat{X}^I,\hat{X}^J])^2 \\ &+\frac{i}{2}\bar{\hat{\Psi}}\Gamma^\mu \hat{D}_\mu \hat{\Psi} + i\bar{\Psi}_0\Gamma^\mu B_\mu \hat{\Psi} - \frac{1}{2}\bar{\Psi}_0 \hat{X}^I[\hat{X}^J,\Gamma_{IJ}\hat{\Psi}] + \frac{1}{2}\bar{\hat{\Psi}} X_0^I[\hat{X}^J,\Gamma_{IJ}\hat{\Psi}] \\ &+\frac{1}{2}\epsilon^{\mu\nu\lambda}\hat{F}_{\mu\nu}B_\lambda - \partial_\mu X_0^I\ B_\mu \hat{X}^I \Big]. \end{aligned} \tag{B.11}$$

In the original formulation of the L-BLG theory, the constraint equations (B.10) are derived by integrating the auxiliary fields X_{-1}^I and Ψ_{-1}:

$$\mathcal{L}_{gh} = (\partial_\mu X_0^I)(\partial^\mu X_{-1}^I) - i\bar{\Psi}_{-1}\Gamma^\mu \partial_\mu \Psi_0. \tag{B.12}$$

Since the above scaling is compatible with the conformal transformations discussed in the previous section, the action (B.11) is invariant under the conformal transformations (see also [56]). The action for the auxiliary fields (B.12) is also invariant if we define the transformations for them as

$$\begin{aligned} \delta X_{-1}^I(x) &= -\epsilon\cdot x X_{-1}^I(x), \\ \delta\Psi_{-1}(x) &= -2\epsilon\cdot x\Psi_{-1}(x) - \epsilon_{\mu\nu\lambda}\epsilon^\nu x^\lambda \Gamma^\mu \Psi_{-1}(x). \end{aligned} \tag{B.13}$$

B.3 Generalized conformal symmetry in D2 branes

Now integrate the B_μ gauge field. It has been discussed that if we pick up a specific solution to the constraint equation (B.10), especially a constant solution

$$X_0^I = v\ \delta^{I,8}, \quad \Psi_0 = 0, \tag{B.14}$$

the L-BLG theory is reduced to the action of the ordinary D2 branes whose Yang-Mills coupling constant is given by $g_{YM} = v$:

$$\mathcal{L} = \mathrm{Tr}\left[-\frac{1}{4v^2}\hat{F}_{\mu\nu}^2 - \frac{1}{2}(\hat{D}_\mu \hat{X}^A)^2 + \frac{1}{4}v^2[\hat{X}^A, \hat{X}^B]^2 + \frac{i}{2}\bar{\hat{\Psi}}\Gamma^\mu \hat{D}_\mu \hat{\Psi} + \frac{1}{2}v\bar{\hat{\Psi}}[\hat{X}^A, \Gamma_{8,A}\hat{\Psi}]\right] \tag{B.15}$$

where $A, B = 1, \cdots, 7$. Then $SO(8)$ is spontaneously broken to $SO(7)$ because we have specialized the 8-th direction. The conformal invariance is also broken. Though the action is the same as that of the D2 branes, we see later that the interpretation of the L-BLG theory as an effective theory of the ordinary D2 branes is not appropriate since the radius of curvature is much smaller than the string scale in the gravity dual.

The constraint equations (B.10) have more general solutions than (B.14) which depend on the spacetime coordinates. Then the resulting action becomes a Yang-Mills theory with a spacetime dependent coupling [15]. As we have shown [13], the action with the spacetime dependent coupling is invariant under the conformal transformations if we consider a set of spacetime dependent solutions. The conformal invariance is discussed in more details in the next section.

We here consider the simplest spacetime dependent solutions:

$$X_0^I = v(x)\,\delta^{I,8}, \quad \Psi_0 = 0, \quad \partial^2 v(x) = 0. \tag{B.16}$$

Then the L-BLG theory is reduced to the same action as that of the D2 branes but with a spacetime varying coupling:

$$\begin{aligned}\mathcal{L} = \mathrm{Tr}\Bigg[&-\frac{1}{4v(x)^2}\hat{F}_{\mu\nu}^2 - \frac{1}{2}(\hat{D}_\mu \hat{X}^A)^2 + \frac{1}{4}v(x)^2[\hat{X}^A, \hat{X}^B]^2 \\ &+\frac{i}{2}\bar{\hat{\Psi}}\Gamma^\mu \hat{D}_\mu \hat{\Psi} + \frac{1}{2}v(x)\bar{\hat{\Psi}}[\hat{X}^A, \Gamma_{8,A}\hat{\Psi}]\Bigg].\end{aligned} \tag{B.17}$$

$SO(8)$ symmetry is spontaneously broken to $SO(7)$ as well, but the action with a varying $v(x)$ has a generalized conformal symmetry if the coupling transforms as

$$\delta v(x) = -(\epsilon \cdot x)\ v(x). \tag{B.18}$$

This transformation is originated in the special conformal transformation of the scalar field (B.3). The generalized conformal transformation for Dp branes were first proposed by Jevicki, Kazama and Yoneya [51]. In the present case, the transformation (B.18) is naturally derived since the coupling constant of the Yang-Mills action is determined by the center of mass coordinates $X_0^I(x)$ of the M2 branes.

It is worth noting that the generalized conformal transformation (B.18) is compatible with the constraint equations (B.10) only when $p = 2$. We will discuss it in the next section.

B.4 Conformal symmetry and $SO(8)$ invariance of L-BLG

The space-time dependent coupling $v(x)$ can be promoted to an $SO(8)$ vector $X_0^I(x)$ by considering general solutions to the constraint equations (B.10) as shown in [15]. Then the resultant action after integrating the B_μ gauge field becomes D2 branes effective action with space-time dependent couplings in a vector representation of the $SO(8)$. In [13] we showed that if we consider space-time dependent solutions the theory has the *generalized conformal symmetry* as well as the manifest $SO(8)$ invariance.

In this section we study more details of the generalized conformal symmetry of the L-BLG theory. Especially we show that the conformal transformations are closed under the constraint equations (B.10).

By integrating the B_μ gauge field, we get the action $S = \int d^3x(\mathcal{L}_0 + \mathcal{L}')$:

$$\begin{aligned}\mathcal{L}_0 = \mathrm{Tr}\Big[&-\frac{1}{2}(\hat{D}_\mu P^I)^2 + \frac{1}{4}X_0^2[P^I,P^J]^2 + \frac{i}{2}\bar{\hat{\Psi}}\Gamma^\mu \hat{D}_\mu \hat{\Psi} + \frac{1}{2}\bar{\hat{\Psi}}[P^I,(X_0^J\Gamma_J)\Gamma_I\hat{\Psi}] \\ &+\frac{1}{2(X_0^I)^2}\big(\frac{1}{2}\epsilon^{\mu\nu\lambda}\hat{F}_{\nu\lambda} + i\bar{\Psi}_0\Gamma^\mu\hat{\Psi} - 2P_I\partial^\mu X_0^I\big)^2 - \frac{1}{2}\bar{\Psi}_0\Gamma_{IJ}\hat{\Psi}[P^I,P^J]\Big],\\ \mathcal{L}' = \frac{1}{X_0^2}&\mathrm{Tr}\Big[\Big(-\bar{\Psi}_0\Gamma_I(X_0^J\Gamma_J)[P^I,\hat{\Psi}] - i\bar{\Psi}_0\Gamma_\mu\hat{D}_\mu\hat{\Psi}\Big)(X_0^K\hat{X}^K)\Big].\end{aligned} \tag{B.19}$$

where we have defined a new scalar field P_I with 7 degrees of freedom by using the projection operator

$$P_I(x) = \left(\delta_{IJ} - \frac{X_{0I}X_{0J}}{X_0^2}\right)X^J. \tag{B.20}$$

The $X_0^I(x)$ field is constrained to satisfy $\partial^2 X_0^I = 0$. This is a generalization of (B.17). We called this theory a Janus field theory of (M)2-branes since the coupling constant is varying with the space-time coordinates.

The action of the gauge field is no longer the Chern-Simons action but we can again show that it is invariant under the conformal transformations. Under the dilatation $x^\mu \to e^\epsilon x^\mu$, each field is multiplied by $e^{-n\epsilon}$ where n is the conformal weight. The weights of $P, X_0, A_\mu, \Psi, \Psi_0$ are $1/2, 1/2, 1, 1, 1$ respectively. The action is evidently invariant.

Special conformal transformation is similarly given by

$$\delta x^\mu = 2\epsilon\cdot x x^\mu - \epsilon^\mu x^2 \tag{B.21}$$

and the fields transform as

$$\begin{aligned}\delta P^I(x) &= -\epsilon\cdot x P^I(x),\\ \delta X_0^I(x) &= -\epsilon\cdot x X_0^I(x),\\ \delta A_\mu(x) &= -2\epsilon\cdot x A_\mu(x) - 2(x\cdot A\ \epsilon_\mu - \epsilon\cdot A\ x_\mu),\\ \delta\hat{\Psi}(x) &= -2\epsilon\cdot x\hat{\Psi}(x) - \epsilon_{\mu\nu\lambda}\epsilon^\nu x^\lambda\Gamma^\mu\hat{\Psi}(x),\\ \delta\Psi_0(x) &= -2\epsilon\cdot x\Psi_0(x) - \epsilon_{\mu\nu\lambda}\epsilon^\nu x^\lambda\Gamma^\mu\Psi_0(x).\end{aligned} \tag{B.22}$$

It is now straightforward to show the invariance of the action. The Lagrangian is not invariant but changes by total derivatives.

Finally we need to check that the transformation is closed within the constraint equations (B.10). Namely if the field $X_0^I(x)$ satisfies $\partial_x^2 X_0^I(x) = 0$, the transformed field $X_0'^I(x')$ must also satisfy $\partial_{x'}^2 X_0'^I(x') = 0$. For an infinitesimal special conformal transformation, this is equivalent to show $\partial^2 \tilde{\delta} X_0^I(x) = 0$ where $\tilde{\delta} X_0^I(x)$ is the transformation at the numerically same point defined by

$$\begin{aligned} \tilde{\delta} X_0^I(x) &= X_0'^I(x) - X_0^I(x) = \delta X_0^I(x) - \delta x^\mu \partial_\mu X_0^I(x), \\ \tilde{\delta} \Psi_0(x) &= \Psi_0'(x) - \Psi_0(x) = \delta \Psi_0(x) - \delta x^\mu \partial_\mu \Psi_0(x). \end{aligned} \tag{B.23}$$

In the following, in order to see the specialty for M2 (or D2)-branes, we generalize the special conformal transformation to Dp-branes [51]:

$$\tilde{\delta} X_0^I(x) = -(3-p)\epsilon \cdot x X_0^I - (2\epsilon \cdot x x^\mu - \epsilon x^2)\partial_\mu X_0^I \tag{B.24}$$

It is easy to show

$$\partial^2(\tilde{\delta} X_0^I(x)) = 2(p-2)\epsilon^\mu \partial_\mu X_0^I \tag{B.25}$$

where we have used the constraint equation $\partial^2 X_0^I = 0$. This vanishes at $p = 2$ only. Similarly, $\tilde{\delta}\Psi_0$ is given by

$$\tilde{\delta}\Psi_0(x) = -2(3-p)\epsilon \cdot x \Psi_0 - \epsilon_{\mu\nu\lambda}\epsilon^\nu x^\lambda \Gamma^\mu \Psi_0 - (2\epsilon \cdot x x^\mu - \epsilon x^2)\partial_\mu \Psi_0 \tag{B.26}$$

and satisfies

$$\Gamma^\alpha \partial_\alpha(\tilde{\delta}\Psi_0(x)) = 2(p-2)\Gamma^\alpha \epsilon_\alpha \Psi_0 \tag{B.27}$$

where we used the constraint equation $\Gamma^\alpha \partial_\alpha \Psi_0 = 0$. Again $\Gamma^\alpha \partial_\alpha(\tilde{\delta}\Psi_0(x)) = 0$ vanishes at $p = 2$ only. Both of the constraints are compatible with the generalized conformal transformations at $p = 2$. It shows a specialty of M2 (or D2) branes.

We have shown that the constraint equations are compatible with the generalized conformal transformations. If the solutions are restricted to constant ones as in (B.14), we no longer have the generalized conformal symmetry. It can be maintained only when we consider a set of space-time dependent solutions to the constraint equations.

Recently H. Verlinde [57] also considered space-time dependent solutions to the constraint equations and discussed the conformal symmetry of the L-BLG theory. In his study the constraint equation is imposed everywhere except at z_i where a local operator $\mathcal{O}_i(z_i)$ is inserted,

$$X_0^I(x) = \sum \frac{q_i^I}{|x - z_i|}. \tag{B.28}$$

This is an inhomogeneous solution to the equation

$$\partial^2 X_0^I = -4\pi \sum q_i^I \delta^3(x - z_i). \tag{B.29}$$

We can add the homogeneous solutions to the above. If q^I and z (omitting the index i) transform as

$$\begin{aligned} \delta q^I &= \epsilon \cdot z q^I \\ \delta z_\mu &= 2(\epsilon \cdot z) z_\mu - \epsilon_\mu z^2, \end{aligned} \tag{B.30}$$

the transformation of X_0^I

$$\delta X_0^I(x) = -(\epsilon \cdot x) X_0^I(x) \tag{B.31}$$

is reproduced and the L-BLG action is invariant under the conformal transformations. Note that q^I cannot be a constant. If q^I is kept fixed, the set of solutions is not closed under the conformal transformations. In order to recover the conformal invariance, q^I should be a position z-dependent charge.

We have shown that the L-BLG theory has both of the $SO(8)$ invariance and the conformal symmetry. In the next section we discuss the symmetry properties of the gravity dual of the ABJM theory.

Appendix C

$SO(8)$ and Conformal Symmetry in Dual Geometry

C.1 Large k limit of ABJM geometry

In the paper [12], it was pointed out that the $U(N)\times U(N)$ ABJM theory is dual to the M-theory on $AdS_4 \times S^7/\mathbf{Z}_k$, which is a $d = 11$ supergravity solution of M2 branes probing the orbifold $\mathbf{C}^4/\mathbf{Z}_k$. We first review the solution of supersymmetric M2 branes in $d = 11$ supergravity.

The $d = 11$ metric of the multiple M2-branes is given by

$$\begin{aligned} ds^2 &= H^{-\frac{2}{3}} \left(\sum_{\mu,\nu=0}^{2} \eta_{\mu\nu} dx^\mu dx^\nu \right) + H^{\frac{1}{3}} \left(dr^2 + r^2 d\Omega_7^2 \right), \\ H(r) &\equiv 1 + \frac{R^6}{r^6}, \end{aligned} \tag{C.1}$$

where $R^6 = 32\pi^2 N' l_p^6$ and $d\Omega_7^2$ is the metric of a unit 7-sphere. N' is the number of the M2 branes and identified with $N' = kN$. The three form field is also given as

$$C^{(3)} = H^{-1} dx^0 \wedge dx^1 \wedge dx^2 \tag{C.2}$$

and the 4-form flux normalized by the world volume is proportional to N'.

By focusing on the near horizon region of the M2-brane, the geometry becomes $AdS_4 \times S^7$ geometry. In the near horizon limit $R \gg r$, $H(r)$ is replaced by $H(r) = (R/r)^6$ and the metric becomes

$$\begin{aligned} ds^2 &= \left(\frac{r}{R}\right)^4 \left(\sum_{\mu,\nu=0}^{2} \eta_{\mu\nu} dx^\mu dx^\nu \right) + \left(\frac{R}{r}\right)^2 dr^2 + R^2 d\Omega_7^2 \\ &= R^2 \left[\frac{1}{4} ds^2_{AdS} + d\Omega_7^2 \right] \end{aligned} \tag{C.3}$$

where we have rescaled the M2 brane world volume coordinates by a factor $2/R^3$. Hence the near horizon geometry of the supersymmetric M2 branes is given by $AdS_4 \times S^7$ with a radius

R. In the large $N' = kN$ limit, the radius becomes much larger than the $d = 11$ Planck length and the $d = 11$ supergravity approximation is valid.

The ABJM theory describes M2 branes on $\mathbf{C}^4/\mathbf{Z}_k$ orbifold. The dual geometry can be obtained by first specifying the polarization (choice of the complex coordinates) in $\mathbf{R}^8$ and then dividing $\mathbf{C}^4$ by $\mathbf{Z}_k$.

Since S^7, parameterized by z^A $(A = 1, \cdots, 4)$ with $|z^A|^2 = 1$, is a $U(1)$-fibration on $\mathbf{CP}^3$, the metric of S^7 is written as

$$d\Omega_7^2 = \left(d\varphi' + \omega\right)^2 + ds^2_{\mathbf{CP}^3} \tag{C.4}$$

where φ' is the overall phase of z^A. The details of the definition of coordinates are written in Appendix E.

We now perform the $\mathbf{Z}_k$ quotient by dividing the overall phase of each z^A, namely the φ' direction. By rewriting $\varphi' = \varphi/k$ with $\varphi \sim \varphi + 2\pi$, the metric of $S^7/\mathbf{Z}_k$ becomes

$$ds^2_{S^7/\mathbf{Z}_k} = \frac{1}{k^2}\left(d\varphi + k\omega\right)^2 + ds^2_{\mathbf{CP}^3}. \tag{C.5}$$

Before performing the $\mathbf{Z}_k$ quotient, the metric has the conformal symmetry associated with the AdS_4 geometry and $SO(8)$ symmetry of S^7. The orbifolding breaks the $SO(8)$ symmetry to $SU(4) \times U(1)$ but the conformal invariance still exists. This is the bosonic symmetry of the ABJM theory.

The L-BLG action can be derived by taking the scaling limit (B.8) of the ABJM theory. In the gravity side, this scaling corresponds to locating the probe M2 branes far from the orbifold singularity and taking the large k limit. As we show in the next section, the former process recovers the $SO(8)$ if the positions of the M2 branes are considered to be dynamical variables. The latter makes the radius of the φ' circle small and $d = 11$ geometry is reduced to $d = 10$.

Now we consider the $k \to \infty$ limit of the dual geometry of the ABJM theory. Following the prescription of ABJM, we shall interprete the coordinate φ as the compact direction in reducing from M-theory to type IIA superstring. Using the reduction formula [58]

$$ds^2_{11} = e^{-\frac{2}{3}\phi} ds^2_{10} + e^{\frac{4}{3}\phi}(l_p)^2\left(d\varphi + A\right)^2 \tag{C.6}$$

we get the $d = 10$ metric and the dilaton field in type IIA supergravity as

$$ds^2_{10} = \frac{r}{kl_p} H^{-\frac{1}{2}} \left(\sum_{\mu,\nu=0}^{2} \eta_{\mu\nu} dx^\mu dx^\nu\right) + \frac{r}{kl_p} H^{\frac{1}{2}} \left(dr^2 + r^2 ds^2_{\mathbf{CP}^3}\right), \tag{C.7}$$

$$e^{2\phi} = \left(\frac{r}{kl_p}\right)^3 H^{\frac{1}{2}} = \left(\frac{R}{kl_p}\right)^3. \tag{C.8}$$

Hence in the $k \to \infty$ limit, the metric becomes $AdS_4 \times \mathbf{CP}^3$:

$$ds^2_{10} = \frac{R^3}{k}\left[\frac{1}{4} ds^2_{AdS_4} + ds^2_{\mathbf{CP}^3}\right] \tag{C.9}$$

where the radius of curvature in string units is

$$R_{\text{str}}^2 = \left(\frac{R}{l_s}\right)^2 = \frac{R^3}{kl_p^3} = 2^{5/2}\pi\sqrt{\frac{N}{k}}. \tag{C.10}$$

The dilaton is a constant and this is the reason why the $d = 10$ metric still has a conformal symmetry associated with the AdS_4 geometry. This is different from the ordinary reduction of the M2 branes to D2 branes by compactifying the 11th direction of the Cartesian coordinate (see Appendix F). Note that in the type IIA picture, in addition to the four-form RR flux F_4, there is a 2-form RR flux:

$$\begin{aligned} F_4 &= \frac{3}{8}\frac{R^3}{l_p^3}\hat{\epsilon}_4, \\ F_2 &= dA = kd\omega \end{aligned} \tag{C.11}$$

where $\hat{\epsilon}_4$ is the volume form in a unit radius AdS_4 space. Hence the geometry is described by the $AdS_4 \times \mathbf{CP}^3$ compactification with N units of the four form flux on AdS_4 and k units of the two-form flux on the $\mathbf{CP}^1$ in $\mathbf{CP}^3$ space.

In the $k \to \infty$ limit with N/k fixed, the compactification radius along the φ-direction R_{11} becomes very small compared to the $d = 11$ Planck length:

$$\frac{R_{11}}{l_p} = \frac{R}{kl_p} \sim \frac{(Nk)^{1/6}}{k} \to 0. \tag{C.12}$$

Thus the theory is reduced to a ten-dimensional type IIA superstring on $AdS_4 \times \mathbf{CP}^3$. However the scaling limit from ABJM to L-BLG is taking large k limit before taking the large N and the 't Hooft coupling N/k becomes 0 in this limit. Since $R_{11} = g_s^{2/3} l_p$, the string coupling constant $g_s = e^{\phi}$ also becomes 0:

$$g_s = e^{\phi} \sim k^{-\frac{5}{4}} N^{\frac{1}{4}} \to 0. \tag{C.13}$$

Since $d = 11$ Planck length l_p and $d = 10$ Planck length $l_p^{(10)}$ are related to the string length as $l_p = g_s^{1/3} l_s$ and $l_p^{(10)} = g_s^{1/4} l_s$, the ratios of the radius of the IIA geometry (C.9) with l_s and $l_p^{(10)}$ are given by

$$\left(\frac{R}{l_s}\right)^2 \sim \sqrt{\frac{N}{k}} \to 0, \qquad \left(\frac{R}{l_p^{(10)}}\right)^2 \sim k^{1/8} N^{3/8} \to \infty. \tag{C.14}$$

Therefore the Type IIA supergravity approximation itself is good but the α' expansion is not good and the theory cannot be considered as the low energy approximation of type IIA superstring. On the other hand, the radius R is much larger than the $d = 11$ Planck length and it may be more appropriately interpreted as a dimensional reduction of M2 branes in the $d = 11$ supergravity.

We summarize the various length scales in the scaling limit of the ABJM theory to the L-BLG theory:

$$R_{11} \ll l_p^{(11)} \ll l_p^{(10)} \ll R_{AdS} \ll l_s. \tag{C.15}$$

The compactification radius R_{11} is much smaller than any other scales and the theory is reduced to $d = 10$. But the radius of the $AdS_4 \times \mathbf{CP}^3$ is smaller than the string length and larger than the $d = 10$ and $d = 11$ Planck scales.

In the ordinary case of the duality between type IIB superstrings on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM in $d = 4$, the radius of curvature R is given by

$$\left(\frac{R}{l_s}\right)^4 \sim g_s N, \qquad \left(\frac{R}{l_p^{(10)}}\right)^4 \sim N. \tag{C.16}$$

Thus it is usually assumed that both of $g_s N$ and N are large so that the type IIB supergravity approximation and the α'-expansion are valid. Unless $g_s N$ is large, α' corrections cannot be neglected and the supergravity description itself is not valid. In the weak coupling limit, the dual field theory is usually considered to be more appropriate. In our case, we can consider the $d = 10$ supergravity as a dimensional reduction of $d = 11$ supergravity. However membranes wrapping the φ direction become very light strings in the unit of the radius of curvature R, and this may invalidate the supergravity approximation of the M-theory.

C.2 Recovery of $SO(8)$ in dual geometry of L-BLG

In taking the scaling limit $k(\gg N) \to \infty$ of the ABJM theory to the L-BLG theory, the eleven-dimensional geometry is reduced to the ten-dimensional $AdS_4 \times \mathbf{CP}^3$:

$$\begin{aligned} ds^2 &= H^{-\frac{2}{3}} \left(\sum \eta_{\mu\nu} dx^\mu dx^\nu\right) + H^{\frac{1}{3}} (dr^2 + r^2 ds^2_{\mathbf{CP}^3}) \\ H(r) &= \frac{R^6}{r^6}. \end{aligned} \tag{C.17}$$

In this section we discuss how the $SO(8)$ can be recovered in the scaling limit of the ABJM geometry to the L-BLG geometry. The L-BLG geometry is obtained by taking $k \to \infty$ limit of $AdS_4 \times S^7/\mathbf{Z}_k$ and simultaneously locating the probe M2 brane far from the origin of the orbifold. In the large k limit, the geometry becomes $d = 10$ $AdS_4 \times \mathbf{CP}^3$, and there are only 7 transverse directions to the M2 brane world volume, However the radial distance in (C.17) is given by the distance in $d = 8$:

$$r^2 = \sum_{I=1}^{8} (X^I)^2. \tag{C.18}$$

It is invariant under the original $SO(8)$ rotation and the $\mathbf{Z}_k$ quotient leaves r invariant.

Now we consider a probe M2 brane in the above geometry. In the static gauge, the M2 brane world volume is identified with the coordinates x^μ ($\mu = 0, 1, 2$) and the position of the M2 brane is given by $X^I(x)$ where $I = 1, \cdots, 8$. There are only 7 independent propagating modes among 8, and the direction that is removed is the φ-direction. Remember that the φ is the overall phase of the complex coordinate z^i of the transverse R^8. Assuming that the probe M2 brane is located far from the source branes, we can separate the probe M2 brane coordinates into the classical background fields $X_0^I(x)$ and the quantum fluctuations $\hat{X}^I(x)$. Since the M2 brane is on $\mathbf{C}^4/U(1)$, all the points on the gauge orbit generated by the φ-rotation are identified. Here the position of the M2 brane is represented by the coordinates of $\mathbf{R}^8$; a point on the gauge orbit is singled out by fixing the gauge (see Appendix E).

If the probe M2 brane is located at

$$X_0^I = v\delta^{I,8} \tag{C.19}$$

where v is much larger than the scale of the fluctuations, the rotation along the φ-direction is approximated by

$$\begin{aligned} \delta X^7 &= -\delta\varphi\ v, \\ \delta X^I &= 0 \quad , \quad I \neq 7. \end{aligned} \tag{C.20}$$

This shows that in the large v limit the φ direction can be identified with the 7th direction X^7 [1]. Since the $\mathbf{Z}_k$ orbifolding with large k corresponds to gauging away the φ-direction, the fluctuation along the 7th direction is killed and the field $\hat{X}^I$ can fluctuate only in the other 7 directions. This means that the $SO(7)$ rotation acts among the other 7 directions around the classical background of (C.19). If the classical background $X_0^I(x)$ takes different directions at different world volume points, the killed direction also changes locally on the world volume.

In order to get a manifest $SO(8)$ covariant formulation of this mechanism, it is convenient to separate the classical background field of the M2 brane and the fluctuations in the complex coordinates as

$$Z^A(x) = Z_0^A(x) + \hat{Z}^A(x). \tag{C.21}$$

If the fluctuations are much smaller than the classical background field, the φ rotation can be approximated as

$$\delta Z^A = i\delta\varphi Z_0^A. \tag{C.22}$$

If we write

$$\begin{aligned} Z_0^A &= X_0^{2A-1} + iX_0^{2A} \\ \hat{Z}^A &= i\hat{X}^{2A-1} - \hat{X}^{2A}, \end{aligned} \tag{C.23}$$

[1] (C.19) has fixed a gauge of the φ rotation and (C.20) is nothing but the direction parallel to the gauge orbit. If we change a gauge,e.g. to $X_0^I = v\delta^{I,7}$, (C.20) is also changed accordingly.

where $A = 1 \cdots 4$, the propagating degrees of freedom along the direction (C.22) are killed and the fluctuations are restricted to obey

$$X_0^I \hat{X}^I = 0. \tag{C.24}$$

Note that the decomposition of the complex fields into the real and the imaginary parts are different between the classical background Z_0^A and the fluctuations $\hat{Z}^A$ in (C.23). With this definition, if $X_0^I = v\delta^{I,8}$, the killed direction becomes the 8th direction of $\hat{X}^I$. We can write the fluctuations perpendicular to $\hat{X}^I$ in (C.24) as

$$P^I = \left(\delta^{IJ} - \frac{X_0^I X_0^J}{(X_0)^2} \right) \hat{X}_J. \tag{C.25}$$

This P^I automatically satisfies the condition (C.24) and 7 degrees of freedom are projected among the 8 degrees of freedom. Now everything is written in a manifestly $SO(8)$ covariant way. The $SO(8)$ covariance is recovered because we have assumed that the fluctuation is much smaller than the classical background fields of the probe M2 brane. This assumption is consistent with the scaling limit of the ABJM theory to the L-BLG theory.

Note here that the $SO(8)$ rotation changes the gauge choice of the φ rotation and $SO(8)$ is mixed with the $U(1)$ gauge transformation. Also note that because of the different assignments of X^I to Z^A for Z_0 and $\hat{Z}$, the $SO(8)$ is different from the original $SO(8)$ before taking the orbifolding.

The analysis here and in the previous section shows why the L-BLG theory has both of the conformal symmetry and the invariance under $SO(8)$. The compactification direction along the φ direction is different from the ordinary reduction to $d = 10$ by compactifying the 11th transverse direction. The dilaton becomes constant and the AdS_4 geometry is preserved. This is the reason why there is a conformal symmetry in the effective field theory of L-BLG.

The $SO(8)$ invariance is more subtle. In the scaling limit of ABJM to L-BLG, we take $k \to \infty$ limit and simultaneously locate the probe M2 brane far from the origin of the orbifold. Then the killed direction of the fluctuations by Z_k $(k \to \infty)$ orbifolding is given by the $SO(8)$ vector of the classical background fields X_0^I after specifying the gauge choice, and defining the projection operator by using X_0^I the manifest $SO(8)$ covariance is obtained.

C.3 Actions of probe branes in $AdS_4 \times \mathbf{CP}^3$

In this section we study possible forms of the effective field theory of probe M2 branes in the background geometry (C.17). The analysis in the section follows the prescription of [59] and [60] that a classical scalar field in the radial direction is interpreted as the Yang-Mills coupling. We will study probe M2 branes in a curved background while flat 11-dimensional background is used there.

By using the metric of (C.17), the generally covariant kinetic term can be written as

$$S_0 = -\frac{1}{2}\int d^3x\,\sqrt{-\det g}g^{\mu\nu}g_{IJ}\mathrm{tr}[D_\mu X^I D_\nu X^J], \tag{C.26}$$

where $\mu,\nu = 0,1,2$ are the world volume indices and $I, J = 1,\cdots,8$ are the target space indices, and $D_\mu = \partial_\mu - iA_\mu$ is the covariant derivative to assure that X^I lies on $\mathbf{C}^4/U(1)$ (see Appendix E).

Both of the world volume metric $g^{\mu\nu}$ and the target space metric g_{IJ} are functions of the position of the M2 branes $X^I(x)$. A static gauge is taken and the world volume metric $g_{\mu\nu}$ is given by the induced metric in the curved space-time (C.17).

This kinetic term can be simplified as follows. The metric $g_{\mu\nu}$ and g^{IJ} are functions of the the M2 brane position through r. As we did in the previous section, we separate the 8 scalar fields $X^I(x)$ of the probe M2 branes into a classical background and quantum fluctuations. If the probe M2 branes are located far from the origin of the orbifold singularity, the position of the M2 branes is approximated by the value of the classical background fields $X_0^I(x)$ and $r \sim \sqrt{(X_0^I(x))^2}$. Inserting the explicit form of the metric, the kinetic term can be simplified (see Appendix E) as

$$S_0 = -\frac{1}{2}\int dx^3 \eta^{\mu\nu}\eta_{IJ}\mathrm{tr}[\partial_\mu P^I \partial_\nu P^J] \tag{C.27}$$

where $P^I(x)$ is the projected fluctuating fields (C.25). In deriving this action, we have used that the classical background fields X_0^I are slowly varying. Note that all the dependence of $H(r)$ vanishes and the kinetic term of the fluctuating fields does not have the explicit dependence on the position of M2 branes.

The position of the M2 branes X_0^I must satisfy the classical equation of motion on the geometry (C.17). Because of the cancellation of $H(r)$, it looks like a free field equation of motion. But the fields X_0^I are restricted to be on the geometry where the φ-direction is killed, and they are slightly different from the constraint equation (B.10) in the L-BLG theory, or that in the scaling limit of the $SU(N)\times SU(N)$ ABJM theory. This is related to the effect of the $U(1)$ gauge field of the ABJM theory. We discuss it in Appendix D.

In the rest of this section, we dare to generalize the discussion of the kinetic term of the scalar field to the other possible terms in the the effective action of the probe M2 branes in the geometry (C.17). First assume that a gauge field is induced on the effective action of the probe M2 branes and its action is given by the ordinary Yang-Mills type. Then the general coordinate invariant YM action in the curved metric (C.17) is given by

$$-\frac{1}{4}\int d^3x\,\sqrt{-\det g}g^{\mu\rho}g^{\nu\sigma}\mathrm{tr}\left[F_{\mu\nu}F_{\rho\sigma}\right] = -\frac{1}{4}\int d^3x\,\left(\frac{R}{r}\right)^2\eta^{\mu\rho}\eta^{\nu\sigma}\mathrm{tr}\left[F_{\mu\nu}F_{\rho\sigma}\right]. \tag{C.28}$$

(Since we are considering the $d = 11$ theory, there is no freedom to multiply a dilaton dependence in the action.) In this case, $H(r)$ dependence remains and the effective Yang-Mills coupling is given by the following field dependent value:

$$g_{YM}^2(x) = \frac{r^2}{R^2} = \frac{(X_0^I(x))^2}{R^2}. \tag{C.29}$$

Similarly if we assume that the scalar field acquires a quartic potential, the general coordinate and $SO(8)$ invariance require its form to be

$$\begin{aligned}
&\frac{1}{4}\int d^3x\,\sqrt{-\det g}\,g_{IK}g_{JL}\mathrm{tr}[P^I,P^J][P^K,P^L] \\
&= \int d^3x\,\frac{1}{4}\frac{(X_0^I)^2}{R^2}\eta_{IK}\eta_{JL}\mathrm{tr}[P^I,P^J][P^K,P^L].
\end{aligned} \tag{C.30}$$

Here P^I are projected scalar fields (C.25).

Summing up these three terms, we have the following forms of the effective action:

$$S = -\frac{1}{2}\int dx^3\left(\mathrm{tr}[\partial_\mu P^I\partial^\mu P^I] - \frac{1}{4}\frac{R^2}{(X_0^I)^2}\mathrm{tr}\left[F_{\mu\nu}F^{\mu\nu}\right] + \frac{1}{4}\frac{(X_0^I)^2}{R^2}\mathrm{tr}[P^I,P^J]^2\right). \tag{C.31}$$

Of course there is little justification of the above analysis but it is amusing to see that this is nothing but the bosonic part of (B.19). The analysis might support an interpretation that the action of L-BLG is the effective action of the probe M2 branes in the geometry of (C.17). The X_0^I dependence of the coefficients will be related to the conformal invariance of the M2 branes. It will be interesting to constrain possible forms of the effective action including fermions, higher derivative terms, or generic potential terms by the generalized conformal invariance.

Appendix D

$U(1)$ part in ABJM theory

In scaling the ABJM theory to the L-BLG theory, we have mainly concerned with the $SU(N) \times SU(N)$ gauge theory. In this appendix we consider the scaling limit of the $U(N) \times U(N)$ ABJM theory, especially the effect of the $U(1)$ part. For simplicity we consider the bosonic terms only. In the presence of the $U(1)$ gauge field, the covariant derivative is modified to

$$D_\mu Y = \hat{D}_\mu \hat{Y} + 2iB_{0\mu}\hat{Y} + i\{\hat{B}_\mu, \hat{Y}\} + \partial_\mu Y_0 + 2i\hat{B}_\mu Y_0 + 2iB_{0\mu}Y_0, \tag{D.1}$$

where $B_{0\mu}$ is the axial combination of the $U(1) \times U(1)$ gauge field

$$B_{0\mu} = \frac{1}{2}(A_\mu^{(L)} - A_\mu^{(R)}). \tag{D.2}$$

The gauge field $B_{0\mu}$ is associated with the gauge transformation of the complex field $Y^A \to e^{i\varphi}Y^A$. Hence if the dual geometry is described by $\mathbf{C}^4/U(1)$, we need the gauge symmetry even after the scaling to L-BLG. Therefore, we do not scale the $B_{0\mu}$ field unlike B_μ. The scaling is given by

$$\hat{B}_\mu \to \lambda\hat{B}_\mu, \qquad Y_0 \to \lambda^{-1}Y_0, \qquad B_{0\mu} \to B_{0\mu} \tag{D.3}$$

and take the limit $\lambda \to 0$. The kinetic term of the scalar fields becomes

$$\begin{aligned} -\frac{1}{2}\mathrm{tr}|D_\mu Y_A|^2 = \mathrm{tr}\Bigg[&-\frac{1}{2}(\hat{D}_\mu\hat{Y}_A + 2i\hat{B}_\mu Y_{0A} + 2iB_{0\mu}\hat{Y}_A)^\dagger(\hat{D}^\mu\hat{Y}^A + 2i\hat{B}^\mu Y_0^A + 2iB_0^\mu\hat{Y}^A) \\ &- \frac{(\partial_\mu Y_{0A} + 2iB_{0\mu}Y_{0A})^\dagger(\partial^\mu Y_0^A + 2iB_0^\mu Y_0^A)}{2\lambda^2} \\ &-i(\partial_\mu Y_{0A} + 2iB_{0\mu}Y_{0A})^\dagger\hat{B}^\mu\hat{Y}^A + i(\partial_\mu Y_0^A + 2iB_{0\mu}Y_0^A)\hat{B}^\mu\hat{Y}_A^\dagger\Bigg]. \end{aligned} \tag{D.4}$$

The difference from the $SU(N)\times SU(N)$ case is that all the derivative is replaced by the covariant derivative with respect to $B_{0\mu}$. Requiring finiteness of the action, one can obtain the modified constraint

$$D_{U(1)}^2 Y_0^A \equiv (\partial_\mu + 2iB_{0\mu})(\partial^\mu + 2iB_0^\mu)Y_0^A = 0. \tag{D.5}$$

The gauge field $B_{0\mu}$ does not have a kinetic term and it is nothing but the auxiliary gauge field A_μ introduced in the $\mathbf{C}^4/U(1)$ gauged model discussed in Appendix E.

In the presence of the vector-like $U(1)$ gauge field

$$A_{0\mu} = \frac{1}{2}(A_\mu^{(L)} + A_\mu^{(R)}), \tag{D.6}$$

there is a coupling of $B_{0\mu}$ to $A_{0\mu}$ through the Chern-Simons term. If we do not scale the $A_{0\mu}$ either, it is given by

$$4\lambda^{-1} K \epsilon^{\mu\nu\rho} \mathrm{tr} B_{0\mu} F_{0\nu\rho}, \tag{D.7}$$

where $F_{0\mu\nu} = \partial_\mu A_{0\nu} - \partial_\nu A_{0\mu}$. Then because of the λ^{-1} coefficient this must vanish too.

If we instead scale the $A_{0\mu}$ gauge field with λ, the coefficient becomes of the order λ^0, and an integration over $B_{0\mu}$ solves it as

$$2B_{0\mu}^{(0)} = -\frac{i}{2|Y_0^A|^2}(Y_0^A \partial_\mu \bar{\hat{Y}}^A - \bar{Y}_0{}^A \partial_\mu \hat{Y}^A) - 2K\epsilon_{\mu\nu\rho} F_0^{\nu\rho}. \tag{D.8}$$

Appendix E

$SO(8)$ recovery in $\mathbf{C}^4/U(1)$ model

In Section C.2 we showed the recovery of $SO(8)$ invariance in the scaling limit of $AdS_4 \times \mathbf{CP}^3$. In this appendix, we study a $\mathbf{C}^4/U(1)$ sigma model and see the recovery of $SO(8)$. This is a generalization of the equivalence of a gauged model on $\mathbf{CP}^1$ and an $O(3)$ nonlinear σ model to a higher dimensional target space.

$\mathbf{C}^4$ is parameterized by the following angular variables:

$$\begin{aligned}
z^1 &= \rho e^{i(\phi_1+\varphi')} \cos\theta, \\
z^2 &= \rho e^{i(\phi_2+\varphi')} \sin\theta \cos\psi, \\
z^3 &= \rho e^{i(\phi_3+\varphi')} \sin\theta \sin\psi \cos\chi, \\
z^4 &= \rho e^{i\varphi'} \sin\theta \sin\psi \sin\chi, \\
0 &\le \varphi' \le 2\pi, \quad 0 \le \theta, \psi, \chi, \phi_1, \phi_2, \phi_3 \le \pi.
\end{aligned} \tag{E.1}$$

We then consider a scalar field on $\mathbf{C}^4/U(1)$ by identifying

$$z_i \sim e^{i\varphi'} z_i. \tag{E.2}$$

The Lagrangian of the scalar field $Z_i(x)$ on $\mathbf{C}^4/U(1)$ must be invariant under the local gauge transformation

$$Z_i(x) \to e^{i\varphi'} Z_i(x) \tag{E.3}$$

and the action can be written by introducing an auxiliary gauge field A_μ as

$$S = \int d^3x |(\partial_\mu - iA_\mu) Z_A|^2. \tag{E.4}$$

In the ABJM theory, the gauge field comes from the $U(1)$ part of the axial combination of the two $U(N)$ gauge fields $B_{0\mu}$ (see Appendix D). The gauge field does not have a kinetic term and and it can be eliminated by solving the equation of motion as

$$A_\mu = \frac{i}{2|Z^A|^2}(Z^A \partial_\mu \bar{Z}^A - \bar{Z}^A \partial_\mu Z^A). \tag{E.5}$$

By substituting the solution to the action, we obtain a nonlinear action which depends on the Z^A fields only. The action (E.4) becomes

$$S = \int d^3x(|\partial Z^A|^2 - A_\mu^2 |Z^A|^2). \tag{E.6}$$

In the case of $\mathbf{CP}^1$ model, it is well known that the model is nothing but the nonlinear σ-model on S^2. In our case, it is a nonlinear model on $\mathbf{C}^4/U(1)$.

Now we expand the field around a classical background and expand the field as

$$Z^A(x) = Z_0^A + \hat{Z}^A. \tag{E.7}$$

The classical background satisfies the equation of motion. Assume that the classical background is very slowly varying and much larger than the fluctuation $\hat{Z}^A$:

$$|Z_0^A| \gg |\hat{Z}^A| \ , \ |dZ_0^A|. \tag{E.8}$$

Under the assumption $(E.8)$, the quadratic terms of the fluctuations in the action (E.6) become

$$S \sim \int d^3x(|\partial \hat{Z}^A|^2 - A_\mu^{(0)2}|Z_0^A|^2) \tag{E.9}$$

where

$$A_\mu^{(0)} = \frac{i}{2|Z_0^A|^2}(Z_0^A \partial_\mu \bar{\hat{Z}}^A - \bar{Z_0}^A \partial_\mu \hat{Z}^A). \tag{E.10}$$

If we decompose the complex fields into real components as

$$\begin{aligned} Z_0^A &= X_0^{2A-1} + iX_0^{2A} \\ \hat{Z}^A &= i\hat{X}^{2A-1} - \hat{X}^{2A}, \end{aligned} \tag{E.11}$$

the gauge field can be written as

$$A_\mu^{(0)} = \frac{1}{(X_0^I)^2} X_0^I \partial_\mu \hat{X}^I. \tag{E.12}$$

Thus the action can be written as a manifestly $SO(8)$ covariant expression:

$$S = \int d^3x\{(\partial\hat{X}^I)^2 - \frac{1}{X_0^2}(X_0^I \partial \hat{X}^I)^2\}. \tag{E.13}$$

In terms of the projected scalar field

$$P^I = \hat{X}^I - \frac{X_0^I X_0^J \hat{X}^J}{(X_0^I)^2}, \tag{E.14}$$

the action is written (under the assumption (E.8))

$$S = \int d^3x(\partial_\mu P^I)^2. \tag{E.15}$$

It is manifestly invariant under the $SO(8)$ transformations. But note that the $SO(8)$ transformation is different from the $SO(8)$ acting on the original $\mathbf{R}^8$ because of the different decompositions of the complex fields into the real components in (E.11).

Appendix F

Ordinary reduction of M2 to D2

In this appendix, we remind the reader of the ordinary reduction of M2 branes in $d = 11$ supergravity to D2 branes in $d = 10$ type IIA supergravity to clarify the difference from the reduction adopted in the ABJM theory. By compactifying x_{11} direction and identifying $x_{11} \sim x_{11}+2\pi R_{11}$ the M2 brane solution is given by replacing the metric (C.1) with a smeared harmonic function [61]

$$H(r) = \sum_{n=-\infty}^{\infty} \frac{R^6}{(r^2 + (x_{11} + 2\pi n R_{11})^2)^3}. \tag{F.1}$$

where r is the radial distance in the 7 non-compact transverse directions. The string coupling constant is given by $R_{11} = g_s l_s$. Then we can get the solution of the multiple D2-branes in the string frame by using the reduction rule and the Poisson resummation at distance much larger than R_{11}:

$$\begin{aligned} ds_{D2} &= H^{-\frac{1}{2}} \left(\sum_{\mu,\nu=0}^{2} \eta_{\mu\nu} dx^\mu dx^\nu \right) + H^{\frac{1}{2}} \left(dr^2 + d\Omega_6^2 \right), \\ e^\phi &= H^{\frac{1}{4}}, \\ H(r) &= \frac{6\pi^2 g_s N l_s^5}{r^5}. \end{aligned} \tag{F.2}$$

It is quite different from (C.9). Especially the dilaton is not a constant and the conformal symmetry of the M2 brane geometry is broken; it is no longer AdS_4. The transverse direction is given by the radial direction and S^6, and therefore it has the $SO(7)$ invariance.

Appendix G

Gamma Matrices

The explicit forms of the antisymmetrized products of the 8×8 Γ matrices we have used in (3.22) are given as $\Gamma_{IJ} = \mathbb{I}_{2\times 2} \otimes \gamma_{IJ}$ where

$$
\begin{aligned}
\gamma_{12} &= \begin{pmatrix} i\sigma^2 & & & \\ & -i\sigma^2 & & \\ & & i\sigma^2 & \\ & & & i\sigma^2 \end{pmatrix}, &
\gamma_{13} &= \begin{pmatrix} & \mathbb{I} & & \\ -\mathbb{I} & & & \\ & & & \sigma^3 \\ & & -\sigma^3 & \end{pmatrix}, \\
\gamma_{14} &= \begin{pmatrix} & i\sigma^2 & & \\ i\sigma^2 & & & \\ & & & \sigma^1 \\ & & -\sigma^1 & \end{pmatrix}, &
\gamma_{15} &= \begin{pmatrix} & & -\sigma^3 & \\ & & & \mathbb{I} \\ \sigma^3 & & & \\ & -\mathbb{I} & & \end{pmatrix}, \\
\gamma_{16} &= \begin{pmatrix} & & -\sigma^1 & \\ & & & -i\sigma^2 \\ \sigma^1 & & & \\ & -i\sigma^2 & & \end{pmatrix}, &
\gamma_{17} &= \begin{pmatrix} & & & -\sigma^3 \\ & & -\mathbb{I} & \\ & \mathbb{I} & & \\ \sigma^3 & & & \end{pmatrix}, \\
\gamma_{18} &= \begin{pmatrix} & & & -\sigma^1 \\ & & i\sigma^2 & \\ & i\sigma^2 & & \\ \sigma^1 & & & \end{pmatrix}, &
\gamma_{52} &= \begin{pmatrix} & & \sigma^1 & \\ & & & -i\sigma^2 \\ -\sigma^1 & & & \\ & -i\sigma^2 & & \end{pmatrix}, \\
\gamma_{53} &= \begin{pmatrix} & & & \mathbb{I} \\ & & \sigma^3 & \\ & -\sigma^3 & & \\ -\mathbb{I} & & & \end{pmatrix}, &
\gamma_{54} &= \begin{pmatrix} & & & i\sigma^2 \\ & & \sigma^1 & \\ & -\sigma^1 & & \\ i\sigma^2 & & & \end{pmatrix}, \\
\gamma_{56} &= \begin{pmatrix} i\sigma^2 & & & \\ & i\sigma^2 & & \\ & & i\sigma^2 & \\ & & & -i\sigma^2 \end{pmatrix}, &
\gamma_{57} &= \begin{pmatrix} & \sigma^3 & & \\ -\sigma^3 & & & \\ & & & \mathbb{I} \\ & & -\mathbb{I} & \end{pmatrix}, \\
\gamma_{58} &= \begin{pmatrix} & \sigma^1 & & \\ -\sigma^1 & & & \\ & & & i\sigma^2 \\ & & i\sigma^2 & \end{pmatrix}
\end{aligned}
\tag{G.1}
$$

and $\mathbb{I}_{2\times 2}$ is a 2×2 identity matrix. We have also defined

$$\Gamma^0 = i\sigma^2 \otimes \mathbb{I}_{8\times 8}. \tag{G.2}$$

The $i\sigma^2$ was used to contract the indices of the 2-component spinor χ and it is the 3 dimensional γ^0 matrix (see the Appendix of [35]). $\mathbb{I}_{8\times 8}$ is an 8×8 identity matrix. They satisfy the following consistency relations as $\Gamma_{12}\Gamma_{13} + \Gamma_{13}\Gamma_{12} = -(\Gamma_2\Gamma_3 + \Gamma_3\Gamma_2) = 0$. At this stage, there is an ambiguity to determine the Γ matrices, but the explicit forms of Γ_I are not necessary here. To fix the ambiguity, we need to consider more general VEVs of X_0^I.

On the other hand, the explicit forms of the antisymmetrized Γ matrices that we used in Section 9 are

$$\Gamma_{12} = \begin{pmatrix} -i\sigma^2 & & & \\ & i\sigma^2 & & \\ & & -i\sigma^2 & \\ & & & -i\sigma^2 \end{pmatrix}, \qquad \Gamma_{13} = \begin{pmatrix} & -\mathbb{I} & & \\ \mathbb{I} & & & \\ & & & -\sigma^3 \\ & & \sigma^3 & \end{pmatrix},$$

$$\Gamma_{14} = \begin{pmatrix} & -i\sigma^2 & & \\ -i\sigma^2 & & & \\ & & & -\sigma^1 \\ & & \sigma^1 & \end{pmatrix}, \qquad \Gamma_{15} = \begin{pmatrix} & & \sigma^3 & \\ & & & -\mathbb{I} \\ -\sigma^3 & & & \\ & \mathbb{I} & & \end{pmatrix},$$

$$\Gamma_{16} = \begin{pmatrix} & & \sigma^1 & \\ & & & i\sigma^2 \\ -\sigma^1 & & & \\ & i\sigma^2 & & \end{pmatrix}, \qquad \Gamma_{17} = \begin{pmatrix} & & & \sigma^3 \\ & & \mathbb{I} & \\ & -\mathbb{I} & & \\ -\sigma^3 & & & \end{pmatrix},$$

$$\Gamma_{18} = \begin{pmatrix} & & & \sigma^1 \\ & & -i\sigma^2 & \\ & -i\sigma^2 & & \\ -\sigma^1 & & & \end{pmatrix}, \qquad \Gamma_{52} = \begin{pmatrix} & & -\sigma^1 & \\ & & & i\sigma^2 \\ \sigma^1 & & & \\ & i\sigma^2 & & \end{pmatrix},$$

$$\Gamma_{53} = \begin{pmatrix} & & & -\mathbb{I} \\ & & -\sigma^3 & \\ & \sigma^3 & & \\ \mathbb{I} & & & \end{pmatrix}, \qquad \Gamma_{54} = \begin{pmatrix} & & & -i\sigma^2 \\ & & -\sigma^1 & \\ & \sigma^1 & & \\ -i\sigma^2 & & & \end{pmatrix},$$

$$\Gamma_{56} = \begin{pmatrix} -i\sigma^2 & & & \\ & -i\sigma^2 & & \\ & & -i\sigma^2 & \\ & & & i\sigma^2 \end{pmatrix}, \qquad \Gamma_{57} = \begin{pmatrix} & -\sigma^3 & & \\ \sigma^3 & & & \\ & & & -\mathbb{I} \\ & & \mathbb{I} & \end{pmatrix},$$

$$\Gamma_{58} = \begin{pmatrix} & -\sigma^1 & & \\ \sigma^1 & & & \\ & & & -i\sigma^2 \\ & & -i\sigma^2 & \end{pmatrix}. \tag{G.3}$$

They indeed satisfy the consistency conditions as $\Gamma_{12}\Gamma_{13} + \Gamma_{13}\Gamma_{12} = -(\Gamma_2\Gamma_3 + \Gamma_3\Gamma_2) = 0$.

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