The Theory of Motion and Ephemerides of The Second Neptunian Satellite Nereid

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Contents

Abstract

The problem of obtaining accurate ephemerides for the second Neptunian satellite Nereid has intrigued many astronomers since its discovery by Kuiper in 1949. That is because of its bizarre orbit. The satellite's orbit has unusually large eccentricity ($e \sim 0.75$) which is considered as the most eccentric known natural satellite in the solar system. This very elongated orbit renders the usage of the classical methods for expanding the disturbing function in terms of the eccentricity, that is because of the slow convergence of the power series solutions especially at higher orders.

In this work we aim to study the dynamical motion of the second Neptunian satellite Nereid using both analytical and numerical methods. We construct an analytical theory of the motion of a highly eccentric Nereid which accurately represents a real satellite system, then we pose emphasis upon comparison with numerical integration of the equations of motion. The theory is elaborated by the use of Lie transformation approach advanced by Hori's device. This method enables us to express the relations between the osculating and the mean elements in an explicit form instead of the implicit form arised by Poincare'-von Zeipel's approach. By the virtue of Hori's perturbations method, we can also get the inverse transformations easily. The main perturbing forces on Nereid which come from the solar

influence are only taken into account through the present theory. The disturbing function is developed in powers of the ratio of the semimajor axes of the satellite and the Sun. To avoid the slow convergence of the power series solution, the disturbing function is put in a closed form with respect to the eccentricity of Nereid. In addition, replacing functions of the true anomaly by expressions involving the mean anomaly is also avoided, and the eccentric anomaly of Nereid has been adopted as independent variable. The present theory includes secular perturbations up to the fourth order, short and long period perturbations up to the third order and small parameter ϵ (which defines the ratio between the orbital period of Nereid and that of Neptune) $\sim 6 \times 10^{-3}$. The results of the present theory satisfy the required accuracy for future observations. We intend to develop this theory to be applied on the retrograde satellites of the major planets. The dissertation is organized as following:

In chapter 1 we give a general introduction which includes the advantages of the use of the analytical techniques and their expected outcome. A review on Nereid, the second Neptunian satellite, and its enigmatic according to different sources are summarized. Chapter one contains also a section about the classification of natural satellites according to their orbits and perturbing forces. Then we pose the motivation and aim of this study.

Chapter 2 contains the method that we have used, equations of motion and the disturbing function. Hori's perturbation method is introduced briefly, and some of the merits and demerits of canonical methods in celestial mechanics have been shown. This chapter includes also procedures for obtaining the osculating orbital elements starting from the mean elements and conversely. We implement each procedure for digital computations by constructing a computational algorithm described by its purpose, input and its computational sequence.

In chapter 3 we dealt with the circular planar restricted three-body problem. In this case, the inclination of Nereid to the orbital plane of Neptune is zero. The osculating orbital elements of the fictitious Nereid are evaluated and given in figures. The results are compared with those computed by the numerical integration of the equations of motion. The residuals are tabulated and showed also by figures. At the end of this chapter we give a short note about d'Alembert characteristics which permit the validity of the analytical expressions based on Lie transform approach.

Chapter 4 is devoted to the circular nonplanar restricted three-body problem. In this case we take the inclination of Nereid into account and deal with the nonplanar solution for a real Nereid. The analytical expressions of the short, intermediate and long periodic perturbations are evaluated. After elimination of the short and intermediate terms, the Hamiltonian system equations are solved in e, I and ω using Jacobi's elliptic function (Kinoshita and Nakai, 1999), whereas the longitude of ascending node and the mean anomaly are expressed in Fourier series expansion. By this solution we got the mean elements which are used for evaluating the osculating orbital elements and ephemerides of Nereid. All these processes are summarized in a computational algorithm and carried out by the powerful MATHEMATICA software package. Moreover, the analytical expressions are transformed into FORTRAN format and programmed to be easy to handle.

We compared the analytical results with those computed by the direct numerical integration of the equations of motion for short and long periodic perturbations. As a result of this comparison, the global internal accuracy of the present theory reached 0.3 km in the semi-major axis, 10^{-7} in the eccentricity and 10^{-5} degree in the angular variables over a period of several hundred years. The behaviour of the orbital motion of the satellite is exhibited in analytical expressions, tables and figures. The way of comparison is discussed briefly.

Finally, we close this research by discussion and conclusions. By this end we provide to the observers an efficient analytical theory, capable of generating accurate ephemerides for the motion prediction of highly eccentric Nereid.

Chapter 1

Introduction

This chapter gives a global view on the subject of the present thesis. The categories of the natural satellites according to the perturbing forces and their orbits are summerized. A review on Neptune system is also included. Then the motivation and aim of this study.

1.1 Analytical and numerical techniques

There are two basic techniques for motion prediction of celestial objects, these are: Analytical techniques and numerical techniques. In fact, it is difficult to obtain analytical solutions of the motion in a complex force model. In contrast, the numerical techniques usually, provide us with the solutions for any number and types of perturbing forces whatever simple or complex but do not allow a global view of the ensemble of solutions and the more information we need to know, the more numerical analysis tools must be used (Breiter, 1997). However, analytical solutions though difficult to obtain for complex force models and limited to relatively simple models, represent a manifold of solutions for a large domain of initial conditions and parameters and find indispensable application to mission planning and qualitative analysis of the motion.

Many advantages of the analytical theories (Chapront 1982, Vakhidov and Vasiliev 1996), they provide a complete set of solutions among which the real solution can be determined by comparison with observations, enable us to understand the character of a system's behaviour and allow a global view of the ensemble solutions. In case of the motion of the satellites with large eccentricities, for instance, the computing time does not depend on the time interval if we use the analytical methods in motion prediction. Therefore at present in many observatories the analytical and semianalytical theories are preferable, especially in the case of satellites with high eccentric orbits. However, if full analytical solutions formulae are utilized with nowadays existing symbols used for manipulating digital computer programs, they definitely invaluable for obtaining solutions with any desired accuracy. Brumberg (1995) answered the question raised what kind of technique -numerical or analytical should be preferred. This controversy means very little, one should combine both kinds of techniques, according to a specific problem and the aim of the research (a particular or general solution, the interval of validity of the solution, accuracy considerations, etc.). In fact celestial mechanics has created in the domain of analytical techniques and applying the numerical techniques to it have no important peculiarities compared with the general techniques of applied mathematics.

1.2 Enigmatic Nereid

Neptune has two known satellites: Triton and Nereid. The innermost satellite Triton has a highly inclined circular retrograde orbit and a mean distance of 354759 km, while the outer satellite, Nereid, has a bizarre orbit. The second Neptunian satellite Nereid was discovered in 1949 by Kuiper at the MacDonald observatory (Kuiper 1949). The satellite's orbit has unusually large eccentricity ($e \sim 0.75$) which is considered as the most eccentric known natural satellite. Because of its high eccentricity, faintness and generally large separation from the host planet Neptune ($1.4 \sim 9.7$ million km), little is known about its orbital motion and physical properties. Its pericenter distance is about four times the distance of Triton (see Fig. 1.1). The main perturbing forces on Nereid are due to the solar infleunce, while the perturbations of Triton and the oblateness of Neptune are very small regarding the Sun perturbations. The velocity of Nereid at the pericenter, 3 km/sec, is just 0.2 km/sec short of escape velocity. Nereid orbits Neptune at a semimajor axis $\approx$ 5,515,000 km with a period of 360 days and inclination of $\sim 10^o$ to the orbital plane of Neptune (Mignard 1975, 1981, Veillet 1982). In his analytic theory, Mignard (1975) showed that the inclination of Nereid may vary by $\pm 3^o$, while the eccentricity has a long-period variation between ~ 0.733 and ~ 0.752.

Nereid has been studied photometrically using two groups of observations: the Earth-based observations (M. Schaefer and B. Schaefer 1988, 1995, William et al 1991, Buratti et al 1997) and Voyager 2 images (Tomas et al 1991). The Voyager 2 path through the Neptune system is shown in Figure 1.2. Schaefer and Schaefer (1988) collected photometric (UBVRI) and astrometric data on Nereid using 0.9 m telescope and CCD camera at Cerro Tololo Inter-American Obs. (CTIAO). Analysing these data, they found an unusual reflectance spectrum for Nereid. Variability of Nereid was observed in all band passes, with an amplitude of greater than 1.5 mg. The variations are likely to be due to rotation effects, with a rotation period between $8 \sim 24$ hours. They explained the unusual orbit of Nereid that it must be have suffered some drastic change in the past. In (1991) William et al, obtained observations of Nereid using CCD camera on 1-m Jacobus Kapteyn telescope, the purpose was to improve the orbital elements of Nereid on La Palma during the period (July $10 \sim 18$ 1990), however, they studied the relative magnitude for Nereid. They confirmed the brightness variations of Nereid which is proved previously by the Schaefers (1988). The amplitude of the light curve to be about 1.3 ± 0.2 mag, and the period about 13.6 ± 0.1 hours. Tomas et al (1991), have analyzed observations of Nereid obtained by Voyager 2 over 12-days interval at solar phase angles of $25^o \sim 96^o$. From these observations they found that the radius of Nereid is about 170 ± 25 km, and there is no evidence of a rotation light curve greater than 15%. They concluded that, Nereid is different from the other small Neptune's satellites and it is difficult to reconcile the Voyager data with some ground-based results. The Earth-based observations of Nereid's light curve were obtained at small phase angles less than 2^o, while those from Voyager are at greater than 25^o.

Later, Schaefer and Schaefer (1995) have continued their analysis using new observations for Nereid and got short-term variability with an amplitude roughly 0.2 mag and secular brightning about o.3 mag over the period from mid-March to early July, 1995. This time they introduced two main reasons for this variability.

1) Nereid's variability is caused by a combination of albedo variations on its surface, an irregular shape and chaotic rotation (this not proved by Voyager data)

2) Nereid is a captured object from the kuiper belt and that its variability is in part caused by outbursts of gases from its surface.

Recently, Buratti et al (1997) have obtained 3 continuous nights of photometric observations of the light curve of Nereid with COSMIC CCD at the Palomar 200-inch telescope. Their preliminary analysis of the data does not show the large amplitude (only 10% mag in a single night) which is reported by the previous authors. Now, there are two groups have different explainations of Nereid's situation:

1- Earth based photometry indicates large brightness variations, however, different observers reported very different light curve amplitudes.

2- Voyager 2 images spanning 12-days show no evidences of variation greater than 10%. In addition, they suggest that either Nereid is nearly spherical or is rotating slowly. We note that Voyager 2 observations did not reveal Nereid's shape nor distinguished whether Nereid is a captured object or not. Since there is no evident agreement between spacecraft and Earth-based observations the problem of Nereid's rotation is still controversial (Buratti et al 1997, Tomas et al 1996). According to Dobrovolskis (1995), if the rotation period of Nereid is over than two weeks, then its high eccentricity will make it in a chaotic spin rate.

The photometric and astrometric observations of Nereid made by Schaefer and Schaefer (1988, 1995), revealed that Nereid is just as puzzling as its larger sibling. Nereid's reddish color is unlike that of any other moon or asteroid, and its brightness varied by at least 1.5 magnitudes during the observing run. It has long been suggested that Nereid may be a captured body (Cruikshank and Brown 1986, Dobrovolskis 1995). An interesting proposal about the case of Nereid is introduced by Farinella et al (1989). In addition to the previous suggestions by many authores, Farinella et al (1989) suggested that Nereid is a quasi-contact synchronized binary system, made of a couple of nearly equal mass, tidally distorted, roughly ellipsoidal components whose shapes would approximately fit equipotential surfaces. In a very recent paper by Brown et al (1998), the origin of Nereid is still mysterious.

1.3 Classification of motion of natural satellites

1.3.1 Classification based on the orbits

The natural satellites may be classified into three categories according to their orbits (Newburn and Gulkis 1973, Burns 1986).

Category 1: Regular satellites

The regular satellites move prograde in nearly circular orbits in the equatorial plane of the mother planet. This sector of satellites represents the four Galilian satellites of Jupiter, the eight classical satellites of Saturn (from Mimas to Iapetus), and all five known Uranian satellites.

Category 2: Irregular satellites

The satellites of this category are moving either prograde or retrograde, and have elongate and highly inclined orbits. Often they lie in the outer region of their parent body. Their orbits are strongly affected by the solar attraction. This is the case for (i) the outer eight satellites of Jupiter (from VI to XIII), (ii) the Saturnian satellite Phoebe, (iii) the two newly discovered satellites of Uranus S1/1997 U1 and S2/1997 U2, and (iv) the two Neptunian satellites Triton and Nereid.

Category 3: Collisional debris

The collisional shards are most always found near the mother planet and mixed with the regular satellites. It has been long thought that these debris are the remnants of the larger satellites. The coorbital pair Janus-Epimetheus, the F ring shepherds, and the Lagrangian satellites of Tethys and Dione are examples within the Saturnian system. While, Adrastea, Thebe and Metis are located in the system of Jupiter. Figure 1.3 shows that the Jovian system is the most convenient representative for the present classification.

1.3.2 Classification according to the perturbing forces

Kovalevsky and Sagnier (1977) classified the satellites according to those for which solar perturbations are most important, those for which higher order planetary terms dominate and those for which satellite interactions control. One can distinguish three classes of problems:

Class 1: Close satellites

For the nearest satellites, especially for those which revolve around very oblate planets, the gravitational potential of the planet is dominant. The theory of motion is the same to those derived for artificial satellites of the Earth. This is the case for (i) Mars' satellites Phobos and Deimos, (ii) the fifth satellite of Jupiter, (iii) Janus, the satellite of Saturn, (iv) all five satellites of Uranus, and (v) Neptune's satellite Triton.

Class 2: Satellites mainly disturbed by the Sun

In this class, the motions of the satellites are mainly governed by the solar influence whatever the planetary effects are negligible or not. This case has two subclasses.

Class 2a. The perturbing force of the Sun is not larger than 1% of the main central force. Perturbations are large, but the general behavior of the motion is elliptic. The typical case is the lunar theory. Other satellites in this class are (i) the Jupiter's satellites VI, VII, X, XIII, and (ii) the satellites of Saturn Titan, Rhea and Iapetus.

Class 2b. Solar perturbations are very strong. In this case the satellite's orbits are more elongated compared with the keplerian ellipses (Grosch 1948, Van Biesbrock 1951). These satellites are (i) the Jovian satellites VIII, IX, XI and XII, (ii) the satellite of Saturn Phoebe, and (iii) the second Neptunian satellite Nereid (see Fig. 1.4 and Fig. 1.5).

Class 3: Satellites disturbed by another satellite

In this case the disturbing function due to the satellite is superior. The theory of motion is similar to that for a planet. However, a few hundred years which are the maximum of planetary span observations must correspond to tens of thousands revolutions in satellite theory. This will cause some difficulities in the satellite theory more than the planetary motion. Such cases of class 3 can be found in: (i) the Galilian satellites (I, II, III, IV), (ii) the Saturnian pairs, Mimas-Enceladus and Tethys-Dione, and (iii) Hyperion as disturbed by Titan.

1.3.3 Classification based on the ratios $\left(\frac{\nu}{n}\right)^2$ and J_2/a^2

Another classification has been done by Kozai (1981) based on the ratios $\left(\frac{\nu}{n}\right)^2$ and J_2/a^2, where, ν and n describe the mean motions of the Sun and the satellite, J_2 and a define the second zonal harmonic of the planet and the semimajor axis of the satellite respectively. According to the values of these ratios, the satellites can be divided into three groups.

(a) Inner satellites, where the solar tidal factor $\left(\frac{\nu}{n}\right)^2$ is much smaller than the oblateness factor of the mother planet J_2/a^2. Most of the resonant satellites can be found among this group, where the mean motions are commensurable to each other.

(b) For the outer satellites group $\left(\frac{\nu}{n}\right)^2$ is much larger than J_2/a^2 and hence the solar influence is superior.

(c) Intermediary satellites, where the ratios $\left(\frac{\nu}{n}\right)^2$ and J_2/a^2 are nearly balanced.

Consequently, from the above classifications we can detect that our object study Nereid may be classified as an irregular satellite (but prograde, classification 1.3.1, category 2), affected by strong solar perturbations (classification 1.3.2, class 2b) and the ratio $\left(\frac{\nu}{n}\right)^2$ is much larger than J_2/a^2 for it (classification 1.3.3, b). A most useful review of the origin, dynamical evolution and physical properties of the natural satellites has been published by Morrison and Cruikshank (1974) and very recently by Peale (1999).

1.4 Motivation of this study

The purpose of studying the dynamics of the natural satellites is threefold

a) Evaluating the orbital elements of the satellites with high accuracy.

b) Determination of the mass and the gravitational moments of the central body from the observed orbital elements.

c) Determining the satellite masses from observation of those orbital parameters which are modified by the gravitational interactions of the individual satellites.

All these studies are almost certainly contribution of understanding the origin of the solar system. Such information on the planetary satellites will provide at least some critical clues to our understanding of the solar system, because the satellites are such diverse bodies, existing in so many different environments, and because most of them are so much less processed than their parents, the planets.

Nereid is one among a few satellites which has inaccurate ephemerides because of its bizarre orbit. This was one of the motivations which make us study the dynamics of the motion of Nereid and its physical properties. From the above discussion and illustration the case of the second Neptunian satellite Nereid (mainly section 1.2), we find that there are many arguments and different opinions about the reliability of Nereid and its orbital motion. We saw also some authors proposed that Nereid may be a binary system. We considered this another motivation to construct our analytic theory. The cornerstone in this study is to construct an efficient analytical theory on the motion of highly eccentric orbits which includes the short, intermediate and long periodic perturbations. Only we considered the solar perturbations effects, since it is the dominant for the outer satellites in general and for Nereid in particular. The theory is applied successfully on the motion of Nereid and compared with our numerical work. The perturbations produced by Triton and the oblateness of Neptune have a very small contribution regarding the solar perturbations (Mignard 1979, Veiga et al, 1996 and references therein).

Moreover, the analytical theory by Mignard (1975, 1981) of Nereid is based on the Von Zeipel's method (1916) which introduces implicit solutions. The secular and long periodic perturbations of Mignard's theory are to m^3, where m defines the ratio of the mean motions of the sun and that of the satellite, and the inclination is restricted to take small values. However, the present theory is based on Lie approach advanced by Hori (1966) which gives solutions in an explicit form. The secular and long periodic perturbations are up to the fourth order. Besides, the disturbing function is developed in the ratio of the semimajor axes of the satellie and the Sun and put in a closed form. This means that there are no any expanssions in eccentricity nor in inclination in contrast to the classical satellite theories. In the present theory, eccentricity and inclination, can take any values. All the analytical expressions of the Hamiltonian equations and the determining functions satisfy d'Alembert characteristics which assures the validity of these formulae. This will add another main advantage to the theory beside the numerical verifications.

1.5 Aim of this study

The main goals of this dissertation can be outlined in the following items

- Constructing an analytical theory on the motion of a satellite with highly eccentric orbit, taking into account the solar attractions.

- Evaluating the osculating orbital elements and ephemerides for Nereid's motion.

- To check the reliability and accuracy of the suggested analytical model, a comparison with the numerical integration of the equations of motion is considered.

- put all the analytical expressions in a way such that they could be applied and utilized easily on a disk digital computers and obtain numerical values.

By these we aimed to establish a symbolic computing package for the perturbations theory of large eccentric orbits in general, and for Nereid in particular. We provide to the observers a computational tool, capable of generating ephemerides for predictions, easy to handle.

Figure Captions

Fig. 1.1. The small circle shows the orbit of Triton, while the dot at its center indicates Neptune. For comparison, the dotted circle represents the semi-major axis s of Nereid's orbit, while the dashed ellipse displays its actual shape. Points were plotted at each whole degree of mean anomaly, and connected in alternating pairs. Since Nereid's orbital period is 360 days, each dash (or gap between dashes) is equivalent to a time interval of one day. Note that Nereid spends a fraction $1/2 - e/\pi \sim 26\%$ of its time ($\sim$ 94 days per orbit) inside the dotted circle (Dobrovolskis 1995).

Fig. 1.2. The Voyager 2 path through the Neptune system is shown in the plane of the spacecraft trajectory. **(A)** the projected orbits of Triton and Nereid are shown, together with the positions of these satellites at the time of Voyager 2's closest approach to Neptune. Tick marks along the trajectory indicate Voyager's position at 1-day intervals. **(B)** An enlarged view covers a 10-hour period that includes closest approaches to Neptune and Triton and passage through Earth and solar shadows (occultation zones) of each. Tick marks along the trajectory are at 1-hour intervals (Stone and Miner 1989).

Fig. 1.3. A sketch of the Jovian system which best illustrates the orbital character of regular satellites, irregular satellites and collisional debris. Orbits are positioned according

to inclination; Orbital eccentricities are indicated by showing apocenter and pericenter distances. Orbital radii for the outer satellites are plotted at 25% of the scale used for the other satellites (Burns 1986).

Fig. 1.4. The orbit of the second Neptunian satellite Nereid as given by Biesbroeck (1951).

Fig. 1.5. The orbit of the eighth satellite of Jupiter pasiphae, upper diagram, projection upon xz plane, lower diagram, projection upon xy plane (Grosch 1948).

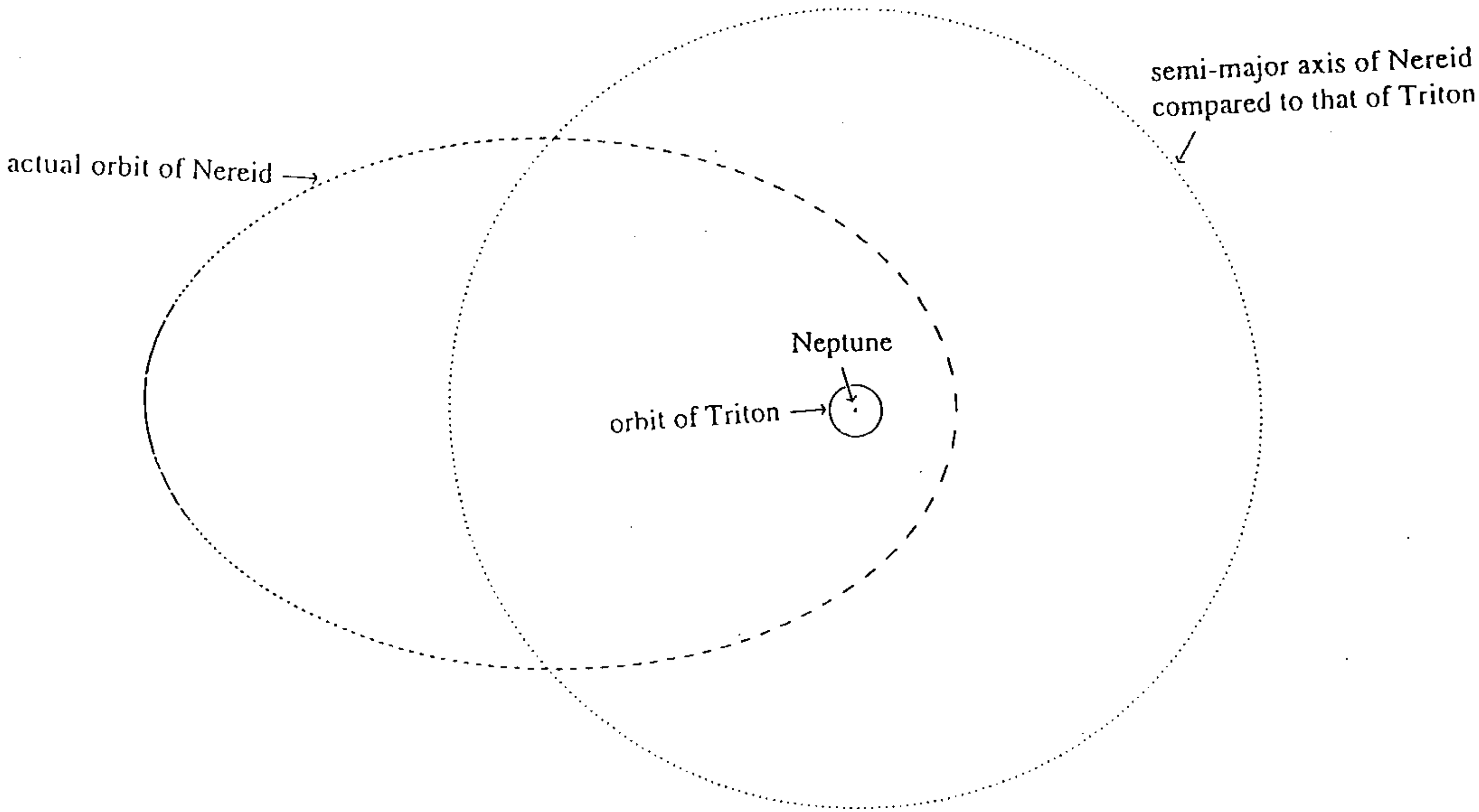

semi-major axis of Nereid
compared to that of Triton
actual orbit of Nereid
Neptune
orbit of Triton

Fig. 1.1

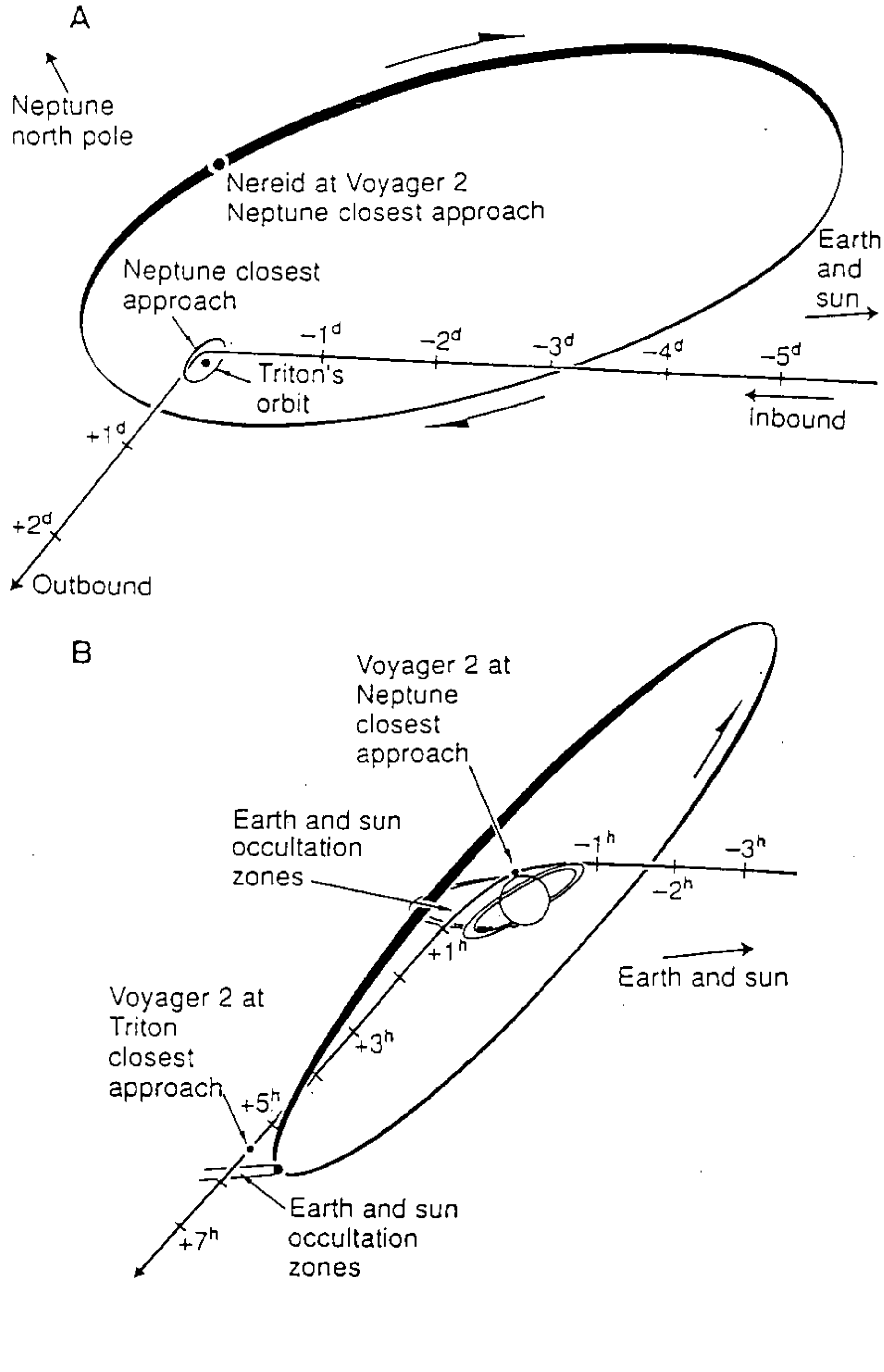
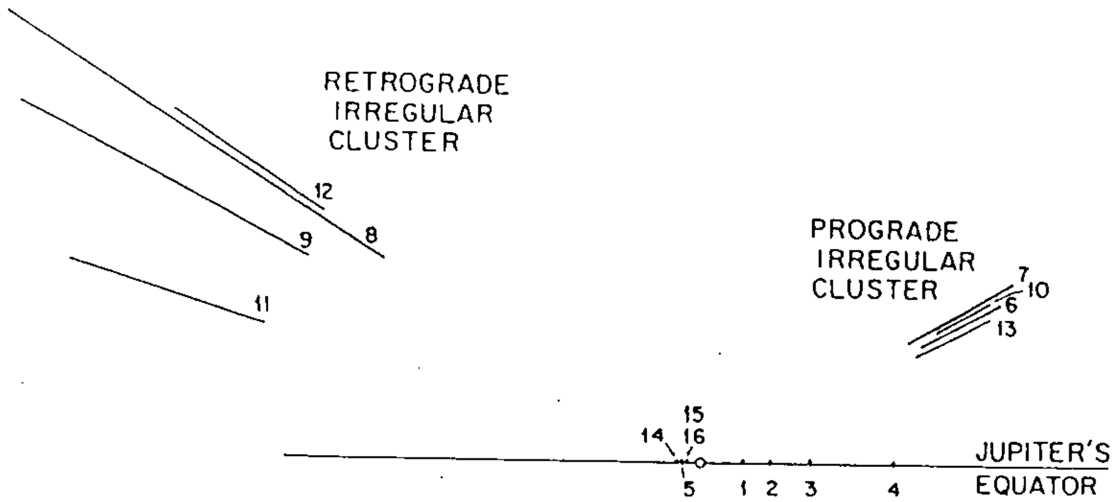

A
Neptune
north pole
Nereid at Voyager 2
Neptune closest approach
Earth
and
sun
Neptune closest
approach
-1^d
-2^d
-3^d
-4^d
-5^d
Triton's
orbit
Inbound
$+1^d$
$+2^d$
Outbound
B
Voyager 2 at
Neptune
closest
approach
Earth and sun
occultation
zones
-1^h
-3^h
-2^h
$+1^h$
Earth and sun
Voyager 2 at
Triton
closest
approach
$+3^h$
$+5^h$
Earth and sun
occultation
zones
$+7^h$

Fig. 1.2



JUPITER'S SATELLITES

Fig. 1.3

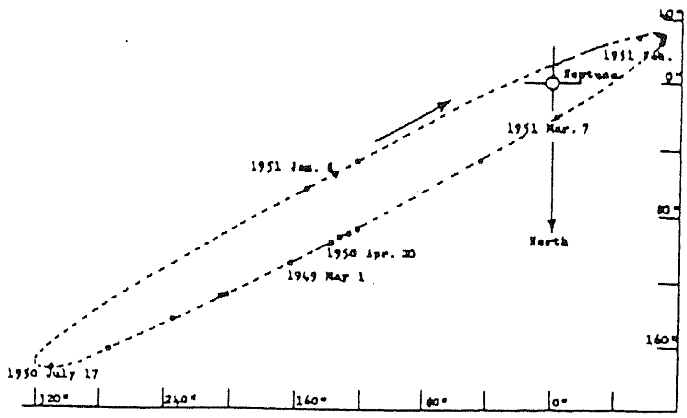

1951 Feb.
Neptune
1951 Mar. 7
1951 Jan. 6
Earth
1950 Apr. 20
1949 Mar 1
1950 July 17
320°
240°
160°
80°
0°
40°
0°
80°
160°

Fig. 1.4

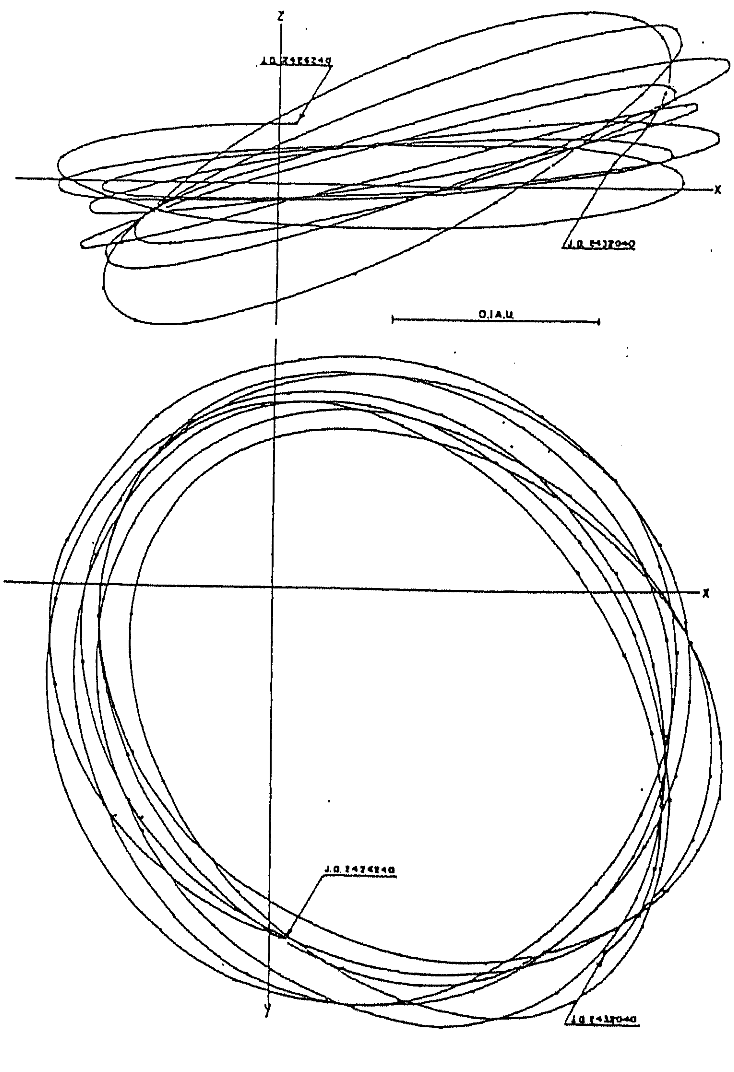

Z
X
0.1 A.U.
X
Y

Fig. 1.5

Chapter 2

Methodology

In the previous chapter we gave a general introduction to the present thesis. In this chapter, we introduce foundations and tools which are necessary for accomplishing this work. Equations of motion and disturbing functions will be given in detail. Procedures for obtaining the osculating elements and ephemerids of Nereid are discussed. Each procedure we used is implemented for digital computations by constructing a computational algorithm to be easy to handle.

2.1 Canonical methods in celestial mechanics

2.1.1 Canonical transformation

The aim of the transformations is to introduce changes of the variables to convert the system of differential equations into a simpler form. We adopt the canonical approach in dealing with our problem, because it provides more understanding of the given problem and it has the possibility of establishing general rules governing transformations from one set of variables to another set. Under these rules the canonical form of the equations is preserved. Briefly, it aimed to eliminate the short-periodic and long-periodic terms from the Hamiltonian system, and the remaining Hamiltonian after the last stage determines the secular perturbations.

2.1.2 Merits and demerits of the canonical methods

As far as we know there are three known canonical methods in celestial mechanics which are used for describing the motion of the celestial objects: Von Zeipel's (1916), Hori's (1966) and Deprit's (1969) method. Von Zeipel's method is based on the principle of separating fast and slow changing variables and it is designated for the canonical systems of differential equations. It reduces to the determination of the two scalar functions: (i) the generating functions and (ii) the new Hamiltonian. However, the determining function depends on the new momenta and the old coordinates. Therefore, the theory will be developed in terms of these mixed variables and this is considered a drawback in the theory (Hori 1966, Brumberg 1995).

On the other hand, Hori's and Deprit's method are elaborated by what is called Poisson brackets which have several advantages over the usual Von Zeipel's approach. The determining function is not a mixed function of old and new coordinates. The two theories are canonically invariant (because this is the merit of Poisson brackets) and give a direct expression of any function of old variables in terms of the new variables. However, Deprit's equations in particular, involve extra terms containing partial derivatives with respect to the small parameter ϵ and thus have greater complexity (Campbell and Jefferys 1970). By an explicit calculation, Campbell and Jefferys (1970) proved that the two theories (of Hori and Deprit) are equivalent through the sixth order in ϵ and should be equivalent to all orders.

Avoiding the implicit relations in Zeipel's approach and the complexity in Deprit's procedure, we adopted Hori's device in constructing the present analytical theory.

2.1.3 Hori's perturbation method

In this section we introduce briefly Hori's perturbation method(1966). The method is mainly based on a theorem by Lie (a convergence power series) and an auxiliary equation for the unperturbed motion in order to get the new Hamiltonian and determining function.

Theorem:

A set of $2n$ variables x_j and y_j defined by the equation

$$f(x,y) = \sum_{n=0}^{\infty} D_S^n f(x',y'), \tag{2.1}$$

is canonical if the series in the right-hand side of the above equation converges, where ϵ is a small parameter independent of the new variables x' and y', the operators D_S^n are defined by

$D_S^0 f = f$, $D_S^1 f = \{f, S\}$, ..., $D_S^n f = D_S^{n-1}(D_S^1 f)$, $n \geq 2$,

the braces stand for Poisson brackets and f and S are arbitrary functions of x' and y'.

To solve such a dynamical system Equations

$$\frac{dx_j}{dt} = \frac{\partial F}{\partial y_j}, \; \frac{dy_j}{dt} = -\frac{\partial F}{\partial x_j}, \tag{2.2}$$

with the Hamiltonian

$$F(x,y) = F_0(x,y) + \sum_{k=1} F_k(x,y), \tag{2.3}$$

and F_k has a factor ϵ^k and independent of time, we consider the canonical transformation

$$x_j, y_j \longrightarrow x_j^{'}, y_j^{'}, j = 1, 2, ...n. \tag{2.4}$$

Then the perturbations of any quantity x_j and y_j can be given by the convergence series

$$x_j = x_j^{'} + \left\{x_j^{'}, S\right\} + \frac{1}{2!}\left\{\left\{x_j^{'}, S\right\}, S\right\} + \frac{1}{3!}\left\{\left\{\left\{x_j^{'}, S\right\}, S\right\}, S\right\} + ... \tag{2.5}$$

$$y_j = y_j^{'} - \left\{y_j^{'}, S\right\} - \frac{1}{2!}\left\{\left\{y_j^{'}, S\right\}, S\right\} - \frac{1}{3!}\left\{\left\{\left\{y_j^{'}, S\right\}, S\right\}, S\right\} - ... \tag{2.6}$$

where,

$$S(x_j^{'}, y_j^{'}) = \sum_{k=1} S_k(x_j^{'}, y_j^{'}), \tag{2.7}$$

is the determining function of the new variables and S_k has a factor ϵ^k. Since the Hamiltonian F is free from the time, we have the energy integral:

$$\sum_{k=0} F_k(x,y) = \sum_{k=0} F_k^*(x', y'). \tag{2.8}$$

where F^* is the new Hamiltonian.

In order to obtain the determining function S_k and the new Hamiltonian F_k^*, a new parameter t^* is introduced to get what is called an auxiliary equation

$$\frac{dx_j^{'}}{dt^*} = \frac{\partial F_0}{\partial y_j^{'}}, \; \frac{dy_j^{'}}{dt^*} = -\frac{\partial F_0}{\partial x_j^{'}}. \tag{2.9}$$

Then we have

$$\{F_0, S_k\} = -\frac{dS_k}{dt^*}. \tag{2.10}$$

The averaging with respect to the parameter t^* eliminates this parameter from the new Hamiltonian F^*. The new equations of motion are given by

$$\frac{dx_j'}{dt} = \frac{\partial F^*}{\partial y_j'}, \frac{dy_j'}{dt} = -\frac{\partial F^*}{\partial x_j'}, \tag{2.11}$$

which have a first integral

$$F_0^*(x', y') = \text{const.}, \tag{2.12}$$

in addition to the energy integral

$$F^*(x', y') = \text{const.} \tag{2.13}$$

Apply the expansion formula (2.1) to the left-hand side of equation (2.8) and equate the terms of equal powers of ϵ in both sides. Then use the results with the help of equation (2.10) to get the following algorithm of the new Hamiltonian and determining function.

Zeroth-order

$$F_0^* = F_0, \tag{2.14}$$

First-order

$$F_1^* = F_{1s}, \tag{2.15}$$

$$S_1 = \int F_{1p} dt^*, \tag{2.16}$$

Second-order

$$F_2^* = F_{2s} + \frac{1}{2}\{F_1 + F_1^*, S_1\}_s, \tag{2.17}$$

$$S_2 = \int \left(F_{2p} + \frac{1}{2}\{F_1 + F_1^*, S_1\}_p\right) dt^*, \tag{2.18}$$

Third-order

$$F_3^* = F_{3s} + \frac{1}{12}\left\{\left\{F_{1p}, S_1\right\}, S_1\right\}_s + \frac{1}{2}\left\{F_2 + F_2^*, S_1\right\}_s + \frac{1}{2}\left\{F_1 + F_1^*, S_2\right\}_s, \tag{2.19}$$

$$S_3 = \int \left(F_{3p} + \frac{1}{12}\left\{\left\{F_{1p}, S_1\right\}, S_1\right\}_p + \frac{1}{2}\left\{F_2 + F_2^*, S_1\right\}_p + \frac{1}{2}\left\{F_1 + F_1^*, S_2\right\}_p\right) dt^*. \tag{2.20}$$

and so on.

The subscripts s and p stand for the secular and periodic parts respectively. This process can be repeated to another set of variables x_j'', y_j'' with introducing a new parameter t^{**}. The new determining function S_k^* and the new Hamiltonian F_k^{**} can be obtained using the algorithm

Zeroth-order

$$F_0^{**} = F_0^*, \tag{2.21}$$

First-order

$$F_1^{**} = F_1^*, \tag{2.22}$$

Second-order

$$F_2^{**} = F_{2s}^*, \tag{2.23}$$

$$S_1^* = \int F_{2p}^* dt^{**}, \tag{2.24}$$

Third-order

$$F_3^{**} = F_{3s}^* + \frac{1}{2}\left\{F_2^* + F_2^{**}, S_1^*\right\}_s, \tag{2.25}$$

$$S_2^* = \int \left(F_{3p}^* + \frac{1}{2}\left\{F_2^* + F_2^{**}, S_1^*\right\}_p\right) dt^{**}, \tag{2.26}$$

Fourth-order

$$F_4^{**} = F_{4s}^* + \frac{1}{12}\left\{\left\{F_{2p}^*, S_1^*\right\}, S_1^*\right\}_s + \frac{1}{2}\left\{F_3^* + F_3^{**}, S_1^*\right\}_s + \frac{1}{2}\left\{F_2^* + F_2^{**}, S_1^*\right\}_s, \tag{2.27}$$

$$S_3^* = \int \left(F_{4p}^* + \frac{1}{12}\left\{\left\{F_{2p}^*, S_1^*\right\}, S_1^*\right\}_p + \frac{1}{2}\left\{F_3^* + F_3^{**}, S_1^*\right\}_p + \frac{1}{2}\left\{F_2^* + F_2^{**}, S_1^*\right\}_p\right) dt^{**}. \quad (2.28)$$

and so on.

Eliminating the parameters t^* and t^{**} from the new Hamiltonian equivalent to eliminating the short and long periodic terms respectively. The remaining Hamiltonian describes the secular perturbations and the new equations of motion will be given by

$$\frac{dx_j''}{dt} = \frac{\partial F^{**}}{\partial y_j''}, \; \frac{dy_j''}{dt} = -\frac{\partial F^{**}}{\partial x_j''}, \quad (2.29)$$

these equations lead to two first integrals

$$F_1^{**}(x'', y'') = \text{const.}, \quad (2.30)$$

$$F_0^{**}(x'', y'') = \text{const.} \quad (2.31)$$

in addition to the energy integral

$$F^{**}(x'', y'') = \text{const.} \quad (2.32)$$

As we already mentioned these procedures can be repeated as many times as necessary.

2.2 Equations of motion and disturbing function

We consider the motion of Nereid around Neptune under the perturbations of the Sun, which is moving in a Keplerian orbit. The origin of coordinates is located at Neptune (Fig. 2.1). The equations of motion of the satellite are given in Delaunay's elements as follows

$$\frac{d(L, G, H, K)}{dt} = \frac{\partial F}{\partial (l, g, h, k)}, \quad (2.33)$$

$$\frac{d(l,g,h,k)}{dt} = -\frac{\partial F}{\partial(L,G,H,K)}, \tag{2.34}$$

with the Hamiltonian

$$F = \frac{\mu^2}{2L^2} - \nu K + R, \tag{2.35}$$

where $\mu = n^2a^3$ (n and a describe the mean motion and the semimajor-axis of the satellite respectively), ν is the mean motion of the sun, k (the mean longitude of the sun) is given by

$$k = \nu t + \text{const.}, \tag{2.36}$$

K is a conjugate momentum of k which is introduced for the Hamiltonian to be independent of time (Brouwer and Clemence, 1961), and R is the disturbing function due to the sun

$$R = R_1 + R_2 + ... \tag{2.37}$$

Delaunay's elements are given by

$L = \sqrt{\mu a}$, l=mean anomaly of the satellite,

$G = L\sqrt{1-e^2}$, g=argument of pericenter of the satellite,

$H = G\cos(i)$, h=longitude of ascending node of the satellite,

i and e define the inclination and eccentricity of the satellite respectively.

Since we deal with the motion of a celestial object with highly eccentric orbit, the conventional techniques in celestial mecanics can not be applied here. So, the disturbing function has to be developed in a power series of the ratio of the semimajor axes of the disturbed body and the Sun (Kozai 1962). However, the convergence of the power series may be slow for some of the outer satellites, for example, the outer Jovian satellites have ratios m between 0.145 and 0.175 (Saha and Tremaine 1993, Solovaya 1995). The case of Nereid represents advantage, the ratio $m \sim 0.006$, and the power series of the perturbing function is convergent. Neglecting the mass of Nereid compared with the Sun, the disturbing function has the form

$$R = k^2 M_\odot \left\{ \frac{1}{\Delta} - \frac{xx' + yy' + zz'}{r'^3} \right\} \tag{2.38}$$

where $M_\odot$ is the mass of the Sun, k^2 is the gravitational constant, r and r' describe the radius vectors Neptune-Nereid and Neptune-Sun respectively and Δ is the distance of the Sun from Nereid

$$\begin{aligned} \Delta^2 &= (x - x')^2 + (y - y')^2 + (z - z')^2 \\ &= r^2 + r'^2 - 2rr' \cos S, \end{aligned} \tag{2.39}$$

S is the angle at Neptune between r and r'

$$\cos S = \frac{xx' + yy' + zz'}{rr'}, \tag{2.40}$$

x, y, z and x', y', z' are the rectangular coordinates of the satellite and the Sun respectively. Let the orbit of Neptune around the Sun is fixed in a plane, xy, so that $z' = 0$.

$$\frac{1}{\Delta} = \frac{1}{r'} \sum_{n=0}^{\infty} \left(\frac{r}{r'}\right)^n P_n(\cos S) \tag{2.41}$$

using equations (2.40) and (2.41) in equation (2.38) we can get

$$R = \frac{\mu}{r'} + \mu \left[\frac{r}{r'^2} P_1(\cos S) + \frac{r^2}{r'^3} P_2(\cos S) + \frac{r^3}{r'^4} P_3(\cos S) + ... - \frac{rr'}{r'^3} \cos S \right] \tag{2.42}$$

Since r' does not depend on the coordinates of Nereid, the first term μ/r' will be neglected, while the second and the last terms cancel with each other because of the Legendre polynomial property $P_1(\cos S) = \cos S$. Then the disturbing function is developed in the ratio of the semimajor axes of the satellite and Sun by the following series

$$R = \nu^2 a^2 \left\{ \left(\frac{r}{a}\right)^2 \left(\frac{a'}{r'}\right)^3 \left(\frac{a}{a'}\right)^0 P_2(\cos S) + \left(\frac{r}{a}\right)^3 \left(\frac{a'}{r'}\right)^4 \left(\frac{a}{a'}\right)^1 P_3(\cos S) \right.$$

$$+\left(\frac{r}{a}\right)^4\left(\frac{a'}{r'}\right)^5\left(\frac{a}{a'}\right)^2 P_4(\cos S)+...\Bigg\}, \tag{2.43}$$

where the primes refer to the Sun. In the present problem, the Hamiltonian of the system, is truncated after the second order term.

Now, we develop the disturbing function in terms of the elliptic elements, by using the cosine formula of the sperical trigonometry

$$\cos S = \cos(\omega + f)\cos(\omega' + f') + \sin(\omega + f)\sin(\omega' + f')\cos i, \tag{2.44}$$

where, ω, ω',f and f' are the argument of pericenter and the true anomaly of Nereid and the Sun respectively. With some mathematical operations using the trigonometric relations, we can easily obtain

$$\cos S = \cos^2\frac{i}{2}\cos(\omega + f - \omega' - f') + \sin^2\frac{i}{2}\cos(\omega + f + \omega' + f'), \tag{2.45}$$

$$\begin{aligned}\cos^2 S &= \frac{1}{4}(\cos^2 i + 1) + \frac{1}{2}\cos^4\frac{i}{2}\cos(2\omega + 2f - 2\omega' - 2f') + \frac{1}{2}\sin^4\frac{i}{2} * \\ &\cos(2\omega + 2f + 2\omega' + 2f') + \frac{1}{4}\sin^2 i\left[\cos(2\omega + 2f) + \cos(2\omega' + 2f')\right],\end{aligned} \tag{2.46}$$

consequently,

$$\begin{aligned}P_2(\cos S) &= \frac{1}{8}(3\cos^2 i - 1) + \frac{3}{4}\cos^4\frac{i}{2}\cos(2\omega + 2f - 2\omega' - 2f') \\ &+\frac{3}{4}\sin^4\frac{i}{2}\cos(2\omega + 2f + 2\omega' + 2f') \\ &+\frac{3}{8}\sin^2 i\left[\cos(2\omega + 2f) + \cos(2\omega' + 2f')\right],\end{aligned} \tag{2.47}$$

$$\begin{aligned}P_3(\cos S) &= \left(-\frac{3}{2}\sin^2\frac{i}{2} + \frac{15}{4}\cos^4\frac{i}{2}\sin^2\frac{i}{2} + \frac{15}{8}\sin^6\frac{i}{2}\right)\cos(\omega + f + \omega' + f') \\ &+\left(-\frac{3}{2}\cos^2\frac{i}{2} + \frac{15}{4}\cos^2\frac{i}{2}\sin^4\frac{i}{2} + \frac{15}{8}\cos^6\frac{i}{2}\right)\cos(\omega + f - \omega' - f')\end{aligned}$$

$$+\frac{5}{8}\sin^6\frac{i}{2}\cos(3\omega+3f+3\omega'+3f')+\frac{5}{8}\cos^6\frac{i}{2}*$$
$$\cos(3\omega+3f-3\omega'-3f')+\frac{15}{8}\cos^2\frac{i}{2}\sin^4\frac{i}{2}\left[\cos(\omega+f+3\omega'+3f')\right.$$
$$+\cos(3\omega+3f+\omega'+f')]+\frac{15}{8}\cos^4\frac{i}{2}\sin^2\frac{i}{2}\left[\cos(\omega+f-3\omega'-3f')\right.$$
$$\left.+\cos(3\omega+3f-\omega'-f')\right]. \tag{2.48}$$

When we consider the effects of P_2 only, the disturbing function R_1 can be expressed in Delaunay's variables as follows

$$\begin{aligned} R_1 &= \nu^2a^2\left(\frac{r}{a}\right)^2\left(\frac{a'}{r'}\right)^3\left\{\frac{1}{8}\left(3\frac{H^2}{G^2}-1\right)+\frac{3}{16}\left(1+\frac{H}{G}\right)^2\cos(2f-2f'+2g+2h)\right. \\ &\quad+\frac{3}{8}\left(1-\frac{H^2}{G^2}\right)[\cos(2f+2g)+\cos(2f'-2h)] \\ &\quad\left.+\frac{3}{16}\left(1-\frac{H}{G}\right)^2\cos(2f+2f'+2g-2h)\right\}, \end{aligned} \tag{2.49}$$

where, $H/G \equiv \cos i$ and $h \equiv -\omega'$ describes the longitude of ascending node of the satellite minus the longitude of perigee of the Sun, the latter being a constant under the assumption of a keplerian motion for the Sun.

2.3 Osculating and mean elements

2.3.1 Procedure for obtaining the osculating elements

Definition

The osculating orbital elements are a combination of secular, long-periodic and short-periodic variations in elements. They are also called instantaneous elements. General speaking , the six orbital elements at any given instant can be computed from the six data for the coordinates and the momenta at that instant. If the actual motion is rigorously Keplerian, then the six orbital elements thus computed are all constant. If it is not Keplerian, then the values of these orbital elements vary from time to time. Such orbital elements are called *osculating elements.*

To reach our goal we applied a succession of canonical transformations on the Hamiltonian equations. At first we intended eliminating the short-periodic term by using the transformation

$$(L, G, H; l, g, h, \lambda_\odot) \longrightarrow (L^*, G^*, H^*; l^*, g^*, h^*, \lambda_\odot), \tag{2.50}$$

where, $\lambda_\odot$ defines the longitude of the Sun. In the present study, the short-period is 360 days, the orbital revolution of Nereid around Neptune. The new Hamiltonian F^* will be free from the mean anomaly as follows

$$F(L, G, H; l, g, h, \lambda_\odot) \longrightarrow F^*(L^*, G^*, H^*; -, g^*, h^*, \lambda_\odot), \tag{2.51}$$

and the set of elements can be given by

$$\left.\begin{aligned}
L &= L^* + F_L^*(L^*, G^*, H^*; -, g^*, h^*, \lambda_\odot), \\
G &= G^* + F_G^*(L^*, G^*, H^*; -, g^*, h^*, \lambda_\odot), \\
H &= H^* + F_H^*(L^*, G^*, H^*; -, g^*, h^*, \lambda_\odot), \\
l &= l^* - F_l^*(L^*, G^*, H^*; -, g^*, h^*, \lambda_\odot), \\
g &= g^* - F_g^*(L^*, G^*, H^*; -, g^*, h^*, \lambda_\odot), \\
h &= h^* - F_h^*(L^*, G^*, H^*; -, g^*, h^*, \lambda_\odot).
\end{aligned}\right\} \tag{2.52}$$

The second step is to eliminate the intermediate terms related the motion of the Sun $\lambda_\odot$, this requires the transformation

$$(L^*, G^*, H^*; l^*, g^*, h^*, \lambda_\odot) \longrightarrow (L^{**}, G^{**}, H^{**}; l^{**}, g^{**}, h^{**}, \lambda_\odot), \tag{2.53}$$

and the new Hamiltonian F^{**} will be free form l, $\lambda_\odot$ and h (since the disturbing potential becomes axial symmetric) as follows

$$F^*(L^*, G^*, H^*; -, g^*, h^*, \lambda_\odot) \longrightarrow F^{**}(L^{**}, G^{**}, H^{**}; -, g^{**}, -, -), \tag{2.54}$$

and the set of elements can be expressed in the new form

$$\left.\begin{aligned}
L^* &= L^{**} + F_L^{**}(L^{**}, G^{**}, H^{**}; -, g^{**}, -, -), \\
G^* &= G^{**} + F_G^{**}(L^{**}, G^{**}, H^{**}; -, g^{**}, -, -), \\
H^* &= H^{**} + F_H^{**}(L^{**}, G^{**}, H^{**}; -, g^{**}, -, -), \\
l^* &= l^{**} - F_l^{**}(L^{**}, G^{**}, H^{**}; -, g^{**}, -, -), \\
g^* &= g^{**} - F_g^{**}(L^{**}, G^{**}, H^{**}; -, g^{**}, -, -), \\
h^* &= h^{**} - F_h^{**}(L^{**}, G^{**}, H^{**}; -, g^{**}, -, -),
\end{aligned}\right\} \tag{2.55}$$

The intermediate period $\lambda_\odot$ is about 165 years which describe the orbital revolution of Neptune around the Sun. Since the Hamiltonian does not include l and h, then the angular momenta L and H are constants and the final equations system is

$$\left.\begin{aligned}
\frac{dG^{**}}{dt} = \frac{\partial F^{**}}{\partial g^{**}}, \frac{dg^{**}}{dt} = -\frac{\partial F^{**}}{\partial G^{**}}, \\
\frac{dl^{**}}{dt} = -\frac{\partial F^{**}}{\partial L^{**}}, \frac{dh^{**}}{dt} = -\frac{\partial F^{**}}{\partial H^{**}}.
\end{aligned}\right\} \tag{2.56}$$

In order to eliminate the angular variable g ($g \sim 13000$ years), we have two choices, the first one is to apply another transformation

$$(L^{**}, G^{**}, H^{**}; l^{**}, g^{**}, h^{**}, \lambda_\odot) \longrightarrow (L^{***}, G^{***}, H^{***}; l^{***}, g^{***}, h^{***}, \lambda_\odot) \tag{2.57}$$

hence,

$$F^{**}(L^{**}, G^*, H^{**}; -, g^{**}, -, -) \longrightarrow F^{***}(L^{***}, G^{***}, H^{***}; -, -, -, -) \tag{2.58}$$

and the final results give what are called the *mean elements*

$$\left.\begin{aligned}
L^{**} &= L^{***} + F_L^{***}\left(L^{**}, G^{**}, H^{***}; -, -, -, -\right), \\
G^{**} &= G^{***} + F_G^{***}\left(L^{***}, G^{***}, H^{***}; -, -, -, -\right), \\
H^{**} &= H^{***} + F_H^{***}\left(L^{***}, G^{***}, H^{***}; -, -, -, -\right), \\
l^{**} &= l^{***} - F_l^{***}\left(L^{***}, G^{***}, H^{***}; -, -, -, -\right), \\
g^{**} &= g^{***} - F_g^{***}\left(L^{***}, G^{***}, H^{***}; -, -, -, -\right), \\
h^{**} &= h^{***} - F_h^{**}\left(L^{***}, G^{***}, H^{***}; -, -, -, -\right),
\end{aligned}\right\} \tag{2.59}$$

The second method (which we have actually used) for eliminating the element g is to solve the system of differential equations (2.56). Kinoshita and Nakai (1999) gave an analytical solution of the eccentricity, inclination and the argument of pericenter using the Jacobian elliptic functions, while the mean anomaly and the longitude of ascending node are expressed in Fourier series expansion. In chapter four, we will introduce these solutions with some details.

In order to get the osculating orbital elements, and hence, evaluate ephemerides for Nereid, we have started with the mean elements. We substituted reversely, from equations (2.59) in equations (2.55), then in equations (2.52). The set of equations (2.52) represents the short periodic variations of the orbital elements, equations (2.55) give the intermediate terms, while equations (2.59) give the mean elements.

2.3.2 Computational Algorithm

In what follows, the implementation of the above symbolic formulations for digital computations will be given through the following algorithm described by its purpose, input and its computational sequence:

- *Purpose*: To compute a_{osc}, e_{osc}, I_{osc}, ω_{osc}, Ω_{osc} and l_{osc} the osculating orbital elements of Nereid moving around Neptune and perturbed by the solar attraction at any time.

- *Input*: the initial values a_0, e_0, I_0, Ω_0, l_0, t_0, t_{end}, and Tol (specified tolerance).

- Units measurements: Masses are given in solar unit, distances are in AU, time in dayes while the angles are given in radian.

- *Computational Sequence*:

(1) Compute the mean elements e_{mean}, I_{mean}, ω_{mean} (equations (2.59)) by solving the system of differential equations (2.56) using Jacobian elliptic functions.

(2) Compute the mean elements Ω_{mean} and l_{mean} (equations (2.59)) using Fouries series expansion.

(3) Evaluate the analytical expressions for the short-periodic variations of the orbital elements (equations (2.52)) with the usage of convenient software packages (e.g. Mathematica).

(4) Evaluate the analytical expressions for the long-periodic variations of the orbital elements (equations (2.55)) using any convenient software packages (e.g. Mathematica).

(5) Transform all the analytical formulae derived in steps (3) and (4) into a Fortran format to be ready in the usage of any Fortran Code.

(6) Compute the long-periodic terms e_{long}, I_{long}, ω_{long}, Ω_{long} and l_{long} in step (4) by inserting both step (1) and step (2) as follows

(a) The variations δe_{long}, δI_{long}, $\delta\omega_{long}$, $\delta\Omega_{long}$ and δl_{long}

$$\left.\begin{aligned}
\delta e_{long} &= \delta_1 e_{long} + \delta_2 e_{long} + \delta_3 e_{long},\\
\delta I_{long} &= \delta_1 I_{long} + \delta_2 I_{long} + \delta_3 I_{long},\\
\delta\omega_{long} &= \delta_1 \omega_{long} + \delta_2 \omega_{long} + \delta_3 \omega_{long},\\
\delta\Omega_{long} &= \delta_1 \Omega_{long} + \delta_2 \Omega_{long} + \delta_3 \Omega_{long},\\
\delta l_{long} &= \delta_1 l_{long} + \delta_2 l_{long} + \delta_3 l_{long},
\end{aligned}\right\} \tag{2.60}$$

(b) The long-periodic terms

Call mean elements

$$\left.\begin{aligned}
e_{long} &= e_{mean} + \delta e_{long},\\
I_{long} &= I_{mean} + \delta I_{long},\\
\omega_{long} &= \omega_{mean} + \delta\omega_{long},\\
\Omega_{long} &= \Omega_{mean} + \delta\Omega_{long},\\
l_{long} &= l_{mean} + \delta l_{long}.
\end{aligned}\right\} \tag{2.61}$$

where δ_1, δ_2 and δ_3 refer to the first, second and third order long-periodic perturbations of the orbital elements respectively.

(7) Compute the short-periodic variations Δa_{sho}, Δe_{sho}, ΔI_{sho}, $\Delta\omega_{sho}$, $\Delta\Omega_{sho}$ and Δl_{sho} in step (3), depending on the results of step (6) and the solution of Kepler equation in eccentric anomaly

Call Kepler

$$\left.\begin{aligned}
\Delta a_{sho} &= \Delta_1 a_{sho} + \Delta_2 a_{sho} + \Delta_3 a_{sho},\\
\Delta e_{sho} &= \Delta_1 e_{sho} + \Delta_2 e_{sho} + \Delta_3 e_{sho},\\
\Delta I_{sho} &= \Delta_1 I_{sho} + \Delta_2 I_{sho} + \Delta_3 I_{sho},\\
\Delta \omega_{sho} &= \Delta_1 \omega_{sho} + \Delta_2 \omega_{sho} + \Delta_3 \omega_{sho},\\
\Delta \Omega_{sho} &= \Delta_1 \Omega_{sho} + \Delta_2 \Omega_{sho} + \Delta_3 \Omega_{sho},\\
\Delta l_{sho} &= \Delta_1 l_{sho} + \Delta_2 l_{sho} + \Delta_3 l_{sho},
\end{aligned}\right\} \tag{2.62}$$

where Δ_1, Δ_2 and Δ_3 describe the first, second and third order short-periodic perturbations in the orbital elements respectively.

(8) Calculate the osculating orbital elements a_{osc}, e_{osc}, I_{osc}, ω_{osc}, Ω_{osc} and l_{osc} according to steps (6) and (7) as follows

$$\left.\begin{aligned}
a_{osc} &= a_0 + \Delta a_{sho},\\
e_{osc} &= e_{long} + \Delta e_{sho},\\
I_{osc} &= I_{long} + \delta I_{sho},\\
\omega_{osc} &= \omega_{long} + \delta \omega_{sho},\\
\Omega_{osc} &= \Omega_{long} + \delta \Omega_{sho},\\
l_{osc} &= l_{long} + \delta l_{sho}.
\end{aligned}\right\} \tag{2.63}$$

(9) The algorithm is completed.

The above algorithm including the analytical expressions has been implemented in FORTRAN CODE. The program permits the calculation of all the secular, short-periodic and long-periodic perturbations, then the osculating elements at any time. One may raise a question, can we get the inverse solution by this theory (mean elements)?. According to the concept of the osculating elements in this Section we can simply say that, the mean elements are osculating elements from which the short-periodic and long-periodic perturbations have been omitted. In fact Hori's perturbations method has the merit of getting the reverse solution by changing the sign of the determining function S to $-S$. However doing this using Von Zeipel's method is complicated since the determining function includes mixed variables.

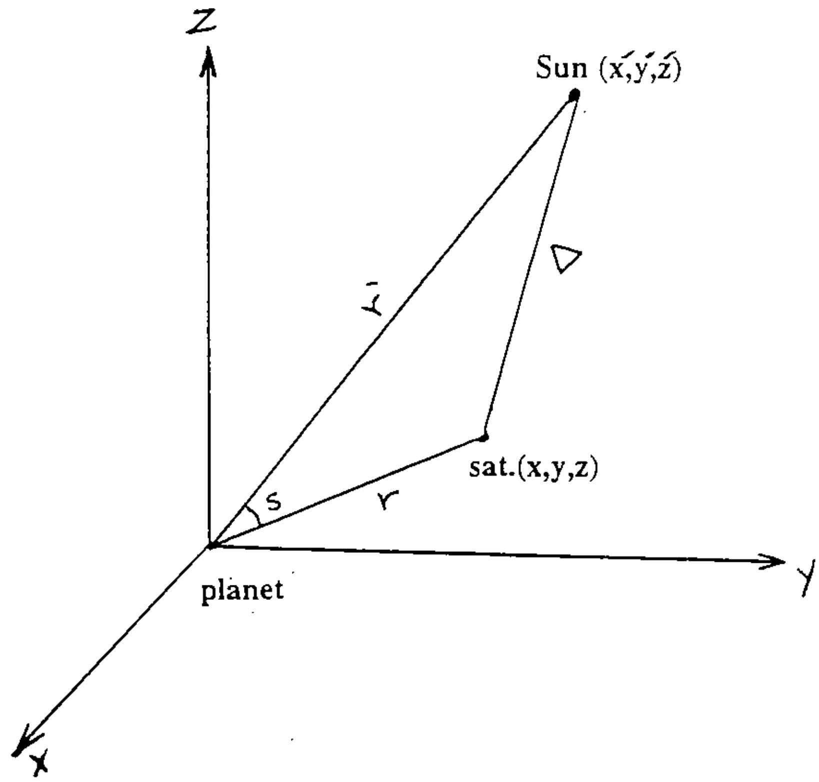


Fig. 2.1. Geometry of the problem of satellite motion

Chapter 3

Planar Motion Solution

In chapter 2 the methods of solution are illustrated. In this chapter, we deal with the motion of Nereid in the frame work of circular planar restricted three-body problem. Although this case of study is constrained to fictitious objects (where inclination equals zero), it is very useful in checking the validity of the non-planar case. The results of comparison between analytical and numerical work are tabulated and shown by figures.

3.1 Introduction

Many instances of high eccentric orbits occur in the solar system, especially among some of minor planets. The second Neptunian satellite Nereid has the most eccentric orbit in all the solar system. Its high eccentricity, renders the usage of the classical analytical methods for expanding the disturbing function in terms of the eccentricity, that is because of the slow convergence of the power series solutions especially at higher orders. This motivated us to construct an efficient analytical theory which can predict the motion of the large eccentric orbits with a high accuracy.

In this chapter we consider the motion of a fictitious Nereid around Neptune under the perturbations of the Sun (Saad & Kinoshita 1999), which is moving in a Keplerian orbit. The origin of coordinates is located at Neptune. We deal with the planar motion solution, where all the three bodies lie in the same orbital plane. We did not make any expansion of the desturbing function in eccentricity or inclination. The use of Lie-Hori device (1966) enables us to express the relations between the osculating and the mean elements in an explicit form instead of the implicit form arised by Poincare'-von Zeipel's approach. The ephemeris which will be obtained by numerical evaluation of both short and long periodic terms from S_i and $S_i^*(i = 1, 2, ...4)$ will be compared with those computed by numerical integration of the equations of motion. In the following Section we give the equations of motion and the disturbing function, while Sections 3.3, 3.4 and 3.5 are devoted to the analytic evaluations of the short-period, long-period and secular perturbations respectively. In Section 3.6 we give the analytical expressions of the osculating orbital elements. At the end of this chapter we summarize the conclusion of the planar motion solution.

3.2 Formulations

According to the assumptions mentioned above, the disturbing function R_1 which is given by equation (2.49) will be reduced to the following form

$$R_1 = \frac{1}{4}\,\nu^2 a^2 \left[\frac{r^2}{a^2} + 3\frac{r^2}{a^2}\cos(2f + 2g + 2h - 2k)\right]. \tag{3.1}$$

Here, R_2 is neglected by considering the ratio a/a' equal to zero, and the Hamiltonian equation will take the form

$$F = \frac{\mu^2}{2L^2} - \nu\,K + \frac{1}{4}\,\nu^2 a^2 \left[\frac{r^2}{a^2} + 3\frac{r^2}{a^2}\cos(2f + 2g + 2h - 2k)\right]. \tag{3.2}$$

Following Hori's approach(1963), we consider the canonical transformation of the variables

$$x_1, x_2, x_3, x_4; y_1, y_2, y_3, y_4 \tag{3.3}$$

where,

$$x_1 = L, x_2 = G, x_3 = H - G, x_4 = K + G, \tag{3.4}$$

and

$$y_1 = l, y_2 = g + h - k, y_3 = h, y_4 = k. \tag{3.5}$$

The new Hamiltonian does not depend on y_3 or y_4, consequently x_3 and x_4 are constants. The canonical system equations (2.33) and (2.34) become

$$\frac{d(x_1, x_2)}{dt} = \frac{\partial F}{\partial(y_1, y_2)}, \tag{3.6}$$

$$\frac{d(y_1, y_2)}{dt} = -\frac{\partial F}{\partial(x_1, x_2)}, \tag{3.7}$$

with the associated Hamiltonian

$$F = \frac{\mu^2}{2L^2} + \nu x_2 + \frac{1}{4}\,\nu^2 a^2 \left[\frac{r^2}{a^2} + 3\frac{r^2}{a^2}\cos(2f + 2y_2)\right]. \tag{3.8}$$

$$F = F_0 + F_1 + F_2 + ... \tag{3.9}$$

where

$$F_0 = \frac{\mu^2}{2{x_1}^2}, F_1 = \nu x_2, \tag{3.10}$$

$$F_2 = \frac{1}{4}\nu^2 a^2 \left[\frac{r^2}{a^2} + 3\frac{r^2}{a^2}\cos(2f + 2y_2)\right]. \tag{3.11}$$

3.3 Short periodic perturbations

To remove the short periodic terms(y_1), we average on the mean anomaly $l \equiv y_1$ by considering the canonical transformation

$$(x_1, x_2, y_1, y_2) \longrightarrow (x_1', x_2', y_1', y_2') \tag{3.12}$$

Then, F will be

$$F(x_1, x_2, y_1, y_2) \longrightarrow F^*(x_1', x_2', -, y_2') \tag{3.13}$$

with the determining function $S \equiv S(x_1',\ x_2',\ y_1',\ y_2')$, where the new Hamiltonian and the determining function are given by

$$F^* = \sum_{j=0}^{5} F_j^*, S = \sum_{j=0}^{4} S_j. \tag{3.14}$$

The normalization is performed such that the new Hamiltonian F^* does not include y_1. We have the following notations

$$\langle\left(\frac{r}{a}\right)^2\rangle = 1 + \frac{3}{2}e^2, \langle\left(\frac{r}{a}\right)^2\cos(2f)\rangle = \frac{5}{2}e^2, \langle\left(\frac{r}{a}\right)^2\sin(2f)\rangle = 0. \tag{3.15}$$

Applying the algorithm of short-periodic terms of Hori (1966), we get new forms of the Hamiltonian and the determining function

$$F_0^* = \frac{\mu^2}{2x_1'^2}, \tag{3.16}$$

$$F_1^* = \nu x_2'^2, \tag{3.17}$$

$$F_2^* = \frac{1}{4}\nu'^2 a'^2 \left(1 + \frac{3}{2}e'^2 + \frac{15}{2}e'^2 \cos(2y_2')\right), \tag{3.18}$$

$S_1 = 0$, since the canonical equations (3.6) and (3.7) are integrable and then, the determining function has the identical transformation. It is convenient to express the formulae in terms of the eccentric anomaly u instead of the mean anomaly $l'(\equiv y_1')$

$$l' = u' - e'\sin(u'). \tag{3.19}$$

Thus, the following notations are useful

$$\left(\frac{r}{a}\right) = 1 - e\cos u, \left(\frac{r}{a}\right)\cos f = \cos u - e, \left(\frac{r}{a}\right)\sin f = \eta \sin u, \tag{3.20}$$

where, $\eta = \sqrt{1-e^2}$. The dashed orbital elements a', e', n', and η' obtained after the elimination of short periodic terms are computed from

$$a' = \frac{x_1'^2}{\mu}, e' = \sqrt{1 - \left(\frac{x_2'}{x_1'}\right)^2}, n' = \frac{\mu^2}{x_1'^3}, \eta' = \frac{x_2'}{x_1'}. \tag{3.21}$$

Hereafter in this section for simplicity the superscript dash attached to a, e, n, η, u and y_2 are omitted, which will not cause confusion.

$$S_2 = \frac{1}{4}\frac{\nu^2 a^2}{n}P, F_3^* = 0, \tag{3.22}$$

$$S_3 = \frac{1}{4}\frac{\nu^3 a^2}{n^2}\left(-2B_2\sin(2y_2) + 2C_2\cos(2y_2)\right), \tag{3.23}$$

where,

$$P = A_1 + B_1\cos(2y_2) + C_1\sin(2y_2), \tag{3.24}$$

$$A_1 = (-2e + \frac{3}{4}e^3)\sin(u) + \frac{3}{4}e^2\sin(2u) - \frac{1}{12}e^3\sin(3u), \tag{3.25}$$

$$B_1 = (-\frac{15}{2}e + \frac{15}{4}e^3)\sin(u) + (\frac{3}{2} + \frac{3}{4}e^2)\sin(2u) + (-\frac{1}{2}e + \frac{1}{4}e^3)\sin(3u), \tag{3.26}$$

$$C_1 = \eta\left(-\frac{15}{4}e^2 - \frac{15}{2}e\cos(u) + (\frac{3}{2} + \frac{3}{2}e^2)\cos(2u) - \frac{1}{2}e\cos(3u)\right), \tag{3.27}$$

$$\begin{aligned} B_2 \;=\;& \frac{33}{8}e^2 - \frac{27}{16}e^4 + (\frac{33}{4}e - \frac{27}{8}e^3)\cos(u) + (-\frac{3}{4} - \frac{19}{8}e^2 + e^4)\cos(2u) \\ &+(\frac{15}{2}e + \frac{1}{24}e^3)\cos(3u) + (-\frac{1}{16}e^2 + \frac{1}{32}e^4)\cos(4u), \end{aligned} \tag{3.28}$$

$$\begin{aligned} C_2 \;=\;& \eta\left[(-\frac{33}{4}e + 3e^3)\sin(u) + (\frac{3}{4} + \frac{11}{4}e^2)\sin(2u) + (-\frac{15}{12}e - \frac{1}{4}e^3)\sin(3u)\right. \\ &\left. + \frac{1}{16}e^2\sin(4u)\right]. \end{aligned} \tag{3.29}$$

Using the relation

$$\{F_0, S_2\} + \{F_1 + F_2^*, S_1\} + F_2 = F_2^* \tag{3.30}$$

and $S_1 = 0$, we can get

$$F_{2p} = -\frac{\partial F_0}{\partial x_1}\frac{\partial S_2}{\partial y_1} \tag{3.31}$$

where, the subscript p stands for the periodic part. After various mathematical operations, the Poisson bracket $\{F_{2p}, S_2\}$ can be simplified to the form

$$\begin{aligned} \{F_{2p}, S_2\} \;=\;& \frac{\nu^4 a^2}{16n^2}\left\{\left(4D + \frac{\eta^2}{e}D_e\right)D - \frac{\eta}{e}D_e(2E) + \frac{\eta}{e}D_{y_2}P_e\right. \\ &\left. - \left(7P + \frac{\eta^2}{e}P_e\right)D_{y_1}\right\}, \end{aligned} \tag{3.32}$$

where,

$$\left.\begin{aligned}
D &= A + B\cos(2y_2) + C\sin(2y_2),\\
D_e &= A_e + B_e\cos(2y_2) + C_e\sin(2y_2),\\
E &= -B_1\sin(2y_2) + C_1\cos(2y_2),\\
D_{y2} &= -2B\sin(2y_2) + 2C\cos(2y_2),\\
P_e &= A_{1e} + B_{1e}\cos(2y_2) + C_{1e}\sin(2y_2),\\
D_{y_1} &= A_{y1} + B_{y1}\cos(2y_2) + C_{y1}\sin(2y_2),
\end{aligned}\right\} \tag{3.33}$$

$$\left.\begin{aligned}
A &= -e^2 - 2e\cos u + \tfrac{1}{2}e^2\cos(2u),\\
B &= -3e^2 - 6e\cos u + (3 - \tfrac{3}{2}e^2)\cos(2u),\\
C &= \eta(6e\sin u - 3\sin(2u).
\end{aligned}\right\} \tag{3.34}$$

the subscripts e and y_1 describe the partial derivatives. The following notations should be followed carefully

$$\left.\begin{aligned}
&\frac{\partial(A,B,C)}{\partial e} = \left(\frac{\partial(A,B,C)}{\partial e}\right) + \frac{\partial(A,B,C)}{\partial u}\left(\frac{a}{r}\right)\sin u,\\
&\frac{\partial(A,B,C)}{\partial l} = \frac{\partial(A,B,C)}{\partial u}\left(\frac{a}{r}\right),\\
&\frac{\partial u}{\partial e} = \left(\frac{a}{r}\right)\sin u,\\
&\frac{\partial u}{\partial l} = \left(\frac{a}{r}\right),\\
&\frac{\partial}{\partial L}\left(\frac{a^2}{n}\right) = \frac{7}{n^2},\\
&\frac{\partial}{\partial L}\left(\frac{a^2}{n^2}\right) = \frac{10}{n^3},\\
&\frac{\partial}{\partial L}\left(\frac{a^2}{n^3}\right) = \frac{13}{n^4},\\
&\frac{\partial}{\partial L}a^2 = \frac{4}{n}.
\end{aligned}\right\} \tag{3.35}$$

$$F_4^* = \frac{1}{2}\left\{F_{2p}, S_2\right\}_s, \tag{3.36}$$

where the subscript s defines the secular part resulting from the normalization, then the Hamiltonian F_4^* is

$$F_4^* = \frac{1}{16}\frac{\nu^4 a^2}{n^2}\left\{-\frac{49}{4}+\frac{873}{4}e^2-\frac{4347}{32}e^4+(\frac{333}{4}e^2-\frac{237}{8}e^4) * \cos(2y_2)+\frac{615}{32}e^4\cos(4y_2)\right\}, \tag{3.37}$$

and the determining function up to the fourth order has the form

$$S_4 = \frac{1}{16}\frac{\nu^4 a^2}{n^3}\{[S_4]_0+[S_4]_{2c}\cos(2y_2)+[S_4]_{2s}\sin(2y_2) + [S_4]_{4c}\cos(4y_2)+[S_4]_{4s}\sin(4y_2)\}, \tag{3.38}$$

where,

$$[S_4]_0 = (-\frac{127}{4}e-\frac{857}{4}e^3+\frac{8383}{96}e^5)\sin(u)+(-\frac{427}{16}e^2+\frac{7739}{96}e^4)\sin(2u) +(-\frac{19}{24}e^3-\frac{547}{96}e^5)\sin(3u)-\frac{7}{384}e^4\sin(4u)+\frac{1}{24}e^5\sin(5u), \tag{3.39}$$

$$[S_4]_{2c} = (-61e-87e^3+\frac{143}{4}e^5)\sin(u)+(4+\frac{21}{8}e^2+\frac{209}{8}e^4)\sin(2u)+(\frac{7}{2}e-\frac{5}{2}e^3 -\frac{21}{8}e^5)\sin(3u)+(-\frac{21}{16}e^2+\frac{25}{32}e^4)\sin(4u)+(\frac{1}{20}e^3-\frac{1}{40}e^5)\sin(5u), \tag{3.40}$$

$$[S_4]_{2s} = -\eta\left\{\frac{61}{2}e^2+\frac{35}{4}e^4+(61e+\frac{35}{2}e^3)\cos(u)+(-4-\frac{37}{8}e^2-\frac{37}{4}e^4)\cos(2u) + (-\frac{7}{2}e+\frac{3}{4}e^3)\cos(3u)+(\frac{21}{16}e^2-\frac{1}{8}e^4)\cos(4u)-\frac{1}{20}e^3\cos(5u)\right\}, \tag{3.41}$$

$$[S_4]_{4c} = (-\frac{99}{2}e^3+\frac{675}{32}e^5)\sin(u)+(-\frac{219}{16}e^2+\frac{459}{32}e^4)\sin(2u)+(\frac{83}{4}e-\frac{147}{8}e^3 +\frac{45}{32}e^5)\sin(3u)+(\frac{9}{16}-\frac{69}{16}e^2+\frac{369}{128}e^4)\sin(4u), \tag{3.42}$$

$$[S_4]_{4s} = -\eta\left\{\frac{99}{4}e^4+\frac{99}{2}e^3\cos(u)+(\frac{219}{16}e^2-\frac{15}{2}e^4)\cos(2u)+(-\frac{83}{4}e+8e^3) * \cos(3u)-(\frac{9}{16}+\frac{129}{32}e^2-\frac{15}{16}e^4)\cos(4u)\right\}. \tag{3.43}$$

Following the same procedure has been done in derivition of F_4^*, the fifth order Hamiltonian F_5^* results in

$$F_5^* = \frac{1}{2}\{F_{2p}, S_3\}_s, \tag{3.44}$$

where, the Poisson bracket $\{F_{2p}, S_3\}$ is reduced to the form

$$\begin{aligned}\{F_{2p}, S_3\} &= \frac{\nu^4 a^2}{16n^3}\left\{\left(4D + \frac{\eta^2}{e}D_e\right)(2E) - \frac{\eta}{e}D_eQ + \frac{\eta}{e}D_{y_2}M_e\right.\\ &\left. - \left(10M + \frac{\eta^2}{e}M_e\right)D_{y_1}\right\},\end{aligned} \tag{3.45}$$

where

$$\left.\begin{aligned} Q &= -4B_2\cos(2y_2) - 4C_2\sin(2y_2),\\ M &= -2B_2\sin(2y_2) + 2C_2\cos(2y_2),\\ M_e &= \tfrac{\partial M}{\partial e} + (2E)\sin u.\end{aligned}\right\} \tag{3.46}$$

thus

$$F_5^* = \frac{1}{16}\frac{\nu^5 a^2}{n^3}\eta\left[-\frac{97}{2} + \frac{2335}{4}e^2 - \frac{1545}{8}e^4 + e^2(101 - 17e^2)\cos(2y_2)\right]. \tag{3.47}$$

The expressions up to S_3 and F_4^* are identical to those derived from the von Zeipel's method (Hori 1963), while the expressions S_4 and F_5^* in this theory are different from their correspondents due to the different methods. It is shown, however, that they are mathematically equivalent to the relationships which are given in Hori(1970) and Yuasa(1970).

3.4 Long periodic perturbations

Removal of long periodic terms($y_2 \equiv 165$ years) from the Hamiltonian F^* requires another canonical transformation

$$(x_1', x_2', y_1', y_2') \longrightarrow (x_1'', x_2'', y_1'', y_2'') \tag{3.48}$$

and the new Hamiltonian will be free from y_1 and y_2 as follows

$$F^*(x_1', x_2', -, y_2') \longrightarrow F^{**}(x_1'', x_2'', -, -) \tag{3.49}$$

where

$$F^{**} = \sum_{k=0}^{5} F_k^{**}, S^* = \sum_{k=0}^{4} S_k^*. \tag{3.50}$$

In this Section the orbital elements a'', e'', n'', and η'' after elimination of the long periodic term are computed as

$$a'' = \frac{x_1''^2}{\mu}, e'' = \sqrt{1 - (\frac{x_2''}{x_1''})^2}, n'' = \frac{\mu^2}{x_1''^3}, \eta'' = \frac{x_2''}{x_1''}. \tag{3.51}$$

As in the previous Subsection, for simplicity we omit the double primes $''$ from the orbital elements a, e, n, η and y_2.

Following the second algorithm in chapter 2 which concerned the long periodic terms, we can get the new Hamiltonian and the determining function in the following forms

$$F_0^{**} = \frac{\mu^2}{2{x_1}^2} = F_0^*, \tag{3.52}$$

$$F_1^{**} = \nu x_2 = F_1^*, \tag{3.53}$$

$$F_2^{**} = \frac{1}{4}\nu^2 a^2 (1 + \frac{3}{2}e^2), \tag{3.54}$$

$$S_1{}^* = -\frac{15}{16}\nu^2 a^2 e^2 \sin(2y_2), \tag{3.55}$$

$$F_3^{**} = \frac{225}{64}\frac{\nu^3 a^2}{n} e^2 \eta, \tag{3.56}$$

$$S_2{}^* = -\frac{45}{64}\frac{\nu^2 a^2}{n} e^2 \eta \sin(2y_2), \tag{3.57}$$

$$F_4^{**} = \frac{1}{128}\frac{\nu^4 a^2}{n^2}\left[-98 + \frac{4167}{2}e^2 - \frac{12069}{8}e^4\right], \tag{3.58}$$

$$S_3{}^* = \frac{1}{128}\frac{\nu^3 a^2}{n^2}\left[-963e^2 + \frac{4119}{2}e^4\right]\sin(2y_2) - \frac{1905}{32}e^4\sin(4y_2), \tag{3.59}$$

$$F_5^{**} = \frac{\nu^5 a^2}{n^3}\eta\left[-\frac{97}{32} + \frac{288085}{4096}e^2 - \frac{872625}{16384}e^4\right], \tag{3.60}$$

$$S_4{}^* = \frac{\nu^4 a^2}{n^3}\eta\left[\left(-\frac{87275}{2048}e^2 + \frac{826975}{16384}e^4\right)\sin(2y_2) - \frac{292185}{65536}e^4\sin(4y_2)\right]. \tag{3.61}$$

3.5 Secular perturbations

The Hamiltonian remaining after the second transformation determines the secular terms. The equations of motion have the form

$$\frac{dx_1''}{dt} = \frac{\partial F^{**}}{\partial y_1''},\ \frac{dy_1''}{dt} = -\frac{\partial F^{**}}{\partial x_1''}, \tag{3.62}$$

$$\frac{dx_2''}{dt} = \frac{\partial F^{**}}{\partial y_2''},\ \frac{dy_2''}{dt} = -\frac{\partial F^{**}}{\partial x_2''} \tag{3.63}$$

where

$$\begin{aligned} F^{**} &= \frac{\mu^2}{2x_1''^2} + \nu x_2'' + \frac{1}{4}\nu^2 a''^2(1+\frac{3}{2}e''^2) + \frac{225}{64}\frac{\nu^3 a''^2}{n''}e''^2\eta'' + \frac{1}{128}\frac{\nu^4 a''^2}{n''^2}[-98 \\ &+ \frac{4167}{2}e''^2 - \frac{12069}{8}e''^4\Big] + \frac{\nu^5 a''^2}{n''^3}\eta''\left[-\frac{97}{32} + \frac{288085}{4096}e''^2 - \frac{872625}{16384}e''^4\right] \end{aligned} \tag{3.64}$$

$$\begin{aligned} -\frac{\partial F^{**}}{\partial x_1''} &= n'' - \frac{1}{4}\frac{\nu^2}{n''}(7+3e''^2) - \frac{225}{32}\frac{\nu^3}{n''^2}\eta''(1+2e''^2) - \frac{1}{128}\frac{\nu^4}{n''^3}(3187 + \frac{21267}{2}e''^2 \\ &- \frac{36207}{4}e''^4) - \frac{1}{1024}\frac{\nu^5}{n''^4}\eta''\left(\frac{213589}{2} + \frac{2008225}{4}e''^2 - \frac{872625}{2}e''^4\right), \end{aligned} \tag{3.65}$$

$$\begin{aligned} -\frac{\partial F^{**}}{\partial x_2''} + \nu &= \frac{3}{4}\frac{\nu^2}{n''}\eta'' + \frac{225}{32}\frac{\nu^3}{n''^2}(1 - \frac{3}{2}e''^2) + \frac{1}{128}\frac{\nu^4}{n''^3}\eta''\left(4167 - \frac{12069}{2}e''^2\right) \\ &+ \frac{1}{1024}\frac{\nu^5}{n''^4}\left(\frac{294293}{2} - 434220e''^2 + \frac{4363125}{16}e''^4\right). \end{aligned} \tag{3.66}$$

Equations (3.65) and (3.66) describe the mean motions of the mean anomaly and the longitude of perigee respectively. Since F^{**} is free from y_1'' and y_2'', then x_1'' and x_2'' are constants, and we have the mean elements

$$l'' = l_0 + \left(\frac{dl}{dt}\right)'' t, (g+h)'' = (g+h)_0 + \frac{d}{dt}(g+h)''t \tag{3.67}$$

where $\left(\frac{dl}{dt}\right)''$ and $\frac{d}{dt}(g+h)''$ are given by Equations (3.65) and (3.66) respectively and l_0 and $(g+h)_0$ are mean elements at the epoch$(t=0)$. Table I gives the secular perturbations in the mean anomaly and the longitude of perigee using the mean elements in Table II.

TABLE I

Secular perturbations are given in radians per day

$O(\frac{\nu}{n})$	Mean an.	Peri.
2	-7.7364E-005	1.7623E-005
3	-2.0983E-006	2.2930E-007
4	-6.2423E-008	4.9778E-009
5	-1.2242E-009	-8.1582E-11

3.6 Analytical expressions of osculating elements

In the previous Sections we got the Hamiltonian and the determining functions for both the short-periodic and long-periodic terms, then the secular perturbations. In this Section we evaluate the osculating orbital elements for the planar problem. The following notations are useful in evaluating the partial derivatives of the determining functions S and S^* with respect to L and G

$$\frac{\partial}{\partial L} = \left(\frac{\partial}{\partial L}\right) + \frac{\eta^2}{ena^2}\frac{\partial}{\partial e}, \tag{3.68}$$

$$\frac{\partial}{\partial G} = \left(\frac{\partial}{\partial G}\right) - \frac{\eta}{ena^2}\frac{\partial}{\partial e}. \tag{3.69}$$

The partial derivatives of the determining function S_2 with respect to x_1, x_2, y_1,y_2 can be simplified in the forms

$$\left.\begin{aligned}
\frac{\partial S_2}{\partial x_1} &= \frac{1}{4}\left(\frac{\nu}{n}\right)^2\left\{7P + \frac{\eta^2}{e}P_e + D\sin u\right\}, \\
\frac{\partial S_2}{\partial x_2} &= -\frac{1}{4}\left(\frac{\nu}{n}\right)^2\frac{\eta}{e}\left\{P_e + D\sin u\right\}, \\
\frac{\partial S_2}{\partial y_1} &= \frac{1}{4}\frac{\nu^2 a^2}{n}D, \\
\frac{\partial S_2}{\partial y_2} &= \frac{1}{4}\frac{\nu^2 a^2}{n}(2E).
\end{aligned}\right\} \tag{3.70}$$

where the symbols P, P_e, D and E have their own definitions in Section 3.3. The derivatives of S_3 are delivered by the set of equations

$$\left.\begin{aligned}
\frac{\partial S_3}{\partial x_1} &= \frac{1}{4}\left(\frac{\nu}{n}\right)^3\left\{10M + \frac{\eta^2}{e}M_e\right\}, \\
\frac{\partial S_3}{\partial x_2} &= -\frac{1}{4}\left(\frac{\nu}{n}\right)^3\frac{\eta}{e}M_e, \\
\frac{\partial S_3}{\partial y_1} &= \frac{1}{4}\frac{\nu^3 a^2}{n^2}(2E), \\
\frac{\partial S_3}{\partial y_2} &= \frac{1}{4}\frac{\nu^3 a^2}{n^2}Q.
\end{aligned}\right\} \tag{3.71}$$

For simplicity, we refer to $[S_4]_0$, $[S_4]_{2c}$, $[S_4]_{4c}$, $[S_4]_{2s}$ and $[S_4]_{4s}$ by the symbols q_1, q_2, q_3, q_4 and q_5 respectively. The derivatives of the fourth order determining function S_4 will be written in the following forms

$$\left.\begin{aligned}
\frac{\partial S_4}{\partial x_1} &= \frac{1}{16}\left(\frac{\nu}{n}\right)^4\left\{13B_4 + \frac{\eta^2}{e}\Xi\right\}, \\
\frac{\partial S_4}{\partial x_2} &= -\frac{1}{16}\left(\frac{\nu}{n}\right)^4\frac{\eta}{e}\Xi, \\
\frac{\partial S_4}{\partial y_1} &= \left(\frac{a}{r}\right)\frac{\partial S_4}{\partial u}, \\
\frac{\partial S_4}{\partial y_2} &= \frac{1}{16}\frac{\nu^4 a^2}{n^3}(B_4)_{y_2},
\end{aligned}\right\} \tag{3.72}$$

where

$$B_4 = q_1 + q_2\cos(2y_2) + q_3\cos(4y_2) + q_4\sin(2y_2) + q_5\sin(4y_2), \tag{3.73}$$

$$\Xi = \chi_1 + \chi_2 + \chi_3 + \chi_4 + \chi_5, \tag{3.74}$$

$$\chi_i = \frac{\partial q_i}{\partial e} + \frac{\partial q_i}{\partial u}\left(\frac{a}{r}\right)\sin u, \tag{3.75}$$

for $i = 1, 2, 3, 4, 5$.

The above partial derivatives of S_2, S_3 and S_4 contribute in obtaining the short-periodic variations of the orbital elements, while that of S_1^*, S_2^*, S_3^* and S_4^* have the contribution of getting the long periodic variations. We may notice that, when we evaluate the perturbations in y_1 and y_2, e appears as a small divisor. However, this small divisor will be disappeared when we compute the perturbations in $y_1 + y_2 (\equiv \ell + g + h)$, that is because

$$\frac{\eta^2}{e} - \frac{\eta}{e} = -\frac{e\eta}{1+\eta} \tag{3.76}$$

Now, we summarize the results of the partial derivatives of S_i^* with respect to x_1, x_2 and y_2 as follows

$$\left.\begin{aligned}
\frac{\partial S_1^*}{\partial x_1} &= -\frac{15}{8}\left(\frac{\nu}{n}\right)\left(1+e^2\right)\sin(2y_2),\\
\frac{\partial S_1^*}{\partial x_2} &= \frac{15}{8}\left(\frac{\nu}{n}\right)\eta\sin(2y_2),\\
\frac{\partial S_1^*}{\partial y_2} &= -\frac{15}{8}\nu a^2 e^2\cos(2y_2),
\end{aligned}\right\} \tag{3.77}$$

for S_2^* we have

$$\left.\begin{aligned}
\frac{\partial S_2^*}{\partial x_1} &= -\frac{45}{32}\left(\frac{\nu}{n}\right)^2\eta\left(1+2e^2\right)\sin(2y_2),\\
\frac{\partial S_2^*}{\partial x_2} &= \frac{45}{64}\left(\frac{\nu}{n}\right)^2\left(2-3e^2\right)\sin(2y_2),\\
\frac{\partial S_2^*}{\partial y_2} &= -\frac{45}{32}\frac{\nu^2 a^2}{n}e^2\eta\cos(2y_2),
\end{aligned}\right\} \tag{3.78}$$

for S_3^* we can write

$$\left.\begin{aligned}
\frac{\partial S_3^*}{\partial x_1} &= \left(\frac{\nu}{n}\right)^3 \left\{\left(-\frac{963}{64} - \frac{3585}{128}e^2 + \frac{12357}{256}e^4\right)\sin(2y_2)\right. \\
&\quad \left. + \left(-\frac{1905}{1024}e^2 - \frac{5715}{2048}e^4\right)\sin(4y_2)\right\}, \\
\frac{\partial S_3^*}{\partial x_2} &= \left(\frac{\nu}{n}\right)^3 \eta \left\{\left(\frac{963}{64} - \frac{4119}{128}e^2\right)\sin(2y_2)\right. \\
&\quad \left. + \frac{1905}{1024}e^2 \sin(4y_2)\right\}, \\
\frac{\partial S_3^*}{\partial y_2} &= \frac{\nu^3 a^2}{n^2} \left\{\left(-\frac{963}{64}e^2 + \frac{4119}{256}e^4\right)\cos(2y_2)\right. \\
&\quad \left. - \frac{1905}{1024}e^4 \cos(4y_2)\right\},
\end{aligned}\right\} \tag{3.79}$$

while for S_4^* we get

$$\left.\begin{aligned}
\frac{\partial S_4^*}{\partial x_1} &= \left(\frac{\nu}{n}\right)^4 \eta \left\{\left(-\frac{20975}{256} - \frac{905025}{4096}e^2 + \frac{772975}{2048}e^4\right)\sin(2y_2)\right. \\
&\quad \left. + \left(-\frac{292185}{16384}e^2 - \frac{292185}{8192}e^4\right)\sin(4y_2)\right\}, \\
\frac{\partial S_4^*}{\partial x_2} &= \left(\frac{\nu}{n}\right)^4 \left\{\left(\frac{20975}{256} - \frac{1276375}{4096}e^2 + \frac{3864875}{16384}e^4\right)\sin(2y_2)\right. \\
&\quad \left. + \left(\frac{292185}{16384}e^2 - \frac{1460925}{65536}e^4\right)\sin(4y_2)\right\}, \\
\frac{\partial S_4^*}{\partial y_2} &= \frac{\nu^4 a^2}{n^3}\eta \left\{\left(-\frac{20975}{256}e^2 + \frac{772975}{8192}e^4\right)\cos(2y_2)\right. \\
&\quad \left. - \frac{292185}{16384}e^4 \cos(4y_2)\right\}.
\end{aligned}\right\} \tag{3.80}$$

For practical calculations, we refer to the partial derivatives of S_j $(j = 2, 3, 4)$ with respect to x_1, x_2, y_1, y_2 by P_i $(i = 1, 2, ..., 12)$ and for the derivatives of S_s^* $(s = 1, 2, 3, 4)$ with respect to x_1, x_2, y_2 by k_i respectively. In what follows, we implement the above analytical expressions for digital computations by constructing the following algorithm described by its purpose, input and its computational sequence:

3.6.1 Computational algorithm

- *Purpose*: To compute the osculating elements a, e, y_1, y_2 of Nereid in the planar motion. Nereid is moving around Neptune and perturbed by the solar attraction.

- *Input*: the initial values a_0'', e_0'', y_{10}'', y_{20}'', t_0, t_{end}, and Tol (specified tolerance).

- Units measurements: Masses are given in solar unit, distances are in AU, time in days while the angles are given in radians.

- *Computational Sequence*:

(1) Compute the secular perturbations, then find y_1'' and y_2'' from the relations

$$y_1'' = y_{10}'' + n_{y_{10}''} t, \quad y_2'' = y_{20}'' + n_{y_{20}''} t, \tag{3.81}$$

(2) Compute the long-periodic variations from the following sequence

(a) for eccentricity:

$$\left.\begin{aligned}
\Psi_1 &= -\frac{\eta}{ena^2} k_3 \equiv \delta_1 e,\\
\Psi_2 &= -\frac{\eta}{ena^2} k_6,\\
\Psi_3 &= -\frac{\eta}{ena^2}\left(\frac{\partial \Psi_1}{\partial e} k_3 - \frac{\partial \Psi_1}{\partial y_2} k_2\right),\\
\Psi_4 &= \Psi_2 + \frac{\Psi_3}{2} \equiv \delta_2 e,\\
\Psi_5 &= -\frac{\eta}{ena^2} k_9,\\
\Psi_6 &= -\frac{\eta}{ena^2}\left(\frac{\partial \Psi_1}{\partial e} k_6 - \frac{\partial \Psi_1}{\partial y_2} k_5\right),\\
\Psi_7 &= -\frac{\eta}{ena^2}\left(\frac{\partial \Psi_2}{\partial e} k_3 - \frac{\partial \Psi_2}{\partial y_2} k_2\right),\\
\Psi_8 &= -\frac{\eta}{ena^2}\left(\frac{\partial \Psi_3}{\partial e} k_3 - \frac{\partial \Psi_3}{\partial y_2} k_2\right),\\
\Psi_9 &= \Psi_5 + \frac{\Psi_6 + \Psi_7}{2} + \frac{\Psi_8}{6} \equiv \delta_3 e,\\
e'_{long} &= e_0'' + \delta_1 e' + \delta_2 e' + \delta_3 e',
\end{aligned}\right\} \tag{3.82}$$

(b) for the argument of pericenter y_2', in the above equations put

$$\left.\begin{aligned}\Psi_1 &= -k_2,\\ \Psi_2 &= -k_5,\\ \Psi_5 &= -k_8,\end{aligned}\right\} \tag{3.83}$$

then apply all the steps mentioned in eccentricity case to get

$$y'_{2long} = y''_{20} + \delta_1 y_2' + \delta_2 y_2' + \delta_3 y_2'. \tag{3.84}$$

(c) in case of the mean anomaly y_1' put

$$\left.\begin{aligned}\Psi_1 &= -k_1,\\ \Psi_2 &= -k_4,\\ \Psi_5 &= -k_7,\end{aligned}\right\} \tag{3.85}$$

then apply all the steps mentioned in the case of eccentricity and the argument of pericenter to get

$$y'_{1long} = y''_{10} + \delta_1 y_1' + \delta_2 y_1' + \delta_3 y_1'. \tag{3.86}$$

(3) Compute the short-periodic variations as follows

Call Kepler

(a) semi-major axis:

$$\left.\begin{aligned}\delta_2 a &= \frac{2}{na} P_3,\\ \delta_3 a &= \frac{2}{na} P_7,\\ \delta a_{sho} &= \delta_2 a + \delta_3 a,\end{aligned}\right\} \tag{3.87}$$

(b) eccentricity:

$$\left.\begin{aligned}
\delta_2 e &= \frac{\eta^2}{ena^2} P_3 - \frac{\eta}{ena^2} P_4, \\
\delta_3 e &= \frac{\eta^2}{ena^2} P_7 - \frac{\eta}{ena^2} P_8, \\
\delta e_{sho} &= \delta_2 e + \delta_3 e,
\end{aligned}\right\} \tag{3.88}$$

(c) Argument of pericenter:

$$\left.\begin{aligned}
\delta_2 y_2 &= -P_2, \\
\delta_3 y_2 &= -P_6, \\
\delta y_{2sho} &= \delta_2 y_2 + \delta_3 y_2
\end{aligned}\right\} \tag{3.89}$$

(d) in case of the mean anomaly

$$\left.\begin{aligned}
\delta_2 y_1 &= -P_1, \\
\delta_3 y_1 &= -P_5, \\
\delta y_{1sho} &= \delta_2 y_1 + \delta_3 y_1.
\end{aligned}\right\} \tag{3.90}$$

(4) Compute the osculating elements from the equations

$$\left.\begin{aligned}
a_{osc} &= a_0'' + \delta a_{sho}, \\
e_{osc} &= e'_{long} + \delta e_{sho}, \\
y_{2osc} &= y'_{2long} + \delta y_{2sho}, \\
y_{1osc} &= y'_{1long} + \delta y_{1sho}.
\end{aligned}\right\} \tag{3.91}$$

(5) The algorithm is completed up to third order. We can continue analogously up to the fourth order.

3.7 Comparison of the analytical solution

We compare the analytical solution obtained in this paper with the numerical integration by applying the analytical solution to a fictitious Nereid(the second satellite of Neptune) with zero inclination. The mean elements of the fictitious Nereid are given in the second column of the Table II. The orbital elements of the sun used in this integration are $a_\odot =$ 30.1104AU, $e_\odot = 0.0$, $k(t = 0) = 10°.0$. The small parameter in the theory, the ratio of the mean motions of the sun and the fictitious Nereid is $\nu/n = 5.98 \times 10^{-3}$. The osculating elements that are the initial conditions for the numerical integration are computed from the analytical solution(Section 3.6) are given in the third column of Table II. We used for the numerical integration the extrapolation method, which has a capability of highly accurate orbital computation.

TABLE II

Mean and osculating elements

orbital elements	mean elements	osculating elements
semi-major axis(km)	5513413.256	5513226.872
eccentricity	0.751201525	0.751270690
long. of pericenter(deg)	254.809177	254.385293
mean anomaly(deg)	359.34112	0.362199662

Figures 3.1 and 3.2 show the osculating orbital elements of the fictitious Nereid over 5 years and 500 years, respectively, which are obtained from the analytical solution. Figures 3.1.3 and 3.2.3 show the periodic variation of the mean anomaly, which is obtained by subtracting the secular part of the osculating mean anomaly. Figures 3.3 and 3.4 show the

differences of the osculating elements between the analytical solution and the orbit obtained by the numerical integration. The secular error in the mean anomaly is $-1°.6 \times 10^{-5}$/year. Figure 3.3.5 and 3.4.5 show the periodic residuals after taking away the secular error. This secular error is removed by the orbital adjustment which increases the semi-major axis by 160m. From this comparison the present theory has the accuracy of the level of 300 m for the semi-major axis, 3×10^{-8} for the eccentricity, and about 0.004 arc second for angle variable. The relative accuracy of them is about 4×10^{-8}, which is between $(\nu/n)^3$ and $(\nu/n)^4$. Now we can say that the present theory with zero inclination has an accuracy to the fourth order, $(\nu/n)^4$, in periodic perturbations.

3.8 d'Alembert characteristics

The d'Alembertian characteristic is one of the most important properties of a series which describes the behavior of the coordinates and Hamiltonians in planetary problems in celestial mechanics. A precise definition of that property with respect to the eccentricity is given by Brouwer and Clemence (1961), when they developed the four functions u, $\left(\frac{r}{a}\right)$, f and $ln\left(\frac{r}{a}\right)$ in Fourier series with multiples of l as arguments and power series of e as coefficients. In these expressions, the lowest power of e occuring in the coefficient of a sine or cosine term equals the multiple of l in the argument. Moreover, in a coefficient of a term, for an odd argument only odd powers of e occur, and with an even argument only even powers of e occur. This property is called *d'Alembert characteristics.*

Consider the motion of a small object (e.g. a minor planet, satellite,...) under the gravitational attraction of the Sun, then the disturbing function R has been developed in a series of cosine terms of the form

$$R = \sum C_{j_1,j_2,j_3,j_1',j_2',j_3'}(a, e, I, a', e', I',) \cos\Theta, \tag{3.92}$$

where the elements $a, e, I, \lambda, \varpi, \Omega$ have the usual definitions for the disturbed body, while the corresponding primed elements are for the Sun. Θ is a combination of two sets of elements

$$\Theta = j_1\lambda + j_2\varpi + j_3\Omega + j_1'\lambda' + j_2'\varpi' + j_3'\Omega'. \tag{3.93}$$

When the elements e, e',I, I' are small, the coefficients C can be developed in the following series

$$C_{j_1,j_2,j_3,j_1',j_2',j_3'} = \sum e^{k_2} e'^{k_2'} I^{k_3} I'^{k_3'} C^*_{k_2,k_2',k_3,k_3'}(a, a'). \tag{3.94}$$

Since the function R does not change by a rotation of the coordinate system about the z axis, and since the angles appearing in the arguments of the cosines are reckoned from a common origin, the coefficients j, j' and k, k' satisfy the relations

$$j_1 + j_2 + j_3 + j_1' + j_2' + j_3' = 0, \tag{3.95}$$

$$\left.\begin{aligned} k_2 &= \mid j_2 \mid +(\text{even}),\\ k_2' &= \mid j_2' \mid +(\text{even}),\\ k_3 &= \mid j_3 \mid +(\text{even}),\\ k_3' &= \mid j_3' \mid +(\text{even}). \end{aligned}\right\} \tag{3.96}$$

$$j_3 + j_3' = (\text{even}), \tag{3.97}$$

$$\min(k_2 + k_2' + k_3 + k_3') \geq \mid j_1 + j_1' \mid . \tag{3.98}$$

Our analytic formulae satisfy the above relations which indicate the correction way we have done in dealing with the present problem although their derivations were laborious. If two quantities A and B that are functions of orbital elements satisfy the d'Alembert characteristics, the Poisson bracket $\{A, B\}$ keeps the d'Alembert characteristics. The original Hamiltonian F (3.2) satisfies d'Alembert characteristics and the operation appeared in Hori's algorithm is only the Poisson bracket. Thus, the Hamiltonian and the determining functions in Hori's method (1966) satisfy d'Alembert characteristics in contrast to Poincare'-von Zeipel's algorithm. In the present theory, the transformed Hamiltonians F^* and the determining functions S, S^*, therefore, should satisfy the d'Alembert characteristics. In fact this is easily verified after F^*, S, S^* are changed in the form of the argument $iu + jy_2'$, where i and j take any numbers.

3.9 Conclusions

In concluding the present chapter, an analytical theory on the motion of a satellite with large eccentricity and zero inclination is constructed. The theory is applied to a fictitious Neptune's satellite Nereid. The secular and periodic perturbations are obtained up to the fifth and fourth order respectively. A comparison to the numerical integration indicates the accuracy 300 m, 3×10^{-8}, and 0.004 arc second for the semi-major axis, eccentricity and the angular variables respectively. Tables III and IV show the amplitudes and the accuracy in the osculating orbital elements for both short and long periodic perturbations respectively.

TABLE III

Amplitudes of the osculating elements

Elements	Short-period	Long-period
semi-major axis	897.587	1047.19
eccentricity	0.00025	0.011
arg. of pericenter	0.01	1.25
mean anomaly	0.0325	0.05

TABLE IV

Accuracy of the osculating elements

Elements	Short-period	Long-period
semi-major axis	0.23	0.3
eccentricity	2×10^{-8}	3×10^{-8}
arg. of pericenter	4×10^{-6}	5×10^{-6}
mean anomaly	1×10^{-5}	1.6×10^{-5}

where the semi-major axis is given in km, eccentricity in radian, and the argument of pericenter and the mean anomaly are given in degree. In the next chapter we'll investigate the solution of the non-planar case.

Figure Captions

Fig. 3.1. The osculating orbital elements of Nereid over 5 years:

(1) eccentricity,

(2) semi-major axis,

(3) periodic part of mean anomaly,

(4) argument of pericenter.

Fig. 3.2. The osculating orbital elements of Nereid over 500 years:

(1) eccentricity,

(2) semi-major axis,

(3) periodic part of mean anomaly,

(4) argument of pericenter.

Fig. 3.3. Difference between analytical and numerical results for the osculating orbital elements of Nereid during 5 years:

(1) eccentricity,

(2) semi-major axis,

(3) mean anomaly,

(4) argument of pericenter,

(5) periodic part of the residuals in the mean anomaly.

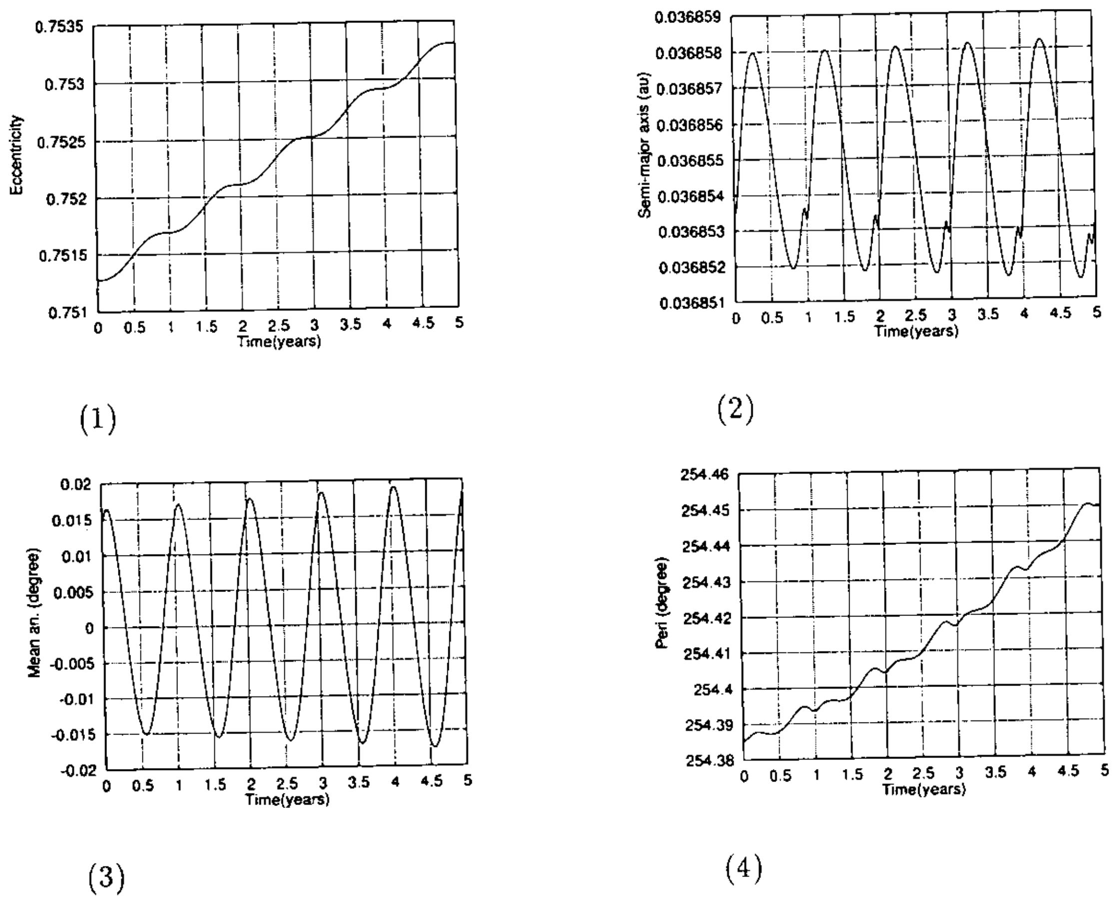
Fig. 3.4. Difference between analytical and numerical results for the osculating orbital elements of Nereid during 500 years:

(1) eccentricity,

(2) argument of pericenter,

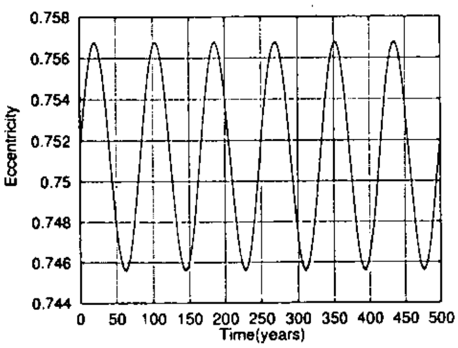
(3) semi-major axis,

(4) mean anomaly.

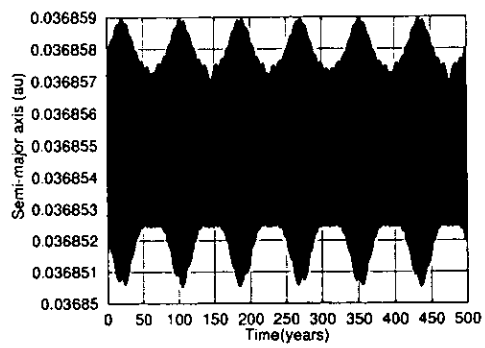


Eccentricity
0.7535
0.753
0.7525
0.752
0.7515
0.751
0 0.5 1 1.5 2 2.5 3 3.5 4 4.5 5
Time(years)
(1)
Semi-major axis (au)
0.036859
0.036858
0.036857
0.036856
0.036855
0.036854
0.036853
0.036852
0.036851
0 0.5 1 1.5 2 2.5 3 3.5 4 4.5 5
Time(years)
(2)
Mean an. (degree)
0.02
0.015
0.01
0.005
0
-0.005
-0.01
-0.015
-0.02
0 0.5 1 1.5 2 2.5 3 3.5 4 4.5 5
Time(years)
(3)
Peri (degree)
254.46
254.45
254.44
254.43
254.42
254.41
254.4
254.39
254.38
0 0.5 1 1.5 2 2.5 3 3.5 4 4.5 5
Time(years)
(4)

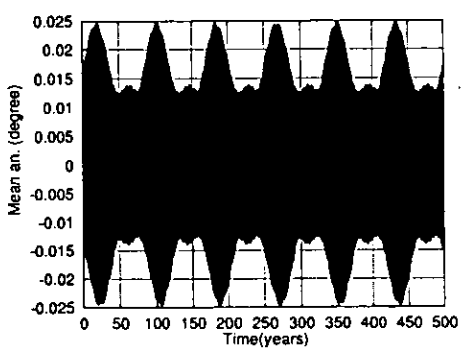
Fig. 3.1


Eccentricity
0.758
0.756
0.754
0.752
0.75
0.748
0.746
0.744
0 50 100 150 200 250 300 350 400 450 500
Time(years)

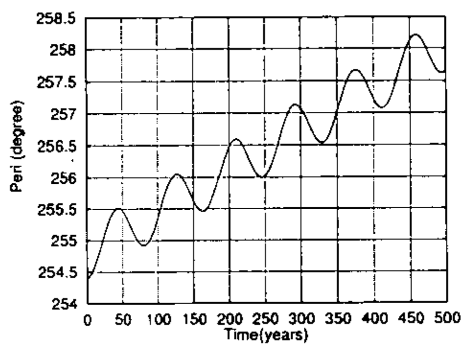
(1)


Semi-major axis (au)
0.036859
0.036858
0.036857
0.036856
0.036855
0.036854
0.036853
0.036852
0.036851
0.03685
0 50 100 150 200 250 300 350 400 450 500
Time(years)

(2)

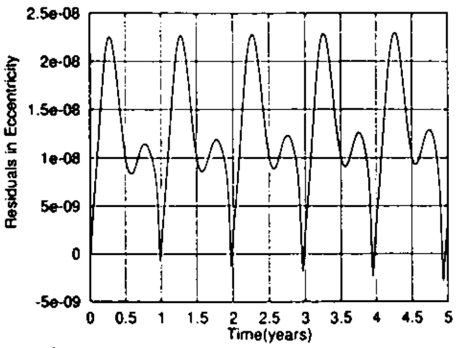

Mean an. (degree)
0.025
0.02
0.015
0.01
0.005
0
-0.005
-0.01
-0.015
-0.02
-0.025
0 50 100 150 200 250 300 350 400 450 500
Time(years)

(3)

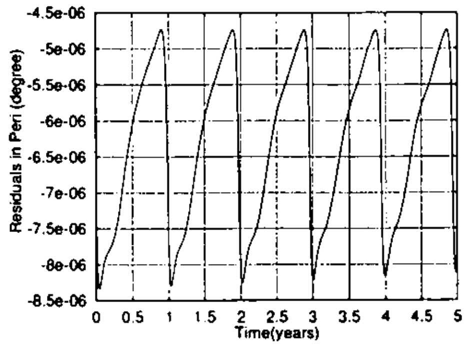

Peri (degree)
258.5
258
257.5
257
256.5
256
255.5
255
254.5
254
0 50 100 150 200 250 300 350 400 450 500
Time(years)

(4)

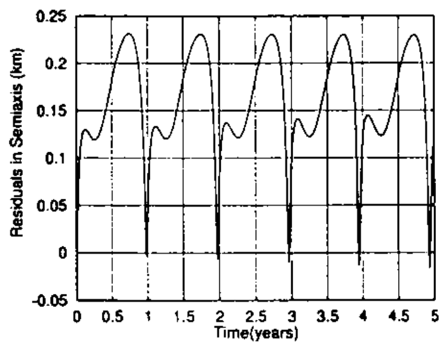
Fig. 3.2


Residuals in Eccentricity
2.5e-08
2e-08
1.5e-08
1e-08
5e-09
0
-5e-09
0 0.5 1 1.5 2 2.5 3 3.5 4 4.5 5
Time(years)

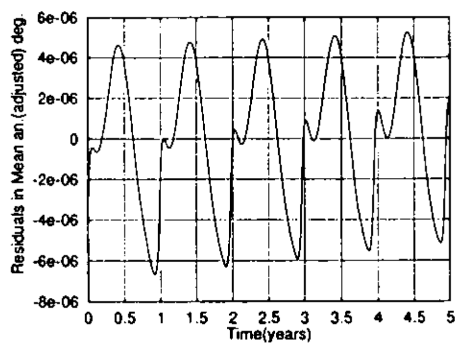
(1)


Residuals in Peri (degree)
-4.5e-06
-5e-06
-5.5e-06
-6e-06
-6.5e-06
-7e-06
-7.5e-06
-8e-06
-8.5e-06
0 0.5 1 1.5 2 2.5 3 3.5 4 4.5 5
Time(years)

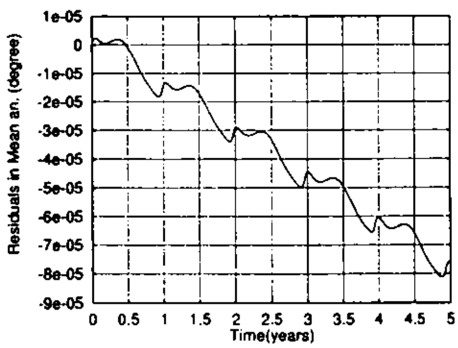
(4)


Residuals in Semiaxis (km)
0.25
0.2
0.15
0.1
0.05
0
-0.05
0 0.5 1 1.5 2 2.5 3 3.5 4 4.5 5
Time(years)

(2)

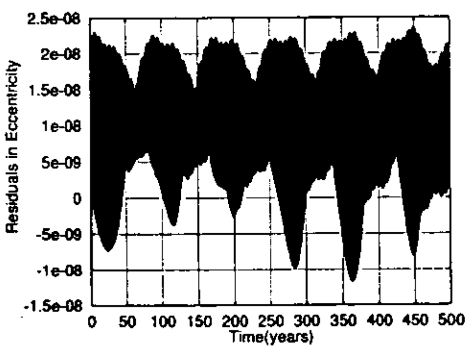

Residuals in Mean an.(adjusted) deg.
6e-06
4e-06
2e-06
0
-2e-06
-4e-06
-6e-06
-8e-06
0 0.5 1 1.5 2 2.5 3 3.5 4 4.5 5
Time(years)

(5)

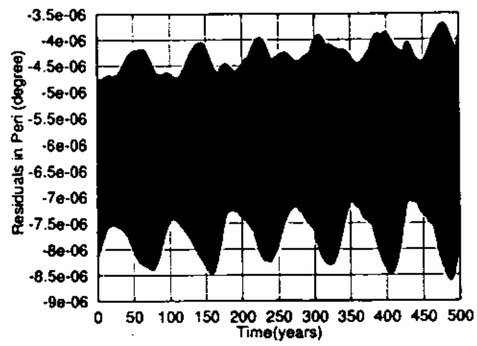

Residuals in Mean an. (degree)
1e-05
0
-1e-05
-2e-05
-3e-05
-4e-05
-5e-05
-6e-05
-7e-05
-8e-05
-9e-05
0 0.5 1 1.5 2 2.5 3 3.5 4 4.5 5
Time(years)

(3)

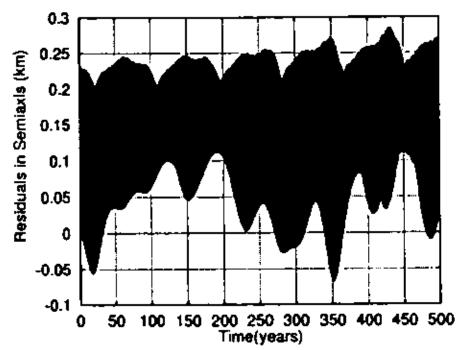
Fig. 3.3


Residuals in Eccentricity
2.5e-08
2e-08
1.5e-08
1e-08
5e-09
0
-5e-09
-1e-08
-1.5e-08
0 50 100 150 200 250 300 350 400 450 500
Time(years)

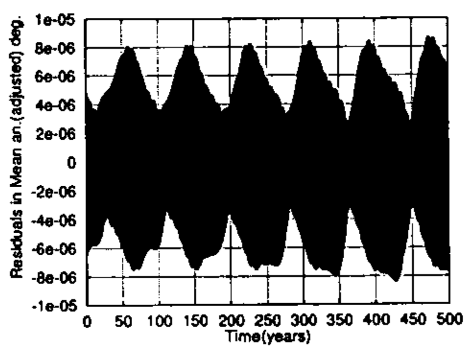
(1)


Residuals in Peri (degree)
-3.5e-06
-4e-06
-4.5e-06
-5e-06
-5.5e-06
-6e-06
-6.5e-06
-7e-06
-7.5e-06
-8e-06
-8.5e-06
-9e-06
0 50 100 150 200 250 300 350 400 450 500
Time(years)

(2)


Residuals in Semiaxis (km)
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.05
-0.1
0 50 100 150 200 250 300 350 400 450 500
Time(years)

(3)


Residuals in Mean an.(adjusted) deg.
1e-05
8e-06
6e-06
4e-06
2e-06
0
-2e-06
-4e-06
-6e-06
-8e-06
-1e-05
0 50 100 150 200 250 300 350 400 450 500
Time(years)

(4)

Fig. 3.4

Chapter 4

Non-planar Motion Solution

In chapter 3, we dealt with the motion of fictitious Nereid. In the present chapter, we investigate the motion of Nereid in the frame work of circular non-planar restricted three-body problem. The solution includes short, intermediate and long periodic perturbations. All the analytical expressions are implemented for digital computations by constructing computational algorithms. Evaluating the analytical expressions have been done by the powerful packages of Mathematica. The Nereid's motion is also studied numerically and compared with analytical results. The behaviour of the motion and the accuracy of the present theory are exhibited by figures and tables.

4.1 Introduction

In the previuos chapter, we considered the circular, planar, restricted three-body problem of Sun-Neptune-Nereid system. In this chapter we take the frame work of the circular, non-planar, restricted three body problem (Saad & Kinoshita 2000). The requirement of circular motion of Neptune-Sun is satisfied, since the eccentricity of the Neptune's orbit is only 0.009. While the requirement of the nonplanar motion is satisfied in the inclination of Nereid (i $\sim 10^o$) to the orbital plane of Neptune.

The restricted three-body problem is called circular or elliptical according to the nature of the primaries' orbits as they execute their Keplerian motion. Similarly, the problem is called planar or nonplanar depending on whether the third body is assumed to move in the plane containing the primaries' orbits. The planar, circular restricted three-body problem consists of two point masses m_1 and m_2 called the primaries moving in circular orbits around their common center of mass. In the plane of their motion moves a third body with infinitesimal mass m_3, not affecting the motion of the primaries, while in the nonplanar case the third body deviates about the orbital plane of the primaries. As a practical matter, the restricted three-body problem provides quite reasonable short-term approximations in situations where one mass is negligible compared with the other two masses. In this case the two primaries are not affected by the attractions of the small body.

By virtue of Voyager 2 encounter, Neptune's known satellite system consisted of one large retrograde highly inclined satellite, Triton, a smaller satellite, Nereid, in a prograde highly eccentric orbit, and six newly discovered satellites in the vicinity of their mother planet Neptune. Triton was discovered by Lassel in 1846, Nereid was discovered by Kuiper in 1949, while the other 6 satellites were discovered in 1989 during the Voyager 2 encounter with the Neptunian system (Stone and Miner 1989). Its highly elliptical orbit and faintness make the motion prediction of Nereid inaccurate by the usage of the classical models. Many authors have dealt with the orbital determinations of Nereid (Rose 1974; Mignard 1975, 1981; Veillet 1982, 1988; Oberti 1990; Jacobson 1990, 1991; Segerman and Richardson 1997). Rose fit van Biesbroeck's (1951, 1957) observations, while Veillet used the theory of Mignard (1975, 1981) which is based on Von Zeipel's method. Jacobson fit the numerically integrated Neptunian satellite orbits (Nereid and Triton) to Earth-based astrometric observations and Voyager spacecraft observations. The second order analytic theory by Oberti gives discripancies about a hundred of kilometers, while in the theory of Segerman and Richardson the disturbing function is expanded in eccentricity. However, most of the analytical theories which are expanded in eccentricity or/and inclination not very accurate for the case of highly eccentric orbits. That is because of the slow convergence of the power series of the disturbing function. In this paper we construct a third order analytical theory on the motion of Nereid which is chiefly perturbed by the Sun. The theory is based on Lie transform approach advanced by Hori(1966). The small parameter is the ratio of the mean motion of the Sun and Nereid $\sim 6 \times 10^{-3}$. The ephemerides evaluated by the analytic expressions of the present theory are compared with those computed by the numerical integrations of the equations of motion. The accuracy and the amplitudes of the osculating orbital elements of Nereid are shown by tables and figures in the last Section of this chapter. Sections 4.3, 4.4 and 4.5 are devoted for the short-period, long-period and secular perturbations respectively.

4.2 Hamiltonian of the motion

The Hamiltonian equation of the nonplanar case is given by

$$F = \frac{\mu^2}{2L^2} + \nu G + F_2, \tag{4.1}$$

where

$$F_0 = \frac{\mu^2}{2L^2}, F_1 = \nu G, \tag{4.2}$$

$$F_2 = F_{21} + F_{22} + F_{23}, \tag{4.3}$$

$$\left.\begin{aligned} F_{21} &= \nu^2 a^2 \left(\frac{r}{a}\right)^2 \left\{(-\frac{1}{8} + \frac{3}{8}\theta^2) + \frac{3}{16}(1+\theta)^2 \cos(2f + 2y_2)\right\}, \\ F_{22} &= \nu^2 a^2 \left(\frac{r}{a}\right)^2 \left\{\frac{3}{8}(1-\theta^2)\left[\cos(2f+2g) + \cos(2g - 2y_2)\right]\right\}, \\ F_{23} &= \nu^2 a^2 \left(\frac{r}{a}\right)^2 \left\{\frac{3}{16}(1-\theta)^2 \cos(2f + 4g - 2y_2)\right\}. \end{aligned}\right\} \tag{4.4}$$

The normalization of F_2 with respect to the mean anomaly of Nereid results in

$$F_{2s} = F_{21s} + F_{22s} + F_{23s}, \tag{4.5}$$

where

$$\left.\begin{aligned} F_{21s} &= \nu^2 a^2 \left\{(1 + \frac{3}{2}e^2)(-\frac{1}{8} + \frac{3}{8}\theta^2) + \frac{15}{32}e^2(1+\theta)^2\cos(2y_2)\right\}, \\ F_{22s} &= \nu^2 a^2 \left\{\frac{15}{16}e^2(1-\theta^2)\cos(2g) + \frac{3}{8}(1+\frac{3}{2}e^2)(1-\theta^2)\cos(2g - 2y_2)\right\}, \\ F_{23s} &= \nu^2 a^2 \left\{\frac{15}{32}e^2(1-\theta)^2\cos(4g - 2y_2)\right\}. \end{aligned}\right\} \tag{4.6}$$

Whence, the periodic part F_{2p} provides

$$F_{2p} = F_{21p} + F_{22p} + F_{23p}, \tag{4.7}$$

where

$$\left.\begin{aligned} F_{21p} &= \nu^2 a^2 \left\{A_{21} + B_{21}\cos(2y_2) + C_{21}\sin(2y_2)\right\}, \\ F_{22p} &= \nu^2 a^2 \left\{A_{22} + B_{22}\cos(2y_2) + C_{22}\sin(2y_2)\right\}, \\ F_{23p} &= \nu^2 a^2 \left\{B_{23}\cos(2y_2) + C_{23}\sin(2y_2)\right\}. \end{aligned}\right\} \tag{4.8}$$

The high eccentric orbit of Nereid precludes replacing functions of the true anomaly by expansions involving the mean anomaly. So, it is convenient to take the eccentric anomaly of Nereid u as independent variable (Hori 1963). In this regard, the expressions of A_{ij} and B_{ij} above yield

$$\left.\begin{aligned}
A_{21} &= \left(-\frac{1}{8}+\frac{3}{8}\theta^2\right)A,\\
B_{21} &= \frac{1}{16}(1+\theta)^2 B,\\
C_{21} &= \frac{1}{16}(1+\theta)^2 C,\\
A_{22} &= \frac{1}{8}(1-\theta^2)\left\{B\cos(2g)+C\sin(2g)\right\},\\
B_{22} &= \frac{3}{8}(1-\theta^2)A\cos(2g),\\
C_{22} &= \frac{3}{8}(1-\theta^2)A\sin(2g),\\
B_{23} &= \frac{1}{16}(1-\theta)^2\left\{B\cos(4g)+C\sin(4g)\right\},\\
C_{23} &= \frac{1}{16}(1-\theta)^2\left\{B\sin(4g)-C\cos(4g)\right\},
\end{aligned}\right\} \tag{4.9}$$

where A, B and C are already defined in the previous chapter. Notable, all the analytical expressions in this case are put in a way such that anyone can easily get their correspondents in the planar case.

4.3 Short periodic perturbations

Elimination of the short periodic terms will be satisfied by finding a canonical transformation

$$(L,G,H;l,g,h,\lambda_\odot) \longrightarrow (L^*,G^*,H^*;l^*,g^*,h^*,\lambda_\odot)\,, \tag{4.10}$$

where, $\lambda_\odot \equiv k$ defines the longitude of the Sun. In order that:

$$F\,(L,G,H;l,g,h,\lambda_\odot) \longrightarrow F^*\,(L^*,G^*,H^*;-,g^*,h^*,\lambda_\odot)\,, \tag{4.11}$$

and the determining function S only includes the new variables $L^*, G^*, H^*; l^*, g^*, h^*, \lambda_\odot$. After eliminating the short-periodic terms, the orbital elements a^*, e^*, n^* and η^* are copmuted from

$$a^* = \frac{L^{*2}}{\mu}, e^* = \sqrt{1 - \left(\frac{G^*}{L^*}\right)^2}, n^* = \frac{\mu^2}{L^{*3}}, \eta^* = \frac{G^*}{L^*}. \tag{4.12}$$

In this Section for simplicity, the superscript * will be omitted from the orbital elements. Following the algorithm concerned the short-period terms, the new Hamiltonian and determining functions deliver

$$F_0^* = \frac{\mu^2}{2L^2}, F_1^* = \nu G, \tag{4.13}$$

$$\begin{aligned} F_2^* &= \nu^2 a^2 \left\{ \left(1 + \frac{3}{2}e^2\right)\left(-\frac{1}{8} + \frac{3}{8}\theta^2\right) + \frac{15}{16}e^2(1-\theta^2)\cos(2g) \right. \\ &+ \frac{3}{8}\left(1 + \frac{3}{2}e^2\right)(1-\theta^2)\cos(2k - 2h) \\ &+ \frac{15}{32}e^2(1+\theta)^2\cos(2k - 2g - 2h) \\ &\left. + \frac{15}{32}e^2(1-\theta)^2\cos(2k + 2g - 2h) \right\}, \end{aligned} \tag{4.14}$$

$S_1 = 0$ since the determining function has the identical transformation, consequently the Hamiltonian $F_3^* = 0$. The determining function S_2 is given by

$$S_2 = S_{21} + S_{22} + S_{23}, \tag{4.15}$$

where

$$\left. \begin{aligned} S_{21} &= \frac{1}{4}\frac{\nu^2 a^2}{n}\left\{A_{21}^{(1)} + B_{21}^{(1)}\cos(2y_2) + C_{21}^{(1)}\sin(2y_2)\right\}, \\ S_{22} &= \frac{1}{4}\frac{\nu^2 a^2}{n}\left\{A_{22}^{(1)} + B_{22}^{(1)}\cos(2y_2) + C_{22}^{(1)}\sin(2y_2)\right\}, \\ S_{23} &= \frac{1}{4}\frac{\nu^2 a^2}{n}\left\{B_{23}^{(1)}\cos(2y_2) + C_{23}^{(1)}\sin(2y_2)\right\}, \end{aligned} \right\} \tag{4.16}$$

the symbols above $A_{2j}^{(1)}$, $B_{2j}^{(1)}$ and $C_{2j}^{(1)}$ have the expressions

$$\left.\begin{aligned} A_{2j}^{(1)} &= \frac{1}{2\pi}\int_0^{2\pi} A_{2j}dl, \\ B_{2j}^{(1)} &= \frac{1}{2\pi}\int_0^{2\pi} B_{2j}dl, \\ C_{2j}^{(1)} &= \frac{1}{2\pi}\int_0^{2\pi} C_{2j}dl, \end{aligned}\right\} \tag{4.17}$$

for $j = 1, 2, 3$, and $A_{23} = 0$. Here, we put $dl = (1 - e\cos u)du$ and consider the average relations

$$\left.\begin{aligned} \langle \cos u \rangle &= -\frac{1}{2}e, \\ \langle \cos ju \rangle &= 0, \\ \langle \sin ju \rangle &= 0, \end{aligned}\right\} \tag{4.18}$$

for $j \geq 2$. Thus, we obtain

$$\begin{aligned} S_{21} = \; & \frac{1}{4}\frac{\nu^2 a^2}{n}\left\{-\frac{1}{32}e(-2+e^2)(1+\theta)^2\sin(2k-2g-2h-3u)\right. \\ & - \frac{3}{32}(2+e^2)(1+\theta)^2\sin(2k-2g-2h-2u) \\ & - \frac{15}{32}e(-2+e^2)(1+\theta)^2\sin(2k-2g-2h-u) \\ & + \frac{1}{8}e(-8+3e^2)(-1+3\theta^2)\sin(u) \\ & + \frac{3}{8}e^2(-1+3\theta^2)\sin(2u) + \frac{1}{24}e^3(1-3\theta^2)\sin(3u) \\ & + \frac{15}{32}e(-2+e^2)(1+\theta)^2\sin(2k-2g-2h+u) \\ & + \frac{3}{32}(2+e^2)(1+\theta)^2\sin(2k-2g-2h+2u) \\ & + \frac{1}{32}e(-2+e^2)(1+\theta)^2\sin(2k-2g-2h+3u) \\ & + \eta\left(\frac{15}{16}e^2(1+\theta)^2\sin(2k-2g-2h)\right. \\ & + \frac{1}{16}e(1+\theta)^2\sin(2k-2g-2h-3u) \\ & - \frac{3}{16}(1+e^2)(1+\theta)^2\sin(2k-2g-2h-2u) \\ & + \frac{15}{16}e(1+\theta)^2\sin(2k-2g-2h-u) \end{aligned}$$

$$
\begin{aligned}
&+ \frac{15}{16}e(1+\theta)^2\sin(2k-2g-2h+u) \\
&- \frac{3}{16}(1+e^2)(1+\theta)^2\sin(2k-2g-2h+2u) \\
&+ \frac{1}{16}e(1+\theta)^2\sin(2k-2g-2h+3u)\Big)\Big\}, \qquad (4.19)
\end{aligned}
$$

$$
\begin{aligned}
S_{22} \;=\;& \frac{1}{4}\frac{\nu^2a^2}{n}\Big\{\frac{1}{16}e(-2+e^2)(-1+\theta^2)\sin(2g-3u) - \frac{1}{16}e^3(-1+\theta^2) \\
&* \sin(2k-2h-3u) + \frac{3}{16}(2+e^2)(-1+\theta^2)\sin(2g-2u) \\
&+ \frac{9}{16}e^2(-1+\theta^2)\sin(2k-2h-2u) + \frac{15}{16}e(-2+e^2)(-1+\theta^2) \\
&* \sin(2g-u) + \frac{3}{16}e(-8+3e^2)(-1+\theta^2)\sin(2k-2h-u) \\
&- \frac{15}{16}e(-2+e^2)(-1+\theta^2)\sin(2g+u) - \frac{3}{16}e(-8+3e^2)(-1+\theta^2) \\
&* \sin(2k-2h+u) - \frac{3}{16}(2+e^2)(-1+\theta^2)\sin(2g+2u) \\
&- \frac{9}{16}e^2(-1+\theta^2)\sin(2k-2h+2u) - \frac{1}{16}e(-2+e^2)(-1+\theta^2) \\
&* \sin(2g+3u) + \frac{1}{16}e^3(-1+\theta^2)\sin(2k-2h+3u) \\
&+ \eta\Big(\frac{15}{8}e^2(-2+e^2)(-1+\theta^2)\sin(2g) + \frac{1}{8}e(-1+\theta^2)\sin(2g-3u) \\
&- \frac{3}{8}(1+e^2)(-1+\theta^2)\sin(2g-2u) + \frac{15}{8}e(-1+\theta^2)\sin(2g-u) \\
&+ \frac{15}{8}e(-1+\theta^2)\sin(2g+u) - \frac{3}{8}(1+e^2)(-1+\theta^2)\sin(2g+2u) \\
&+ \frac{1}{8}e(-1+\theta^2)\sin(2g+3u)\Big)\Big\}, \qquad (4.20)
\end{aligned}
$$

finally S_{23} is given by

$$
\begin{aligned}
S_{23} \;=\;& \frac{1}{4}\frac{\nu^2a^2}{n}\Big\{-\frac{1}{32}e(-2+e^2)(-1+\theta)^2\sin(2k+2g-2h-3u) \\
&- \frac{3}{32}(2+e^2)(-1+\theta)^2\sin(2k+2g-2h-2u) \\
&- \frac{15}{32}e(-2+e^2)(-1+\theta)^2\sin(2k+2g-2h-u) \\
&+ \frac{15}{32}e(-2+e^2)(-1+\theta)^2\sin(2k+2g-2h+u)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{3}{32}(2+e^2)(-1+\theta)^2\sin(2k+2g-2h+2u) \\
&+ \frac{1}{32}e(-2+e^2)(-1+\theta)^2\sin(2k+2g-2h+3u) \\
&+ \eta\left(-\frac{15}{16}e^2(-1+\theta)^2\sin(2k+2g-2h)\right. \\
&- \frac{1}{16}e(-1+\theta)^2\sin(2k+2g-2h-3u) \\
&+ \frac{3}{16}(1+e^2)(-1+\theta)^2\sin(2k+2g-2h-2u) \\
&- \frac{15}{16}e(-1+\theta)^2\sin(2k+2g-2h-u) \\
&- \frac{15}{16}e(-1+\theta)^2\sin(2k+2g-2h+u) \\
&+ \frac{3}{16}(1+e^2)(-1+\theta)^2\sin(2k+2g-2h+2u) \\
&\left.\left.- \frac{1}{16}e(-1+\theta)^2\sin(2k+2g-2h+3u)\right)\right\}.
\end{aligned} \tag{4.21}
$$

We intended removing any secular terms from the determining function by applying the mathematical operations

$$
\left.\begin{aligned}
A_{2j}^{(1)} &= A_{2j}^{(1)} - \langle A_{2j}^{(1)}\rangle, \\
B_{2j}^{(1)} &= B_{2j}^{(1)} - \langle B_{2j}^{(1)}\rangle, \\
C_{2j}^{(1)} &= C_{2j}^{(1)} - \langle C_{2j}^{(1)}\rangle,
\end{aligned}\right\} \tag{4.22}
$$

for $j = 1, 2, 3$.

Substituting from equations (4.19), (4.20) and (4.21) in equation (4.15) we get the analytical expression of the determining function S_2. Here, $A_{2j}^{(1)}$, $B_{2j}^{(1)}$ and $C_{2j}^{(1)}$ are free from any angular variables, however they are factorized by $(1-\theta^2) \equiv \sin^2 i$. The small parameter in this theory is roughly of the order of 10^{-3}, this means that if the inclination of Nereid is $\sim 10^o$, then $\sin^2 i \sim \sqrt{\epsilon}$ (semi order). Now we are going to the derivation of the fourth order Hamiltonian F_4^*. It can be given by the simple expression

$$
F_4^* = F_{41}^* + F_{42}^* + F_{43}^*, \tag{4.23}
$$

where

$$\left.\begin{aligned} F^*_{41} &= \frac{1}{2}\left(\{F_{21p}, S_{21}\}_s + \{F_{21p}, S_{22}\}_s + \{F_{21p}, S_{23}\}_s\right), \\ F^*_{42} &= \frac{1}{2}\left(\{F_{22p}, S_{21}\}_s + \{F_{22p}, S_{22}\}_s + \{F_{22p}, S_{23}\}_s\right), \\ F^*_{43} &= \frac{1}{2}\left(\{F_{23p}, S_{21}\}_s + \{F_{23p}, S_{22}\}_s + \{F_{23p}, S_{23}\}_s\right), \end{aligned}\right\} \tag{4.24}$$

the subscripts s and p define the secular and periodic parts respectively, F_{2jp} and S_{2j} for $j = 1, 2, 3$ are given by equations (4.8) and (4.16). Proceeding various mathematical derivations, the Poisson bracket $\{F_{2p}, S_2\}$ can be reduced to the form

$$\begin{aligned} \{F_{2p}, S_2\} &= \frac{\nu^4 a^2}{16n^2}\left\{\left(4\mathcal{D} + \frac{\eta^2}{e}\mathcal{D}_e\right)\mathcal{D} - \frac{\eta}{e}\mathcal{D}_e(2\mathcal{E}) + \frac{\eta}{e}\mathcal{D}_g\mathcal{P}_e\right. \\ &\quad - \left(7\mathcal{P} + \frac{\eta^2}{e}P_e\right)\mathcal{D}_l + L\left(2\mathcal{E}\mathcal{D}_G - \mathcal{D}_g\mathcal{P}_G\right) \\ &\quad \left. - L\left(\mathcal{P}_H\mathcal{D}_h - \mathcal{P}_h\mathcal{D}_H\right)\right\}, \end{aligned} \tag{4.25}$$

where

$$\mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_3, \tag{4.26}$$

$$\left.\begin{aligned} \mathcal{D}_1 &= A_{21} + B_{21}\cos(2y_2) + C_{21}\sin(2y_2), \\ \mathcal{D}_2 &= A_{22} + B_{22}\cos(2y_2) + C_{22}\sin(2y_2), \\ \mathcal{D}_3 &= B_{23}\cos(2y_2) + C_{23}\sin(2y_2), \end{aligned}\right\} \tag{4.27}$$

$$\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3, \tag{4.28}$$

$$\left.\begin{aligned} \mathcal{E}_1 &= -B^{(1)}_{21}\sin(2y_2) + C^{(1)}_{21}\cos(2y_2), \\ \mathcal{E}_2 &= -B^{(1)}_{22}\sin(2y_2) + C^{(1)}_{22}\cos(2y_2), \\ \mathcal{E}_3 &= -B^{(1)}_{23}\sin(2y_2) + C^{(1)}_{23}\cos(2y_2), \end{aligned}\right\} \tag{4.29}$$

similarly, $\mathcal{P}$ can be defined by the following form

$$\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3, \tag{4.30}$$

where

$$
\left.\begin{aligned}
\mathcal{P}_1 &= A_{21}^{(1)} + B_{21}^{(1)}\cos(2y_2) + C_{21}^{(1)}\sin(2y_2),\\
\mathcal{P}_2 &= A_{22}^{(1)} + B_{22}^{(1)}\cos(2y_2) + C_{22}^{(1)}\sin(2y_2),\\
\mathcal{P}_3 &= B_{23}^{(1)}\cos(2y_2) + C_{23}^{(1)}\sin(2y_2).
\end{aligned}\right\} \tag{4.31}
$$

Since $\mathcal{D}$ and $\mathcal{P}$ are functions of the eccentric anomaly u, then their partial derivatives $\mathcal{D}_e$ and $\mathcal{P}_e$ with respect to e deliver

$$
\frac{\partial(\mathcal{D},\mathcal{P})}{\partial e} = \left(\frac{\partial(\mathcal{D},\mathcal{P})}{\partial e}\right) + \frac{\partial(\mathcal{D},\mathcal{P})}{\partial u}\frac{\partial u}{\partial e}, \tag{4.32}
$$

It should be to follow the rules of the partial derivatives in the set of equations (3.35). Taking the secular part of the Poisson bracket $\{F_{2p}, S_2\}$, the analytical form of F_4^* arises

$$
F_4^* = F_{4s}^* + F_{4p}^*, \tag{4.33}
$$

where

$$
\begin{aligned}
F_{4s}^* \;=\; & \frac{\nu^4 a^2}{n^2}\left\{\frac{1}{4096}\left(-8\left(47+282\theta^2+63\theta^4\right)-63e^2\left(239+170\theta^2+143\theta^4\right)\right.\right.\\
&+\left.72e^2\left(377+190\theta^2+209\theta^4\right)\right)+\frac{1}{1024}\left(3e^2\left(-1+\theta^2\right)\left(-6\left(83+195\theta^2\right)\right.\right.\\
&+\left.\left.\left. e^2\left(131+555\theta^2\right)\right)\right)\cos(2g)+\frac{1845}{4096}e^4\left(-1+\theta^2\right)^2\cos(4g)\right\},
\end{aligned} \tag{4.34}
$$

and the periodic part has the form

$$
\begin{aligned}
F_{4p}^* \;=\; & \frac{\nu^4 a^2}{n^2}\left\{-\frac{3}{4096}\left(56-1672e^2+1001e^4\right)\left(-1+\theta^2\right)^2\cos(4k-4h)\right.\\
&+\frac{615}{8192}e^4\left(1+\theta\right)^4\cos(4k-4g-4h)+\frac{15}{2048}e^2\left(-78+37e^2\right)\left(-1+\theta\right)\\
&*\left(1+\theta\right)^3\cos(4k-2g-4h)+\frac{15}{2048}e^2\left(-78+37e^2\right)\left(-1+\theta\right)^3\left(1+\theta\right)\\
&*\cos(4k+2g-4h)+\frac{615}{8192}e^4\left(-1+\theta\right)^4\cos(4k+4g-4h)+\frac{3}{1024}\left(-1+\theta^2\right)\\
&*\left(216+56\theta^2-152e^2\left(13+11\theta^2\right)+7e^4\left(199+143\theta^2\right)\right)\cos(2k-2h)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{615}{2048} e^4 (-1+\theta)(1+\theta)^3 \cos(2k-4g-2h) - \frac{3}{512} e^2 (1+\theta)^2 \\
&* \left(-222 + 390\theta - 390\theta^2 + e^2 \left(79 - 185\theta + 185\theta^2\right)\right) \cos(2k-2g-2h) \\
&- \frac{3}{512} e^2 (-1+\theta)^2 \left(-6\left(37 + 65\theta + 65\theta^2\right) + e^2 \left(79 + 185\theta + 185\theta^2\right)\right) \\
&* \cos(2k+2g-2h) - \frac{615}{2048} e^4 (-1+\theta)^3 \cos(2k+4g-2h) \Big\},
\end{aligned} \tag{4.35}
$$

where $g = \varpi - \Omega$. By the virtue of the powerful Mathematica software, we could evaluate and simplify the analytic formulae of both the Hamiltonian and the determining function of the present theory although their derivations were laborious. We give S_3 as follows

$$
S_3 = S_{31} + S_{32} + S_{33}, \tag{4.36}
$$

where

$$
\left.
\begin{aligned}
S_{31} &= \frac{1}{4} \frac{\nu^3 a^2}{n^2} \mathcal{M}_1, \\
S_{32} &= \frac{1}{4} \frac{\nu^3 a^2}{n^2} \mathcal{M}_2, \\
S_{33} &= \frac{1}{4} \frac{\nu^3 a^2}{n^2} \mathcal{M}_3,
\end{aligned}
\right\} \tag{4.37}
$$

$$
\left.
\begin{aligned}
\mathcal{M}_1 &= -2B_{21}^{(2)} \sin(2y_2) + 2C_{21}^{(2)} \cos(2y_2), \\
\mathcal{M}_2 &= -2B_{22}^{(2)} \sin(2y_2) + 2C_{22}^{(2)} \cos(2y_2), \\
\mathcal{M}_3 &= -2B_{23}^{(2)} \sin(2y_2) + 2C_{23}^{(2)} \cos(2y_2),
\end{aligned}
\right\} \tag{4.38}
$$

and

$$
\left.
\begin{aligned}
B_{2j}^{(2)} &= \frac{1}{2\pi} \int_0^{2\pi} B_{2j}^{(1)} dl, \\
C_{2j}^{(2)} &= \frac{1}{2\pi} \int_0^{2\pi} C_{2j}^{(1)} dl,
\end{aligned}
\right\} \tag{4.39}
$$

for $j = 1, 2, 3$. For the purpose of removing any secular term from the determining function, we subtracted the averages from the original expressions as follows

$$
\left.
\begin{aligned}
B_{2j}^{(2)} &= B_{2j}^{(2)} - \langle B_{2j}^{(2)} \rangle, \\
C_{2j}^{(2)} &= C_{2j}^{(2)} - \langle C_{2j}^{(2)} \rangle.
\end{aligned}
\right\} \tag{4.40}
$$

Then, the final form of S_3 is represented by its three parts S_{31}, S_{32} and S_{33}

$$\begin{aligned}
S_{31} \;=\;& \frac{1}{4}\frac{\nu^3 a^2}{n^2}(1+\theta)^2 \Big\{ -\frac{3}{32}e^2(-22+9e^2)\sin(2k-2g-2h) + \frac{1}{128}e^2(-2+e^2) \\
& * \sin(2k-2g-2h-4u) + \frac{1}{96}e(10+e^2)\sin(2k-2g-2h-3u) \\
& + \frac{1}{32}(-6-19e^2+8e^4)\sin(2k-2g-2h-2u) - \frac{3}{32}e(-22+9e^2) \\
& \sin(2k-2g-2h-u) - \frac{3}{32}e(-22+9e^2)\sin(2k-2g-2h+u) \\
& + \frac{1}{32}(-6-19e^2+8e^4)\sin(2k-2g-2h+2u) \\
& + \frac{1}{96}e(10+e^2)\sin(2k-2g-2h+3u) + \frac{1}{128}e^2(-2+e^2) \\
& * \sin(2k-2g-2h+4u) + \eta\Big(-\frac{1}{64}e^2\sin(2k-2g-2h-4u) \\
& + \frac{1}{48}e(5+3e^2)\sin(2k-2g-2h-3u) - \frac{1}{16}(3+11e^2) \\
& * \sin(2k-2g-2h-2u) - \frac{3}{16}e(-11+4e^2)\sin(2k-2g-2h-u) \\
& + \frac{3}{16}e(-11+4e^2)\sin(2k-2g-2h+u) + \frac{1}{16}(3+11e^2) \\
& * \sin(2k-2g-2h+2u) - \frac{1}{48}e(5+3e^2)\sin(2k-2g-2h+3u) \\
& + \frac{1}{64}e^2\sin(2k-2g-2h+4u)\Big)\Big\}, \qquad (4.41)
\end{aligned}$$

$$\begin{aligned}
S_{32} \;=\;& \frac{1}{4}\frac{\nu^3 a^2}{n^2}(-1+\theta^2)\Big\{ \frac{3}{16}e^2(-16+3e^2)\sin(2k-2h) + \frac{1}{64}e^4\sin(2k-2h-4u) \\
& - \frac{11}{48}e^3\sin(2k-2h-3u) - \frac{1}{16}e^2(-21+4e^2)\sin(2k-2h-2u) \\
& + \frac{3}{16}e(-16+3e^2)\sin(2k-2h-u) + \frac{3}{16}e(-16+3e^2)\sin(2k-2h+u) \\
& - \frac{1}{16}e^2(-21+4e^2)\sin(2k-2h+2u) - \frac{11}{48}e^3\sin(2k-2h+3u) \\
& + \frac{1}{64}e^4\sin(2k-2h+4u)\Big\}, \qquad (4.42)
\end{aligned}$$

and S_{33} is given by

$$S_{33} \;=\; \frac{1}{4}\frac{\nu^3 a^2}{n^2}(-1+\theta)^2\Big\{ -\frac{3}{32}e^2(-22+9e^2)\sin(2k+2g-2h) + \frac{1}{128}e^2(-2+e^2)$$

$$
\begin{aligned}
&* \sin(2k+2g-2h-4u) + \frac{1}{96}e(10+e^2)\sin(2k+2g-2h-3u) \\
&+ \frac{1}{32}(-6-19e^2+8e^4)\sin(2k+2g-2h-2u) - \frac{3}{32}e(-22+9e^2) \\
&\sin(2k+2g-2h-u) - \frac{3}{32}e(-22+9e^2)\sin(2k+2g-2h+u) \\
&+ \frac{1}{32}(-6-19e^2+8e^4)\sin(2k+2g-2h+2u) \\
&+ \frac{1}{96}e(10+e^2)\sin(2k+2g-2h+3u) + \frac{1}{128}e^2(-2+e^2) \\
&* \sin(2k+2g-2h+4u) + \eta\left(-\frac{1}{64}e^2\sin(2k+2g-2h-4u)\right. \\
&+ \frac{1}{48}e(5+3e^2)\sin(2k+2g-2h-3u) - \frac{1}{16}(3+11e^2) \\
&* \sin(2k+2g-2h-2u) + \frac{3}{16}e(-11+4e^2)\sin(2k-2g-2h-u) \\
&- \frac{3}{16}e(-11+4e^2)\sin(2k+2g-2h+u) - \frac{1}{16}(3+11e^2) \\
&* \sin(2k+2g-2h+2u) + \frac{1}{48}e(5+3e^2)\sin(2k+2g-2h+3u) \\
&\left.\left. - \frac{1}{64}e^2\sin(2k+2g-2h+4u)\right)\right\}, \qquad (4.43)
\end{aligned}
$$

where $u = k - \varpi$.

4.4 Long periodic perturbations

In this Section, we remove the long (intermediate) periodic terms which are related to the motion of the Sun. This will be achieved by building the canonical transformation

$$(L^*, G^*, H^*; l^*, g^*, h^*, \lambda_\odot) \longrightarrow (L^{**}, G^{**}, H^{**}; l^{**}, g^{**}, h^{**}, \lambda_\odot), \qquad (4.44)$$

In order that:

$$F^*(L^*, G^*, H^*; l^*, g^*, h^*, \lambda_\odot) \longrightarrow F^{**}(L^{**}, G^{**}, H^{**}; -, g^{**}, -, -), \qquad (4.45)$$

and the determining function S only includes the new variables $L^{**}, G^{**}, H^{**}; l^{**}, g^{**}, h^{**}, \lambda_{\odot}$. The new Hamiltonian F^{**} will be also free from h, since the disturbing potential becomes axial symmetric. After eliminating the long (intermediate) terms, the orbital elements a^{**}, e^{**}, n^{**} and η^{**} are copmuted from

$$a^{**} = \frac{L^{**2}}{\mu}, e^{**} = \sqrt{1 - \left(\frac{G^{**}}{L^{**}}\right)^2}, n^{**} = \frac{\mu^2}{L^{**3}}, \eta^{**} = \frac{G^{**}}{L^{**}}. \tag{4.46}$$

In this Section right now, the superscript ** will be omitted from the orbital elements. We follow the algorithm concerned the long-period terms to get the new Hamiltonian and determining functions as follows

$$F_0^{**} = \frac{\mu^2}{2L^2}, F_1^{**} = \nu G, \tag{4.47}$$

$$F_2^{**} = \nu^2 a^2 \left\{ \left(1 + \frac{3}{2}e^2\right) + \left(\frac{-1}{8} + \frac{3}{8}\theta^2\right) + \frac{15}{16}e^2(1-\theta^2)\cos(2g) \right\}, \tag{4.48}$$

$$\begin{aligned} S_1^* &= \nu a^2 \left\{ \frac{3}{16}\left(1 + \frac{3}{2}e^2\right)(1-\theta^2)\sin(2k-2h) + \frac{15}{64}e^2(1+\theta)^2 \right. \\ &\quad \left. * \sin(2k-2g-2h) + \frac{15}{64}e^2(1-\theta)^2\sin(2k+2g-2h) \right\}, \end{aligned} \tag{4.49}$$

$$F_3^{**} = \frac{\nu^3 a^2}{n}\eta \left\{ \frac{9}{128}\theta(2 - 2\theta^2 + e^2(33 + 17\theta^2)) + \frac{135}{128}e^2\theta(1-\theta^2)\cos(2g) \right\}, \tag{4.50}$$

$$\begin{aligned} S_2^* &= \frac{\nu^2 a^2}{n}\eta \left\{ \frac{9}{128}(-2 + 17e^2)\theta(1-\theta^2)\sin(2k-2h) \right. \\ &\quad + \frac{45}{256}e^2(1+\theta)^2(-2+3\theta)\sin(2k-2g-2h) \\ &\quad \left. + \frac{45}{256}e^2(1-\theta)^2(2+3\theta)\sin(2k+2g-2h) \right\}, \end{aligned} \tag{4.51}$$

$$\begin{aligned} F_4^{**} &= \frac{\nu^4 a^2}{n^2} \left\{ \frac{1}{8192}(8(-67 - 726\theta^2 + 9\theta^4) + 144e^2(329 + 253\theta^2 + 344\theta^4) \right. \\ &\quad - 9e^4(2527 + 27940\theta^2 + 54070\theta^4)) + \frac{3}{2048}e^2(-1+\theta^2)(-12(68 + 345\theta^2) \\ &\quad \left. + e^2(307 + 4035\theta^2))\cos(2g) + \frac{315}{8192}e^4(-1+\theta^2)^2\cos(4g) \right\}, \end{aligned} \tag{4.52}$$

while S_3^* has the analytical expression

$$
\begin{aligned}
S_3^* = \; & \frac{\nu^3 a^2}{n^2}\Big\{(-1+\theta^2)^2\left(-\frac{69}{1024}+\frac{3}{4}e^2+\frac{123}{8192}e^4\right)\sin(4k-4h) \\
& + \frac{1905}{16384}e^4(1+\theta)^4\sin(4k-4g-4h) \\
& + (-1+\theta)(1+\theta)^3\left(-\frac{315}{1024}e^2-\frac{645}{4096}e^4\right)\sin(4k-2g-4h) \\
& + (-1+\theta)(1+\theta)^3\left(-\frac{315}{1024}e^2-\frac{645}{4096}e^4\right)\sin(4k+2g-4h) \\
& + \frac{1905}{16384}e^4(1+\theta)^4\sin(4k+4g-4h)+(-1+\theta^2) \\
& * \left(\frac{171}{256}-\frac{39}{256}\theta^2-\frac{3}{256}e^2(215+904\theta^2)+\frac{3}{2048}e^4(1355+8131\theta^2)\right) \\
& * \sin(2k-2h)+\frac{795}{4096}e^4(-1+\theta)(1+\theta)^3\sin(2k-4g-2h)+(1+\theta)^2 \\
& * \left(\frac{9}{512}e^2(79-175\theta+310\theta^2)-\frac{3}{1024}e^4(563-1315\theta+2125\theta^2)\right) \\
& * \sin(2k-2g-2h)+(-1+\theta)^2\left(\frac{9}{512}e^2(79+175\theta+310\theta^2)\right. \\
& \left. -\frac{3}{1024}e^4(563+1315\theta+2125\theta^2)\right)\sin(2k+2g-2h) \\
& + \frac{795}{4096}e^4(-1+\theta)^3(1+\theta)\sin(2k+4g-2h)\Big\}. \qquad (4.53)
\end{aligned}
$$

Notice that all the above analytical expressions satisfy d'Alembert characteristics mentioned in the precedent chapter. This may prove the validity of these expressions from the analytical point of view. Up to this stage, the Hamiltonian system is still include the long terms g. Omitting these terms can be done by two ways, the first one (which is analogue to the previous procedures) is to build a new canonical transformation

$$(L^{**}, G^{**}, H^{**}; l^{**}, g^{**}, h^{**}, \lambda_\odot) \longrightarrow (L^{***}, G^{***}, H^{***}; l^{***}, g^{***}, h^{***}, \lambda_\odot) \qquad (4.54)$$

in order that:

$$F^{**}(L^{**}, G^{*}, H^{**}; -, g^{**}, -, -) \longrightarrow F^{***}(L^{***}, G^{***}, H^{***}; -, -, -, -) \qquad (4.55)$$

and the final results give the mean elements. The second way is to use the Jacobian elliptic functions (Kinoshita and Nakai 1991, 1999) in solving the Hamiltonian equations system

$$\left.\begin{array}{l} L^{**} = \text{const.}, \dfrac{dl^{**}}{dt} = -\dfrac{\partial F^{**}}{\partial L^{**}}, \\ \dfrac{dG^{**}}{dt} = \dfrac{\partial F^{**}}{\partial g^{**}}, \dfrac{dg^{**}}{dt} = -\dfrac{\partial F^{**}}{\partial G^{**}}, \\ H^{**} = \text{const.}, \dfrac{dh^{**}}{dt} = -\dfrac{\partial F^{**}}{\partial H^{**}}, \end{array}\right\} \tag{4.56}$$

and the final results are also mean elements. In this theory we adopt the latter's method to remove g from the new Hamiltonian system.

4.5 Secular perturbations

4.5.1 Solution for the elements e, I and g

The Hamiltonian equations system (4.56) delivers the following forms

$$\begin{aligned}\frac{dG^{**}}{dt} &= \frac{15}{8}\nu^2 a^2 e^2(-1+\theta^2)\sin(2g) + \frac{135}{64}\frac{\nu^3 a^2}{n}\eta e^2\theta(-1+\theta^2) \\ &* \sin(2g) + \frac{\nu^4 a^2}{n^2}\left\{-\frac{1}{1024}\left(3e^2(-1+\theta^2)(-12(68+345\theta^2)\right.\right. \\ &\left.\left. + e^2(307+4035\theta^2))\right)\sin(2g) - \frac{315}{2048}e^4(-1+\theta^2)^2\sin(4g)\right\}, \end{aligned} \tag{4.57}$$

and

$$\begin{aligned}\frac{dg^{**}}{dt} &= -\frac{3}{8}\frac{\nu^2}{n}\frac{1}{\eta}\left\{1-e^2-5\theta^2+5(-1+e^2+\theta^2)\right\}\cos(2g) \\ &-\frac{27}{64}\frac{\nu^3}{n^2}\theta\left\{-11+11e^2-5\theta^2+5(-1+e^2+\theta^2)\right\}\cos(2g) \\ &+\frac{3}{2048}\frac{\nu^4}{n^3}\frac{1}{\eta}\left\{7896-15477e^2+7581e^4+5104\theta^2-8382e^2\theta^2\right. \\ &+4191e^4\theta^2+8280\theta^4-7965e^2\theta^4+4(408+1662\theta^2-2070\theta^4 \\ &+e^4(307+1864\theta^2)+e^2(-715-3728\theta^2+1965\theta^4))\cos(2g) \\ &\left.+105e^2(-1+\theta^2)(-1+e^2+\theta^2)\cos(4g)\right\}. \end{aligned} \tag{4.58}$$

We introduce the analytical solution of the elements e, I, g where the circulation case of the argument of pericenter (see Kinoshita and Nakai 1999, for further details). For simplicity, the two differential equations above will be truncated up to the second order and can be rewritten in the following forms

$$\frac{d\eta}{dt^*} = \frac{15}{8}e^2(-1+\theta^2)\sin(2g), \tag{4.59}$$

and

$$\frac{dg}{dt^*} = -\frac{3}{8}\frac{1}{\eta}\left\{1-e^2-5\theta^2+5(-1+e^2+\theta^2)\right\}\cos(2g), \tag{4.60}$$

where

$$t^* = \gamma^* t, \gamma^* = \frac{m_\odot}{m_\odot + m_{nep}}(1 - e_\odot^2)^{-3/2}\frac{\nu^2}{n}, \tag{4.61}$$

note that in the previous analytical expressions the ratio $\dfrac{m_\odot}{m_\odot + m_{nep}} \equiv 1$. The energy integral of the second order takes the form

$$C = (5 - 3x)(3h/x - 1) + 15(1 - x)(1 - h/x)\cos(2g), \tag{4.62}$$

where $x = \eta^2$, and $\eta \cos I = \sqrt{h} = \text{const}$. Considering the initial conditions $I = I_0$ and $x = x_0$ at $g = 0$, then we have

$$C = 10 - 12x_0 + 6h. \tag{4.63}$$

Then we can easily write

$$\cos(2g) = \frac{\mathcal{F}(x)}{5(1 - x)(x - h)}, \tag{4.64}$$

where

$$\mathcal{F}(x) = -x^2 + x(5(1 + h) - 4x_0) - 5h. \tag{4.65}$$

With the helping of the trigonometric relations

$$\left.\begin{aligned} \sin^2 g &= \frac{1}{2}(1 - \cos(2g)), \\ \cos^2 g &= \frac{1}{2}(1 + \cos(2g)), \end{aligned}\right\} \tag{4.66}$$

we can deduce

$$\left.\begin{aligned} \sin^2 g &= \frac{2x(x_0 - x)}{5(1 - x)(x - h)}, \\ \sin^2 g &= \frac{y}{5(1 - x)(x - h)}, \end{aligned}\right\} \tag{4.67}$$

where

$$y = -3x^2 + x(5 + 5h - 2x_0) - 5h. \tag{4.68}$$

Since $x = \eta^2$, then $dx/dt^* = 2\eta d\eta/dt^*$, consequently

$$\frac{dx}{dt^*} = \frac{15}{4}\eta c^2(-1 + \theta^2)\sin(2g). \tag{4.69}$$

Using the notations

$$\left.\begin{aligned} e^2 &= (1-x), \\ \cos^2 I &= \frac{h}{\eta^2}, \\ \sin^2 I &= \frac{(x-h)}{\eta^2}, \\ (-1+\theta^2) &= -\sin^2 I, \\ \sin(2g) &= 2\sin g\cos g, \end{aligned}\right\} \tag{4.70}$$

together with equations (4.67) and (4.68) in equation (4.69), then the equation of motion of x reduces to the form

$$\frac{dx}{dt^*} = -\frac{3}{2}\sqrt{2(x_0-x)y}. \tag{4.71}$$

The quadratic equation $y=0$ has the solution

$$x_1, x_2 = \frac{1}{6}(5+5h-2x_0 \pm \sqrt{-60+(-5-5h+2x_0)^2}, \tag{4.72}$$

in order that:

$$\left.\begin{aligned} x_1+x_2 &= \frac{1}{3}(5+5h-2x_0), \\ x_1x_2 &= \frac{5}{3}h. \end{aligned}\right\} \tag{4.73}$$

Thus, the equation of x delivers

$$\frac{dx}{dt^*} = -\frac{3\sqrt{6}}{2}\sqrt{(x-x_1)(x-x_2)(x_0-x)}. \tag{4.74}$$

The solution of the above equation will be expressed in terms of the Jacobian elliptic function as follows

$$x = x_0 + (x_1-x_0)\cos^2 \mathcal{Z}, \tag{4.75}$$

where

$$\mathcal{Z} = \frac{2\mathcal{K}}{\pi}(g^{**}+\frac{\pi}{2}), \tag{4.76}$$

$$g^{**} = n_{g^{**}} t, \tag{4.77}$$

$$n_{g^{**}} = \frac{3\sqrt{6}\pi}{8\mathcal{K}} \sqrt{x_2 - x_1} \gamma^*. \tag{4.78}$$

The symbol $\mathcal{K}$ defines the complete elliptic integral of the first kind, whereas n_g^{**} is the mean motion of the angular variable g^{**}. If we chose $x_1 < x_2$, the maximum eccentricity arises as

$$e_{max} = \sqrt{1 - x_1}, \mathrm{for} g^{**} = \frac{\pi}{2}, \tag{4.79}$$

and the corresponding inclination takes the form

$$I_{min} = \cos^{-1} \sqrt{h/(1 - e_{max}^2)}, \mathrm{for} I_0 < \frac{\pi}{2}. \tag{4.80}$$

4.5.2 Solution for the elements h and l

In this Subsection, we address briefly the solution for the longitude of ascending node and the mean anomaly, by the usage of the Fourier series expansions. The fourth order differential equations of both h^{**} and l^{**} are given sequently as follows

$$\begin{aligned} \frac{dh^{**}}{dt} &= \frac{3}{8}\frac{\nu^2}{n}\frac{1}{\eta}\theta \left\{-2 - 3e^2 + 5e^2 \cos(2g)\right\} \\ &+ \frac{9}{128}\frac{\nu^3}{n^2} \left\{-2 + 6\theta^2 - 3e^2(11 + 17\theta^2) + 15e^2(-1 + 3\theta^2)\cos(2g)\right\} \\ &- \frac{3}{2048}\frac{\nu^4}{n^3}\frac{1}{\eta}\theta \left\{-968 + 6072e^2 - 4191e^4 + 24\theta^2 + 16152e^2\theta^2 \right. \\ &- 16221e^4\theta^2 + 4e^2(1662 - 4140\theta^2 + e^2(-1864 + 40350\theta^2)) \\ &\left. * \cos(2g) + 105e^4(-1 + \theta^2)\cos(4g)\right\}, \end{aligned} \tag{4.81}$$

$$\begin{aligned} \frac{dl^{**}}{dt} - n &= \frac{1}{8}\frac{\nu^2}{n} \left\{(-7 - 3e^2)(-1 + 3\theta^2) + 15(1 + e^2)(-1 + \theta^2)\cos(2g)\right\} \\ &+ \frac{9}{64}\frac{\nu^3}{n^2}\eta\theta \left\{-39 - 11\theta^2 - 2e^2(33 + 17\theta^2) + 15(1 + 2e^2)(-1 + \theta^2)\cos(2g)\right\} \\ &- \frac{1}{4096}\frac{\nu^4}{n^3} \left\{44696 + 144018e^2 - 68229e^4 + 7392\theta^2 + 95436e^2\theta^2 - 75438e^4\theta^2 \right. \end{aligned}$$

$$+ 498960\theta^4 + 100818e^2\theta^4 - 145989e^4\theta^4 + 12(-1+\theta^2)(-12(68+345\theta^2)$$

$$- 10e^2(265+849\theta^2) + 3e^4(307+4035\theta^2))\cos(2g)$$

$$+ 315e^2(2+3e^2)(-1+\theta^2)^2\cos(4g)\Big\}. \tag{4.82}$$

As in the case of the elements e, I and g, we truncate the above differential equations after the second order (since the solution up to the fourth order, expressed in Jacobian elliptic functions is cumbersome) and rewrite them in terms of x, then we have

$$\frac{dh}{dt} = \frac{3}{8}\gamma^* \frac{\sqrt{h}}{x}\left\{3x - 5 + \frac{\mathcal{F}(x)}{x-h}\right\}, \tag{4.83}$$

$$\frac{dl}{dt} = \frac{1}{8}\frac{1}{x}\gamma^*\left\{(3x-10)(3h-x) + 3(x-2)\frac{\mathcal{F}(x)}{1-x}\right\}, \tag{4.84}$$

where $\mathcal{F}(x)$, γ^* and t^* are already defined in the previous Subsection. Considering the circulation case of the argument of pericenter, the equations (4.83) and (4.84) yield

$$\frac{dh}{dt} = \frac{3}{4}\gamma^*\sqrt{h}\frac{(x+h-2x_0)}{x-h}, \tag{4.85}$$

and

$$\frac{dl}{dt} = \frac{1}{4}\gamma^*\frac{1}{1-x}\left\{x(4+3h-6x_0) + 12x_0 - 3h - 10\right\}. \tag{4.86}$$

In what follows we just give the basic equations of h and l, expressed in Fourier series expansion, while their derivation is given in (Kinoshita and Nakai 1999). At first we write the analytical expressions of h:

$$\left.\begin{aligned}
h &= h^* + \textstyle\sum_{m=1}^{\infty} b_m \sin(2mg^*),\\
h^* &= n_{h^*}t + h_0^*,\\
b_m &= -\frac{2(-q)^m}{m(1-q^{2m})}\sinh m\frac{\pi c}{\mathcal{K}},\\
c &= F(\xi, k'),\\
\sin\xi &= \sqrt{\frac{x_2 - x_1}{x_2 - h}},\\
k' &= \sqrt{1-k^2},\\
k^2 &= \frac{x_0 - x_1}{x_2 - x_1},
\end{aligned}\right\} \tag{4.87}$$

where b_m defines the amplitude of periodic terms, $F(\xi, k')$ is the normal elliptic integral of the second kind, q refers to Jacobi's nome and k^2 describes the modulus of the complete elliptic integral of the first kind $\mathcal{K}$. While the mean motion of the longitude of the ascending node has the form

$$\left.\begin{aligned}
n_{h^*} &= -\frac{3}{4}\sqrt{h}\gamma^*(-1 + 2\frac{x_0 - h}{x_2 - h}) - \Lambda_0(\xi, k)n_{g^*},\\
\Lambda_0(\xi, k) &= \frac{2}{\pi}\left\{EF(\xi, k') + \mathcal{K}E(\xi, k') - \mathcal{K}F(\xi, k')\right\},
\end{aligned}\right\} \tag{4.88}$$

where $\Lambda_0(\xi, k)$ is Heuman's Lambda functions and E is the complete elliptic integral of the second kind. Similarly, the correspondence basic equations of the mean anomaly deliver

$$\left.\begin{aligned}
l &= l^* + \textstyle\sum_{m=1}^{\infty} f_m \sin(2mg^*),\\
l^* &= n_{l^*}t + l_0^*,\\
f_m &= -\frac{2q^m}{m(1-q^{2m})}\sinh m\frac{\pi c'}{\mathcal{K}},\\
c' &= F(\zeta, k'),\\
\sin\xi &= \sqrt{\frac{(x_2 - 1)(1 - x_0)}{(1 - x_1)(x_2 - x_0)}},
\end{aligned}\right\} \tag{4.89}$$

and the secular perturbations in l is given by

$$n_{l^*} = \frac{1}{4}\gamma^* (-4 + 6x_0 - 3h) - \Lambda_0(\zeta, k) n_{g^*}. \tag{4.90}$$

All the formulae in this Subsection and the previous one were for the purpose of evaluating the mean orbital elements which are used in computing the osculating elements.

4.6 The osculating orbital elements

In Sections 4.3, 4.4 and 4.5 we discussed the analytical expressions of the short-period, long (intermediate)-period and the secular perturbations respectively. This Section is devoted for evaluating the osculating orbital elements of the nonplanar problem. For this purpose, we first find the partial derivatives of the determining functions with respect to the Delaunay's elements. The derivatives of S_2 can be simplified to

$$\left.\begin{aligned}
\frac{\partial S_2}{\partial L} &= \frac{1}{4}\frac{\nu^2}{n^2}\left(7\mathcal{P} + \frac{\eta^2}{e}\mathcal{P}_e\right), \\
\frac{\partial S_2}{\partial G} &= \frac{1}{4}\frac{\nu^2 a^2}{n}\left(\frac{\partial \mathcal{P}}{\partial G}\right) - \frac{1}{4}\frac{\nu^2}{n^2}\frac{\eta}{e}\mathcal{P}_e, \\
\frac{\partial S_2}{\partial H} &= \frac{1}{4}\frac{\nu^2 a^2}{n}\mathcal{P}_H, \\
\frac{\partial S_2}{\partial l} &= \frac{1}{4}\frac{\nu^2 a^2}{n}\mathcal{D}, \\
\frac{\partial S_2}{\partial g} &= \frac{1}{4}\frac{\nu^2 a^2}{n}\mathcal{P}_g, \\
\frac{\partial S_2}{\partial h} &= \frac{1}{4}\frac{\nu^2 a^2}{n}\mathcal{P}_h,
\end{aligned}\right\} \tag{4.91}$$

where $\mathcal{D}$ and $\mathcal{P}$ are given by equations (4.26) and (4.30) respectively. While $\mathcal{P}_e$ is defined in equation (4.32). The partial derivatives of the determining function S_3 arise from the set of equations

$$\left.\begin{aligned}
\frac{\partial S_3}{\partial L} &= \frac{1}{4}\frac{\nu^3}{n^3}\left(10\mathcal{M} + \frac{\eta^2}{e}\mathcal{M}_e\right),\\
\frac{\partial S_3}{\partial G} &= \frac{1}{4}\frac{\nu^3 a^2}{n^2}\left(\frac{\partial \mathcal{M}}{\partial G}\right) - \frac{1}{4}\frac{\nu^3}{n^3}\frac{\eta}{e}\mathcal{M}_e,\\
\frac{\partial S_3}{\partial H} &= \frac{1}{4}\frac{\nu^3 a^2}{n^2}\mathcal{M}_H,\\
\frac{\partial S_3}{\partial l} &= \frac{1}{4}\frac{\nu^3 a^2}{n^2}(2\mathcal{E}),\\
\frac{\partial S_3}{\partial g} &= \frac{1}{4}\frac{\nu^3 a^2}{n^2}\mathcal{M}_g,\\
\frac{\partial S_3}{\partial h} &= \frac{1}{4}\frac{\nu^3 a^2}{n^2}\mathcal{M}_h,
\end{aligned}\right\} \tag{4.92}$$

where

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3, \tag{4.93}$$

$$\mathcal{M}_e = \left(\frac{\partial \mathcal{M}}{\partial e}\right) + (2\mathcal{E})\sin u, \tag{4.94}$$

$\mathcal{E}$ is defined by equation (4.28), whereas $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ are given by the set of equations (4.38). The partial derivatives of the disturbing functions $S_i (i = 1, 2, 3, 4)$ are simplified to take the following configurations

$$\left.\begin{aligned}
\frac{\partial S_1^*}{\partial L} &= \frac{\nu}{n}\Big\{-\frac{3}{16}(7+3e^2)(-1+\theta^2)\sin(2k-2h) + \frac{15}{32}(1+e^2)(1+\theta)^2\\
&\quad * \sin(2k-2g-2h) + \frac{15}{32}(1+e^2)(-1+\theta)^2\sin(2k+2g-2h)\Big\},\\
\frac{\partial S_1^*}{\partial G} &= \frac{\nu}{n}\frac{1}{\eta}\Big\{\frac{3}{16}(-3+3e^2+5\theta^2)\sin(2k-2h) + \frac{15}{32}(-1+e^2-\theta)(1+\theta)\\
&\quad * \sin(2k-2g-2h) - \frac{15}{32}(-1+\theta)(-1+e^2+\theta)\sin(2k+2g-2h)\Big\},\\
\frac{\partial S_1^*}{\partial H} &= \frac{\nu}{n}\frac{1}{\eta}\Big\{-\frac{3}{16}(2+3e^2)\theta\sin(2k-2h) + \frac{15}{32}e^2(1+\theta)\sin(2k-2g-2h)\\
&\quad + \frac{15}{32}e^2(-1+\theta)\sin(2k+2g-2h)\Big\},\\
\frac{\partial S_1^*}{\partial g} &= \nu a^2\Big\{-\frac{15}{32}e^2(1+\theta)^2\cos(2k-2g-2h) + \frac{15}{32}e^2(1+\theta)^2\cos(2k+2g-2h)\Big\},\\
\frac{\partial S_1^*}{\partial h} &= \nu a^2\Big\{\frac{3}{16}(2+3e^2)(-1+\theta^2)\cos(2k-2h) - \frac{15}{32}e^2(1+\theta)^2\\
&\quad * \cos(2k-2g-2h) - \frac{15}{32}e^2(-1+\theta)^2\cos(2k+2g-2h)\Big\},
\end{aligned}\right\} \tag{4.95}$$

where the superscript * is omitted from the orbital elements, for simplicity. As we already mentioned in chapter 3 the determining functions S^* have contributions in the long period, whereas S for short periodic perturbations. The partial derivatives of S_2^* are evaluated and given by the group equations

$$\left.\begin{aligned}
\frac{\partial S_2^*}{\partial L} &= \frac{\nu^2}{n^2}\eta\Big\{-\frac{9}{64}(11+34e^2)\theta(-1+\theta^2)\sin(2k-2h)\\
&\quad+\frac{45}{128}(1+2e^2)(1+\theta)^2(-2+3\theta)\sin(2k-2g-2h)\\
&\quad+\frac{45}{128}(1+2e^2)(-1+\theta)^2(2+3\theta)\sin(2k+2g-2h)\Big\},\\
\frac{\partial S_2^*}{\partial G} &= \frac{\nu^2}{n^2}\Big\{\frac{9}{64}\theta(-17+17e^2+15\theta^2)\sin(2k-2h)\\
&\quad-\frac{45}{128}(1+\theta)(-2+e^2(3-2\theta)+\theta+3\theta^2)\sin(2k-2g-2h)\\
&\quad-\frac{45}{128}(-1+\theta)(-2-\theta+3\theta^2+e^2(3+2\theta))\sin(2k+2g-2h)\Big\},\\
\frac{\partial S_2^*}{\partial H} &= \frac{\nu^2}{n^2}\Big\{-\frac{9}{128}(-2+17e^2)(-1+3\theta^2)\sin(2k-2h)\\
&\quad+\frac{45}{256}e^2(-1+8\theta+9\theta^2)\sin(2k-2g-2h)\\
&\quad+\frac{45}{256}e^2(-1-8\theta+9\theta^2)\sin(2k+2g-2h)\Big\},\\
\frac{\partial S_2^*}{\partial g} &= \frac{\nu^2 a^2}{n}\eta\Big\{-\frac{45}{128}e^2(1+\theta)^2(-2+3\theta)\cos(2k-2g-2h)\\
&\quad+\frac{45}{128}e^2(-1+\theta)^2(2+3\theta)\cos(2k+2g-2h)\Big\},\\
\frac{\partial S_2^*}{\partial h} &= \frac{\nu^2 a^2}{n}\eta\Big\{\frac{9}{64}(-2+17e^2)\theta(-1+\theta^2)\cos(2k-2h)\\
&\quad-\frac{45}{128}e^2(1+\theta)^2(-2+3\theta)\cos(2k-2g-2h)\\
&\quad-\frac{45}{128}e^2(-1+\theta)^2(2+3\theta)\cos(2k+2g-2h)\Big\},
\end{aligned}\right\} \quad (4.96)$$

Finally, we give the derivatives of the third order S_3^* in spite of their bit little long forms. However, by the knowledge of these derivatives and those of the Hamiltonian, we can evaluate the Poisson brackets with the help of MATHEMATICA package (Wolfram 1996). The derivatives deliver

$$
\left.
\begin{aligned}
\frac{\partial S_3^*}{\partial L} &= \frac{\nu^3}{n^3}\{L_1\sin(4k-4h)+L_2\sin(4k-4g-4h)+L_3\sin(4k-2g-4h)\\
&\quad +L_4\sin(4k+2g-4h)+L_5\sin(4k+4g-4h)+L_6\sin(2k-2h)\\
&\quad +L_7\sin(2k-4g-2h)+L_8\sin(2k-2g-2h)\\
&\quad +L_9\sin(2k+2g-2h)+L_{10}\sin(2k+4g-2h)\},\\
\frac{\partial S_3^*}{\partial G} &= \frac{\nu^3}{n^3}\frac{1}{\eta}\{G_1\sin(4k-4h)+G_2\sin(4k-4g-4h)+G_3\sin(4k-2g-4h)\\
&\quad +G_4\sin(4k+2g-4h)+G_5\sin(4k+4g-4h)+G_6\sin(2k-2h)\\
&\quad +G_7\sin(2k-4g-2h)+G_8\sin(2k-2g-2h)\\
&\quad +G_9\sin(2k+2g-2h)+G_{10}\sin(2k+4g-2h)\},\\
\frac{\partial S_3^*}{\partial H} &= \frac{\nu^3}{n^3}\frac{1}{\eta}\{H_1\sin(4k-4h)+H_2\sin(4k-4g-4h)+H_3\sin(4k-2g-4h)\\
&\quad +H_4\sin(4k+2g-4h)+H_5\sin(4k+4g-4h)+H_6\sin(2k-2h)\\
&\quad +H_7\sin(2k-4g-2h)+H_8\sin(2k-2g-2h)\\
&\quad +H_9\sin(2k+2g-2h)+H_{10}\sin(2k+4g-2h)\},\\
\frac{\partial S_3^*}{\partial g} &= \frac{\nu^3 a^2}{n^2}\{g_1\cos(4k-4g-4h)+g_2\cos(4k-2g-4h)\\
&\quad +g_3\cos(4k+2g-4h)+g_4\cos(4k+4g-4h)\\
&\quad +g_5\cos(2k-4g-2h)+g_6\cos(2k-2g-2h)\\
&\quad +g_7\cos(2k+2g-2h)+g_8\cos(2k+4g-2h)\},\\
\frac{\partial S_3^*}{\partial h} &= \frac{\nu^3 a^2}{n^2}\{h_1\cos(4k-4h)+h_2\cos(4k-4g-4h)+h_3\cos(4k-2g-4h)\\
&\quad +h_4\cos(4k+2g-4h)+h_5\cos(4k+4g-4h)+h_6\cos(2k-2h)\\
&\quad +h_7\cos(2k-4g-2h)+h_8\cos(2k-2g-2h)\\
&\quad +h_9\cos(2k+2g-2h)+h_{10}\cos(2k+4g-2h)\},
\end{aligned}
\right\} \tag{4.97}
$$

where the coefficients L_j, G_j, H_j, g_i, h_j ($j = 1,2,\ldots,10$ and $i = 1,2,\ldots,8$) are given in Appendix.

In order to get the osculating orbital elements we consider for simplicity that, the partial derivatives of S_j $(j = 2,3)$ with respect to L', G', H', l', g', h' refer to $\mathcal{P}_i$ $(i = 1,2,...,12)$ and the derivatives of S^*_ℓ $(\ell = 1,2,3)$ with respect to L'', G'', H'', g'', h'' are given by $\mathcal{K}_s$ $(s = 1,2,...,15)$. In what follows, we implement the above analytical expressions for digital computations by constructing the following algorithm described by its purpose, input and its computational sequence:

4.6.1 Computational algorithm

- *Purpose*: To compute the osculating orbital elements a, e, I, ω, Ω, l of Nereid for the nonplanar case. Nereid is inclined to the orbital plane of Neptune and perturbed by the solar effects.

- *Input*: the initial values a''_0, e''_0, I''_0, Ω''_0, l''_0, t_0, t_{end}, and *tol* (specified tolerance).

- Units measurement: Masses are given in solar unit, distances are in AU, time in days while the angles are given in radians.

- *Computational Sequence*:

(1) Compute the mean elements e'', I'', ω'', Ω'', l'', from equations (4.75), (4.76), (4.77), (4.78), (4.87) and (4.89).

(2) Compute the long (intermediate)-periodic variations from the following sequence

(a) As for the eccentricity:

$$\left.\begin{aligned}
\Phi_1 &= -\frac{\eta}{ena^2}\mathcal{K}_4 \equiv \delta_1 e,\\
\Phi_2 &= -\frac{\eta}{ena^2}\mathcal{K}_9,\\
\Phi_3 &= \left(\left(\frac{\partial\Phi_1}{\partial G} - \frac{\eta}{ena^2}\frac{\partial\Phi_1}{\partial e}\right)\mathcal{K}_4 - \frac{\partial\Phi_1}{\partial g}\mathcal{K}_2\right) + \left(\frac{\partial\Phi_1}{\partial H}\mathcal{K}_5 - \frac{\partial\Phi_1}{\partial h}\mathcal{K}_3\right),\\
\Phi_4 &= \Phi_2 + \frac{\Phi_3}{2} \equiv \delta_2 e,\\
\Phi_5 &= -\frac{\eta}{ena^2}k_{14},\\
\Phi_6 &= \left(\left(\frac{\partial\Phi_1}{\partial G} - \frac{\eta}{ena^2}\frac{\partial\Phi_1}{\partial e}\right)k_9 - \frac{\partial\Phi_1}{\partial g}k_7\right) + \left(\frac{\partial\Phi_1}{\partial H}k_{10} - \frac{\partial\Phi_1}{\partial h}k_8\right),\\
\Phi_7 &= \left(\left(\frac{\partial\Phi_2}{\partial G} - \frac{\eta}{ena^2}\frac{\partial\Phi_2}{\partial e}\right)k_4 - \frac{\partial\Phi_2}{\partial g}k_2\right) + \left(\frac{\partial\Phi_2}{\partial H}k_5 - \frac{\partial\Phi_2}{\partial h}k_3\right),\\
\Phi_8 &= \left(\left(\frac{\partial\Phi_3}{\partial G} - \frac{\eta}{ena^2}\frac{\partial\Phi_3}{\partial e}\right)k_4 - \frac{\partial\Phi_3}{\partial g}k_2\right) + \left(\frac{\partial\Phi_3}{\partial H}k_5 - \frac{\partial\Phi_3}{\partial h}k_3\right),\\
\Phi_9 &= \Phi_5 + \frac{(\Phi_6 + \Phi_7)}{2} + \frac{\Phi_8}{6} \equiv \delta_3 e,\\
e'_{long} &= e'' + \delta_1 e' + \delta_2 e' + \delta_3 e',
\end{aligned}\right\} \tag{4.98}$$

(b) Inclination: in this case we use the notation $I = \arccos(H/G)$ as follows

(b1) to get the angular momentum G' make the following changes in item (a)

$$\left.\begin{aligned}
\Phi_1 &= \mathcal{K}_4,\\
\Phi_2 &= \mathcal{K}_9,\\
\Phi_5 &= \mathcal{K}_{14},
\end{aligned}\right\} \tag{4.99}$$

then apply all the steps in (a) to get

$$G'_{long} = G'' + \delta_1 G' + \delta_2 G' + \delta_3 G'. \tag{4.100}$$

(b2) to get the angular momentum H' make the following changes in item (a)

$$\left.\begin{aligned}
\Phi_1 &= \mathcal{K}_5,\\
\Phi_2 &= \mathcal{K}_{10},\\
\Phi_5 &= \mathcal{K}_{15},
\end{aligned}\right\} \tag{4.101}$$

then apply all the steps in (a) to get

$$H'_{long} = H'' + \delta_1 H' + \delta_2 H' + \delta_3 H'. \tag{4.102}$$

(b3) then we can get I'_{long} using the definition

$$I'_{long} = \arccos\left(H'_{long}/G'_{long}\right). \tag{4.103}$$

(c) for the argument of pericenter ω', in the above equations put

$$\left.\begin{aligned} \Phi_1 &= -\mathcal{K}_2, \\ \Phi_2 &= -\mathcal{K}_7, \\ \Phi_5 &= -\mathcal{K}_{12}, \end{aligned}\right\} \tag{4.104}$$

then apply all the steps mentioned in eccentricity case to get

$$\omega'_{long} = \omega'' + \delta_1\omega' + \delta_2\omega' + \delta_3\omega'. \tag{4.105}$$

(d) for the longitude of ascending node Ω', change

$$\left.\begin{aligned} \Phi_1 &= -\mathcal{K}_3, \\ \Phi_2 &= -\mathcal{K}_8, \\ \Phi_5 &= -\mathcal{K}_{13}, \end{aligned}\right\} \tag{4.106}$$

then apply all the steps mentioned in item (a) to get

$$\Omega'_{long} = \Omega'' + \delta_1\Omega' + \delta_2\Omega' + \delta_3\Omega'. \tag{4.107}$$

(e) in case of the mean anomaly l' put

$$\left.\begin{aligned} \Phi_1 &= -\mathcal{K}_1, \\ \Phi_2 &= -\mathcal{K}_6, \\ \Phi_5 &= -\mathcal{K}_{11}, \end{aligned}\right\} \tag{4.108}$$

then apply all the steps mentioned in the case (a) to get:

$$l'_{long} = l'' + \delta_1 l' + \delta_2 l' + \delta_3 l'. \tag{4.109}$$

(3) Compute the short-periodic variations as follows

Call Kepler (*solve Kepler equation at every time t and substitute in the items below*)

(a) semi-major axis:

$$\left.\begin{aligned} \delta_2 a &= \frac{2}{na}\mathcal{P}_4, \\ \delta_3 a &= \frac{2}{na}\mathcal{P}_{10}, \\ \delta a_{sho} &= \delta_2 a + \delta_3 a, \end{aligned}\right\} \tag{4.110}$$

(b) eccentricity:

$$\left.\begin{aligned} \delta_2 e &= \frac{\eta^2}{ena^2}\mathcal{P}_4 - \frac{\eta}{ena^2}\mathcal{P}_5, \\ \delta_3 e &= \frac{\eta^2}{ena^2}\mathcal{P}_{10} - \frac{\eta}{ena^2}\mathcal{P}_{11}, \\ \delta e_{sho} &= \delta_2 e + \delta_3 e, \end{aligned}\right\} \tag{4.111}$$

(c) Inclination: in this case we use also the notation $I = \arccos(H/G)$

(c1) to get the angular momentum G

$$\left.\begin{aligned} \delta_2 G &= \mathcal{P}_5, \\ \delta_3 G &= \mathcal{P}_{11}, \\ \delta G_{sho} &= \delta_2 G + \delta_3 G \end{aligned}\right\} \tag{4.112}$$

(c2) to get the angular momentum H

$$\left.\begin{aligned} \delta_2 H &= \mathcal{P}_6, \\ \delta_3 H &= \mathcal{P}_{12}, \\ \delta H_{sho} &= \delta_2 H + \delta_3 H \end{aligned}\right\} \tag{4.113}$$

(d) Argument of pericenter:

$$\left.\begin{aligned} \delta_2\omega &= -\mathcal{P}_2, \\ \delta_3\omega &= -\mathcal{P}_8, \\ \delta\omega_{sho} &= \delta_2\omega + \delta_3\omega \end{aligned}\right\} \tag{4.114}$$

(e) Longitude of ascending node:

$$\left.\begin{aligned} \delta_2\Omega &= -\mathcal{P}_3, \\ \delta_3\Omega &= -\mathcal{P}_9, \\ \delta\Omega_{sho} &= \delta_2\Omega + \delta_3\Omega \end{aligned}\right\} \tag{4.115}$$

(f) in case of the mean anomaly

$$\left.\begin{aligned} \delta_2 l &= -\mathcal{P}_1, \\ \delta_3 l &= -\mathcal{P}_7, \\ \delta l_{sho} &= \delta_2 l + \delta_3 l. \end{aligned}\right\} \tag{4.116}$$

(4) Compute the osculating orbital elements from the equations

$$\left.\begin{aligned} a_{osc} &= a_0'' + \delta a_{sho}, \\ e_{osc} &= e'_{long} + \delta e_{sho}, \\ G_{osc} &= G'_{long} + \delta G_{sho}, \\ H_{osc} &= H'_{long} + \delta H_{sho}, \\ I_{osc} &= \arccos\left(H_{osc}/G_{osc}\right), \\ \omega_{osc} &= \omega'_{long} + \delta\omega_{sho}, \\ \Omega_{osc} &= \Omega'_{long} + \delta\Omega_{sho}, \\ l_{osc} &= l'_{long} + \delta l_{sho}. \end{aligned}\right\} \tag{4.117}$$

(5) The algorithm is completed up to the third order.

4.7 Comparison with Numerical Integration

In this Section we give in more details the way of comparison between the analytical and numerical experiments. The residuals and the amplitudes of the osculating orbital elements of real Nereid are shown by figures and tables. To check the present theory, we adopted Bulirsch-Stoer extrapolation method because it has a capability of highly accurate orbital computation.

4.7.1 The way of comparison

In order to compare the analytical solution with the numerical integration of the equations of motion we performed the following steps

- Analytical evaluation of the osculating elements by inserting the initial mean elements coming from Jacobson (1990, 1991) in the analytical program:

  $a_0^{***} = 0.036854 = a$ (au)

  $e_0^{***} = 0.751201525$ (rad)

  $I_0^{***} = 7.23242919$ (deg)

  $\omega_0^{***} = 0.0$

  $\Omega_0^{***} = 333.979128$ (deg)

  $l_0^{***} = 359.341112$ (deg)

- Numerical computation of the osculating elements by inserting the initial conditions $\underline{\mathbf{r}}_0$, $\underline{\mathbf{v}}_0$ coming from the osculating elements of the above analytical solution

$$\begin{Bmatrix} a_{osc} \\ e_{osc} \\ I_{osc} \\ \omega_{osc} \\ \Omega_{osc} \\ l_{osc} \end{Bmatrix} \longrightarrow \begin{Bmatrix} x_0 \\ y_0 \\ z_0 \\ \dot{x}_0 \\ \dot{y}_0 \\ \dot{z}_0 \end{Bmatrix}$$

- Then get the direct difference $(O - C)$ between analytical and numerical results (e.g. see figure 4.5). In case of nonzero inclination, we make some corrections to improve the accuracy.

- Remove the linear part from the elements in figure 4.5 namely the mean anomaly, the argument of pericenter and the longitude of the ascending node, then the results are shown in figure 4.6.

- The corrections in the mean motions of ℓ, g and h have the values

  $\delta n_\ell = -3.78855 \times 10^{-8}$ (rad/day)

  $\delta n_g = 1.11885 \times 10^{-8}$ (rad/day)

  $\delta n_h = -7.17298 \times 10^{-9}$ (rad/day)

- Put the new corrections

  $n_\ell \longrightarrow n_\ell + \delta n_\ell$

  $n_g \longrightarrow n_g + \delta n_g$

  $n_h \longrightarrow n_h + \delta n_h$

  in the analytical program and recalculate the osculating elements.

- Compute the residuals in the elements $(O^* - C)$, which can be seen in figure 4.7.
- Use the least square fitting to remove the secular part from figure 4.7 and the final results are expressed by figure 4.8.
- The comparison is completed.

Tables V and VI show the actual amplitudes and the accuracy in the osculating orbital elements of the nonplanar case, for both short and long periodic perturbations respectively. The osculating elements are listed in table VII.

TABLE V

Amplitudes of the osculating elements (non-planar case)

Elements	Short-period	Long-period
semi-major axis	747.989	1196.78
eccentricity	0.0004	0.0115
arg. of pericenter	0.006	0.7
inclination	0.0025	0.16
long. of asc. node	0.01	0.17
mean anomaly	0.0325	2.0

TABLE VI

Accuracy of the osculating elements (non-planar case)

Elements	Short-period	Long-period
semi-major axis	0.3	0.3
eccentricity	3×10^{-8}	1×10^{-7}
arg. of pericenter	3×10^{-6}	7×10^{-4}
inclination	1.5×10^{-6}	1×10^{-4}
long. of asc. node	3×10^{-6}	7×10^{-4}
mean anomaly	2.5×10^{-5}	6×10^{-5}

where the semi-major axis is given in km, eccentricity in radian, and the argument of pericenter, longitude of ascending node and the mean anomaly are given in degree. The results of the present theory satisfy the required accuracy for the observations.

TABLE VII

Osculating elements of the orbit of Nereid at Julian ephemerides date 2433680.5 referred to the mean orbital plane of Neptune

Elements	Value	Units
semi-major axis	5513376.2332	(km)
eccentricity	0.749139307372	(rad)
arg. of pericenter	0.3205009745	(deg)
inclination	7.2023147577	(deg)
long. of asc. node	334.0527164964	(deg)
mean anomaly	358.3922274885	(deg)
mean motion	0.999825034	(deg/day)
orbital period of Nereid	360.0629988	(day)
orbital period of Neptune	165.223494	(year)
period of arg. of pericenter	13606.468483	(year)
period of node	-17901.900839	(year)
mean motion of arg. of pericenter	7.24381×10^{-5}	(deg/day)
mean motion of node	-5.505707×10^{-5}	(deg/day)

Perturbed mean motion of mean anomaly ($n_{l^{**}}$) $=-7.691307\times10^{-5}$ (deg/day),

Disturbed mean motion of mean anomaly ($n_l+n_{l^{**}}$) $=1.744889\times10^{-2}$ (rad/day),

$=0.999748121$ (deg/day).

The above results arised according to inserting certain initial mean elements. So, if we used different initial conditions, of course the results of the above table will be changed.

4.8 Summary and conclusions

We have constructed an analytical theory for the motion of the second Neptunian satellite Nereid in the frame work of the circular non-planar restricted three body problem using Lie transforms approach. The main perturbing forces due to the solar effects are only taken into account. The disturbing function is developed in powers of the ratio of the semimajor axes of the satellite and the sun and put in a closed form with respect to the eccentricity. In the present chapter we offered a complete theory which includes the short, intermediate and long periodic perturbations. The osculating orbital elements which describe the orbital motion of Nereid are evaluated analytically and got ephemerides of Nereid. The comparison with the numerical integration of the equations of motion gives an accuracy on the level of 0.3 km in the semi-major axis, 3×10^{-7} in the eccentricity and 10^{-5} degree in the angular variables over several hundred years.

Figures 4.1 and 4.2 show the behaviour of the osculating elements of Nereid over 5 and 300 years respectively. The direct difference between the analytical and numerical results for short period interval is given by figure 4.3, while figure 4.4 exhibits the residuals in the elements using least square fitting. The check of the reliability and accuracy of the theory for a relatively long interval is given in figures 4.5 and 4.6. Figure 4.7 represents the accuracy in the elements after making corrections in the mean motions of ℓ, g and h. These corrections are coming from the linear part of figure 4.5. Finally, the residuals in the osculating elements are adjusted and exhibited in figure 4.8.

Figure Captions

Fig. 4.1. The osculating orbital elements of Nereid for the nonplanar case over 5 years:

(1) semi-major axis,

(2) eccentricity,

(3) argument of pericenter,

(4) inclination,

(5) longitude of ascending node,

(6) periodic part of the mean anomaly.

Fig. 4.2. The osculating orbital elements of Nereid for the nonplanar case over 300 years:

(1) semi-major axis,

(2) eccentricity,

(3) periodic part of the argument of pericenter,

(4) inclination,

(5) periodic part of the longitude of ascending node,

(6) periodic part of the mean anomaly.

Fig. 4.3. Difference between analytical and numerical results of the orbital elements of Nereid during 5 years:

(1) semi-major axis (in km),

(2) eccentricity (in radians),

(3) argument of pericenter (in degree),

(4) inclination (in degree),

(5) longitude of ascending node (in degree),

(6) the mean anomaly (in degree).

Fig. 4.4. Residuals in the orbital elements of Nereid during 5 years by using least-square fitting :

(1) semi-major axis (in km),

(2) eccentricity (in radians),

(3) argument of pericenter (in degree),

(4) inclination (in degree),

(5) longitude of ascending node (in degree),

(6) the mean anomaly (in degree).

Fig. 4.5. Difference between analytical and numerical results of the orbital elements of Nereid over 300 years:

(1) semi-major axis (in km),

(2) eccentricity (in radians),

(3) argument of pericenter (in degree),

(4) inclination (in degree),

(5) longitude of ascending node (in degree),

(6) the mean anomaly (in degree).

Fig. 4.6. Residuals in the orbital elements of Nereid over 300 years by using least-square

fitting :

(1) semi-major axis (in km),

(2) eccentricity (in radians),

(3) argument of pericenter (in degree),

(4) inclination (in degree),

(5) longitude of ascending node (in degree),

(6) the mean anomaly (in degree).

Fig. 4.7. Difference between analytical and numerical results of the orbital elements of Nereid over 300 years after making corrections in the mean motions of ℓ, g and h:

(1) semi-major axis (in km),

(2) eccentricity (in radians),

(3) argument of pericenter (in degree),

(4) inclination (in degree),

(5) longitude of ascending node (in degree),

(6) the mean anomaly (in degree).

Fig. 4.8. Residuals in the osculating orbital elements of Nereid over 300 years by using least-squares fitting, and after making corrections in the mean motions of ℓ, g and h:

(1) semi-major axis (in km),

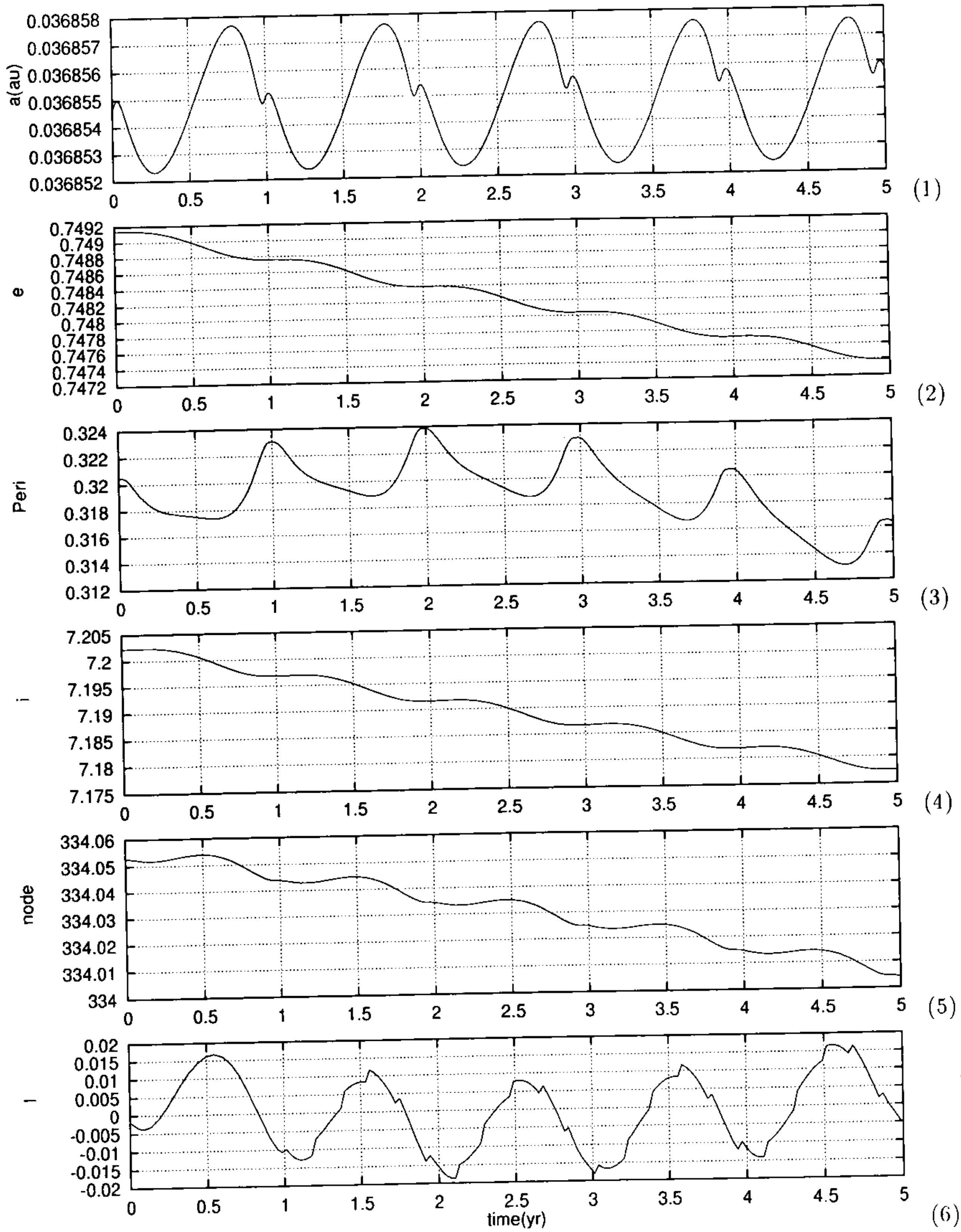
(2) eccentricity (in radians),

(3) argument of pericenter (in degree),

(4) inclination (in degree),

(5) longitude of ascending node (in degree),

(6) the mean anomaly (in degree).



a(au)
0.036858
0.036857
0.036856
0.036855
0.036854
0.036853
0.036852
(1)
e
0.7492
0.749
0.7488
0.7486
0.7484
0.7482
0.748
0.7478
0.7476
0.7474
0.7472
(2)
Peri
0.324
0.322
0.32
0.318
0.316
0.314
0.312
(3)
i
7.205
7.2
7.195
7.19
7.185
7.18
7.175
(4)
node
334.06
334.05
334.04
334.03
334.02
334.01
334
(5)
l
0.02
0.015
0.01
0.005
0
-0.005
-0.01
-0.015
-0.02
0 0.5 1 1.5 2 2.5 3 3.5 4 4.5 5
time(yr)
(6)

Fig. 4.1

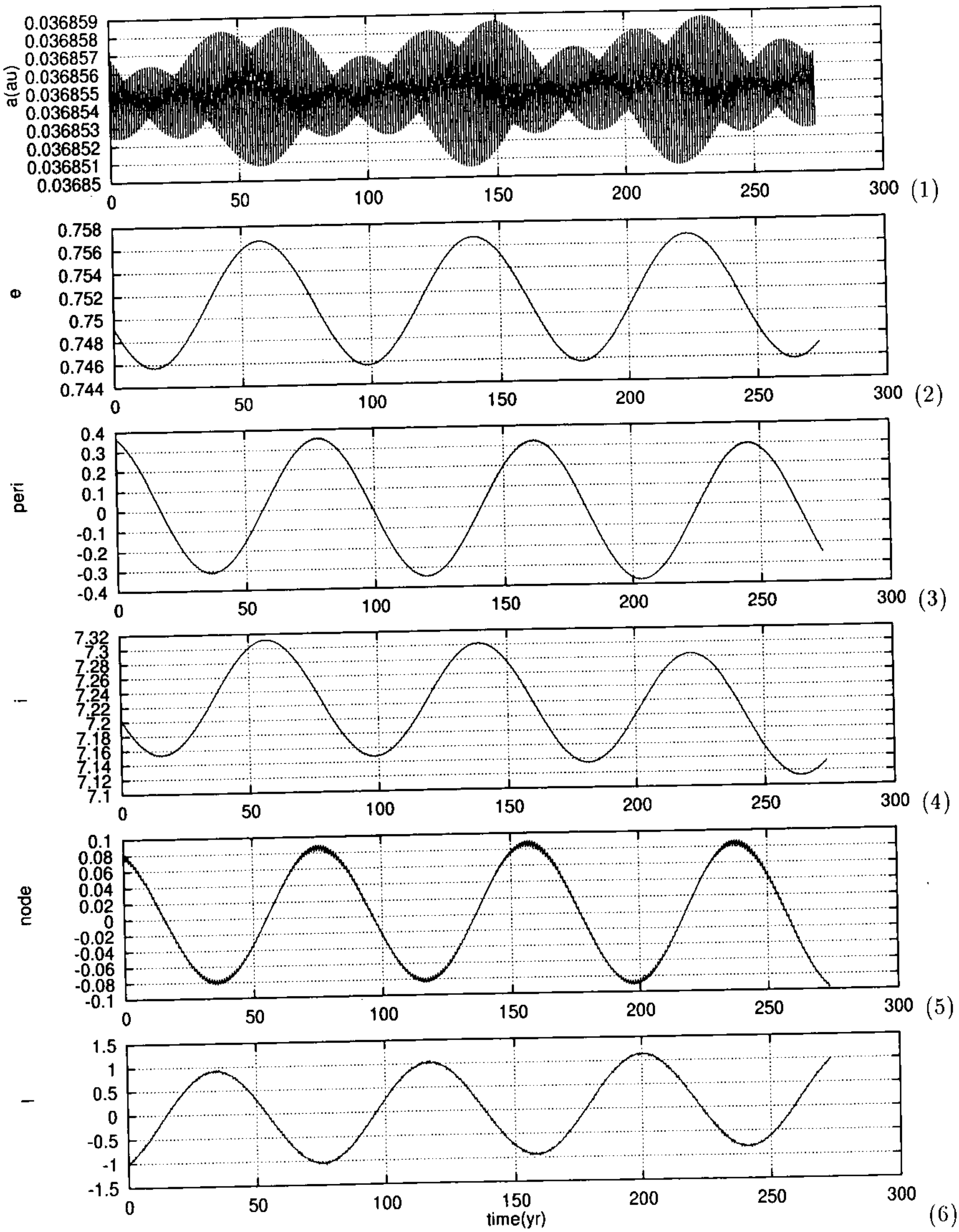
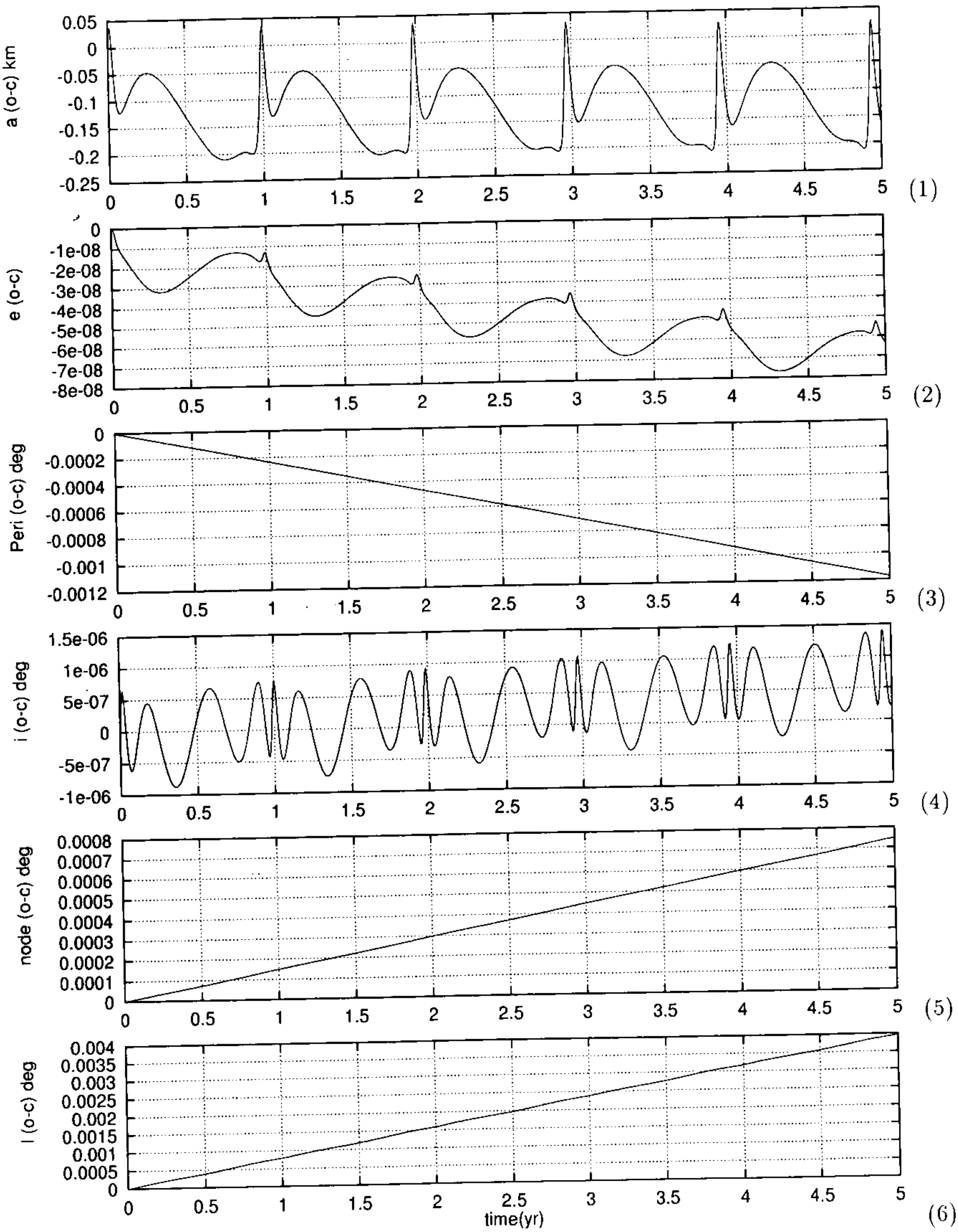

a(au)
0.036859
0.036858
0.036857
0.036856
0.036855
0.036854
0.036853
0.036852
0.036851
0.03685
0 50 100 150 200 250 300
(1)
e
0.758
0.756
0.754
0.752
0.75
0.748
0.746
0.744
0 50 100 150 200 250 300
(2)
peri
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
0 50 100 150 200 250 300
(3)
i
7.32
7.3
7.28
7.26
7.24
7.22
7.2
7.18
7.16
7.14
7.12
7.1
0 50 100 150 200 250 300
(4)
node
0.1
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
-0.1
0 50 100 150 200 250 300
(5)
l
1.5
1
0.5
0
-0.5
-1
-1.5
0 50 100 150 200 250 300
time(yr)
(6)

Fig. 4.2



a (o-c) km
e (o-c)
Peri (o-c) deg
i (o-c) deg
node (o-c) deg
l (o-c) deg
time(yr)
(1)
(2)
(3)
(4)
(5)
(6)

Fig. 4.3

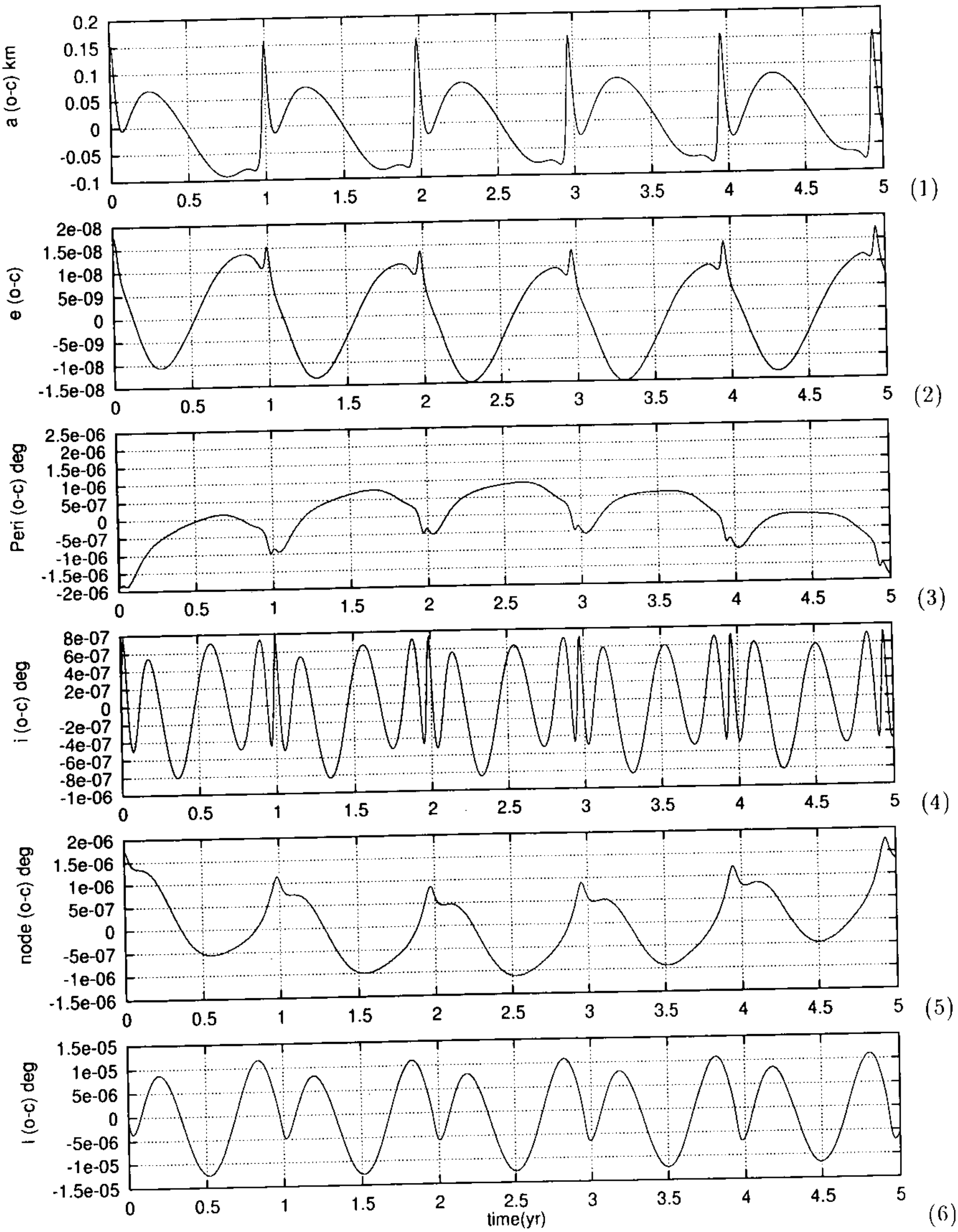

a (o-c) km
(1)
e (o-c)
(2)
Peri (o-c) deg
(3)
i (o-c) deg
(4)
node (o-c) deg
(5)
l (o-c) deg
time(yr)
(6)

Fig. 4.4

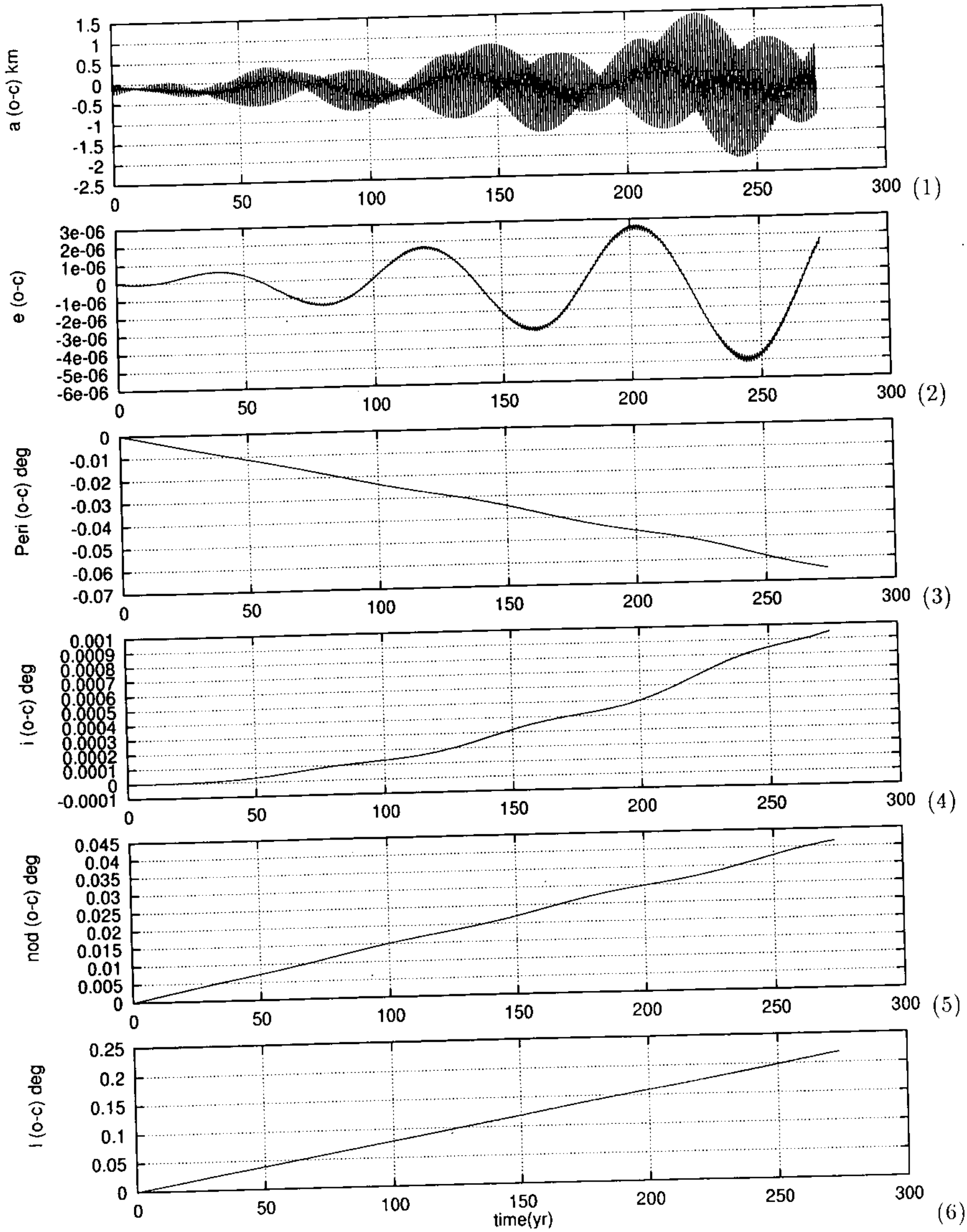


Fig. 4.5

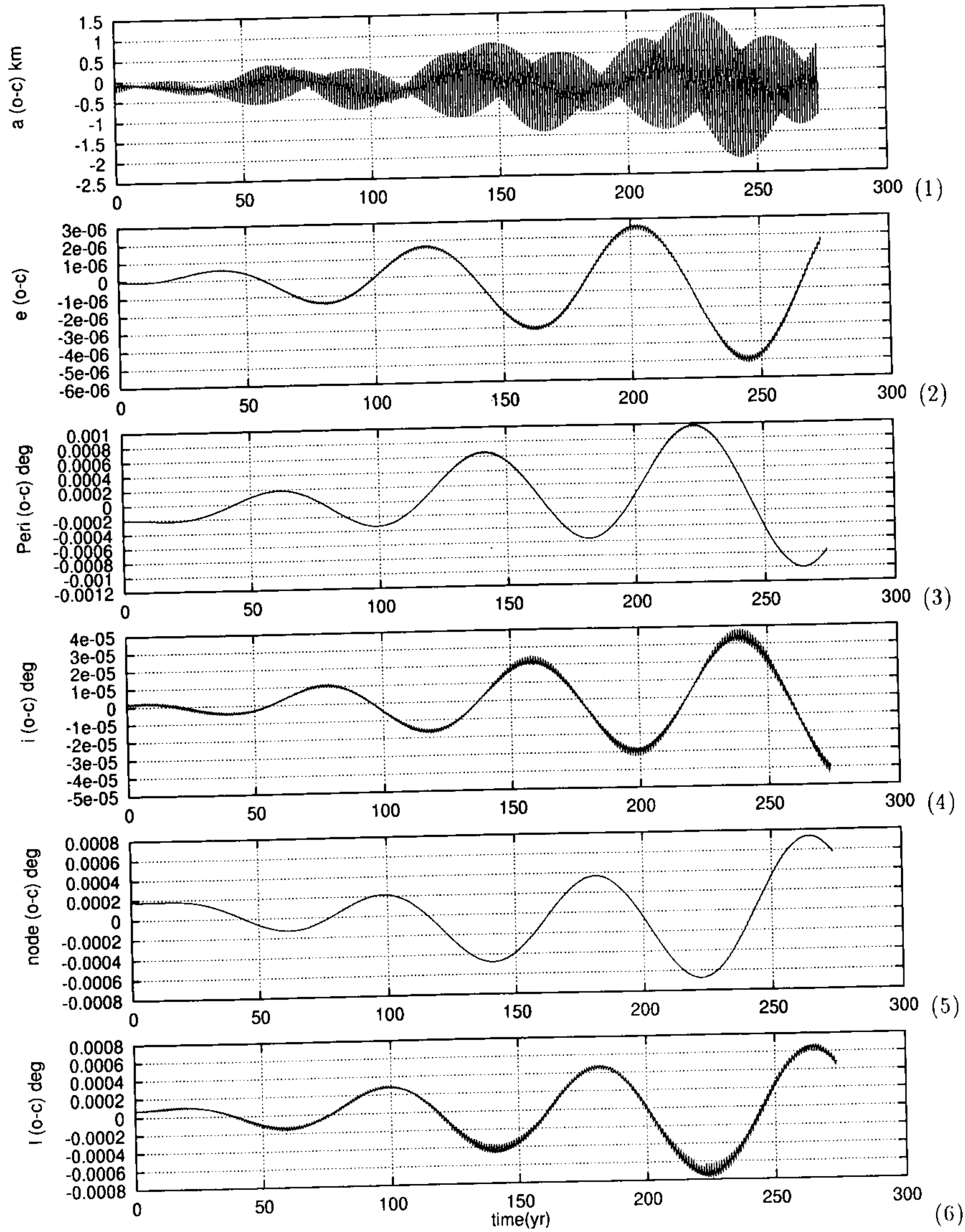

a (o-c) km
e (o-c)
Peri (o-c) deg
i (o-c) deg
node (o-c) deg
l (o-c) deg
time(yr)
(1)
(2)
(3)
(4)
(5)
(6)

Fig. 4.6

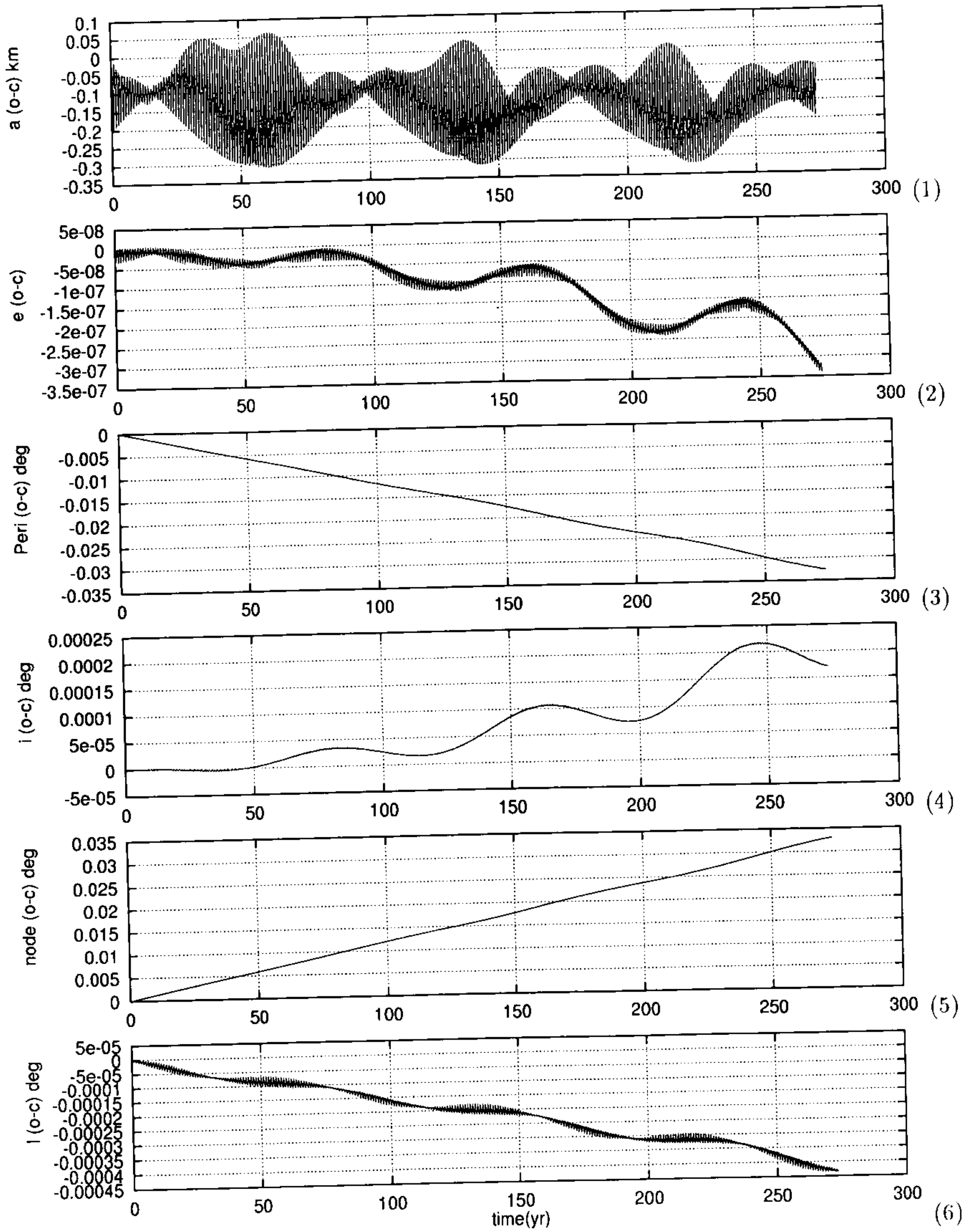

a (o-c) km
(1)
e (o-c)
(2)
Peri (o-c) deg
(3)
i (o-c) deg
(4)
node (o-c) deg
(5)
l (o-c) deg
(6)
time(yr)

Fig. 4.7

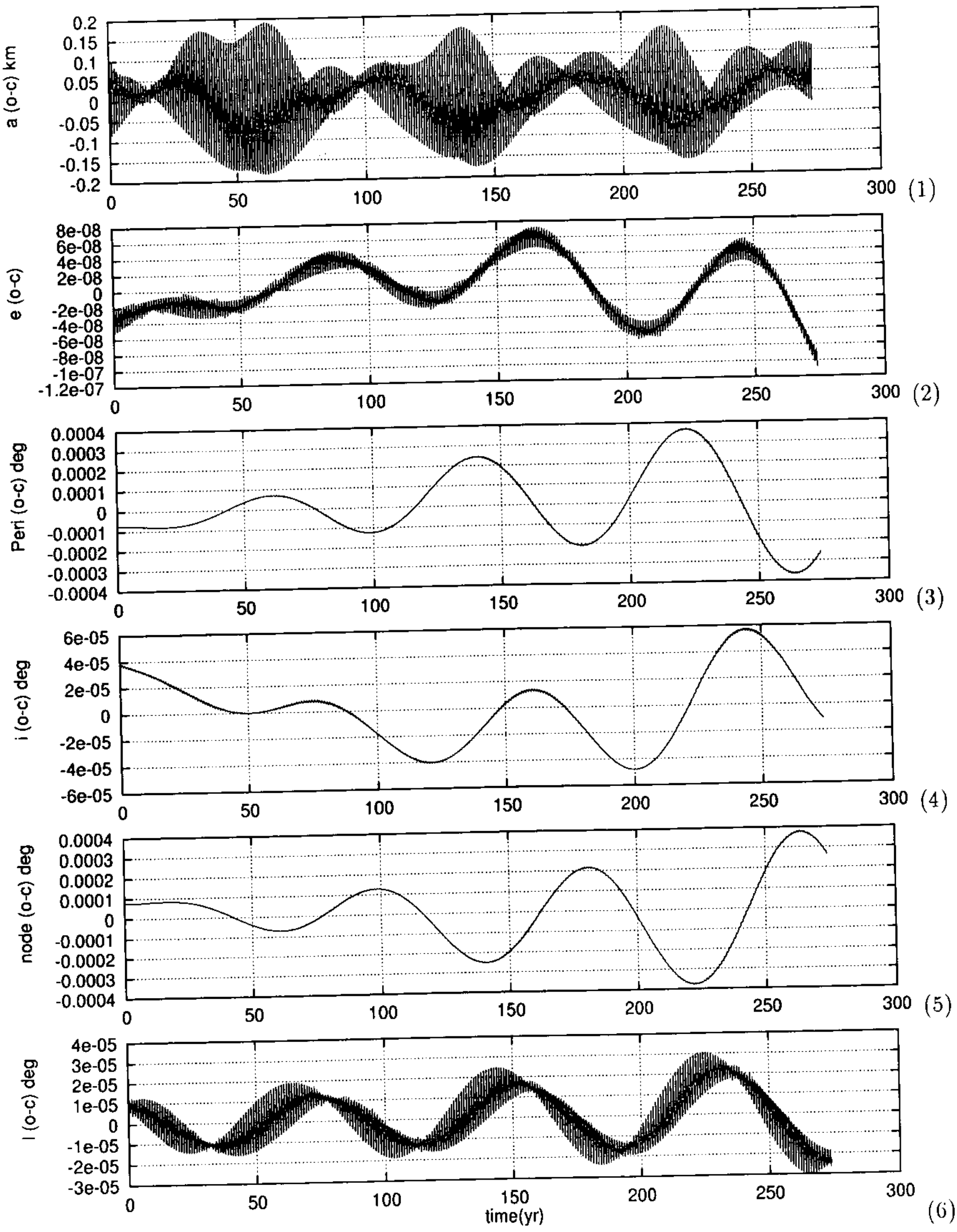

a (o-c) km
(1)
e (o-c)
(2)
Peri (o-c) deg
(3)
i (o-c) deg
(4)
node (o-c) deg
(5)
l (o-c) deg
time(yr)
(6)

Fig. 4.8

Chapter 5

Discussion and Conclusions

Planetary satellites display a rich variety of orbital configurations and surface characteristics that have fascinated astronomers over several centuries. Dynamical studies of these satellites are almost certainly contribution of understanding the origin of the solar system. Such information on the planetary satellites will provide at least some critical clues to our understanding of the solar system, because the satellites are such diverse bodies, existing in so many different environments of their parent planets.

In this research, an analytical theory of the motion of the second Neptunian satellite Nereid is constructed using Lie transformation approach. The main perturbing forces which come from the solar influence are only taken into account. The disturbing function is developed in powers of the ratio of the semimajor axes of the satellite and the Sun and put in a closed form with respect to the eccentricity. The convergence of the power series may be slow for some of the outer satellites, for example, the outer Jovian satellites have ratios m between 0.145 and 0.175 (Saha and Tremaine 1993, Solovaya 1995). The case of Nereid represents advantage, the ratio $m \sim 0.006$, and the power series of the perturbing function is convergent. The theory includes secular perturbations up to the fourth order, short, intermediate and long period perturbations up to the third order. In the planar case, the periodic terms are performed to the fourth order while the secular terms are to the fifth order. The osculating orbital elements which describe the orbital motion of Nereid are evaluated analytically. Since the high eccentric orbit of Nereid precludes replacing functions of the true anomaly by expansions involving the mean anomaly, it is convenient to take the eccentric anomaly of Nereid u as an independent variable.

The global internal accuracy of the theory is obtained by direct comparison with the numerical integration of the equations of motion of the satellite. The maximum discrepancies reached 0.3 km in the semimajor axis, 10^{-7} in the eccentricity and 10^{-4} degree in the angular variables over a period of several hundred years. Looking to figures 4.1 and 4.2, we see that the amplitudes of the osculating elements of Nereid are increasing with time. This means that, studying the motion of Nereid for short-period interval ($\ell = 360$ days) is not enough. We extended our analytical study for several hundred years and got ephemerides of Nereid. The direct difference between the analytical and numerical results for short period interval is given by figure 4.3, while figure 4.4 exhibits the residuals in the elements using least square fitting. The test of the reliability and accuracy of the theory for a relatively long interval is given in figures 4.5 and 4.6. Figure 4.7 represents the accuracy in the elements after making corrections in the mean motions of ℓ, g and h. These corrections are coming from the linear part of figure 4.5.

General spaeking, correction is made to overcome secular error which grows linearly with time (Kinoshita 1968, Kinoshita & Nakai 1992) during calculations and to reach the true orbit as possible. If after a correction has been made, the resulting deviations from the true orbit are small, then the calculated orbit will always closely follow the true orbit. However, if the resulting deviations are systematic, then the calculated orbit will eventually diverge from the true orbit. I would like to refer the fact that, the corrections are exceedingly small so that, even over the course of a long integration, the difference between the calculated orbit and the true one is negligible. This what happened when we added the corrections of the mean motions of ℓ, g and h. The corrections values were -3.78774×10^{-8}, 1.1188×10^{-8} and -7.173324×10^{-9} degree per day in ℓ, g and h respectively. However, when I tried to make another corrections in these mean motions, I got values much smaller than that in the previous correction. Consequently, their contribution in improving the osculating elements this time is insignificant. Although the present theory satisfies the required accuracy for future observations from space, further improvement can be made to increase this accuracy for very long period, if the corrections in the other three elements are included. Finally, the residuals in the osculating elements are adjusted and exhibited in figure 4.8. The amplitudes and accuracy both for short and long period are given in tables V and VI respectively. In case of zero inclination, however, the residuals in the elements of a fictitious Nereid for 5 and 500 years are shown in figures 3.3 and 3.4 respectively. The amplitudes and accuracy magnitude are given in tables III and IV respectively. The accuracy in semi-major axis does not exceed 230 and 300 meter over 5 and 500 years respectively. Although the zero inclination case has meaningless for a real Nereid, it was informative and a good test for the non-zero case.

Different values of Nereid's inclination are found in text-books and published papers. One may read $i = 27^o.5$, $i = 10^o$, $i = 6^o.7$, and $i = 7^o.23$. I'll give interpretation of the mentioned values to avoid the reader any confusion. The value $i = 27^o.5$ (or $i = 28^o$) used by Mignard

(1981) and veillet (1982) is an osculating element referred to the Earth mean equator and equinox of 1950.0, epoch 1981.0. As for the value $i = 10^o$, it is a mean element referred to the mean orbital plane of Neptune (see also Mignard 1981 and Veillet 1982, 1988). The osculating element $i = 6°.7$ is evaluated by Rose (1974), Veillet (1988) and Jacobson (1990). It is also called Nereid barycentric mean element at Julian ephemerides date 2433680.5, referred to the mean orbital plane of Neptune (Jacobson 1990). However, Jacobson (1991) improved the orbit of Nereid (numerically) using spacecraft and Earth-based observations. According to this improvement the inclination of Nereid became $i = 7°.23$, referred to Nereid invariable plane and the orbit is changed about 65 km. Here the invariable plane of Nereid is that plane on which the orbit of Nereid precesses almost uniformaly. In fact the previous orbit predicted by jacobson (1990) has error about 200,000 km. This error was due to the limited accuracy of the observations used in the orbit determination and the poorly known physical constants used in the orbital motion model. The model proposed by Jacobson is fit the numerically integrated Neptunian satellite orbits (Nereid and Triton) to Earth-based astrometric observations and Voyager spacecraft observations.

The present theory has not been fitted to the observations. We intend to do that after including the perturbations of Triton and the oblateness of Neptune although the effects of the latter is very small. Then the integrations constants of the theory can have a real meaning. However, the comparison with the numerical integration of the equations of motion gives a great accuracy which is much consistent with the observations. The perturbation potentials magnitudes of Sun, Triton and Neptune relative to the two-body potential are given by $V_{S2} = 7.6 \times 10^{-5}$, $V_{T2} = 5.3 \times 10^{-7}$ and $V_{J2} = 4.3 \times 10^{-8}$ respectively. It is easy to know roughly both the perturbation of Triton and the oblateness of Neptune compared to the solar effects. We can write $V_{T2}/V_{S2} \sim 6.9 \times 10^{-3}$ and $V_{J2}/V_{S2} \sim 5.6 \times 10^{-4}$. The effect of the third harmonic of the Sun $V_{S3} = 1.2 \times 10^{-7}$ is nearly has the same order of the second harmonic of Triton. So, one may develop this theory by including both V_{S3} and V_{T2} for further motion prediction of Nereid. Since our theory is pure analytical, a direct comparison with observations using real data should determine the integration constants accurately and hence, provide their real meaning. Nereid was discovered by Kuiper in 1949. This means that the observations period for Nereid does not exceed 50 years. The present theory has been elaborated to predict the motion of Nereid over several hundered years.

By this end we provide to the observers a computational tool, capable of generating ephemerides for predictions, easy to handle.

Appendix A

The coefficients of $\sin/\cos(ik+jg+sh)$ in the partial derivatives of S_3^* with respect to L'', G'', H'', l'', g'', h''

$$
\begin{aligned}
L_1 &= \frac{9}{16384}(376 + 2758e^2 + 41e^4)(-1+\theta^2)^2,\\
L_2 &= \frac{1905}{32768}e^2(2+3e^2)(1+\theta)^4,\\
L_3 &= -\frac{15}{8192}(84+422e^2+129e^4)(-1+\theta)(1+\theta)^3,\\
L_4 &= -\frac{15}{8192}(84+422e^2+129e^4)(-1+\theta)^3(1+\theta),\\
L_5 &= \frac{1905}{32768}e^2(2+3e^2)(-1+\theta)^4,\\
L_6 &= \frac{1}{2048}(3(-1+\theta^2)(560-772\theta^2-6e^2(695+2111\theta^2)\\
&\quad +e^4(4065+24393\theta^2))),\\
L_7 &= \frac{795}{4096}e^2(2+3e^2)(-1+\theta)(1+\theta)^3,\\
L_8 &= -\frac{1}{1024}(3(1+\theta)^2(-6(79-175\theta+310\theta^2)-10e^2(77-157\theta+319\theta^2)\\
&\quad +3e^4(563-1315\theta+2125\theta^2))),\\
L_9 &= -\frac{1}{1024}(3(-1+\theta)^2(-6(79+175\theta+310\theta^2)-10e^2(77+157\theta+319\theta^2)\\
&\quad +3e^4(563+1315\theta+2125\theta^2))),
\end{aligned}
$$

$$
\begin{aligned}
L_{10} &= 6\frac{795}{4096}e^2(2+3e^2)(-1+\theta)^3(1+\theta),\\
G_1 &= -\frac{3}{8192}(1+\theta^2)(-1024+41e^4+840\theta^2+e^2(983\theta^2)),\\
G_2 &= \frac{1905}{16384}(-1+e^2-\theta)(1+\theta)^3,\\
G_3 &= -\frac{1}{8192}(15(1+\theta)^2(43e^4(-2+\theta)+e^2(2+84\theta-170\theta^2)\\
&\quad -84(-1+\theta^2))),\\
G_4 &= -\frac{1}{8192}(15(-1+\theta)^2(43e^4(2+\theta)+84(-1+\theta^2)\\
&\quad +2e^2(-1+42\theta+85\theta^2))),\\
G_5 &= \frac{1905}{16384}e^2(-1+\theta)^3(-1+e^2+\theta),\\
G_6 &= -\frac{1}{1024}(3(860+3036\theta^2-3720\theta^4+e^4(1355+3388\theta^2),\\
&\quad +e^2(-2215-6776\theta^2+4515\theta^4))),\\
G_7 &= \frac{795}{4096}e^2(1+\theta)^2(2+e^2(-2+\theta)-2\theta^2),\\
G_8 &= -\frac{1}{2048}(3(1+\theta)(e^4(2252-2819\theta+2935\theta^2)\\
&\quad +12(79-96\theta+135\theta^2+310\theta^3)\\
&\quad -2e^2(1600-2029\theta+2145\theta^2+2390\theta^3))),\\
G_9 &= \frac{1}{2048}(3(-1+\theta)(e^4(2252+2819\theta+2935\theta^2)\\
&\quad +12(79+96\theta+135\theta^2-310\theta^3)\\
&\quad +2e^2(-1600-2029\theta-2145\theta^2+2390\theta^3))),\\
G_{10} &= \frac{795}{4096}e^2(-1+\theta)^2(e^2(2+\theta)+2(-1+\theta^2)),\\
H_1 &= \frac{3}{8192}(-184+2048e^2+41e^4)\theta(-1+\theta^2),\\
H_2 &= \frac{1905}{16384}e^4(1+\theta)^3,
\end{aligned}
$$

$$
\begin{aligned}
H_3 &= -\frac{15}{8192}(84+43e^2)(1+\theta)^2(-1+2\theta),\\
H_4 &= -\frac{15}{8192}(84+43e^2)(-1+\theta)^2(1+2\theta),\\
H_5 &= \frac{1905}{16384}e^4(-1+\theta)^3,\\
H_6 &= \frac{3}{1024}\theta(280-104\theta^2+e^2(2756-7232\theta^2)+e^4(-3388+8131\theta^2)),\\
H_7 &= \frac{795}{4096}e^4(1+\theta)^2(-1+2\theta),\\
H_8 &= -\frac{1}{2048}(3e^2(1+\theta)(-6(-17+95\theta+1240\theta^2)\\
&\quad +e^2(-189+305\theta+8500\theta^2))),\\
H_9 &= -\frac{1}{2048}(3e^2(-1+\theta)(6(17+95\theta-1240\theta^2)\\
&\quad +e^2(-189-305\theta+8500\theta^2))),\\
H_{10} &= \frac{795}{4096}e^4(-1+\theta)^2(1+2\theta),\\
g_1 &= -\frac{1905}{16384}e^4(1+\theta)^4,\\
g_2 &= \frac{15}{8192}e^2(84+43e^2)(-1+\theta)(1+\theta)^3,\\
g_3 &= -\frac{15}{8192}e^2(84+43e^2)(-1+\theta)^3(1+\theta),\\
g_4 &= \frac{1905}{16384}e^4(1+\theta)^4,\\
g_5 &= -\frac{795}{2048}e^4(-1+\theta)(1+\theta)^3,\\
g_6 &= \frac{1}{1024}(3e^2(1+\theta)^2(-6(79-175\theta+310\theta^2)+e^2(563-1315\theta+2125\theta^2))),\\
g_7 &= -\frac{1}{1024}(3e^2(-1+\theta)^2(-6(79+175\theta+310\theta^2)+e^2(563+1315\theta+2125\theta^2))),\\
g_8 &= \frac{795}{2048}e^4(-1+\theta)^3(1+\theta),\\
h_1 &= -\frac{3}{8192}(-184+2048e^2+41e^4)(-1+\theta)^2,\\
h_2 &= -\frac{1905}{16384}e^4(1+\theta)^4,
\end{aligned}
$$

$$
\begin{aligned}
h_3 &= \frac{15}{4096}e^2(84+43e^2)(-1+\theta)(1+\theta)^3,\\
h_4 &= \frac{15}{4096}e^2(84+43e^2)(-1+\theta)^3(1+\theta),\\
h_5 &= \frac{1905}{16384}e^4(-1+\theta)^4,\\
h_6 &= -\frac{1}{2048}(3(-1+\theta^2)(456-104\theta^2-8e^2(215+904\theta^2)\\
&\quad +e^4(1355+8131\theta^2))),\\
h_7 &= \frac{795}{4096}e^4(-1+\theta)(1+\theta)^3,\\
h_8 &= \frac{1}{1024}(3e^2(1+\theta)^2(-6(79-175\theta+310\theta^2)+e^2(563-1315\theta+2125\theta^2))),\\
h_9 &= \frac{1}{1024}(3e^2(-1+\theta)^2(-6(79+175\theta+310\theta^2)+e^2(563+1315\theta+2125\theta^2))),\\
h_{10} &= -\frac{795}{4096}e^4(-1+\theta)^3(1+\theta).
\end{aligned}
$$

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