Motion around Triangular Lagrange Points Perturbed by Other Bodies

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Abstract

We discovered a dramatically large difference in the the motion of a test particle in the vicinity of triangular Lagrange points, L_4 and L_5, between the restricted three body and the N-body ($N \geq 4$) problems. We revealed that this is caused by the forced oscillation term due to the third and other perturbing bodies since the net of gravity force of the primary and secondary bodies almost vanishes around the Lagrange points in the corotational coordinate system. Taking into account the direct effects of the other perturbing bodies including the effects of their eccentricities up to the second order, we constructed an analytical theory of the motion of the test particle being linear with respect to the magnitude of departure from the Lagrange points, in the planar restricted N-body problem. We compared our analytical solution with a numerical integration and confirmed that the solution represents the linear part of the true solution so well that the residuals are only due to the non-linear effect of the primary and the secondary system mainly which we ignored. By using the solution, we discussed the global aspects of the orbit such as the existence region or the averaged position. The results will be useful in designing the orbit of near-future space missions to be located in the vicinity of the triangular Lagrange points.

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Chapter 1

Introduction

1.1 Historical Background

The five Lagrange points from L_1 to L_5 are special solutions of the three body problem. See Fig. 1.1. The collinear solutions, L_1, L_2, and L_3, were found by Euler in 1767 and the rest triangular solutions, L_4 and L_5, were added by Lagrange in 1772. Fig. 1.2 shows the effective potential surface in the corotational coordinate system and we find that L_1, L_2, and L_3 lie on the saddle points and L_4 and L_5 do on the summit of potential. Especially the latter points, L_4 and L_5, are important in terms of their linear stability because the centrifugal force and the Coriolis force balance. Since then, there have been vast investigations of the Lagrange points, especially after the discovery of Trojan asteroids as natural examples. Recently this tendency has been accelerated because they have high potentiality for the long-lived space missions and the astronautical applications such as the gravitational wave detection or the space telescope for observing the near Earth crossing objects. Among them, many efforts have been devoted to examine the motion around the Lagrange points. Most studies were developed in the framework of the restricted and general three body problems. In fact, the simplest approach to obtain an approximate solution is to linearize the equation of motion around the Lagrange point in the restricted circular and planar three body problem [44, 60]. This analytical solution quite well coincides with the result of numerical integration.

Of course, the stability of orbits have been discussed intensively. Obviously the next step was to include additional physical effects; the non-linear effect by Bhatnagar [3], Deprit [12], Gozdziewski [32], Hagel [33], and Papadakis [63]; the effect of eccentricity of the primary and secondary bodies by Danby [9], Erdi [18], Ichtiaroglou [41], Kinoshita

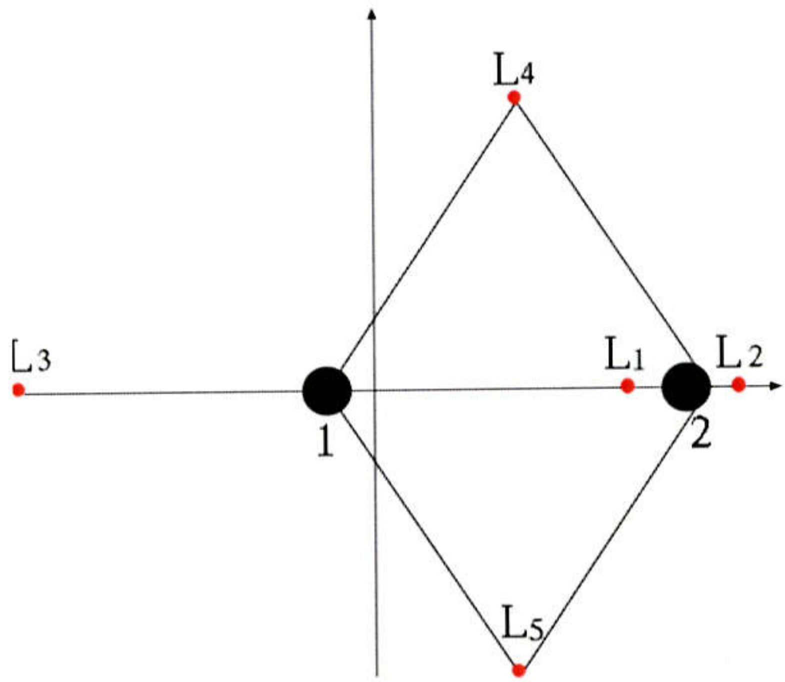

Figure 1.1: Configuration of Lagrange Equilibrium Points

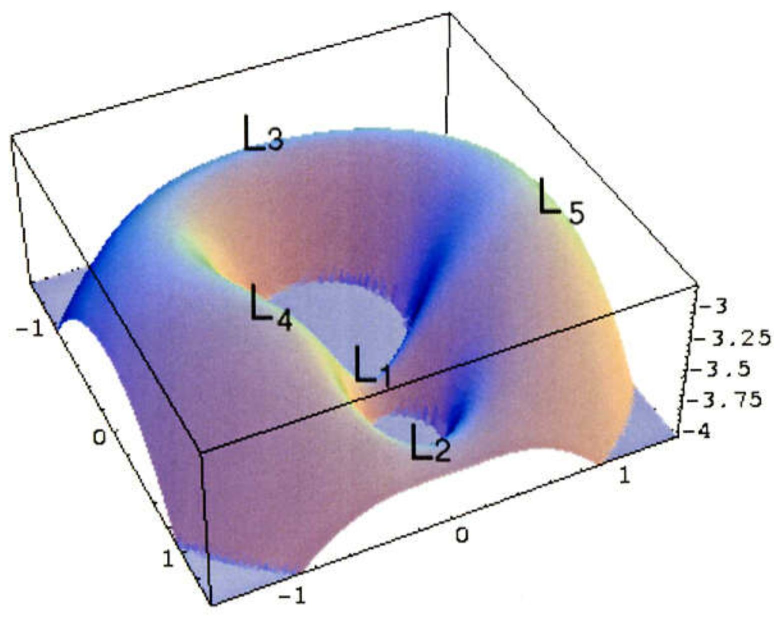

Figure 1.2: Effective Potential Surface

[45], Selaru [70], and Todoran [81]; the effect of high inclination of the orbit of the test particle by Zhang *et al.* [91]; the effect of radiation pressure due to the primary body by Kumar [49], Lukyanov [53, 54], Ragos [67], Simmons [72], and Todoran [80]; the effect of dragging force by Murray [61]; the effect of J_2 and other higher order gravitational field of the primary body by Kondurar [48], Shrivastava *et al.* [71], and Sharma *et al.* [76, 75]; the effect of variability of mass of the primary body by Horedt [37, 39], Horedt *et al.* [38], and Singh *et al.* [73]; the general relativistic effect of the primary body by Maindl [55] and Maindl *et al.* [56]; and the effect of electro-magnetic force of a charged primary body by Dionysiou [16].

On the other hand, the stability of the motion was analytically studied by Celletti [7], Danby [9], Deprit *et al.* [13, 15], Giorgilli [28], Howard [40], Kinoshita [46], Roels [68], Whipple [85], and Zagouras *et al.* [88]. Garfinkel considered the motion of Trojan asteroids, mainly taking notice of the tadpoles and horseshoe orbits in the three body problem ([22] to [27]).

In terms of the orbit design of the space missions, the Lagrange points are quite attractive. Especially the collinear points L_1 and L_2 were discussed by Breakwell *et al.* [5], Hiday-Johnston *et al.* [36], and Palutan [62]. Actually, a few space missions were launched into the position; the solar observation satellite ISEE-3 (later ICE), and SOHO by NASA and ESA at L_1 of the Sun-Earth system. Also many missions are planned; CMB observation satellite MAP, 8m-size space telescope NGST by NASA, space Astrometry satellite GAIA by ESA, an infrared telescope satellite SPICA by ISAS at L_2 of the Sun-Earth system.

While the long term behavior of the motion around the Lagrange points has been mostly studied by numerical integrations. The purposes of these studies are mainly theoretical; the population of the long-lived asteroids by Melita *et al.* [59], the search for stable orbits of planets by Weibel *et al.* [83], Innanen *et al.* [42], Erdi [18], Zhang *et al.* [89, 90], and Markellos *et al.* [57], and the escape probability from the triangular region by Tsiganis *et al.* [82]. Unfortunately, in the framework of N-body problem, there are few analytical researches, especially for the perturbations on the third and other perturbing bodies.

1.2 Discovery of Different Behavior of Motion

As we briefly summarized in the previous section, the most of the analytical researches on the dynamical behavior around the Lagrange points have been conducted in the framework of the restricted or general three body problems. These treatises rather correspond to solve the free oscillation problem around an equilibrium point in terms of the oscillation dynamics. However, when we consider an actual problems in the N-body system such as our solar system, there exist not only the primary and the secondary bodies but other perturbing bodies. In the existing works, very few authors discussed the applicability of the results of the restricted and general three body problems to the actual N-body system.

In order to clarify the applicability, we numerically examined how accurately the solution of the restricted three body problem represents the solution in the real N-body system. As the test problem, we consider the motion around the triangular Lagrange point of Sun-Jupiter system with and without the influence of the perturbation of Saturn (see Fig. 1.3). We regarded that the all the bodies move on the same orbital plane. To our surprise, the obtained results indicate quite large differences in the motion around L_4. Fig. 1.4 shows the orbit of a Trojan-like asteroid in the corotational coordinate system of the Sun-Jupiter system. The blue line expresses the result of the restricted three body problem and the red line does the result including the effect of Saturn. Note that both are obtained by a precise numerical integration so that excluded are the error of analytical treatments. In the experiment, we assumed that both Jupiter and Saturn are moving on the circular orbit. The initial condition of the asteroid is expressed in terms of the deviation from L_4 in the rectangular coordinate in the adopted corotating coordinate system as $dX = 5.2 \times 10^{-5}$ [AU], $dY = 0$ [AU], $d\dot{X} = d\dot{Y} = 0$ at $t = 0$. We put Saturn on the positive X axis at $t = 0$. Fig. 1.5 is the same as Fig. 1.4 but plotted are the time evolution of dX. Obviously the solution in the case of Sun-Jupiter-Saturn system is quite different from that of Sun-Jupiter system; about 10 times larger than the latter in the magnitude. Therefore the result obtained from the three body problem does not become the first approximation of the N-body problem. This is a new discovery. The differences are no other than the effect of forced oscillation term due to Saturn. Of course, the dominant gravity force acting on the asteroid is evidently those due to the Sun and Jupiter. However the closer to L_4 the asteroid is, the smaller the influence of the gravity of these two bodies becomes. Therefore, in the vicinity of L_4, the effect of Saturn plays

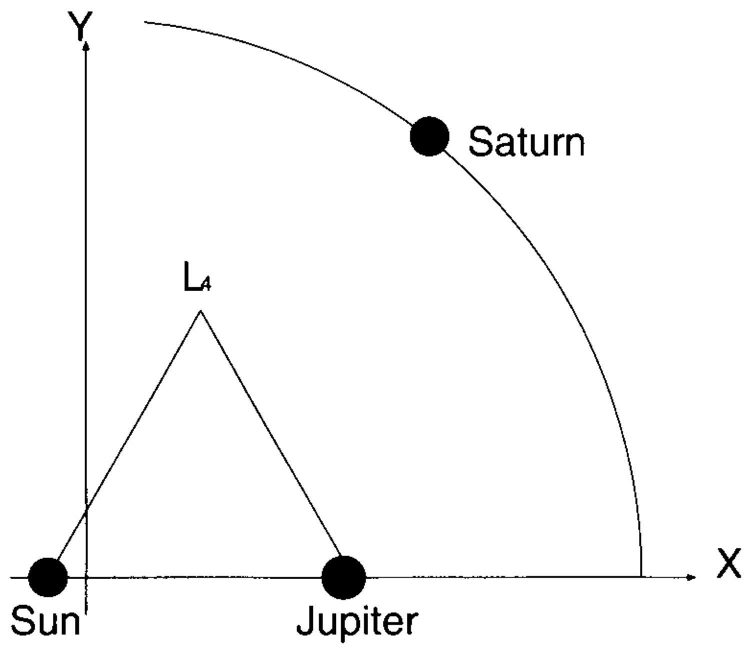

Figure 1.3: Configuration.

the key role. In terms of oscillation dynamics, we must first consider the forced oscillation caused by the external bodies like Saturn in this case.

1.3 Existing Treatises in N-Body System

As we mentioned earlier, there are only a few investigations that dealt analytically with the dynamical behaviors around Lagrange points in the framework of N-body problem (Jorba [43], and Gómez *et al.* [29, 30, 31]). In the series of their study, they tried to investigate motion of the collinear and the triangular Lagrange points by both the numerical and the semi-analytical approaches. Their interests is mainly paid to the motion around the Earth-Moon system including the effect of the Sun, other planets, and the solar radiation pressure. In their works, the corresponding approach to our study is the bicircular problem in which the motion of test particle moves under the gravitational forces of Earth, Moon, and Sun. They assumed that the orbit of Moon around the Earth is circular, and the Sun moves around the Earth-Moon barycenter in another circular orbit. In that simplified model, they interpreted the procedure to obtain the analytical solution based on the Lie transformation in coordinate. However, they did not solve it completely, and the

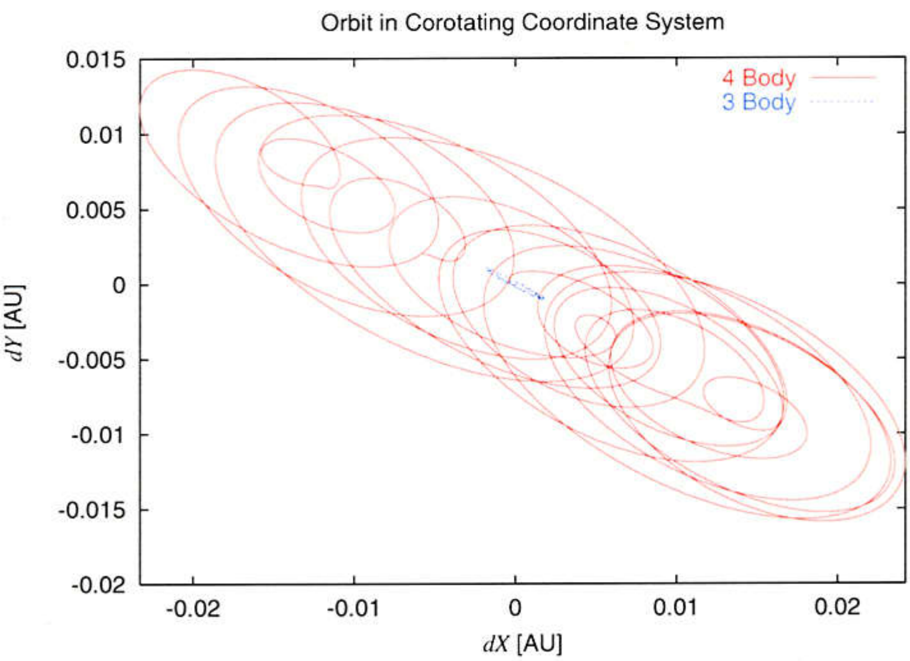

Figure 1.4: Orbits of Trojan-like asteroid in the corotational coordinate system. The blue line (3 Body) expresses the orbit of restricted three body problem when the Sun and Jupiter are the primary and the secondary perturbers. The red line (4 Body) does that including the perturbation of Saturn.

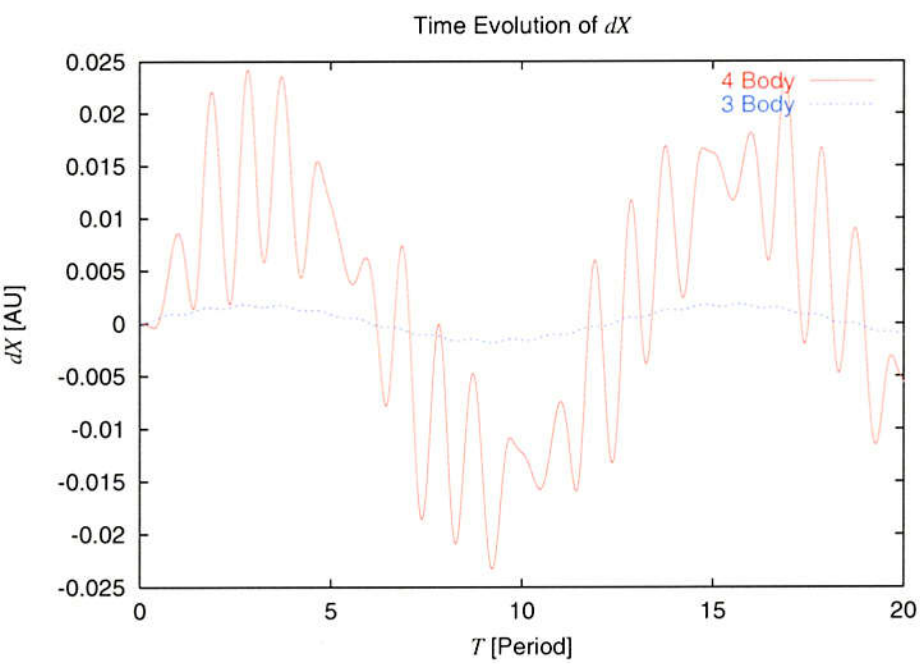

Figure 1.5: Time Evolution of dX. Same as Fig. 1.4 but plotted are the time evolution of dX.

	Our Solution	Gómez *et al.* (2001)
Main interests	Sun-Planetary system	Earth-Moon system
Approach	Purely analytical	Semi-analytical
	Expansion of disturbing force	Lie transformation
Non-linear effect	No	Yes
Eccentricity of primary and secondary	No	No
Eccentricity of other bodies	Second order	No
Comparison with numerical integration	Yes	No
Discussion of application limit	Yes	No
Global aspect of orbit	Yes	No

Table 1.1: Comparison of our theory and that of Gómez *et al.* (2001)

several figures inserted in their textbook were described with the aid of some numerical processes. Hence their approach is not satisfactory in many points as illustrated in Table 1.1. Thus as yet, there does exist the purely analytical solution around the Lagrange point. Further, as their advanced research, they considered the motion of test particle around the Lagrange point in the real solar system. They calculated the solution by Newton method using the preselected frequencies which are related to the motion of the perturbers. These frequencies are obtained by the FFT based on the JPL's planetary ephemeris DE405. But as they mentioned in their text, this approach sometimes faces a convergence problem and is not unsuitable for the long period orbital calculations.

1.4 Purpose of This Thesis

In this thesis, we will construct a purely analytical theory of the motion of the test particle around the triangular Lagrange points in the framework of N-body system. We include the effect of direct gravitational force of the third and other perturbing bodies. And we express the solution as an explicit function of time. We also investigate our solution in terms of the time evolution behavior as well as the global aspects of orbit such as the width, the thickness, and their ratio of the orbital region, respectively, and the barycenter of the orbit as the time averaged value. Then we compare our analytical solution with the results of numerical integration.

It is beneficial to construct an analytical theory such as we do, especially for designing the orbits of some space missions. As we mentioned before, some space missions located on the triangular Lagrange are planned. But the present orbital design must carry out the vast of numerical integration in the huge initial condition space. Then the analytical solution is expected that it reduces the considerable time of the numerical integration and restricts the initial condition. In fact the calculation and prediction of the orbit behavior, our solution is much faster than the numerical integration or the semi-analytical approach of Jorba and Gómez *et al.*. Among them, the re-evaluation of the orbital region is much easy especially when the spacecraft was put into a wrong orbit. Then it is possible to provide the preliminary constrain for the initial condition of the orbital design.

Chapter 2

Analytical Solution

Let us construct an analytical theory of the motion of a test particle around the triangular Lagrangian point when there exist the perturbations due to extra perturbing bodies.

We assume that (1) the orbits of all bodies are coplanar, (2) the orbits of the primary and the secondary around their barycenter are circular, and (3) the extra perturbing bodies are moving around the barycenter of the primary and secondary in non-circular Keplerian orbit.

As we showed in Fig. 1.4, the solution of the three body problem does not become the first approximation of the motion of N-body problem. In fact, in the vicinity of the Lagrange point, the effect of the direct gravitational force of the other bodies dominates that of the primary and the secondary bodies. Therefore we can no longer regard the effect of the other bodies as the "perturbation" around the Lagrange point as the usual approach of the perturbation theory. Thus we start from the equation of motion in the inertial coordinate system and expand it around the Lagrange point in the coordinate. Then we estimate the magnitude of the expanded terms and include the dominate terms.

The equation of motion of the test particle is written in the inertial coordinate system as,

$$\frac{d^2\boldsymbol{r}}{dt^2} = \boldsymbol{F}_L(\boldsymbol{r}, t) + \boldsymbol{F}_E(\boldsymbol{r}, t) \tag{2.1}$$
$$\boldsymbol{r} = \boldsymbol{r}_0, \quad \dot{\boldsymbol{r}} = \dot{\boldsymbol{r}}_0, \quad t = 0,$$

where

$$\boldsymbol{F}_L(\boldsymbol{r}, t) = GM_1 \frac{\boldsymbol{r}_1 - \boldsymbol{r}}{|\boldsymbol{r}_1 - \boldsymbol{r}|^3} + GM_2 \frac{\boldsymbol{r}_2 - \boldsymbol{r}}{|\boldsymbol{r}_2 - \boldsymbol{r}|^3} \tag{2.2}$$

$$\boldsymbol{F}_E(\boldsymbol{r}, t) = \sum_{I=3}^{N} GM_I \frac{\boldsymbol{r}_I - \boldsymbol{r}}{|\boldsymbol{r}_I - \boldsymbol{r}|^3}. \tag{2.3}$$

The subscript I denotes the perturbing bodies, G is Newton's gravitational constant, and M_I is the mass of perturbing bodies. In Eq. (2.1), we separated the perturbing force into two parts. The one is the net effect of the primary and the secondary bodies which vanishes at the Lagrange points in the corotational coordinate frame. The other is the contribution due to the extra perturbing bodies which remain finite at the Lagrange point.

Let us introduce a new variable $\delta\boldsymbol{r} = \boldsymbol{r} - \boldsymbol{r}_L$ where $\boldsymbol{r}_L$ denotes the Lagrange point, Then we expand Eq. (2.1) around $\boldsymbol{r}_L$ as,

$$\begin{aligned} \frac{d^2(\boldsymbol{r}_L + \delta\boldsymbol{r})}{dt^2} &= \boldsymbol{F}_L(\boldsymbol{r}_L + \delta\boldsymbol{r}, t) + \boldsymbol{F}_E(\boldsymbol{r}_L + \delta\boldsymbol{r}, t) \\ &\approx \boldsymbol{F}_L(\boldsymbol{r}_L, t) + \boldsymbol{F}_E(\boldsymbol{r}_L, t) + \left(\frac{\partial \boldsymbol{F}_L(\boldsymbol{r}, t)}{\partial \boldsymbol{r}}\right)_{\boldsymbol{r}_L} \delta\boldsymbol{r} \\ &\quad + \left(\frac{\partial \boldsymbol{F}_E(\boldsymbol{r}, t)}{\partial \boldsymbol{r}}\right)_{\boldsymbol{r}_L} \delta\boldsymbol{r}, \end{aligned} \tag{2.4}$$

Noting that the Lagrange point satisfies the relation,

$$\frac{d^2\boldsymbol{r}_L}{dt^2} - \boldsymbol{F}_L(\boldsymbol{r}_L, t) = 0,$$

we rewrite the above equation of motion as,

$$\begin{aligned} \frac{d^2\delta\boldsymbol{r}}{dt^2} &= \boldsymbol{F}_E(\boldsymbol{r}_L, t) + \left(\frac{\partial \boldsymbol{F}_L(\boldsymbol{r}, t)}{\partial \boldsymbol{r}}\right)_{\boldsymbol{r}_L} \delta\boldsymbol{r} + \left(\frac{\partial \boldsymbol{F}_E(\boldsymbol{r}, t)}{\partial \boldsymbol{r}}\right)_{\boldsymbol{r}_L} \delta\boldsymbol{r} \\ &\approx \boldsymbol{F}_E(\boldsymbol{r}_L, t) + \left(\frac{\partial \boldsymbol{F}_L(\boldsymbol{r}, t)}{\partial \boldsymbol{r}}\right)_{\boldsymbol{r}_L} \delta\boldsymbol{r}. \end{aligned} \tag{2.5}$$

Here we estimate the magnitude of each term of Eq. (2.5) in the Sun-Jupiter-Saturn system. The ratios of $(\partial\boldsymbol{F}_E(\boldsymbol{r}, t)/\partial\boldsymbol{r})_{\boldsymbol{r}_L}\delta\boldsymbol{r}/\boldsymbol{F}_E(\boldsymbol{r}_L, t)$ and $(\partial\boldsymbol{F}_E(\boldsymbol{r}, t)/\partial\boldsymbol{r})_{\boldsymbol{r}_L}\delta\boldsymbol{r}/\boldsymbol{F}_E(\boldsymbol{r}_L, t)$ are,

$$\begin{aligned} \frac{(\partial\boldsymbol{F}_E(\boldsymbol{r}, t)/\partial\boldsymbol{r})_{\boldsymbol{r}_L}\delta\boldsymbol{r}}{\boldsymbol{F}_E(\boldsymbol{r}_L, t)} &= 0.014\varepsilon \\ \frac{(\partial\boldsymbol{F}_E(\boldsymbol{r}, t)/\partial\boldsymbol{r})_{\boldsymbol{r}_L}\delta\boldsymbol{r}}{\boldsymbol{F}_E(\boldsymbol{r}_L, t)} &= 4.6 \times 10^{-5}\varepsilon, \end{aligned}$$

where

$$\varepsilon = \frac{\delta r}{r_L}.$$

Therefore we ignore the third term in the first line in Eq. (2.5). Next we assume that the solution is split as,

$$\delta \boldsymbol{r} = \delta \boldsymbol{r}_L + \delta \boldsymbol{r}_E, \tag{2.6}$$

where $\delta \boldsymbol{r}_L$ and $\delta \boldsymbol{r}_E$ satisfy the following equations, respectively,

$$\frac{d^2\delta \boldsymbol{r}_L}{dt^2} = \left(\frac{\partial \boldsymbol{F}_L}{\partial \boldsymbol{r}}\right)_{\boldsymbol{r}_L} \delta \boldsymbol{r}_L, \quad \delta \boldsymbol{r}_L = \delta \boldsymbol{r}_0, \delta \dot{\boldsymbol{r}}_L = \delta \dot{\boldsymbol{r}}_0, \text{at} \quad t = 0, \tag{2.7}$$

$$\frac{d^2\delta \boldsymbol{r}_E}{dt^2} = \boldsymbol{F}(\boldsymbol{r}_L, t)_E + \left(\frac{\partial \boldsymbol{F}(\boldsymbol{r}, t)_L}{\partial \boldsymbol{r}}\right)_{\boldsymbol{r}_L} \delta \boldsymbol{r}_E \quad \delta \boldsymbol{r}_E = 0, \delta \dot{\boldsymbol{r}}_E = 0, \text{at} \quad t = 0. \tag{2.8}$$

In the following sections, we will specifically derive the expression of the solution.

2.1 Solution of Free Oscillation

First, we derive the solution of the free oscillation part in the inertial coordinate system. The equation of motion of $\delta \boldsymbol{r}_L$ is expressed as

$$\frac{\partial^2\delta \boldsymbol{r}_L}{dt^2} = \left(\frac{\partial \boldsymbol{F}(\boldsymbol{r}, t)}{\partial \boldsymbol{r}}\right)_{\boldsymbol{r}_L} \delta \boldsymbol{r}_L. \tag{2.9}$$

Usually the solution is given in the corotational coordinate system. See the derivation given in the literature such as Kinoshita [44] or Murray & Dermott [60]. The equation of motion in the case of the restricted three body problem is express in the corotational coordinate system as,

$$\frac{d^2\delta\bar{X}}{dt^2} - 2n\frac{d\delta\bar{Y}}{dt} = \left(\frac{\partial^2 U}{\partial X^2}\right)_{\boldsymbol{r}_L} \delta\bar{X}, \tag{2.10}$$

$$\frac{d^2\delta\bar{Y}}{dt^2} + 2n\frac{d\delta\bar{X}}{dt} = \left(\frac{\partial^2 U}{\partial Y^2}\right)_{\boldsymbol{r}_L} \delta\bar{Y}, \tag{2.11}$$

in which U is the potential,

$$U = -\frac{1}{2}n^2(X^2 + Y^2) - \frac{GM_1}{|\boldsymbol{r}_1 - \boldsymbol{r}|} - \frac{GM_2}{|\boldsymbol{r}_2 - \boldsymbol{r}|}. \tag{2.12}$$

The solution has a form of harmonic oscillator of two modes,

$$\delta\bar{X} = \sum_{\beta=1}^{2} C_\beta \cos(\omega_\beta t + \gamma_\beta), \quad \delta\bar{Y} = \sum_{\beta=1}^{2} S_\beta \sin(\omega_\beta t + \gamma_\beta). \tag{2.13}$$

where ω_1, ω_2 are the eigenfrequencies expressed as,

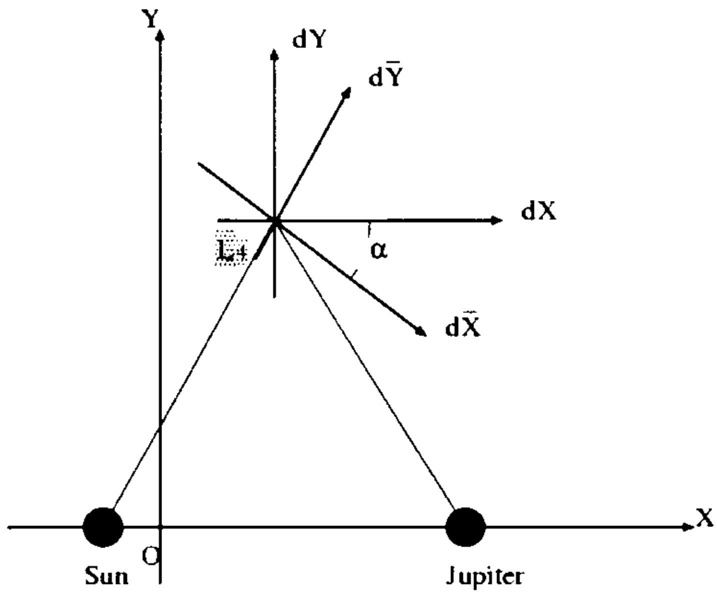

Figure 2.1: Relations among the coordinates

$$\omega_1 = \sqrt{\frac{1}{2}\left\{1+\sqrt{1-27\nu(1-\nu)}\right\}}n \tag{2.14}$$

$$\omega_2 = \sqrt{\frac{1}{2}\left\{1-\sqrt{1-27\nu(1-\nu)}\right\}}n, \tag{2.15}$$

where

$$\nu = \frac{M_2}{M_1+M_2}, \quad n = \sqrt{\frac{G(M_1+M_2)}{a^3}}.$$

In the case of Sun-Jupiter system, ω_1 and ω_2 have the period of 11.901 year and 147.42 year, respectively. The amplitude and initial phase depend on the initial condition. We put $(\delta\bar{x}, \delta\bar{y}, \delta\dot{\bar{x}}, \delta\dot{\bar{y}}) = (\delta\bar{x}_0, \delta\bar{y}_0, \delta\dot{\bar{x}}_0, \delta\dot{\bar{y}}_0)$ at $t = 0$ as the initial condition and the amplitude and initial phase are given by

$$C_1 = \frac{2n\delta\bar{y}_0 + (\omega_2^2 - a^*)\delta\bar{x}_0}{(\omega_2^2 - \omega_1^2)}\sqrt{\frac{a^{*2}(2n\delta\bar{y}_0 + (\omega_2^2 - a^*)\delta\bar{x}_0)^2}{2n\omega_1\omega_2^2\delta\bar{y}_0 + \omega_1(a^* - \omega_2^2)\delta\bar{x}_0^2} + 1} \tag{2.16}$$

$$C_2 = \frac{2n\omega_1^2\omega_2\delta\bar{y}_0 + (a^* - \omega_1^2)\omega_2\delta\bar{x}_0}{a^*(\omega_1^2 - \omega_2^2)}\sqrt{\frac{a^{*2}(2n\delta\bar{y}_0 + (\omega_1^2 - a^*)\delta\bar{x}_0)^2}{2n\omega_1^2\omega_2\delta\bar{y}_0 + (a^* - \omega_1^2)\omega_2\delta\bar{x}_0^2} + 1} \tag{2.17}$$

$$\gamma_1 = \arctan\left(\frac{-(\omega_2^2 - a^*)\omega_1\delta\dot{\bar{x}}_0 + 2n\omega_1\omega_2^2\delta\bar{y}_0}{a^*(2n\delta\dot{\bar{y}}_0 + (\omega_2^2 - a^*)\delta\bar{x}_0)}\right) \tag{2.18}$$

$$\gamma_2 = \arctan\left(\frac{-(\omega_1^2 - a^*)\omega_2\delta\dot{\bar{x}}_0 + 2n\omega_1^2\omega_2\delta\bar{y}_0}{a^*(2n\delta\dot{\bar{y}}_0 + (\omega_1^2 - a^*)\delta\bar{x}_0)}\right) \tag{2.19}$$

$$\frac{S_\beta}{C_\beta} = -\frac{\omega_\beta^2 - a^*}{2n\omega_\beta}, \quad \beta = 1, 2, \tag{2.20}$$

where

$$a^* = -\frac{3}{2}(1 - \sqrt{1 - 3\nu(1-\nu)})n^2.$$

Thus the solution is expressed in the inertial system as

$$x = X\cos\ell - Y\sin\ell, \quad y = X\sin\ell + Y\cos\ell \tag{2.21}$$

where ℓ is the longitude of the secondary body and

$$\begin{aligned} X &= a\left\{\cos\left(\frac{\pi}{3}\right) - \mu + \delta\bar{X}\cos\alpha - \delta\bar{Y}\sin\alpha\right\} \\ Y &= a\left\{\sin\left(\frac{\pi}{3}\right) + \delta\bar{X}\sin\alpha + \delta\bar{Y}\cos\alpha\right\}. \end{aligned}$$

Here α is the rotation angle (see, Fig. 2.1) and defined by the relation.

$$\alpha = -\frac{\arctan(-\sqrt{3}(1-2\nu))}{2}.$$

Note that the the effect of the initial condition of the motion is completely absorbed by the solution of free oscillation. Fig. 2.2 illustrates the orbit of the free oscillation in the corotational coordinate system where $\varepsilon = 10^{-5}$.

2.2 Solution of Forced Oscillation

From Eq. (2.5), the equation of motion of $\delta\boldsymbol{r}_E$ becomes

$$\frac{d^2\delta\boldsymbol{r}_E}{dt^2} = \boldsymbol{F}_E(\boldsymbol{r}_L, t) + \left(\frac{\partial\boldsymbol{F}_L(\boldsymbol{r}, t)}{\partial\boldsymbol{r}}\right)_{\boldsymbol{r}_L}\delta\boldsymbol{r}_E. \tag{2.22}$$

As we examined the order of terms in the right-hand side, the first term is the main part of the forced oscillation. In evaluating it, we consider the effect of eccentricity of the extra bodies up to the second order. Note that the eccentricity of Saturn is as small as $e = 0.0555$. We expand the position of the perturbing bodies in the coordinate system where x-axis is the direction of the perihelion, up to the order of e_I^2 as

$$\begin{aligned} \frac{\eta_I}{a_I} &\equiv \sqrt{1-e_I^2}\sin u_I = \left(1 - \frac{5}{8}e_I^2\right)\sin\ell_I - \frac{1}{8}e_I\sin 2\ell_I + \frac{3}{8}e_I^2\sin 3\ell_I + \cdots \quad (2.23) \\ \frac{\xi_I}{a_I} &\equiv \cos u_I = -\frac{1}{2}e_I + \left(1 - \frac{3}{8}e_I^2\right)\cos\ell_I + \frac{1}{2}e_I\cos 2\ell_I + \frac{3}{8}e_I^2\cos 3\ell_I + \cdots. \quad (2.24) \end{aligned}$$

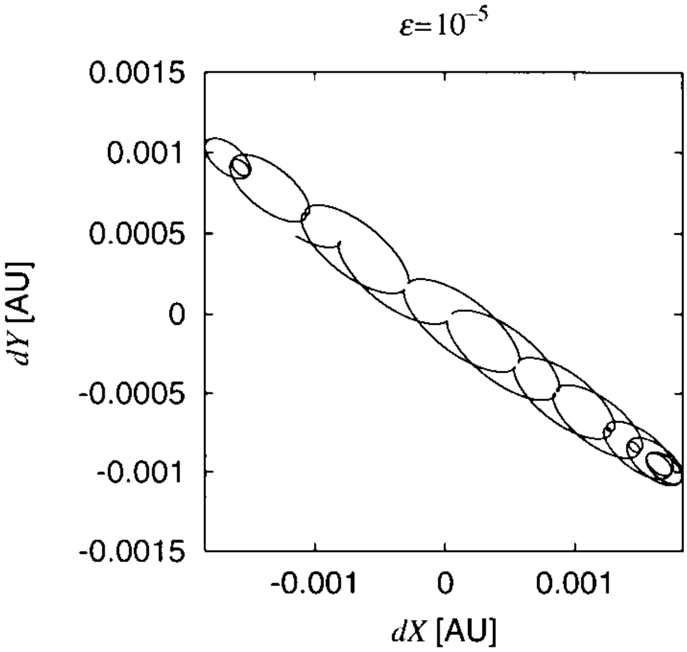

Figure 2.2: Solution of Free Oscillation around Lagrange Point

By rotating them by the angle ϖ, the longitude of perihelion, we obtain the expression in the inertial coordinate system as,

$$x_I = \xi_I \cos\varpi - \eta_I \sin\varpi, \quad y_I = \xi_I \sin\varpi + \eta_I \cos\varpi. \tag{2.25}$$

Since we assume that the orbit of the secondary body is circular, we approximate

$$\begin{aligned} \frac{r}{r_I} &= \left(\frac{a}{a_I}\right)\frac{1}{1-e_I\cos u_I} \approx \frac{a}{a_I}(1+e_I\cos u_I + e_I^2\cos^2 u_I) + \cdots \\ &\approx \frac{a}{a_I}\left(1-\frac{e_I^2}{2}+e_I\cos\ell_I+\frac{3}{8}e_I^2\cos 2\ell_I\right). \end{aligned} \tag{2.26}$$

Then we expand the denominator of $\boldsymbol{F}_E$ by using Legendre polynomials up to the order of $(r/r_I)^7$ as,

$$\begin{aligned} \frac{1}{|\boldsymbol{r}_I-\boldsymbol{r}|^3} &= \frac{1}{r_I^3}\left[1+3\cos S_i\frac{r}{r_I}+\frac{3}{2}\left(5\cos^2 S_I-1\right)\left(\frac{r}{r_I}\right)^2\right. \\ &\quad +\frac{5}{2}\left(7\cos^3 S_I-3\cos S_I\right)\left(\frac{r}{r_I}\right)^3 \\ &\quad +\frac{15}{8}\left(21\cos^4 S_I-14*\cos^2 S_I+1\right)\left(\frac{r}{r_I}\right)^4 \end{aligned}$$

$$+\frac{21}{8}\left(33\cos^5 S_I - 30\cos^3 S_I + 5\cos S_I\right)\left(\frac{r}{r_I}\right)^5$$
$$+\frac{7}{17}\left(429\cos^6 S_I - 495\cos^4 S_I + 135\cos^2 S_I - 5\right)\left(\frac{r}{r_I}\right)^6$$
$$+\frac{9}{17}\left(715\cos^7 S_I - 1001\cos^5 S_I + 385\cos_I^3 - 35\cos S_I\right)\left(\frac{r}{r_I}\right)^7\Bigg], \tag{2.27}$$

where $S_I = \ell_I - \ell = n_I(t - t_{I0}) - n(t - t_0)$, n_I is the mean motion of I-th perturbing body, and t_0, t_{I0} are the times of perihelion passage of Lagrange point and I-th perturbing body, respectively. We expanded $1/|\boldsymbol{r}_I - \boldsymbol{r}|^3$ up to the order of $(r/r_I)^7$ and ignore the third and higher terms with respect to e_I. Since we assume that the orbit of the secondary body is circular, the position of the triangular Lagrange point (L_4) becomes

$$x_L = a\left(\cos\left(\ell + \frac{\pi}{3}\right) - \mu\cos\ell\right), \quad y_L = a\left(\sin\left(\ell + \frac{\pi}{3}\right) - \mu\sin\ell\right) \tag{2.28}$$

Thus the first term of the right-hand side of Eq. (2.22) is explicitly expressed as the function of time t as,

$$\begin{aligned}
\begin{pmatrix} F_x \\ F_y \end{pmatrix} &= \frac{1}{a_I^3}\Bigg[1 + 3\cos S_i\left(1 - \frac{e_I^2}{2} + e_I\cos\ell + \frac{3}{8}e_I^2\cos 2\ell\right)\frac{a}{a_I} \\
&+ \frac{3}{2}\left(5\cos^2 S_i - 1\right)\left(1 - \frac{2e_I^2}{2} + 2e_I\cos\ell + \frac{6}{8}e_I^2\cos 2\ell\right)\left(\frac{a}{a_I}\right)^2 \\
&+ \frac{5}{2}\left(7\cos^3 S_i - 3\cos S_i\right)\left(\frac{a}{a_I}\right)^3 \\
&+ \frac{15}{8}\left(21\cos^4 S - 14 * \cos^2 S + 1\right)\left(\frac{a}{a_I}\right)^4 \\
&+ \frac{21}{8}\left(33\cos^5 S - 30\cos^3 S + 5\cos S\right)\left(\frac{a}{a_I}\right)^5 \\
&+ \frac{7}{17}\left(429\cos^6 S - 495\cos^4 S + 135\cos^2 S - 5\right)\left(\frac{a}{a_I}\right)^6 \\
&+ \frac{9}{17}\left(715\cos^7 S - 1001\cos^5 S + 385\cos^3 S - 35\cos S\right)\left(\frac{a}{a_I}\right)^7\Bigg] \\
&\times \begin{pmatrix}
a_I\left[\left\{\left(1 - \frac{5}{8}e_I^2\right)\sin\ell_I - \frac{1}{8}e_I\sin 2\ell_I + \frac{3}{8}e_I^2\sin 3\ell_I\right\}\cos\varpi\right. \\
\left. + \left\{\frac{1}{2}e_I - \left(1 - \frac{3}{8}e_I^2\right)\cos\ell_I - \frac{1}{2}e_I\cos 2\ell_I - \frac{3}{8}e_I^2\cos 3\ell_I\right\}\sin\varpi\right] \\
-a\left(\cos\left(\ell + \frac{\pi}{3}\right) - \mu\cos\ell\right) \\
a_I\left[\left\{\left(1 - \frac{5}{8}e_I^2\right)\sin\ell_I - \frac{1}{8}e_I\sin 2\ell_I + \frac{3}{8}e_I^2\sin 3\ell_i\right\}\sin\varpi\right. \\
\left. - \left\{\frac{1}{2}e_I - \left(1 - \frac{3}{8}e_I^2\right)\cos\ell_I - \frac{1}{2}e_I\cos 2\ell_I - \frac{3}{8}e_I^2\cos 3\ell_I\right\}\cos\varpi\right] \\
-a\left(\sin\left(\ell + \frac{\pi}{3}\right) - \mu\sin\ell\right)
\end{pmatrix}
\end{aligned} \tag{2.29}$$

The partial derivative in Eq. (2.22) is given by

$$\frac{\partial F_{Lj}(\boldsymbol{r},t)}{\partial r_k} = \sum_{I=1}^{2} GM_I \left[-\frac{\delta_{jk}}{|r_{Ik}-r_k|^3} + \frac{3(r_{Ij}-r_j)\otimes(r_{Ik}-r_k)}{|r_{Ik}-r_k|^5} \right] \tag{2.30}$$

where the subscripts j and k represent the coordinate and δ_{jk} is the Kronecker's delta. The components of the right-hand side of Eq. (2.30) become

$$\begin{aligned}
\frac{\partial F_{Lx}}{\partial x} &= \frac{1}{a^3}\left[\frac{\mu}{2}\left\{1+3\cos\left(2\left(\ell+\frac{\pi}{3}\right)\right)\right\} + 3GM_2\left\{\frac{\cos 2\ell}{2} - \cos\left(2\ell+\frac{\pi}{3}\right)\right\}\right] && (2.31)\\
\frac{\partial F_{Ly}}{\partial y} &= \frac{1}{a^3}\left[\frac{\mu}{2}\left\{1-3\cos\left(2\left(\ell+\frac{\pi}{3}\right)\right)\right\} - 3GM_2\left\{\frac{\cos 2\ell}{2} - \cos\left(2\ell+\frac{\pi}{3}\right)\right\}\right] && (2.32)\\
\frac{\partial F_{Lx}}{\partial y} &= \frac{3}{a^3}\left[\frac{\mu}{2}\sin\left(2\left(\ell+\frac{\pi}{3}\right)\right) + GM_2\left\{\frac{\sin 2\ell}{2} - \sin\left(2\ell-\frac{\pi}{3}\right)\right\}\right] && (2.33)\\
\frac{\partial F_{Ly}}{\partial x} &= \frac{\partial F_{Lx}}{\partial y} && (2.34)
\end{aligned}$$

Now it has become straightforward to solve Eq. (2.22) since the main term contains only t. Actually we solve it iteratively. Namely we expand the solution as,

$$\delta\boldsymbol{r}_E \equiv \sum_{n=0}^{\infty} \delta\boldsymbol{r}_E^{(n)} = \delta\boldsymbol{r}_E^{(0)} + \delta\boldsymbol{r}_E^{(1)} + \delta\boldsymbol{r}_E^{(2)} + \cdots, \tag{2.35}$$

where $\delta\boldsymbol{r}_E^{(n)}$ satisfy the following equations

$$\begin{aligned}
\frac{d^2\delta\boldsymbol{r}_E^{(0)}}{dt^2} &= \boldsymbol{F}_E(\boldsymbol{r}_L,t), && (2.36)\\
\frac{d^2\delta\boldsymbol{r}_E^{(1)}}{dt^2} &= \frac{\partial\boldsymbol{F}_L(\boldsymbol{r}_L,t)}{\partial\boldsymbol{r}}\delta\boldsymbol{r}_E^{(0)}, && (2.37)\\
\frac{d^2\delta\boldsymbol{r}_E^{(2)}}{dt^2} &= \frac{\partial\boldsymbol{F}_L(\boldsymbol{r}_L,t)}{\partial\boldsymbol{r}}\delta\boldsymbol{r}_E^{(1)}, && (2.38)\\
&\cdots .
\end{aligned}$$

These equations are directly solved by the double integration as

$$\begin{aligned}
\delta\boldsymbol{r}_E^{(0)} &= \int\left[\int \boldsymbol{F}_E(\boldsymbol{r}_L,t)dt\right]dt, && (2.39)\\
\delta\boldsymbol{r}_E^{(1)} &= \int\left[\int\left(\frac{\partial\boldsymbol{F}_L(\boldsymbol{r},t)}{\partial\boldsymbol{r}}\right)_{\boldsymbol{r}_L}\delta\boldsymbol{r}_E^{(0)}dt\right]dt, && (2.40)\\
&\cdots .
\end{aligned}$$

Here we only consider $\delta\boldsymbol{r}_E^{(0)}$ and $\delta\boldsymbol{r}_E^{(1)}$ because $\delta\boldsymbol{r}_E^{(2)}$ and the higher correspond to the solve the nonlinear terms. Note that the initial condition is given in the form $\delta\boldsymbol{r}_E^{(j)} = 0$ at $t = 0$

since the initial conditions of the whole equation of motion is satisfied by the solution of the free oscillation part. In the above expression, we ignored the second and higher terms. Thus the final solution is expressed in the series of a/a_I as,

$$\delta \boldsymbol{r}_E = \sum_{j=0}^{\infty} \left[\delta \boldsymbol{r}_E^{(0j)} + \delta \boldsymbol{r}_E^{(1j)} \right] \left(\frac{a}{a_I} \right)^j . \tag{2.41}$$

Actually we expanded the series up to $(a/a_I)^7$. The obtained solutions are explicitly given in Appendix B.

Chapter 3

Numerical Comparisons

3.1 Condition of Numerical Experiments

From now on, we will show the comparisons between our analytical solution and the numerical integrations.

As a test problem of the N-body system, we consider the motion of test particle in the vicinity of L_4 point of the Sun-Jupiter system under the gravitational forces of the Sun, Jupiter, and Saturn. We assume that the orbits of all the bodies are coplanar. We compare our analytical solution (hereafter noted Analytical) with the numerical solution of the restricted four body problem (hereafter 4 Body), and with the numerical solution of restricted three body problem (hereafter 3 Body). We also assume that the Jupiter's orbit is circular and its semi-major axis is 5.2026 AU. For the Analytical and 4 Body cases, we include the perturbation of Saturn, whose semi-major axis and eccentricity are chosen as 9.5549 AU and 0.0555, respectively. Initially the test particle is located at a point departed by the radius δr from L_4 and the initial position angle Φ (see Fig. 3.1). The numerical solutions were obtained by using the method of variation of parameter based on the KS regularization developed by us [2]. The details of the KS regularization and its method of variation of parameter are discussed in Appendix A. And we adopted the Adams method as the numerical integrator.

3.2 Time Evolution of Orbits

First we show the differences in the corotational coordinate system. Fig. 3.2 shows the time evolution for the 20 revolutions with respect to the Jupiter. This is for the initial

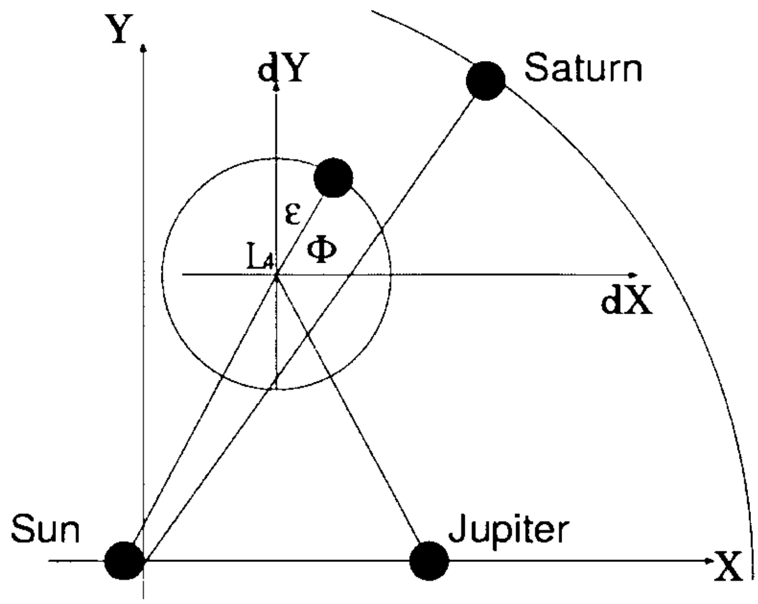

Figure 3.1: Initial Condition

condition $\delta r = 5.2 \times 10^{-5}$ AU, and $\Phi = 0$ degree. This figure illustrates that the analytical solution agrees well with the numerical integration of 4 Body case. Figs. 3.3, and 3.4 illustrate the difference between the analytical and numerical (4 Body) solutions for the 20 revolutions of Jupiter's orbit. And Figs. 3.5, and 3.6 are the same as Figs. 3.3, and 3.4 but for a longer time span, 1000 revolutions of Jupiter. For the short span, the maximum error is of the order of 10^{-4}. For the long term, the error is of the form of the mixed secular term and finally becomes of the same order of the size of orbit. Figs. 3.7, and 3.8 are the same as Figs. 3.3, and 3.4 but plotted are for a longer period. From these figures, the beating period is almost 3250 orbital periods of Jupiter. The feature of the error in the short term supports an idea that the error is caused by the non-linear effect of the free oscillation term. The observed frequency of the error is around 0.16, which is quite close to the double of the longer eigenfrequency of the free oscillation, $\omega_2 = 0.0805$. Fig. 3.9 plots the frequency analysis of the residual of dX by Fast Fourier Transform (FFT) and Fig. 3.10 is the same as Fig. 3.9 but represented are the close up. Clearly, the effect of $2\omega_2$ stand out and we can find that the error growth of our analytical solution is mainly occurred by elimination of the nonlinear effect of the tidal force of the primary and the secondary bodies. Also the amplitude of the relative error is of the order of the square of the relative amplitude of the linear solution, $C_{1,2}$ or $S_{1,2}$. Figs. 3.11 and 3.15 show the error in radius. Figs. 3.12 and 3.16 are the same as Figs. 3.11 and 3.15 but for the longitude. Figs. 3.13 and 3.14 are the same as Figs. 3.11 and 3.12 but plotted are the close-up of Analytical. Unchanged is the error increasing in a mixed secular manner. While Figs. 3.17, and 3.18 represent the ε- and Φ-dependence where we depicted only the dependence of the initial position. Shown are the deviations referred to the $p = -5$ for Fig. 3.17, and to the $\Phi = 0$ for Fig. 3.18, respectively. Confirmed was an almost similar dependence with the initial velocity. These figures indicate that, when perturbed by other bodies, the behavior of the test particle moving around the triangular Lagrange point hardly depends on the initial condition. Namely the solution of the main part remains the forced oscillation due to the other bodies as we expected.

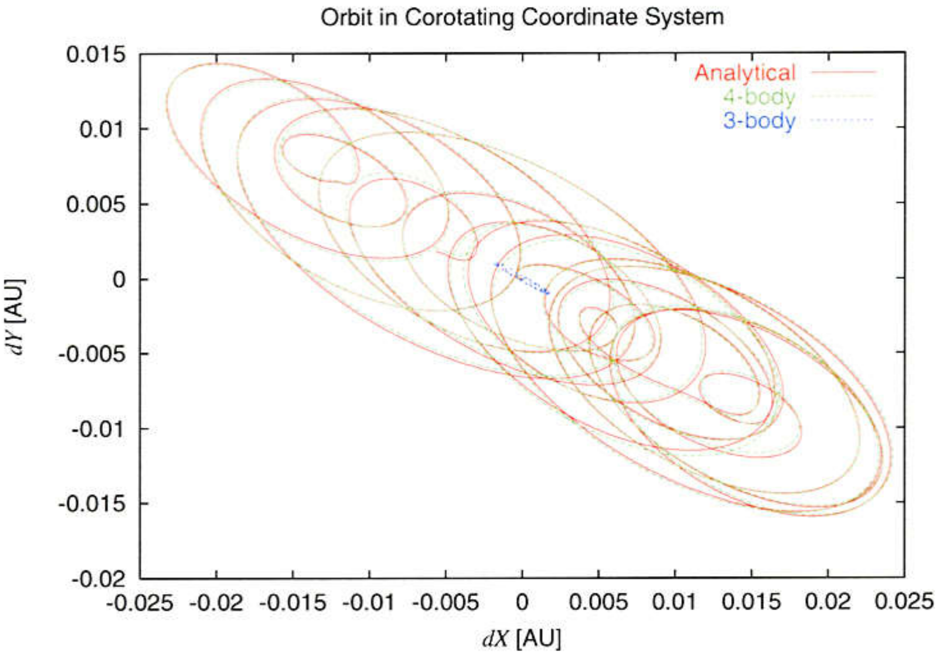

Figure 3.2: Orbit in the corotational coordinate system : $\theta = 0$ (deg.)

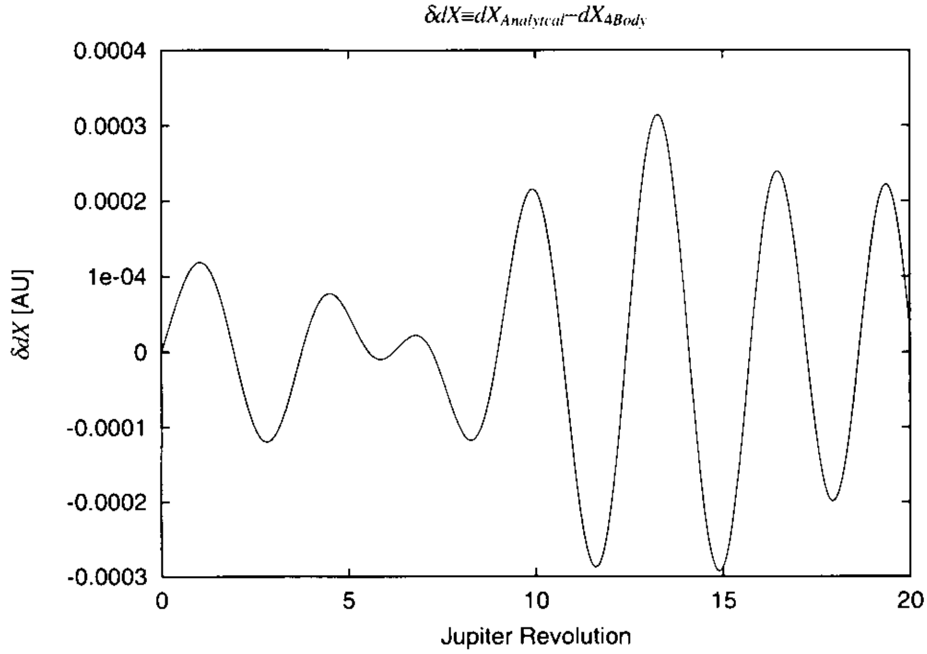

Figure 3.3: Error of Analytical Solution with respect to Numerical Solution

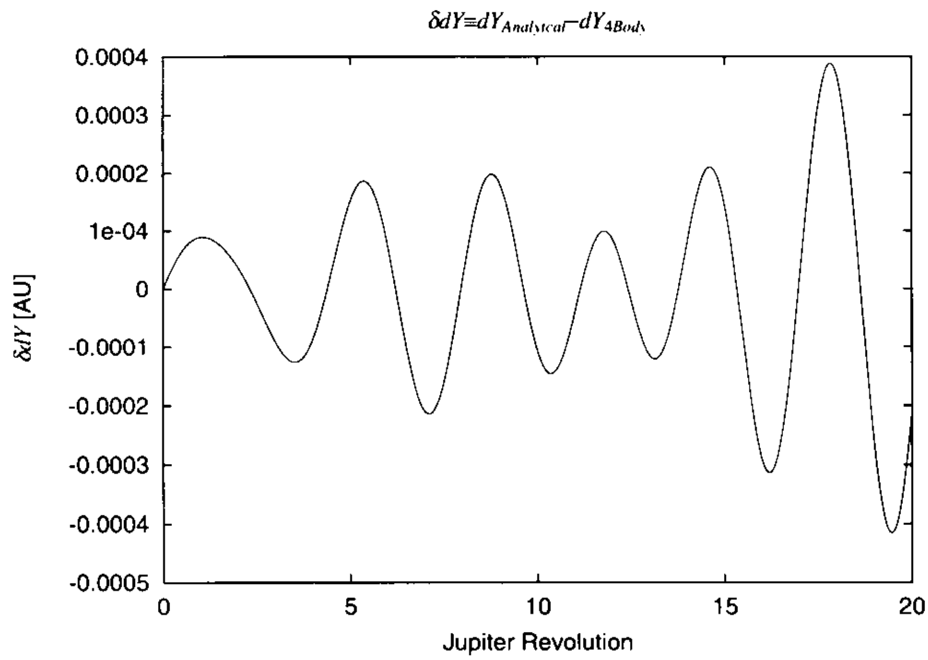

Figure 3.4: Error of Analytical Solution with respect to Numerical Solution

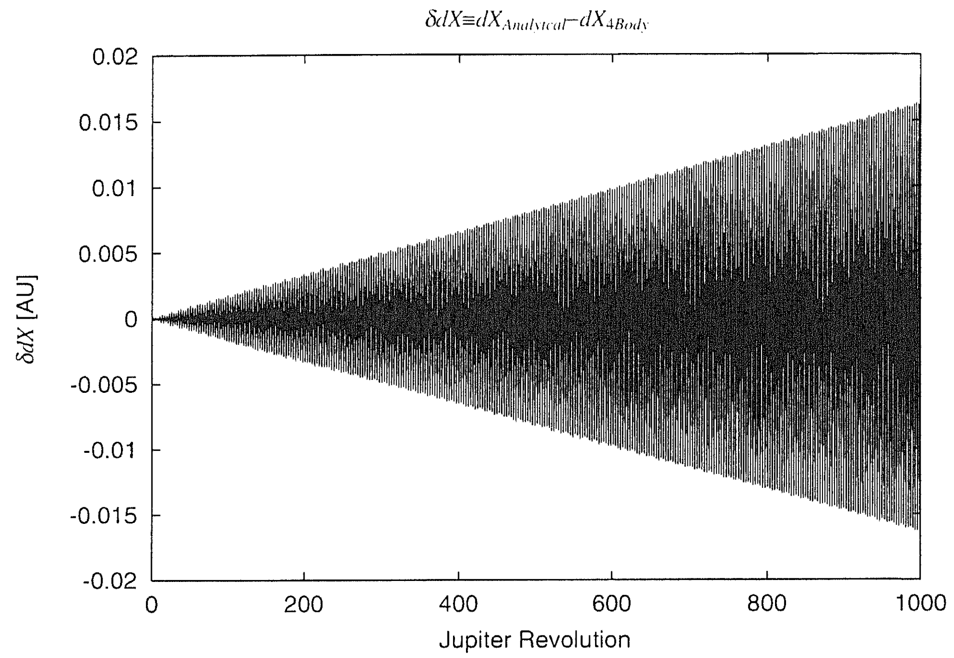

Figure 3.5: Error of Analytical Solution with respect to Numerical Solution

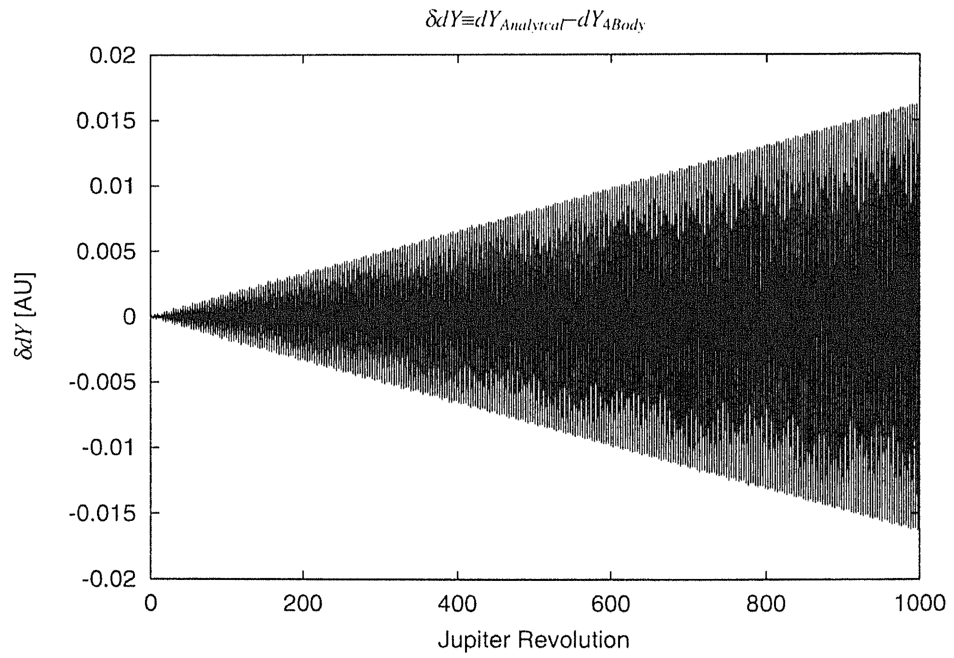

Figure 3.6: Error of Analytical Solution with respect to Numerical Solution

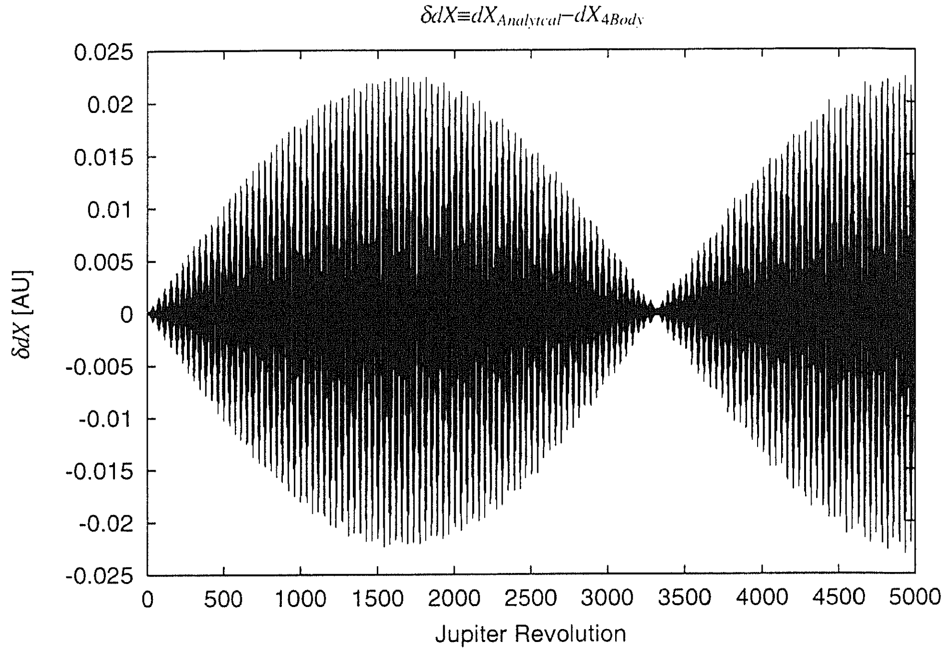

Figure 3.7: Error of Analytical Solution with respect to Numerical Solution

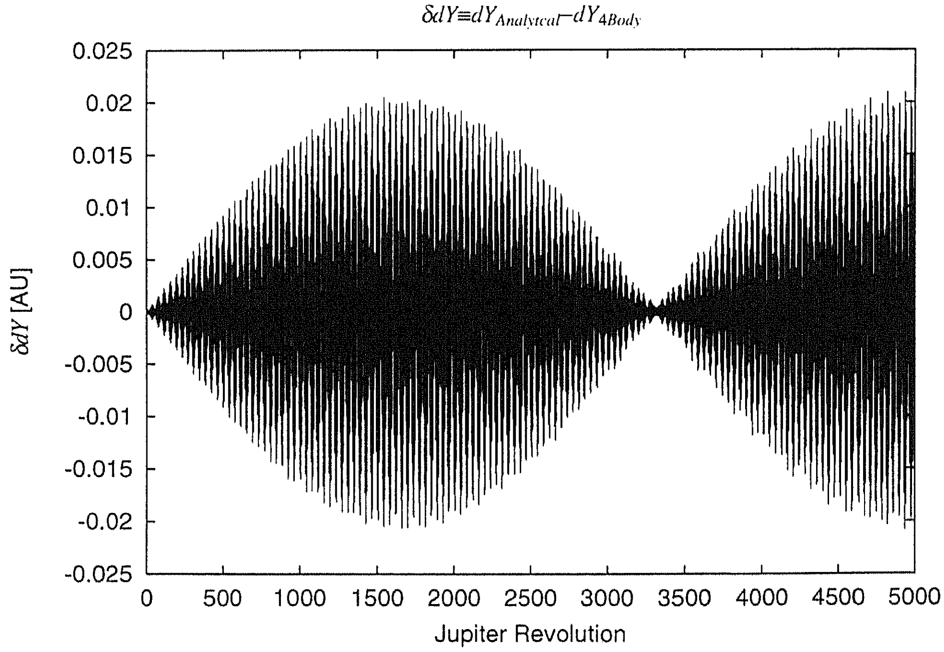

Figure 3.8: Error of Analytical Solution with respect to Numerical Solution

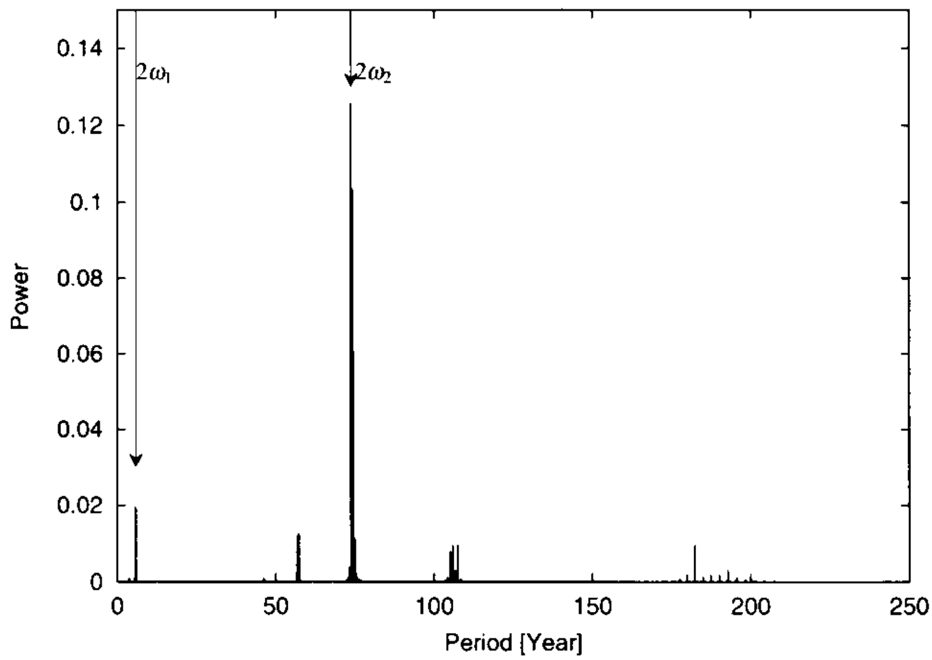

Figure 3.9: Frequency analysis of residual.

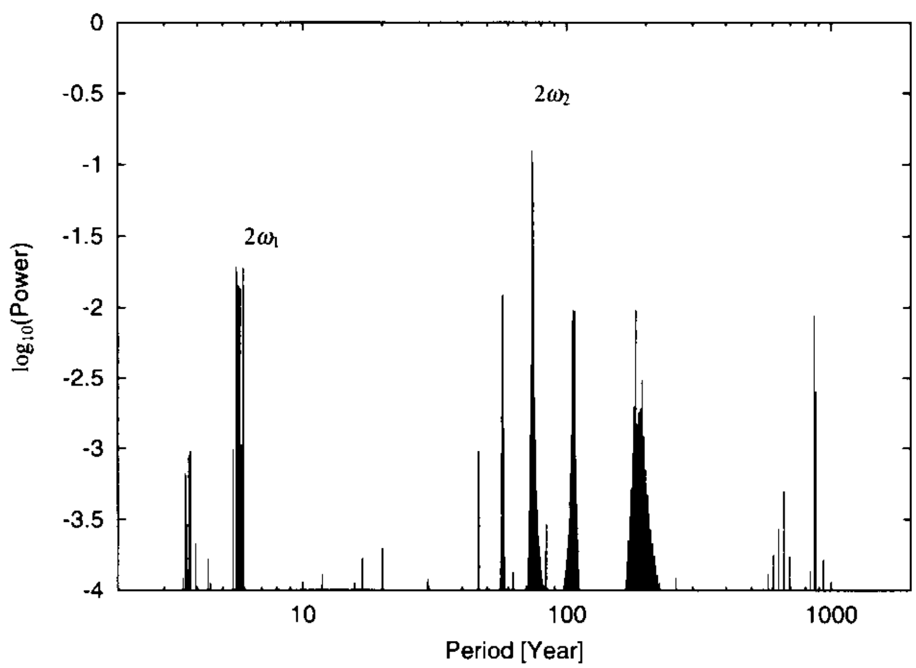

Figure 3.10: Frequency analysis of residual : Close Up.

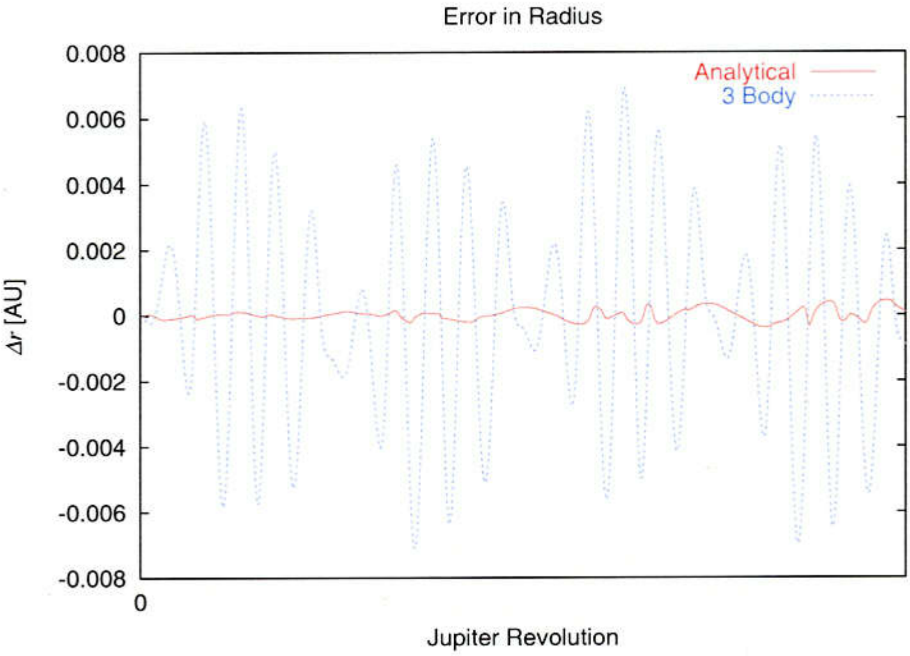

Figure 3.11: Radial Error of Analytical Solution with respect to Numerical Solution

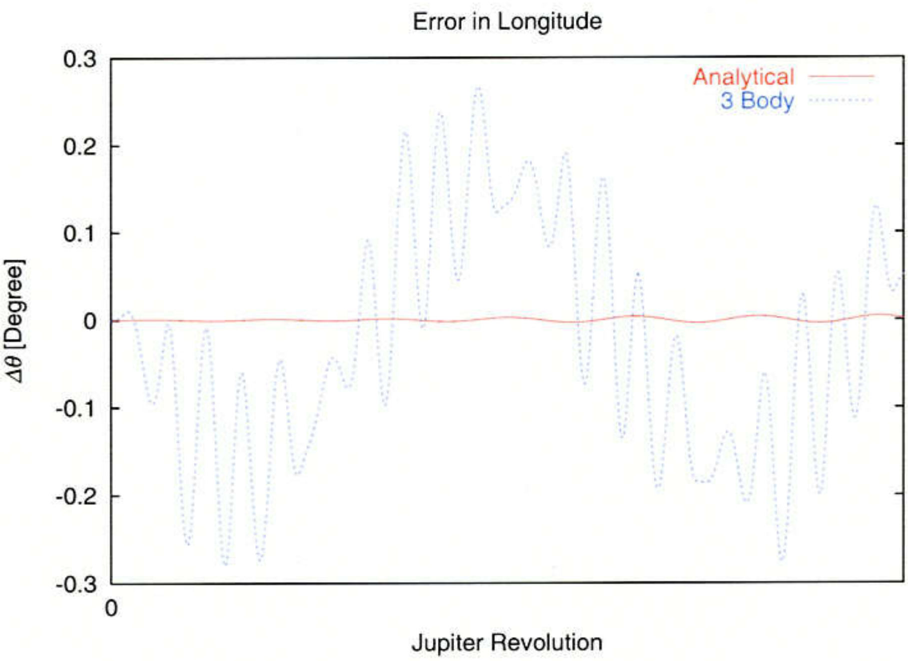

Figure 3.12: Longitude Error of Analytical Solution with respect to Numerical Solution

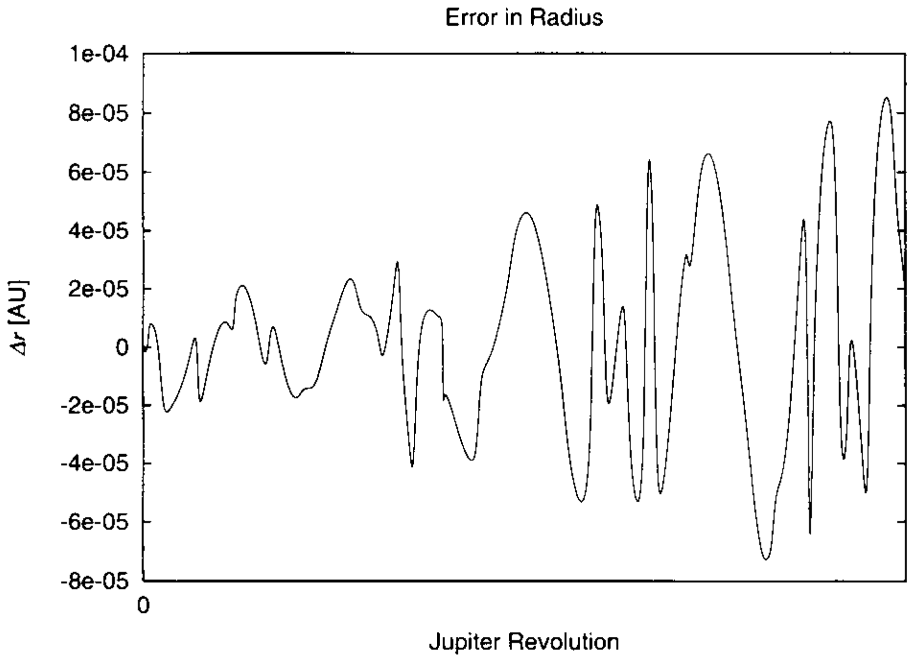

Figure 3.13: Radial Error of Analytical Solution with respect to Numerical Solution

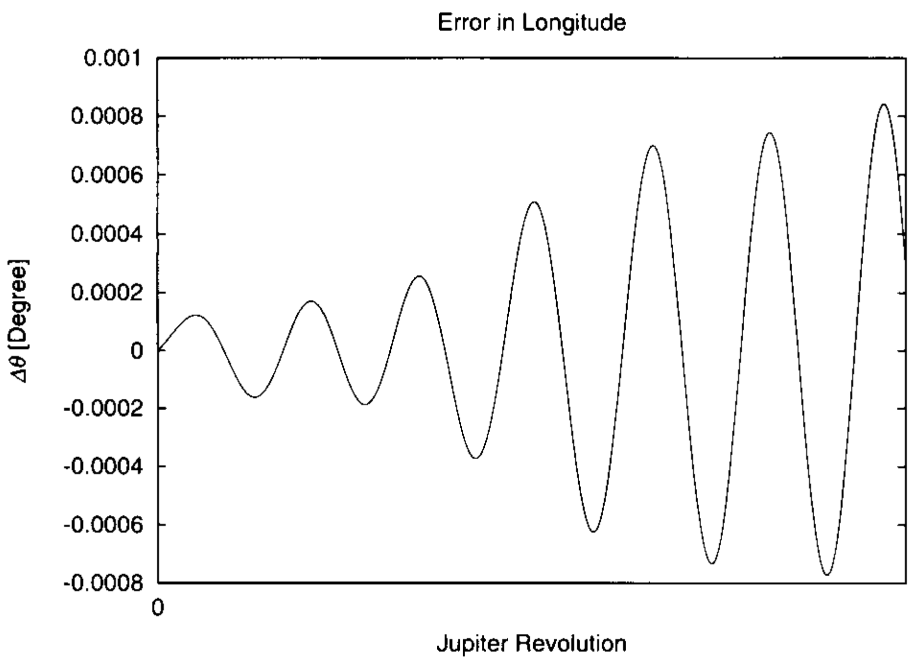

Figure 3.14: Longitude Error of Analytical Solution with respect to Numerical Solution

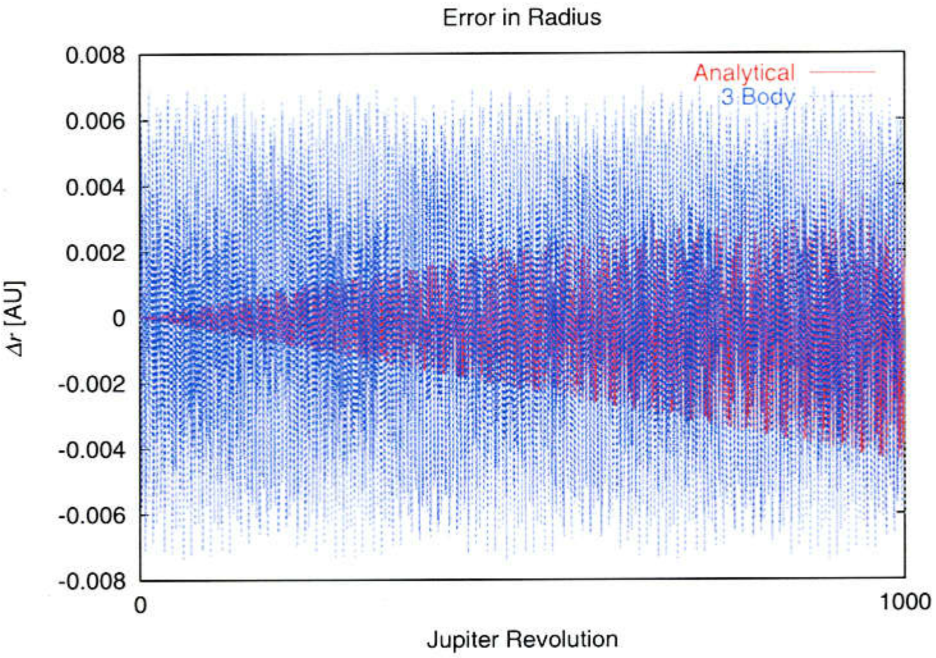

Figure 3.15: Radial Error of Analytical Solution with respect to Numerical Solution

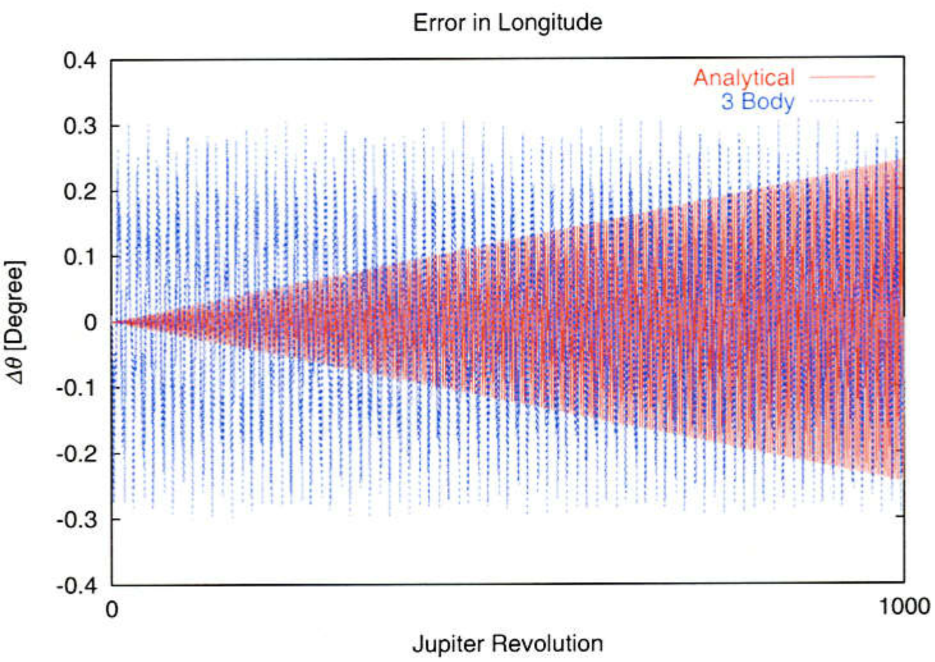

Figure 3.16: Longitude Error of Analytical Solution with respect to Numerical Solution

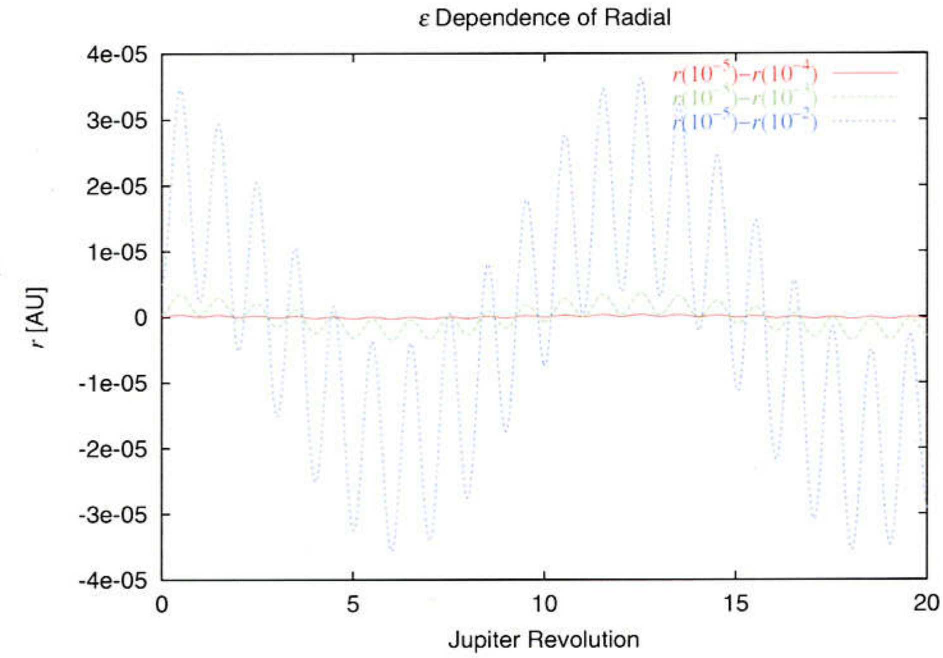

Figure 3.17: Initial Condition ε Dependence of Radial

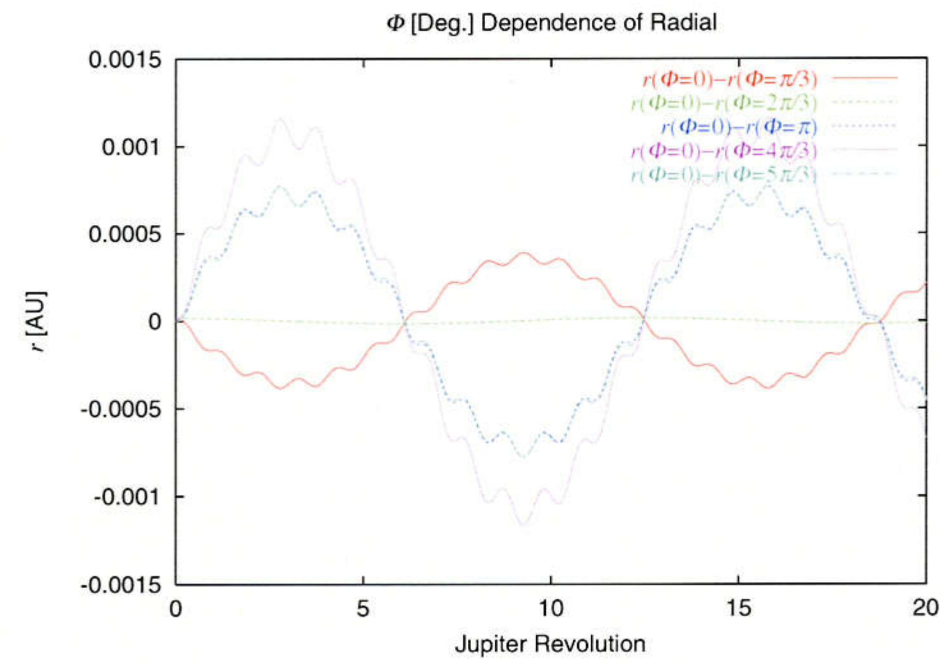

Figure 3.18: Initial Condition Φ Dependence of Radial

3.3 Global Aspects of Orbits

Now it has become clear that the dynamical behavior of the motion around L_4 is mostly independent on the initial condition. Let us investigate its global aspects, especially the size and shape of the region which the orbit occupies, as well as the barycenter in the sense of the long-term average. The information on the feature is beneficial in designing the space mission and some astronautical application of the triangular Lagrange points. Our analytical solution is useful in evaluating the orbital region or in estimating the averaged quantities.

First we discuss the shape of the orbital region. Fig. 1.4 suggests that the region resembles an ellipsoid. Then we define the width and the thickness by the major and the minor axes of an approximate ellipse. Also we introduce the aspect ratio as

$$\text{Ratio} = \frac{\text{Width}}{\text{Thickness}}. \tag{3.1}$$

Figs. 3.20, and 3.21 plot the initial position angle dependence of the width and the

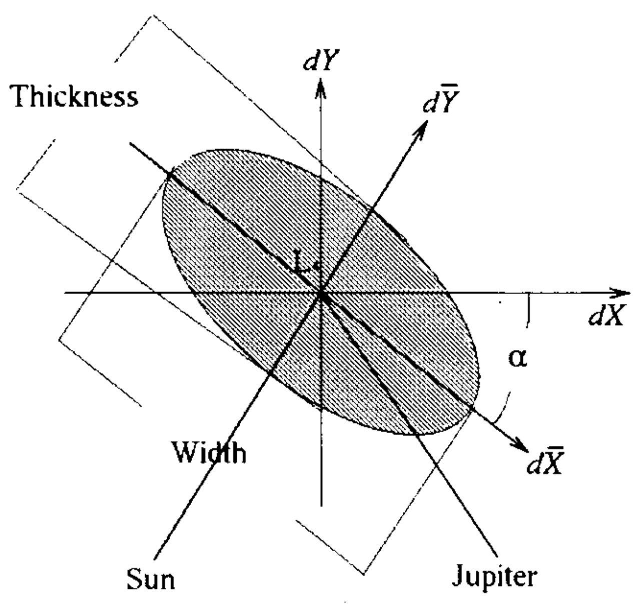

Figure 3.19: Definition of Width and Thickness of Orbit

thickness while the initial magnitude of the departure is fixed as $\delta r = 5.2 \times 10^{-5}$ AU. Fig.

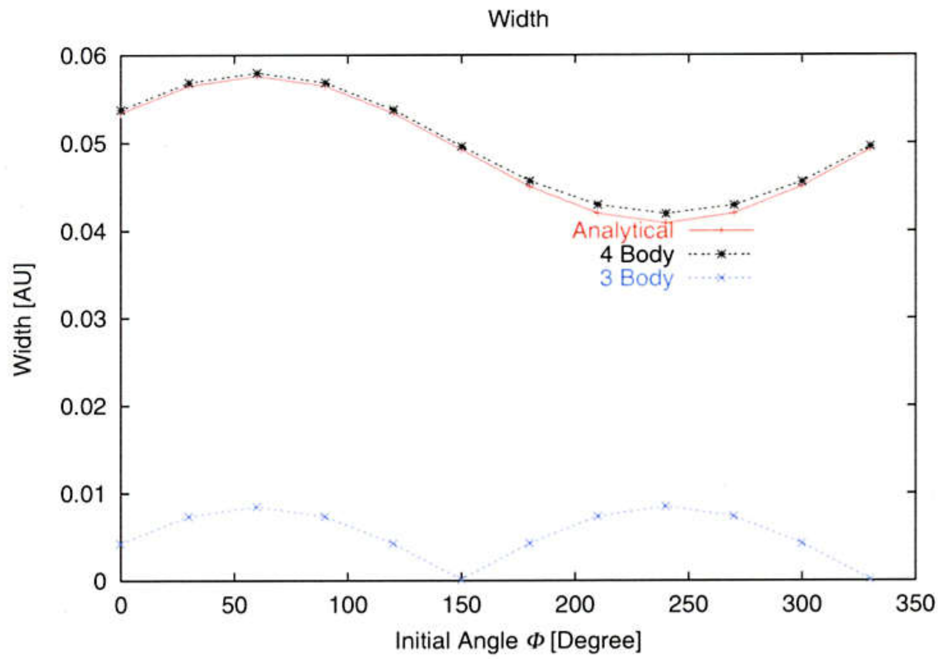

Figure 3.20: Width of Orbital Region

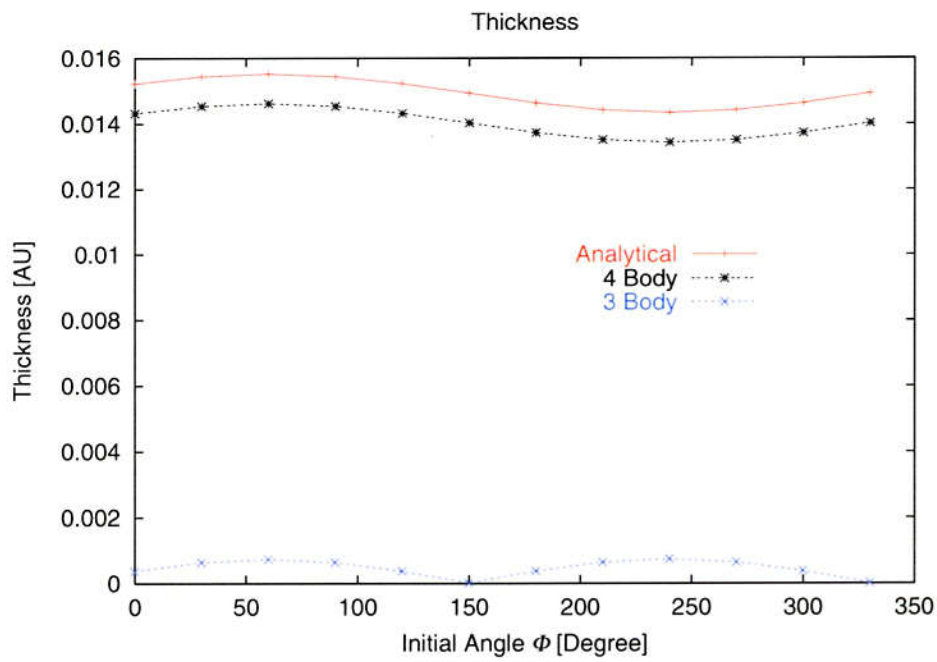

Figure 3.21: Thickness of Orbital Region

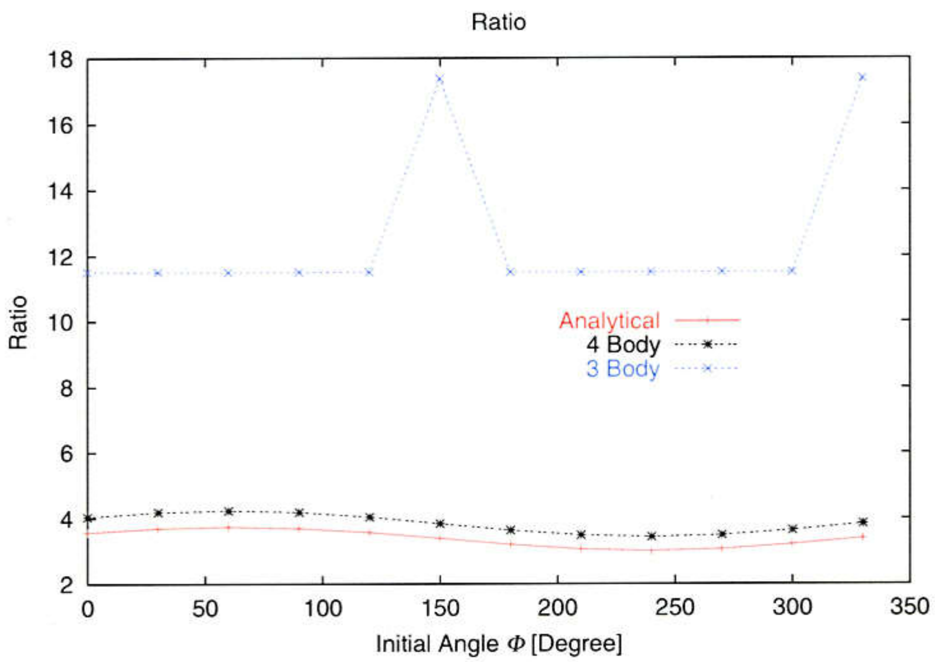

Figure 3.22: Ratio of Width and Thickness of Orbital Region

3.22 is a similar plot of the aspect ratio. It is obvious that our analytical solution shows an excellent agreement with the numerical integration.

Next we deal with the barycenter of the orbit. Using the analytical solution, we evaluate it by the time average

$$\delta\bar{\boldsymbol{R}} = \frac{1}{T}\int_0^T \delta\boldsymbol{R}dt, \tag{3.2}$$

where T is the interval of calculation. This procedure was done analytically for our analytical solution while numerically for the numerical integration. Fig. 3.23 shows the barycenter in the corotational coordinate system for various initial position angle Φ. Fig. 3.24 is the close-up of Fig. 3.23 focused on the difference between the cases of the Analytical and 4 Body. Note that, the barycenters of the cases Analytical and the case 4 Body remain in almost the same region. While that of the case 3 Body wander according to the initial position angle. Figs. 3.25 and 3.26 illustrate the initial position angle dependence of the barycenter. Clearly the case of 3 Body depends on the initial position angle much more than those of Analytical and of 4 Body. From the result of Figs. 3.23 through 3.26, we again confirm that the motion of the test particle in the vicinity of the triangular Lagrange points does depend on the direct gravitational force due to the third

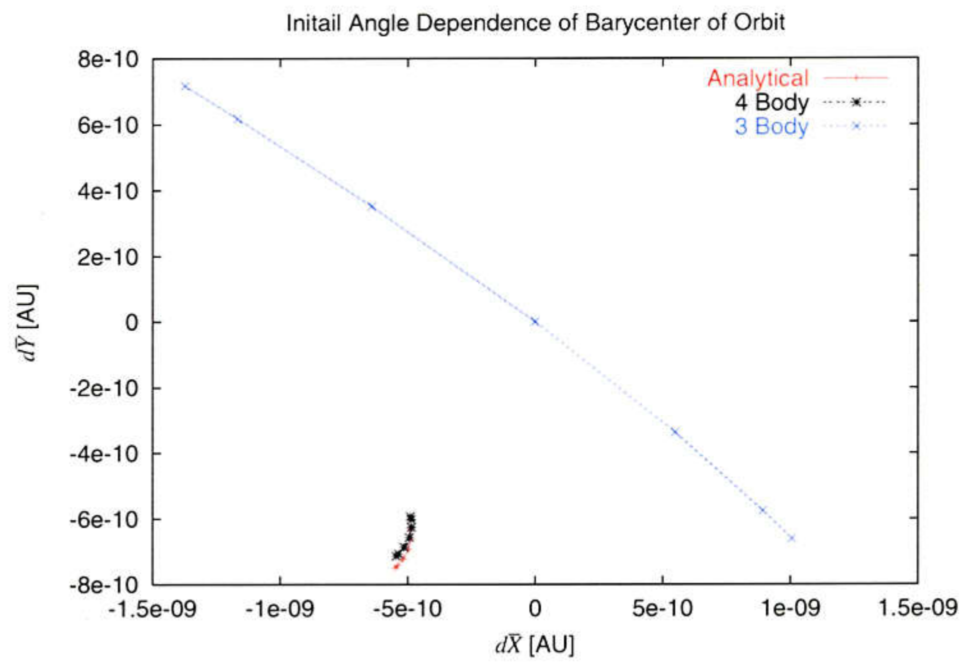

Figure 3.23: Initial Angle Dependence of Barycenter of Orbit

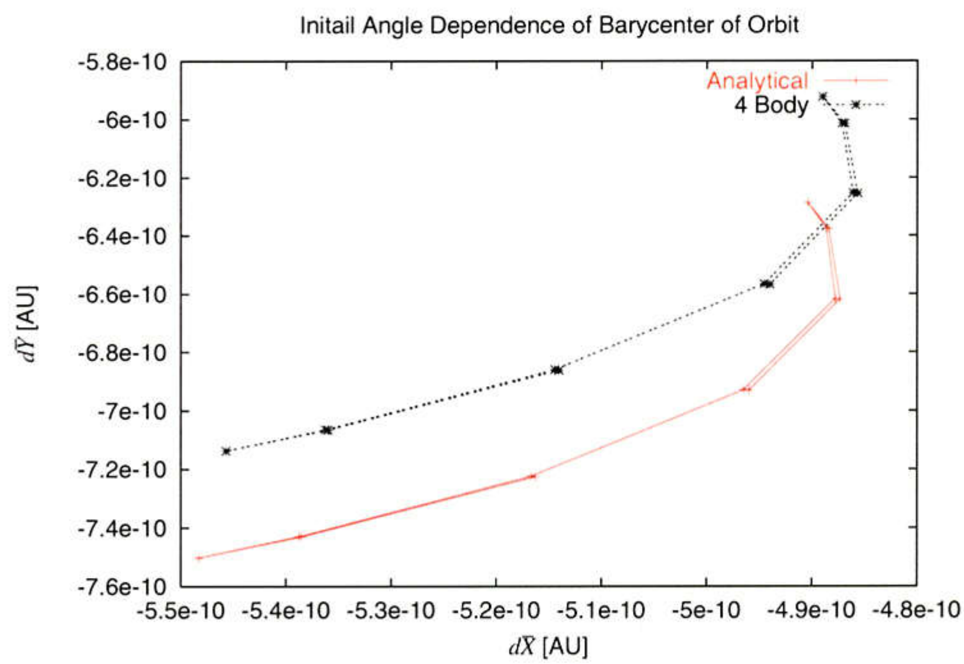

Figure 3.24: Initial Angle Dependence of Barycenter of Orbit: Close-Up

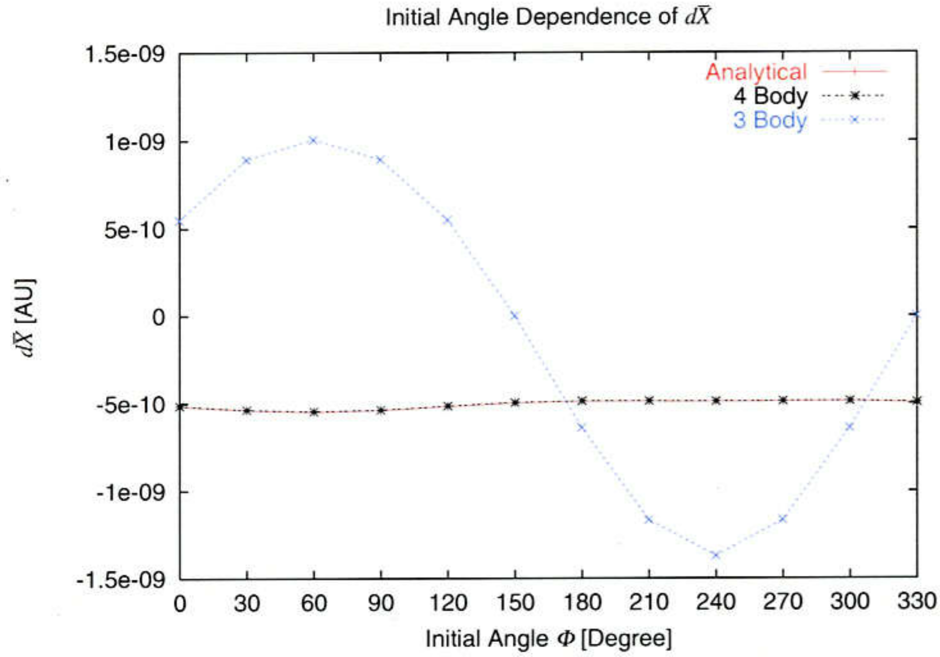

Figure 3.25: Initial Angle Dependence of Barycenter $d\bar{X}$.

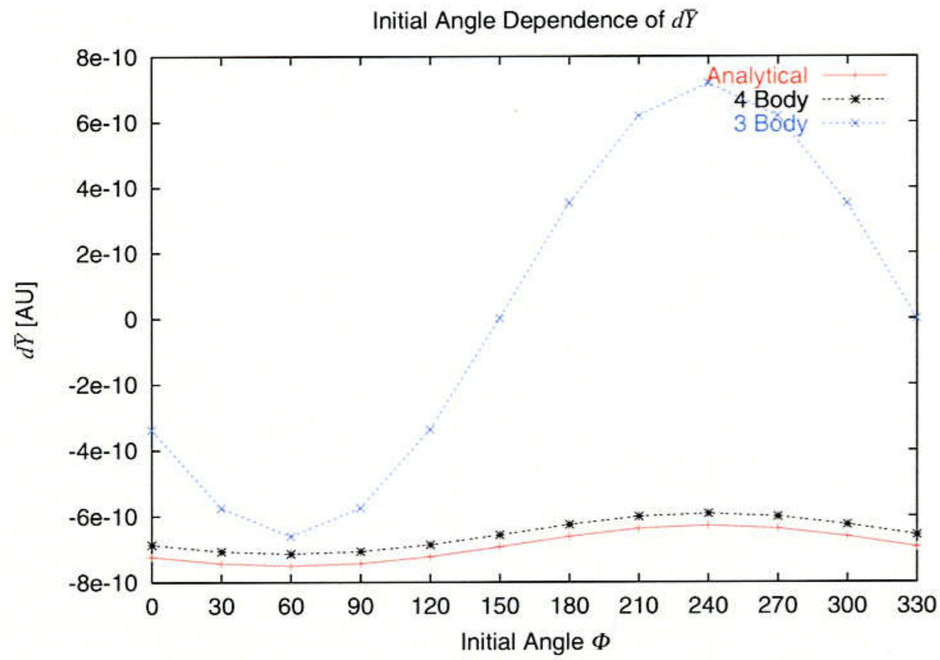

Figure 3.26: Initial Phase Dependence of Barycenter $d\bar{X}$.

and other perturbing bodies rather than the tidal force of the primary and secondary bodies.

3.4 Comparison with Real System

As the next examination, we compare our solution with the numerical integration where the motion of perturbers is given not by pure Keplerian ones but by the actual planetary ephemeris, DE 405 (hereafter cited by DE). Our analytical solution is constructed based on the simple physical model where the orbit of all the bodies is on the same plane and all the perturbers move on the Keplerian orbit. However in the actual system such as our solar system, the motion of the perturbers is more complicated; the effect of the eccentricity of the secondary, the effect of the inclination of the other perturbers, the influences of other planets as Uranus and Neptune, that say, the orbit of the perturbers is not completely closed. Therefore it is useful to examine how much our solution represent the real motion.

We chose the epoch as JD 2305424.50, and adopted the osculating elements at the epoch in constructing the analytical theory. Since our theory is planar, we regard the Sun-Jupiter plane of DE as our fundamental plane and projected the actual orbit of Saturn on this plane (see Fig. 3.27). The initial condition of the test particle was set as $\delta r = 5.2 \times 10^{-5}$ AU and $\Phi = 0$. Figs. 3.28, and 3.29 show the deviation in radius and in longitude between the cases of Analytical and DE, respectively. Figs. 3.30, and 3.31 are the same as Figs. 3.28, and 3.29 but plotted are in the long run. The maximum values of the relative errors in radius and in longitude are 0.0025 and 0.001, respectively. Fig. 3.32 illustrates the frequency analysis of the residual in dX. Comparing with Figs. 3.9 and 3.10, there appear the various frequencies, however, we can realize that the main contribution to the error growth is produced by the lack of the nonlinear effect of the primary and the secondary bodies and then the elimination of the effect of the eccentricity of the secondary body or the inclination of other perturbers is relatively less contribution to the residual. Figs. 3.33, 3.34, and 3.35 are the plots of the width, the thickness of the orbital region, and these ratio, respectively. The width of the orbital region of the analytical case is almost the same as that of DE, while the thickness of DE is slightly larger than that of analytical ones. Figs. 3.36, 3.37, and 3.38 illustrate the initial position angle dependence of the barycenter of the orbit. These behaviors are almost the same between the cases of DE and Analytical. Then, we conclude that our analytical solution

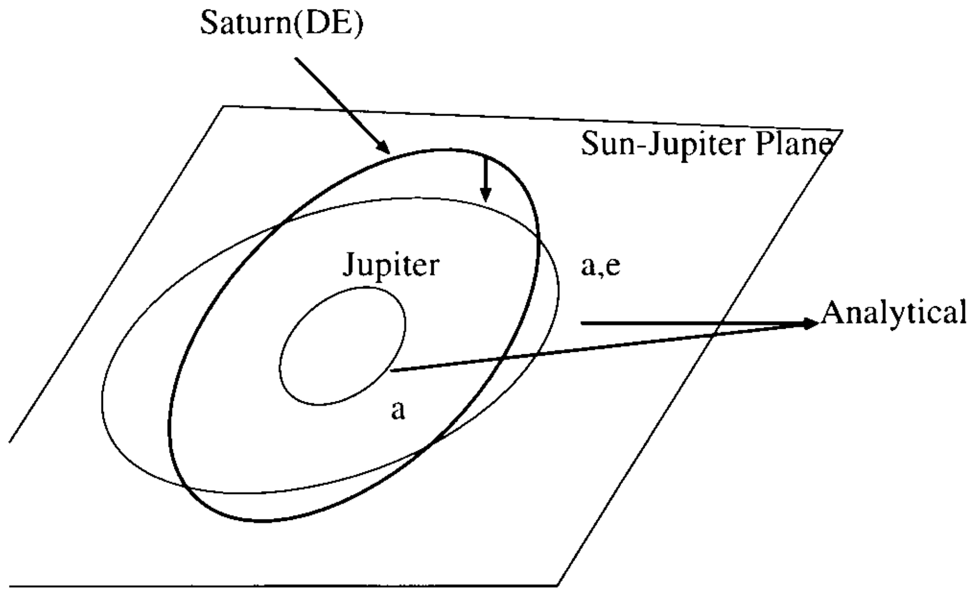

Figure 3.27: Osculating element from DE 405

represents the quite well the features of true orbit.

3.5 Limitation of Application

Finally, we examine the limitation of our analytical solution. Figs. 3.39, 3.40, and 3.41 are the plots of ϵ-dependence of the width of orbit for the cases of Analytical, 4 Body, and 3 Body, respectively. Figs. 3.42, 3.43, and 3.44, and Figs.3.45, 3.46, and 3.47 are the same as Figs. 3.39, 3.40, and 3.41 but plotted are the thickness of orbit and the aspect ratio, respectively. For the cases of Analytical and 4 Body, the behaviors are the same up to $\epsilon = 10^{-4}$. At $\epsilon = 10^{-3}$, a bit of difference occurs, and at $\epsilon = 10^{-2}$, the behavior of 4 Body is rather similar to the that of 3 Body. The order of magnitude 10^{-3} is almost the same as the value at which the nonlinear effect of the tidal force of the primary and the secondary bodies and the direct gravitation due to Saturn balance,

$$\boldsymbol{F}_E(\boldsymbol{r}_L, t) \sim \frac{1}{2}\frac{\partial^2 F_{Li}(\boldsymbol{r}_L, t)}{\partial r_j \partial r_k}\delta r_j \delta r_k.$$

For the Sun-Jupiter-Saturn system, this critical value is,

$$\epsilon_c \sim 4.1 \times 10^{-3}.$$

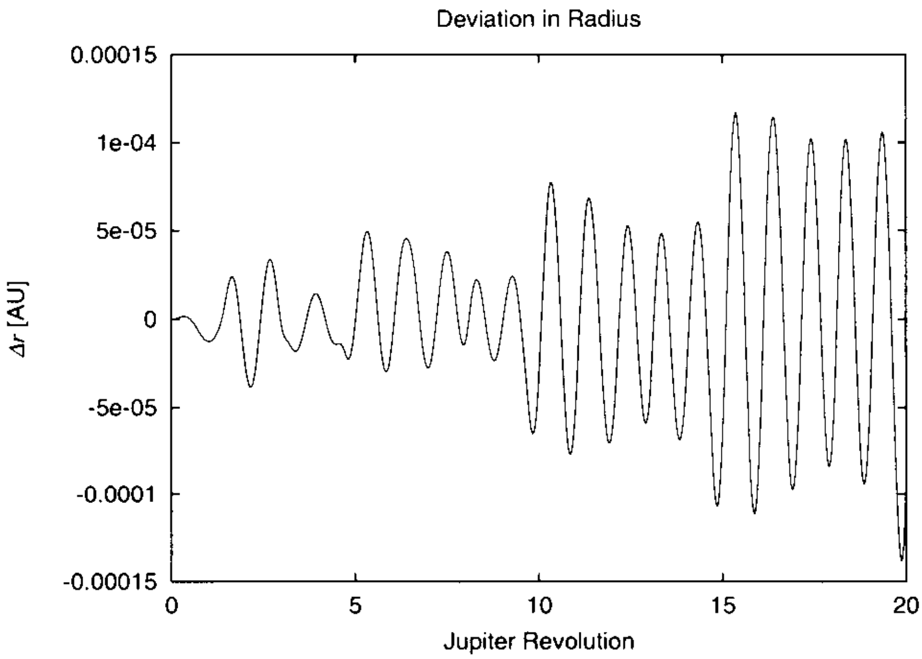

Figure 3.28: Deviation of Radius between Analytical and DE

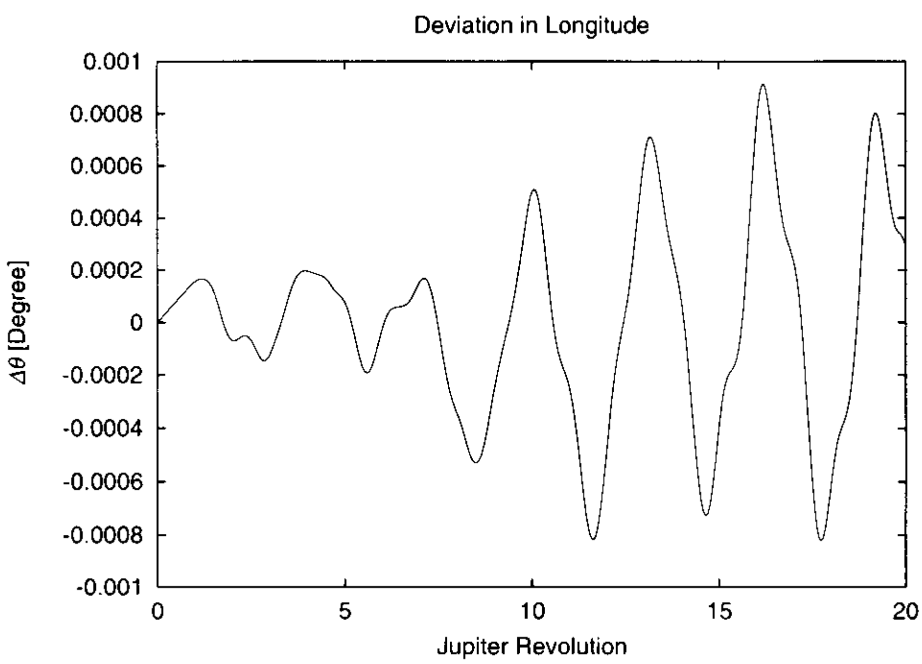

Figure 3.29: Deviation of Longitude between Analytical and DE

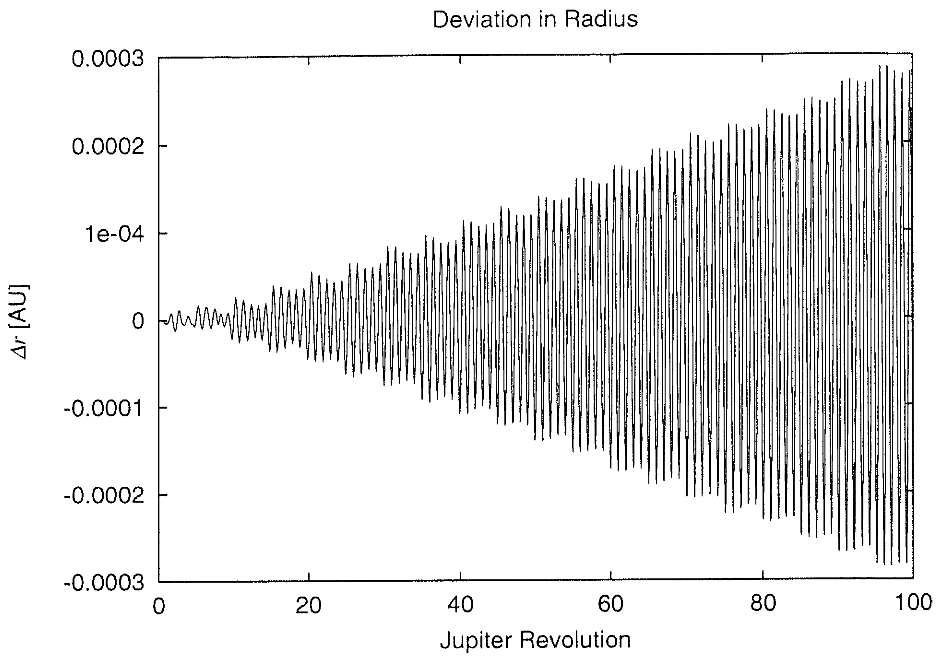

Figure 3.30: Deviation of Radius between Analytical and DE

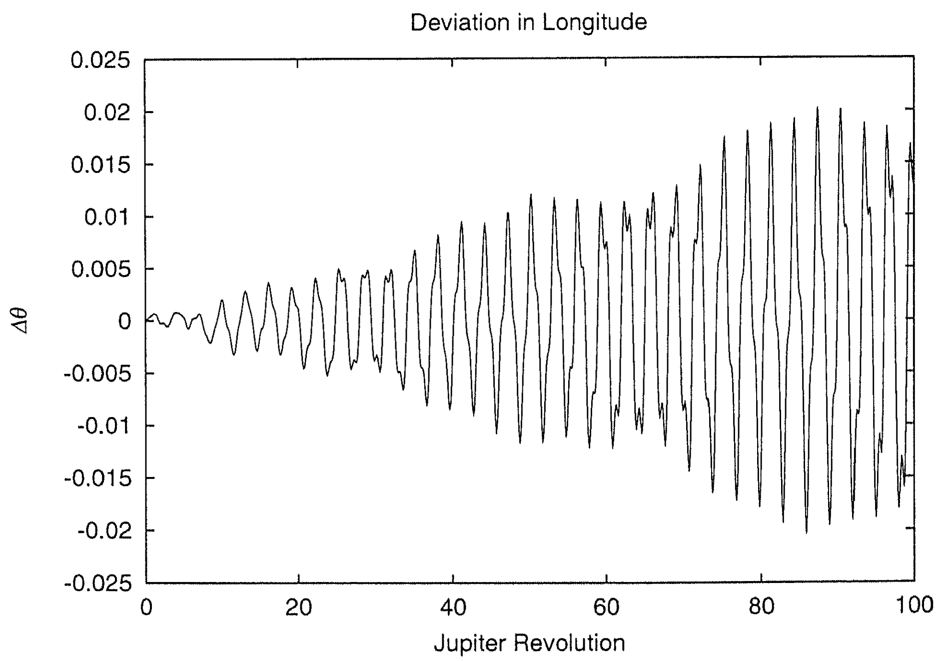

Figure 3.31: Deviation of Longitude between Analytical and DE

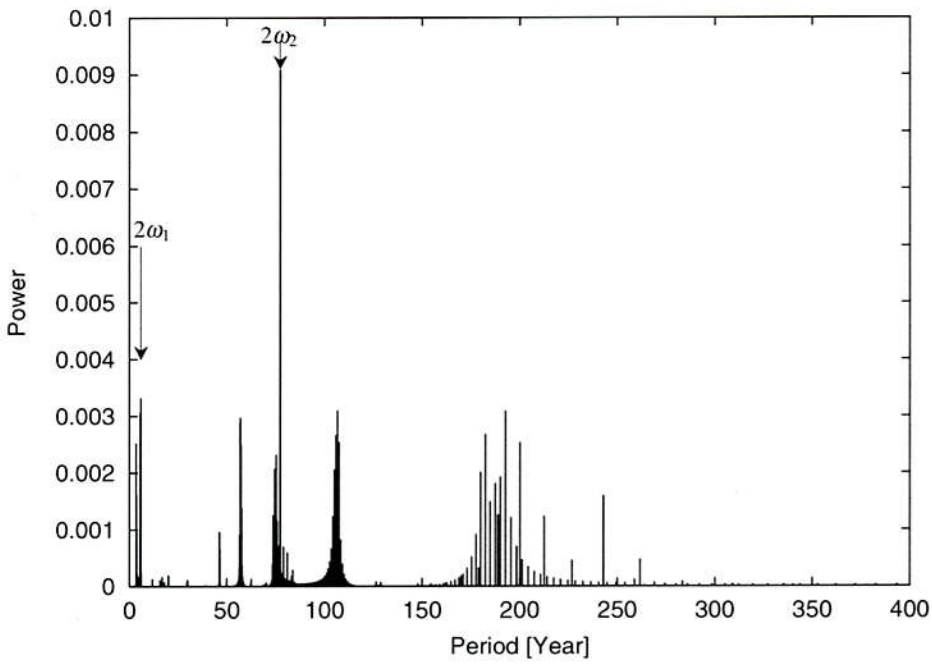

Figure 3.32: Frequency analysis of residual

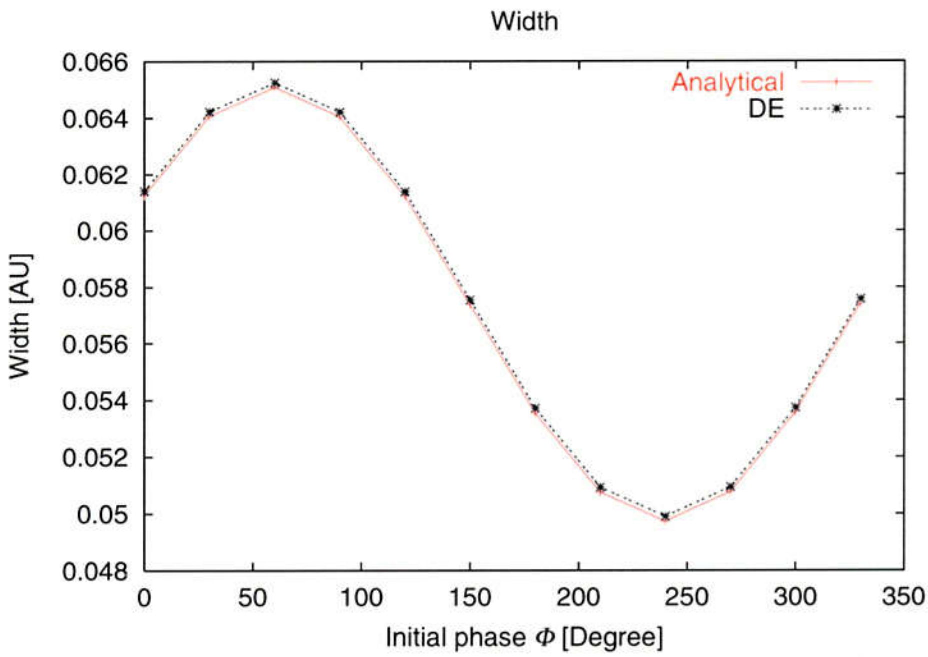

Figure 3.33: Width of Orbital Region

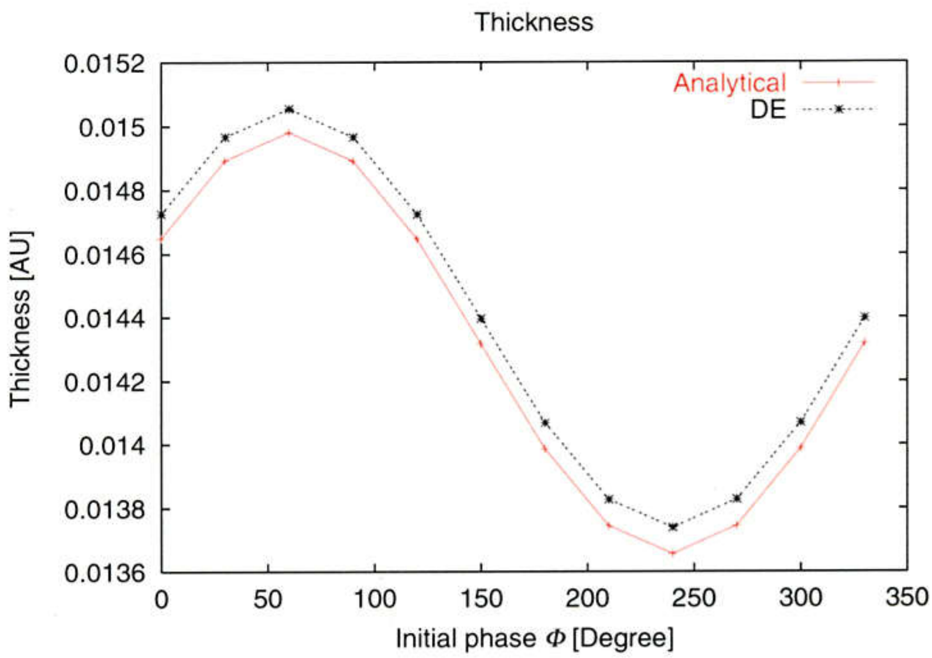

Figure 3.34: Thickness of Orbital Region

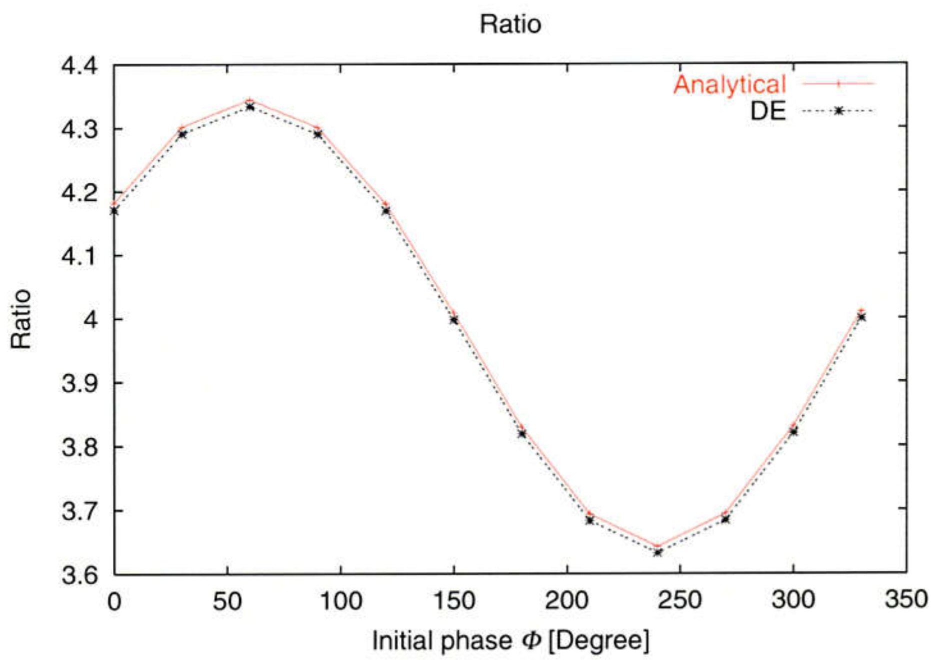

Figure 3.35: Ratio of Orbital Region

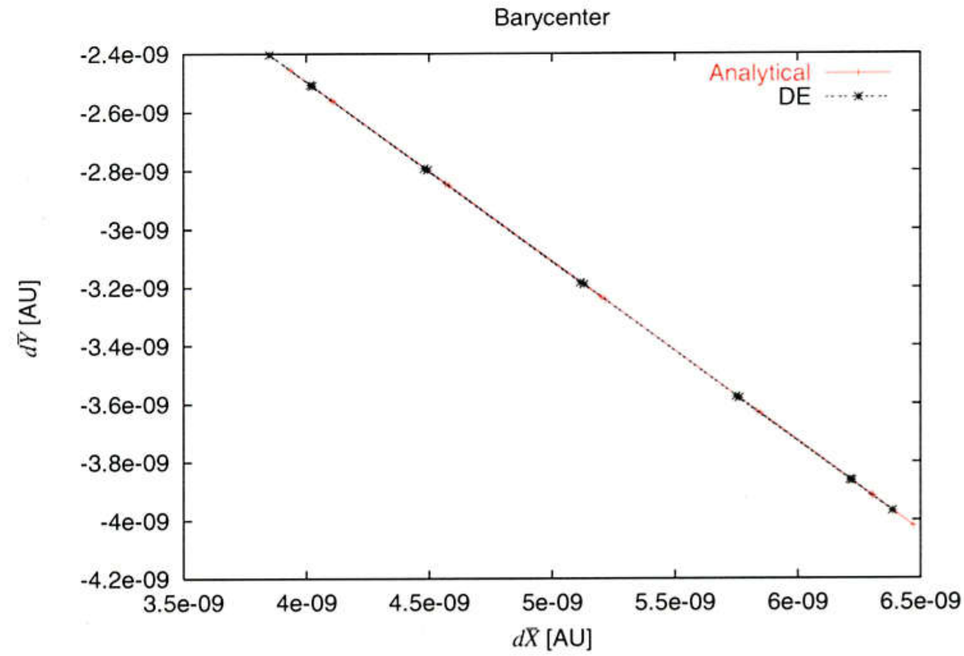

Figure 3.36: Initial Angle Dependence of Barycenter of Orbit

Therefore this value almost coincides with the observed value. Fig. 3.48 shows the orbits in the corotating coordinate system where $\epsilon = 10^{-2}$. Obviously the orbits of 4 Body and of 3 Body have the similar feature. This is because the effect of the non-linearity of the tidal force of the primary and the secondary dominates over the forced oscillation term due to the third and other perturbing bodies. Since our theory is linear, it cannot deal with the non-linear effect and this is the limitation.

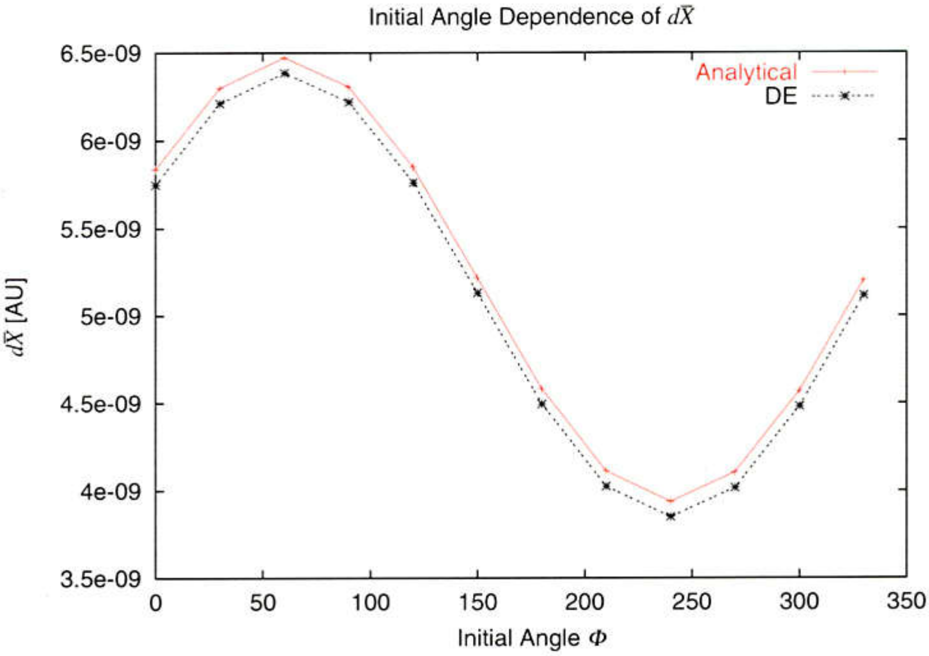

Figure 3.37: Initial Angle Dependence of Barycenter of Orbit: $d\bar{X}$

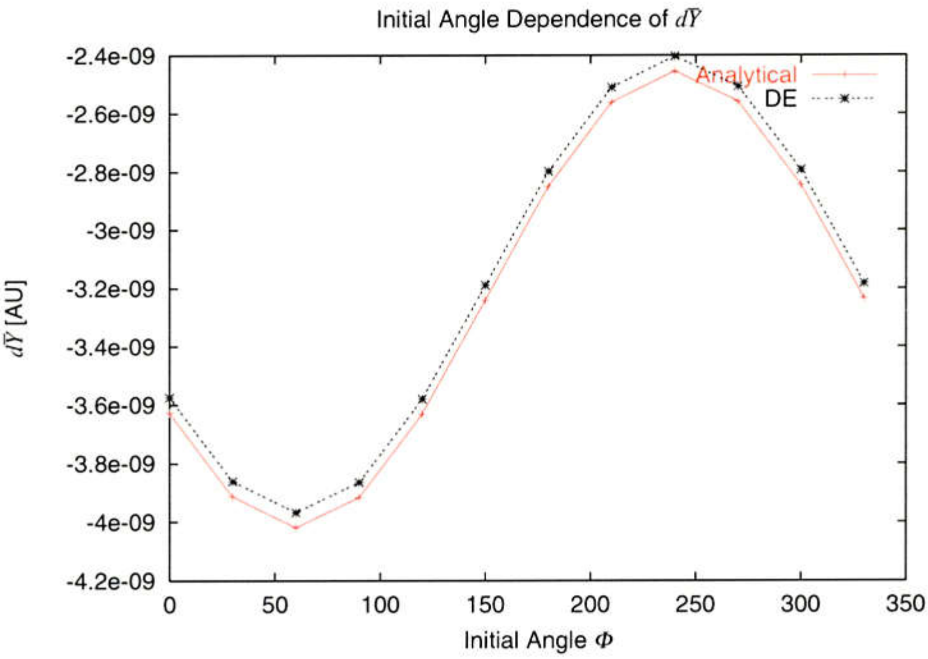

Figure 3.38: Initial Angle Dependence of Barycenter of Orbit: $d\bar{Y}$

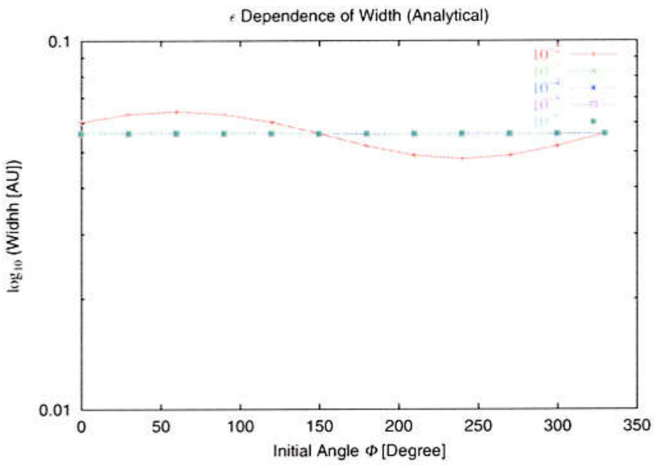

Figure 3.39: ϵ Dependence of Width (Analytical)

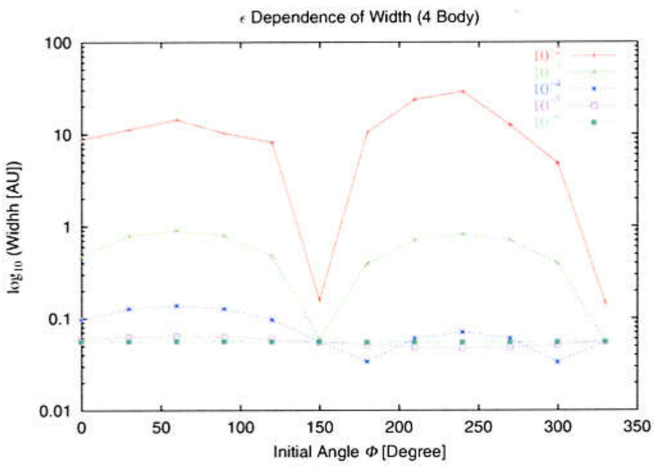

Figure 3.40: ϵ Dependence of Width (4 Body)

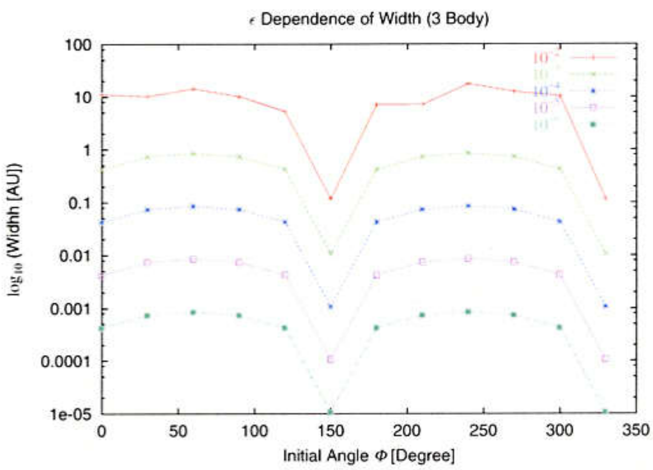

Figure 3.41: ϵ Dependence of Width (3 Body)

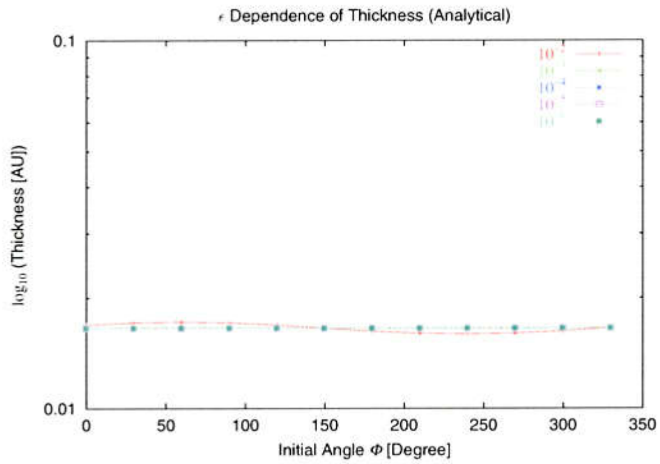

Figure 3.42: ϵ Dependence of Thickness (Analytical)

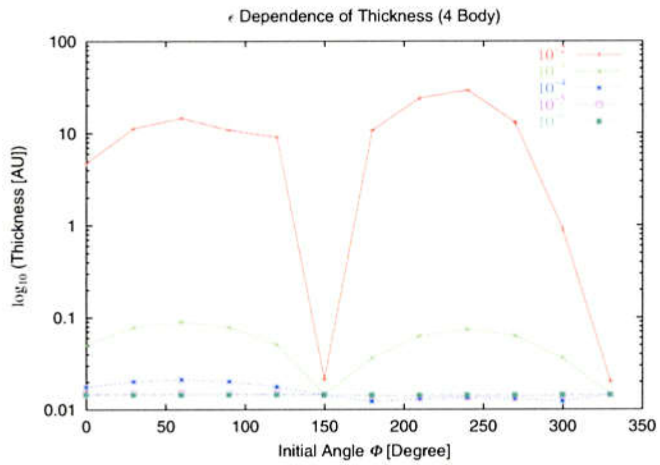

Figure 3.43: ϵ Dependence of Thickness (4 Body)

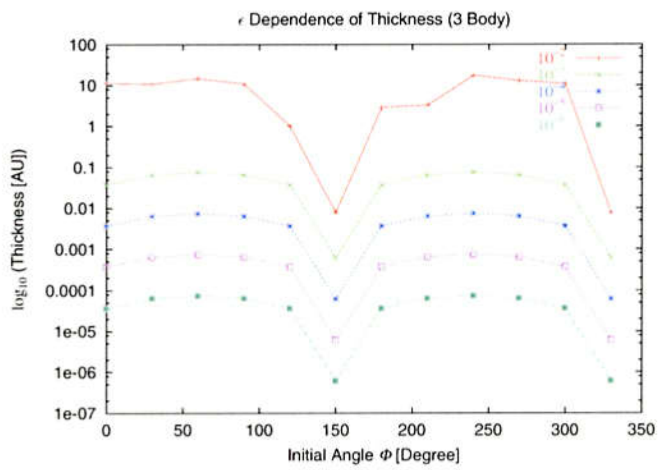

Figure 3.44: ϵ Dependence of Thickness (3 Body)

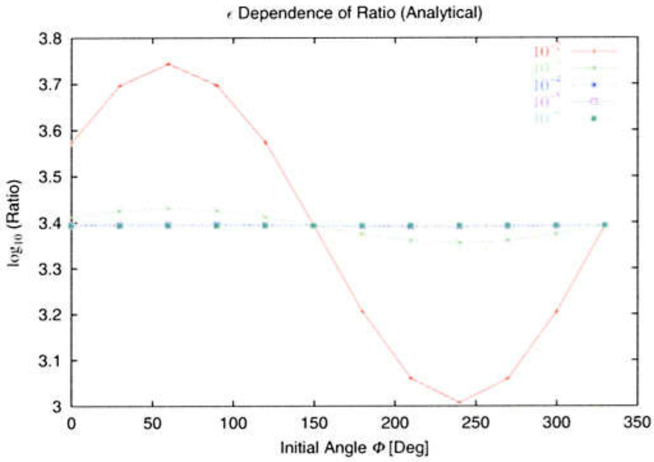

Figure 3.45: ϵ Dependence of Ratio (Analytical)

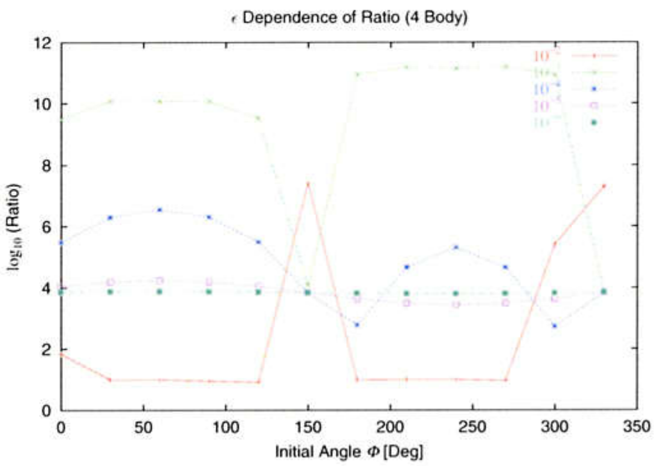

Figure 3.46: ϵ Dependence of Ratio (4 Body)

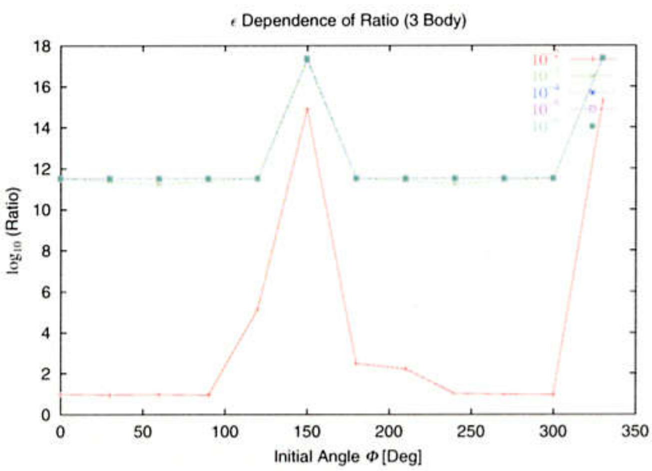

Figure 3.47: ϵ Dependence of Ratio (3 Body)

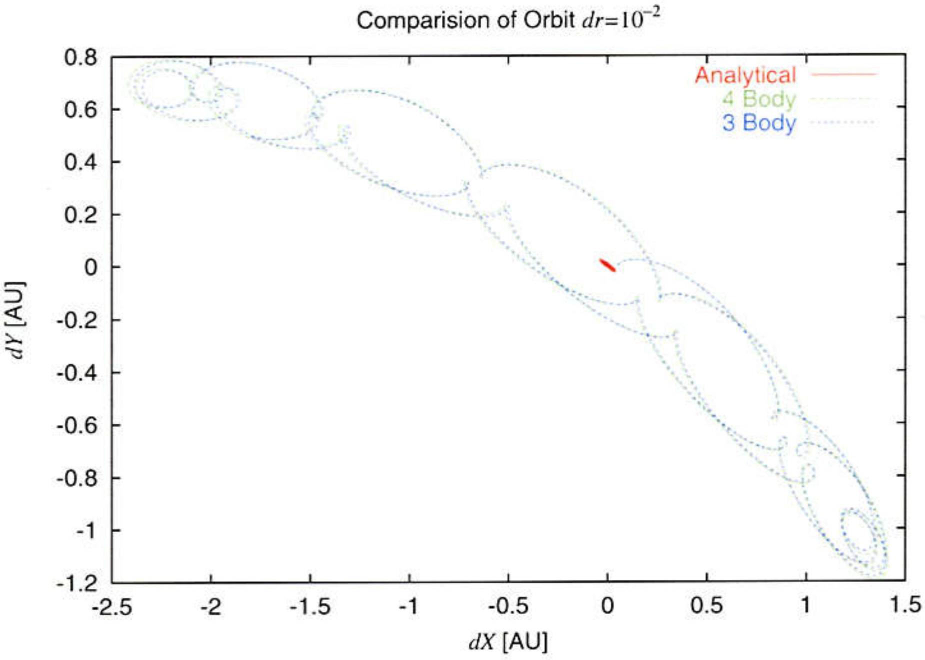

Figure 3.48: Comparison of Orbit $dr = 10^{-2}$

Chapter 4

Conclusion

4.1 Summary

We discovered that there exists a large difference in the dynamical behavior of the test particle around the triangular Lagrange points between the restricted three body problem and the restricted N-body problem. We revealed that this difference arises from the direct gravitational perturbation due to the third and other bodies. We created a purely analytical theory of the motion around the triangular Lagrangian point in the framework of the planar N-body system. We considered the effect of the forced oscillation term due to the third and other perturbing bodies by expanding not the disturbing potential but the disturbing force. We represented the disturbing force as an explicit function of time and obtained the correction in position due to the forced oscillation by its double integration. The effect of the eccentricity of the third and other perturbers was taken into account up to the second order. Here we emphasize that our analytical solution is linear theory such that it is easily applicable to calculate the perturbations due to the any number of perturbing bodies though we limit our discussion to the restricted 4 body problem in the main text.

For the short period, our solution well coincides with the numerical solution with a relative maximum error less than 10^{-4}. For the long period, the residual between the analytical solution and the numerical solution grows as the mixed secular manner and then beat. By the frequency analysis of the residual, we found that the deviation is mainly caused by the non-linear effect of the tidal force of the primary and the secondary bodies. This fact indicates that the analytical solution we derived is well express the effect of the direct gravitational force due to the other bodies in the linear theory. Then the improvement for the solution must be performed not to the part of the forced oscillation but to that of free oscillation of the primary and the secondary bodies. We also provided some global quantities of the orbit, such as the width, the thickness, and the aspect ratio of the orbital region as well as the barycenter in the sense of long-term time average. Further we compared our solution with the numerical one based on the real solar system by using JPL's planetary ephemeris DE 405. From this comparison, the main contribution to the residual between the analytical solution and the numerical one is also mainly caused by the elimination of the nonlinear effect of the tidal force in the part of the free oscillation of the primary and the secondary bodies. Finally we examined the limitation of the application of our analytical solution and realized that when the direct gravitation of the other bodies and the nonlinear effect of the tidal force of the primary and the secondary bodies balance, our solution reach the limit.

In conclusion, for the motion of the test particles around triangular Lagrange points in the restricted N-body system, the key role is played by the direct gravitational force due to the third and other bodies rather than the non-linear effect of the tidal force of the primary and the secondary bodies. Practically speaking, this range of allowance is quite large. For example, in the case of the Sun-Earth system, it corresponds to 450000 km in the deviation from L_4 by the gravitational influence of Jupiter.

Our solution is especially effective for designing the orbit of some space missions to be put near the triangular Lagrange points such as the gravitational wave detection or the space telescope for observing the near Earth crossing objects. This is because it is expected that the analytical solution limits the initial condition and then reduce considerably the vast of numerical integration and also it makes us estimate easily the orbital region of the spacecrafts without the numerical integration.

4.2 Future Work

As a future work, it is necessary to improve the part of the free oscillation of the primary and the secondary bodies including the nonlinear effect of their tidal force for better agreement with the numerical integration. Since our formalism is restricted to the planar case, the introduction of the inclination of the test particle will be significant in the applying it to the real system such as the motion of the Trojan asteroids.

Acknowledgment

I would like to express the deepest gratitude to my supervisor, Professor Toshio Fukushima at National Astronomical Observatory of Japan for his constant support and guidance throughout this study. Without his continuous assistances, this thesis would have never been accomplished completely. I would like to thank Professor Hiroshi Kinoshita for his fruitful comments and suggestions which led this thesis to the better direction.

I would like to thank the members of the seminar on celestial mechanics at National Astronomical Observatory of Japan for deepening my understanding and extending my view of the celestial mechanics.

I wish to acknowledge the all those who encourage, cheer and support me mentally in my daily life.

Finally, I would like to express my grateful appreciation to my family for their constant support and encouragement in my life.

Appendix A

Numerical Integration in KS Element

In this appendix, we will show that the KS regularization can reduce the orbital integration error drastically. Further, we will indicate that the KS regularization stabilize the symmetric multistep method of the special second order ordinary differential equations (ODEs) and avoid its stepsize resonance/instability. And, as application of the KS regularization, we will discuss the effectiveness of the method of variation of parameter of the KS regularization and construct the third set of the element of the KS regularization based on the Stiefel's formulation. This third set of the KS element is the complete one in the sense that all the elements become constant when unperturbed case. The element formulations are applicable for the any kind of perturbation types and reduce the numerical integration error of both the position and the physical time. Thus it is useful to integrate the perturbed two body problem.

In this thesis, all the numerical solutions were integrated by using this third set of element. This appendix is composed mainly based on Arakida & Fukushima, 2000, 2001 [1, 2].

A.1 Reduction of Integration Error

Studying the celestial mechanics, the numerical integrations are the strong tools for investigating the long term behavior of dynamical systems, constructing highly precision planetary ephemeris, and so on. Unfortunately when we integrate the Kepler problem by using the traditional integrators such as, the Runge-Kutta method, the Adams method, the Störmer-Cowell method, and the extrapolation method[1] , the error of the conserved quantities grows linearly with respect to the time, and then the positional error increase quadratically. This is because in the Kepler problem, the mean motion n which corresponds to the orbital frequency is related to the energy of the orbital motion. Namely,

$$\Delta E \sim O(t), \quad \Delta n \sim O(t), \quad \Delta\theta = \int \Delta n dt \sim O(t^2). \tag{A.1}$$

This quadratic positional error growth has been the limitation for the long term orbital integration.

However, in early '90s, the new two integrators have overcome this barrier; the symplectic integrator [47, 86] and the symmetric multistep method [20, 21, 50, 65]. These new integrators do not produce the secular error in the conserved quantities and therefore the increase of the positional error reduce linear growth;

$$\Delta E \sim O(1), \quad \Delta n \sim O(1), \quad \Delta\theta = \int \Delta n dt \sim O(t). \tag{A.2}$$

Hence the symplectic integrator and the symmetric multistep method enable us to carry out the very long term integration of the dynamical system for the age of solar system. Until now, developing the highly accurate integration schemes has mainly devoted to an reduction of numerical integration error.

On the other hand, it is also effective for the reduction of the integration error to transform the nonlinear equation of motion of Kepler problem into the form that lead to less integration error. The KS (Kustaanheimo-Stiefel) regularization [79] is the one of the such transformation and by this transformation the equation of motion of the perturbed two body problem is rewritten in the linear and regular form, namely the form of the perturbed harmonic oscillator. However, so far, the KS regularization has properly been used to avoid the two body close approach of celestial bodies and it has been regarded as less effective in terms of the long term numerical integration since it was thought that the KS regularization cannot reduce the integration error significantly but increase the computational time and the complexity of coding.

[1] See the details for each traditional integrators, Hairer *et al.* [34].

A.2 KS Regularization

A.2.1 Equation of Motion

The KS regularization contains both the space transformation and the time one. The 3-dimensional coordinate $\vec{r} = (x, y, z, 0)$, and the physical time t are transformed into $\boldsymbol{u} = (u_1, u_2, u_3, u_4)$, and the fictitious time s which is the independent variable in the KS regularization. These are related by

$$\left\{ \begin{array}{l} \vec{r} = L\boldsymbol{u}, \quad \dfrac{dt}{ds} = r \\ L = \begin{pmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ -u_3 & -u_4 & u_1 & u_2 \\ u_4 & -u_3 & u_2 & -u_1 \end{pmatrix}, \quad r = |\vec{r}| = |\boldsymbol{u}|^2. \end{array} \right. \tag{A.3}$$

According to these relations, the equation of motion of the perturbed two body problem,

$$\frac{d^2\vec{r}}{dt^2} + \frac{\mu\vec{r}}{r^3} = \vec{F}, \tag{A.4}$$

is expressed as,

$$\frac{d^2\boldsymbol{u}}{dt^2} + \frac{h_k}{2}\boldsymbol{u} = \frac{1}{2}|\boldsymbol{u}|^2 L^T \vec{F}, \quad \frac{dh_k}{ds} = -2L^T\vec{F}\frac{d\boldsymbol{u}}{ds}, \quad \frac{dt}{ds} = r, \tag{A.5}$$

where superscript T denotes the transpose matrix, $\vec{F}$ is the 3-dimensional acceleration, and h_k is the Kepler energy defined as,

$$h_k = \frac{\mu - 2|\boldsymbol{u}'|^2}{r}. \tag{A.6}$$

A.2.2 Numerical Experiments

Let us show the results of the numerical comparison and indicate that the KS regularization dramatically reduce the numerical integration error. As the test problems, we adopted (1) the pure Kepler problem (hereafter K), (2) the perturbation due to the air dragging force (A), (3) the 3-dimensional circular restricted three body problem (R3), and (4) the 3-dimensional general three body problem. Table A.1 lists the integrators tested. We investigated the growth of errors in position and the physical time for all the cases. We also did the error growth of some quantities such as the Jacobi integral, the

Integrator	Perturbation			
	K	R3	A	G3
Adams-Bashforth (ABn)	$\surd$	$\surd$	$\surd$	$\surd$
Extrapolation (EXn)	$\surd$	$\surd$	$\surd$	
Fourth order Runge-Kutta (RK4)	$\surd$	$\surd$	$\surd$	$\surd$
Eighth order Runge-Kutta-Fehlberg (RKF8)	$\surd$	$\surd$	$\surd$	
Symmetric Multistep (ET) & Adams-Bashforth (ET4+AB4)	$\surd$	$\surd$	$\surd$	
Störmer & Adams-Bashforth (Sn+ABn)	$\surd$	$\surd$		
Symmetric Multistep (LW & QT) & Adams-Bashforth (SMn+ABn)	$\surd$	$\surd$		

Table A.1: Numerical Integrator Tested for Restricted Two Body Problem. The Stoörmer and symmetric multistep method (Quinlan & Tremaine 1990) are not suitable in the case of air dragging force because perturbation depends on the velocity v. Though the formula of Evans & Tremaine is for the general first order ODEs, it showed the instability when integrating Kepler energy h_k. Therefore we adopt it only for integrating the perturbed harmonic oscillator. We denote Lambert and Watson (1976) for 2nd, 4th, and 6th order for the special second order ODEs as LWn, Quinlan and Tremaine (1990) for 8th order for the special second order ODEs as QTn, and Evans and Tremaine (1999) for 4th order for the general first order ODEs as ETn.

total energy, and the total angular momentum when they were conserved. In the pure Kepler problem, we evaluated the error by the deviation from the analytical solution

$$\Delta X = X_{\text{numerical}} - X_{\text{analytical}}$$

where the analytical solution was that given in the explicit function of the fictitious time. In the other cases when the analytical solutions are not available easily, we conducted the numerical integrations by fixed step sizes and evaluated the error by a difference between two runs of numerical integrations with different step sizes as,

$$\Delta X = X_{h_1=h} - X_{h_2=\frac{h}{2}}.$$

We note that the numerical integrations were carried out as the total number of the function calls per unit time are the same. Fig. A.1 represents the integrator type dependence of the positional error with respect to the fictitious time s where the perturbation is fixed as R3. Fig. A.2 is the same as Fig. A.1 but plotted is the perturbation type dependence of positional error where integrated by the fourth order Adams-Bashforth method. From these two figures, the integration error only grows linearly with respect to the s. Fig. A.3 shows the eccentricity dependence of the positional error after 120 revolution of K where the integrator was fixed as the eighth-order symmetric multistep method for the KS regularized and unregularized equation of motion with same steps pre period. In the case of the KS regularized, the error growth hardly depend on the eccentricity, while in the unregularized case, the error drastically increase according to the eccentricity. From Figs. A.1, A.2, and A.3, the feature of error growth of KS regularization is independent on the perturbation type, the type of integrator adopted, and the nominal eccentricity. In the KS regularization, the physical time must be calculated by the numerical integration because of the time transformation. Figs. A.4 and A.5 are the same as Fig. A.1 but show the error growth of time evolution of the traditional integrators and the time symmetric ones, respectively. The error of physical time grows linearly when the harmonic oscillator part of equation of motion are integrated by the time symmetric integrators; the leapfrog or symmetric multistep method, while proportional to s^2 when integrating the same parts by the traditional ones. Figs. A.6 and A.7 are the same as Fig. A.1 but represented are the error of Jacobi Integral of the traditional integrator and the time symmetric ones, respectively. In the case of the traditional integrators, there appears the linear error growth, while in the case of the time symmetric integrators the error oscillates and remains almost constant.

A.2.3 Error Analysis of Physical Time Evolution

In the KS regularization, the physical time evolution is obtained by the numerical integration. Let us consider its error propagation. To simplify the situation, we restrict ourselves to its 2-dimensional subset, the Levi-Civita transformation. Without losing a generality, we assume the transformed equation of motion is written as,

$$\frac{d^2u_k}{ds^2} + u_k = 0, \quad (k = 1,\ 2) \tag{A.7}$$

under the initial condition,

$$u_1(0) = 1, u_2(0) = 0, \frac{du_1(0)}{ds} = 0, \frac{du_2(0)}{ds} = 1,$$

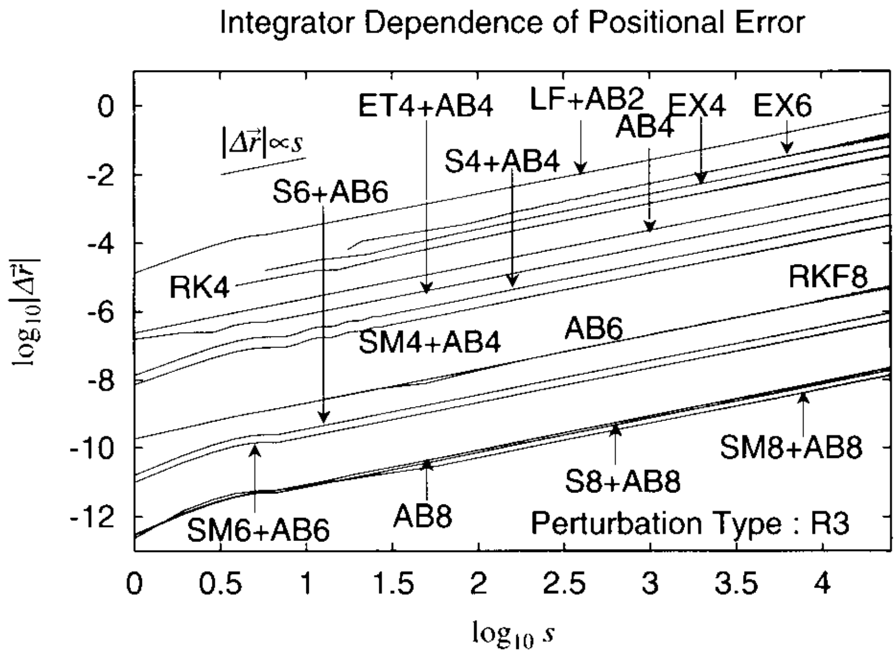

Figure A.1: Integration Type Dependence of Positional Error

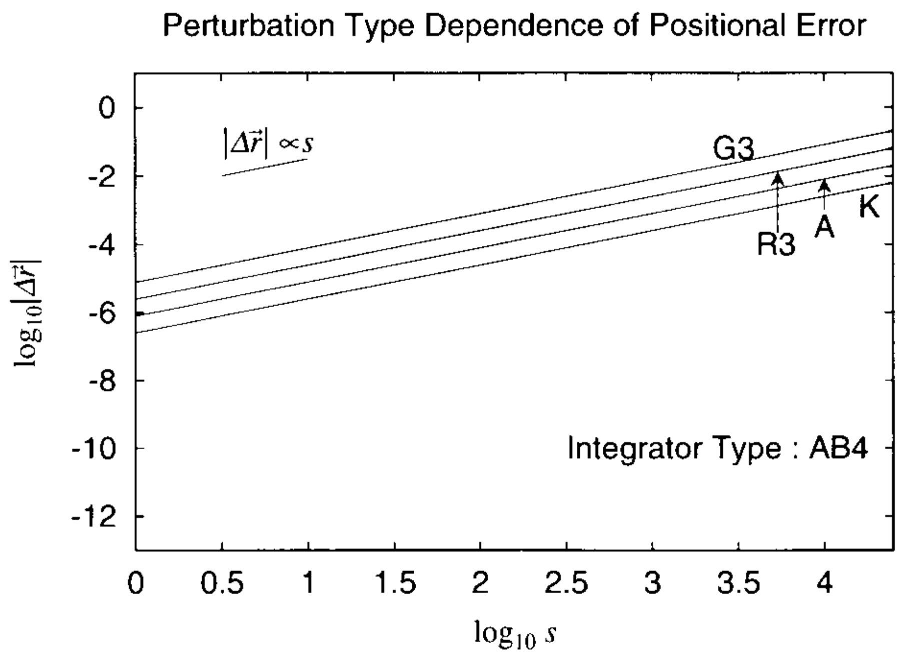

Figure A.2: Perturbation Type Dependence of Positional Error

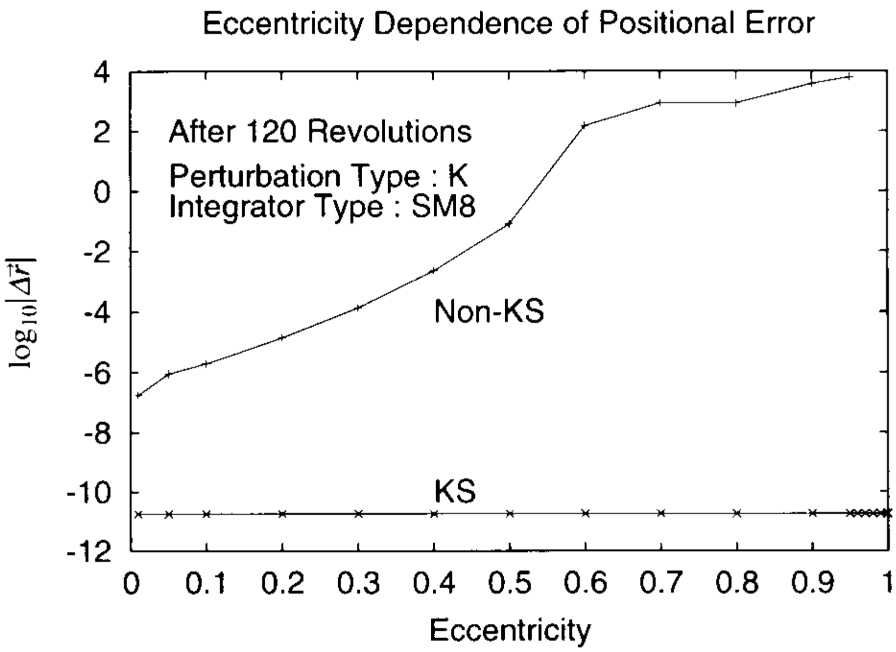

Figure A.3: Eccentricity Dependence of Positional Error

where u_k were the new variables connecting to the position vector $(x,\ y)$ as $x = u_1^2 - u_2^2$, and $y = 2u_1u_2$. The analytical solution of Eq.(A.7) is expressed as,

$$u_1 = \cos s, \quad u_2 = \sin s. \tag{A.8}$$

Consider the truncation error of the numerical integration. According to Henrici [35], the differential equation of error, Δu_k is given by,

$$\frac{d\Delta u_k^2}{ds^2} + \Delta u_k = \sum_{j=p}^{\infty} C_j h^p \left(\frac{d}{ds}\right)^p u_k,\ (k = 1,\ 2). \tag{A.9}$$

This is rewritten as;

$$\frac{d^2}{ds^2}\begin{pmatrix}\Delta u_1 \\ \Delta u_2\end{pmatrix} + \begin{pmatrix}\Delta u_1 \\ \Delta u_2\end{pmatrix} = \begin{pmatrix} C_{even} & C_{odd} \\ -C_{odd} & C_{even}\end{pmatrix}\begin{pmatrix}\cos s \\ \sin s\end{pmatrix} \tag{A.10}$$

where

$$C_{even} = \sum_{i=[p/2]}^{\infty} C_{2i}h^{2i}(-1)^i, \quad C_{odd} = \sum_{i=[p/2]}^{\infty} C_{2i-1}h^{2i-1}(-1)^i, \tag{A.11}$$

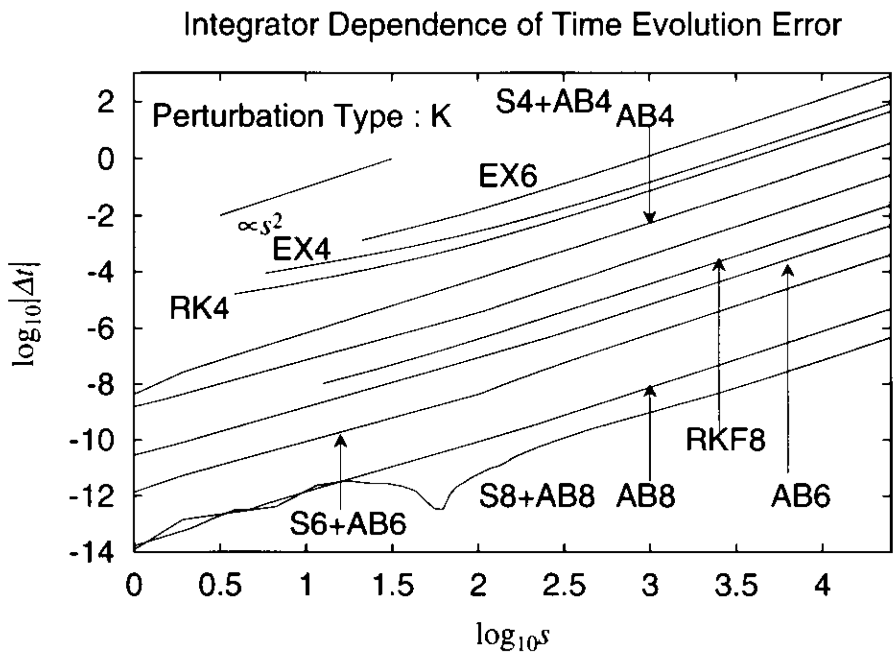

Figure A.4: Integrator Type Dependence of Time Evolution Error : Traditional Integrator

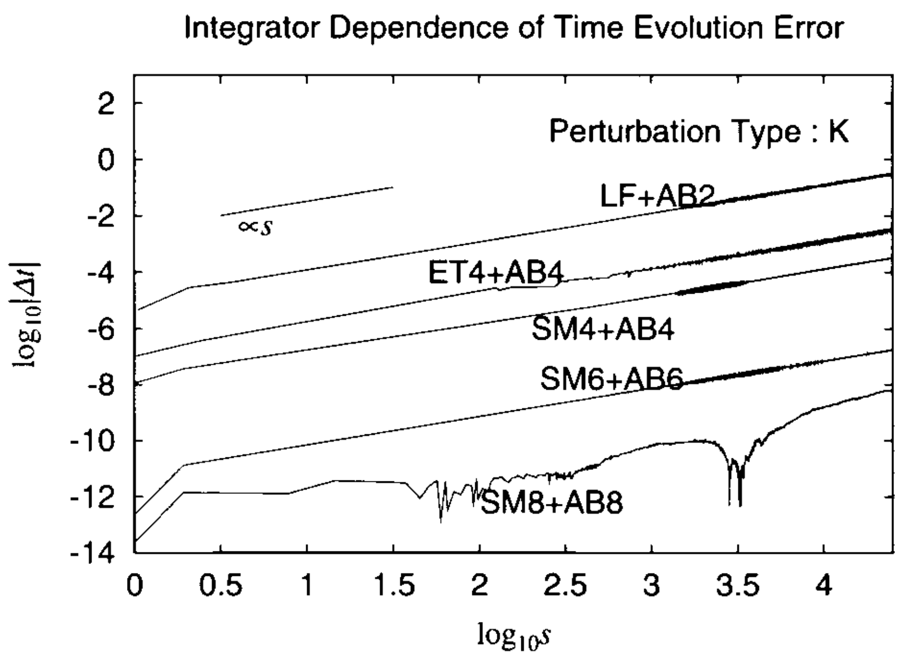

Figure A.5: Integrator Type Dependence of Time Evolution Error : Time Symmetric Integrator

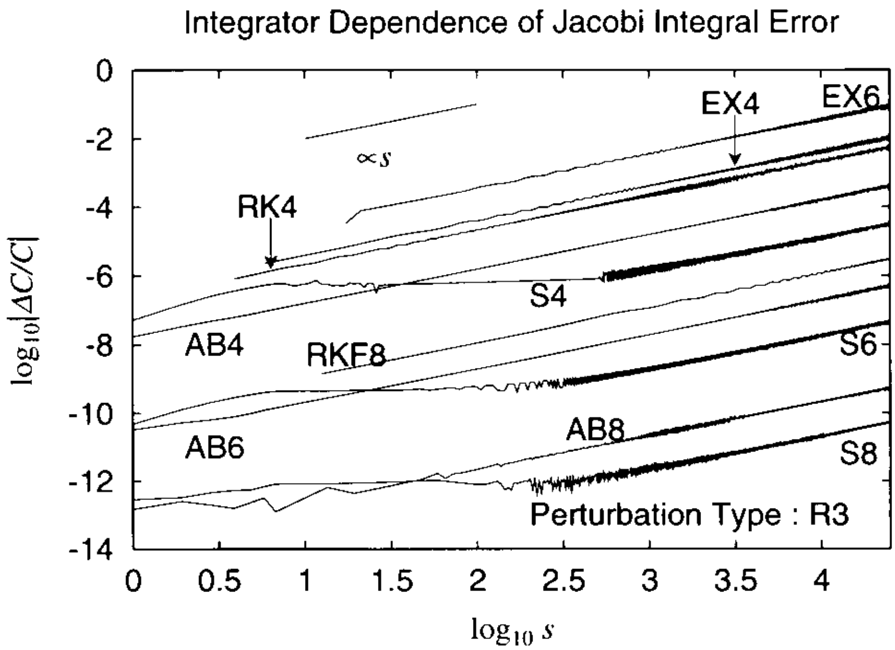

Figure A.6: Integrator Type Dependence of Jacobi Integral Error : Traditional Integrator

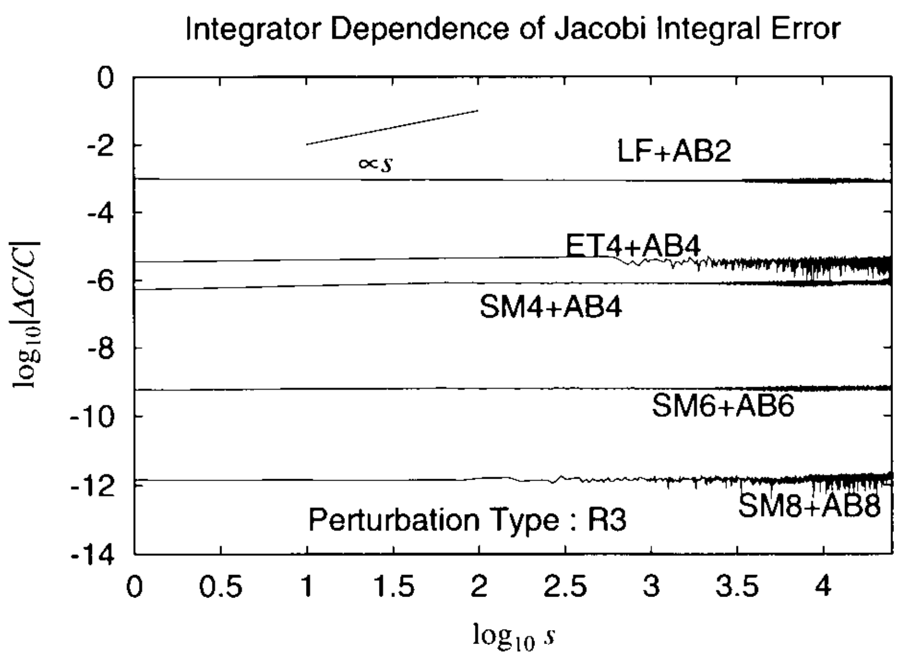

Figure A.7: Integrator Type Dependence of Jacobi Integral Error : Traditional Integrator and C_j are the error constants proper to the integrator, p is the order of integration and h is the stepsize. In general, if the initial conditions are given by,

$$\begin{pmatrix} \Delta u_1 & d\Delta u_1/ds \\ \Delta u_2 & d\Delta u_2/ds \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}, \quad (s=0) \tag{A.12}$$

then the solutions of Eq.(A.10) become

$$\begin{pmatrix} \Delta u_1 \\ \Delta u_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} C_{even} & C_{odd} \\ -C_{odd} & C_{even} \end{pmatrix}\begin{pmatrix} s\sin s \\ \sin s - s\cos s \end{pmatrix} + \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} \cos s \\ \sin s \end{pmatrix}. \tag{A.13}$$

The differential equation describing the time evolution is written as

$$\frac{dt}{ds} = r = u_1^2 + u_2^2, \tag{A.14}$$

and its variational equation becomes

$$\frac{d\Delta t}{ds} = 2(u_1\Delta u_1 + u_2\Delta u_2). \tag{A.15}$$

This equation is explicitly integrated as

$$\begin{aligned} \Delta t &= C_{even}\left(\frac{s}{2} - \frac{\sin 2s}{4}\right) - C_{odd}\left(\frac{s^2}{2} + \frac{\cos 2s}{4}\right) \\ &\quad +2\left\{(a_1+b_2)\frac{s}{2} - (b_2+a_1)\frac{\cos 2s}{2} + (a_1-b_2)\frac{\sin 2s}{4}\right\}. \end{aligned} \tag{A.16}$$

In general, there appear the error terms of both odd and even powers of h. Thus Δt grows in proportion to s^2. However, in the case of the time symmetric integrators, Δu_k contains only the even powers of h. Therefore the error Δt only grows linearly. Even if there are errors corresponding to the initial values, its contribution only appears as a linear growth. We note that Stiefel [77] evaluated that the error of physical time propagates in proportion to s in the case of the fourth order Runge-Kutta method, however, as we proved above, all kind of the traditional integrators produce the quadratic error growth with respect to s.

A.2.4 Reduction of Stepsize Resonance/Instability of Symmetric Multistep Method

As Quinlan [66] and Fukushima [21] have shown, the some symmetric multistep methods for the special second order ODEs face the stepsize resonance/instability when integrating the nonlinear ODEs, such as the Kepler problem. Quinlan revealed the mechanism of such resonance/instability by linear stability analysis. He concluded that the stability condition for the potential $\phi(r)$ in terms of dispersion relation is,

$$\omega_2(r) = \frac{1}{2}\left[4\omega^2(r) - \kappa^2(r)\right] = 0, \tag{A.17}$$

where $\omega(r)$ is the circular frequency defined as,

$$\omega(r) \equiv \frac{1}{r}\phi'(r), \tag{A.18}$$

and $\kappa(r)$ is the epicycle frequency defined as,

$$\kappa^2(r) \equiv r\frac{d\omega^2}{dr} + 4\omega^2 = \phi''(r) + \frac{3}{r}\phi'(r). \tag{A.19}$$

From these relations, we can realize that the stability condition of the potential $\phi(r)$ becomes,

$$\phi(r) = \omega_0^2 r^2 + \phi_0, \tag{A.20}$$

where ω_0 and ϕ_0 are the integration constants. Thus we have found that the stable form of potential $\phi(r)$ essentially limited to that of the harmonic oscillator. This indicates that the harmonic oscillator potential can only avoid the stepsize resonance/instability. Fig. A.8 shows the relative energy error of one-dimensional perturbed harmonic oscillator after 1000 nominal revolutions as a function of steps per period. The integrators adopted are the different order of the symmetric multistep methods. In this figure, we adopted the perturbation type as $F = \epsilon x$. Fig. A.9 is the same as Fig. A.8 but plotted are the stepsize dependence of Jacobi Integral error of the circular restricted three-body problem. Here we fixed the integrator as the eighth-order symmetric multistep method of Quinlan & Tremaine for both the KS regularized and unregularized cases. In the case of KS regularized, the error decrease with respect to stepsize, while in the case of unregularized, many spikes appear. Therefore we have numerically confirmed that the stepsize resonance/instability of the symmetric multistep method is avoided by the KS transformation.

A.2.5 Comparison of CPU time

For applying the KS regularization, the most practical concern is the increase of computational time. Fig. A.10 plots the CPU time ratio between the KS regularized and

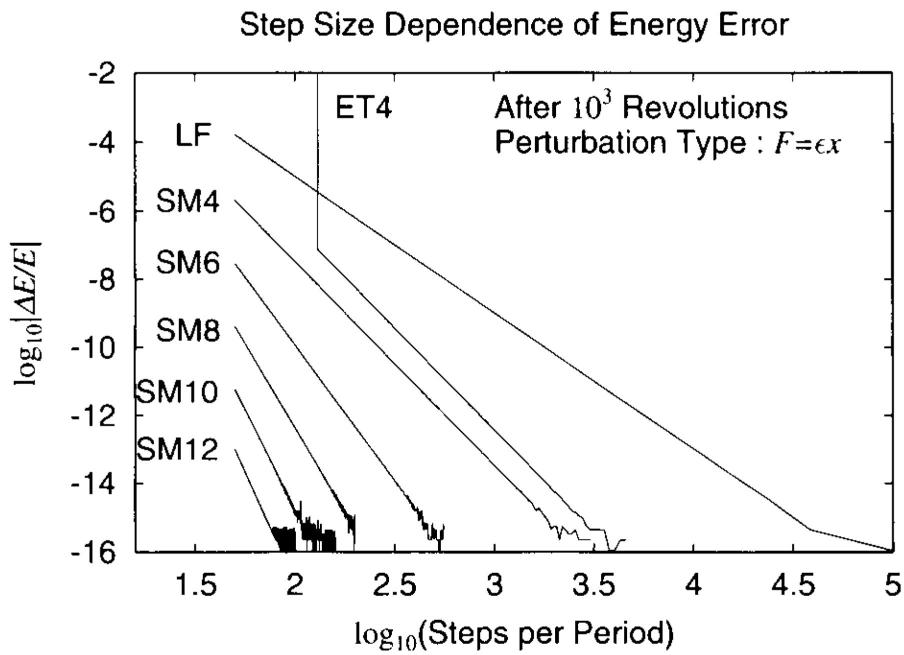

Figure A.8: Step Size Dependence of Energy Error of Perturbed Harmonic Oscillator.

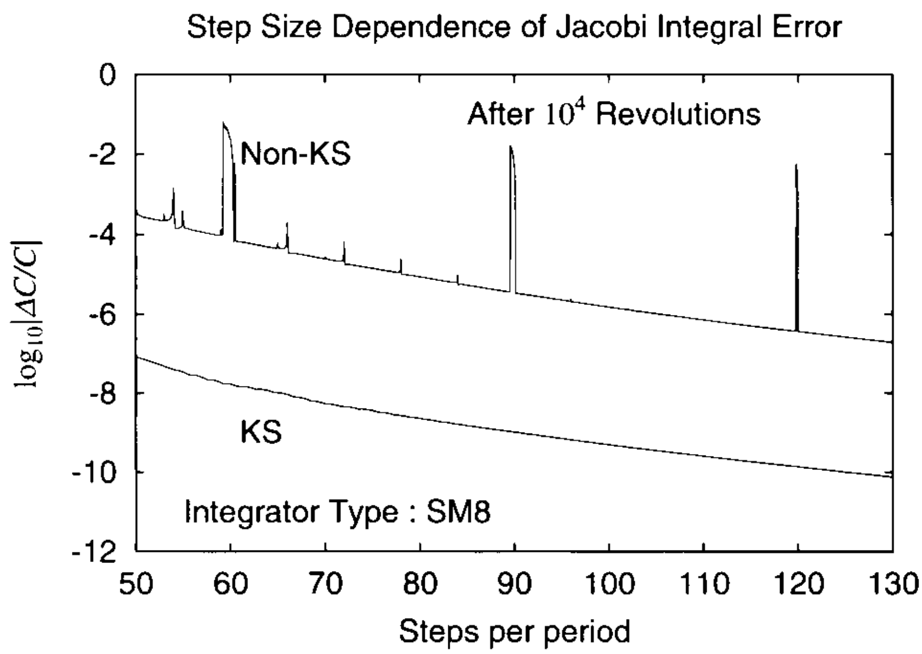

Figure A.9: Stepsize Dependence of Jacobi Integral Error

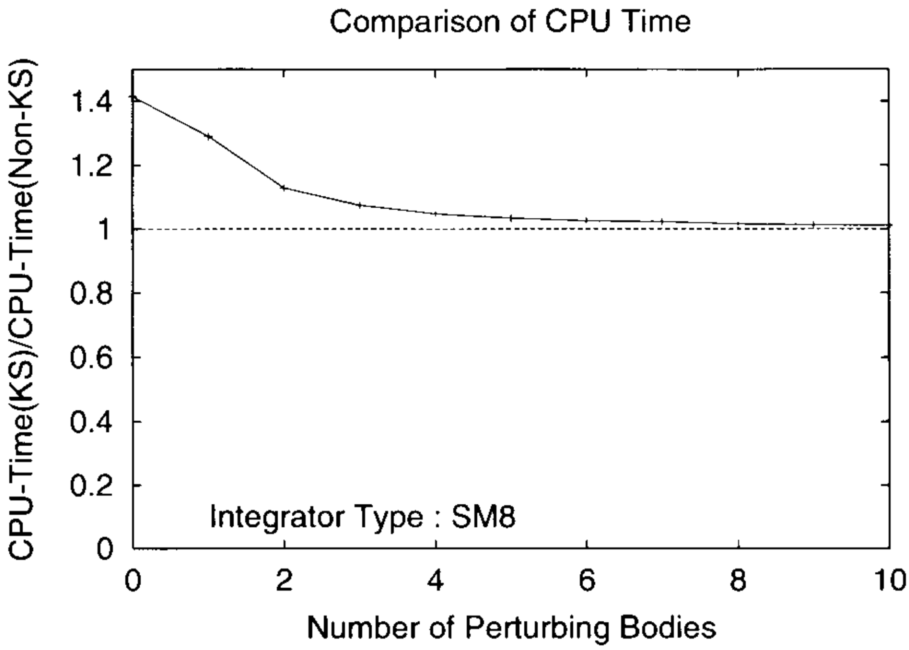

Figure A.10: Comparison of CPU time.

unregularized equation of motion as the function of the number of perturbing bodies. In the case of no perturbing bodies, the CPU time of regularized case is about 40 % larger than that of unregularized one. However the CPU time for the regularized case becomes about 10 % larger than that of the unregularized case for two perturbers, and then the difference of the CPU time becomes a few % for more perturbers. Hence, although the KS regularization requires the integration of 10 variables instead of six in rectangular coordinates, the actual increase in CPU time is not significant if the force calculation is sufficiently complicated, this fact corresponds to more than six perturbers in the case of an asteroid motion or the more than a three degree or higher order gravitational potential of Earth in the case of an artificial satellite motion.

A.3 KS Element

As we have shown in the previous section, the KS regularization can avoid the stepsize resonance/instability of the symmetric multistep method of special second order ODEs, then it can enable us to perform the fast and highly accurate integration of the restricted

N-body problem under the condition, $\Delta x \propto t$.

Nevertheless, the KS regularization itself have still limits; the symmetric multistep method for special second order ODEs cannot deal with the acceleration depending on velocity v, and the symmetric multistep method for general first order ODEs often faces the numerical instability. Then, as the one possibility to overcome this limitation and to find a scheme that reduces the integration error in both the position and the physical time for the any types of perturbation, we considered the application of the method of variation of parameter to the KS regularization. Since the equation of motion in the KS regularization is expressed in the form of perturbed harmonic oscillator, we can define the KS element as the amplitudes and the phases. The KS element was first introduced by Stiefel [78] and after modified by Stiefel & Scheifele [79]. However Stiefel's element is not complete in the sense that the time element is not given. While the element of Stiefel & Scheifele includes the time element, τ but this time element is new variable rather than element. Thus we will introduce the time element t_0 based on Stiefel's approach and construct the complete third set of the KS element.

A.3.1 Complete Set of Element

Stiefel started to rewrite the equation of motion in the KS variable, $\boldsymbol{u}$ as,

$$\frac{d^2\boldsymbol{u}}{ds^2} + \omega_0^2\boldsymbol{u} = \boldsymbol{F}, \tag{A.21}$$

where

$$\boldsymbol{F} = \frac{1}{2}rL^T\vec{F} - W\boldsymbol{u}, \quad W \equiv \frac{1}{2}h_k - \omega_0^2, \quad \omega_0 \equiv \sqrt{\frac{h_{k0}}{2}}, \tag{A.22}$$

Here, L^T is the transpose of the KS matrix, ω_0 is a constant representing the initial value of the angular velocity of the associated harmonic oscillator, and the Kepler energy h_k is replaced by W, which corresponds to the work done by the perturbation. Next, he introduced the new variables, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ defined as,

$$\boldsymbol{\alpha} \equiv \boldsymbol{u}\cos\omega_0 s - \frac{\boldsymbol{u}'}{\omega_0}\sin\omega_0 s, \quad \boldsymbol{\beta} \equiv \boldsymbol{u}\sin\omega_0 s + \frac{\boldsymbol{u}'}{\omega_0}\cos\omega_0 s. \tag{A.23}$$

In other words, these are the coefficients of the solution, $\boldsymbol{u}$ in terms of the trigonometric functions. From these relations, the exact solution of the physical time t in the unperturbed case is expressed as,

$$t - t_0 = \int r ds = \frac{\boldsymbol{\alpha}^2 + \boldsymbol{\beta}^2}{2}s + \frac{\boldsymbol{\alpha}^2 - \boldsymbol{\beta}^2}{4\omega_0}\sin 2\omega_0 s - \frac{\boldsymbol{\alpha}\cdot\boldsymbol{\beta}^2}{2\omega_0}\cos 2\omega_0 s, \tag{A.24}$$

here t_0 is the integration constant. Based on this form, we name t_0 a "time element" in the perturbed case, which is defined as,

$$t_0 \equiv t - \left(\frac{\boldsymbol{\alpha}^2 + \boldsymbol{\beta}^2}{2} s + \frac{\boldsymbol{\alpha}^2 - \boldsymbol{\beta}^2}{4\omega_0} \sin 2\omega_0 s - \frac{\boldsymbol{\alpha} \cdot \boldsymbol{\beta}^2}{2\omega_0} \cos 2\omega_0 s \right). \tag{A.25}$$

Then the equation of motion of the resulting new set of elements is written as,

$$\begin{cases} \dfrac{d\boldsymbol{\alpha}}{ds} = -\dfrac{\boldsymbol{F}}{\omega_0} \sin \omega_0 s, \quad \dfrac{d\boldsymbol{\beta}}{ds} = \dfrac{\boldsymbol{F}}{\omega_0} \cos \omega_0 s, \quad \dfrac{dW}{ds} = L^T \vec{F} \cdot \boldsymbol{u}', \\ \dfrac{dt_0}{ds} = \dfrac{1}{\omega_0} \boldsymbol{F} \cdot \left[(\boldsymbol{\alpha} \sin \omega_0 s - \boldsymbol{\beta} \cos \omega_0 s) s + \dfrac{1}{2\omega_0} (\boldsymbol{\alpha} \cos \omega_0 s + \boldsymbol{\beta} \sin \omega_0 s) \right]. \end{cases} \tag{A.26}$$

The set $(\boldsymbol{\alpha}, \boldsymbol{\beta}, W, t_0)$ is the complete set of elements since all of the s-derivatives vanish in the unperturbed case.

Here we note that it seems to be also possible to apply the time element to the framework of Stiefel & Scheifele. However, when we tried to introduce the new time element τ_0, we found that this does not work well; the quadratic error growth in the physical time appears.

A.3.2 Numerical Experiments

From now on, we compare the error growth of four formulations; (1) the original KS variables, (2) Stiefel's variables, (3) the variables of Stiefel and Scheifele, and (4) the complete set of elements that we developed above. As the test problem, we adopted the 3-dimensional circular restricted three-body problem. For the numerical integrator, we used the eighth-order Adams-Bashforth method (predictor formula) where the starting values was determined by using the extrapolation method. We note that the results obtained did not change when the integrator was replaced by other traditional ones, such as Runge-Kutta-type integrator or the extrapolation method.

Fig. A.11 plots the growth of numerical integration error in position with respect to s. In all cases, the positional error is in proportion to s. Fig. A.12 is the same as Fig. A.11 but represents the error of physical time. In the case of the original KS variable, the error grows quadratically with respect to s. Otherwise, the error is proportional to s. Fig. A.13 is the same as Fig. A.11 but illustrates the error of Jacobi integral. In the cases of the KS variable and Stiefel & Scheifele formulation, the error increase linearly, while in the cases of Stiefel and our new scheme, the error remains almost constant. However, over the longer period, the effect of accumulation of rounding-off error appears.

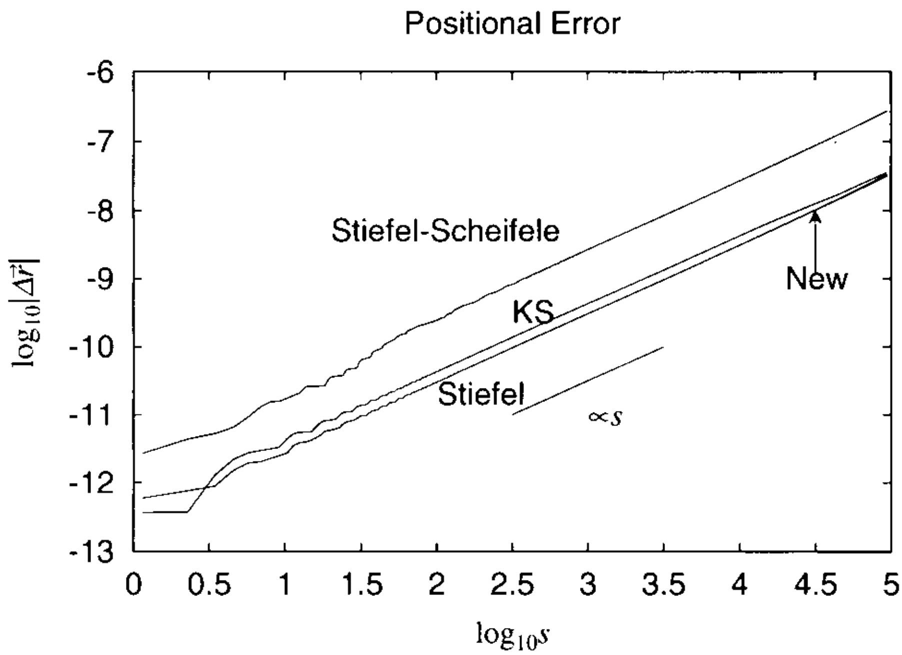

Figure A.11: Growth of numerical integration error in position.

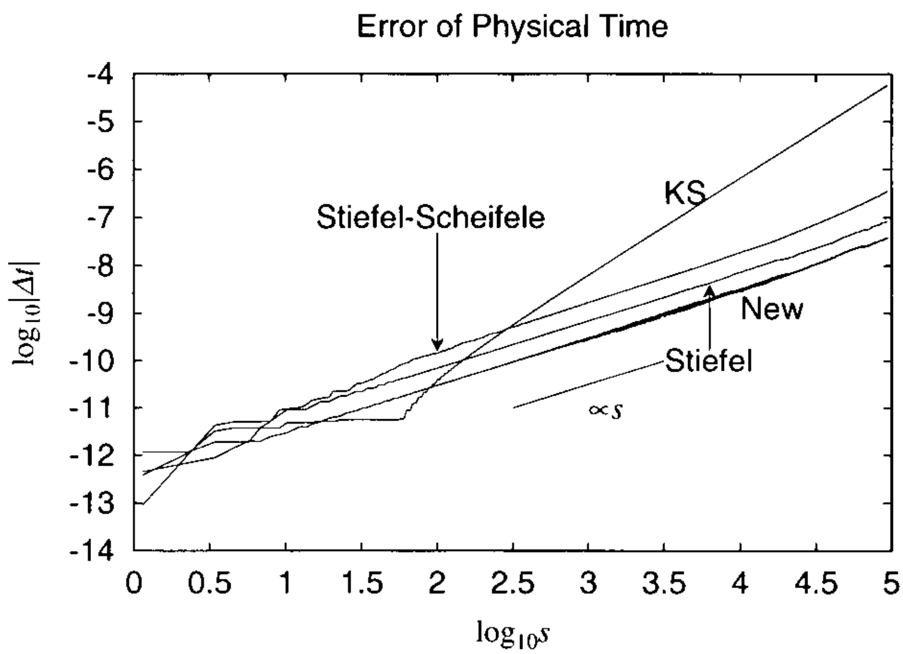

Figure A.12: Growth of numerical integration error in physical time.

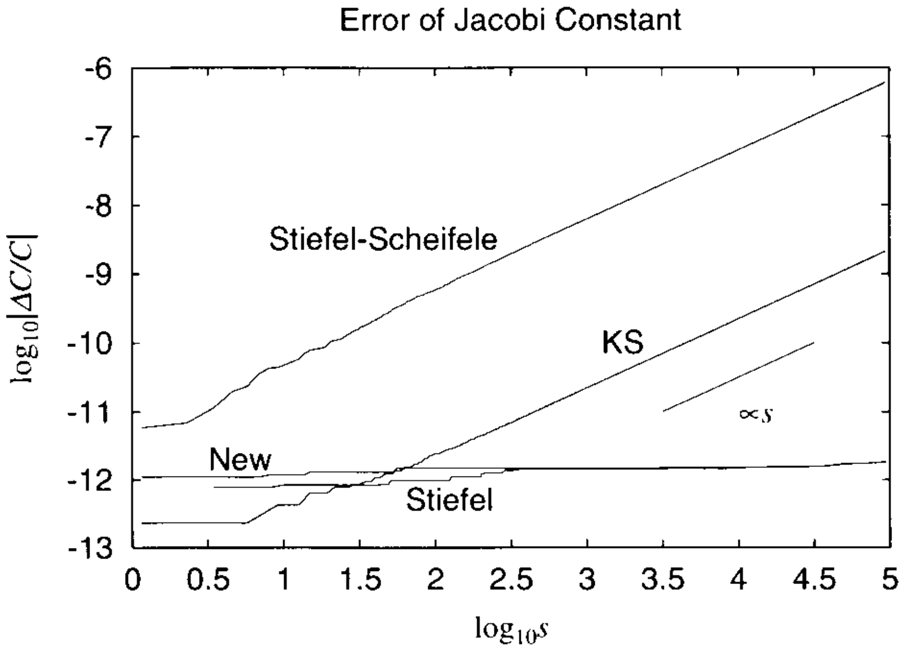

Figure A.13: Growth of numerical integration error in Jacobi integral.

A.3.3 Error Growth of Physical Time

Let us examine the reasons why the element formulations can reduce the integration error of the physical time. It is difficult to derive the analytical expression of the error propagation of the physical time in the element formulations, so that we evaluate the error of variations of the coefficients that appear in the equation of motion of the physical time by two run of the numerical integrations with different stepsize. Fig. A.14 illustrates the integration error of various coefficients of the physical time appeared in the approaches of Stiefel and ours,

$$\begin{cases} \Delta C_1 \equiv \boldsymbol{\alpha} \cdot \Delta\boldsymbol{\alpha} + \boldsymbol{\beta} \cdot \Delta\boldsymbol{\beta}, \\ \Delta C_2 \equiv \boldsymbol{\alpha} \cdot \Delta\boldsymbol{\alpha} - \boldsymbol{\beta} \cdot \Delta\boldsymbol{\beta}, \\ \Delta C_3 \equiv \boldsymbol{\alpha} \cdot \Delta\boldsymbol{\beta} + \boldsymbol{\beta} \cdot \Delta\boldsymbol{\alpha}, \\ \Delta C_4 \equiv \boldsymbol{F} \cdot (\Delta\boldsymbol{\alpha} \sin\omega_0 s - \Delta\boldsymbol{\beta} \cos\omega_0 s) \\ \qquad +\Delta\boldsymbol{F} \cdot (\boldsymbol{\alpha} \sin\omega_0 s - \boldsymbol{\beta} \cos\omega_0 s), \\ \Delta C_5 \equiv \boldsymbol{F} \cdot (\Delta\boldsymbol{\alpha} \cos\omega_0 s + \Delta\boldsymbol{\beta} \sin\omega_0 s) \\ \qquad +\Delta\boldsymbol{F} \cdot (\boldsymbol{\alpha} \cos\omega_0 s + \boldsymbol{\beta} \sin\omega_0 s). \end{cases} \tag{A.27}$$

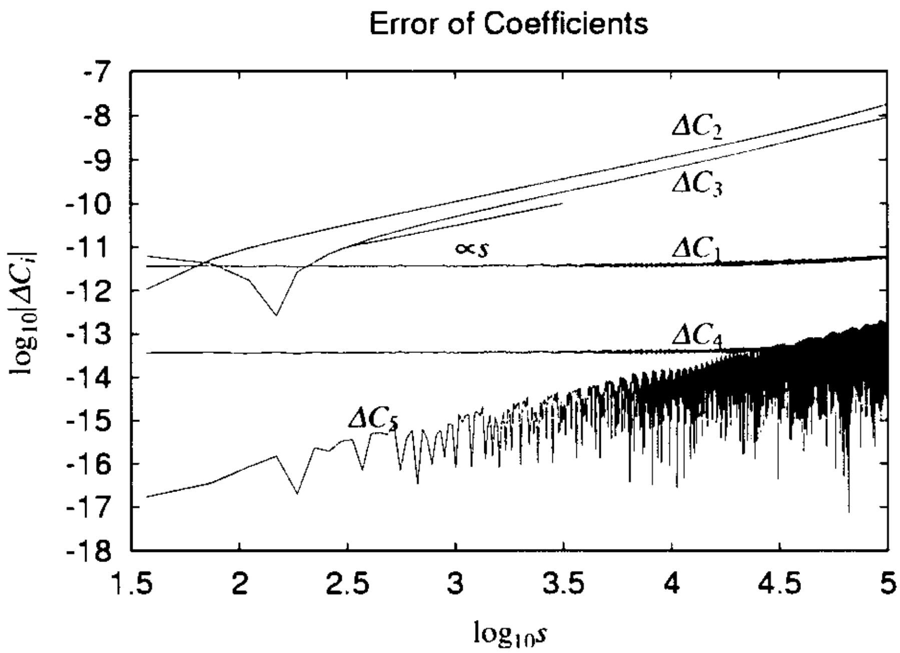

Figure A.14: Error of Coefficients.

Note that the error of linear term coefficient, ΔC_1 is simply oscillating and becomes almost constant when averaged. While ΔC_2 and ΔC_3 grow linearly. Since ΔC_1 is the error of $\frac{1}{2}(\boldsymbol{\alpha}^2 + \boldsymbol{\beta}^2)$ corresponding to the Kepler energy, ΔC_1 remains almost constant. This is similar to the fact that the error of Jacobi constant almost constant in the approaches of Stiefel's and ours (see Fig. A.13). The error of another linear term coefficient, ΔC_4, is also oscillating and remains almost constant in the long run. Also ΔC_5 is oscillating but its amplitude increases as a mixed secular term. Since the integral of a mixed secular term is the sum of another mixed secular term and a periodic term, it is concluded that the magnitude of error of the physical time propagates in proportion to s. Fig. A.15 is the same as Fig. A.14 but plotted are the error of coefficients of the equation of time element τ in the case of Stiefel & Scheifele[2] ;

$$\begin{cases} \Delta D_1 \equiv \dfrac{\Delta\boldsymbol{\xi}\cdot\boldsymbol{\xi} - \Delta\boldsymbol{\eta}\cdot\boldsymbol{\eta}}{2\omega} - \dfrac{\boldsymbol{\xi}\cdot\boldsymbol{\xi} - \boldsymbol{\eta}\cdot\boldsymbol{\eta}}{2\omega^2}\Delta\omega, \\ \Delta D_2 \equiv \dfrac{\Delta\boldsymbol{\xi}\cdot\boldsymbol{\eta} - \Delta\boldsymbol{\eta}\cdot\boldsymbol{\xi}}{2\omega} - \dfrac{\boldsymbol{\xi}\cdot\boldsymbol{\eta}}{2\omega^2}\Delta\omega, \\ \Delta D_3 \equiv \dfrac{1}{8\omega^3}\Delta H, \quad \Delta D_4 \equiv -\dfrac{3}{8\omega^4}H\Delta\omega. \end{cases} \tag{A.28}$$

[2] In the textbook of Stiefel & Scheifele, they used the symbols $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ in place of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ used here.

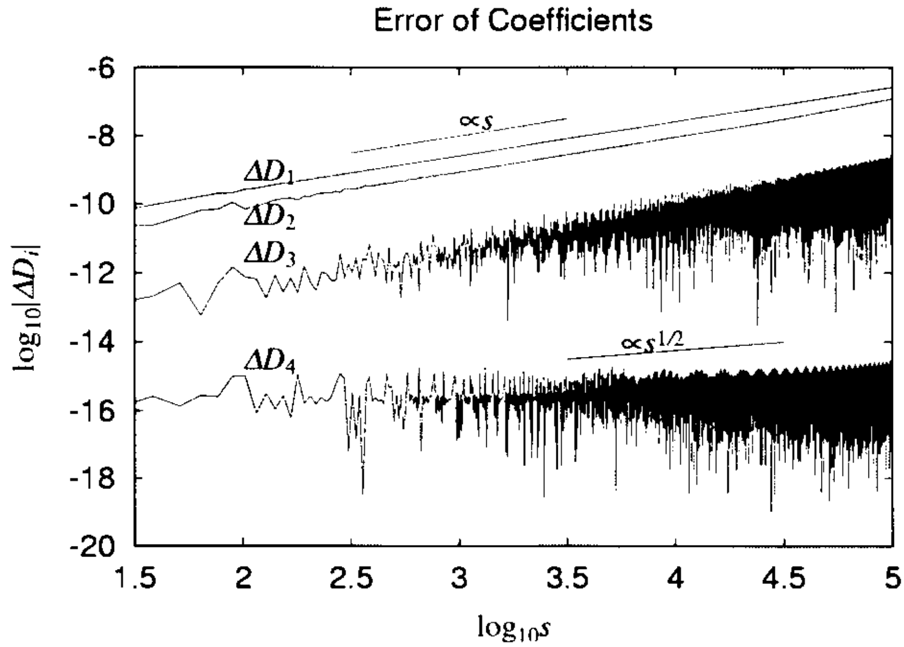

Figure A.15: Error of Coefficients.

where

$$H \equiv \mu + r\boldsymbol{u} \cdot \boldsymbol{f} + 8\left(\frac{d\boldsymbol{u}}{dE} \cdot \boldsymbol{f}\right)\left(\frac{d\boldsymbol{u}}{dE} \cdot \boldsymbol{u}\right).$$

The errors ΔD_1 and ΔD_2 are in proportion to s. While ΔD_3 has the same property as ΔC_5. Note that the error of ΔD_4 shows the effect of rounding off since it grows in proportion to $\sqrt{s}$. Anyway the error of physical time also grows linearly with respect to s.

A.3.4 Comparison of CPU Time

As we have found above, the application of method of variation of parameter to the KS regularization works quite well, namely the both of the position and the physical time error grow only linearly. As the most practical concern in using the element formulations, we compared the increase in CPU time due to their introduction. Fig. A.16 shows the CPU time ration of each element formulation with respect to that of the original KS variable. As clearly be seen, the increase in CPU time is very small, at most 2 % larger than that of the original KS variable. This is because the additional operation required

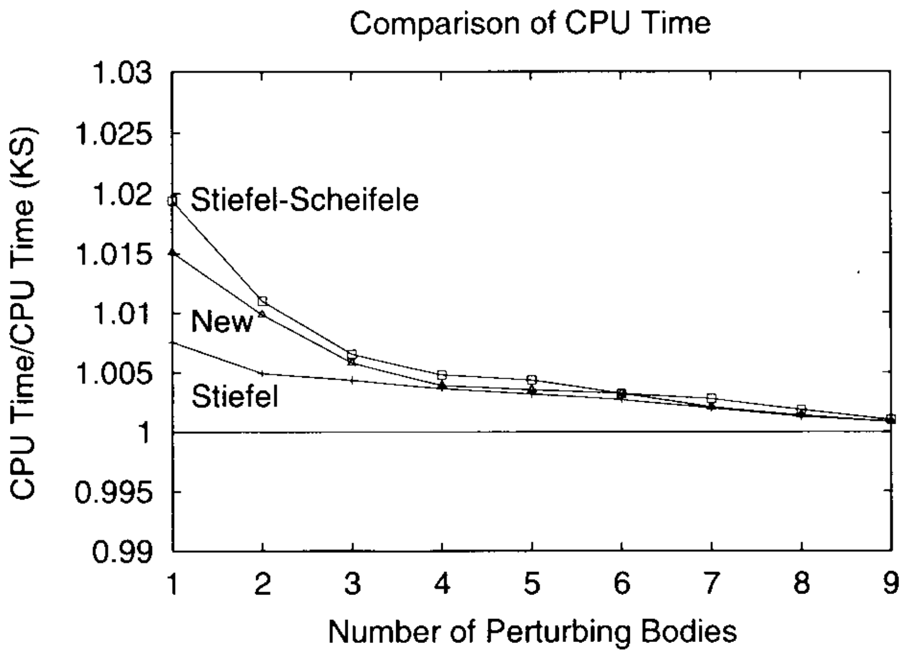

Figure A.16: Growth of numerical integration error in Jacobi integral.

by the element formulation mainly consists of the evaluation of trigonometric functions such as $\sin \omega_0 s$, which reduces to simple arithmetic by using the additional theorem. As we have shown in Fig. A.10, the increase in CPU time caused by the introduction of KS regularization itself become negligible for the practical problems. Therefore, combing these two facts, we can conclude that there is no significant difference in CPU time among the unregularized, the KS regularization, and the KS element formulations.

A.4 Summary

We confirmed that the KS regularization contribute to the reduction of the numerical integration error in position dramatically. This feature does not depend on the type of integrator adopted, the perturbation type, and of course the nominal eccentricity. And the KS regularization avoid the stepsize resonance/instability of the symmetric multistep method of the special second order ODEs when integrating the Kepler problem. Further we indicated that the application of the method of variation of parameter to the KS regularization products only linear integration error growth in both the position and the physical time for any kind of perturbation. And we constructed the complete set of the KS element based on the Stiefel's approach. However we note that these good property in the reduction of the numerical integration error fails when we apply the KS regularization and its element approach to the general N-body problem since the fictitious time is proper to each body. As the practical concern to use the KS regularization and its element approach, we compared the increase in CPU time due to the their introduction and found that there appear no significant increase in CPU time when we deal with the practical problem. Therefore the KS regularization and its method of variation of parameter is effective to study the long term behavior of perturbed two body problems; especially the dynamics of comets, minor planets, the Moon, and natural/artificial satellites.

Appendix B

Expression of Analytical Solution

Our analytical solution $\delta\boldsymbol{r}_E^{(j)}$ are formally written as,

$$\delta\boldsymbol{r}_E^{(0)} = \int\int \boldsymbol{F}_E(\boldsymbol{r}_L, t)dtdt, \quad \delta\boldsymbol{r}_E^{(1)} = \int\int \left(\frac{\partial \boldsymbol{F}_L(\boldsymbol{r}, t)}{\partial \boldsymbol{r}}\right)_{\boldsymbol{r}_L} dtdt\ nonumber. \tag{B.1}$$

In the below, we will expand them with respect to the parameter, $q = a/a_I$ and list the expanded terms up to the seventh order.

B.1 Order of q^0

$$\begin{aligned}
&\delta x_E^{(0)} \\
= &\left[\left\{-\frac{e_I^2(\sin(3n_I(t-t_{I0}))-\sin(3n_I(-t_{I0}))}{3n_I^2}+\frac{e_I(\sin(2n_I(t-t_{I0}))-\sin(2n_I(-t_{I0})))}{32n_I^2}\right.\right. \\
&\left.-\frac{(5e_I^2/8+1)(\sin(n_I(t-t_{I0}))-\sin(n_I(-t_{I0})))}{n_I^2}\right\}\cos\varpi \\
&-\left\{-\frac{e_I^2(\cos(3n_I(t-t_{I0}))-\cos(3n_I(-t_{I0}))}{3n_I^2}-\frac{e_I(\cos(2n_I(t-t_{I0}))-\cos(2n_I(-t_{I0}))}{8n_I^2}\right. \\
&\left.\left.-\frac{(5e_I^2/8+1)(\cos(n_I(t-t_{I0}))-\cos(n_I(-t_{I0}))}{n_I^2}\right\}\sin\varpi\right] \\
&-\frac{a}{n^2}\left\{\mu\left(\cos(nt-nt_0)-\cos nt_0\right)-\left(\cos\left(nt-nt_0+\frac{\pi}{3}\right)-\cos\left(nt_0-\frac{\pi}{3}\right)\right)\right\}
\end{aligned} \tag{B.2}$$

$$\begin{aligned}
&\delta y_E^{(0)} \\
= &\ a_I\left[\left\{\frac{-e_I^2(\sin(3n_I(t-t_{0I}))-\sin(3n_I(-t_{0I})))}{3n_I^2}+\frac{e_I(\sin(2n_I(t-t_{0I}))-\sin(2n_I(-t_{0I})))}{32n_I^2}\right.\right. \\
&\left.-\frac{(5e_I^2/8+1)(\sin(n_I(t-t_{0I}))-\sin(n_I(-t_{0I}))}{n_I^2}\right\}\sin\varpi \\
&+\left\{-\frac{e_I^2(\cos(3n_I(t-t_{0I}))-\cos(3n_I(-t_{0I}))}{3n_I^2}-\frac{e_I(\cos(2n_I(t-t_{0I}))-\cos(2n_I(-t_{0I})))}{8n_I^2}\right.
\end{aligned}$$

$$- \frac{(5e_I^2/8 + 1)(\cos(n_I(t - t_{0I})) - \cos(n_I(-t_{0I})))}{n_I^2} \Bigg\} \cos \varpi \Bigg]$$

$$- \frac{a}{n^2} \left\{ \mu(\sin(nt - nt_0) + \sin(nt_0)) - \left(\sin\left(nt - nt_0 + \frac{\pi}{3} \right) + \sin\left(nt_0 - \frac{\pi}{3} \right) \right) \right\} \tag{B.3}$$

B.2 Order of q^1

$$
\begin{aligned}
&\delta x_E^{(1)} \\
= \quad & a_I \left[\frac{3e_I^2 \sin\varpi}{64n_I^2 + 16nn_I - 8n^2} \right. \\
& \left\{ \frac{(-2n_I - n)\sin(4n_I t_{I0} + nt_0)\sin(8n_I t_{I0} + (n-4n_I)t)}{n - 4n_I} - \frac{(2n_I + n)\cos(4n_I t_{I0} + nt_0)\cos(8n_I t_{I0} + (n-4n_I)t)}{n-4n_I} \right. \\
& + \frac{(n-4n_I)\sin(4n_I t_{I0} + nt_0)\sin(6n_I t_{I0} + 2nt_0 + (-2n_I - n)t)}{-2n_I - n} \\
& - \frac{(4n_I - n)\cos(4n_I t_{I0} + nt_0)\cos(6n_I t_{I0} + 2nt_0 + (-2n_I - n)t)}{-2n_I - n} \\
& + \left(\frac{(4n_I - n)\sin(2n_I t_{I0} + (2n_I + n)t)}{2n_I + n} + \frac{(2n_I + n)\sin(2nt_0 + (4n_I - n)t)}{4n_I - n} \right) \sin(4n_I t_{I0} + nt_0) \\
& \left. + \left(-\frac{(n-4n_I)\cos(2n_I t_{I0} + (2n_I + n)t)}{2n_I + n} - \frac{(-2n_I - n)\cos(2nt_0 + (4n_I - n)t)}{4n_I - n} \right) \cos(4n_I t_{I0} + nt_0) \right\} \\
& - \frac{3e_I^2 \cos\varpi}{64n_I^2 + 16nn_I - 8n^2} \\
& \left\{ \frac{(2n_I + n)\cos(4n_I t_{I0} + nt_0)\sin(8n_I t_{I0} + (n-4n_I)t)}{n - 4n_I} - \frac{(2n_I + n)\sin(4n_I t_{I0} + nt_0)\cos(8n_I t_{I0} + (n-4n_I)t)}{n-4n_I} \right. \\
& + \frac{(4n_I - n)\cos(4n_I t_{I0} + nt_0)\sin(6n_I t_{I0} + 2nt_0 + (-2n_I - n)t)}{-2n_I - n} \\
& - \frac{(4n_I - n)\sin(4n_I t_{I0} + nt_0)\cos(6n_I t_{I0} + 2nt_0 + (-2n_I - n)t)}{-2n_I - n} \\
& + \left(-\frac{(4n_I - n)\cos(2n_I t_{I0} + (2n_I + n)t)}{2n_I + n} - \frac{(2n_I + n)\cos(2nt_0 + (4n_I - n)t)}{4n_I - n} \right) \sin(4n_I t_{I0} + nt_0) \\
& \left. + \left(\frac{(4n_I - n)\sin(2n_I t_{I0} + (2n_I + n)t)}{2n_I + n} + \frac{(2n_I + n)\sin(2nt_0 + (4n_I - n)t)}{4n_I - n} \right) \cos(4n_I t_{I0} + nt_0) \right\} \\
& + \frac{e_I \sin\varpi}{48n_I^2 + 32nn_I - 16n^2} \\
& \left\{ \frac{(-n_I - n)\sin(3n_I t_{I0} + nt_0)\sin(6n_I t_{I0} + (n-3n_I)t)}{n - 3n_I} - \frac{(n_I + n)\cos(3n_I t_{I0} + nt_0)\cos(6n_I t_{I0} + (n-3n_I)t)}{n-3n_I} \right. \\
& + \frac{(n-3n_I)\sin(3n_I t_{I0} + nt_0)\sin(4n_I t_{I0} + 2nt_0 + (-n_I - n)t)}{-n_I - n} \\
& - \frac{(3n_I - n)\cos(3n_I t_{I0} + nt_0)\cos(4n_I t_{I0} + 2nt_0 + (-n_I - n)t)}{-n_I - n} \\
& + \left(\frac{(3n_I - n)\sin(2n_I t_{I0} + (n_I + n)t)}{n_I + n} + \frac{(n_I + n)\sin(2nt_0 + (3n_I - n)t)}{3n_I - n} \right) \sin(3n_I t_{I0} + nt_0) \\
& \left. + \left(-\frac{(n-3n_I)\cos(2n_I t_{I0} + (n_I + n)t)}{n_I + n} - \frac{(-n_I - n)\cos(2nt_0 + (3n_I - n)t)}{3n_I - n} \right) \cos(3n_I t_{I0} + nt_0) \right\} \\
& + \frac{e_I \cos\varpi}{192n_I^2 + 128nn_I - 64n^2} \\
& \left\{ \frac{(n_I + n)\cos(3n_I t_{I0} + nt_0)\sin(6n_I t_{I0} + (n-3n_I)t)}{n - 3n_I} - \frac{(n_I + n)\sin(3n_I t_{I0} + nt_0)\cos(6n_I t_{I0} + (n-3n_I)t)}{n-3n_I} \right. \\
& + \frac{(3n_I - n)\cos(3n_I t_{I0} + nt_0)\sin(4n_I t_{I0} + 2nt_0 + (-n_I - n)t)}{-n_I - n} \\
& - \frac{(3n_I - n)\sin(3n_I t_{I0} + nt_0)\cos(4n_I t_{I0} + 2nt_0 + (-n_I - n)t)}{-n_I - n} \\
& + \left(-\frac{(3n_I - n)\cos(2n_I t_{I0} + (n_I + n)t)}{n_I + n} - \frac{(n_I + n)\cos(2nt_0 + (3n_I - n)t)}{3n_I - n} \right) \sin(3n_I t_{I0} + nt_0) \\
& \left. + \left(\frac{(3n_I - n)\sin(2n_I t_{I0} + (n_I + n)t)}{n_I + n} + \frac{(n_I + n)\sin(2nt_0 + (3n_I - n)t)}{3n_I - n} \right) \cos(3n_I t_{I0} + nt_0) \right\} \\
& + \frac{5e_I^2 \sin\varpi}{128nn_I - 64n^2}
\end{aligned}
$$

$$
\begin{aligned}
&\left\{-\frac{n\sin(2n_It_{I0}+nt_0)\sin(4n_It_{I0}+(n-2n_I)t)}{n-2n_I}-\frac{n\cos(2n_It_{I0}+nt_0)\cos(4n_It_{I0}+(n-2n_I)t)}{-2n-n_I}\right.\\
&-\frac{(n-2n_I)\sin(2n_It_{I0}+nt_0)\sin(2n_It_{I0}+2nt_0-nt)}{n}\\
&+\frac{(2n_I-n)\cos(2n_It_{I0}+nt_0)\cos(2n_It_{I0}+2nt_0-nt)}{n}\\
&+\left(\frac{(2n_I-n)\sin(2n_It_{I0}+nt)}{n}+\frac{n\sin(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\sin(2n_It_{I0}+nt_0)\\
&\left.+\left(\frac{n\cos(2nt_0+(2n_I-n)t)}{2n_I-n}-\frac{(n-2n_I)\cos(2n_It_{I0}+nt)}{n}\right)\cos(2n_It_{I0}+nt_0)\right\}\\
&+\frac{\sin\varpi}{16nn_I-8n^2}\\
&\left\{-\frac{n\sin(2n_It_{I0}+nt_0)\sin(4n_It_{I0}+(n-2n_I)t)}{n-2n_I}-\frac{n\cos(2n_It_{I0}+nt_0)\cos(4n_It_{I0}+(n-2n_I)t)}{n-2n_I}\right.\\
&-\frac{(n-2n_I)\sin(2n_It_{I0}+nt_0)\sin(2n_It_{I0}+2nt_0-nt)}{n}\\
&+\frac{(2n_I-n)\cos(2n_It_{I0}+nt_0)\cos(2n_It_{I0}+2nt_0-nt)}{/}n\\
&+\left(\frac{(2n_I-n)\sin(2n_It_{I0}+nt)}{n}+\frac{n\sin(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\sin(2n_It_{I0}+nt_0)\\
&\left.+\left(\frac{n\cos(2nt_0+(2n_I-n)t)}{2n_I-n}-\frac{(n-2n_I)\cos(2n_It_{I0}+nt)}{n}\right)\cos(2n_It_{I0}+nt_0)\right\}\\
&-\frac{5e_I^2\cos\varpi}{128nn_I-64n^2}\left\{\frac{n\cos(2n_It_{I0}+nt_0)\sin(4n_It_{I0}+(n-2n_I)t)}{n-2n_I}-\frac{n\sin(2n_It_{I0}+nt_0)\cos(4n_It_{I0}+(n-2n_I)t)}{n-2n_I}\right.\\
&-\frac{(2n_I-n)\cos(2n_It_{I0}+nt_0)\sin(2n_It_{I0}+2nt_0-nt)}{n}\\
&+\frac{(2n_I-n)\sin(2n_It_{I0}+nt_0)\cos(2n_It_{I0}+2nt_0-nt)}{n}\\
&+\left(-\frac{(2n_I-n)\cos(2n_It_{I0}+nt)}{n}-\frac{n\cos(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\sin(2n_It_{I0}+nt_0)\\
&\left.+\left(\frac{(2n_I-n)\sin(2n_It_{I0}+nt)}{n}+\frac{n\sin(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\cos(2n_It_{I0}+nt_0)\right\}\\
&-\frac{\cos\varpi}{16*n*n_I-8*n^2}\\
&\left\{\frac{n\cos(2n_It_{I0}+nt_0)\sin(4n_It_{I0}+(n-2n_I)t)}{2n-n_I}-\frac{n\sin(2n_It_{I0}+nt_0)\cos(4n_It_{I0}+(n-2n_I)t)}{2n-n_I}\right.\\
&-\frac{(2n_I-n)\cos(2n_It_{I0}+nt_0)\sin(2n_It_{I0}+2nt_0-nt)}{n}\\
&+\frac{(2n_I-n)\sin(2n_It_{I0}+nt_0)\cos(2n_It_{I0}+2nt_0-nt)}{n}\\
&+\left(-\frac{(2n_I-n)\cos(2n_It_{I0}+nt)}{n}-\frac{n\cos(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\sin(2n_It_{I0}+nt_0)\\
&\left.+\left(\frac{(2n_I-n)\sin(2n_It_{I0}+nt)}{n}+\frac{n\sin(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\cos(2n_It_{I0}+nt_0)\right\}\\
&\left.-e_I\sin\varpi\frac{\cos(n_It_{I0}-nt_0-n_It+nt)}{2(n-n_I)^2}\right]
\end{aligned}
$$

$$
\begin{aligned}
&\left[-\frac{n_i}{n_i-2n}\left\{\left(\cos\left(\left(n_i-2n\right)t-n_it_{i0}+2nt_0-\frac{\pi}{3}\right)-\cos\left(n_it_{i0}-2nt_0+\frac{\pi}{3}\right)\right)\right.\right.\\
&\left.-\mu(\cos((n_i-2n)t-n_it_{i0}+2nt_0)-\cos(-n_it_{i0}+2nt_0))\right\}-\frac{n_i-2n}{n_i}\left\{\left(\cos\left(n_it-n_it_{i0}+\frac{\pi}{3}\right)-\cos\left(n_it_{i0}-\frac{\pi}{3}\right)\right)\right.\\
&\left.\left.+\mu(\cos(n_it-n_it_{i0})-\cos(-n_it_{i0}))\right\}\right] \qquad \text{(B.}
\end{aligned}
$$

$\delta y_E^{(1)}$

$$
\begin{aligned}
= \quad & a_I \left[\frac{3e_I^2 \sin\varpi}{64n_I^2 + 16nn_I - 8n^2} \right. \\
& \left\{ \frac{(-2n_I - n)\sin(4n_I t_{I0} + nt_0)\sin(8n_I t_{I0} + (n - 4n_I)t)}{n - 4n_I} - \frac{(2n_I + n)\cos(4n_I t_{I0} + nt_0)\cos(8n_I t_{I0} + (n - 4n_I)t)}{n - 4n_I} \right. \\
& + \frac{(n - 4n_I)\sin(4n_I t_{I0} + nt_0)\sin(6n_I t_{I0} + 2nt_0 + (-2n_I - n)t)}{-2n_I - n} \\
& - \frac{(4n_I - n)\cos(4n_I t_{I0} + nt_0)\cos(6n_I t_{I0} + 2nt_0 + (-2n_I - n)t)}{-2n_I - n} \\
& + \left(\frac{(4n_I - n)\sin(2n_I t_{I0} + (2n_I + n)t)}{2n_I + n} + \frac{(2n_I + n)\sin(2nt_0 + (4n_I - n)t)}{4n_I - n} \right) \sin(4n_I t_{I0} + nt_0) \\
& \left. + \left(- \frac{(n - 4n_I)\cos(2n_I t_{I0} + (2n_I + n)t)}{2n_I + n} - \frac{(-2n_I - n)\cos(2nt_0 + (4n_I - n)t)}{4n_I - n} \right) \cos(4n_I t_{I0} + nt_0) \right\} \\
& - \frac{3e_I^2 \cos\varpi}{64n_I^2 + 16nn_I - 8n^2} \\
& \left\{ \frac{(2n_I + n)\cos(4n_I t_{I0} + nt_0)\sin(8n_I t_{I0} + (n - 4n_I)t)}{n - 4n_I} - \frac{(2n_I + n)\sin(4n_I t_{I0} + nt_0)\cos(8n_I t_{I0} + (n - 4n_I)t)}{n - 4n_I} \right. \\
& + \frac{(4n_I - n)\cos(4n_I t_{I0} + nt_0)\sin(6n_I t_{I0} + 2nt_0 + (-2n_I - n)t)}{-2n_I - n} \\
& - \frac{(4n_I - n)\sin(4n_I t_{I0} + nt_0)\cos(6n_I t_{I0} + 2nt_0 + (-2n_I - n)t)}{-2n_I - n} \\
& + \left(- \frac{(4n_I - n)\cos(2n_I t_{I0} + (2n_I + n)t)}{2n_I + n} - \frac{(2n_I + n)\cos(2nt_0 + (4n_I - n)t)}{4n_I - n} \right) \sin(4n_I t_{I0} + nt_0) \\
& \left. + \left(\frac{(4n_I - n)\sin(2n_I t_{I0} + (2n_I + n)t)}{2n_I + n} + \frac{(2n_I + n)\sin(2nt_0 + (4n_I - n)t)}{4n_I - n} \right) \cos(4n_I t_{I0} + nt_0) \right\} \\
& + \frac{e_I \sin\varpi}{48n_I^2 + 32nn_I - 16n^2} \\
& \left\{ \frac{(-n_I - n)\sin(3n_I t_{I0} + nt_0)\sin(6n_I t_{I0} + (n - 3n_I)t)}{n - 3n_I} - \frac{(n_I + n)\cos(3n_I t_{I0} + nt_0)\cos(6n_I t_{I0} + (n - 3n_I)t)}{n - 3n_I} \right. \\
& + \frac{(n - 3n_I)\sin(3n_I t_{I0} + nt_0)\sin(4n_I t_{I0} + 2nt_0 + (-n_I - n)t)}{-n_I - n} \\
& - \frac{(3n_I - n)\cos(3n_I t_{I0} + nt_0)\cos(4n_I t_{I0} + 2nt_0 + (-n_I - n)t)}{-n_I - n} \\
& + \left(\frac{(3n_I - n)\sin(2n_I t_{I0} + (n_I + n)t)}{n_I + n} + \frac{(n_I + n)\sin(2nt_0 + (3n_I - n)t)}{3n_I - n} \right) \sin(3n_I t_{I0} + nt_0) \\
& \left. + \left(- \frac{(n - 3n_I)\cos(2n_I t_{I0} + (n_I + n)t)}{n_I + n} - \frac{(-n_I - n)\cos(2nt_0 + (3n_I - n)t)}{3n_I - n} \right) \cos(3n_I t_{I0} + nt_0) \right\} \\
& + \frac{e_I \cos\varpi}{192n_I^2 + 128nn_I - 64n^2} \\
& \left\{ \frac{(n_I + n)\cos(3n_I t_{I0} + nt_0)\sin(6n_I t_{I0} + (n - 3n_I)t)}{n - 3n_I} - \frac{(n_I + n)\sin(3n_I t_{I0} + nt_0)\cos(6n_I t_{I0} + (n - 3n_I)t)}{n - 3n_I} \right. \\
& + \frac{(3n_I - n)\cos(3n_I t_{I0} + nt_0)\sin(4n_I t_{I0} + 2nt_0 + (-n_I - n)t)}{-n_I - n} \\
& - \frac{(3n_I - n)\sin(3n_I t_{I0} + nt_0)\cos(4n_I t_{I0} + 2nt_0 + (-n_I - n)t)}{-n_I - n} \\
& + \left(- \frac{(3n_I - n)\cos(2n_I t_{I0} + (n_I + n)t)}{n_I + n} - \frac{(n_I + n)\cos(2nt_0 + (3n_I - n)t)}{3n_I - n} \right) \sin(3n_I t_{I0} + nt_0) \\
& \left. + \left(\frac{(3n_I - n)\sin(2n_I t_{I0} + (n_I + n)t)}{n_I + n} + \frac{(n_I + n)\sin(2nt_0 + (3n_I - n)t)}{3n_I - n} \right) \cos(3n_I t_{I0} + nt_0) \right\} \\
& + \frac{5e_I^2 \sin\varpi}{128nn_I - 64n^2} \\
& \left\{ - \frac{n\sin(2n_I t_{I0} + nt_0)\sin(4n_I t_{I0} + (n - 2n_I)t)}{n - 2n_I} - \frac{n\cos(2n_I t_{I0} + nt_0)\cos(4n_I t_{I0} + (n - 2n_I)t)}{-2n - n_I} \right. \\
& - \frac{(n - 2n_I)\sin(2n_I t_{I0} + nt_0)\sin(2n_I t_{I0} + 2nt_0 - nt)}{n}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(2n_I-n)\cos(2n_It_{I0}+nt_0)\cos(2n_It_{I0}+2nt_0-nt)}{n}\\
&+\left(\frac{(2n_I-n)\sin(2n_It_{I0}+nt)}{n}+\frac{n\sin(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\sin(2n_It_{I0}+nt_0)\\
&+\left(\frac{n\cos(2nt_0+(2n_I-n)t)}{2n_I-n}-\frac{(n-2n_I)\cos(2n_It_{I0}+nt)}{n}\right)\cos(2n_It_{I0}+nt_0)\Big\}\\
&+\frac{\sin\varpi}{16nn_I-8n^2}\\
&\Big\{-\frac{n\sin(2n_It_{I0}+nt_0)\sin(4n_It_{I0}+(n-2n_I)t)}{n-2n_I}-\frac{n\cos(2n_It_{I0}+nt_0)\cos(4n_It_{I0}+(n-2n_I)t)}{n-2n_I}\\
&-\frac{(n-2n_I)\sin(2n_It_{I0}+nt_0)\sin(2n_It_{I0}+2nt_0-nt)}{n}\\
&+\frac{(2n_I-n)\cos(2n_It_{I0}+nt_0)\cos(2n_It_{I0}+2nt_0-nt)}{/}n\\
&+\left(\frac{(2n_I-n)\sin(2n_It_{I0}+nt)}{n}+\frac{n\sin(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\sin(2n_It_{I0}+nt_0)\\
&+\left(\frac{n\cos(2nt_0+(2n_I-n)t)}{2n_I-n}-\frac{(n-2n_I)\cos(2n_It_{I0}+nt)}{n}\right)\cos(2n_It_{I0}+nt_0)\Big\}\\
&-\frac{5e_I^2\cos\varpi}{128nn_I-64n^2}\Big\{\frac{n\cos(2n_It_{I0}+nt_0)\sin(4n_It_{I0}+(n-2n_I)t)}{n-2n_I}-\frac{n\sin(2n_It_{I0}+nt_0)\cos(4n_It_{I0}+(n-2n_I)t)}{n-2n_I}\\
&-\frac{(2n_I-n)\cos(2n_It_{I0}+nt_0)\sin(2n_It_{I0}+2nt_0-nt)}{n}\\
&+\frac{(2n_I-n)\sin(2n_It_{I0}+nt_0)\cos(2n_It_{I0}+2nt_0-nt)}{n}\\
&+\left(-\frac{(2n_I-n)\cos(2n_It_{I0}+nt)}{n}-\frac{n\cos(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\sin(2n_It_{I0}+nt_0)\\
&+\left(\frac{(2n_I-n)\sin(2n_It_{I0}+nt)}{n}+\frac{n\sin(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\cos(2n_It_{I0}+nt_0)\Big\}\\
&-\frac{\cos\varpi}{16nn_I-8n^2}\\
&\Big\{\frac{n\cos(2n_It_{I0}+nt_0)\sin(4n_It_{I0}+(n-2n_I)t)}{2n-n_I}-\frac{n\sin(2n_It_{I0}+nt_0)\cos(4n_It_{I0}+(n-2n_I)t)}{2n-n_I}\\
&-\frac{(2n_I-n)\cos(2n_It_{I0}+nt_0)\sin(2n_It_{I0}+2nt_0-nt)}{n}\\
&+\frac{(2n_I-n)\sin(2n_It_{I0}+nt_0)\cos(2n_It_{I0}+2nt_0-nt)}{n}\\
&+\left(-\frac{(2n_I-n)\cos(2n_It_{I0}+nt)}{n}-\frac{n\cos(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\sin(2n_It_{I0}+nt_0)\\
&+\left(\frac{(2n_I-n)\sin(2n_It_{I0}+nt)}{n}+\frac{n\sin(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\cos(2n_It_{I0}+nt_0)\Big\}\\
&-e_I\sin\varpi\frac{\cos(n_It_{I0}-nt_0-n_It+nt)}{2(n-n_I)^2}\Bigg]\\
&+\frac{a}{2(n_i^2-2nn_i)}\\
&\left[\frac{n_i}{n_i-2n}\left\{\left(\sin\left(\left(n_i-2n\right)t-n_it_{i0}+2nt_0-\frac{\pi}{3}\right)+\sin\left(n_it_{i0}-2nt_0+\frac{\pi}{3}\right)\right)\right.\right.\\
&\left.-\mu\left(\sin\left(\left(n_i-2n\right)t-n_it_{i0}+2nt_0\right)+\sin\left(n_it_{i0}-2nt_0\right)\right)\right\}+\frac{2n-n_i}{n_i}\left\{\left(\sin\left(n_it-n_it_{i0}+\frac{\pi}{3}\right)+\sin\left(n_it_{i0}-\frac{\pi}{3}\right)\right.\right.\\
&\left.\left.-\mu(\sin(n_it-n_it_{i0})+\sin(n_it_{i0}))\right\}\right] \qquad \text{(B.}
\end{aligned}
$$

B.3 Order of q^2

$$
\begin{aligned}
&\delta x_E^{(2)} \\
= \; & 45a_I \Bigg[\frac{3e_I^2 \sin\varpi}{240n_I^3 + 384nn_I^2 - 192n^2n_I} \\
& \Bigg\{ \frac{(-3n_I^2 - 6nn_I)\sin(5n_It_{I0} + 2nt_0)\sin(10n_It_{I0} + (2n - 5n_I)t)}{2n - 5n_I} \\
& - \frac{(3n_I^2 + 6nn_I)\cos(5n_It_{I0} + 2nt_0)\cos(10n_It_{I0} + (2n - 5n_I)t)}{2n - 5n_I} \\
& - \frac{(-10n_I^2 - 16nn_I + 8n^2)\sin(5n_It_{I0} + 2nt_0)\sin(8n_It_{I0} + 2nt_0 - 3n_It)}{3n_I} \\
& + \frac{(10n_I^2 + 16nn_I - 8n^2)\cos(5n_It_{I0} + 2nt_0)\cos(8n_It_{I0} + 2nt_0 - 3n_It)}{3n_I} \\
& + \frac{(6nn_I - 15n_I^2)\sin(5n_It_{I0} + 2nt_0)\sin(6n_It_{I0} + 4nt_0 + (-n_I - 2n)t)}{-n_I - 2n} \\
& - \frac{(15n_I^2 - 6nn_I)\cos(5n_It_{I0} + 2nt_0)\cos(6n_It_{I0} + 4nt_0 + (-n_I - 2n)t)}{-n_I - 2n} \\
& + \Bigg(\frac{(15n_I^2 - 6nn_I)\sin(4n_It_{I0} + (n_I + 2n)t)}{n_I + 2n} + \frac{(10n_I^2 + 16nn_I - 8n^2)\sin(2n_It_{I0} + 2nt_0 + 3n_It)}{3n_I} \\
& + \frac{(3n_I^2 + 6nn_I)\sin(4nt_0 + (5n_I - 2n)t)}{5n_I - 2n} \Bigg) \sin(5n_It_{I0} + 2nt_0) \\
& + \Bigg(-\frac{(6nn_I - 15n_I^2)\cos(4n_It_{I0} + (n_I + 2n)t)}{n_I + 2n} - \frac{(-10n_I^2 - 16nn_I + 8n^2)\cos(2n_It_{I0} + 2nt_0 + 3n_It)}{3n_I} \\
& - \frac{(-3n_I^2 - 6nn_I)\cos(4nt_0 + (5n_I - 2n)t)}{5n_I - 2n} \Bigg) \cos(5n_It_{I0} + 2nt_0) \Bigg\} \\
& - \frac{3e_I^2 \cos\varpi}{240n_I^3 + 384nn_I^2 - 192n^2n_I} \\
& \Bigg\{ \frac{(3n_I^2 + 6nn_I)\cos(5n_It_{I0} + 2nt_0)\sin(10n_It_{I0} + (2n - 5n_I)t)}{2n - 5n_I} \\
& - \frac{(3n_I^2 + 6nn_I)\sin(5n_It_{I0} + 2nt_0)\cos(10n_It_{I0} + (2n - 5n_I)t)}{2n - 5n_I} \\
& - \frac{(10n_I^2 + 16nn_I - 8n^2)\cos(5n_It_{I0} + 2nt_0)\sin(8n_It_{I0} + 2nt_0 - 3n_It)}{3n_I} \\
& + \frac{(10n_I^2 + 16nn_I - 8n^2)\sin(5n_It_{I0} + 2nt_0)\cos(8n_It_{I0} + 2nt_0 - 3n_It)}{3n_I} \\
& + \frac{(15n_I^2 - 6nn_I)\cos(5n_It_{I0} + 2nt_0)\sin(6n_It_{I0} + 4nt_0 + (-n_I - 2n)t)}{-n_I - 2n} \\
& - \frac{(15n_I^2 - 6nn_I)\sin(5n_It_{I0} + 2nt_0)\cos(6n_It_{I0} + 4nt_0 + (-n_I - 2n)t)}{-n_I - 2n} \\
& + \Bigg(-\frac{(15n_I^2 - 6nn_I)\cos(4n_It_{I0} + (n_I + 2n)t)}{n_I + 2n} - \frac{(10n_I^2 + 16nn_I - 8n^2)\cos(2n_It_{I0} + 2nt_0 + 3n_It)}{3n_I} \\
& - \frac{(3n_I^2 + 6nn_I)\cos(4nt_0 + (5n_I - 2n)t)}{5n_I - 2n} \Bigg) \sin(5n_It_{I0} + 2nt_0) \\
& + \Bigg(\frac{(15n_I^2 - 6nn_I)\sin(4n_It_{I0} + (n_I + 2n)t)}{n_I + 2n} + \frac{(10n_I^2 + 16nn_I - 8n^2)\sin(2n_It_{I0} + 2nt_0 + 3n_It)}{3n_I} \\
& + \frac{(3n_I^2 + 6nn_I)\sin(4nt_0 + (5n_I - 2n)t)}{5n_I - 2n} \Bigg) \cos(5n_It_{I0} + 2nt_0) \Bigg\} \\
& + \frac{e_I \sin\varpi}{128nn_I^2 - 64n^2n_I}
\end{aligned}
$$

$$
\begin{aligned}
&\Bigg\{-\frac{nn_I\sin(4n_It_{I0}+2nt_0)\sin(8n_It_{I0}+(2n-4n_I)t)}{2n-4n_I}\\
&-\frac{nn_I\cos(4n_It_{I0}+2nt_0)\cos(8n_It_{I0}+(2n-4n_I)t)}{2n-4n_I}\\
&-\frac{(2n^2-4nn_I)\sin(4n_It_{I0}+2nt_0)\sin(6n_It_{I0}+2nt_0-2n_It)}{2n_I}\\
&+\frac{(4nn_I-2n^2)\cos(4n_It_{I0}+2nt_0)\cos(6n_It_{I0}+2nt_0-2n_It)}{2n_I}\\
&-\frac{(nn_I-2n_I^2)\sin(4n_It_{I0}+2nt_0)\sin(4n_It_{I0}+4nt_0-2nt)}{2n}\\
&+\frac{(2n_I^2-nn_I)\cos(4n_It_{I0}+2nt_0)\cos(4n_It_{I0}+4nt_0-2nt)}{2n}\\
&+\left(\frac{(2n_I^2-nn_I)\sin(4n_It_{I0}+2nt)}{2n}+\frac{(4nn_I-2n^2)\sin(2n_It_{I0}+2nt_0+2n_It)}{2n_I}\right.\\
&\left.+\frac{nn_I\sin(4nt_0+(4n_I-2n)t)}{4n_I-2n}\right)\sin(4n_It_{I0}+2nt_0)\\
&+\left(-\frac{(nn_I-2n_I^2)\cos(4n_It_{I0}+2nt)}{2n}-\frac{(2n^2-4nn_I)\cos(2n_It_{I0}+2nt_0+2n_It)}{2n_I}\right.\\
&\left.+\frac{nn_I\cos(4nt_0+(4n_I-2n)t)}{4n_I-2n}\right)\cos(4n_It_{I0}+2nt_0)\Bigg\}\\
&+\frac{e_I\cos\varpi}{512nn_I^2-256n^2n_I}\\
&\Bigg\{\frac{nn_I\cos(4n_It_{I0}+2nt_0)\sin(8n_It_{I0}+(2n-4n_I)t)}{2n-4n_I}\\
&-\frac{nn_I\sin(4n_It_{I0}+2nt_0)\cos(8n_It_{I0}+(2n-4n_I)t)}{2n-4n_I}\\
&-\frac{(4nn_I-2n^2)\cos(4n_It_{I0}+2nt_0)\sin(6n_It_{I0}+2nt_0-2n_It)}{2n_I}\\
&+\frac{(4nn_I-2n^2)\sin(4n_It_{I0}+2nt_0)\cos(6n_It_{I0}+2nt_0-2n_It)}{2n_I}\\
&-\frac{(2n_I^2-nn_I)\cos(4n_It_{I0}+2nt_0)\sin(4n_It_{I0}+4nt_0-2nt)}{2n}\\
&+\frac{(2n_I^2-nn_I)\sin(4n_It_{I0}+2nt_0)\cos(4n_It_{I0}+4nt_0-2nt)}{2n}\\
&+\left(-\frac{(2n_I^2-nn_I)\cos(4n_It_{I0}+2nt)}{2n}-\frac{(4nn_I-2n^2)\cos(2n_It_{I0}+2nt_0+2n_It)}{2n_I}\right.\\
&\left.-\frac{nn_I\cos(4nt_0+(4n_I-2n)t)}{4n_I-2n}\right)\sin(4n_It_{I0}+2nt_0)\\
&+\left(\frac{(2n_I^2-nn_I)\sin(4n_It_{I0}+2nt)}{(2n}+\frac{(4nn_I-2n^2)\sin(2n_It_{I0}+2nt_0+2n_It)}{2n_I}\right.\\
&\left.+\frac{nn_I\sin(4nt_0+(4n_I-2n)t)}{4n_I-2n}\right)\cos(4n_It_{I0}+2nt_0)\Bigg\}\\
&+\frac{5e_I^2\sin\varpi}{384n_I^3-1024nn_I^2+512n^2n_I}\\
&\Bigg\{\frac{(2nn_I-n_I^2)\sin(3n_It_{I0}+2nt_0)\sin(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\\
&-\frac{(n_I^2-2nn_I)\cos(3n_It_{I0}+2nt_0)\cos(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\\
&-\frac{(-6n_I^2+16nn_I-8n^2)\sin(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+2nt_0-n_It)}{n_I}\\
&+\frac{(6n_I^2-16nn_I+8n^2)\cos(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+2nt_0-n_It)}{n_I}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(2nn_I-3n_I^2)\sin(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}\\
&-\frac{(3n_I^2-2nn_I)\cos(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}\\
&+\left(\frac{(3n_I^2-2nn_I)\sin(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}+\frac{(6n_I^2-16nn_I+8n^2)\sin(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.\\
&\left.+\frac{(n_I^2-2nn_I)\sin(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\sin(3n_It_{I0}+2nt_0)\\
&+\left(-\frac{(2nn_I-3n_I^2)\cos(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}-\frac{(-6n_I^2+16nn_I-8n^2)\cos(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.\\
&\left.-\frac{(2nn_I-n_I^2)\cos(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\cos(3n_It_{I0}+2nt_0)\Bigg\}\\
&+\frac{\sin\varpi}{48n_I^3-128nn_I^2+64n^2n_I}\\
&\Bigg\{\frac{(2nn_I-n_I^2)\sin(3n_It_{I0}+2nt_0)\sin(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\\
&-\frac{(n_I^2-2nn_I)\cos(3n_It_{I0}+2nt_0)\cos(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\\
&-\frac{(-6n_I^2+16nn_I-8n^2)\sin(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+2nt_0-n_It)}{n_I}\\
&+\frac{(6n_I^2-16nn_I+8n^2)\cos(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+2nt_0-n_It)}{n_I}\\
&+\frac{(2nn_I-3n_I^2)\sin(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}\\
&-\frac{(3n_I^2-2nn_I)\cos(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}\\
&+\left(\frac{(3n_I^2-2nn_I)\sin(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}+\frac{(6n_I^2-16nn_I+8n^2)\sin(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.\\
&\left.+\frac{(n_I^2-2nn_I)\sin(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\sin(3n_It_{I0}+2nt_0)\\
&+\left(-\frac{(2nn_I-3n_I^2)\cos(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}-\frac{(-6n_I^2+16nn_I-8n^2)\cos(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.\\
&\left.-\frac{(2nn_I-n_I^2)\cos(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\cos(3n_It_{I0}+2nt_0)\Bigg\}\\
&-\frac{5e_I^2\cos\varpi}{84n_I^3-1024nn_I^2+512n^2n_I}\\
&\Bigg\{\frac{(n_I^2-2nn_I)\cos(3n_It_{I0}+2nt_0)\sin(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\\
&-\frac{(n_I^2-2nn_I)\sin(3n_It_{I0}+2nt_0)\cos(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\\
&-\frac{(6n_I^2-16nn_I+8n^2)\cos(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+2nt_0-n_It)}{n_I}\\
&+\frac{(6n_I^2-16nn_I+8n^2)\sin(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+2nt_0-n_It)}{n_I}\\
&+\frac{(2nn_I-3n_I^2)\cos(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}\\
&-\frac{(2nn_I-3n_I^2)\sin(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}
\end{aligned}
$$

$$+\left(-\frac{(2nn_I-3n_I^2)\cos(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}-\frac{(6n_I^2-16nn_I+8n^2)\cos(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.$$

$$\left.-\frac{(n_I^2-2nn_I)\cos(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\sin(3n_It_{I0}+2nt_0)$$

$$+\left(\frac{(2nn_I-3n_I^2)\sin(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}+\frac{(6n_I^2-16nn_I+8n^2)\sin(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.$$

$$\left.\left.+\frac{(n_I^2-2nn_I)\sin(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\cos(3n_It_{I0}+2nt_0)\right\}$$

$$-\frac{\cos\varpi}{48n_I^3-128nn_I^2+64n^2n_I}$$

$$\left\{\frac{(n_I^2-2nn_I)\cos(3n_It_{I0}+2nt_0)\sin(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\right.$$

$$-\frac{(n_I^2-2nn_I)\sin(3n_It_{I0}+2nt_0)\cos(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}$$

$$-\frac{(6n_I^2-16nn_I+8n^2)\cos(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+2nt_0-n_It)}{n_I}$$

$$+\frac{(6n_I^2-16nn_I+8n^2)\sin(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+2nt_0-n_It)}{n_I}$$

$$+\frac{(2nn_I-3n_I^2)\cos(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}$$

$$-\frac{(2nn_I-3n_I^2)\sin(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}$$

$$+\left(-\frac{(2nn_I-3n_I^2)\cos(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}-\frac{(6n_I^2-16nn_I+8n^2)\cos(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.$$

$$\left.-\frac{(n_I^2-2nn_I)\cos(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\sin(3n_It_{I0}+2nt_0)$$

$$+\left(\frac{(2nn_I-3n_I^2)\sin(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}+\frac{(6n_I^2-16nn_I+8n^2)\sin(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.$$

$$\left.\left.+\frac{(n_I^2-2nn_I)\sin(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\cos(3n_It_{I0}+2nt_0)\right\}$$

$$-\frac{e_I\sin\varpi}{32n_I-32n}$$

$$\left(-\frac{\sin(2n_It_{I0}+2nt_0)\sin(4n_It_{I0}+(2n-2n_I)t)}{2n-2n_I}-\frac{\cos(2n_It_{I0}+2nt_0)\cos(4n_It_{I0}+(2n-2n_I)t)}{2n-2n_I}\right.$$

$$+\frac{(4n-4n_I)t^2\sin(2n_It_{I0}+2nt_0)^2}{2}+\frac{\sin(4nt_0+(2n_I-2n)t)\sin(2n_It_{I0}+2nt_0)}{2n_I-2n}$$

$$\left.+\frac{(4n-4n_I)t^2\cos(2n_It_{I0}+2nt_0)^2}{2}+\frac{\cos(4nt_0+(2n_I-2n)t)\cos(2n_It_{I0}+2nt_0)}{2n_I-2n}\right)$$

$$-e_I^2\cos\varpi\frac{\sin(3n_It_{I0}-3n_It)}{n_I^2}-e_I^2\sin\varpi\frac{\cos(3n_It_{I0}-3n_It)}{n_I^2}+3e_I\cos\varpi\frac{\sin(2n_It_{I0}-2n_It)}{32n_I^2}$$

$$-3e_I\sin\varpi\frac{\cos(2n_It_{I0}-2n_It)}{8n_I^2}-15e_I^2\cos\varpi\frac{\sin(n_It_{I0}-n_It)}{8n_I^2}-3\cos\varpi\frac{\sin(n_It_{I0}-n_It)}{n_I^2}$$

$$\left.-15e_I^2\sin\varpi\frac{\cos(n_It_{I0}-n_It)}{8n_I^2}-3\sin\varpi\frac{\cos(n_It_{I0}-n_It)}{n_I^2}\right]$$

$$+\frac{a}{4(4nn_i^2-8n^2n_i+3n^3)}$$

$$\left[-\frac{5(2nn_i-3n^2)}{(2n_i-n)}\right.$$

$$\left\{\left(\cos\left(\left(2n_i - n\right)t - 2n_i t_{i0} + nt_0 + \frac{\pi}{3}\right) - \cos\left(2n_i t_{i0} - nt_0 - \frac{\pi}{3}\right)\right)\right.$$

$$\left. - \mu\left(\cos((2n_i - n)t - 2n_i t_{i0} + nt_0) - \cos(2n_i t_{i0} - nt_0)\right)\right\}$$

$$- \frac{5(2nn_i - n^2)}{2n_i - 3n}$$

$$\left\{\left(\cos\left(\left(2n_i - 3n\right)t - 2n_i t_{i0} + 3nt_0 - \frac{\pi}{3}\right) - \cos\left(2n_i t_{i0} - 3nt_0 + \frac{\pi}{3}\right)\right)\right.$$

$$\left. - \mu(\cos((2n_i - 3n)t - 2n_i t_{i0} + 3nt_0) - \cos(2n_i t_{i0} - 3nt_0))\right\}$$

$$- \frac{6(4n_i^2 - 8nn_i + 3n^2)}{n}$$

$$\left.\left\{(\cos\left(nt - nt_0 + \frac{\pi}{3}\right) - \cos\left(nt_0 - \frac{\pi}{3}\right)) - \mu(\cos(nt - nt_0) - \cos(nt_0))\right\}\right] \tag{B.6}$$

$$\delta y_E^{(2)}$$

$$= 45a_I \left[\frac{3e_I^2 \sin \varpi}{240n_I^3 + 384nn_I^2 - 192n^2 n_I}\right.$$

$$\left\{\frac{(-3n_I^2 - 6nn_I)\sin(5n_I t_{I0} + 2nt_0)\sin(10n_I t_{I0} + (2n - 5n_I)t)}{2n - 5n_I}\right.$$

$$- \frac{(3n_I^2 + 6nn_I)\cos(5n_I t_{I0} + 2nt_0)\cos(10n_I t_{I0} + (2n - 5n_I)t)}{2n - 5n_I}$$

$$- \frac{(-10n_I^2 - 16nn_I + 8n^2)\sin(5n_I t_{I0} + 2nt_0)\sin(8n_I t_{I0} + 2nt_0 - 3n_I t)}{3n_I}$$

$$+ \frac{(10n_I^2 + 16nn_I - 8n^2)\cos(5n_I t_{I0} + 2nt_0)\cos(8n_I t_{I0} + 2nt_0 - 3n_I t)}{3n_I}$$

$$+ \frac{(6nn_I - 15n_I^2)\sin(5n_I t_{I0} + 2nt_0)\sin(6n_I t_{I0} + 4nt_0 + (-n_I - 2n)t)}{-n_I - 2n}$$

$$- \frac{(15n_I^2 - 6nn_I)\cos(5n_I t_{I0} + 2nt_0)\cos(6n_I t_{I0} + 4nt_0 + (-n_I - 2n)t)}{-n_I - 2n}$$

$$+ \left(\frac{(15n_I^2 - 6nn_I)\sin(4n_I t_{I0} + (n_I + 2n)t)}{n_I + 2n} + \frac{(10n_I^2 + 16nn_I - 8n^2)\sin(2n_I t_{I0} + 2nt_0 + 3n_I t)}{3n_I}\right.$$

$$\left. + \frac{(3n_I^2 + 6nn_I)\sin(4nt_0 + (5n_I - 2n)t)}{5n_I - 2n}\right)\sin(5n_I t_{I0} + 2nt_0)$$

$$+ \left(-\frac{(6nn_I - 15n_I^2)\cos(4n_I t_{I0} + (n_I + 2n)t)}{n_I + 2n} - \frac{(-10n_I^2 - 16nn_I + 8n^2)\cos(2n_I t_{I0} + 2nt_0 + 3n_I t)}{3n_I}\right.$$

$$\left.\left. - \frac{(-3n_I^2 - 6nn_I)\cos(4nt_0 + (5n_I - 2n)t)}{5n_I - 2n}\right)\cos(5n_I t_{I0} + 2nt_0)\right\}$$

$$- \frac{3e_I^2 \cos \varpi}{240n_I^3 + 384nn_I^2 - 192n^2 n_I}$$

$$\left\{\frac{(3n_I^2 + 6nn_I)\cos(5n_I t_{I0} + 2nt_0)\sin(10n_I t_{I0} + (2n - 5n_I)t)}{2n - 5n_I}\right.$$

$$- \frac{(3n_I^2 + 6nn_I)\sin(5n_I t_{I0} + 2nt_0)\cos(10n_I t_{I0} + (2n - 5n_I)t)}{2n - 5n_I}$$

$$- \frac{(10n_I^2 + 16nn_I - 8n^2)\cos(5n_I t_{I0} + 2nt_0)\sin(8n_I t_{I0} + 2nt_0 - 3n_I t)}{3n_I}$$

$$+\frac{(10n_I^2+16nn_I-8n^2)\sin(5n_It_{I0}+2nt_0)\cos(8n_It_{I0}+2nt_0-3n_It)}{3n_I}$$
$$+\frac{(15n_I^2-6nn_I)\cos(5n_It_{I0}+2nt_0)\sin(6n_It_{I0}+4nt_0+(-n_I-2n)t)}{-n_I-2n}$$
$$-\frac{(15n_I^2-6nn_I)\sin(5n_It_{I0}+2nt_0)\cos(6n_It_{I0}+4nt_0+(-n_I-2n)t)}{-n_I-2n}$$
$$+\left(-\frac{(15n_I^2-6nn_I)\cos(4n_It_{I0}+(n_I+2n)t)}{n_I+2n}-\frac{(10n_I^2+16nn_I-8n^2)\cos(2n_It_{I0}+2nt_0+3n_It)}{3n_I}\right.$$
$$\left.-\frac{(3n_I^2+6nn_I)\cos(4nt_0+(5n_I-2n)t)}{5n_I-2n}\right)\sin(5n_It_{I0}+2nt_0)$$
$$+\left(\frac{(15n_I^2-6nn_I)\sin(4n_It_{I0}+(n_I+2n)t)}{n_I+2n}+\frac{(10n_I^2+16nn_I-8n^2)\sin(2n_It_{I0}+2nt_0+3n_It)}{3n_I}\right.$$
$$\left.\left.+\frac{(3n_I^2+6nn_I)\sin(4nt_0+(5n_I-2n)t)}{5n_I-2n}\right)\cos(5n_It_{I0}+2nt_0)\right\}$$
$$+\frac{e_I\sin\varpi}{128nn_I^2-64n^2n_I}$$
$$\left\{-\frac{nn_I\sin(4n_It_{I0}+2nt_0)\sin(8n_It_{I0}+(2n-4n_I)t)}{2n-4n_I}\right.$$
$$-\frac{nn_I\cos(4n_It_{I0}+2nt_0)\cos(8n_It_{I0}+(2n-4n_I)t)}{2n-4n_I}$$
$$-\frac{(2n^2-4nn_I)\sin(4n_It_{I0}+2nt_0)\sin(6n_It_{I0}+2nt_0-2n_It)}{2n_I}$$
$$+\frac{(4nn_I-2n^2)\cos(4n_It_{I0}+2nt_0)\cos(6n_It_{I0}+2nt_0-2n_It)}{2n_I}$$
$$-\frac{(nn_I-2n_I^2)\sin(4n_It_{I0}+2nt_0)\sin(4n_It_{I0}+4nt_0-2nt)}{2n}$$
$$+\frac{(2n_I^2-nn_I)\cos(4n_It_{I0}+2nt_0)\cos(4n_It_{I0}+4nt_0-2nt)}{2n}$$
$$+\left(\frac{(2n_I^2-nn_I)\sin(4n_It_{I0}+2nt)}{2n}+\frac{(4nn_I-2n^2)\sin(2n_It_{I0}+2nt_0+2n_It)}{2n_I}\right.$$
$$\left.+\frac{nn_I\sin(4nt_0+(4n_I-2n)t)}{4n_I-2n}\right)\sin(4n_It_{I0}+2nt_0)$$
$$+\left(-\frac{(nn_I-2n_I^2)\cos(4n_It_{I0}+2nt)}{2n}-\frac{(2n^2-4nn_I)\cos(2n_It_{I0}+2nt_0+2n_It)}{2n_I}\right.$$
$$\left.\left.+\frac{nn_I\cos(4nt_0+(4n_I-2n)t)}{4n_I-2n}\right)\cos(4n_It_{I0}+2nt_0)\right\}$$
$$+\frac{e_I\cos\varpi}{512nn_I^2-256n^2n_I}$$
$$\left\{\frac{nn_I\cos(4n_It_{I0}+2nt_0)\sin(8n_It_{I0}+(2n-4n_I)t)}{2n-4n_I}\right.$$
$$-\frac{nn_I\sin(4n_It_{I0}+2nt_0)\cos(8n_It_{I0}+(2n-4n_I)t)}{2n-4n_I}$$
$$-\frac{(4nn_I-2n^2)\cos(4n_It_{I0}+2nt_0)\sin(6n_It_{I0}+2nt_0-2n_It)}{2n_I}$$
$$+\frac{(4nn_I-2n^2)\sin(4n_It_{I0}+2nt_0)\cos(6n_It_{I0}+2nt_0-2n_It)}{2n_I}$$
$$-\frac{(2n_I^2-nn_I)\cos(4n_It_{I0}+2nt_0)\sin(4n_It_{I0}+4nt_0-2nt)}{2n}$$
$$+\frac{(2n_I^2-nn_I)\sin(4n_It_{I0}+2nt_0)\cos(4n_It_{I0}+4nt_0-2nt)}{2n}$$
$$+\left(-\frac{(2n_I^2-nn_I)\cos(4n_It_{I0}+2nt)}{2n}-\frac{(4nn_I-2n^2)\cos(2n_It_{I0}+2nt_0+2n_It)}{2n_I}\right.$$

$$
\begin{aligned}
&-\frac{nn_I\cos(4nt_0+(4n_I-2n)t)}{4n_I-2n}\Bigg)\sin(4n_It_{I0}+2nt_0)\\
&+\Bigg(\frac{(2n_I^2-nn_I)\sin(4n_It_{I0}+2nt)}{(2n}+\frac{(4nn_I-2n^2)\sin(2n_It_{I0}+2nt_0+2n_It)}{2n_I}\\
&+\frac{nn_I\sin(4nt_0+(4n_I-2n)t)}{4n_I-2n}\Bigg)\cos(4n_It_{I0}+2nt_0)\Bigg\}\\
&+\frac{5e_I^2\sin\varpi}{384n_I^3-1024nn_I^2+512n^2n_I}\\
&\Bigg\{\frac{(2nn_I-n_I^2)\sin(3n_It_{I0}+2nt_0)\sin(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\\
&-\frac{(n_I^2-2nn_I)\cos(3n_It_{I0}+2nt_0)\cos(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\\
&-\frac{(-6n_I^2+16nn_I-8n^2)\sin(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+2nt_0-n_It)}{n_I}\\
&+\frac{(6n_I^2-16nn_I+8n^2)\cos(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+2nt_0-n_It)}{n_I}\\
&+\frac{(2nn_I-3n_I^2)\sin(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}\\
&-\frac{(3n_I^2-2nn_I)\cos(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}\\
&+\Bigg(\frac{(3n_I^2-2nn_I)\sin(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}+\frac{(6n_I^2-16nn_I+8n^2)\sin(2n_It_{I0}+2nt_0+n_It)}{n_I}\\
&+\frac{(n_I^2-2nn_I)\sin(4nt_0+(3n_I-2n)t)}{3n_I-2n}\Bigg)\sin(3n_It_{I0}+2nt_0)\\
&+\Bigg(-\frac{(2nn_I-3n_I^2)\cos(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}-\frac{(-6n_I^2+16nn_I-8n^2)\cos(2n_It_{I0}+2nt_0+n_It)}{n_I}\\
&-\frac{(2nn_I-n_I^2)\cos(4nt_0+(3n_I-2n)t)}{3n_I-2n}\Bigg)\cos(3n_It_{I0}+2nt_0)\Bigg\}\\
&+\frac{\sin\varpi}{48n_I^3-128nn_I^2+64n^2n_I}\\
&\Bigg\{\frac{(2nn_I-n_I^2)\sin(3n_It_{I0}+2nt_0)\sin(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\\
&-\frac{(n_I^2-2nn_I)\cos(3n_It_{I0}+2nt_0)\cos(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\\
&-\frac{(-6n_I^2+16nn_I-8n^2)\sin(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+2nt_0-n_It)}{n_I}\\
&+\frac{(6n_I^2-16nn_I+8n^2)\cos(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+2nt_0-n_It)}{n_I}\\
&+\frac{(2nn_I-3n_I^2)\sin(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}\\
&-\frac{(3n_I^2-2nn_I)\cos(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}\\
&+\Bigg(\frac{(3n_I^2-2nn_I)\sin(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}+\frac{(6n_I^2-16nn_I+8n^2)\sin(2n_It_{I0}+2nt_0+n_It)}{n_I}\\
&+\frac{(n_I^2-2nn_I)\sin(4nt_0+(3n_I-2n)t)}{3n_I-2n}\Bigg)\sin(3n_It_{I0}+2nt_0)\\
&+\Bigg(-\frac{(2nn_I-3n_I^2)\cos(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}-\frac{(-6n_I^2+16nn_I-8n^2)\cos(2n_It_{I0}+2nt_0+n_It)}{n_I}\\
&-\frac{(2nn_I-n_I^2)\cos(4nt_0+(3n_I-2n)t)}{3n_I-2n}\Bigg)\cos(3n_It_{I0}+2nt_0)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{5e_I^2\cos\varpi}{384n_I^3-1024nn_I^2+512n^2n_I}\\
&\left\{\frac{(n_I^2-2nn_I)\cos(3n_It_{I0}+2nt_0)\sin(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\right.\\
&-\frac{(n_I^2-2nn_I)\sin(3n_It_{I0}+2nt_0)\cos(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\\
&-\frac{(6n_I^2-16nn_I+8n^2)\cos(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+2nt_0-n_It)}{n_I}\\
&+\frac{(6n_I^2-16nn_I+8n^2)\sin(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+2nt_0-n_It)}{n_I}\\
&+\frac{(2nn_I-3n_I^2)\cos(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}\\
&-\frac{(2nn_I-3n_I^2)\sin(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}\\
&+\left(-\frac{(2nn_I-3n_I^2)\cos(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}-\frac{(6n_I^2-16nn_I+8n^2)\cos(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.\\
&\left.-\frac{(n_I^2-2nn_I)\cos(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\sin(3n_It_{I0}+2nt_0)\\
&+\left(\frac{(2nn_I-3n_I^2)\sin(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}+\frac{(6n_I^2-16nn_I+8n^2)\sin(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.\\
&\left.\left.+\frac{(n_I^2-2nn_I)\sin(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\cos(3n_It_{I0}+2nt_0)\right\}\\
&-\frac{\cos\varpi}{48n_I^3-128nn_I^2+64n^2n_I}\\
&\left\{\frac{(n_I^2-2nn_I)\cos(3n_It_{I0}+2nt_0)\sin(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\right.\\
&-\frac{(n_I^2-2nn_I)\sin(3n_It_{I0}+2nt_0)\cos(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\\
&-\frac{(6n_I^2-16nn_I+8n^2)\cos(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+2nt_0-n_It)}{n_I}\\
&+\frac{(6n_I^2-16nn_I+8n^2)\sin(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+2nt_0-n_It)}{n_I}\\
&+\frac{(2nn_I-3n_I^2)\cos(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}\\
&-\frac{(2nn_I-3n_I^2)\sin(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}\\
&+\left(-\frac{(2nn_I-3n_I^2)\cos(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}-\frac{(6n_I^2-16nn_I+8n^2)\cos(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.\\
&\left.-\frac{(n_I^2-2nn_I)\cos(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\sin(3n_It_{I0}+2nt_0)\\
&+\left(\frac{(2nn_I-3n_I^2)\sin(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}+\frac{(6n_I^2-16nn_I+8n^2)\sin(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.\\
&\left.\left.+\frac{(n_I^2-2nn_I)\sin(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\cos(3n_It_{I0}+2nt_0)\right\}\\
&-\frac{e_I\sin\varpi}{2n_I-32n}\\
&\left(-\frac{\sin(2n_It_{I0}+2nt_0)\sin(4n_It_{I0}+(2n-2n_I)t)}{2n-2n_I}-\frac{\cos(2n_It_{I0}+2nt_0)\cos(4n_It_{I0}+(2n-2n_I)t)}{2n-2n_I}\right.\\
&+\frac{(4n-4n_I)t^2\sin(2n_It_{I0}+2nt_0)^2}{2}+\frac{\sin(4nt_0+(2n_I-2n)t)\sin(2n_It_{I0}+2nt_0)}{2n_I-2n}
\end{aligned}
$$

$$+\frac{(4n-4n_I)t^2\cos(2n_It_{I0}+2nt_0)^2}{2}+\frac{\cos(4nt_0+(2n_I-2n)t)\cos(2n_It_{I0}+2nt_0)}{2n_I-2n}\Bigg)$$

$$-e_I^2\cos\varpi\frac{\sin(3n_It_{I0}-3n_It)}{n_I^2}-e_I^2\sin\varpi\frac{\cos(3n_It_{I0}-3n_It)}{n_I^2}+3e_I\cos\varpi\frac{\sin(2n_It_{I0}-2n_It)}{32n_I^2}$$

$$-3e_I\sin\varpi\frac{\cos(2n_It_{I0}-2n_It)}{8n_I^2}-15e_I^2\cos\varpi\frac{\sin(n_It_{I0}-n_It)}{8n_I^2}-3\cos\varpi\frac{\sin(n_It_{I0}-n_It)}{n_I^2}$$

$$-15e_I^2\sin\varpi\frac{\cos(n_It_{I0}-n_It)}{8n_I^2}-3\sin\varpi\frac{\cos(n_It_{I0}-n_It)}{n_I^2}\Bigg]$$

$$-\frac{a}{4(4nn_i^2-8n^2n_i+3n^3)}$$

$$\Bigg[\frac{5(2nn_i-3n^2)}{(2n_i-n)}$$

$$\Bigg\{\Bigg(\sin\Bigg(\Bigg(2n_i-n\Bigg)t-2n_it_{i0}+nt_0+\frac{\pi}{3}\Bigg)+\sin\Bigg(2n_it_{i0}-nt_0-\frac{\pi}{3}\Bigg)\Bigg)$$

$$-\mu\Bigg(\sin(2n_i-n)t-2n_it_{i0}+nt_0)-\sin(-2n_it_{i0}+nt_0)\Bigg)\Bigg\}$$

$$+\frac{5(n^2-2nn_i)}{(2n_i-3n)}$$

$$\Bigg\{\Bigg(\sin\Bigg(\Bigg(2n_i-3n\Bigg)t-2n_it_{i0}+3nt_0-\frac{\pi}{3}\Bigg)+\sin\Bigg(2n_it_{i0}-3nt_0+\frac{\pi}{3}\Bigg)\Bigg)$$

$$-\mu(\sin((2n_i-3n)t-2n_it_{i0}+3nt_0)-\sin(-2n_it_{i0}+3nt_0))\Bigg\}$$

$$+\frac{6(4n_i^2-8nn_i+3n^2)}{n}$$

$$\Bigg\{(\sin(nt-nt_0+\frac{\pi}{3})+\sin(nt_0-\frac{\pi}{3}))-\mu(\sin(nt-nt_0)+\sin(nt_0)\Bigg\}\Bigg] \tag{B.7}$$

B.4 Order of q^3

$$
\begin{aligned}
&\delta x_E^{(3)} \\
= \ & a_I \Bigg[\frac{3e_I^2 \sin\varpi}{32n_I^2 + 16nn_I - 8n^2} \\
&\left\{ \frac{(-2n_I - n)\sin(4n_I t_{I0} + nt_0)\sin(8n_I t_{I0} + (n - 4n_I)t)}{n - 4n_I} - \frac{(2n_I + n)\cos(4n_I t_{I0} + nt_0)\cos(8n_I t_{I0} + (n - 4n_I)t)}{n - 4n_I} \right. \\
&+ \frac{(n - 4n_I)\sin(4n_I t_{I0} + nt_0)\sin(6n_I t_{I0} + 2nt_0 + (-2n_I - n)t)}{-2n_I - n} \\
&- \frac{(4n_I - n)\cos(4n_I t_{I0} + nt_0)\cos(6n_I t_{I0} + 2nt_0 + (-2n_I - n)t)}{-2n_I - n} \\
&+ \left(\frac{(4n_I - n)\sin(2n_I t_{I0} + (2n_I + n)t)}{2n_I + n} + \frac{(2n_I + n)\sin(2nt_0 + (4n_I - n)t)}{4n_I - n} \right) \sin(4n_I t_{I0} + nt_0) \\
&\left. + \left(-\frac{(n - 4n_I)\cos(2n_I t_{I0} + (2n_I + n)t)}{2n_I + n} - \frac{(-2n_I - n)\cos(2nt_0 + (4n_I - n)t)}{4n_I - n} \right) \cos(4n_I t_{I0} + nt_0) \right\} \\
&- \frac{3e_I^2 \cos\varpi}{64 * n_I^2 + 16 * n * n_I - 8 * n^2} \\
&\left\{ \frac{(2n_I + n)\cos(4n_I t_{I0} + nt_0)\sin(8n_I t_{I0} + (n - 4n_I)t)}{n - 4n_I} - \frac{(2n_I + n)\sin(4n_I t_{I0} + nt_0)\cos(8n_I t_{I0} + (n - 4n_I)t)}{n - 4n_I} \right. \\
&+ \frac{(4n_I - n)\cos(4n_I t_{I0} + nt_0)\sin(6n_I t_{I0} + 2nt_0 + (-2n_I - n)t)}{-2n_I - n} \\
&- \frac{(4n_I - n)\sin(4n_I t_{I0} + nt_0)\cos(6n_I t_{I0} + 2nt_0 + (-2n_I - n)t)}{-2n_I - n} \\
&+ \left(-\frac{(4n_I - n)\cos(2n_I t_{I0} + (2n_I + n)t)}{2n_I + n} - \frac{(2n_I + n)\cos(2nt_0 + (4n_I - n)t)}{4n_I - n} \right) \sin(4n_I t_{I0} + nt_0) \\
&\left. + \left(\frac{(4n_I - n)\sin(2n_I t_{I0} + (2n_I + n)t)}{2n_I + n} + \frac{(2n_I + n)\sin(2nt_0 + (4n_I - n)t)}{4n_I - n} \right) \cos(4n_I t_{I0} + nt_0) \right\} \\
&+ \frac{e_I \sin\varpi}{48n_I^2 + 32nn_I - 16n^2} \\
&\left\{ \frac{(-n_I - n)\sin(3n_I t_{I0} + nt_0)\sin(6n_I t_{I0} + (n - 3n_I)t)}{n - 3n_I} - \frac{(n_I + n)\cos(3n_I t_{I0} + nt_0)\cos(6n_I t_{I0} + (n - 3n_I)t)}{n - 3n_I} \right. \\
&+ \frac{(n - 3n_I)\sin(3n_I t_{I0} + nt_0)\sin(4n_I t_{I0} + 2nt_0 + (-n_I - n)t)}{-n_I - n} \\
&- \frac{(3n_I - n)\cos(3n_I t_{I0} + nt_0)\cos(4n_I t_{I0} + 2nt_0 + (-n_I - n)t)}{-n_I - n} \\
&+ \left(\frac{(3n_I - n)\sin(2n_I t_{I0} + (n_I + n)t)}{n_I + n} + \frac{(n_I + n)\sin(2nt_0 + (3n_I - n)t)}{3n_I - n} \right) \sin(3n_I t_{I0} + nt_0) \\
&\left. + \left(-\frac{(n - 3n_I)\cos(2n_I t_{I0} + (n_I + n)t)}{n_I + n} - \frac{(-n_I - n)\cos(2nt_0 + (3n_I - n)t)}{3n_I - n} \right) \cos(3n_I t_{I0} + nt_0) \right\} \\
&+ \frac{e_I \cos\varpi}{192n_I^2 + 128nn_I - 64n^2} \\
&\left\{ \frac{(n_I + n)\cos(3n_I t_{I0} + nt_0)\sin(6n_I t_{I0} + (n - 3n_I)t)}{n - 3n_I} - \frac{(n_I + n)\sin(3n_I t_{I0} + nt_0)\cos(6n_I t_{I0} + (n - 3n_I)t)}{n - 3n_I} \right. \\
&+ \frac{(3n_I - n)\cos(3n_I t_{I0} + nt_0)\sin(4n_I t_{I0} + 2nt_0 + (-n_I - n)t)}{-n_I - n} \\
&- \frac{(3n_I - n)\sin(3n_I t_{I0} + nt_0)\cos(4n_I t_{I0} + 2nt_0 + (-n_I - n)t)}{-n_I - n} \\
&+ \left(-\frac{(3n_I - n)\cos(2n_I t_{I0} + (n_I + n)t)}{n_I + n} - \frac{(n_I + n)\cos(2nt_0 + (3n_I - n)t)}{3n_I - n} \right) \sin(3n_I t_{I0} + nt_0) \\
&\left. + \left(\frac{(3n_I - n)\sin(2n_I t_{I0} + (n_I + n)t)}{n_I + n} + \frac{(n_I + n)\sin(2nt_0 + (3n_I - n)t)}{3n_I - n} \right) \cos(3n_I t_{I0} + nt_0) \right\} \\
&+ \frac{5e_I^2 \sin\varpi}{128nn_I - 64n^2}
\end{aligned}
$$

$$
\begin{aligned}
&\Big\{-\frac{n\sin(2n_It_{I0}+nt_0)\sin(4n_It_{I0}+(n-2n_I)t)}{n-2n_I}-\frac{n\cos(2n_It_{I0}+nt_0)\cos(4n_It_{I0}+(n-2n_I)t)}{-2n-n_I}\\
&-\frac{(n-2n_I)\sin(2n_It_{I0}+nt_0)\sin(2n_It_{I0}+2nt_0-nt)}{n}\\
&+\frac{(2n_I-n)\cos(2n_It_{I0}+nt_0)\cos(2n_It_{I0}+2nt_0-nt)}{n}\\
&+\left(\frac{(2n_I-n)\sin(2n_It_{I0}+nt)}{n}+\frac{n\sin(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\sin(2n_It_{I0}+nt_0)\\
&+\left(\frac{n\cos(2nt_0+(2n_I-n)t)}{2n_I-n}-\frac{(n-2n_I)\cos(2n_It_{I0}+nt)}{n}\right)\cos(2n_It_{I0}+nt_0)\Big\}\\
&+\frac{\sin\varpi}{16nn_I-8n^2}\\
&\Big\{-\frac{n\sin(2n_It_{I0}+nt_0)\sin(4n_It_{I0}+(n-2n_I)t)}{n-2n_I}-\frac{n\cos(2n_It_{I0}+nt_0)\cos(4n_It_{I0}+(n-2n_I)t)}{n-2n_I}\\
&-\frac{(n-2n_I)\sin(2n_It_{I0}+nt_0)\sin(2n_It_{I0}+2nt_0-nt)}{n}\\
&+\frac{(2n_I-n)\cos(2n_It_{I0}+nt_0)\cos(2n_It_{I0}+2nt_0-nt)}{/}n\\
&+\left(\frac{(2n_I-n)\sin(2n_It_{I0}+nt)}{n}+\frac{n\sin(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\sin(2n_It_{I0}+nt_0)\\
&+\left(\frac{n\cos(2nt_0+(2n_I-n)t)}{2n_I-n}-\frac{(n-2n_I)\cos(2n_It_{I0}+nt)}{n}\right)\cos(2n_It_{I0}+nt_0)\Big\}\\
&-\frac{5e_I^2\cos\varpi}{128nn_I-64n^2}\Big\{\frac{n\cos(2n_It_{I0}+nt_0)\sin(4n_It_{I0}+(n-2n_I)t)}{n-2n_I}-\frac{n\sin(2n_It_{I0}+nt_0)\cos(4n_It_{I0}+(n-2n_I)t)}{n-2n_I}\\
&-\frac{(2n_I-n)\cos(2n_It_{I0}+nt_0)\sin(2n_It_{I0}+2nt_0-nt)}{n}\\
&+\frac{(2n_I-n)\sin(2n_It_{I0}+nt_0)\cos(2n_It_{I0}+2nt_0-nt)}{n}\\
&+\left(-\frac{(2n_I-n)\cos(2n_It_{I0}+nt)}{n}-\frac{n\cos(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\sin(2n_It_{I0}+nt_0)\\
&+\left(\frac{(2n_I-n)\sin(2n_It_{I0}+nt)}{n}+\frac{n\sin(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\cos(2n_It_{I0}+nt_0)\Big\}\\
&-\frac{\cos\varpi}{16nn_I-8n^2}\\
&\Big\{\frac{n\cos(2n_It_{I0}+nt_0)\sin(4n_It_{I0}+(n-2n_I)t)}{2n-n_I}-\frac{n\sin(2n_It_{I0}+nt_0)\cos(4n_It_{I0}+(n-2n_I)t)}{2n-n_I}\\
&-\frac{(2n_I-n)\cos(2n_It_{I0}+nt_0)\sin(2n_It_{I0}+2nt_0-nt)}{n}\\
&+\frac{(2n_I-n)\sin(2n_It_{I0}+nt_0)\cos(2n_It_{I0}+2nt_0-nt)}{n}\\
&+\left(-\frac{(2n_I-n)\cos(2n_It_{I0}+nt)}{n}-\frac{n\cos(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\sin(2n_It_{I0}+nt_0)\\
&+\left(\frac{(2n_I-n)\sin(2n_It_{I0}+nt)}{n}+\frac{n\sin(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\cos(2n_It_{I0}+nt_0)\Big\}\\
&-e_I\sin\varpi\frac{\cos(n_It_{I0}-nt_0-n_It+nt)}{2(n-n_I)^2}\Big]\\
&+\frac{a}{8(9n_i^4-36nn_i^3+44n^2n_i^2-16n^3n_i)}\\
&\Big[-\frac{7(3n_i^3-10nn_i^2+8n^2n_i)}{3n_i-2n}\\
&\Big\{\left(\cos\left(\left(3n_i-2n\right)t-3n_it_{i0}+2nt_0+\frac{\pi}{3}\right)-\cos\left(3n_it_{i0}-2nt_0-\frac{\pi}{3}\right)\right)\\
&-\mu(\cos((3n_i-2n)t-3n_it_{i0}+2nt_0)-\cos(-3n_it_{i0}+2nt_0))\Big\}
\end{aligned}
$$

$$-\frac{7(3n_i^3 - 8nn_i^2 + 4n^2n_i)}{(3n_i - 4n)}$$

$$\left\{\left(\cos\left(\left(3n_i - 4n\right)t - 3n_it_{i0} + 4nt_0 - \frac{\pi}{3}\right) - \cos\left(3n_it_{i0} - 4nt_0 + \frac{\pi}{3}\right)\right)\right.$$

$$\left. - \mu(\cos((3n_i - 4n)t - 3n_it_{i0} + 4nt_0) - \cos(-3n_it_{i0} + 4nt_0))\right\}$$

$$-\frac{9(9n_i^3 - 18nn_i^2 + 8n^2n_i)}{n}$$

$$\left\{\left(\cos\left(\left(n_i - 2n\right)t - n_it_{i0} + 2nt_0 - \frac{\pi}{3}\right) - \cos\left(n_it_{i0} - 2nt_0 + \frac{\pi}{3}\right)\right)\right.$$

$$\left. - \mu(\cos((n_i - 2n)t - n_it_{i0} + 2nt_0) - \cos(-n_it_{i0} + 2nt_0))\right\}$$

$$-\frac{9(9n_i^3 - 36nn_i^2 + 44n^2n_i - 16n^3)}{n_i}$$

$$\left\{\left(\cos\left(n_it - n_it_{i0} + \frac{\pi}{3}\right) - \cos\left(n_it_{i0} - \frac{\pi}{3}\right)\right)\right.$$

$$\left.\left. - \mu(\cos(n_it - n_it_{i0}) - \cos(n_it - n_it_{i0}))\right\}\right] \tag{B.8}$$

$$\delta y_E^{(3)}$$

$$= -a_I\left[\frac{3e_I^2 \sin\varpi}{64n_I^2 + 16nn_I - 8n^2}\right.$$

$$\left\{\frac{(-2n_I - n)\sin(4n_It_{I0} + nt_0)\sin(8n_It_{I0} + (n - 4n_I)t)}{n - 4n_I} - \frac{(2n_I + n)\cos(4n_It_{I0} + nt_0)\cos(8n_It_{I0} + (n - 4n_I)t)}{n - 4n_I}\right.$$

$$+\frac{(n - 4n_I)\sin(4n_It_{I0} + nt_0)\sin(6n_It_{I0} + 2nt_0 + (-2n_I - n)t)}{-2n_I - n}$$

$$-\frac{(4n_I - n)\cos(4n_It_{I0} + nt_0)\cos(6n_It_{I0} + 2nt_0 + (-2n_I - n)t)}{-2n_I - n}$$

$$+\left(\frac{(4n_I - n)\sin(2n_It_{I0} + (2n_I + n)t)}{2n_I + n} + \frac{(2n_I + n)\sin(2nt_0 + (4n_I - n)t)}{4n_I - n}\right)\sin(4n_It_{I0} + nt_0)$$

$$\left.+\left(-\frac{(n - 4n_I)\cos(2n_It_{I0} + (2n_I + n)t)}{2n_I + n} - \frac{(-2n_I - n)\cos(2nt_0 + (4n_I - n)t)}{4n_I - n}\right)\cos(4n_It_{I0} + nt_0)\right\}$$

$$-\frac{3e_I^2 \cos\varpi}{64n_I^2 + 16nn_I - 8n^2}$$

$$\left\{\frac{(2n_I + n)\cos(4n_It_{I0} + nt_0)\sin(8n_It_{I0} + (n - 4n_I)t)}{n - 4n_I} - \frac{(2n_I + n)\sin(4n_It_{I0} + nt_0)\cos(8n_It_{I0} + (n - 4n_I)t)}{n - 4n_I}\right.$$

$$+\frac{(4n_I - n)\cos(4n_It_{I0} + nt_0)\sin(6n_It_{I0} + 2nt_0 + (-2n_I - n)t)}{-2n_I - n}$$

$$-\frac{(4n_I - n)\sin(4n_It_{I0} + nt_0)\cos(6n_It_{I0} + 2nt_0 + (-2n_I - n)t)}{-2n_I - n}$$

$$+\left(-\frac{(4n_I - n)\cos(2n_It_{I0} + (2n_I + n)t)}{2n_I + n} - \frac{(2n_I + n)\cos(2nt_0 + (4n_I - n)t)}{4n_I - n}\right)\sin(4n_It_{I0} + nt_0)$$

$$\left.+\left(\frac{(4n_I - n)\sin(2n_It_{I0} + (2n_I + n)t)}{2n_I + n} + \frac{(2n_I + n)\sin(2nt_0 + (4n_I - n)t)}{4n_I - n}\right)\cos(4n_It_{I0} + nt_0)\right\}$$

$$+\frac{e_I \sin\varpi}{48n_I^2 + 32nn_I - 16n^2}$$

$$\left\{\frac{(-n_I - n)\sin(3n_It_{I0} + nt_0)\sin(6n_It_{I0} + (n - 3n_I)t)}{n - 3n_I} - \frac{(n_I + n)\cos(3n_It_{I0} + nt_0)\cos(6n_It_{I0} + (n - 3n_I)t)}{n - 3n_I}\right.$$

$$+\frac{(n - 3n_I)\sin(3n_It_{I0} + nt_0)\sin(4n_It_{I0} + 2nt_0 + (-n_I - n)t)}{-n_I - n}$$

$$-\frac{(3n_I - n)\cos(3n_I t_{I0} + nt_0)\cos(4n_I t_{I0} + 2nt_0 + (-n_I - n)t)}{-n_I - n}$$

$$+\left(\frac{(3n_I - n)\sin(2n_I t_{I0} + (n_I + n)t)}{n_I + n} + \frac{(n_I + n)\sin(2nt_0 + (3n_I - n)t)}{3n_I - n}\right)\sin(3n_I t_{I0} + nt_0)$$

$$+\left(-\frac{(n - 3n_I)\cos(2n_I t_{I0} + (n_I + n)t)}{n_I + n} - \frac{(-n_I - n)\cos(2nt_0 + (3n_I - n)t)}{3n_I - n}\right)\cos(3n_I t_{I0} + nt_0)\Bigg\}$$

$$+\frac{e_I \cos\varpi}{192n_I^2 + 128nn_I - 64n^2}$$

$$\Bigg\{\frac{(n_I + n)\cos(3n_I t_{I0} + nt_0)\sin(6n_I t_{I0} + (n - 3n_I)t)}{n - 3n_I} - \frac{(n_I + n)\sin(3n_I t_{I0} + nt_0)\cos(6n_I t_{I0} + (n - 3n_I)t)}{n - 3n_I}$$

$$+\frac{(3n_I - n)\cos(3n_I t_{I0} + nt_0)\sin(4n_I t_{I0} + 2nt_0 + (-n_I - n)t)}{-n_I - n}$$

$$-\frac{(3n_I - n)\sin(3n_I t_{I0} + nt_0)\cos(4n_I t_{I0} + 2nt_0 + (-n_I - n)t)}{-n_I - n}$$

$$+\left(-\frac{(3n_I - n)\cos(2n_I t_{I0} + (n_I + n)t)}{n_I + n} - \frac{(n_I + n)\cos(2nt_0 + (3n_I - n)t)}{3n_I - n}\right)\sin(3n_I t_{I0} + nt_0)$$

$$+\left(\frac{(3n_I - n)\sin(2n_I t_{I0} + (n_I + n)t)}{n_I + n} + \frac{(n_I + n)\sin(2nt_0 + (3n_I - n)t)}{3n_I - n}\right)\cos(3n_I t_{I0} + nt_0)\Bigg\}$$

$$+\frac{5e_I^2 \sin\varpi}{128nn_I - 64n^2}$$

$$\Bigg\{-\frac{n\sin(2n_I t_{I0} + nt_0)\sin(4n_I t_{I0} + (n - 2n_I)t)}{n - 2n_I} - \frac{n\cos(2n_I t_{I0} + nt_0)\cos(4n_I t_{I0} + (n - 2n_I)t)}{-2n - n_I}$$

$$-\frac{(n - 2n_I)\sin(2n_I t_{I0} + nt_0)\sin(2n_I t_{I0} + 2nt_0 - nt)}{n}$$

$$+\frac{(2n_I - n)\cos(2n_I t_{I0} + nt_0)\cos(2n_I t_{I0} + 2nt_0 - nt)}{n}$$

$$+\left(\frac{(2n_I - n)\sin(2n_I t_{I0} + nt)}{n} + \frac{n\sin(2nt_0 + (2n_I - n)t)}{2n_I - n}\right)\sin(2n_I t_{I0} + nt_0)$$

$$+\left(\frac{n\cos(2nt_0 + (2n_I - n)t)}{2n_I - n} - \frac{(n - 2n_I)\cos(2n_I t_{I0} + nt)}{n}\right)\cos(2n_I t_{I0} + nt_0)\Bigg\}$$

$$+\frac{\sin\varpi}{16nn_I - 8n^2}$$

$$\Bigg\{-\frac{n\sin(2n_I t_{I0} + nt_0)\sin(4n_I t_{I0} + (n - 2n_I)t)}{n - 2n_I} - \frac{n\cos(2n_I t_{I0} + nt_0)\cos(4n_I t_{I0} + (n - 2n_I)t)}{n - 2n_I}$$

$$-\frac{(n - 2n_I)\sin(2n_I t_{I0} + nt_0)\sin(2n_I t_{I0} + 2nt_0 - nt)}{n}$$

$$+\frac{(2n_I - n)\cos(2n_I t_{I0} + nt_0)\cos(2n_I t_{I0} + 2nt_0 - nt)}{/} n$$

$$+\left(\frac{(2n_I - n)\sin(2n_I t_{I0} + nt)}{n} + \frac{n\sin(2nt_0 + (2n_I - n)t)}{2n_I - n}\right)\sin(2n_I t_{I0} + nt_0)$$

$$+\left(\frac{n\cos(2nt_0 + (2n_I - n)t)}{2n_I - n} - \frac{(n - 2n_I)\cos(2n_I t_{I0} + nt)}{n}\right)\cos(2n_I t_{I0} + nt_0)\Bigg\}$$

$$-\frac{5e_I^2 \cos\varpi}{128nn_I - 64n^2}\Bigg\{\frac{n\cos(2n_I t_{I0} + nt_0)\sin(4n_I t_{I0} + (n - 2n_I)t)}{n - 2n_I} - \frac{n\sin(2n_I t_{I0} + nt_0)\cos(4n_I t_{I0} + (n - 2n_I)t)}{n - 2n_I}$$

$$-\frac{(2n_I - n)\cos(2n_I t_{I0} + nt_0)\sin(2n_I t_{I0} + 2nt_0 - nt)}{n}$$

$$+\frac{(2n_I - n)\sin(2n_I t_{I0} + nt_0)\cos(2n_I t_{I0} + 2nt_0 - nt)}{n}$$

$$+\left(-\frac{(2n_I - n)\cos(2n_I t_{I0} + nt)}{n} - \frac{n\cos(2nt_0 + (2n_I - n)t)}{2n_I - n}\right)\sin(2n_I t_{I0} + nt_0)$$

$$+\left(\frac{(2n_I - n)\sin(2n_I t_{I0} + nt)}{n} + \frac{n\sin(2nt_0 + (2n_I - n)t)}{2n_I - n}\right)\cos(2n_I t_{I0} + nt_0)\Bigg\}$$

$$-\frac{\cos\varpi}{16 * n * n_I - 8 * n^2}$$

$$\Bigg\{\frac{n\cos(2n_I t_{I0} + nt_0)\sin(4n_I t_{I0} + (n - 2n_I)t)}{2n - n_I} - \frac{n\sin(2n_I t_{I0} + nt_0)\cos(4n_I t_{I0} + (n - 2n_I)t)}{2n - n_I}$$

$$
\begin{aligned}
&-\frac{(2n_I - n)\cos(2n_I t_{I0} + nt_0)\sin(2n_I t_{I0} + 2nt_0 - nt)}{n} \\
&+\frac{(2n_I - n)\sin(2n_I t_{I0} + nt_0)\cos(2n_I t_{I0} + 2nt_0 - nt)}{n} \\
&+\left(-\frac{(2n_I - n)\cos(2n_I t_{I0} + nt)}{n} - \frac{n\cos(2nt_0 + (2n_I - n)t)}{2n_I - n}\right)\sin(2n_I t_{I0} + nt_0) \\
&+\left(\frac{(2n_I - n)\sin(2n_I t_{I0} + nt)}{n} + \frac{n\sin(2nt_0 + (2n_I - n)t)}{2n_I - n}\right)\cos(2n_I t_{I0} + nt_0)\Bigg\} \\
&-e_I \sin\varpi \frac{\cos(n_I t_{I0} - nt_0 - n_I t + nt)}{2(n - n_I)^2}\Bigg] \\
&-\frac{a}{8(9n_i^4 - 36nn_i^3 + 44n^2n_i^2 - 16n^3n_i)} \\
&\Bigg[\frac{7(3n_i^3 - 10nn_i^2 + 8n^2n_i)}{3n_i - 2n} \\
&\Bigg\{\left(\sin\left(\left(3n_i - 2n\right)t - 3n_i t_{i0} + 2nt_0 + \frac{\pi}{3}\right) + \sin\left(3n_i t_{i0} - 2nt_0 - \frac{\pi}{3}\right)\right) \\
&- \mu(\sin((3n_i - 2n)t - 3n_i t_{i0} + 2nt_0) - \sin(-3n_i t_{i0} + 2nt_0))\Bigg\} \\
&+\frac{7(-3n_i^3 + 8nn_i^2 - 4n^2n_i)}{3n_i - 4n} \\
&\Bigg\{\left(\sin\left(\left(3n_i - 4n\right)t - 3n_i t_{i0} + 4nt_0 - \frac{\pi}{3}\right) + \sin\left(3n_i t_{i0} - 4nt_0 + \frac{\pi}{3}\right)\right) \\
&- \mu(\sin((3n_i - 4n)t - 3n_i t_{i0} + 4nt_0) + \sin(3n_i t_{i0} - 4nt_0))\Bigg\} \\
&+\frac{9(-9n_i^3 + 18nn_i^2 - 8n^2n_i)}{n_i - 2n} \\
&\Bigg\{\left(\sin\left(\left(n_i - 2n\right)t - n_i t_{i0} + 2nt_0 - \frac{\pi}{3}\right) + \sin\left(n_i t_{i0} - 2nt_0 + \frac{\pi}{3}\right)\right) \\
&- \mu(\sin((n_i - 2n)t - n_i t_{i0} + 2nt_0) - \sin(-n_i t_{i0} + 2nt_0))\Bigg\} \\
&+\frac{9(9n_i^3 - 36nn_i^2 + 44n^2n_i - 16n^3)}{n_i} \\
&\Bigg\{\left(\sin\left(n_i t - n_i t_{i0} + \frac{\pi}{3}\right) + \sin\left(n_i t_{i0} - \frac{\pi}{3}\right)\right) \\
&- \mu(\sin(n_i t - n_i t_{i0}) - \sin(-n_i t_{i0}))\Bigg\}\Bigg]
\end{aligned} \tag{B.9}
$$

B.5 Order of q^4

$$
\begin{aligned}
&\delta x_E^{(4)} \\
= \; & a_I \left[\frac{3e_I^2 \sin\varpi}{240n_I^3 + 384nn_I^2 - 192n^2 n_I} \right. \\
&\left\{ \frac{(-3n_I^2 - 6nn_I)\sin(5n_I t_{I0} + 2nt_0)\sin(10n_I t_{I0} + (2n - 5n_I)t)}{2n - 5n_I} \right. \\
&- \frac{(3n_I^2 + 6nn_I)\cos(5n_I t_{I0} + 2nt_0)\cos(10n_I t_{I0} + (2n - 5n_I)t)}{2n - 5n_I} \\
&- \frac{(-10n_I^2 - 16nn_I + 8n^2)\sin(5n_I t_{I0} + 2nt_0)\sin(8n_I t_{I0} + 2nt_0 - 3n_I t)}{3n_I} \\
&+ \frac{(10n_I^2 + 16nn_I - 8n^2)\cos(5n_I t_{I0} + 2nt_0)\cos(8n_I t_{I0} + 2nt_0 - 3n_I t)}{3n_I} \\
&+ \frac{(6nn_I - 15n_I^2)\sin(5n_I t_{I0} + 2nt_0)\sin(6n_I t_{I0} + 4nt_0 + (-n_I - 2n)t)}{-n_I - 2n} \\
&- \frac{(15n_I^2 - 6nn_I)\cos(5n_I t_{I0} + 2nt_0)\cos(6n_I t_{I0} + 4nt_0 + (-n_I - 2n)t)}{-n_I - 2n} \\
&+ \left(\frac{(15n_I^2 - 6nn_I)\sin(4n_I t_{I0} + (n_I + 2n)t)}{n_I + 2n} + \frac{(10n_I^2 + 16nn_I - 8n^2)\sin(2n_I t_{I0} + 2nt_0 + 3n_I t)}{3n_I} \right. \\
&\left. + \frac{(3n_I^2 + 6nn_I)\sin(4nt_0 + (5n_I - 2n)t)}{5n_I - 2n} \right) \sin(5n_I t_{I0} + 2nt_0) \\
&+ \left(- \frac{(6nn_I - 15n_I^2)\cos(4n_I t_{I0} + (n_I + 2n)t)}{n_I + 2n} - \frac{(-10n_I^2 - 16nn_I + 8n^2)\cos(2n_I t_{I0} + 2nt_0 + 3n_I t)}{3n_I} \right. \\
&\left.\left. - \frac{(-3n_I^2 - 6nn_I)\cos(4nt_0 + (5n_I - 2n)t)}{5n_I - 2n} \right) \cos(5n_I t_{I0} + 2nt_0) \right\} \\
&- \frac{3e_I^2 \cos\varpi}{240n_I^3 + 384nn_I^2 - 192n^2 n_I} \\
&\left\{ \frac{(3n_I^2 + 6nn_I)\cos(5n_I t_{I0} + 2nt_0)\sin(10n_I t_{I0} + (2n - 5n_I)t)}{2n - 5n_I} \right. \\
&- \frac{(3n_I^2 + 6nn_I)\sin(5n_I t_{I0} + 2nt_0)\cos(10n_I t_{I0} + (2n - 5n_I)t)}{2n - 5n_I} \\
&- \frac{(10n_I^2 + 16nn_I - 8n^2)\cos(5n_I t_{I0} + 2nt_0)\sin(8n_I t_{I0} + 2nt_0 - 3n_I t)}{3n_I} \\
&+ \frac{(10n_I^2 + 16nn_I - 8n^2)\sin(5n_I t_{I0} + 2nt_0)\cos(8n_I t_{I0} + 2nt_0 - 3n_I t)}{3n_I} \\
&+ \frac{(15n_I^2 - 6nn_I)\cos(5n_I t_{I0} + 2nt_0)\sin(6n_I t_{I0} + 4nt_0 + (-n_I - 2n)t)}{-n_I - 2n} \\
&- \frac{(15n_I^2 - 6nn_I)\sin(5n_I t_{I0} + 2nt_0)\cos(6n_I t_{I0} + 4nt_0 + (-n_I - 2n)t)}{-n_I - 2n} \\
&+ \left(- \frac{(15n_I^2 - 6nn_I)\cos(4n_I t_{I0} + (n_I + 2n)t)}{n_I + 2n} - \frac{(10n_I^2 + 16nn_I - 8n^2)\cos(2n_I t_{I0} + 2nt_0 + 3n_I t)}{3n_I} \right. \\
&\left. - \frac{(3n_I^2 + 6nn_I)\cos(4nt_0 + (5n_I - 2n)t)}{5n_I - 2n} \right) \sin(5n_I t_{I0} + 2nt_0) \\
&+ \left(\frac{(15n_I^2 - 6nn_I)\sin(4n_I t_{I0} + (n_I + 2n)t)}{n_I + 2n} + \frac{(10n_I^2 + 16nn_I - 8n^2)\sin(2n_I t_{I0} + 2nt_0 + 3n_I t)}{3n_I} \right. \\
&\left.\left. + \frac{(3n_I^2 + 6nn_I)\sin(4nt_0 + (5n_I - 2n)t)}{5n_I - 2n} \right) \cos(5n_I t_{I0} + 2nt_0) \right\} \\
&+ \frac{e_I \sin\varpi}{128nn_I^2 - 64n^2 n_I}
\end{aligned}
$$

$$
\begin{aligned}
&\Bigg\{-\frac{nn_I\sin(4n_It_{I0}+2nt_0)\sin(8n_It_{I0}+(2n-4n_I)t)}{2n-4n_I}\\
&-\frac{nn_I\cos(4n_It_{I0}+2nt_0)\cos(8n_It_{I0}+(2n-4n_I)t)}{2n-4n_I}\\
&-\frac{(2n^2-4nn_I)\sin(4n_It_{I0}+2nt_0)\sin(6n_It_{I0}+2nt_0-2n_It)}{2n_I}\\
&+\frac{(4nn_I-2n^2)\cos(4n_It_{I0}+2nt_0)\cos(6n_It_{I0}+2nt_0-2n_It)}{2n_I}\\
&-\frac{(nn_I-2n_I^2)\sin(4n_It_{I0}+2nt_0)\sin(4n_It_{I0}+4nt_0-2nt)}{2n}\\
&+\frac{(2n_I^2-nn_I)\cos(4n_It_{I0}+2nt_0)\cos(4n_It_{I0}+4nt_0-2nt)}{2n}\\
&+\Bigg(\frac{(2n_I^2-nn_I)\sin(4n_It_{I0}+2nt)}{2n}+\frac{(4nn_I-2n^2)\sin(2n_It_{I0}+2nt_0+2n_It)}{2n_I}\\
&+\frac{nn_I\sin(4nt_0+(4n_I-2n)t)}{4n_I-2n}\Bigg)\sin(4n_It_{I0}+2nt_0)\\
&+\Bigg(-\frac{(nn_I-2n_I^2)\cos(4n_It_{I0}+2nt)}{2n}-\frac{(2n^2-4nn_I)\cos(2n_It_{I0}+2nt_0+2n_It)}{2n_I}\\
&+\frac{nn_I\cos(4nt_0+(4n_I-2n)t)}{4n_I-2n}\Bigg)\cos(4n_It_{I0}+2nt_0)\Bigg\}\\
&+\frac{e_I\cos\varpi}{512nn_I^2-256n^2n_I}\\
&\Bigg\{\frac{nn_I\cos(4n_It_{I0}+2nt_0)\sin(8n_It_{I0}+(2n-4n_I)t)}{2n-4n_I}\\
&-\frac{nn_I\sin(4n_It_{I0}+2nt_0)\cos(8n_It_{I0}+(2n-4n_I)t)}{2n-4n_I}\\
&-\frac{(4nn_I-2n^2)\cos(4n_It_{I0}+2nt_0)\sin(6n_It_{I0}+2nt_0-2n_It)}{2n_I}\\
&+\frac{(4nn_I-2n^2)\sin(4n_It_{I0}+2nt_0)\cos(6n_It_{I0}+2nt_0-2n_It)}{2n_I}\\
&-\frac{(2n_I^2-nn_I)\cos(4n_It_{I0}+2nt_0)\sin(4n_It_{I0}+4nt_0-2nt)}{2n}\\
&+\frac{(2n_I^2-nn_I)\sin(4n_It_{I0}+2nt_0)\cos(4n_It_{I0}+4nt_0-2nt)}{2n}\\
&+\Bigg(-\frac{(2n_I^2-nn_I)\cos(4n_It_{I0}+2nt)}{2n}-\frac{(4nn_I-2n^2)\cos(2n_It_{I0}+2nt_0+2n_It)}{2n_I}\\
&-\frac{nn_I\cos(4nt_0+(4n_I-2n)t)}{4n_I-2n}\Bigg)\sin(4n_It_{I0}+2nt_0)\\
&+\Bigg(\frac{(2n_I^2-nn_I)\sin(4n_It_{I0}+2nt)}{(2n}+\frac{(4nn_I-2n^2)\sin(2n_It_{I0}+2nt_0+2n_It)}{2n_I}\\
&+\frac{nn_I\sin(4nt_0+(4n_I-2n)t)}{4n_I-2n}\Bigg)\cos(4n_It_{I0}+2nt_0)\Bigg\}\\
&+\frac{5e_I^2\sin\varpi}{384n_I^3-1024nn_I^2+512n^2n_I}\\
&\Bigg\{\frac{(2nn_I-n_I^2)\sin(3n_It_{I0}+2nt_0)\sin(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\\
&-\frac{(n_I^2-2nn_I)\cos(3n_It_{I0}+2nt_0)\cos(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\\
&-\frac{(-6n_I^2+16nn_I-8n^2)\sin(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+2nt_0-n_It)}{n_I}\\
&+\frac{(6n_I^2-16nn_I+8n^2)\cos(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+2nt_0-n_It)}{n_I}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(2nn_I-3n_I^2)\sin(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}\\
&-\frac{(3n_I^2-2nn_I)\cos(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}\\
&+\left(\frac{(3n_I^2-2nn_I)\sin(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}+\frac{(6n_I^2-16nn_I+8n^2)\sin(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.\\
&\left.+\frac{(n_I^2-2nn_I)\sin(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\sin(3n_It_{I0}+2nt_0)\\
&+\left(-\frac{(2nn_I-3n_I^2)\cos(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}-\frac{(-6n_I^2+16nn_I-8n^2)\cos(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.\\
&\left.-\frac{(2nn_I-n_I^2)\cos(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\cos(3n_It_{I0}+2nt_0)\Bigg\}\\
&+\frac{\sin\varpi}{48n_I^3-128nn_I^2+64n^2n_I}\\
&\Bigg\{\frac{(2nn_I-n_I^2)\sin(3n_It_{I0}+2nt_0)\sin(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\\
&-\frac{(n_I^2-2nn_I)\cos(3n_It_{I0}+2nt_0)\cos(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\\
&-\frac{(-6n_I^2+16nn_I-8n^2)\sin(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+2nt_0-n_It)}{n_I}\\
&+\frac{(6n_I^2-16nn_I+8n^2)\cos(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+2nt_0-n_It)}{n_I}\\
&+\frac{(2nn_I-3n_I^2)\sin(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}\\
&-\frac{(3n_I^2-2nn_I)\cos(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}\\
&+\left(\frac{(3n_I^2-2nn_I)\sin(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}+\frac{(6n_I^2-16nn_I+8n^2)\sin(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.\\
&\left.+\frac{(n_I^2-2nn_I)\sin(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\sin(3n_It_{I0}+2nt_0)\\
&+\left(-\frac{(2nn_I-3n_I^2)\cos(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}-\frac{(-6n_I^2+16nn_I-8n^2)\cos(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.\\
&\left.-\frac{(2nn_I-n_I^2)\cos(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\cos(3n_It_{I0}+2nt_0)\Bigg\}\\
&-\frac{5e_I^2\cos\varpi}{384n_I^3-1024nn_I^2+512n^2n_I}\\
&\Bigg\{\frac{(n_I^2-2nn_I)\cos(3n_It_{I0}+2nt_0)\sin(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\\
&-\frac{(n_I^2-2nn_I)\sin(3n_It_{I0}+2nt_0)\cos(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\\
&-\frac{(6n_I^2-16nn_I+8n^2)\cos(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+2nt_0-n_It)}{n_I}\\
&+\frac{(6n_I^2-16nn_I+8n^2)\sin(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+2nt_0-n_It)}{n_I}\\
&+\frac{(2nn_I-3n_I^2)\cos(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}\\
&-\frac{(2nn_I-3n_I^2)\sin(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}
\end{aligned}
$$

$$+\left(-\frac{(2nn_I-3n_I^2)\cos(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}-\frac{(6n_I^2-16nn_I+8n^2)\cos(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.$$
$$\left.-\frac{(n_I^2-2nn_I)\cos(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\sin(3n_It_{I0}+2nt_0)$$
$$+\left(\frac{(2nn_I-3n_I^2)\sin(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}+\frac{(6n_I^2-16nn_I+8n^2)\sin(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.$$
$$\left.+\frac{(n_I^2-2nn_I)\sin(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\cos(3n_It_{I0}+2nt_0)\Bigg\}$$
$$-\frac{\cos\varpi}{48n_I^3-128nn_I^2+64n^2n_I}$$
$$\Bigg\{\frac{(n_I^2-2nn_I)\cos(3n_It_{I0}+2nt_0)\sin(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}$$
$$-\frac{(n_I^2-2nn_I)\sin(3n_It_{I0}+2nt_0)\cos(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}$$
$$-\frac{(6n_I^2-16nn_I+8n^2)\cos(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+2nt_0-n_It)}{n_I}$$
$$+\frac{(6n_I^2-16nn_I+8n^2)\sin(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+2nt_0-n_It)}{n_I}$$
$$+\frac{(2nn_I-3n_I^2)\cos(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}$$
$$-\frac{(2nn_I-3n_I^2)\sin(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}$$
$$+\left(-\frac{(2nn_I-3n_I^2)\cos(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}-\frac{(6n_I^2-16nn_I+8n^2)\cos(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.$$
$$\left.-\frac{(n_I^2-2nn_I)\cos(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\sin(3n_It_{I0}+2nt_0)$$
$$+\left(\frac{(2nn_I-3n_I^2)\sin(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}+\frac{(6n_I^2-16nn_I+8n^2)\sin(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.$$
$$\left.+\frac{(n_I^2-2nn_I)\sin(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\cos(3n_It_{I0}+2nt_0)\Bigg\}$$
$$-\frac{e_I\sin\varpi}{32n_I-32n}$$
$$\left(-\frac{\sin(2n_It_{I0}+2nt_0)\sin(4n_It_{I0}+(2n-2n_I)t)}{2n-2n_I}-\frac{\cos(2n_It_{I0}+2nt_0)\cos(4n_It_{I0}+(2n-2n_I)t)}{2n-2n_I}\right.$$
$$+\frac{(4n-4n_I)t^2\sin(2n_It_{I0}+2nt_0)^2}{2}+\frac{\sin(4nt_0+(2n_I-2n)t)\sin(2n_It_{I0}+2nt_0)}{2n_I-2n}$$
$$\left.+\frac{(4n-4n_I)t^2\cos(2n_It_{I0}+2nt_0)^2}{2}+\frac{\cos(4nt_0+(2n_I-2n)t)\cos(2n_It_{I0}+2nt_0)}{2n_I-2n}\right)$$
$$-e_I^2\cos\varpi\frac{\sin(3n_It_{I0}-3n_It)}{n_I^2}-e_I^2\sin\varpi\frac{\cos(3n_It_{I0}-3n_It)}{n_I^2}+3e_I\cos\varpi\frac{\sin(2n_It_{I0}-2n_It)}{32n_I^2}$$
$$-3e_I\sin\varpi\frac{\cos(2n_It_{I0}-2n_It)}{8n_I^2}-15e_I^2\cos\varpi\frac{\sin(n_It_{I0}-n_It)}{8n_I^2}-3\cos\varpi\frac{\sin(n_It_{I0}-n_It)}{n_I^2}$$
$$-15e_I^2\sin\varpi\frac{\cos(n_It_{I0}-n_It)}{8n_I^2}\;-3\sin\varpi\frac{\cos(n_It_{I0}-n_It)}{n_I^2}\Bigg]$$
$$+\frac{a}{16(64nn_i^4-256n^2n_i^3+364n^3n_i^2-216n^4n_i+45n^5)}$$
$$\Bigg[-\frac{21(16nn_i^3-52n^2n_i^2+52n^3n_i-15n^4)}{4n_i-3n}$$

$$\left\{\left(\cos\left(\left(4n_i-3n\right)t-4n_it_{i0}+3nt_0+\frac{\pi}{3}\right)-\cos\left(4n_it_{i0}-3nt_0-\frac{\pi}{3}\right)\right)\right.$$

$$\left.-\mu(\cos((4n_i-3n)t-4n_it_{i0}+3nt_0)-\cos(-4n_it_{i0}+3nt_0))\right\}$$

$$-\frac{21(16nn_i^3-44n^2n_i^2+36n^3n_i-9n^4)}{4n_i-5n}$$

$$\left\{\left(\cos\left(\left(4n_i-5n\right)t-4n_it_{i0}+5nt_0-\frac{\pi}{3}\right)-\cos\left(4n_it_{i0}-5nt_0+\frac{\pi}{3}\right)\right)\right.$$

$$\left.-\mu(\cos((4n_i-5n)t-4n_it_{i0}+5nt_0)-\cos(-4n_it_{i0}+5nt_0))\right\}$$

$$-\frac{28(32nn_i^3-112n^2n_i^2+126n^3n_i-45n^4}{2n_i-n}$$

$$\left\{\left(\cos\left(\left(2n_i-n\right)t-2n_it_{i0}+nt_0+\frac{\pi}{3}\right)-\cos\left(2n_it_{i0}-nt_0-\frac{\pi}{3}\right)\right)\right.$$

$$\left.-\mu(\cos((2n_i-n)t-2n_it_{i0}+nt_0)-\cos(-2n_it_{i0}+nt_0))\right\}$$

$$-\frac{28(32nn_i^3-80n^2n_i^2+62n^3n_i-15n^4)}{2n_i-3n}$$

$$\left\{\left(\cos\left(\left(2n_i-3n\right)t-2n_it_{i0}+3nt_0-\frac{\pi}{3}\right)-\cos\left(2n_it_{i0}-3nt_0+\frac{\pi}{3}\right)\right)\right.$$

$$\left.-\mu(\cos((2n_i-3n)t-2n_it_{i0}+3nt_0)-\cos(-2n_it_{i0}+3nt_0))\right\}$$

$$-\frac{30(64n_i^4-256nn_i^3+364n^2n_i^2-216n^3n_i+45n^4)}{n}$$

$$\left\{\left(\cos\left(nt-nt_0+\frac{\pi}{3}\right)-\cos\left(nt_0-\frac{\pi}{3}\right)\right)\right.$$

$$\left.\left.-\mu(\cos(nt-nt_0)-\cos(-nt_0))\right\}\right] \tag{B.10}$$

$$\delta y_E^{(4)}$$

$$= a_I\left[\frac{3e_I^2\sin\varpi}{240n_I^3+384nn_I^2-192n^2n_I}\right.$$

$$\left\{\frac{(-3n_I^2-6nn_I)\sin(5n_It_{I0}+2nt_0)\sin(10n_It_{I0}+(2n-5n_I)t)}{2n-5n_I}\right.$$

$$-\frac{(3n_I^2+6nn_I)\cos(5n_It_{I0}+2nt_0)\cos(10n_It_{I0}+(2n-5n_I)t)}{2n-5n_I}$$

$$-\frac{(-10n_I^2-16nn_I+8n^2)\sin(5n_It_{I0}+2nt_0)\sin(8n_It_{I0}+2nt_0-3n_It)}{3n_I}$$

$$+\frac{(10n_I^2+16nn_I-8n^2)\cos(5n_It_{I0}+2nt_0)\cos(8n_It_{I0}+2nt_0-3n_It)}{3n_I}$$

$$+\frac{(6nn_I-15n_I^2)\sin(5n_It_{I0}+2nt_0)\sin(6n_It_{I0}+4nt_0+(-n_I-2n)t)}{-n_I-2n}$$

$$-\frac{(15n_I^2-6nn_I)\cos(5n_It_{I0}+2nt_0)\cos(6n_It_{I0}+4nt_0+(-n_I-2n)t)}{-n_I-2n}$$

$$
\begin{aligned}
&+\left(\frac{(15n_I^2-6nn_I)\sin(4n_It_{I0}+(n_I+2n)t)}{n_I+2n}+\frac{(10n_I^2+16nn_I-8n^2)\sin(2n_It_{I0}+2nt_0+3n_It)}{3n_I}\right.\\
&\left.+\frac{(3n_I^2+6nn_I)\sin(4nt_0+(5n_I-2n)t)}{5n_I-2n}\right)\sin(5n_It_{I0}+2nt_0)\\
&+\left(-\frac{(6nn_I-15n_I^2)\cos(4n_It_{I0}+(n_I+2n)t)}{n_I+2n}-\frac{(-10n_I^2-16nn_I+8n^2)\cos(2n_It_{I0}+2nt_0+3n_It)}{3n_I}\right.\\
&\left.\left.-\frac{(-3n_I^2-6nn_I)\cos(4nt_0+(5n_I-2n)t)}{5n_I-2n}\right)\cos(5n_It_{I0}+2nt_0)\right\}\\
&-\frac{3e_I^2\cos\varpi}{240n_I^3+384nn_I^2-192n^2n_I}\\
&\left\{\frac{(3n_I^2+6nn_I)\cos(5n_It_{I0}+2nt_0)\sin(10n_It_{I0}+(2n-5n_I)t)}{2n-5n_I}\right.\\
&-\frac{(3n_I^2+6nn_I)\sin(5n_It_{I0}+2nt_0)\cos(10n_It_{I0}+(2n-5n_I)t)}{2n-5n_I}\\
&-\frac{(10n_I^2+16nn_I-8n^2)\cos(5n_It_{I0}+2nt_0)\sin(8n_It_{I0}+2nt_0-3n_It)}{3n_I}\\
&+\frac{(10n_I^2+16nn_I-8n^2)\sin(5n_It_{I0}+2nt_0)\cos(8n_It_{I0}+2nt_0-3n_It)}{3n_I}\\
&+\frac{(15n_I^2-6nn_I)\cos(5n_It_{I0}+2nt_0)\sin(6n_It_{I0}+4nt_0+(-n_I-2n)t)}{-n_I-2n}\\
&-\frac{(15n_I^2-6nn_I)\sin(5n_It_{I0}+2nt_0)\cos(6n_It_{I0}+4nt_0+(-n_I-2n)t)}{-n_I-2n}\\
&+\left(-\frac{(15n_I^2-6nn_I)\cos(4n_It_{I0}+(n_I+2n)t)}{n_I+2n}-\frac{(10n_I^2+16nn_I-8n^2)\cos(2n_It_{I0}+2nt_0+3n_It)}{3n_I}\right.\\
&\left.-\frac{(3n_I^2+6nn_I)\cos(4nt_0+(5n_I-2n)t)}{5n_I-2n}\right)\sin(5n_It_{I0}+2nt_0)\\
&+\left(\frac{(15n_I^2-6nn_I)\sin(4n_It_{I0}+(n_I+2n)t)}{n_I+2n}+\frac{(10n_I^2+16nn_I-8n^2)\sin(2n_It_{I0}+2nt_0+3n_It)}{3n_I}\right.\\
&\left.\left.+\frac{(3n_I^2+6nn_I)\sin(4nt_0+(5n_I-2n)t)}{5n_I-2n}\right)\cos(5n_It_{I0}+2nt_0)\right\}\\
&+\frac{e_I\sin\varpi}{128nn_I^2-64n^2n_I}\\
&\left\{-\frac{nn_I\sin(4n_It_{I0}+2nt_0)\sin(8n_It_{I0}+(2n-4n_I)t)}{2n-4n_I}\right.\\
&-\frac{nn_I\cos(4n_It_{I0}+2nt_0)\cos(8n_It_{I0}+(2n-4n_I)t)}{2n-4n_I}\\
&-\frac{(2n^2-4nn_I)\sin(4n_It_{I0}+2nt_0)\sin(6n_It_{I0}+2nt_0-2n_It)}{2n_I}\\
&+\frac{(4nn_I-2n^2)\cos(4n_It_{I0}+2nt_0)\cos(6n_It_{I0}+2nt_0-2n_It)}{2n_I}\\
&-\frac{(nn_I-2n_I^2)\sin(4n_It_{I0}+2nt_0)\sin(4n_It_{I0}+4nt_0-2nt)}{2n}\\
&+\frac{(2n_I^2-nn_I)\cos(4n_It_{I0}+2nt_0)\cos(4n_It_{I0}+4nt_0-2nt)}{2n}\\
&+\left(\frac{(2n_I^2-nn_I)\sin(4n_It_{I0}+2nt)}{2n}+\frac{(4nn_I-2n^2)\sin(2n_It_{I0}+2nt_0+2n_It)}{2n_I}\right.\\
&\left.+\frac{nn_I\sin(4nt_0+(4n_I-2n)t)}{4n_I-2n}\right)\sin(4n_It_{I0}+2nt_0)\\
&+\left(-\frac{(nn_I-2n_I^2)\cos(4n_It_{I0}+2nt)}{2n}-\frac{(2n^2-4nn_I)\cos(2n_It_{I0}+2nt_0+2n_It)}{2n_I}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{nn_I\cos(4nt_0+(4n_I-2n)t)}{4n_I-2n}\Bigg)\cos(4n_It_{I0}+2nt_0)\Bigg\}\\
&+\frac{e_I\cos\varpi}{512nn_I^2-256n^2n_I}\\
&\Bigg\{\frac{nn_I\cos(4n_It_{I0}+2nt_0)\sin(8n_It_{I0}+(2n-4n_I)t)}{2n-4n_I}\\
&-\frac{nn_I\sin(4n_It_{I0}+2nt_0)\cos(8n_It_{I0}+(2n-4n_I)t)}{2n-4n_I}\\
&-\frac{(4nn_I-2n^2)\cos(4n_It_{I0}+2nt_0)\sin(6n_It_{I0}+2nt_0-2n_It)}{2n_I}\\
&+\frac{(4nn_I-2n^2)\sin(4n_It_{I0}+2nt_0)\cos(6n_It_{I0}+2nt_0-2n_It)}{2n_I}\\
&-\frac{(2n_I^2-nn_I)\cos(4n_It_{I0}+2nt_0)\sin(4n_It_{I0}+4nt_0-2nt)}{2n}\\
&+\frac{(2n_I^2-nn_I)\sin(4n_It_{I0}+2nt_0)\cos(4n_It_{I0}+4nt_0-2nt)}{2n}\\
&+\Bigg(-\frac{(2n_I^2-nn_I)\cos(4n_It_{I0}+2nt)}{2n}-\frac{(4nn_I-2n^2)\cos(2n_It_{I0}+2nt_0+2n_It)}{2n_I}\\
&-\frac{nn_I\cos(4nt_0+(4n_I-2n)t)}{4n_I-2n}\Bigg)\sin(4n_It_{I0}+2nt_0)\\
&+\Bigg(\frac{(2n_I^2-nn_I)\sin(4n_It_{I0}+2nt)}{(2n}+\frac{(4nn_I-2n^2)\sin(2n_It_{I0}+2nt_0+2n_It)}{2n_I}\\
&+\frac{nn_I\sin(4nt_0+(4n_I-2n)t)}{4n_I-2n}\Bigg)\cos(4n_It_{I0}+2nt_0)\Bigg\}\\
&+\frac{5e_I^2\sin\varpi}{384n_I^3-1024nn_I^2+512n^2n_I}\\
&\Bigg\{\frac{(2nn_I-n_I^2)\sin(3n_It_{I0}+2nt_0)\sin(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\\
&-\frac{(n_I^2-2nn_I)\cos(3n_It_{I0}+2nt_0)\cos(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\\
&-\frac{(-6n_I^2+16nn_I-8n^2)\sin(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+2nt_0-n_It)}{n_I}\\
&+\frac{(6n_I^2-16nn_I+8n^2)\cos(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+2nt_0-n_It)}{n_I}\\
&+\frac{(2nn_I-3n_I^2)\sin(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}\\
&-\frac{(3n_I^2-2nn_I)\cos(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}\\
&+\Bigg(\frac{(3n_I^2-2nn_I)\sin(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}+\frac{(6n_I^2-16nn_I+8n^2)\sin(2n_It_{I0}+2nt_0+n_It)}{n_I}\\
&+\frac{(n_I^2-2nn_I)\sin(4nt_0+(3n_I-2n)t)}{3n_I-2n}\Bigg)\sin(3n_It_{I0}+2nt_0)\\
&+\Bigg(-\frac{(2nn_I-3n_I^2)\cos(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}-\frac{(-6n_I^2+16nn_I-8n^2)\cos(2n_It_{I0}+2nt_0+n_It)}{n_I}\\
&-\frac{(2nn_I-n_I^2)\cos(4nt_0+(3n_I-2n)t)}{3n_I-2n}\Bigg)\cos(3n_It_{I0}+2nt_0)\Bigg\}\\
&+\frac{\sin\varpi}{48n_I^3-128nn_I^2+64n^2n_I}\\
&\Bigg\{\frac{(2nn_I-n_I^2)\sin(3n_It_{I0}+2nt_0)\sin(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{(n_I^2-2nn_I)\cos(3n_It_{I0}+2nt_0)\cos(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\\
&-\frac{(-6n_I^2+16nn_I-8n^2)\sin(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+2nt_0-n_It)}{n_I}\\
&+\frac{(6n_I^2-16nn_I+8n^2)\cos(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+2nt_0-n_It)}{n_I}\\
&+\frac{(2nn_I-3n_I^2)\sin(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}\\
&-\frac{(3n_I^2-2nn_I)\cos(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}\\
&+\left(\frac{(3n_I^2-2nn_I)\sin(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}+\frac{(6n_I^2-16nn_I+8n^2)\sin(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.\\
&\left.+\frac{(n_I^2-2nn_I)\sin(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\sin(3n_It_{I0}+2nt_0)\\
&+\left(-\frac{(2nn_I-3n_I^2)\cos(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}-\frac{(-6n_I^2+16nn_I-8n^2)\cos(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.\\
&\left.-\frac{(2nn_I-n_I^2)\cos(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\cos(3n_It_{I0}+2nt_0)\Bigg\}\\
&-\frac{5e_I^2\cos\varpi}{384n_I^3-1024nn_I^2+512n^2n_I}\\
&\Bigg\{\frac{(n_I^2-2nn_I)\cos(3n_It_{I0}+2nt_0)\sin(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\\
&-\frac{(n_I^2-2nn_I)\sin(3n_It_{I0}+2nt_0)\cos(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\\
&-\frac{(6n_I^2-16nn_I+8n^2)\cos(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+2nt_0-n_It)}{n_I}\\
&+\frac{(6n_I^2-16nn_I+8n^2)\sin(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+2nt_0-n_It)}{n_I}\\
&+\frac{(2nn_I-3n_I^2)\cos(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}\\
&-\frac{(2nn_I-3n_I^2)\sin(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}\\
&+\left(-\frac{(2nn_I-3n_I^2)\cos(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}-\frac{(6n_I^2-16nn_I+8n^2)\cos(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.\\
&\left.-\frac{(n_I^2-2nn_I)\cos(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\sin(3n_It_{I0}+2nt_0)\\
&+\left(\frac{(2nn_I-3n_I^2)\sin(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}+\frac{(6n_I^2-16nn_I+8n^2)\sin(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.\\
&\left.+\frac{(n_I^2-2nn_I)\sin(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\cos(3n_It_{I0}+2nt_0)\Bigg\}\\
&-\frac{\cos\varpi}{48n_I^3-128nn_I^2+64n^2n_I}\\
&\Bigg\{\frac{(n_I^2-2nn_I)\cos(3n_It_{I0}+2nt_0)\sin(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\\
&-\frac{(n_I^2-2nn_I)\sin(3n_It_{I0}+2nt_0)\cos(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\\
&-\frac{(6n_I^2-16nn_I+8n^2)\cos(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+2nt_0-n_It)}{n_I}\\
&+\frac{(6n_I^2-16nn_I+8n^2)\sin(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+2nt_0-n_It)}{n_I}
\end{aligned}
$$

$$+\frac{(2nn_I-3n_I^2)\cos(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}$$
$$-\frac{(2nn_I-3n_I^2)\sin(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}$$
$$+\Bigg(-\frac{(2nn_I-3n_I^2)\cos(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}-\frac{(6n_I^2-16nn_I+8n^2)\cos(2n_It_{I0}+2nt_0+n_It)}{n_I}$$
$$-\frac{(n_I^2-2nn_I)\cos(4nt_0+(3n_I-2n)t)}{3n_I-2n}\Bigg)\sin(3n_It_{I0}+2nt_0)$$
$$+\Bigg(\frac{(2nn_I-3n_I^2)\sin(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}+\frac{(6n_I^2-16nn_I+8n^2)\sin(2n_It_{I0}+2nt_0+n_It)}{n_I}$$
$$+\frac{(n_I^2-2nn_I)\sin(4nt_0+(3n_I-2n)t)}{3n_I-2n}\Bigg)\cos(3n_It_{I0}+2nt_0)\Bigg\}$$
$$-\frac{e_I\sin\varpi}{32n_I-32n}$$
$$\Bigg(-\frac{\sin(2n_It_{I0}+2nt_0)\sin(4n_It_{I0}+(2n-2n_I)t)}{2n-2n_I}-\frac{\cos(2n_It_{I0}+2nt_0)\cos(4n_It_{I0}+(2n-2n_I)t)}{2n-2n_I}$$
$$+\frac{(4n-4n_I)t^2\sin(2n_It_{I0}+2nt_0)^2}{2}+\frac{\sin(4nt_0+(2n_I-2n)t)\sin(2n_It_{I0}+2nt_0)}{2n_I-2n}$$
$$+\frac{(4n-4n_I)t^2\cos(2n_It_{I0}+2nt_0)^2}{2}+\frac{\cos(4nt_0+(2n_I-2n)t)\cos(2n_It_{I0}+2nt_0)}{2n_I-2n}\Bigg)$$
$$-e_I^2\cos\varpi\frac{\sin(3n_It_{I0}-3n_It)}{n_I^2}-e_I^2\sin\varpi\frac{\cos(3n_It_{I0}-3n_It)}{n_I^2}+3e_I\cos\varpi\frac{\sin(2n_It_{I0}-2n_It)}{32n_I^2}$$
$$-3e_I\sin\varpi\frac{\cos(2n_It_{I0}-2n_It)}{8n_I^2}-15e_I^2\cos\varpi\frac{\sin(n_It_{I0}-n_It)}{8n_I^2}-3\cos\varpi\frac{\sin(n_It_{I0}-n_It)}{n_I^2}$$
$$-15e_I^2\sin\varpi\frac{\cos(n_It_{I0}-n_It)}{8n_I^2}-3\sin\varpi\frac{\cos(n_It_{I0}-n_It)}{n_I^2}\Bigg]$$
$$-\frac{a}{16(64nn_i^4-256n^2n_i^3+364n^3n_i^2-216n^4n_i+45n^5)}$$
$$\Bigg[\frac{21(16nn_i^3-52n^2n_i^2+52n^3n_i-15n^4}{4n_i-3n}$$
$$\Bigg\{\Bigg(\sin\Bigg(\Bigg(4n_i-3n\Bigg)t-4n_it_{i0}+3nt_0+\frac{\pi}{3}\Bigg)+\sin\Bigg(4n_it_{i0}-3nt_0-\frac{\pi}{3}\Bigg)\Bigg)$$
$$-\mu(\sin((4n_i-3n)t-4n_it_{i0}+3nt_0)-\sin(-4n_it_{i0}+3nt_0))\Bigg\}$$
$$+\frac{21(-16nn_i^3+44n^2n_i^2-36n^3n_i+9n^4)}{4n_i-5n}$$
$$\Bigg\{\Bigg(\sin\Bigg(\Bigg(4n_i-5n\Bigg)t-4n_it_{i0}+5nt_0-\frac{\pi}{3}\Bigg)-\sin\Bigg(-4n_it_{i0}+5nt_0-\frac{\pi}{3}\Bigg)\Bigg)$$
$$-\mu(\sin((4n_i-5n)t-4n_it_{i0}+5nt_0)-\sin(-4n_it_{i0}+5nt_0))\Bigg\}$$
$$+\frac{28(32nn_i^3-112n^2n_i^2+126n^3n_i-45n^4)}{2n_i-n}$$
$$\Bigg\{\Bigg(\sin\Bigg(\Bigg(2n_i-n\Bigg)t-2n_it_{i0}+nt_0+\frac{\pi}{3}\Bigg)-\sin\Bigg(-2n_it_{i0}+nt_0+\frac{\pi}{3}\Bigg)\Bigg)$$
$$-\mu(\sin((2n_i-n)t-2n_it_{i0}+nt_0)-\sin(-2n_it_{i0}+nt_0))\Bigg\}$$

$$
\begin{aligned}
&+\frac{28(-32nn_i^3+80n^2n_i^2-62n^3n_i+15n^4}{2n_i-3n}\\
&\left\{\left(\sin\left(\left(2n_i-3n\right)t-2n_it_{i0}+3nt_0-\frac{\pi}{3}\right)-\sin\left(-2n_it_{i0}+3nt_0-\frac{\pi}{3}\right)\right)\right.\\
&\left.-\mu(\sin((2n_i-3n)t-2n_it_{i0}+3nt_0)-\sin(-2n_it_{i0}+3nt_0))\right\}\\
&+\frac{30(64n_i^4-256nn_i^3+364n^2n_i^2-216n^3n_i+45n^4)}{n}\\
&\left\{\left(\sin\left(nt-nt_0+\frac{\pi}{3}\right)-\sin\left(-nt_0+\frac{\pi}{3}\right)\right)\right.\\
&\left.-\mu(\sin(nt-nt_0)-\sin(-nt_0))\right\}\Bigg]
\end{aligned}
\tag{B.11}
$$

B.6 Order of q^5

$$
\begin{aligned}
&\delta x_E^{(5)} \\
= \; & a_I \left[\frac{3e_I^2 \sin \varpi}{64 * n_I^2 + 16 * n * n_I - 8 * n^2} \right. \\
& \left\{ \frac{(-2n_I - n)\sin(4n_I t_{I0} + nt_0)\sin(8n_I t_{I0} + (n - 4n_I)t)}{n - 4n_I} - \frac{(2n_I + n)\cos(4n_I t_{I0} + nt_0)\cos(8n_I t_{I0} + (n - 4n_I)t)}{n - 4n_I} \right. \\
& + \frac{(n - 4n_I)\sin(4n_I t_{I0} + nt_0)\sin(6n_I t_{I0} + 2nt_0 + (-2n_I - n)t)}{-2n_I - n} \\
& - \frac{(4n_I - n)\cos(4n_I t_{I0} + nt_0)\cos(6n_I t_{I0} + 2nt_0 + (-2n_I - n)t)}{-2n_I - n} \\
& + \left(\frac{(4n_I - n)\sin(2n_I t_{I0} + (2n_I + n)t)}{2n_I + n} + \frac{(2n_I + n)\sin(2nt_0 + (4n_I - n)t)}{4n_I - n} \right) \sin(4n_I t_{I0} + nt_0) \\
& \left. + \left(- \frac{(n - 4n_I)\cos(2n_I t_{I0} + (2n_I + n)t)}{2n_I + n} - \frac{(-2n_I - n)\cos(2nt_0 + (4n_I - n)t)}{4n_I - n} \right) \cos(4n_I t_{I0} + nt_0) \right\} \\
& - \frac{3e_I^2 \cos \varpi}{64n_I^2 + 16nn_I - 8n^2} \\
& \left\{ \frac{(2n_I + n)\cos(4n_I t_{I0} + nt_0)\sin(8n_I t_{I0} + (n - 4n_I)t)}{n - 4n_I} - \frac{(2n_I + n)\sin(4n_I t_{I0} + nt_0)\cos(8n_I t_{I0} + (n - 4n_I)t)}{n - 4n_I} \right. \\
& + \frac{(4n_I - n)\cos(4n_I t_{I0} + nt_0)\sin(6n_I t_{I0} + 2nt_0 + (-2n_I - n)t)}{-2n_I - n} \\
& - \frac{(4n_I - n)\sin(4n_I t_{I0} + nt_0)\cos(6n_I t_{I0} + 2nt_0 + (-2n_I - n)t)}{-2n_I - n} \\
& + \left(- \frac{(4n_I - n)\cos(2n_I t_{I0} + (2n_I + n)t)}{2n_I + n} - \frac{(2n_I + n)\cos(2nt_0 + (4n_I - n)t)}{4n_I - n} \right) \sin(4n_I t_{I0} + nt_0) \\
& \left. + \left(\frac{(4n_I - n)\sin(2n_I t_{I0} + (2n_I + n)t)}{2n_I + n} + \frac{(2n_I + n)\sin(2nt_0 + (4n_I - n)t)}{4n_I - n} \right) \cos(4n_I t_{I0} + nt_0) \right\} \\
& + \frac{e_I \sin \varpi}{48n_I^2 + 32nn_I - 16n^2} \\
& \left\{ \frac{(-n_I - n)\sin(3n_I t_{I0} + nt_0)\sin(6n_I t_{I0} + (n - 3n_I)t)}{n - 3n_I} - \frac{(n_I + n)\cos(3n_I t_{I0} + nt_0)\cos(6n_I t_{I0} + (n - 3n_I)t)}{n - 3n_I} \right. \\
& + \frac{(n - 3n_I)\sin(3n_I t_{I0} + nt_0)\sin(4n_I t_{I0} + 2nt_0 + (-n_I - n)t)}{-n_I - n} \\
& - \frac{(3n_I - n)\cos(3n_I t_{I0} + nt_0)\cos(4n_I t_{I0} + 2nt_0 + (-n_I - n)t)}{-n_I - n} \\
& + \left(\frac{(3n_I - n)\sin(2n_I t_{I0} + (n_I + n)t)}{n_I + n} + \frac{(n_I + n)\sin(2nt_0 + (3n_I - n)t)}{3n_I - n} \right) \sin(3n_I t_{I0} + nt_0) \\
& \left. + \left(- \frac{(n - 3n_I)\cos(2n_I t_{I0} + (n_I + n)t)}{n_I + n} - \frac{(-n_I - n)\cos(2nt_0 + (3n_I - n)t)}{3n_I - n} \right) \cos(3n_I t_{I0} + nt_0) \right\} \\
& + \frac{e_I \cos \varpi}{192n_I^2 + 128nn_I - 64n^2} \\
& \left\{ \frac{(n_I + n)\cos(3n_I t_{I0} + nt_0)\sin(6n_I t_{I0} + (n - 3n_I)t)}{n - 3n_I} - \frac{(n_I + n)\sin(3n_I t_{I0} + nt_0)\cos(6n_I t_{I0} + (n - 3n_I)t)}{n - 3n_I} \right. \\
& + \frac{(3n_I - n)\cos(3n_I t_{I0} + nt_0)\sin(4n_I t_{I0} + 2nt_0 + (-n_I - n)t)}{-n_I - n} \\
& - \frac{(3n_I - n)\sin(3n_I t_{I0} + nt_0)\cos(4n_I t_{I0} + 2nt_0 + (-n_I - n)t)}{-n_I - n} \\
& + \left(- \frac{(3n_I - n)\cos(2n_I t_{I0} + (n_I + n)t)}{n_I + n} - \frac{(n_I + n)\cos(2nt_0 + (3n_I - n)t)}{3n_I - n} \right) \sin(3n_I t_{I0} + nt_0) \\
& \left. + \left(\frac{(3n_I - n)\sin(2n_I t_{I0} + (n_I + n)t)}{n_I + n} + \frac{(n_I + n)\sin(2nt_0 + (3n_I - n)t)}{3n_I - n} \right) \cos(3n_I t_{I0} + nt_0) \right\} \\
& + \frac{5e_I^2 \sin \varpi}{128nn_I - 64n^2}
\end{aligned}
$$

$$\left\{-\frac{n\sin(2n_It_{I0}+nt_0)\sin(4n_It_{I0}+(n-2n_I)t)}{n-2n_I}-\frac{n\cos(2n_It_{I0}+nt_0)\cos(4n_It_{I0}+(n-2n_I)t)}{-2n-n_I}\right.$$
$$-\frac{(n-2n_I)\sin(2n_It_{I0}+nt_0)\sin(2n_It_{I0}+2nt_0-nt)}{n}$$
$$+\frac{(2n_I-n)\cos(2n_It_{I0}+nt_0)\cos(2n_It_{I0}+2nt_0-nt)}{n}$$
$$+\left(\frac{(2n_I-n)\sin(2n_It_{I0}+nt)}{n}+\frac{n\sin(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\sin(2n_It_{I0}+nt_0)$$
$$\left.+\left(\frac{n\cos(2nt_0+(2n_I-n)t)}{2n_I-n}-\frac{(n-2n_I)\cos(2n_It_{I0}+nt)}{n}\right)\cos(2n_It_{I0}+nt_0)\right\}$$
$$+\frac{\sin\varpi}{16nn_I-8n^2}$$
$$\left\{-\frac{n\sin(2n_It_{I0}+nt_0)\sin(4n_It_{I0}+(n-2n_I)t)}{n-2n_I}-\frac{n\cos(2n_It_{I0}+nt_0)\cos(4n_It_{I0}+(n-2n_I)t)}{n-2n_I}\right.$$
$$-\frac{(n-2n_I)\sin(2n_It_{I0}+nt_0)\sin(2n_It_{I0}+2nt_0-nt)}{n}$$
$$+\frac{(2n_I-n)\cos(2n_It_{I0}+nt_0)\cos(2n_It_{I0}+2nt_0-nt)}{/}n$$
$$+\left(\frac{(2n_I-n)\sin(2n_It_{I0}+nt)}{n}+\frac{n\sin(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\sin(2n_It_{I0}+nt_0)$$
$$\left.+\left(\frac{n\cos(2nt_0+(2n_I-n)t)}{2n_I-n}-\frac{(n-2n_I)\cos(2n_It_{I0}+nt)}{n}\right)\cos(2n_It_{I0}+nt_0)\right\}$$
$$-\frac{5e_I^2\cos\varpi}{128nn_I-64n^2}\left\{\frac{n\cos(2n_It_{I0}+nt_0)\sin(4n_It_{I0}+(n-2n_I)t)}{n-2n_I}-\frac{n\sin(2n_It_{I0}+nt_0)\cos(4n_It_{I0}+(n-2n_I)t)}{n-2n_I}\right.$$
$$-\frac{(2n_I-n)\cos(2n_It_{I0}+nt_0)\sin(2n_It_{I0}+2nt_0-nt)}{n}$$
$$+\frac{(2n_I-n)\sin(2n_It_{I0}+nt_0)\cos(2n_It_{I0}+2nt_0-nt)}{n}$$
$$+\left(-\frac{(2n_I-n)\cos(2n_It_{I0}+nt)}{n}-\frac{n\cos(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\sin(2n_It_{I0}+nt_0)$$
$$\left.+\left(\frac{(2n_I-n)\sin(2n_It_{I0}+nt)}{n}+\frac{n\sin(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\cos(2n_It_{I0}+nt_0)\right\}$$
$$-\frac{\cos\varpi}{16nn_I-8n^2}$$
$$\left\{\frac{n\cos(2n_It_{I0}+nt_0)\sin(4n_It_{I0}+(n-2n_I)t)}{2n-n_I}-\frac{n\sin(2n_It_{I0}+nt_0)\cos(4n_It_{I0}+(n-2n_I)t)}{2n-n_I}\right.$$
$$-\frac{(2n_I-n)\cos(2n_It_{I0}+nt_0)\sin(2n_It_{I0}+2nt_0-nt)}{n}$$
$$+\frac{(2n_I-n)\sin(2n_It_{I0}+nt_0)\cos(2n_It_{I0}+2nt_0-nt)}{n}$$
$$+\left(-\frac{(2n_I-n)\cos(2n_It_{I0}+nt)}{n}-\frac{n\cos(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\sin(2n_It_{I0}+nt_0)$$
$$\left.+\left(\frac{(2n_I-n)\sin(2n_It_{I0}+nt)}{n}+\frac{n\sin(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\cos(2n_It_{I0}+nt_0)\right\}$$
$$\left.-e_I\sin\varpi\frac{\cos(n_It_{I0}-nt_0-n_It+nt)}{2(n-n_I)^2}\right]$$
$$+\frac{a}{32(225n_i^6-1350nn_i^5+3116n^2n_i^4-3464n^3n_i^3+1856n^4n_i^2-384n^5n_i)}$$
$$\left[-\frac{33(45n_i^5-234nn_i^4+436n^2n_i^3-344n^3n_i^2+96n^4n_i)}{5n_i-4n}\right.$$
$$\left\{\left(\cos\left(\left(5n_i-4n\right)t-5n_it_{i0}+4nt_0+\frac{\pi}{3}\right)-\cos\left(-5n_it_{i0}+4nt_0+\frac{\pi}{3}\right)\right)\right.$$
$$\left.-\mu(\cos((5n_i-4n)t-5n_it_{i0}+4nt_0)-\cos(-5n_it_{i0}+4nt_0))\right\}$$

$$
-\frac{33(45n_i^5-216nn_i^4+364n^2n_i^3-256n^3n_i^2+64n^4n_i)}{5n_i-6n}
$$
$$
\left\{\left(\cos\left(\left(5n_i-6n\right)t-5n_it_{i0}+6nt_0-\frac{\pi}{3}\right)-\cos\left(-5n_it_{i0}+6nt_0-\frac{\pi}{3}\right)\right)\right.
$$
$$
\left.-\mu(\cos((5n_i-6n)t-5n_it_{i0}+6nt_0)-\cos(-5n_it_{i0}+6nt_0))\right\}
$$
$$
-\frac{45(75n_i^5-400nn_i^4+772n^2n_i^3-640n^3n_i^2+192n^4n_i)}{3n_i-2n}
$$
$$
\left\{\left(\cos\left(\left(3n_i-2n\right)t-3n_it_{i0}+2nt_0+\frac{\pi}{3}\right)-\cos\left(-3n_it_{i0}+2nt_0+\frac{\pi}{3}\right)\right)\right.
$$
$$
\left.-\mu(\cos((3n_i-2n)t-3n_it_{i0}+2nt_0)-\cos(-3n_it_{i0}+2nt_0))\right\}
$$
$$
-\frac{45(75n_i^5-350nn_i^4+572n^2n_i^3-392n^3n_i^2+96n^4n_i)}{3n_i-4n}
$$
$$
\left\{\left(\cos\left(\left(3n_i-4n\right)t-3n_it_{i0}+4nt_0-\frac{\pi}{3}\right)-\cos\left(-3n_it_{i0}+4nt_0-\frac{\pi}{3}\right)\right)\right.
$$
$$
\left.-\mu(\cos((3n_i-4n)t-3n_it_{i0}+4nt_0)-\cos(-3n_it_{i0}+4nt_0))\right\}
$$
$$
-\frac{50(225n_i^5-900nn_i^4+1316n^2n_i^3-832n^3n_i^2+192n^4n_i)}{n_i-2n}
$$
$$
\left\{\left(\cos\left(\left(n_i-2n\right)t-n_it_{i0}+2nt_0-\frac{\pi}{3}\right)-\cos\left(-n_it_{i0}+2nt_0-\frac{\pi}{3}\right)\right)\right.
$$
$$
\left.-\mu(\cos((n_i-2n)t-n_it_{i0}+2nt_0)-\cos(-n_it_{i0}+2nt_0))\right\}
$$
$$
-\frac{50(225n_i^5-1350nn_i^4+3116n^2n_i^3-3464n^3n_i^2+1856n^4n_i-384n^5)}{n_i}
$$
$$
\left\{\left(\cos\left(n_it-n_it_{i0}+\frac{\pi}{3}\right)-\cos\left(-n_it_{i0}+\frac{\pi}{3}\right)\right)\right.
$$
$$
\left.\left.-\mu(\cos(n_it-n_it_{i0})-\cos(-n_it_{i0}))\right\}\right] \tag{B.12}
$$

$$
\begin{aligned}
&\delta y_E^{(5)}\\
= \;& a_I\left[\frac{3e_I^2\sin\varpi}{32n_I^2+16nn_I-8n^2}\right.\\
&\left\{\frac{(-2n_I-n)\sin(4n_It_{I0}+nt_0)\sin(8n_It_{I0}+(n-4n_I)t)}{n-4n_I}-\frac{(2n_I+n)\cos(4n_It_{I0}+nt_0)\cos(8n_It_{I0}+(n-4n_I)t)}{n-4n_I}\right.\\
&+\frac{(n-4n_I)\sin(4n_It_{I0}+nt_0)\sin(6n_It_{I0}+2nt_0+(-2n_I-n)t)}{-2n_I-n}\\
&-\frac{(4n_I-n)\cos(4n_It_{I0}+nt_0)\cos(6n_It_{I0}+2nt_0+(-2n_I-n)t)}{-2n_I-n}\\
&+\left(\frac{(4n_I-n)\sin(2n_It_{I0}+(2n_I+n)t)}{2n_I+n}+\frac{(2n_I+n)\sin(2nt_0+(4n_I-n)t)}{4n_I-n}\right)\sin(4n_It_{I0}+nt_0)\\
&\left.+\left(-\frac{(n-4n_I)\cos(2n_It_{I0}+(2n_I+n)t)}{2n_I+n}-\frac{(-2n_I-n)\cos(2nt_0+(4n_I-n)t)}{4n_I-n}\right)\cos(4n_It_{I0}+nt_0)\right\}
\end{aligned}
$$

$$-\frac{3e_I^2\cos\varpi}{64n_I^2+16nn_I-8n^2}$$

$$\Big\{\frac{(2n_I+n)\cos(4n_It_{I0}+nt_0)\sin(8n_It_{I0}+(n-4n_I)t)}{n-4n_I}-\frac{(2n_I+n)\sin(4n_It_{I0}+nt_0)\cos(8n_It_{I0}+(n-4n_I)t)}{n-4n_I}$$

$$+\frac{(4n_I-n)\cos(4n_It_{I0}+nt_0)\sin(6n_It_{I0}+2nt_0+(-2n_I-n)t)}{-2n_I-n}$$

$$-\frac{(4n_I-n)\sin(4n_It_{I0}+nt_0)\cos(6n_It_{I0}+2nt_0+(-2n_I-n)t)}{-2n_I-n}$$

$$+\left(-\frac{(4n_I-n)\cos(2n_It_{I0}+(2n_I+n)t)}{2n_I+n}-\frac{(2n_I+n)\cos(2nt_0+(4n_I-n)t)}{4n_I-n}\right)\sin(4n_It_{I0}+nt_0)$$

$$+\left(\frac{(4n_I-n)\sin(2n_It_{I0}+(2n_I+n)t)}{2n_I+n}+\frac{(2n_I+n)\sin(2nt_0+(4n_I-n)t)}{4n_I-n}\right)\cos(4n_It_{I0}+nt_0)\Big\}$$

$$+\frac{e_I\sin\varpi}{48n_I^2+32nn_I-16n^2}$$

$$\Big\{\frac{(-n_I-n)\sin(3n_It_{I0}+nt_0)\sin(6n_It_{I0}+(n-3n_I)t)}{n-3n_I}-\frac{(n_I+n)\cos(3n_It_{I0}+nt_0)\cos(6n_It_{I0}+(n-3n_I)t)}{n-3n_I}$$

$$+\frac{(n-3n_I)\sin(3n_It_{I0}+nt_0)\sin(4n_It_{I0}+2nt_0+(-n_I-n)t)}{-n_I-n}$$

$$-\frac{(3n_I-n)\cos(3n_It_{I0}+nt_0)\cos(4n_It_{I0}+2nt_0+(-n_I-n)t)}{-n_I-n}$$

$$+\left(\frac{(3n_I-n)\sin(2n_It_{I0}+(n_I+n)t)}{n_I+n}+\frac{(n_I+n)\sin(2nt_0+(3n_I-n)t)}{3n_I-n}\right)\sin(3n_It_{I0}+nt_0)$$

$$+\left(-\frac{(n-3n_I)\cos(2n_It_{I0}+(n_I+n)t)}{n_I+n}-\frac{(-n_I-n)\cos(2nt_0+(3n_I-n)t)}{3n_I-n}\right)\cos(3n_It_{I0}+nt_0)\Big\}$$

$$+\frac{e_I\cos\varpi}{192n_I^2+128nn_I-64n^2}$$

$$\Big\{\frac{(n_I+n)\cos(3n_It_{I0}+nt_0)\sin(6n_It_{I0}+(n-3n_I)t)}{n-3n_I}-\frac{(n_I+n)\sin(3n_It_{I0}+nt_0)\cos(6n_It_{I0}+(n-3n_I)t)}{n-3n_I}$$

$$+\frac{(3n_I-n)\cos(3n_It_{I0}+nt_0)\sin(4n_It_{I0}+2nt_0+(-n_I-n)t)}{-n_I-n}$$

$$-\frac{(3n_I-n)\sin(3n_It_{I0}+nt_0)\cos(4n_It_{I0}+2nt_0+(-n_I-n)t)}{-n_I-n}$$

$$+\left(-\frac{(3n_I-n)\cos(2n_It_{I0}+(n_I+n)t)}{n_I+n}-\frac{(n_I+n)\cos(2nt_0+(3n_I-n)t)}{3n_I-n}\right)\sin(3n_It_{I0}+nt_0)$$

$$+\left(\frac{(3n_I-n)\sin(2n_It_{I0}+(n_I+n)t)}{n_I+n}+\frac{(n_I+n)\sin(2nt_0+(3n_I-n)t)}{3n_I-n}\right)\cos(3n_It_{I0}+nt_0)\Big\}$$

$$+\frac{5e_I^2\sin\varpi}{128nn_I-64n^2}$$

$$\Big\{-\frac{n\sin(2n_It_{I0}+nt_0)\sin(4n_It_{I0}+(n-2n_I)t)}{n-2n_I}-\frac{n\cos(2n_It_{I0}+nt_0)\cos(4n_It_{I0}+(n-2n_I)t)}{-2n-n_I}$$

$$-\frac{(n-2n_I)\sin(2n_It_{I0}+nt_0)\sin(2n_It_{I0}+2nt_0-nt)}{n}$$

$$+\frac{(2n_I-n)\cos(2n_It_{I0}+nt_0)\cos(2n_It_{I0}+2nt_0-nt)}{n}$$

$$+\left(\frac{(2n_I-n)\sin(2n_It_{I0}+nt)}{n}+\frac{n\sin(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\sin(2n_It_{I0}+nt_0)$$

$$+\left(\frac{n\cos(2nt_0+(2n_I-n)t)}{2n_I-n}-\frac{(n-2n_I)\cos(2n_It_{I0}+nt)}{n}\right)\cos(2n_It_{I0}+nt_0)\Big\}$$

$$+\frac{\sin\varpi}{6nn_I-8n^2}$$

$$\Big\{-\frac{n\sin(2n_It_{I0}+nt_0)\sin(4n_It_{I0}+(n-2n_I)t)}{n-2n_I}-\frac{n\cos(2n_It_{I0}+nt_0)\cos(4n_It_{I0}+(n-2n_I)t)}{n-2n_I}$$

$$-\frac{(n-2n_I)\sin(2n_It_{I0}+nt_0)\sin(2n_It_{I0}+2nt_0-nt)}{n}$$

$$+\frac{(2n_I-n)\cos(2n_It_{I0}+nt_0)\cos(2n_It_{I0}+2nt_0-nt)}{/}n$$

$$+\left(\frac{(2n_I-n)\sin(2n_It_{I0}+nt)}{n}+\frac{n\sin(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\sin(2n_It_{I0}+nt_0)$$
$$+\left(\frac{n\cos(2nt_0+(2n_I-n)t)}{2n_I-n}-\frac{(n-2n_I)\cos(2n_It_{I0}+nt)}{n}\right)\cos(2n_It_{I0}+nt_0)\Bigg\}$$
$$-\frac{5e_I^2\cos\varpi}{128nn_I-64n^2}\Bigg\{\frac{n\cos(2n_It_{I0}+nt_0)\sin(4n_It_{I0}+(n-2n_I)t)}{n-2n_I}-\frac{n\sin(2n_It_{I0}+nt_0)\cos(4n_It_{I0}+(n-2n_I)t)}{n-2n_I}$$
$$-\frac{(2n_I-n)\cos(2n_It_{I0}+nt_0)\sin(2n_It_{I0}+2nt_0-nt)}{n}$$
$$+\frac{(2n_I-n)\sin(2n_It_{I0}+nt_0)\cos(2n_It_{I0}+2nt_0-nt)}{n}$$
$$+\left(-\frac{(2n_I-n)\cos(2n_It_{I0}+nt)}{n}-\frac{n\cos(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\sin(2n_It_{I0}+nt_0)$$
$$+\left(\frac{(2n_I-n)\sin(2n_It_{I0}+nt)}{n}+\frac{n\sin(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\cos(2n_It_{I0}+nt_0)\Bigg\}$$
$$-\frac{\cos\varpi}{16nn_I-8n^2}$$
$$\Bigg\{\frac{n\cos(2n_It_{I0}+nt_0)\sin(4n_It_{I0}+(n-2n_I)t)}{2n-n_I}-\frac{n\sin(2n_It_{I0}+nt_0)\cos(4n_It_{I0}+(n-2n_I)t)}{2n-n_I}$$
$$-\frac{(2n_I-n)\cos(2n_It_{I0}+nt_0)\sin(2n_It_{I0}+2nt_0-nt)}{n}$$
$$+\frac{(2n_I-n)\sin(2n_It_{I0}+nt_0)\cos(2n_It_{I0}+2nt_0-nt)}{n}$$
$$+\left(-\frac{(2n_I-n)\cos(2n_It_{I0}+nt)}{n}-\frac{n\cos(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\sin(2n_It_{I0}+nt_0)$$
$$+\left(\frac{(2n_I-n)\sin(2n_It_{I0}+nt)}{n}+\frac{n\sin(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\cos(2n_It_{I0}+nt_0)\Bigg\}$$
$$-e_I\sin\varpi\frac{\cos(n_It_{I0}-nt_0-n_It+nt)}{2(n-n_I)^2}\Bigg]$$
$$-\frac{a}{32(225n_i^6-1350nn_i^5+3116n^2n_i^4-3464n^3n_i^3+1856n^4n_i^2-384n^5n_i)}$$
$$\Bigg[\frac{33(45n_i^5-234nn_i^4+436n^2n_i^3-344n^3n_i^2+96n^4n_i)}{5n_i-4n}$$
$$\left\{\left(\sin\left(\left(5n_i-4n\right)t-5n_it_{i0}+4nt_0+\frac{\pi}{3}\right)-\sin\left(-5n_it_{i0}+4nt_0+\frac{\pi}{3}\right)\right)\right.$$
$$\left.-\mu(\sin((5n_i-4n)t-5n_it_{i0}+4nt_0)-\sin(-5n_it_{i0}+4nt_0))\right\}$$
$$+33(-45n_i^5+216nn_i^4-364n^2n_i^3+256n^3n_i^2-64n^4n_i)(5n_i-6n)$$
$$\left\{\left(\sin\left(\left(5n_i-6n\right)t-5n_it_{i0}+6nt_0-\frac{\pi}{3}\right)-\sin\left(-5n_it_{i0}+6nt_0-\frac{\pi}{3}\right)\right)\right.$$
$$\left.-\mu(\sin((5n_i-6n)t-5n_it_{i0}+6nt_0)-\sin(-5n_it_{i0}+6nt_0))\right\}$$
$$+\frac{45(75n_i^5-400nn_i^4+772n^2n_i^3-640n^3n_i^2+192n^4n_i)}{3n_i-2n}$$
$$\left\{(\sin((3n_i-2n)t-3n_it_{i0}+2nt_0+\frac{\pi}{3})-\sin(-3n_it_{i0}+2nt_0+\frac{\pi}{3}))\right.$$
$$\left.-mu(\sin((3n_i-2n)t-3n_it_{i0}+2nt_0)-\sin(-3n_it_{i0}+2nt_0))\right\}$$
$$+\frac{45(-75n_i^5+350nn_i^4-572n^2n_i^3+392n^3n_i^2-96n^4n_i)}{3n_i-4n}$$

$$
\begin{aligned}
&\Bigg\{\Bigg(\sin\Bigg(\Bigg(3n_i - 4n\Bigg)t - 3n_i t_{i0} + 4nt_0 - \frac{\pi}{3}\Bigg) - \sin\Bigg(- 3n_i t_{i0} + 4nt_0 - \frac{\pi}{3}\Bigg)\Bigg)\\
&- \mu(\sin((3n_i - 4n)t - 3n_i t_{i0} + 4nt_0) - \sin(-3n_i t_{i0} + 4nt_0))\Bigg\}\\
&+ \frac{50(-225n_i^5 + 900nn_i^4 - 1316n^2n_i^3 + 832n^3n_i^2 - 192n^4n_i)}{(}n_i - 2n)\\
&\Bigg\{\Bigg(\sin\Bigg(\Bigg(n_i - 2n\Bigg)t - n_i t_{i0} + 2nt_0 - \frac{\pi}{3}\Bigg) - \sin\Bigg(- n_i t_{i0} + 2nt_0 - \frac{\pi}{3}\Bigg)\Bigg)\\
&- \mu(\sin((n_i - 2n)t - n_i t_{i0} + 2nt_0) - \sin(-n_i t_{i0} + 2nt_0))\Bigg\}\\
&+ \frac{50(225n_i^5 - 1350nn_i^4 + 3116n^2n_i^3 - 3464n^3n_i^2 + 1856n^4n_i - 384n^5)}{n_i}\\
&\Bigg\{\Bigg(\sin\Bigg(n_i t - n_i t_{i0} + \frac{\pi}{3}\Bigg) - \sin\Bigg(- n_i t_{i0} + \frac{\pi}{3}\Bigg)\Bigg)\\
&- \mu(\sin(n_i t - n_i t_{i0}) - \sin(-n_i t_{i0}))\Bigg\}\Bigg]
\end{aligned}
\tag{B.13}
$$

B.7 Order of q^6

$$
\begin{aligned}
&\delta x_E^{(6)} \\
= \quad & a_I \Bigg[\frac{3e_I^2 \sin \varpi}{240n_I^3 + 384nn_I^2 - 192n^2 n_I} \\
&\Bigg\{ \frac{(-3n_I^2 - 6nn_I)\sin(5n_I t_{IO} + 2nt_O)\sin(10n_I t_{IO} + (2n - 5n_I)t)}{2n - 5n_I} \\
&- \frac{(3n_I^2 + 6nn_I)\cos(5n_I t_{IO} + 2nt_O)\cos(10n_I t_{IO} + (2n - 5n_I)t)}{2n - 5n_I} \\
&- \frac{(-10n_I^2 - 16nn_I + 8n^2)\sin(5n_I t_{IO} + 2nt_O)\sin(8n_I t_{IO} + 2nt_O - 3n_I t)}{3n_I} \\
&+ \frac{(10n_I^2 + 16nn_I - 8n^2)\cos(5n_I t_{IO} + 2nt_O)\cos(8n_I t_{IO} + 2nt_O - 3n_I t)}{3n_I} \\
&+ \frac{(6nn_I - 15n_I^2)\sin(5n_I t_{IO} + 2nt_O)\sin(6n_I t_{IO} + 4nt_O + (-n_I - 2n)t)}{-n_I - 2n} \\
&- \frac{(15n_I^2 - 6nn_I)\cos(5n_I t_{IO} + 2nt_O)\cos(6n_I t_{IO} + 4nt_O + (-n_I - 2n)t)}{-n_I - 2n} \\
&+ \Bigg(\frac{(15n_I^2 - 6nn_I)\sin(4n_I t_{IO} + (n_I + 2n)t)}{n_I + 2n} + \frac{(10n_I^2 + 16nn_I - 8n^2)\sin(2n_I t_{IO} + 2nt_O + 3n_I t)}{3n_I} \\
&+ \frac{(3n_I^2 + 6nn_I)\sin(4nt_O + (5n_I - 2n)t)}{5n_I - 2n} \Bigg) \sin(5n_I t_{IO} + 2nt_O) \\
&+ \Bigg(- \frac{(6nn_I - 15n_I^2)\cos(4n_I t_{IO} + (n_I + 2n)t)}{n_I + 2n} - \frac{(-10n_I^2 - 16nn_I + 8n^2)\cos(2n_I t_{IO} + 2nt_O + 3n_I t)}{3n_I} \\
&- \frac{(-3n_I^2 - 6nn_I)\cos(4nt_O + (5n_I - 2n)t)}{5n_I - 2n} \Bigg) \cos(5n_I t_{IO} + 2nt_O) \Bigg\} \\
&- \frac{3e_I^2 \cos \varpi}{240n_I^3 + 384nn_I^2 - 192n^2 n_I} \\
&\Bigg\{ \frac{(3n_I^2 + 6nn_I)\cos(5n_I t_{IO} + 2nt_O)\sin(10n_I t_{IO} + (2n - 5n_I)t)}{2n - 5n_I} \\
&- \frac{(3n_I^2 + 6nn_I)\sin(5n_I t_{IO} + 2nt_O)\cos(10n_I t_{IO} + (2n - 5n_I)t)}{2n - 5n_I} \\
&- \frac{(10n_I^2 + 16nn_I - 8n^2)\cos(5n_I t_{IO} + 2nt_O)\sin(8n_I t_{IO} + 2nt_O - 3n_I t)}{3n_I} \\
&+ \frac{(10n_I^2 + 16nn_I - 8n^2)\sin(5n_I t_{IO} + 2nt_O)\cos(8n_I t_{IO} + 2nt_O - 3n_I t)}{3n_I} \\
&+ \frac{(15n_I^2 - 6nn_I)\cos(5n_I t_{IO} + 2nt_O)\sin(6n_I t_{IO} + 4nt_O + (-n_I - 2n)t)}{-n_I - 2n} \\
&- \frac{(15n_I^2 - 6nn_I)\sin(5n_I t_{IO} + 2nt_O)\cos(6n_I t_{IO} + 4nt_O + (-n_I - 2n)t)}{-n_I - 2n} \\
&+ \Bigg(- \frac{(15n_I^2 - 6nn_I)\cos(4n_I t_{IO} + (n_I + 2n)t)}{n_I + 2n} - \frac{(10n_I^2 + 16nn_I - 8n^2)\cos(2n_I t_{IO} + 2nt_O + 3n_I t)}{3n_I} \\
&- \frac{(3n_I^2 + 6nn_I)\cos(4nt_O + (5n_I - 2n)t)}{5n_I - 2n} \Bigg) \sin(5n_I t_{IO} + 2nt_O) \\
&+ \Bigg(\frac{(15n_I^2 - 6nn_I)\sin(4n_I t_{IO} + (n_I + 2n)t)}{n_I + 2n} + \frac{(10n_I^2 + 16nn_I - 8n^2)\sin(2n_I t_{IO} + 2nt_O + 3n_I t)}{3n_I} \\
&+ \frac{(3n_I^2 + 6nn_I)\sin(4nt_O + (5n_I - 2n)t)}{5n_I - 2n} \Bigg) \cos(5n_I t_{IO} + 2nt_O) \Bigg\} \\
&+ \frac{e_I \sin \varpi}{128nn_I^2 - 64n^2 n_I}
\end{aligned}
$$

$$
\begin{aligned}
&\Big\{-\frac{nn_I\sin(4n_It_{I0}+2nt_0)\sin(8n_It_{I0}+(2n-4n_I)t)}{2n-4n_I}\\
&-\frac{nn_I\cos(4n_It_{I0}+2nt_0)\cos(8n_It_{I0}+(2n-4n_I)t)}{2n-4n_I}\\
&-\frac{(2n^2-4nn_I)\sin(4n_It_{I0}+2nt_0)\sin(6n_It_{I0}+2nt_0-2n_It)}{2n_I}\\
&+\frac{(4nn_I-2n^2)\cos(4n_It_{I0}+2nt_0)\cos(6n_It_{I0}+2nt_0-2n_It)}{2n_I}\\
&-\frac{(nn_I-2n_I^2)\sin(4n_It_{I0}+2nt_0)\sin(4n_It_{I0}+4nt_0-2nt)}{2n}\\
&+\frac{(2n_I^2-nn_I)\cos(4n_It_{I0}+2nt_0)\cos(4n_It_{I0}+4nt_0-2nt)}{2n}\\
&+\Big(\frac{(2n_I^2-nn_I)\sin(4n_It_{I0}+2nt)}{2n}+\frac{(4nn_I-2n^2)\sin(2n_It_{I0}+2nt_0+2n_It)}{2n_I}\\
&+\frac{nn_I\sin(4nt_0+(4n_I-2n)t)}{4n_I-2n}\Big)\sin(4n_It_{I0}+2nt_0)\\
&+\Big(-\frac{(nn_I-2n_I^2)\cos(4n_It_{I0}+2nt)}{2n}-\frac{(2n^2-4nn_I)\cos(2n_It_{I0}+2nt_0+2n_It)}{2n_I}\\
&+\frac{nn_I\cos(4nt_0+(4n_I-2n)t)}{4n_I-2n}\Big)\cos(4n_It_{I0}+2nt_0)\Big\}\\
&+\frac{e_I\cos\varpi}{512nn_I^2-256n^2n_I}\\
&\Big\{\frac{nn_I\cos(4n_It_{I0}+2nt_0)\sin(8n_It_{I0}+(2n-4n_I)t)}{2n-4n_I}\\
&-\frac{nn_I\sin(4n_It_{I0}+2nt_0)\cos(8n_It_{I0}+(2n-4n_I)t)}{2n-4n_I}\\
&-\frac{(4nn_I-2n^2)\cos(4n_It_{I0}+2nt_0)\sin(6n_It_{I0}+2nt_0-2n_It)}{2n_I}\\
&+\frac{(4nn_I-2n^2)\sin(4n_It_{I0}+2nt_0)\cos(6n_It_{I0}+2nt_0-2n_It)}{2n_I}\\
&-\frac{(2n_I^2-nn_I)\cos(4n_It_{I0}+2nt_0)\sin(4n_It_{I0}+4nt_0-2nt)}{2n}\\
&+\frac{(2n_I^2-nn_I)\sin(4n_It_{I0}+2nt_0)\cos(4n_It_{I0}+4nt_0-2nt)}{2n}\\
&+\Big(-\frac{(2n_I^2-nn_I)\cos(4n_It_{I0}+2nt)}{2n}-\frac{(4nn_I-2n^2)\cos(2n_It_{I0}+2nt_0+2n_It)}{2n_I}\\
&-\frac{nn_I\cos(4nt_0+(4n_I-2n)t)}{4n_I-2n}\Big)\sin(4n_It_{I0}+2nt_0)\\
&+\Big(\frac{(2n_I^2-nn_I)\sin(4n_It_{I0}+2nt)}{(2n}+\frac{(4nn_I-2n^2)\sin(2n_It_{I0}+2nt_0+2n_It)}{2n_I}\\
&+\frac{nn_I\sin(4nt_0+(4n_I-2n)t)}{4n_I-2n}\Big)\cos(4n_It_{I0}+2nt_0)\Big\}\\
&+\frac{5e_I^2\sin\varpi}{384n_I^3-1024nn_I^2+512n^2n_I}\\
&\Big\{\frac{(2nn_I-n_I^2)\sin(3n_It_{I0}+2nt_0)\sin(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\\
&-\frac{(n_I^2-2nn_I)\cos(3n_It_{I0}+2nt_0)\cos(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\\
&-\frac{(-6n_I^2+16nn_I-8n^2)\sin(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+2nt_0-n_It)}{n_I}\\
&+\frac{(6n_I^2-16nn_I+8n^2)\cos(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+2nt_0-n_It)}{n_I}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(2nn_I-3n_I^2)\sin(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}\\
&-\frac{(3n_I^2-2nn_I)\cos(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}\\
&+\left(\frac{(3n_I^2-2nn_I)\sin(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}+\frac{(6n_I^2-16nn_I+8n^2)\sin(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.\\
&\left.+\frac{(n_I^2-2nn_I)\sin(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\sin(3n_It_{I0}+2nt_0)\\
&+\left(-\frac{(2nn_I-3n_I^2)\cos(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}-\frac{(-6n_I^2+16nn_I-8n^2)\cos(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.\\
&\left.\left.-\frac{(2nn_I-n_I^2)\cos(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\cos(3n_It_{I0}+2nt_0)\right\}\\
&+\frac{\sin\varpi}{48n_I^3-128nn_I^2+64n^2n_I}\\
&\left\{\frac{(2nn_I-n_I^2)\sin(3n_It_{I0}+2nt_0)\sin(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\right.\\
&-\frac{(n_I^2-2nn_I)\cos(3n_It_{I0}+2nt_0)\cos(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\\
&-\frac{(-6n_I^2+16nn_I-8n^2)\sin(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+2nt_0-n_It)}{n_I}\\
&+\frac{(6n_I^2-16nn_I+8n^2)\cos(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+2nt_0-n_It)}{n_I}\\
&+\frac{(2nn_I-3n_I^2)\sin(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}\\
&-\frac{(3n_I^2-2nn_I)\cos(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}\\
&+\left(\frac{(3n_I^2-2nn_I)\sin(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}+\frac{(6n_I^2-16nn_I+8n^2)\sin(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.\\
&\left.+\frac{(n_I^2-2nn_I)\sin(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\sin(3n_It_{I0}+2nt_0)\\
&+\left(-\frac{(2nn_I-3n_I^2)\cos(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}-\frac{(-6n_I^2+16nn_I-8n^2)\cos(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.\\
&\left.\left.-\frac{(2nn_I-n_I^2)\cos(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\cos(3n_It_{I0}+2nt_0)\right\}\\
&-\frac{5e_I^2\cos\varpi}{384n_I^3-1024nn_I^2+512n^2n_I}\\
&\left\{\frac{(n_I^2-2nn_I)\cos(3n_It_{I0}+2nt_0)\sin(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\right.\\
&-\frac{(n_I^2-2nn_I)\sin(3n_It_{I0}+2nt_0)\cos(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\\
&-\frac{(6n_I^2-16nn_I+8n^2)\cos(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+2nt_0-n_It)}{n_I}\\
&+\frac{(6n_I^2-16nn_I+8n^2)\sin(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+2nt_0-n_It)}{n_I}\\
&+\frac{(2nn_I-3n_I^2)\cos(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}\\
&-\frac{(2nn_I-3n_I^2)\sin(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}
\end{aligned}
$$

$$+\left(-\frac{(2nn_I-3n_I^2)\cos(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}-\frac{(6n_I^2-16nn_I+8n^2)\cos(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.$$
$$\left.-\frac{(n_I^2-2nn_I)\cos(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\sin(3n_It_{I0}+2nt_0)$$
$$+\left(\frac{(2nn_I-3n_I^2)\sin(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}+\frac{(6n_I^2-16nn_I+8n^2)\sin(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.$$
$$\left.\left.+\frac{(n_I^2-2nn_I)\sin(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\cos(3n_It_{I0}+2nt_0)\right\}$$
$$-\frac{\cos\varpi}{48n_I^3-128nn_I^2+64n^2n_I}$$
$$\left\{\frac{(n_I^2-2nn_I)\cos(3n_It_{I0}+2nt_0)\sin(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\right.$$
$$-\frac{(n_I^2-2nn_I)\sin(3n_It_{I0}+2nt_0)\cos(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}$$
$$-\frac{(6n_I^2-16nn_I+8n^2)\cos(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+2nt_0-n_It)}{n_I}$$
$$+\frac{(6n_I^2-16nn_I+8n^2)\sin(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+2nt_0-n_It)}{n_I}$$
$$+\frac{(2nn_I-3n_I^2)\cos(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}$$
$$-\frac{(2nn_I-3n_I^2)\sin(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}$$
$$+\left(-\frac{(2nn_I-3n_I^2)\cos(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}-\frac{(6n_I^2-16nn_I+8n^2)\cos(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.$$
$$\left.-\frac{(n_I^2-2nn_I)\cos(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\sin(3n_It_{I0}+2nt_0)$$
$$+\left(\frac{(2nn_I-3n_I^2)\sin(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}+\frac{(6n_I^2-16nn_I+8n^2)\sin(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.$$
$$\left.\left.+\frac{(n_I^2-2nn_I)\sin(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\cos(3n_It_{I0}+2nt_0)\right\}$$
$$-\frac{e_I\sin\varpi}{32n_I-32n}$$
$$\left(-\frac{\sin(2n_It_{I0}+2nt_0)\sin(4n_It_{I0}+(2n-2n_I)t)}{2n-2n_I}-\frac{\cos(2n_It_{I0}+2nt_0)\cos(4n_It_{I0}+(2n-2n_I)t)}{2n-2n_I}\right.$$
$$+\frac{(4n-4n_I)t^2\sin(2n_It_{I0}+2nt_0)^2}{2}+\frac{\sin(4nt_0+(2n_I-2n)t)\sin(2n_It_{I0}+2nt_0)}{2n_I-2n}$$
$$\left.+\frac{(4n-4n_I)t^2\cos(2n_It_{I0}+2nt_0)^2}{2}+\frac{\cos(4nt_0+(2n_I-2n)t)\cos(2n_It_{I0}+2nt_0)}{2n_I-2n}\right)$$
$$-e_I^2\cos\varpi\frac{\sin(3n_It_{I0}-3n_It)}{n_I^2}-e_I^2\sin\varpi\frac{\cos(3n_It_{I0}-3n_It)}{n_I^2}+3e_I\cos\varpi\frac{\sin(2n_It_{I0}-2n_It)}{32n_I^2}$$
$$-3e_I\sin\varpi\frac{\cos(2n_It_{I0}-2n_It)}{8n_I^2}-15e_I^2\cos\varpi\frac{\sin(n_It_{I0}-n_It)}{8n_I^2}-3\cos\varpi\frac{\sin(n_It_{I0}-n_It)}{n_I^2}$$
$$\left.-15e_I^2\sin\varpi\frac{\cos(n_It_{I0}-n_It)}{8n_I^2}-3\sin\varpi\frac{\cos(n_It_{I0}-n_It)}{n_I^2}\right]$$
$$+\frac{a}{64(2304nn_i^6-13824n^2n_i^5+33776n^3n_i^4-42944n^4n_i^3+29912n^5n_i^2-10800n^6n_i+1575n^7)}$$
$$\left[-\frac{429(384nn_i^5-1984n^2n_i^4+3976n^3n_i^3-3844n^4n_i^2+1782n^5n_i-315n^6)}{6n_i-5n}\right.$$

$$\left\{\left(\cos\left(\left(6n_i - 5n\right)t - 6n_i t_{i0} + 5nt_0 + \frac{\pi}{3}\right) - \cos\left(-6n_i t_{i0} + 5nt_0 + \frac{\pi}{3}\right)\right)\right.$$

$$\left. - \mu(\cos((6n_i - 5n)t - 6n_i t_{i0} + 5nt_0) - \cos(-6n_i t_{i0} + 5nt_0))\right\}$$

$$- \frac{429(384nn_i^5 - 1856n^2n_i^4 + 3464n^3n_i^3 - 3116n^4n_i^2 + 1350n^5n_i - 225n^6)}{6n_i - 7n}$$

$$\left\{\left(\cos\left(\left(6n_i - 7n\right)t - 6n_i t_{i0} + 7nt_0 - \frac{\pi}{3}\right) - \cos\left(-6n_i t_{i0} + 7nt_0 - \frac{\pi}{3}\right)\right)\right.$$

$$\left. - \mu(\cos((6n_i - 7n)t - 6n_i t_{i0} + 7nt_0) - \cos(-6n_i t_{i0} + 7nt_0))\right\}$$

$$- \frac{594(576nn_i^5 - 3024n^2n_i^4 + 6176n^3n_i^3 - 6104n^4n_i^2 + 2900n^5n_i - 525n^6)}{4n_i - 3n}$$

$$\left\{\left(\cos\left(\left(4n_i - 3n\right)t - 4n_i t_{i0} + 3nt_0 + \frac{\pi}{3}\right) - \cos\left(-4n_i t_{i0} + 3nt_0 + \frac{\pi}{3}\right)\right)\right.$$

$$\left. - \mu(\cos((4n_i - 3n)t - 4n_i t_{i0} + 3nt_0) - \cos(-4n_i t_{i0} + 3nt_0))\right\}$$

$$- \frac{594(576nn_i^5 - 2736n^2n_i^4 + 5024n^3n_i^3 - 4456n^4n_i^2 + 1908n^5n_i - 315n^6)}{4n_i - 5n}$$

$$\left\{\left(\cos\left(\left(4n_i - 5n\right)t - 4n_i t_{i0} + 5nt_0 - \frac{\pi}{3}\right) - \cos\left(-4n_i t_{i0} + 5nt_0 - \frac{\pi}{3}\right)\right)\right.$$

$$\left. - mu(\cos((4n_i - 5n)t - 4n_i t_{i0} + 5nt_0) - \cos(-4n_i t_{i0} + 5nt_0))\right\}$$

$$- \frac{675(1152nn_i^5 - 6336n^2n_i^4 + 13720n^3n_i^3 - 14612n^4n_i^2 + 7650n^5n_i - 1575n^6)}{2n_i - n}$$

$$\left\{\left(\cos\left(\left(2n_i - n\right)t - 2n_i t_{i0} + nt_0 + \frac{\pi}{3}\right) - \cos\left(-2n_i t_{i0} + nt_0 + \frac{\pi}{3}\right)\right)\right.$$

$$\left. - \mu(\cos((2n_i - n)t - 2n_i t_{i0} + nt_0) - \cos(-2n_i t_{i0} + nt_0))\right\}$$

$$- \frac{675(1152nn_i^5 - 5184n^2n_i^4 + 9112n^3n_i^3 - 7804n^4n_i^2 + 3250n^5n_i - 525n^6)}{2n_i - 3n}$$

$$\left\{\left(\cos\left(\left(2n_i - 3n\right)t - 2n_i t_{i0} + 3nt_0 - \frac{\pi}{3}\right) - \cos\left(-2n_i t_{i0} + 3nt_0 - \frac{\pi}{3}\right)\right)\right.$$

$$\left. - \mu(\cos((2n_i - 3n)t - 2n_i t_{i0} + 3nt_0) - \cos(-2n_i t_{i0} + 3nt_0))\right\}$$

$$- \frac{700(2304n_i^6 - 13824nn_i^5 + 33776n^2n_i^4 - 42944n^3n_i^3 + 29912n^4n_i^2 - 10800n^5n_i + 1575n^6)}{n}$$

$$\left\{\left(\cos\left(nt - nt_0 + \frac{\pi}{3}\right) - \cos\left(-nt_0 + \frac{\pi}{3}\right)\right)\right.$$

$$\left.\left. - \mu(\cos(nt - nt_0) - \cos(-nt_0))\right\}\right] \tag{B.14}$$

$\delta x_E^{(6)}$

$$= a_I \Bigg[\frac{3e_I^2 \sin \varpi}{240n_I^3 + 384nn_I^2 - 192n^2 n_I}$$

$$\Bigg\{ \frac{(-3n_I^2 - 6nn_I)\sin(5n_I t_{I0} + 2nt_0)\sin(10n_I t_{I0} + (2n - 5n_I)t)}{2n - 5n_I}$$

$$- \frac{(3n_I^2 + 6nn_I)\cos(5n_I t_{I0} + 2nt_0)\cos(10n_I t_{I0} + (2n - 5n_I)t)}{2n - 5n_I}$$

$$- \frac{(-10n_I^2 - 16nn_I + 8n^2)\sin(5n_I t_{I0} + 2nt_0)\sin(8n_I t_{I0} + 2nt_0 - 3n_I t)}{3n_I}$$

$$+ \frac{(10n_I^2 + 16nn_I - 8n^2)\cos(5n_I t_{I0} + 2nt_0)\cos(8n_I t_{I0} + 2nt_0 - 3n_I t)}{3n_I}$$

$$+ \frac{(6nn_I - 15n_I^2)\sin(5n_I t_{I0} + 2nt_0)\sin(6n_I t_{I0} + 4nt_0 + (-n_I - 2n)t)}{-n_I - 2n}$$

$$- \frac{(15n_I^2 - 6nn_I)\cos(5n_I t_{I0} + 2nt_0)\cos(6n_I t_{I0} + 4nt_0 + (-n_I - 2n)t)}{-n_I - 2n}$$

$$+ \Bigg(\frac{(15n_I^2 - 6nn_I)\sin(4n_I t_{I0} + (n_I + 2n)t)}{n_I + 2n} + \frac{(10n_I^2 + 16nn_I - 8n^2)\sin(2n_I t_{I0} + 2nt_0 + 3n_I t)}{3n_I}$$

$$+ \frac{(3n_I^2 + 6nn_I)\sin(4nt_0 + (5n_I - 2n)t)}{5n_I - 2n} \Bigg) \sin(5n_I t_{I0} + 2nt_0)$$

$$+ \Bigg(- \frac{(6nn_I - 15n_I^2)\cos(4n_I t_{I0} + (n_I + 2n)t)}{n_I + 2n} - \frac{(-10n_I^2 - 16nn_I + 8n^2)\cos(2n_I t_{I0} + 2nt_0 + 3n_I t)}{3n_I}$$

$$- \frac{(-3n_I^2 - 6nn_I)\cos(4nt_0 + (5n_I - 2n)t)}{5n_I - 2n} \Bigg) \cos(5n_I t_{I0} + 2nt_0) \Bigg\}$$

$$- \frac{3e_I^2 \cos \varpi}{240n_I^3 + 384nn_I^2 - 192n^2 n_I}$$

$$\Bigg\{ \frac{(3n_I^2 + 6nn_I)\cos(5n_I t_{I0} + 2nt_0)\sin(10n_I t_{I0} + (2n - 5n_I)t)}{2n - 5n_I}$$

$$- \frac{(3n_I^2 + 6nn_I)\sin(5n_I t_{I0} + 2nt_0)\cos(10n_I t_{I0} + (2n - 5n_I)t)}{2n - 5n_I}$$

$$- \frac{(10n_I^2 + 16nn_I - 8n^2)\cos(5n_I t_{I0} + 2nt_0)\sin(8n_I t_{I0} + 2nt_0 - 3n_I t)}{3n_I}$$

$$+ \frac{(10n_I^2 + 16nn_I - 8n^2)\sin(5n_I t_{I0} + 2nt_0)\cos(8n_I t_{I0} + 2nt_0 - 3n_I t)}{3n_I}$$

$$+ \frac{(15n_I^2 - 6nn_I)\cos(5n_I t_{I0} + 2nt_0)\sin(6n_I t_{I0} + 4nt_0 + (-n_I - 2n)t)}{-n_I - 2n}$$

$$- \frac{(15n_I^2 - 6nn_I)\sin(5n_I t_{I0} + 2nt_0)\cos(6n_I t_{I0} + 4nt_0 + (-n_I - 2n)t)}{-n_I - 2n}$$

$$+ \Bigg(- \frac{(15n_I^2 - 6nn_I)\cos(4n_I t_{I0} + (n_I + 2n)t)}{n_I + 2n} - \frac{(10n_I^2 + 16nn_I - 8n^2)\cos(2n_I t_{I0} + 2nt_0 + 3n_I t)}{3n_I}$$

$$- \frac{(3n_I^2 + 6nn_I)\cos(4nt_0 + (5n_I - 2n)t)}{5n_I - 2n} \Bigg) \sin(5n_I t_{I0} + 2nt_0)$$

$$+ \Bigg(\frac{(15n_I^2 - 6nn_I)\sin(4n_I t_{I0} + (n_I + 2n)t)}{n_I + 2n} + \frac{(10n_I^2 + 16nn_I - 8n^2)\sin(2n_I t_{I0} + 2nt_0 + 3n_I t)}{3n_I}$$

$$+ \frac{(3n_I^2 + 6nn_I)\sin(4nt_0 + (5n_I - 2n)t)}{5n_I - 2n} \Bigg) \cos(5n_I t_{I0} + 2nt_0) \Bigg\}$$

$$+ \frac{e_I \sin \varpi}{128nn_I^2 - 64n^2 n_I}$$

$$\Bigg\{ - \frac{nn_I \sin(4n_I t_{I0} + 2nt_0)\sin(8n_I t_{I0} + (2n - 4n_I)t)}{2n - 4n_I}$$

$$- \frac{nn_I \cos(4n_I t_{I0} + 2nt_0)\cos(8n_I t_{I0} + (2n - 4n_I)t)}{2n - 4n_I}$$

$$-\frac{(2n^2-4nn_I)\sin(4n_It_{I0}+2nt_0)\sin(6n_It_{I0}+2nt_0-2n_It)}{2n_I}$$
$$+\frac{(4nn_I-2n^2)\cos(4n_It_{I0}+2nt_0)\cos(6n_It_{I0}+2nt_0-2n_It)}{2n_I}$$
$$-\frac{(nn_I-2n_I^2)\sin(4n_It_{I0}+2nt_0)\sin(4n_It_{I0}+4nt_0-2nt)}{2n}$$
$$+\frac{(2n_I^2-nn_I)\cos(4n_It_{I0}+2nt_0)\cos(4n_It_{I0}+4nt_0-2nt)}{2n}$$
$$+\left(\frac{(2n_I^2-nn_I)\sin(4n_It_{I0}+2nt)}{2n}+\frac{(4nn_I-2n^2)\sin(2n_It_{I0}+2nt_0+2n_It)}{2n_I}\right.$$
$$\left.+\frac{nn_I\sin(4nt_0+(4n_I-2n)t)}{4n_I-2n}\right)\sin(4n_It_{I0}+2nt_0)$$
$$+\left(-\frac{(nn_I-2n_I^2)\cos(4n_It_{I0}+2nt)}{2n}-\frac{(2n^2-4nn_I)\cos(2n_It_{I0}+2nt_0+2n_It)}{2n_I}\right.$$
$$\left.+\frac{nn_I\cos(4nt_0+(4n_I-2n)t)}{4n_I-2n}\right)\cos(4n_It_{I0}+2nt_0)\Bigg\}$$
$$+\frac{e_I\cos\varpi}{512nn_I^2-256n^2n_I}$$
$$\Bigg\{\frac{nn_I\cos(4n_It_{I0}+2nt_0)\sin(8n_It_{I0}+(2n-4n_I)t)}{2n-4n_I}$$
$$-\frac{nn_I\sin(4n_It_{I0}+2nt_0)\cos(8n_It_{I0}+(2n-4n_I)t)}{2n-4n_I}$$
$$-\frac{(4nn_I-2n^2)\cos(4n_It_{I0}+2nt_0)\sin(6n_It_{I0}+2nt_0-2n_It)}{2n_I}$$
$$+\frac{(4nn_I-2n^2)\sin(4n_It_{I0}+2nt_0)\cos(6n_It_{I0}+2nt_0-2n_It)}{2n_I}$$
$$-\frac{(2n_I^2-nn_I)\cos(4n_It_{I0}+2nt_0)\sin(4n_It_{I0}+4nt_0-2nt)}{2n}$$
$$+\frac{(2n_I^2-nn_I)\sin(4n_It_{I0}+2nt_0)\cos(4n_It_{I0}+4nt_0-2nt)}{2n}$$
$$+\left(-\frac{(2n_I^2-nn_I)\cos(4n_It_{I0}+2nt)}{2n}-\frac{(4nn_I-2n^2)\cos(2n_It_{I0}+2nt_0+2n_It)}{2n_I}\right.$$
$$\left.-\frac{nn_I\cos(4nt_0+(4n_I-2n)t)}{4n_I-2n}\right)\sin(4n_It_{I0}+2nt_0)$$
$$+\left(\frac{(2n_I^2-nn_I)\sin(4n_It_{I0}+2nt)}{(2n}+\frac{(4nn_I-2n^2)\sin(2n_It_{I0}+2nt_0+2n_It)}{2n_I}\right.$$
$$\left.+\frac{nn_I\sin(4nt_0+(4n_I-2n)t)}{4n_I-2n}\right)\cos(4n_It_{I0}+2nt_0)\Bigg\}$$
$$+\frac{5e_I^2\sin\varpi}{384n_I^3-1024nn_I^2+512n^2n_I}$$
$$\Bigg\{\frac{(2nn_I-n_I^2)\sin(3n_It_{I0}+2nt_0)\sin(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}$$
$$-\frac{(n_I^2-2nn_I)\cos(3n_It_{I0}+2nt_0)\cos(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}$$
$$-\frac{(-6n_I^2+16nn_I-8n^2)\sin(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+2nt_0-n_It)}{n_I}$$
$$+\frac{(6n_I^2-16nn_I+8n^2)\cos(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+2nt_0-n_It)}{n_I}$$
$$+\frac{(2nn_I-3n_I^2)\sin(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}$$
$$-\frac{(3n_I^2-2nn_I)\cos(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}$$

$$
\begin{aligned}
&+\left(\frac{(3n_I^2-2nn_I)\sin(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}+\frac{(6n_I^2-16nn_I+8n^2)\sin(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.\\
&\left.+\frac{(n_I^2-2nn_I)\sin(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\sin(3n_It_{I0}+2nt_0)\\
&+\left(-\frac{(2nn_I-3n_I^2)\cos(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}-\frac{(-6n_I^2+16nn_I-8n^2)\cos(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.\\
&\left.\left.-\frac{(2nn_I-n_I^2)\cos(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\cos(3n_It_{I0}+2nt_0)\right\}\\
&+\frac{\sin\varpi}{48n_I^3-128nn_I^2+64n^2n_I}\\
&\left\{\frac{(2nn_I-n_I^2)\sin(3n_It_{I0}+2nt_0)\sin(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\right.\\
&-\frac{(n_I^2-2nn_I)\cos(3n_It_{I0}+2nt_0)\cos(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\\
&-\frac{(-6n_I^2+16nn_I-8n^2)\sin(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+2nt_0-n_It)}{n_I}\\
&+\frac{(6n_I^2-16nn_I+8n^2)\cos(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+2nt_0-n_It)}{n_I}\\
&+\frac{(2nn_I-3n_I^2)\sin(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}\\
&-\frac{(3n_I^2-2nn_I)\cos(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}\\
&+\left(\frac{(3n_I^2-2nn_I)\sin(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}+\frac{(6n_I^2-16nn_I+8n^2)\sin(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.\\
&\left.+\frac{(n_I^2-2nn_I)\sin(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\sin(3n_It_{I0}+2nt_0)\\
&+\left(-\frac{(2nn_I-3n_I^2)\cos(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}-\frac{(-6n_I^2+16nn_I-8n^2)\cos(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.\\
&\left.\left.-\frac{(2nn_I-n_I^2)\cos(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\cos(3n_It_{I0}+2nt_0)\right\}\\
&-\frac{5e_I^2\cos\varpi}{384n_I^3-1024nn_I^2+512n^2n_I}\\
&\left\{\frac{(n_I^2-2nn_I)\cos(3n_It_{I0}+2nt_0)\sin(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\right.\\
&-\frac{(n_I^2-2nn_I)\sin(3n_It_{I0}+2nt_0)\cos(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}\\
&-\frac{(6n_I^2-16nn_I+8n^2)\cos(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+2nt_0-n_It)}{n_I}\\
&+\frac{(6n_I^2-16nn_I+8n^2)\sin(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+2nt_0-n_It)}{n_I}\\
&+\frac{(2nn_I-3n_I^2)\cos(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}\\
&-\frac{(2nn_I-3n_I^2)\sin(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}\\
&+\left(-\frac{(2nn_I-3n_I^2)\cos(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}-\frac{(6n_I^2-16nn_I+8n^2)\cos(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.\\
&\left.-\frac{(n_I^2-2nn_I)\cos(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\sin(3n_It_{I0}+2nt_0)
\end{aligned}
$$

$$+\left(\frac{(2nn_I-3n_I^2)\sin(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}+\frac{(6n_I^2-16nn_I+8n^2)\sin(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.$$
$$\left.+\frac{(n_I^2-2nn_I)\sin(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\cos(3n_It_{I0}+2nt_0)\Bigg\}$$
$$-\frac{\cos\varpi}{48n_I^3-128nn_I^2+64n^2n_I}$$
$$\Bigg\{\frac{(n_I^2-2nn_I)\cos(3n_It_{I0}+2nt_0)\sin(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}$$
$$-\frac{(n_I^2-2nn_I)\sin(3n_It_{I0}+2nt_0)\cos(6n_It_{I0}+(2n-3n_I)t)}{2n-3n_I}$$
$$-\frac{(6n_I^2-16nn_I+8n^2)\cos(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+2nt_0-n_It)}{n_I}$$
$$+\frac{(6n_I^2-16nn_I+8n^2)\sin(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+2nt_0-n_It)}{n_I}$$
$$+\frac{(2nn_I-3n_I^2)\cos(3n_It_{I0}+2nt_0)\sin(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}$$
$$-\frac{(2nn_I-3n_I^2)\sin(3n_It_{I0}+2nt_0)\cos(4n_It_{I0}+(2n-n_I)t)}{2n-n_I}$$
$$+\left(-\frac{(2nn_I-3n_I^2)\cos(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}-\frac{(6n_I^2-16nn_I+8n^2)\cos(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.$$
$$\left.-\frac{(n_I^2-2nn_I)\cos(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\sin(3n_It_{I0}+2nt_0)$$
$$+\left(\frac{(2nn_I-3n_I^2)\sin(2n_It_{I0}+4nt_0+(n_I-2n)t)}{n_I-2n}+\frac{(6n_I^2-16nn_I+8n^2)\sin(2n_It_{I0}+2nt_0+n_It)}{n_I}\right.$$
$$\left.+\frac{(n_I^2-2nn_I)\sin(4nt_0+(3n_I-2n)t)}{3n_I-2n}\right)\cos(3n_It_{I0}+2nt_0)\Bigg\}$$
$$-\frac{e_I\sin\varpi}{32n_I-32n}$$
$$\left(-\frac{\sin(2n_It_{I0}+2nt_0)\sin(4n_It_{I0}+(2n-2n_I)t)}{2n-2n_I}-\frac{\cos(2n_It_{I0}+2nt_0)\cos(4n_It_{I0}+(2n-2n_I)t)}{2n-2n_I}\right.$$
$$+\frac{(4n-4n_I)t^2\sin(2n_It_{I0}+2nt_0)^2}{2}+\frac{\sin(4nt_0+(2n_I-2n)t)\sin(2n_It_{I0}+2nt_0)}{2n_I-2n}$$
$$\left.+\frac{(4n-4n_I)t^2\cos(2n_It_{I0}+2nt_0)^2}{2}+\frac{\cos(4nt_0+(2n_I-2n)t)\cos(2n_It_{I0}+2nt_0)}{2n_I-2n}\right)$$
$$-e_I^2\cos\varpi\frac{\sin(3n_It_{I0}-3n_It)}{n_I^2}-e_I^2\sin\varpi\frac{\cos(3n_It_{I0}-3n_It)}{n_I^2}+3e_I\cos\varpi\frac{\sin(2n_It_{I0}-2n_It)}{32n_I^2}$$
$$-3e_I\sin\varpi\frac{\cos(2n_It_{I0}-2n_It)}{8n_I^2}-15e_I^2\cos\varpi\frac{\sin(n_It_{I0}-n_It)}{8n_I^2}-3\cos\varpi\frac{\sin(n_It_{I0}-n_It)}{n_I^2}$$
$$\left.-15e_I^2\sin\varpi\frac{\cos(n_It_{I0}-n_It)}{8n_I^2}-3\sin\varpi\frac{\cos(n_It_{I0}-n_It)}{n_I^2}\right]$$
$$-\frac{a}{64(2304nn_i^6-13824n^2n_i^5+33776n^3n_i^4-42944n^4n_i^3+29912n^5n_i^2-10800n^6n_i+1575n^7)}$$
$$\left[\frac{429(384nn_i^5-1984n^2n_i^4+3976n^3n_i^3-3844n^4n_i^2+1782n^5n_i-315n^6)}{6n_i-5n}\right.$$
$$\Bigg\{\left(\cos\left(\left(6n_i-5n\right)t-6n_it_{i0}+5nt_0+\frac{\pi}{3}\right)-\cos\left(-6n_it_{i0}+5nt_0+\frac{\pi}{3}\right)\right)$$
$$-\mu(\cos((6n_i-5n)t-6n_it_{i0}+5nt_0)-\cos(-6n_it_{i0}+5nt_0))\Bigg\}$$

$$
\begin{aligned}
&-\frac{429(384nn_i^5-1856n^2n_i^4+3464n^3n_i^3-3116n^4n_i^2+1350n^5n_i-225n^6)}{6n_i-7n}\\
&\left\{\left(\cos\left(\left(6n_i-7n\right)t-6n_it_{i0}+7nt_0-\frac{\pi}{3}\right)-\cos\left(-6n_it_{i0}+7nt_0-\frac{\pi}{3}\right)\right)\right.\\
&\left.-\mu(\cos((6n_i-7n)t-6n_it_{i0}+7nt_0)-\cos(-6n_it_{i0}+7nt_0))\right\}\\
&+\frac{594(576nn_i^5-3024n^2n_i^4+6176n^3n_i^3-6104n^4n_i^2+2900n^5n_i-525n^6)}{4n_i-3n}\\
&\left\{\left(\cos\left(\left(4n_i-3n\right)t-4n_it_{i0}+3nt_0+\frac{\pi}{3}\right)-\cos\left(-4n_it_{i0}+3nt_0+\frac{\pi}{3}\right)\right)\right.\\
&\left.-\mu(\cos((4n_i-3n)t-4n_it_{i0}+3nt_0)-\cos(-4n_it_{i0}+3nt_0))\right\}\\
&-\frac{594(576nn_i^5-2736n^2n_i^4+5024n^3n_i^3-4456n^4n_i^2+1908n^5n_i-315n^6)}{4n_i-5n}\\
&\left\{\left(\cos\left(\left(4n_i-5n\right)t-4n_it_{i0}+5nt_0-\frac{\pi}{3}\right)-\cos\left(-4n_it_{i0}+5nt_0-\frac{\pi}{3}\right)\right)\right.\\
&\left.-\mu(\cos((4n_i-5n)t-4n_it_{i0}+5nt_0)-\cos(-4n_it_{i0}+5nt_0))\right\}\\
&+\frac{675(1152nn_i^5-6336n^2n_i^4+13720n^3n_i^3-14612n^4n_i^2+7650n^5n_i-1575n^6)}{2n_i-n}\\
&\left\{\left(\cos\left(\left(2n_i-n\right)t-2n_it_{i0}+nt_0+\frac{\pi}{3}\right)-\cos\left(-2n_it_{i0}+nt_0+\frac{\pi}{3}\right)\right)\right.\\
&\left.-\mu(\cos((2n_i-n)t-2n_it_{i0}+nt_0)-\cos(-2n_it_{i0}+nt_0))\right\}\\
&-\frac{675(1152nn_i^5-5184n^2n_i^4+9112n^3n_i^3-7804n^4n_i^2+3250n^5n_i-525n^6)}{2n_i-3n}\\
&\left\{\left(\cos\left(\left(2n_i-3n\right)t-2n_it_{i0}+3nt_0-\frac{\pi}{3}\right)-\cos\left(-2n_it_{i0}+3nt_0-\frac{\pi}{3}\right)\right)\right.\\
&\left.-\mu(\cos((2n_i-3n)t-2n_it_{i0}+3nt_0)-\cos(-2n_it_{i0}+3nt_0))\right\}\\
&+\frac{700(2304n_i^6-13824nn_i^5+33776n^2n_i^4-42944n^3n_i^3+29912n^4n_i^2-10800n^5n_i+1575n^6)}{n}\\
&\left\{\left(\cos\left(nt-nt_0+\frac{\pi}{3}\right)-\cos\left(-nt_0+\frac{\pi}{3}\right)\right)\right.\\
&\left.\left.-\mu(\cos(nt-nt_0)-\cos(-nt_0))\right\}\right]
\end{aligned}
\tag{B.15}
$$

B.8 Order of q^7

$$
\begin{aligned}
&\delta x_E^{(7)} \\
= \quad & a_I \left[\frac{3e_I^2 \sin\varpi}{64n_I^2 + 16nn_I - 8n^2} \right. \\
& \left\{ \frac{(-2n_I - n)\sin(4n_I t_{I0} + nt_0)\sin(8n_I t_{I0} + (n - 4n_I)t)}{n - 4n_I} - \frac{(2n_I + n)\cos(4n_I t_{I0} + nt_0)\cos(8n_I t_{I0} + (n - 4n_I)t)}{n - 4n_I} \right. \\
& + \frac{(n - 4n_I)\sin(4n_I t_{I0} + nt_0)\sin(6n_I t_{I0} + 2nt_0 + (-2n_I - n)t)}{-2n_I - n} \\
& - \frac{(4n_I - n)\cos(4n_I t_{I0} + nt_0)\cos(6n_I t_{I0} + 2nt_0 + (-2n_I - n)t)}{-2n_I - n} \\
& + \left(\frac{(4n_I - n)\sin(2n_I t_{I0} + (2n_I + n)t)}{2n_I + n} + \frac{(2n_I + n)\sin(2nt_0 + (4n_I - n)t)}{4n_I - n} \right) \sin(4n_I t_{I0} + nt_0) \\
& \left. + \left(-\frac{(n - 4n_I)\cos(2n_I t_{I0} + (2n_I + n)t)}{2n_I + n} - \frac{(-2n_I - n)\cos(2nt_0 + (4n_I - n)t)}{4n_I - n} \right) \cos(4n_I t_{I0} + nt_0) \right\} \\
& - \frac{3e_I^2 \cos\varpi}{64n_I^2 + 16nn_I - 8n^2} \\
& \left\{ \frac{(2n_I + n)\cos(4n_I t_{I0} + nt_0)\sin(8n_I t_{I0} + (n - 4n_I)t)}{n - 4n_I} - \frac{(2n_I + n)\sin(4n_I t_{I0} + nt_0)\cos(8n_I t_{I0} + (n - 4n_I)t)}{n - 4n_I} \right. \\
& + \frac{(4n_I - n)\cos(4n_I t_{I0} + nt_0)\sin(6n_I t_{I0} + 2nt_0 + (-2n_I - n)t)}{-2n_I - n} \\
& - \frac{(4n_I - n)\sin(4n_I t_{I0} + nt_0)\cos(6n_I t_{I0} + 2nt_0 + (-2n_I - n)t)}{-2n_I - n} \\
& + \left(-\frac{(4n_I - n)\cos(2n_I t_{I0} + (2n_I + n)t)}{2n_I + n} - \frac{(2n_I + n)\cos(2nt_0 + (4n_I - n)t)}{4n_I - n} \right) \sin(4n_I t_{I0} + nt_0) \\
& \left. + \left(\frac{(4n_I - n)\sin(2n_I t_{I0} + (2n_I + n)t)}{2n_I + n} + \frac{(2n_I + n)\sin(2nt_0 + (4n_I - n)t)}{4n_I - n} \right) \cos(4n_I t_{I0} + nt_0) \right\} \\
& + \frac{e_I \sin\varpi}{48n_I^2 + 32nn_I - 16n^2} \\
& \left\{ \frac{(-n_I - n)\sin(3n_I t_{I0} + nt_0)\sin(6n_I t_{I0} + (n - 3n_I)t)}{n - 3n_I} - \frac{(n_I + n)\cos(3n_I t_{I0} + nt_0)\cos(6n_I t_{I0} + (n - 3n_I)t)}{n - 3n_I} \right. \\
& + \frac{(n - 3n_I)\sin(3n_I t_{I0} + nt_0)\sin(4n_I t_{I0} + 2nt_0 + (-n_I - n)t)}{-n_I - n} \\
& - \frac{(3n_I - n)\cos(3n_I t_{I0} + nt_0)\cos(4n_I t_{I0} + 2nt_0 + (-n_I - n)t)}{-n_I - n} \\
& + \left(\frac{(3n_I - n)\sin(2n_I t_{I0} + (n_I + n)t)}{n_I + n} + \frac{(n_I + n)\sin(2nt_0 + (3n_I - n)t)}{3n_I - n} \right) \sin(3n_I t_{I0} + nt_0) \\
& \left. + \left(-\frac{(n - 3n_I)\cos(2n_I t_{I0} + (n_I + n)t)}{n_I + n} - \frac{(-n_I - n)\cos(2nt_0 + (3n_I - n)t)}{3n_I - n} \right) \cos(3n_I t_{I0} + nt_0) \right\} \\
& + \frac{e_I \cos\varpi}{192n_I^2 + 128nn_I - 64n^2} \\
& \left\{ \frac{(n_I + n)\cos(3n_I t_{I0} + nt_0)\sin(6n_I t_{I0} + (n - 3n_I)t)}{n - 3n_I} - \frac{(n_I + n)\sin(3n_I t_{I0} + nt_0)\cos(6n_I t_{I0} + (n - 3n_I)t)}{n - 3n_I} \right. \\
& + \frac{(3n_I - n)\cos(3n_I t_{I0} + nt_0)\sin(4n_I t_{I0} + 2nt_0 + (-n_I - n)t)}{-n_I - n} \\
& - \frac{(3n_I - n)\sin(3n_I t_{I0} + nt_0)\cos(4n_I t_{I0} + 2nt_0 + (-n_I - n)t)}{-n_I - n} \\
& + \left(-\frac{(3n_I - n)\cos(2n_I t_{I0} + (n_I + n)t)}{n_I + n} - \frac{(n_I + n)\cos(2nt_0 + (3n_I - n)t)}{3n_I - n} \right) \sin(3n_I t_{I0} + nt_0) \\
& \left. + \left(\frac{(3n_I - n)\sin(2n_I t_{I0} + (n_I + n)t)}{n_I + n} + \frac{(n_I + n)\sin(2nt_0 + (3n_I - n)t)}{3n_I - n} \right) \cos(3n_I t_{I0} + nt_0) \right\} \\
& + \frac{5e_I^2 \sin\varpi}{128nn_I - 64n^2}
\end{aligned}
$$

$$\Big\{-\frac{n\sin(2n_It_{I0}+nt_0)\sin(4n_It_{I0}+(n-2n_I)t)}{n-2n_I}-\frac{n\cos(2n_It_{I0}+nt_0)\cos(4n_It_{I0}+(n-2n_I)t)}{-2n-n_I}$$

$$-\frac{(n-2n_I)\sin(2n_It_{I0}+nt_0)\sin(2n_It_{I0}+2nt_0-nt)}{n}$$

$$+\frac{(2n_I-n)\cos(2n_It_{I0}+nt_0)\cos(2n_It_{I0}+2nt_0-nt)}{n}$$

$$+\left(\frac{(2n_I-n)\sin(2n_It_{I0}+nt)}{n}+\frac{n\sin(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\sin(2n_It_{I0}+nt_0)$$

$$+\left(\frac{n\cos(2nt_0+(2n_I-n)t)}{2n_I-n}-\frac{(n-2n_I)\cos(2n_It_{I0}+nt)}{n}\right)\cos(2n_It_{I0}+nt_0)\Big\}$$

$$+\frac{\sin\varpi}{16nn_I-8n^2}$$

$$\Big\{-\frac{n\sin(2n_It_{I0}+nt_0)\sin(4n_It_{I0}+(n-2n_I)t)}{n-2n_I}-\frac{n\cos(2n_It_{I0}+nt_0)\cos(4n_It_{I0}+(n-2n_I)t)}{n-2n_I}$$

$$-\frac{(n-2n_I)\sin(2n_It_{I0}+nt_0)\sin(2n_It_{I0}+2nt_0-nt)}{n}$$

$$+\frac{(2n_I-n)\cos(2n_It_{I0}+nt_0)\cos(2n_It_{I0}+2nt_0-nt)}{/}n$$

$$+\left(\frac{(2n_I-n)\sin(2n_It_{I0}+nt)}{n}+\frac{n\sin(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\sin(2n_It_{I0}+nt_0)$$

$$+\left(\frac{n\cos(2nt_0+(2n_I-n)t)}{2n_I-n}-\frac{(n-2n_I)\cos(2n_It_{I0}+nt)}{n}\right)\cos(2n_It_{I0}+nt_0)\Big\}$$

$$-\frac{5e_I^2\cos\varpi}{128nn_I-64n^2}\Big\{\frac{n\cos(2n_It_{I0}+nt_0)\sin(4n_It_{I0}+(n-2n_I)t)}{n-2n_I}-\frac{n\sin(2n_It_{I0}+nt_0)\cos(4n_It_{I0}+(n-2n_I)t)}{n-2n_I}$$

$$-\frac{(2n_I-n)\cos(2n_It_{I0}+nt_0)\sin(2n_It_{I0}+2nt_0-nt)}{n}$$

$$+\frac{(2n_I-n)\sin(2n_It_{I0}+nt_0)\cos(2n_It_{I0}+2nt_0-nt)}{n}$$

$$+\left(-\frac{(2n_I-n)\cos(2n_It_{I0}+nt)}{n}-\frac{n\cos(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\sin(2n_It_{I0}+nt_0)$$

$$+\left(\frac{(2n_I-n)\sin(2n_It_{I0}+nt)}{n}+\frac{n\sin(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\cos(2n_It_{I0}+nt_0)\Big\}$$

$$-\frac{\cos\varpi}{16nn_I-8n^2}$$

$$\Big\{\frac{n\cos(2n_It_{I0}+nt_0)\sin(4n_It_{I0}+(n-2n_I)t)}{2n-n_I}-\frac{n\sin(2n_It_{I0}+nt_0)\cos(4n_It_{I0}+(n-2n_I)t)}{2n-n_I}$$

$$-\frac{(2n_I-n)\cos(2n_It_{I0}+nt_0)\sin(2n_It_{I0}+2nt_0-nt)}{n}$$

$$+\frac{(2n_I-n)\sin(2n_It_{I0}+nt_0)\cos(2n_It_{I0}+2nt_0-nt)}{n}$$

$$+\left(-\frac{(2n_I-n)\cos(2n_It_{I0}+nt)}{n}-\frac{n\cos(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\sin(2n_It_{I0}+nt_0)$$

$$+\left(\frac{(2n_I-n)\sin(2n_It_{I0}+nt)}{n}+\frac{n\sin(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\cos(2n_It_{I0}+nt_0)\Big\}$$

$$-e_I\sin\varpi\frac{\cos(n_It_{I0}-nt_0-n_It+nt)}{2(n-n_I)^2}\Big]$$

$$+\frac{a}{128(11025n_i^8-88200nn_i^7+295784n^2n_i^6-539904n^3n_i^5+579984n^4n_i^4-366976n^5n_i^3+126720n^6n_i^2-18432n^7n_i)}$$

$$\Bigg[-\frac{2145(1575n_i^7-11250nn_i^6+32612n^2n_i^5-49176n^3n_i^4+40704n^4n_i^3-17536n^5n_i^2+3072n^6n_i)}{7n_i-6n}$$

$$\Bigg\{\Bigg(\cos\Bigg(\Bigg(7n_i-6n\Bigg)t-7n_it_{i0}+6nt_0+\frac{\pi}{3}\Bigg)-\cos\Bigg(-7n_it_{i0}+6nt_0+\frac{\pi}{3}\Bigg)\Bigg)$$

$$-\mu(\cos((7n_i-6n)t-7n_it_{i0}+6nt_0)-\cos(-7n_it_{i0}+6nt_0))\Bigg\}$$

$$-\frac{2145(1575n_i^7 - 10800nn_i^6 + 29912n^2n_i^5 - 42944n^3n_i^4 + 33776n^4n_i^3 - 13824n^5n_i^2 + 2304n^6n_i)}{7n_i - 8n}$$

$$\Bigg\{\Bigg(\cos\Bigg(\Bigg(7n_i - 8n\Bigg)t - 7n_it_{i0} + 8nt_0 - \frac{\pi}{3}\Bigg) - \cos\Bigg(- 7n_it_{i0} + 8nt_0 - \frac{\pi}{3}\Bigg)\Bigg)$$

$$- \mu(\cos((7n_i - 8n)t - 7n_it_{i0} + 8nt_0) - \cos(-7n_it_{i0} + 8nt_0))\Bigg\}$$

$$-\frac{3003(2205n_i^7 - 15876nn_i^6 + 46456n^2n_i^5 - 70816n^3n_i^4 + 59344n^4n_i^3 - 25920n^5n_i^2 + 4608n^6n_i)}{5n_i - 4n}$$

$$\Bigg\{\Bigg(\cos\Bigg(\Bigg(5n_i - 4n\Bigg)t - 5n_it_{i0} + 4nt_0 + \frac{\pi}{3}\Bigg) - \cos\Bigg(- 5n_it_{i0} + 4nt_0 + \frac{\pi}{3}\Bigg)\Bigg)$$

$$- \mu(\cos((5n_i - 4n)t - 5n_it_{i0} + 4nt_0) - \cos(-5n_it_{i0} + 4nt_0))\Bigg\}$$

$$-\frac{3003(2205n_i^7 - 14994nn_i^6 + 41164n^2n_i^5 - 58584n^3n_i^4 + 45696n^4n_i^3 - 18560n^5n_i^2 + 3072n^6n_i)}{5n_i - 6n}$$

$$\Bigg\{\Bigg(\cos\Bigg(\Bigg(5n_i - 6n\Bigg)t - 5n_it_{i0} + 6nt_0 - \frac{\pi}{3}\Bigg) - \cos\Bigg(- 5n_it_{i0} + 6nt_0 - \frac{\pi}{3}\Bigg)\Bigg)$$

$$- \mu(\cos((5n_i - 6n)t - 5n_it_{i0} + 6nt_0) - \cos(-5n_it_{i0} + 6nt_0))\Bigg\}$$

$$-\frac{3465(3675n_i^7 - 26950nn_i^6 + 80628n^2n_i^5 - 126216n^3n_i^4 + 109184n^4n_i^3 - 49536n^5n_i^2 + 9216n^6n_i)}{3n_i - 2n}$$

$$\Bigg\{\Bigg(\cos\Bigg(\Bigg(3n_i - 2n\Bigg)t - 3n_it_{i0} + 2nt_0 + \frac{\pi}{3}\Bigg) - \cos\Bigg(- 3n_it_{i0} + 2nt_0 + \frac{\pi}{3}\Bigg)\Bigg)$$

$$- \mu(\cos((3n_i - 2n)t - 3n_it_{i0} + 2nt_0) - \cos(-3n_it_{i0} + 2nt_0))\Bigg\}$$

$$-\frac{3465(3675n_i^7 - 24500nn_i^6 + 65928n^2n_i^5 - 92064n^3n_i^4 + 70576n^4n_i^3 - 28224n^5n_i^2 + 4608n^6n_i)}{3n_i - 4n}$$

$$\Bigg\{\Bigg(\cos\Bigg(\Bigg(3n_i - 4n\Bigg)t - 3n_it_{i0} + 4nt_0 - \frac{\pi}{3}\Bigg) - \cos\Bigg(- 3n_it_{i0} + 4nt_0 - \frac{\pi}{3}\Bigg)\Bigg)$$

$$- \mu(\cos((3n_i - 4n)t - 3n_it_{i0} + 4nt_0) - \cos(-3n_it_{i0} + 4nt_0))\Bigg\}$$

$$-\frac{3675(11025n_i^7 - 66150nn_i^6 + 163484n^2n_i^5 - 212936n^3n_i^4 + 154112n^4n_i^3 - 58752n^5n_i^2 + 9216n^6n_i)}{n_i - 2n}$$

$$\Bigg\{\Bigg(\cos\Bigg(\Bigg(n_i - 2n\Bigg)t - n_it_{i0} + 2nt_0 - \frac{\pi}{3}\Bigg) - \cos\Bigg(- n_it_{i0} + 2nt_0 - \frac{\pi}{3}\Bigg)\Bigg)$$

$$- \mu(\cos((n_i - 2n)t - n_it_{i0} + 2nt_0) - \cos(-n_it_{i0} + 2nt_0))\Bigg\}$$

$$-\frac{3675(11025n_i^7 - 88200nn_i^6 + 295784n^2n_i^5 - 539904n^3n_i^4 + 579984n^4n_i^3 - 366976n^5n_i^2 + 126720n^6n_i - 18432n^7)}{n_i}$$

$$\Bigg\{\Bigg(\cos\Bigg(n_it - n_it_{i0} + \frac{\pi}{3}\Bigg) - \cos\Bigg(- n_it_{i0} + \frac{\pi}{3}\Bigg)\Bigg)$$

$$
\left. \left. - \mu(\cos(n_i t - n_i t_{i0}) - \cos(-n_i t_{i0})) \right\} \right] \tag{B.16}
$$

$$
\begin{aligned}
&\delta y_E^{(7)} \\
= \quad & -a_I \left[\frac{3e_I^2 \sin\varpi}{64n_I^2 + 16nn_I - 8n^2} \right. \\
&\left\{ \frac{(-2n_I - n)\sin(4n_I t_{I0} + nt_0)\sin(8n_I t_{I0} + (n - 4n_I)t)}{n - 4n_I} - \frac{(2n_I + n)\cos(4n_I t_{I0} + nt_0)\cos(8n_I t_{I0} + (n - 4n_I)t)}{n - 4n_I} \right. \\
&+ \frac{(n - 4n_I)\sin(4n_I t_{I0} + nt_0)\sin(6n_I t_{I0} + 2nt_0 + (-2n_I - n)t)}{-2n_I - n} \\
&- \frac{(4n_I - n)\cos(4n_I t_{I0} + nt_0)\cos(6n_I t_{I0} + 2nt_0 + (-2n_I - n)t)}{-2n_I - n} \\
&+ \left(\frac{(4n_I - n)\sin(2n_I t_{I0} + (2n_I + n)t)}{2n_I + n} + \frac{(2n_I + n)\sin(2nt_0 + (4n_I - n)t)}{4n_I - n} \right) \sin(4n_I t_{I0} + nt_0) \\
&+ \left. \left(-\frac{(n - 4n_I)\cos(2n_I t_{I0} + (2n_I + n)t)}{2n_I + n} - \frac{(-2n_I - n)\cos(2nt_0 + (4n_I - n)t)}{4n_I - n} \right) \cos(4n_I t_{I0} + nt_0) \right\} \\
&- \frac{3e_I^2 \cos\varpi}{64n_I^2 + 16nn_I - 8n^2} \\
&\left\{ \frac{(2n_I + n)\cos(4n_I t_{I0} + nt_0)\sin(8n_I t_{I0} + (n - 4n_I)t)}{n - 4n_I} - \frac{(2n_I + n)\sin(4n_I t_{I0} + nt_0)\cos(8n_I t_{I0} + (n - 4n_I)t)}{n - 4n_I} \right. \\
&+ \frac{(4n_I - n)\cos(4n_I t_{I0} + nt_0)\sin(6n_I t_{I0} + 2nt_0 + (-2n_I - n)t)}{-2n_I - n} \\
&- \frac{(4n_I - n)\sin(4n_I t_{I0} + nt_0)\cos(6n_I t_{I0} + 2nt_0 + (-2n_I - n)t)}{-2n_I - n} \\
&+ \left(-\frac{(4n_I - n)\cos(2n_I t_{I0} + (2n_I + n)t)}{2n_I + n} - \frac{(2n_I + n)\cos(2nt_0 + (4n_I - n)t)}{4n_I - n} \right) \sin(4n_I t_{I0} + nt_0) \\
&+ \left. \left(\frac{(4n_I - n)\sin(2n_I t_{I0} + (2n_I + n)t)}{2n_I + n} + \frac{(2n_I + n)\sin(2nt_0 + (4n_I - n)t)}{4n_I - n} \right) \cos(4n_I t_{I0} + nt_0) \right\} \\
&+ \frac{e_I \sin\varpi}{48n_I^2 + 32nn_I - 16n^2} \\
&\left\{ \frac{(-n_I - n)\sin(3n_I t_{I0} + nt_0)\sin(6n_I t_{I0} + (n - 3n_I)t)}{n - 3n_I} - \frac{(n_I + n)\cos(3n_I t_{I0} + nt_0)\cos(6n_I t_{I0} + (n - 3n_I)t)}{n - 3n_I} \right. \\
&+ \frac{(n - 3n_I)\sin(3n_I t_{I0} + nt_0)\sin(4n_I t_{I0} + 2nt_0 + (-n_I - n)t)}{-n_I - n} \\
&- \frac{(3n_I - n)\cos(3n_I t_{I0} + nt_0)\cos(4n_I t_{I0} + 2nt_0 + (-n_I - n)t)}{-n_I - n} \\
&+ \left(\frac{(3n_I - n)\sin(2n_I t_{I0} + (n_I + n)t)}{n_I + n} + \frac{(n_I + n)\sin(2nt_0 + (3n_I - n)t)}{3n_I - n} \right) \sin(3n_I t_{I0} + nt_0) \\
&+ \left. \left(-\frac{(n - 3n_I)\cos(2n_I t_{I0} + (n_I + n)t)}{n_I + n} - \frac{(-n_I - n)\cos(2nt_0 + (3n_I - n)t)}{3n_I - n} \right) \cos(3n_I t_{I0} + nt_0) \right\} \\
&+ \frac{e_I \cos\varpi}{192n_I^2 + 128nn_I - 64n^2} \\
&\left\{ \frac{(n_I + n)\cos(3n_I t_{I0} + nt_0)\sin(6n_I t_{I0} + (n - 3n_I)t)}{n - 3n_I} - \frac{(n_I + n)\sin(3n_I t_{I0} + nt_0)\cos(6n_I t_{I0} + (n - 3n_I)t)}{n - 3n_I} \right. \\
&+ \frac{(3n_I - n)\cos(3n_I t_{I0} + nt_0)\sin(4n_I t_{I0} + 2nt_0 + (-n_I - n)t)}{-n_I - n} \\
&- \frac{(3n_I - n)\sin(3n_I t_{I0} + nt_0)\cos(4n_I t_{I0} + 2nt_0 + (-n_I - n)t)}{-n_I - n} \\
&+ \left(-\frac{(3n_I - n)\cos(2n_I t_{I0} + (n_I + n)t)}{n_I + n} - \frac{(n_I + n)\cos(2nt_0 + (3n_I - n)t)}{3n_I - n} \right) \sin(3n_I t_{I0} + nt_0) \\
&+ \left. \left(\frac{(3n_I - n)\sin(2n_I t_{I0} + (n_I + n)t)}{n_I + n} + \frac{(n_I + n)\sin(2nt_0 + (3n_I - n)t)}{3n_I - n} \right) \cos(3n_I t_{I0} + nt_0) \right\} \\
&+ \frac{5e_I^2 \sin\varpi}{128nn_I - 64n^2}
\end{aligned}
$$

$$
\begin{aligned}
&\left\{-\frac{n\sin(2n_It_{I0}+nt_0)\sin(4n_It_{I0}+(n-2n_I)t)}{n-2n_I}-\frac{n\cos(2n_It_{I0}+nt_0)\cos(4n_It_{I0}+(n-2n_I)t)}{-2n-n_I}\right.\\
&-\frac{(n-2n_I)\sin(2n_It_{I0}+nt_0)\sin(2n_It_{I0}+2nt_0-nt)}{n}\\
&+\frac{(2n_I-n)\cos(2n_It_{I0}+nt_0)\cos(2n_It_{I0}+2nt_0-nt)}{n}\\
&+\left(\frac{(2n_I-n)\sin(2n_It_{I0}+nt)}{n}+\frac{n\sin(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\sin(2n_It_{I0}+nt_0)\\
&\left.+\left(\frac{n\cos(2nt_0+(2n_I-n)t)}{2n_I-n}-\frac{(n-2n_I)\cos(2n_It_{I0}+nt)}{n}\right)\cos(2n_It_{I0}+nt_0)\right\}\\
&+\frac{\sin\varpi}{16nn_I-8n^2}\\
&\left\{-\frac{n\sin(2n_It_{I0}+nt_0)\sin(4n_It_{I0}+(n-2n_I)t)}{n-2n_I}-\frac{n\cos(2n_It_{I0}+nt_0)\cos(4n_It_{I0}+(n-2n_I)t)}{n-2n_I}\right.\\
&-\frac{(n-2n_I)\sin(2n_It_{I0}+nt_0)\sin(2n_It_{I0}+2nt_0-nt)}{n}\\
&+\frac{(2n_I-n)\cos(2n_It_{I0}+nt_0)\cos(2n_It_{I0}+2nt_0-nt)}{/}n\\
&+\left(\frac{(2n_I-n)\sin(2n_It_{I0}+nt)}{n}+\frac{n\sin(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\sin(2n_It_{I0}+nt_0)\\
&\left.+\left(\frac{n\cos(2nt_0+(2n_I-n)t)}{2n_I-n}-\frac{(n-2n_I)\cos(2n_It_{I0}+nt)}{n}\right)\cos(2n_It_{I0}+nt_0)\right\}\\
&-\frac{5e_I^2\cos\varpi}{128nn_I-64n^2}\left\{\frac{n\cos(2n_It_{I0}+nt_0)\sin(4n_It_{I0}+(n-2n_I)t)}{n-2n_I}-\frac{n\sin(2n_It_{I0}+nt_0)\cos(4n_It_{I0}+(n-2n_I)t)}{n-2n_I}\right.\\
&-\frac{(2n_I-n)\cos(2n_It_{I0}+nt_0)\sin(2n_It_{I0}+2nt_0-nt)}{n}\\
&+\frac{(2n_I-n)\sin(2n_It_{I0}+nt_0)\cos(2n_It_{I0}+2nt_0-nt)}{n}\\
&+\left(-\frac{(2n_I-n)\cos(2n_It_{I0}+nt)}{n}-\frac{n\cos(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\sin(2n_It_{I0}+nt_0)\\
&\left.+\left(\frac{(2n_I-n)\sin(2n_It_{I0}+nt)}{n}+\frac{n\sin(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\cos(2n_It_{I0}+nt_0)\right\}\\
&-\frac{\cos\varpi}{16nn_I-8n^2}\\
&\left\{\frac{n\cos(2n_It_{I0}+nt_0)\sin(4n_It_{I0}+(n-2n_I)t)}{2n-n_I}-\frac{n\sin(2n_It_{I0}+nt_0)\cos(4n_It_{I0}+(n-2n_I)t)}{2n-n_I}\right.\\
&-\frac{(2n_I-n)\cos(2n_It_{I0}+nt_0)\sin(2n_It_{I0}+2nt_0-nt)}{n}\\
&+\frac{(2n_I-n)\sin(2n_It_{I0}+nt_0)\cos(2n_It_{I0}+2nt_0-nt)}{n}\\
&+\left(-\frac{(2n_I-n)\cos(2n_It_{I0}+nt)}{n}-\frac{n\cos(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\sin(2n_It_{I0}+nt_0)\\
&\left.+\left(\frac{(2n_I-n)\sin(2n_It_{I0}+nt)}{n}+\frac{n\sin(2nt_0+(2n_I-n)t)}{2n_I-n}\right)\cos(2n_It_{I0}+nt_0)\right\}\\
&\left.-e_I\sin\varpi\frac{\cos(n_It_{I0}-nt_0-n_It+nt)}{2(n-n_I)^2}\right]\\
&-\frac{a}{128(11025n_i^8-88200nn_i^7+295784n^2n_i^6-539904n^3n_i^5+579984n^4n_i^4-366976n^5n_i^3+126720n^6n_i^2-18432n^7n_i)}\\
&\left[\frac{2145(1575n_i^7-11250nn_i^6+32612n^2n_i^5-49176n^3n_i^4+40704n^4n_i^3-17536n^5n_i^2+3072n^6n_i)}{7n_i-6n}\right.\\
&\left\{\left(\sin\left(\left(7n_i-6n\right)t-7n_it_{i0}+6nt_0+\frac{\pi}{3}\right)-\sin\left(-7n_it_{i0}+6nt_0+\frac{\pi}{3}\right)\right)\right.\\
&\left.-\mu(\sin((7n_i-6n)t-7n_it_{i0}+6nt_0)-\sin(-7n_it_{i0}+6nt_0))\right\}
\end{aligned}
$$

$$+\frac{2145(-1575n_i^7+10800nn_i^6-29912n^2n_i^5+42944n^3n_i^4-33776n^4n_i^3+13824n^5n_i^2-2304n^6n_i)}{7n_i-8n}$$
$$\Bigg\{\Bigg(\sin\Bigg(\Bigg(7n_i-8n\Bigg)t-7n_it_{i0}+8nt_0-\frac{\pi}{3}\Bigg)-\sin\Bigg(-7n_it_{i0}+8nt_0-\frac{\pi}{3}\Bigg)\Bigg)$$
$$-\mu(\sin((7n_i-8n)t-7n_it_{i0}+8nt_0)-\sin(-7n_it_{i0}+8nt_0))\Bigg\}$$
$$+\frac{3003(2205n_i^7-15876nn_i^6+46456n^2n_i^5-70816n^3n_i^4+59344n^4n_i^3-25920n^5n_i^2+4608n^6n_i)}{5n_i-4n}$$
$$\Bigg\{\Bigg(\sin\Bigg(\Bigg(5n_i-4n\Bigg)t-5n_it_{i0}+4nt_0+\frac{\pi}{3}\Bigg)-\sin\Bigg(-5n_it_{i0}+4nt_0+\frac{\pi}{3}\Bigg)\Bigg)$$
$$-\mu(\sin((5n_i-4n)t-5n_it_{i0}+4nt_0)-\sin(-5n_it_{i0}+4nt_0))\Bigg\}$$
$$+\frac{3003(-2205n_i^7+14994nn_i^6-41164n^2n_i^5+58584n^3n_i^4-45696n^4n_i^3+18560n^5n_i^2-3072n^6n_i)}{5n_i-6n}$$
$$\Bigg\{\Bigg(\sin\Bigg(\Bigg(5n_i-6n\Bigg)t-5n_it_{i0}+6nt_0-\frac{\pi}{3}\Bigg)-\sin\Bigg(-5n_it_{i0}+6nt_0-\frac{\pi}{3}\Bigg)\Bigg)$$
$$-\mu(\sin((5n_i-6n)t-5n_it_{i0}+6nt_0)-\sin(-5n_it_{i0}+6nt_0))\Bigg\}$$
$$+\frac{3465(3675n_i^7-26950nn_i^6+80628n^2n_i^5-126216n^3n_i^4+109184n^4n_i^3-49536n^5n_i^2+9216n^6n_i)}{3n_i-2n}$$
$$\Bigg\{\Bigg(\sin\Bigg(\Bigg(3n_i-2n\Bigg)t-3n_it_{i0}+2nt_0+\frac{\pi}{3}\Bigg)-\sin\Bigg(-3n_it_{i0}+2nt_0+\frac{\pi}{3}\Bigg)\Bigg)$$
$$-\mu(\sin((3n_i-2n)t-3n_it_{i0}+2nt_0)-\sin(-3n_it_{i0}+2nt_0))/(3n_i-2n)\Bigg\}$$
$$+\frac{3465(-3675n_i^7+24500nn_i^6-65928n^2n_i^5+92064n^3n_i^4-70576n^4n_i^3+28224n^5n_i^2-4608n^6n_i)}{3n_i-4n}$$
$$\Bigg\{\Bigg(\sin\Bigg(\Bigg(3n_i-4n\Bigg)t-3n_it_{i0}+4nt_0-\frac{\pi}{3}\Bigg)-\sin\Bigg(-3n_it_{i0}+4nt_0-\frac{\pi}{3}\Bigg)\Bigg)$$
$$-\mu(\sin((3n_i-4n)t-3n_it_{i0}+4nt_0)-\sin(-3n_it_{i0}+4nt_0))\Bigg\}$$
$$+\frac{3675(-11025n_i^7+66150nn_i^6-163484n^2n_i^5+212936n^3n_i^4-154112n^4n_i^3+58752n^5n_i^2-9216n^6n_i)}{n_i-2n}$$
$$\Bigg\{\Bigg(\sin\Bigg(\Bigg(n_i-2n\Bigg)t-n_it_{i0}+2nt_0-\frac{\pi}{3}\Bigg)-\sin\Bigg(-n_it_{i0}+2nt_0-\frac{\pi}{3}\Bigg)\Bigg)$$
$$-\mu(\sin((n_i-2n)t-n_it_{i0}+2nt_0)-\sin(-n_it_{i0}+2nt_0))\Bigg\}$$
$$+\frac{3675(11025n_i^7-88200nn_i^6+295784n^2n_i^5-539904n^3n_i^4+579984n^4n_i^3-366976n^5n_i^2+126720n^6n_i-18432n^7)}{n_i}$$
$$\Bigg\{\Bigg(\sin\Bigg(n_it-n_it_{i0}+\frac{\pi}{3}\Bigg)-\sin\Bigg(-n_it_{i0}+\frac{\pi}{3}\Bigg)\Bigg)$$

$$- \mu(\sin(n_i t - n_i t_{i0}) - \sin(-n_i t_{i0}))\Bigg\}\Bigg] \tag{B.17}$$

Bibliography

[1] Arakida, H., & Fukushima, T., 2000, AJ, **120**, 3333

[2] Arakida, H., & Fukushima, T., 2001, AJ, **121**, 1764

[3] Bhatnagar, K. B., & Gupta, U., 1986, CeMec, **39**, 67

[4] Bhatnagar, K. B., Gupta, U., & Bhardwaj, R., 1994, CeMDA, **59**, 345

[5] Breakwell, J. V., & Brown, J. V., 1979, CeMec, **20**, 389

[6] Brouwer, D., & Clemence, G. M., 1961, "Methods of Celestial Mechanics", Academic Press, New York & London

[7] Celletti, A., & Giorgilli, A., 1991, CeMDA, **50**, 31

[8] Danby, J. M. A., 1964, AJ, **69**, 294

[9] Danby, J. M. A.,1992, "Fundamentals of Celestial Mechanics" (2nd ed.), Willmann-Bell Inc.

[10] Danby, J. M. A., 1964, AJ, **69**, 165

[11] Dermott, S. F., & Murray, C. D., 1981, Icarus, **48**, 12

[12] Deprit, A., & Deldvoie, A., 1965, Icarus, **4**, 242

[13] Deprit, A., & Deprit-Bartholome, A., 1967, AJ, **72**, 173

[14] Deprit, A., Henrard, J., & Rom, A. R. M., 1967, Icarus, **6**, 381

[15] Deprit, A., & Palmore, J., 1966, AJ, **71** 94

[16] Dionysiou, D. D., & Stamou, G. G., 1989, Ap&SS, **152**, 1

[17] Dvorak, R., 1992, CeMDA, **54**, 195

[18] Erdi, B., 1978, CeMec, **18**, 141

[19] Erdi, B., 1983, Proceedings of the Seventy-fourth Colloquium "Dynamical trapping and evolution in the solar system" , Gerakini, Greece, August 30-September 2, 1982. Reidel Publishing Co., 165

[20] Evans, N. W., and Tremaine, S., 1999, AJ, **118**, 1888

[21] Fukushima, T., 1999, Proc. IAU Coll. 173, 304, eds: Svoreň, J., Pittich, E.M., and Rickman, H., Astron. Inst, Slovak. Acad. Sci., Tatranskä Lomnica

[22] Garfinkel, B., 1976, CeMec, **14**, 301

[23] Garfinkel, B., 1977, AJ, **82**, 368

[24] Garfinkel, B., 1978, CeMec, **19**, 259

[25] Garfinkel, B., 1980, CeMec, **22**, 267

[26] Garfinkel, B., 1983, CeMec, **30**, 373

[27] Garfinkel, B., 1985, CeMec, **36**, 19

[28] Giorgilli, A., & Skokos, C., 1997, A&A, **317**, 254

[29] Gómez, G., Jorba, À., Masdemont, J., & Simó, C., 2001, "Dynamics and Mission Design Near Libration Points, Vol. III Advanced Methods for Collinear Points", World Scientific, Singapore, New Jersey, London and Hong Kong

[30] Gómez, G., Jorba, À., Masdemont, J., & Simó, C., 2001, "Dynamics and Mission Design Near Libration Points, Vol. IV Advanced Methods for Triangular Points", World Scientific, Singapore, New Jersey, London and Hong Kong

[31] Gómez, G., Llibre, J., Martínez, R., & Simó, C., 2001, "Dynamics and Mission Design Near Libration Points, Vol. II Fundamentals: The Case of Triangular Libration Points", World Scientific, Singapore, New Jersey, London and Hong Kong

[32] Gozdziewski, K., 1998, CeMDA, **70**, 41

[33] Hagel, J., 1995, CeMDA, **63**, 205

[34] Hairer, H., Nørsett, S. P., and Wanner, G., 1987, Solving Ordinary Differential Equation I, Springer-Verlag, Berlin

[35] Henrici, P., 1962, Discrete Variable Methods in Ordinary Differential Equations, Wiley & Sons, Inc., New York

[36] Hiday-Johnston, L. A., & Howell, K. C., 1994, CeMDA, **58**, 317

[37] Horedt, G., 1974, CeMec, **10**, 319

[38] Horedt, G. P., 1984, CeMec, **33**, 367

[39] Horedt, G., & Mioc, V., 1976, CeMec, **13**, 421

[40] Howard, J. E., 1990, CeMDA, **48**, 267

[41] Ichtiaroglou, S., & Voyatzis, G., 1990, JApA, **11**, 11

[42] Innanen, K. A., & Mikkola, S., 1989, AJ, **97**, 900

[43] Jorba, À., 2000, A&A, **364**, 327

[44] Kinoshita, H., 1998, Celestial Mechanics and Orbital Dynamics, University of Tokyo Press (In Japanese)

[45] Kinoshita, H., 1970, PASJ, **22**, 373

[46] Kinoshita, H., 1979, Proceedings of the Advanced Study "Instabilities in dynamical systems: Applications to celestial mechanics" Cortina d'Ampezzo, Italy, July 30-August 12, 1978, D. Reidel Publishing Co., 229

[47] Kinoshita, H., Yoshida, H., and Nakai, H., 1991, Celest. Mech., 50, 59

[48] Kondurar, V. T., 1974, CeMec, **10**, 327

[49] Kumar, V., & Choudhry, R. K., 1987, CeMec, **40**, 155

[50] Lambert, J. D., and Watson, I. A., 1976, J. Inst. Math. Applic., **18**, 189

[51] Lohinger, E.,& Dvorak, R., 1993, A&A, **280**, 683

[52] Lukianov, L. G., 1976, CeMec, **13**, 455

[53] Lukyanov, L. G., 1986, SvA, **30**, 720

[54] Lukyanov, L. G., 1986, AZh, **63**, 1222

[55] Maindl, T. I. 1996, Completing the Inventory of the Solar System, Astronomical Society of the Pacific Conference Proceedings, volume 107, T.W. Rettig and J.M. Hahn, Eds., 147

[56] Maindl, Th. I., & Hagel, H., 1996, "Proceedings of the 3rd International Workshop on Positional Astronomy and Celestial Mechanics", held in Cuenca, Spain, October 17 - 21, 1994, Valencia: Universitat, Observatorio Astronomico, edited by Eleonora I. Yagudina, Maria J. Martinez Uso, and Alicia Cordero Barbero, 183

[57] Markellos, V. V., & Moran, P. E., & Black, W., 1975, Ap&SS, **33**, 385

[58] McKenzie, R., & Szebehely, V., 1981, CeMec, **23**, 223

[59] Melita, M. D., & Brunini, A., 2001, MNRAS, **311**,, L12

[60] Murray, C. D., 1994, Icarus, **112**, 465

[61] Murray, C. D., & Dermott, S. F., 1999, "Solar System Dynamics", Cambrigde Univ. Press

[62] Palutan, F., 1994, JBIS, **47**, 497

[63] Papadakis, K. E., 1998, CeMDA, **72**, 235

[64] Perdios, E., & Zagouras, C. G., 1991, CeMDA, **51**, 75

[65] Quinlan, G .D., 1999, astro-ph/9901136

[66] Quinlan, G. D., and Tremaine, S., 1990, AJ, **100**, 1694

[67] Ragos, O., & Zagouras, C., 1991, CeMDA, **50**, 325

[68] Roels, J., 1975, CeMec, **12**, 327

[69] Richardson, D. L., 1980, CeMec, **22**, 231

[70] Selaru, D., & Cucu-Dimitrescu, C., 1995, CeMDA, **61**, 333

[71] Shrivastava, A. K. & Garain, D., 1991, **51**, 67

[72] Simmons, J. F. L., McDonald, A. J. C., & Brown, J. C., 1985, CeMec, **35**, 145

[73] Singh, J., & Ishwar, B., 1984, CeMec, **32**, 297

[74] Skokos, Ch. & Dokoumetzidis, A., 2001, A&A, **367**, 729

[75] Sharma, R. K., & Rao, P. V. S., 1975, CeMec, **12**, 189

[76] Sharma, R. K., & Rao, P. V. S., 1978, CeMec, **18**, 185

[77] Stiefel, E. L., 1970, CeMec, **2**, 274

[78] Stiefel, E., Rössler, Waldvogel, J., & Burdet, C. A., 1967, Method of Regularization for computing Orbits in Celestial Mechanics, NASA Constractor Report NASA CR-769

[79] Stiefel, E., L., & Sheifele, G., 1971, Linear and Regular Celestial Mecchanics, Springer

[80] Todoran, I., 1994, Ap&SS, **215**, 237

[81] Todoran, I., & Roman, R., 1992, Astron. Nachr., **313**, 315

[82] Tsiganis, K., Dvorak, R., & Pilat-Lohinger, E., 2000, A&A, **354**, 1091

[83] Weibel, W. M., Kaula, W. M., & Newman, W. I., 1990, Icarus, **83**, 382

[84] Whipple, A. L., 1983, CeMec, **30**, 385

[85] Whipple, A. L., 1984, CeMec, **33**, 271

[86] Yoshida, H., 1993, CeMDA, **56**, 27

[87] Zagouras, C. G., 1991, CeMDA, **51**, 331

[88] Zagouras, C. G., & Kazantzis, P. G., 1979, Ap&SS, **61**, 389

[89] Zhang, S.-P., & Innanen, K. A., 1988, AJ, **96**,1983

[90] Zhang, S.-P., & Innanen, K. A., 1988, AJ, **96**, 1988

[91] Zhang, S.-P., & Innanen, K. A., 1988, AJ, **96**, 1995