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Quasi-periodic structures in the large-scale galaxy distribution and three-dimensional Voronoi tessellation

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SUMMARY

It has been suggested that the void and wall structure associated with the large-scale galaxy distribution might be qualitatively, or perhaps even physically, modelled by a Voronoi tessellation, and that such structure might account for the surprisingly regular, sharp peaks in the galaxy redshift distributions obtained from 'pencil beam' surveys. Taking cell wall crossings by random line segments to correspond to such redshift peaks, we derive an exact expression for the distribution of spacings of these intersections in a three-dimensional Voronoi tessellation. This result verifies that the spacings are non-random and quasi-periodic, qualitatively resembling the observed pattern, even though the cell wall structure is generated from randomly placed seeds. Finally, we use moments of the spacing distribution to show that apparently periodic samples, similar to those recently reported, represent only one to two σ fluctuations in a Voronoi tessellation.

1 INTRODUCTION

The Voronoi tessellation (Voronoi 1908) is a scheme for dividing an N-dimensional space into cells. Points, called 'seeds', are placed at random within the space; the technique is trivially generalized to any placement of seeds, in fact. The cell associated with each seed is the locus of points closer to that seed than to any other seed. The cell boundaries or walls are, therefore, the locus of points equidistant from two seeds and which are not closer to any third seed. Each cell is then an irregular convex polygonal region of the space, and the tessellation fills the space exactly.

The Voronoi tessellation model is advantageous in that it is mathematically simple and well defined and that it clearly exhibits some of the same *qualitative* features as the large-scale galaxy distribution (Yoshioka & Ikeuchi 1989; Icke & van de Weygaert 1990). Moreover, it may give a reasonably accurate *quantitative* description of some specific physical models of structure formation, such as explosive scenarios (Ikeuchi 1981; Ostriker & Cowie 1981; Bertschinger 1985) and even some aspects of purely gravitational models (Icke 1984; Icke & van de Weygaert 1987; van de Weygaert & Icke 1989). In any case, Coles (1990) has recently used the results of Moller's (1989) extensive mathematical analysis of Voronoi tessellation to argue that such large-scale structure may well explain the quasi-periodicity observed by Broadhurst *et al.* (1990). Monte Carlo simulations have led van de Weygaert (1990) to the same conclusion.

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In order to gain some further insight into the 'void and wall' structure associated with the large-scale galaxy distribution (de Lapparent, Geller & Huchra 1986; Geller & Huchra 1989), we here develop some analytic results concerning the statistical properties of a three-dimensional Voronoi tessellation taken to be a qualitative model of the cosmic large-scale structure. In particular, we concentrate on the statistics of cell wall crossings by line segments which we identify with the sharp peaks in the galaxy redshift distribution seen in 'pencil beam' (i.e., narrow field of view) redshift surveys (Broadhurst *et al.* 1990).

The most interesting result of this analysis is the confirmation that the distribution of cell wall crossings along an arbitrary line-of-sight is highly non-random despite the fact that it is generated (via the Voronoi prescription) from randomly placed seeds. Moreover, the nature of the non-randomness is that the cell wall crossings are much more regularly spaced than would be expected if they were statistically independent, at least *qualitatively* the same sort of non-randomness as observed by Broadhurst *et al.* (1990).

The analysis assumes that all distances are measured in units of a distance a defined by

$$\frac{4\pi}{3} a^3 n = 1, \tag{1}$$

where n is the mean seed density. Thus, a sphere of radius R will contain R^3 seeds on average. This convention simply reduces the number of numerical constants in the equations.

2 PROBABILITY OF NO WALL CROSSINGS

Consider a random point in the three-dimensional Voronoi tessellated space and a line segment extending a distance x from the point in a random direction. Now let $P_0(x)$ be defined as the probability that this line segment does not intersect any cell walls. In general, the initial point will be a distance r from the nearest seed and the line segment will be at an angle α to the direction to that nearest seed. The other end of the line segment will then lie at a distance

$$d=(r^2+x^2-2rx\cos\alpha)^{1/2} \tag{2}$$

from that seed. If there are no other seeds closer than d to the other end of the line segment, then due to the convex nature of a Voronoi cell, the segment cannot have crossed a boundary. However, if there is some other seed at a distance less than d, then the line segment must have crossed one or more cell walls.

The expected number of points inside a *randomly* chosen sphere of radius d is just d^3, and the probability that there are no seeds in this sphere is just $\exp(-d^3)$, according to Poisson's law. However, there is a complication. Some part of the volume within distance d of the end of the line segment is also within a distance less than r of the beginning point; it is therefore already guaranteed that this sub-region is empty of seeds. Its volume must be subtracted before Poisson's law can be applied. A little solid geometry then shows that the expected number of points in the volume closer than d to the end of the line segment, but not closer than r to the beginning, is just

$$V(x,r,\alpha)=d^3\left(\frac{1}{2}-\frac{\cos^3\beta}{4}+\frac{3}{4}\cos\beta\right)-r^3\left(\frac{1}{2}+\frac{\cos^3\alpha}{4}-\frac{3}{4}\cos\alpha\right), \tag{3}$$

where

$$\cos\beta=\frac{x-r\cos\alpha}{d}. \tag{4}$$

Now, in order to obtain $P_0(x)$ we must integrate $\exp(-V)$ over the probability distributions of r and α. Since

$$P(r)\,dr=\tfrac{3}{2}\,r^2e^{-r^3}\,dr \tag{5}$$

and

$$P(\alpha)\,d\alpha=\sin\alpha\,d\alpha \tag{6}$$

we obtain the integral

$$P_0(x)=\frac{3}{2}\int_0^\infty\int_0^\pi r^2\sin\alpha\; e^{-V(x,r,\alpha)-r^3}\,d\alpha\,dr \tag{7}$$

which is easily evaluated numerically. The result is shown in Fig. 1.

3 WALL SEPARATION DISTRIBUTION

A line (or long line segment) passing through a three-dimensional Voronoi tessellation will pass through many cell walls. If we call the distance between two such crossings s then the probability distribution of s values is denoted by $P_s(s)\,ds$.

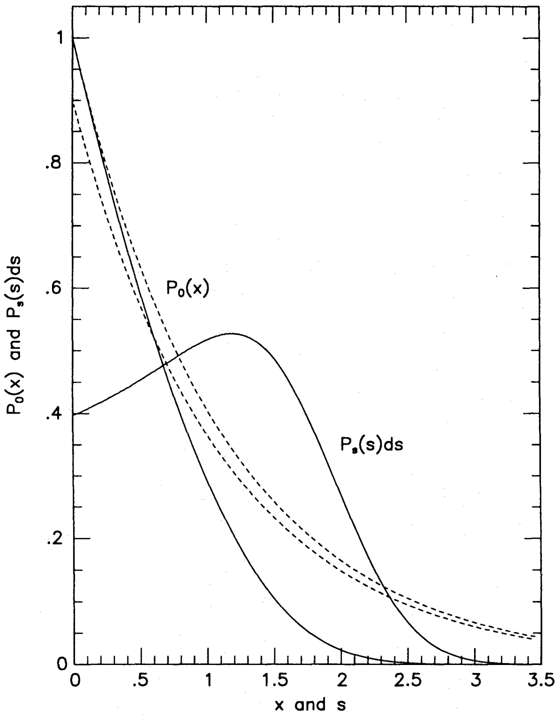

Figure 1. The probability $P_0(x)$ that a randomly placed line segment of length x will not cross a cell wall and $P_s(s)\,ds$ the probability distribution of cell wall separations s along a random line-of-sight. The solid curves are for the Voronoi tessellation, and the dashed curves show the same functions for the case of random independently placed walls. The two $P_0(x)$ functions have values of unity at $x=0$, and the $P_s(s)\,ds$ functions are obtained by twice differentiating the $P_0(x)$ curves (see equation 12). The most significant feature of these curves is that the Voronoi tessellation produces more regular wall separations (fewer unusually large or small separations) than the random case.

Since a random point along such a line is equivalent to a random point in the space, it is possible to relate $P_s(s)\,ds$ to $P_0(x)$ derived in the previous section via Bayes's Theorem. This is achieved by writing

$$P_0(x)=\int_0^\infty P(x|s)\,P(s)\,ds, \tag{8}$$

where $P(s)\,ds$ is simply the probability that the random point is between two walls separated by a distance s, and $P(x|s)$ is the probability that a line segment of length x placed at random within a linear region of length s will lie entirely within the region. It is trivial to see that

$$P(s)\,ds=sP_s(s)\,ds, \tag{9}$$

(where the factor of s accounts for the fact that a random point is more likely to fall between walls with large separations) and that

$$P(x|s)=\begin{cases}(s-x)/s & x<s\\ 0 & x>s.\end{cases} \tag{10}$$

Substituting equations (9) and (10) back into (8) gives

$$P_0(x)=\int_x^\infty sP_s(s)\,ds-x\int_x^\infty P_s(s)\,ds. \tag{11}$$

If this expression is then twice differentiated with respect to x, we have the desired relation

$$\frac{d^2P_0(x)}{dx^2}=P_s(x) \tag{12}$$

which is fully general (i.e., not particularly associated with Voronoi tessellation).

Using this relation, $P_s(s)\,ds$ may be extracted from the $P_0(x)$ curve determined in the previous section. It is also shown in Fig. 1. As is shown in a succeeding paper (Ikeuchi, Yoshioka & Turner, in preparation), results of a Monte Carlo simulation of a Voronoi tessellation agree with these analytic results very well.

4 DISCUSSION

The most striking feature of the $P_s(s)\,ds$ distribution derived above is that it has a definite characteristic scale. Wall separations of ~1.2 in our units are the most probable and both smaller and larger ones are less likely. Despite the fact that the seeds which generate the cell wall structure are randomly placed, the walls themselves have a more regular than random spacing. This can be seen clearly in Fig. 1 by comparing the previously plotted set of Voronoi tessellation functions to those which would apply if the cell walls were placed independently along the line-of-sight with the same mean density. The distribution $P_s(s)\,ds$ of wall separations is much broader for the independently placed walls; both smaller than average and larger than average wall separations are substantially more probable than in the Voronoi case. This is the same qualitative feature shown by the distribution of galaxy redshifts in the Broadhurst *et al.* (1990) survey, an unusually regular spacing of the peaks.

This feature of the distribution can be roughly quantified by considering the ratio of the dispersion in wall separations σ_s to the mean separation $\langle s\rangle$. For independently placed cell walls, this ratio is just $\sigma_s/\langle s\rangle=1$, but for the three-dimensional Voronoi tessellation it is significantly smaller $\sigma_s/\langle s\rangle\approx 0.58$. In other words, the dispersion in wall separations at the same mean density is nearly a factor of 2 smaller than if the cell walls were independently placed. Note also that because the Voronoi $P_s(s)\,ds$ falls well below the one for the Poisson case (Fig. 1) near $s=0$, the cell wall covariance function will be significantly negative at small separations. Ikeuchi *et al.* (in preparation) discuss the cell wall covariance function in detail.

While qualitatively suggestive, the three-dimensional Voronoi tessellation considered here is still almost certainly quantitatively too random to explain the strong regularity (Kurki-Suonio, Matthews, and Fuller 1990) in the Broadhurst *et al.* (1990) results. Nevertheless, as a counter-example, this result shows that a comparison of the galaxy data with the simple null hypothesis of randomly placed walls is far too weak a test to reject the general idea that large-scale structure derives from underlying random (or random phase) perturbations. The Voronoi tessellation prescription can be regarded as a very crude approximation to the interactions which one would expect between adjacent evolving perturbations as large-scale structure forms. The details of such interactions depend, of course, on the specific model but might include such things as competition for mass and volume or collision of expanding blast waves.

As shown in Ikeuchi *et al.* (in preparation), the probability of finding a void with the volume $V=y\cdot a^3$ is analytically given by Nozakura (1985, private communication)

$$P_V(y)\,dy=\frac{1}{5!}(6y)^5\,e^{-6y}\,dy. \tag{13}$$

Then, the maximum probability is expected for a void volume $V_m=5a^3/6$. If the population of voids with various volumes is obtained we may compare it with the above expression.

Furthermore, even within the framework of Voronoi tessellations, one can easily imagine slightly more realistic models. For example, since the ratio of cell volume V to wall area scales as $V^{1/3}$, one would expect the walls of larger cells to be more densely populated than those of small ones and thus to be more likely to be detected (i.e., produce a peak containing a significant excess of galaxies) in 'pencil beam' surveys. To the extent that small wall separations are produced by small volume cells (they can also be produced by passing through the 'corners' of large cells), this means that cases of small wall separations will be likely to be missed thus producing an even narrower $P_s(s)\,ds$ distribution. This effect

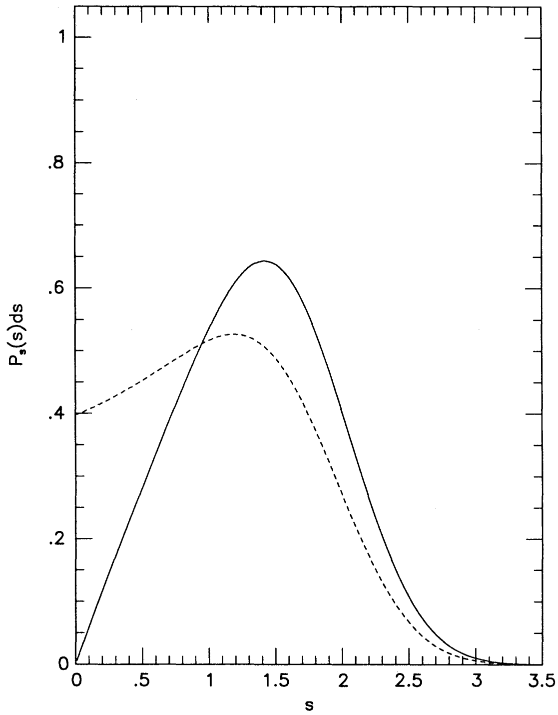

Figure 2. The possible effect of a detection bias on the wall separation distribution. The dashed curve shows the same wall separation distribution for the Voronoi tessellation displayed in Fig. 1. The solid curve is derived by multiplying it by the detection probability function given by equation (14) and renormalizing. The parameter D was set to a value of 1.2 for illustrative purposes. Such detection biases may occur if the walls of large voids are more densely populated with galaxies than those of small ones, thus producing even more regular wall separation distributions.

is illustrated in Fig. 2 which shows a wall separation distribution produced by multiplying the Voronoi curve by a function

$$P_{\mathrm{D}}(s)=1-\mathrm{e}^{-s/D} \tag{14}$$

taken to be the probability of detection for a wall separation s. The value of D might be determined explicitly for a particular survey and some structure model but has simply been adjusted to give an interesting result in this case ($D=1.2$). It should be clear that this procedure has not been rigorously justified and that it is simply intended to illustrate the sorts of effects which such a detection bias might produce; it is, however, not entirely unrealistic. As Fig. 2 shows, such biases could produce an impressively narrow $P_s(s)\,ds$ distribution of separations of *detected* walls, further accentuating the sort of regularity already displayed by the Voronoi tessellation itself. For $D=1.2$, $\sigma_s/\langle s\rangle$ is reduced to 0.42 for example.

Beyond the qualitative implications discussed above, it may be appropriate to more closely consider a quantitative comparison of the observations and Voronoi tessellation if one takes it to give a relatively accurate description of the large scale structure predicted by some physical models (Icke & van de Weygaert 1990). In the present context, this corresponds to asking if it could reproduce structure like that observed by Broadhurst *et al.* (1990). The observed value of $\sigma_s/\langle s\rangle$ cannot be precisely determined from the so far published data, but it is clearly quite small, certainly ≤ 0.1, compared to either the expected Voronoi value of 0.58 or even the detection biased value of 0.42 discussed above. Nevertheless, the observations cover only a fairly small number of cell wall spacings, and substantial fluctuations about the expected values are likely. In particular, Coles (1990) points out that a sample of N cell wall spacings will show rms fluctuations in $\sigma_s/\langle s\rangle$ of at least

$$\delta=\left[\frac{\langle s^4\rangle-\langle s^2\rangle^2}{N\langle s\rangle^2}\right]^{1/2}. \tag{15}$$

Taking the appropriate moments of $P_s(s)\,ds$, we find $\delta=1.44/N^{1/2}$. Also, for the detection bias modified s distribution shown in Fig. 2 (with $D=1.2$), we find $\delta=1.21/N^{1/2}$. Thus, in either case, quite small values of $\sigma_s/\langle s\rangle$ such as those observed by Broadhurst *et al.* (1990) represent only one to two standard deviations fluctuations, not highly improbable events, since N (the number of observed cell wall separations) is probably no more than 10–20. Essentially, the same conclusion was obtained by Coles (1990) based on rather indirect estimates of the moments of $P_s(s)\,ds$ and by van de Weygaert (1990) based on numerical simulations.

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