Gravity and Hydrodynamics

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Abstract

We study relationships between gravitational theories and hydrodynamic systems with several approaches. There are at least three approaches which realize this idea: the membrane paradigm, the AdS/CFT duality, the BKLS approach. In this thesis, we focus on each the AdS/CFT duality and the BKLS approach individually.

First, we examine the AdS/CFT duality and the universality of the shear viscosity to the entropy density ratio η/s for various holographic superfluids. In the study of the AdS/CFT duality, fluids corresponding to a large class of geometries ensure the universality $\eta/s = 1/(4\pi)$. The universality has been extensively studied, and this holds for all known examples which have been studied. We study three types of the holographic superfluids as yet another example of the universality: s-wave, p-eave and $(p + ip)$-wave holographic superfluids. For the s-wave case, the ratio has the universal value $1/(4\pi)$ as in various holographic models. For the p-wave case, there are two shear viscosity coefficients because of the anisotropic boundary spacetime, and one coefficient has the universal value. For the other viscosity coefficient, the existing technique is not applicable since there is no tensor mode of metric perturbations which decouples from Yang-Mills perturbations. For the $(p + ip)$-wave case, the situation is the same as the case of the latter component in the p-wave. These results imply that p-wave and $(p+ip)$-wave holographic superfluids may not have the universality, and in fact, they are the first examples of the non-universal shear viscosity to the entropy density ratio.

Second, we study another realization, the BKLS approach. The BKLS approach is proposed by Bredberg *et al.* (1006.1902), where the fluid is defined by the Brown-York tensor on a timelike surface at $r = r_c$ in black hole backgrounds. We consider both Rindler space and the Schwarzschild-AdS (SAdS) black hole. The former describes an incompressible fluid, whereas the latter describes the vanishing bulk viscosity at arbitrary r_c, but these two results do not contradict with each other. We also find an interesting "coincidence" with the black hole membrane paradigm which gives a negative bulk viscosity. In order to show these results, we rewrite the hydrodynamic stress tensor via metric perturbations using the conservation equation. The resulting expressions are suitable to compare with the Brown-York tensor.

Contents

Introduction

In theoretical physics, there are a large number of dualities which relate a theory to another theory. The dualities can often transform a difficult problem to an easy problem. One example of the dualities is a correspondence between $(d+1)$-dimensional gravitational theories and d-dimensional hydrodynamics. There are at least three approaches which realize this duality: (i) the membrane paradigm, (ii) the AdS/CFT duality, (iii) the BKLS approach. They have been studied in different contexts. Let us explain the history of these approaches.

The oldest realization is (i) the membrane paradigm [1, 2], which has been studied in the context of the black hole physics. They tried to map the dynamics of the black hole to hydrodynamics and focused on the black hole's event horizon. Once an object passes through the black hole's event horizon, the object cannot affect an outside observer. In other words, the observer cannot see the inside of the black hole but can see the surface of the black hole. Therefore, the black hole dynamics should be effectively described by a dynamical membrane on the stretched horizon, a timelike surface located slightly outside the true horizon. The membrane dynamics is described by the Einstein equation on the stretched horizon and the equation is the same as the Navier-Stokes equations, mathematically. However, the membrane paradigm has the unpleasant features as a fluid such as a negative bulk viscosity. In addition, the microscopic realization of the membrane paradigm is not clear.

On the other hand, (ii) the AdS/CFT duality [3, 4, 5, 6], based on string theory, provides several explicit realizations of microscopic understandings of the corresponding hydrodynamics since the D-branes[7] provide both the asymptotic AdS black brane geometries and the corresponding strongly coupled field theories, which live on the boundary of the AdS geometries. For example, the D3-branes provides the $\mathrm{AdS}_5 \times S^5$ geometry and the strongly coupled $\mathcal{N}=4$ Super Yang-Mills theory in the large-N_c limit. One of the most important features of the AdS/CFT duality is that exact correlation functions of the strongly coupled field theories can be derived from the classical gravity, which is easy to calculate.

Therefore, the AdS/CFT duality has been applied to real-world physics *e.g.*, the quark-gluon plasma. The quark-gluon plasma can be described as a strongly coupled viscous fluid according to the heavy-ion collision at RHIC. Using the AdS/CFT duality, it turned out that the strongly coupled $\mathcal{N} = 4$ Super Yang-Mills theory and the quark-gluon plasma have almost same value of the shear viscosity to the entropy density ratio η/s. It may sound strange at the first glance since they are quite different field theories. But the agreement of the η/s would be because of the universality of the strong coupled field theories. (See the next section for more detail on the universality.) Therefore, it is important to find such robust and universal features to apply the AdS/CFT duality to real-world physics. More recently, the AdS/CFT duality has been applied to the condensed matter physics. The condensed matter physics is a low energy effective theory, so it is important to find the IR fixed point. This idea is realized by the holographic renormalization group that the field theory lives on arbitrary timelike surface, and the position of the surface corresponds to the energy scale of the field theory. The boundary and the horizon of the geometry correspond to the UV limit and the IR limit of the field theory, respectively. Therefore, the dependence on the position of the timelike surface is interpreted as the Wilsonian renormalization group flow of the field theory.

The UV limit of the field theory should be a conformal field theory as long as the corresponding geometry is asymptotic AdS. However, real world materials are not confromally invariant in the UV limit. In the context of the Wilsonian renormalization group, the renormalized field theory doesn't depend on the physics above the cutoff scale. So, (iii) Bredberg, Keeler, Lysov, and Strominger proposed an approach which doesn't depend on the asymptotics of the geometry [27, 28]. They introduced a timelike surface at arbitrary position for the "boundary" where the fluid lives. This approach doesn't provide microscopic understanding of the fluid but should describe robust features of the correspondence between the gravity and hydrodynamics, instead.

In this thesis, we focus on each (ii) the AdS/CFT duality and (iii) the BKLS approach, and study the relationship between them.

The AdS/CFT duality (Chapter 2)

In the context of the relationship between gravitational theories and hydrodynamics, the shear viscosity, one of the transport coefficient, has been extensively studied. This is because η/s, the ratio of the shear viscosity to the entropy density, is universal, *i.e.*,

$$\frac{\eta}{s} = \frac{1}{4\pi} \ ,$$

according to the AdS/CFT duality, the membrane paradigm and the BKLS approach. Especially, in the context of AdS/CFT, the universality has been extensively studied, and this holds for all known examples which have been studied (See, *e.g.*, Ref. [8, 9]).

The shear viscosity to the entropy density ratio η/s was first derived for the D3-brane [10], which is the dual of $\mathcal{N} = 4$ SYM. Then the same results were obtained for the M2- and M5-branes [11]. These three branes are the duals of conformal theories. Moreover, the universality holds even if theories are non-conformal, *e.g.*, Dp-branes for $p \neq 3$, the Klebanov-Tseytlin geometry, the Maldacena-Nunez geometry, and the $\mathcal{N} = 2^*$ system [12, 13, 14, 15, 16]. For the application to real QCD, each the finite density theories [17, 18, 19, 20] and the theories with fundamental fermions has been studied [21]. They are the duals of the charged black holes and the D3-D7 system, respectively. They ensure the universality. Even for a time-depending systems, the universality is held [22]. For the application to condensed matter physics, the Lifshitz-like geometry have the universality [23] Although there exists several arguments to generally support the universality [24, 25], it is still unclear why the universality holds microscopically and how generic the universality is.

In this Thesis, first, we study the holographic superfluids, which provide yet another example of the universality. The holographic superfluids exhibit a second-order phase transition. We study three types of the holographic superfluids, s-wave, p-wave, and $(p + ip)$-wave. They are characterized by the order parameter of the phase transition, *i.e.*, in the bulk gravitational theories, the order parameter of the s-wave holographic superfluids is a scalar field, and the one of both p-wave and $(p+ip)$-wave holographic superfluids is a SU(2) gauge field. (The difference between the p-wave and $(p + ip)$-wave is condensing components of the gauge field.) We found following results: (i) the s-wave holographic superfluids holds the universality of η/s, (ii) the p-wave and $(p + ip)$-wave holographic superfluids may have non-universal η/s because of the spacial anisotropy coming from the gauge field. (We will discuss the relationship between the universality violation and the spacial anisotropy in Sec. 2.2.) These are the first examples of the non-universal system. Actually, our work triggered detailed studies of the non-universal shear viscosity [86, 87, 88, 89].

The BKLS approach (Chapter 3)

Let us summarize the three approaches again in order to realize the feature of the BKLS approach.

1. Historically, the membrane paradigm [1, 2] is the oldest one. In this case, the fluid lives on the stretched horizon $r \to r_0$. However, the membrane paradigm has the unpleasant features as a fluid such as a *negative* bulk viscosity. The membrane paradigm originally

focuses on the $(3+1)$-dimensional asymptotically flat black holes, but asymptotics should not matter much since it focuses on the near-horizon limit.

2. In the AdS/CFT duality, the dual fluid "lives" at the AdS boundary $r \to \infty$. The advantage of the AdS/CFT duality is a clear microscopic interpretation for the dual fluid. The AdS/CFT results are widely used for real-world applications such as the quark-gluon plasma. (See, *e.g.*, Refs. [8, 9, 26] for reviews.)

3. More recently, Bredberg, Keeler, Lysov, and Strominger (BKLS) [27, 28] proposed the timelike surface at arbitrary position $r = r_c$ for the "boundary" where the fluid lives (See also, *e.g.*, Refs. [29, 30]). The BKLS approach is analogous to the holographic renormalization. In the near-horizon limit, the BKLS approach describes an *incompressible* fluid.

Another closely related idea is a "black hole in a cavity" [31]. This idea was proposed to obtain a well-defined thermal equilibrium for asymptotically flat black holes such as the Schwarzschild black hole. The Schwarzschild black hole has a negative heat capacity, so it is unstable by the Hawking radiation. However, if the black hole is surrounded by a finite-temperature cavity, and if the cavity is close enough to the horizon, a thermal equilibrium is achieved. In a sense, the BKLS approach is an AdS black hole in a cavity.

While each approach has a different motivation and physical interpretation, one thing is common: they all employ the Brown-York tensor [32] as the fluid stress tensor. Thus, they are somehow related to each other.

Both in the membrane paradigm and in the BKLS approach (in particular in Ref. [28]), one often starts to identify the velocity field of the fluid in the bulk spacetime. This has its own advantage that the relationship between the Einstein equation and the Navier-Stokes equation is direct and transparent. On the other hand, this brings us an immediate problem why a particular vector field should be regarded as the velocity field. So, we do not take such a path.

- Instead, we consider metric perturbations and study the (linear) response of the Brown-York tensor by the perturbations *à la* AdS/CFT duality.

- In hydrodynamics, the velocity field is determined from the metric perturbations (Sec. 3.1). Then, one can eliminate the velocity field completely in the hydrodynamic stress tensor. The resulting expression contains metric perturbations only, which is suitable to compare with the Brown-York tensor. In our approach, the velocity field is a consequence of metric perturbations.

One purpose of this chapter is to reexamine the BKLS approach using the above formulation.

In particular, we study the issue of the bulk viscosity ζ, which is non-negative in the AdS/CFT duality, negative in the membrane paradigm, and is irrelevant in the BKLS approach (because of an incompressible fluid). For that purpose, we consider the sound mode perturbations whose analysis was somewhat incomplete in Ref. [27]. We study Rindler space, which is the near-horizon limit of black holes with nondegenerate horizon, and the five-dimensional Schwarzschild-AdS black hole ($SAdS_5$)[1]. Our results are summarized as follows:

1. For Rindler space, the Brown-York tensor gives an incompressible fluid in accordance with the BKLS result (Sec. 3.2).
2. For the $SAdS_5$ black hole, the Brown-York tensor always gives the vanishing bulk viscosity irrespective of the boundary position r_c (Sec. 3.3).
3. There are no contradictions between two results since the hydrodynamic regime used for the SAdS black hole "differs" from the hydrodynamic regime used for Rindler space (when expressed in terms of the SAdS variables) (Sec. 3.4).

In addition, we obtain one of the second-order hydrodynamic transport coefficient τ_π for the $SAdS_5$ black hole in the BKLS approach.

The framework of this thesis

This thesis is organized as follows. In Chapter 1 we shall give a short review of the linear response theory, which provides the response of an operator induced by small perturbations on the thermal equilibrium. It is necessary in order to obtain the transport coefficients from perturbations. We shall review the basics of the hydrodynamics, which include the isotropic first order hydrodynamics and the second order hydrodynamics.The anisotropic fluid is dual to the p-wave and $(p + ip)$-wave holographic superfluids. In the context of the AdS/CFT duality, the corresponding fluid usually has the conformal symmetry. The SAdS black hole with a finite cutoff, however, might not have conformal symmetry. So the non-conformal second order fluid should be considered. Using the linear response theory and the hydrodynamic constitutive equation, we will find the response of the hydrodynamic stress tensor induced by the gravitational perturbations, and the viscosities finally.

[1]While our work was in progress, there appeared preprints which study Rindler hydrodynamics [33, 34, 35].

Chapter 2 contains a brief review of AdS/CFT and the study of the ratio η/s of the holographic superfluid, simultaneously. First, we shall show the s-wave holographic superfluids have a universal shear viscosity. This proof includes a large class of geometries and is the most general theorem of the universality at present. Then we discuss anisotropic viscosities of the p-wave and $(p + ip)$-wave holographic superfluids and they can have non-universal value. The study is based on our paper [36].

In Chapter 3, which is based on our research [37], we shall present another realization, the BKLS approach. We define the approach in terms of the Brown-York tensor and metric perturbations. First, we write down the linear response of the hydrodynamic stress tensor in terms of the metric perturbations. Then we show the linear perturbation of the Brown-York tensor in the Rindler space. Comparing the Brown-York tensor and the hydrodynamics stress tensor, we shall reproduce the results of the BKLS approach. In order to compare with the $SAdS_5$ black hole, we shall derive the viscosities, especially the bulk viscosity, of $SAdS_5$ with arbitrary r_c and show that the bulk viscosity vanishes even for arbitrary r_c. We shall discuss how they can be compatible. We also compute a second-order hydrodynamic coefficient τ_π for arbitrary r_c.

Chapter 1

Hydrodynamics and Linear Response Theory

Here, the minimal formalism of hydrodynamics and the linear response theory are explained.

1.1 Linear Response Theory

Linear response theory is based on the following conditions:

- Hamiltonian in the system can be decomposed into the time independent part and the time depending part.
- The time depending part is made of a small fluctuation which is perturbed from the outside of the system.
- The system was thermal equilibrium in the past.

The time depending density of state $\rho(t)$ and time depending Hamiltonian $H(t)$ satisfies the von-Neumann equation

$$i\frac{\partial}{\partial t}\rho(t) = [H(t), \rho(t)] \ . \tag{1.1.1}$$

This means the time evolution of the density of state is

$$\rho(t) = U(t, t_0)\rho(t_0)U^\dagger(t, t_0) \ . \tag{1.1.2}$$

Even if the system depends on time, the expectation value of an arbitrary observable B at time t can be represented by

$$\langle B \rangle_t = Tr\rho(t)B \ . \tag{1.1.3}$$

It is, however, difficult to calculate the time depending expectation value exactly. So we take the most simple assumption that the system was in thermal equilibrium and then perturbed linearly.

First, we take Schrödinger picture. Assume that $H(t)$ can be decomposed into the time independent part H_0 and the time dependent part $\delta H(t)$, and the time dependent part is made of the time independent operator $\mathcal{O}_A(x)$ and its time dependent source $\phi_A(t,x)$:

$$H(t) = H_0 + \delta H(t), \quad \delta H(t) = -\int dx^{d-1} \phi_A(t,x)\mathcal{O}_A(x) \ . \tag{1.1.4}$$

The time evolution operator from t_0 to t satisfies the following differential equation.

$$i\frac{\partial}{\partial t}U(t,t_0) = H(t)U(t,t_0), \quad U(t_0,t_0) = 1 \ . \tag{1.1.5}$$

The solution is

$$U(t,t_0) = T\left[\exp\left(-i\int_{t_0}^{t} dsH(s)\right)\right] \ . \tag{1.1.6}$$

This time evolution operator also should be decomposed into the H_0 part and the $\delta H(t)$ part. First, the H_0 part can be clearly defined as

$$U_0(t,t_0) = \exp\left(-i(t-t_0)H_0\right) \ . \tag{1.1.7}$$

Then we naively define the $\delta H(t)$ part as[1]

$$U(t,t_0) = U_0(t,t_0)U_i(t) \ . \tag{1.1.8}$$

This definition and the time evolution equation for the full Hamiltonian lead the time evolution equation for $\delta H(t)$.

$$\begin{aligned} i\frac{d}{dt}U_i(t) &= U_0^\dagger(t,t_0)\delta H(t)U_0(t,t_0)U_i(t) \\ &= \delta H_I(t)U_i(t) \ . \end{aligned} \tag{1.1.9}$$

[1]The subscript "i" in U_i means "Interaction part". Don't confuse this with the subscript "I", which means "Interaction pictuer".

Here, the time dependent part of the Hamiltonian in the interaction picture is defiened as

$$\delta H_I(t) := U_0^\dagger(t,t_0)\delta H(t) U_0(t,t_0) \ . \tag{1.1.10}$$

Note that the time evolution of states is under U_i and of operators is under U_0 in the interaction picture. The solution of $U_i(t)$ is

$$U_i(t) = T\left[\exp\left(-i\int_{t_0}^t ds \delta H_I(s)\right)\right] \ . \tag{1.1.11}$$

So the naive definition reads the proper expression eventually.

Now using the expansion of the full time evolution operator

$$U(t,t_0) = U_0(t,t_0)\left(1 - i\int_{t_0}^t ds\delta H_I(s) + \mathcal{O}(\delta H_I^2)\right) \ , \tag{1.1.12}$$

one can derive the first order approximation of the expectation value of the arbitrary observable B

$$\begin{aligned}\langle B\rangle_t &= Tr U(t,t_0)\rho(t_0)U^\dagger(t,t_0)B \\ &= Tr\rho(t_0)B_I(t,t_0) + i\int_{t_0}^t dt_1 Tr\rho(t_0)[\delta H_I(t_1), B_I(t,t_0)] + \mathcal{O}(\delta H_I^2) \ .\end{aligned} \tag{1.1.13}$$

Let the system be thermal equilibrium at t_0 so $\delta H_I(t_0) = 0$ and the density of state becomes

$$\rho(t_0) = \frac{e^{-\beta H_0}}{Tr e^{-\beta H_0}} =: \rho_0 \ . \tag{1.1.14}$$

Therefore, the state density commute with $U_0(t,t_0)$ and so $Tr\rho(t_0)B_I(t,t_0) = Tr\rho_0 B$. The first order approximation become

$$\langle B\rangle_t = Tr\rho_0 B + i\int_{t_0}^t dt_1 Tr\rho_0[\delta H_I(t_1), B_I(t,t_0)] + \mathcal{O}(\delta H_I^2) \ . \tag{1.1.15}$$

If we take the arbitrary observable $B = \mathcal{O}_A(x)$, the response of the expectation value can be written as

$$\begin{aligned}\delta\langle\mathcal{O}_A\rangle(t,x) &= i\int_{t_0}^t dt_1 \int d^{d-1}x' Tr\rho_0[\mathcal{O}_A^I(t,x), \mathcal{O}_B^I(t',x')]\phi_B(t'x') \\ &= i\int_{-\infty}^{\infty} dt' \int d^{d-1}x' \theta(t-t')\langle[\mathcal{O}_A^I(t,x), \mathcal{O}_B^I(t',x')]\rangle\phi_B(t'x') \ ,\end{aligned} \tag{1.1.16}$$

Using $\int d^{d-1}xdte^{i\omega t-ikx}$ to transform it into the Fourier space expression,

$$\delta\langle\tilde{\mathcal{O}}_A\rangle(\omega,k) = -G^R_{AB}(\omega,k)\tilde{\phi}_B(\omega,k) \ , \tag{1.1.17}$$

where we defined the Retarded Green's function

$$G^R_{AB}(\omega,k) = -i\int_{-\infty}^{\infty} d^dx e^{i\omega t-ikx}\theta(t)\langle[\mathcal{O}_A(t,x),\mathcal{O}_B(0,0)]\rangle \ . \tag{1.1.18}$$

This expression means that even if the system depends on time, the first order approximation in the fluctuation can be understood in terms of the information of thermal equilibrium quantities.

1.2 Hydrodynamics

The hydrodynamics is an effective theory under long wave length limit, *i.e.*, the derivative expansion. The referential scale in the hydrodynamics is the mean free path ℓ. Therefore, the validity of the hydrodynamic approximation is controlled by the combination of the characteristic momentum scale and the mean free path $(k\cdot\ell)$.

One of the most distinctive features of hydrodynamics is that it is based on the conservation equations and the constitutive equation for the hydrodynamic stress tensor, not certain Lagrangian. The reason why the constitutive equation should be introduced is that the differential equations from the conservation equations is not closed themselves. The constitutive equation for the hydrodynamic stress tensor reduces the degrees of freedom and then the differential equations can be closed.

The constitutive equation is constructed by the derivative expansion. So the hydrodynamic stress tensor of the perfect fluid does not include any derivatives.

$$T^{\mu\nu} = T^{\mu\nu}_{\text{perfect}} + \mathcal{O}(\partial^1) \ . \tag{1.2.1}$$

If $(k\cdot\ell)$ is small enough to be neglected, the system can be described by the perfect fluid.

Since the hydrodynamics is an effective theory, each fluid is characterized by several coefficients, which is called *transport coefficients*. The number of the transport coefficients can be determined by the symmetry of each system. For example, the isotropic fluid has the two first order transport coefficients η and ζ. This is because the symmetric rank two tensor transforming under $SO(d-1)$ can be decomposed into the two irreducible representations, which are the symmetric traceless part and the trace part. Therefore, to fix the transport coefficient in a hydrodynamic system is to fix the equation of motion in the hydrodynamic system.

In this section, we construct the constitutive equations for several fluids, derive Kubo formula for the first order transport coefficients and find the dispersion relations using the equation of motions in an arbitrary curved background. Following discussion is based on [38] and [9] for the first order fluids, and [39] for the second order fluids. (See [40, 41, 42, 43] for more detail on the second order hydrodynamics.) Note that the readers should mind that, *in this section, we discuss the boundary theory from the point of view of AdS/CFT although we discuss the fluid on the curved spacetime.*

1.2.1 Perfect Fluid

The equation of motion of the hydrodynamics is based on the conservation equation.

$$\nabla_\mu T^{\mu\nu} = 0 \ , \tag{1.2.2}$$

where $T^{\mu\nu}$ is the hydrodynamic stress tensor of the fluid. Here, we impose that the system is locally equilibrium: the system is thermal equilibrium in the neighborhood of an arbitrary position $\mathbf{x}$. So the thermodynamic quantities can be defined there and the state of the neighborhood can be specified by the thermodynamic quantities and the fluid velocity u^μ, which is normalized as $u_\mu u^\mu = -1$. This means that the constitutive equation is $T^{\mu\nu} = T^{\mu\nu}(u^\mu, P, \cdots)$, where P is the pressure of the system. They are locally defined.

Now we derive the constitutive equation of the perfect fluid. Let the dimension of the spacetime d-dimension. Consider an infinitesimal volume element and its surface element $d\Sigma_\mu$. The momentum flux through the μ surface is

$$dq^i = T^{i\nu} d\Sigma_\nu \ . \tag{1.2.3}$$

Especially in the rest flame, the volume element doesn't carry total momentum, so $T^{i0} = 0$ and the pressure against each surface is same and perpendicular since the volume element doesn't move. In addition, this volume element is not affected by the next volume element since the dissipative momentum transfer is not exist. Therefore,

$$T^{ij} = P\delta^{ij} \ . \tag{1.2.4}$$

The energy density in the volume element is $T^{00} = \epsilon$, so the generally covariant form of the hydrodynamic stress tensor is

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + P^{\mu\nu} P \ , \tag{1.2.5}$$

where $P^{\mu\nu}$ is the projection operator defined by

$$P^{\mu\nu} = u^\mu u^\nu + g^{\mu\nu} \ . \tag{1.2.6}$$

One can show that the fluid described by (1.2.5) doesn't have dissipation, using the thermodynamic relations. Projecting the divergence of the hydrodynamic stress tensor along u_μ

$$\begin{aligned} 0 &= u_\nu \nabla_\mu \left(w u^\mu u^\nu + P g^{\mu\nu} \right) \\ &= -T \nabla_\mu \left(\sigma u^\mu \right) \ , \end{aligned} \tag{1.2.7}$$

where we introduced the enthalpy $w = \epsilon + P$ and the entropy density σ, and we used the thermodynamic equation $dw = T d\sigma + dp$ and $\epsilon + P = T\sigma$. The entropy flux is conserved.

$$\nabla_\mu (\sigma u^\mu) = 0 \ . \tag{1.2.8}$$

This means the system represented by the hydrodynamic stress tensor (1.2.5) does not have any dissipation.

1.2.2 First Order Hydrodynamics

In order to pick up the effect of the dissipation, we examine the derivative term $\tau^{\mu\nu}$.

$$T^{\mu\nu} = (\epsilon + P) u^\mu u^\nu + P g^{\mu\nu} - \tau^{\mu\nu} \ . \tag{1.2.9}$$

Let us consider $\tau^{\mu\nu}$ up to the first derivative in the isotropic system. This approximation is valid when $(k \cdot \ell)^2 \ll 1$. Now we should reconsider the definition of the fluid velocity. We define the velocity in the condition that, in the rest flame of any given volume element, the momentum of the volume element is zero and its energy is expressed in terms of the same formulae as when dissipative processes are absent. This means that $\tau^{\mu\nu} u_\nu = 0$ and, in the rest flame, the spatial direction of the fluid velocity is absent $u^i = 0$. Therefore, in the proper coordinate only τ_{ij} should be exist.

Decomposing τ_{ij} into the irreducible representations of $SO(d-1)$,

$$\tau_{ij} = \eta \left(\partial_i u_j + \partial_j u_i - \frac{2}{p} \delta_{ij} \partial_k u^k \right) + \zeta \delta_{ij} \partial_k u^k \ , \tag{1.2.10}$$

where ζ is the bulk viscosity and η is the shear viscosity, and p denotes the spatial dimension $d-1$. Since the bulk viscosity is the trace part of the hydrodynamic stress tensor, it provides the force which change the volume of the element. In the conformal fluid, the bulk viscosity is absent since its hydrodynamic stress tensor is traceless. On the other hand, the shear viscosity provides the force which change the shape of the volume element without its volume unchanging. In the general covariant form,

$$\tau^{\mu\nu} = P^{\mu\alpha} P^{\nu\beta} \left[\eta \left(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - \frac{2}{p} g_{\alpha\beta} \nabla_\gamma u^\gamma \right) + \zeta g_{\alpha\beta} \partial_\gamma u^\gamma \right] \ . \tag{1.2.11}$$

Note that the projection operator is introduced in order to preserve the condition $\tau^{\mu\nu} u_\nu$=0.

The positivity of the entropy production restricts the transport coefficients. As the perfect fluid, the time component of the conservation equation in the proper coordinate $(\nabla_\mu T^{\mu\nu}) u_\nu = 0$ leads

$$\nabla_\mu (\sigma u^\mu) = \frac{1}{T} \left(\tau^{<\mu\nu>} u_{<\mu\nu>} + \frac{1}{p} P_{\alpha\beta} \tau^{\alpha\beta} P^{\mu\nu} u_{\mu\nu} \right) \ . \tag{1.2.12}$$

Here, the spacially projected symmetric traceless symbol

$$A^{<\mu\nu>} := \frac{1}{2} P^{\mu\alpha} P^{\nu\beta} (A_{\alpha\beta} + A_{\beta\alpha}) - \frac{1}{p} P^{\mu\nu} P^{\alpha\beta} A_{\alpha\beta} \ , \tag{1.2.13}$$

and the symmetrized first derivative of fluid velocity

$$u_{\mu\nu} := \frac{1}{2} \left(\nabla_\mu u_\nu + \nabla_\nu u_\mu \right) \ , \tag{1.2.14}$$

are defined. Eq. (1.2.12) means that, in order to keep the entropy production positive for any given velocity, the symmetric traceless part of the viscous tensor $\tau^{<\mu\nu>}$ and trace part of that $P_{\mu\nu} \tau^{\mu\nu}$ must proportional to $u_{<\mu\nu>}$ and $P^{\mu\nu} u_{\mu\nu}$, respectively. In addition, the coefficient of each term must positive. Therefore, the transport coefficients η and ζ should be positive.

Since the equation of motion is closed, one can derive the poles of the linearized hydrodynamics. We derive them in Sec. 3.1.

1.2.3 Second Order Hydrodynamics

We proceed the derivative expansion of the hydrodynamic stress tensor up to second order. Therefore, the viscous tensor $\tau^{\mu\nu}$ in eqviscousstresstensor must include all possible second-order terms [2].

Basically, the second order hydrodynamics should be introduced when the characteristic momentum scale of the system compared with the mean free path, *i.e.*, $(k \cdot \ell)^2$, cannot be neglected. There is another understanding to introduce the second order term. In the first order "relativistic" hydrodynamics, the speed of propagation for heat and viscosity are infinite since the EOMs for the propagation obey parabolic differential equation, which is the same as the traditional thermal diffusion equation. Eventually, the first order "relativistic" hydrodynamics is only applicable to the system slowly varying on space and time scales characterized by the mean free path ℓ and the momentum k. Israel introduced the second order derivative terms because of the latter motivation [40], and then he generalized the theory in the curved background [41]. (See appendix of Ref.[42] for review.)

From the AdS/CFT duality's point of view, conformal symmetry is so important, so the second order hydrodynamics with conformal invariance is required. Baier *et al.* constructed the theory and found additional terms [39] which were neglected in Refs.[40, 41] [3]. Then the most general isotropic second order hydrodynamics without charges are constructed [43].

In this thesis, we don't need the non-linear terms. So we introduce a non-conformal viscous tensor:

$$\begin{aligned}\tau^{\mu\nu} =& -\eta\sigma^{\mu\nu} - \zeta(\nabla\cdot u)P^{\mu\nu} \\ &+ \eta\tau_\pi{}^{<}\nabla_u\sigma^{\mu\nu>} + \zeta\tau_\Pi\nabla_u(\nabla\cdot u)P^{\mu\nu} \\ &+ \kappa_1 R^{\alpha<\mu\nu>\beta}u_\alpha u_\beta + \kappa_2 R^{<\mu\nu>} + \left(\kappa_3 R^{\alpha\beta}u_\alpha u_\beta + \kappa_4 R\right)P^{\mu\nu}\ ,\end{aligned} \tag{1.2.15}$$

where the antisymmetric traceless tensor $\sigma^{\mu\nu}$ is defined by $\sigma^{\mu\nu} := 2^{<}\nabla^\mu u^{\nu>}$. We have introduced six second order coefficients, *i.e.*, τ_π, τ_Π, κ_1, κ_2, κ_3, κ_4. Note that, for the conformal fluids, τ_Π, κ_3 and κ_4 are absent, and $\kappa_1 = -(d-2)\kappa_2$. (*cf.* Ref.[39].)

[2] In this thesis, we consider only the *isotropic* second order hydrodynamics.

[3] There are two types of the additional terms. First, Baier *et al.* found the terms made of Riemann tensor and Ricci tensor. Second, in order to preserve conformal invariance, additional non-linear terms are required.

1.2.4 Kubo Relations for the Viscosity Coefficients

The viscosity coefficients represent the dissipation of the fluid, so they can be obtained by linear response theory. Linear response theory is based on the operator of interest and its corresponding source. Since we are now interested in the response of the hydrodynamic stress tensor $T^{\mu\nu}$, the appropriate source is the metric $g_{\mu\nu}$.

In order to apply linear response theory to the viscous hydrodynamics, the state where the spacial component of the velocity is absent $u^\mu = (1, 0, 0, \cdots, 0)$ and the spacetime is perturbed by the spatially homogeneous metric perturbation around the flat spacetime $\eta_{\mu\nu}$[4],

$$g_{ij} = \delta_{ij} + h_{ij}(t), \quad h_{ij} \ll 1, \quad h_{ii} = 0 \tag{1.2.16}$$

$$g_{00} = -1, \quad g_{0i}(t, x) = 0 \; . \tag{1.2.17}$$

The response of the velocity δu_μ from the metric perturbation is absent since the homogeneous perturbation preserve the spatial rotational isometry and one can show the normalization $u_\mu u^\mu = -1$ preserve $\delta u_0 = 0$.

The response of the energy momentum tensor is

$$\delta T_{ij} = P h_{ij} + \eta \partial_0 h_{ij}, \quad (i \neq j) \; . \tag{1.2.18}$$

Using (1.1.17), one can find the Green's function of the off-diagonal hydrodynamic stress tensor,

$$G_R^{12,12}(\omega, 0) = -i \int_{-\infty}^{\infty} d^d x e^{i\omega t} \theta(t) \langle [T^{12}(t, x), T^{12}(0, 0)] \rangle = -i\eta\omega + \mathcal{O}(\omega^2) \; . \tag{1.2.19}$$

Here, we took $i = 1, j = 2$ without loss of generality because of the $SO(p)$ spatial isometry. The first term is called contact term, which is independent of the frequency. Removing the contact term by hand, one can obtain the shear viscosity from the Green's function.

$$\eta = -\lim_{\omega \to 0} \frac{1}{\omega} \mathrm{Im} G_R^{12,12}(\omega, 0) \; . \tag{1.2.20}$$

This is the Kubo relation for the shear viscosity.

We use Eq. (1.2.20) in Chapter 2. Although there are several Kubo relations for the other viscosities, we don't discuss them here. We discuss the response of the hydrodynamic

[4] The homogeneous perturbation correspond to the long wave length limit $k \to 0$. In general, we should take care of the order of the limit. Sometimes, the limit and another limit are non-commutative.

stress tensor from the sound mode metric perturbation in Sec. 3.1.

Chapter 2

AdS/CFT and Hydrodynamics

Historically, AdS/CFT is proposed by Maldacena[3] in 1997. He proposed that the strong coupling and the large N_c-limit of the $\mathcal{N} = 4$ super Yang-Mills theory, which is the low energy effective theory of D3-brane, corresponds to the near horizon limit of the black D3-brane solution. And then Gubser, Klebanov, and Polyakov[6], and Witten[4] proposed the prescription for obtaining the correlation function of operators in the field field theory from the calculus in the corresponding bulk gravitational theory. The prescription is called GKP-Witten relation.

Although they formulate the relation in Euclidean signature, in order to find time dependent dynamics, especially dynamics with dissipation, the real time formalism is needed. Son and Starinets found the prescription to obtain Retarded Green's functions exactly in real time Lorentzian signature[44], and they derived the shear viscosity to the entropy density ratio for $\mathcal{N} = 4$ super Yang-Mills (SYM) plasma[10, 45]. The ratio has been studied widely in order to understand real quark gluon plasma (QGP).

The prescription enables us to find the dissipative dynamics in the boundary theory through retarded Green's functions. It is interesting to study the dissipative dynamics in not only QGP but condensed matter physics as well. In this thesis, we focus on the holographic superfluids, which exhibit the second order phase transition and superfluid-like behaviors.

In the studies of holographic superfluids [47, 48, 49, 50, 51, 52, 53] (See, *e.g.*, Refs. [54, 55, 56] for reviews), one often uses numerical computations or some approximations. This is because the holographic superfluids are Einstein-matter systems and it is in general difficult to solve such systems. One approximation often employed is the "probe approximation," where the backreaction of matter fields onto the geometry can be ignored. While the approximation is enough to see the phase transition and to compute properties such as the conductivity, gravitational properties of these systems, in particular analytic results are largely intact. It is desirable to obtain gravitational properties of these systems analytically. We investigate this issue in this thesis. We study η/s, the ratio of the shear viscosity to the entropy density for holographic superfluids.

Technically, the universality largely depends on the following two properties of the bulk theory:

1. One can use the Kubo formula to compute η and carry out the tensor mode computation of gravitational perturbations. There are no other fields which transform as a tensor even if matter fields are present.

2. The entropy density is given by the Bekenstein-Hawking formula as long as the gravitational action takes the Einstein-Hilbert form.

In this thesis, we consider three class of holographic superfluids, the s-wave, p-wave, and $(p+ip)$-wave holographic superfluids in $(d+1)$-dimensional bulk spacetime. Our results are summarized as follows:

(i) The s-wave holographic superfluids are described by Einstein-Maxwell-complex scalar systems [47, 48, 51]. In this case, the universality holds with a modification of the technique in Ref. [25].

(ii) The p-wave holographic superfluids are described by Einstein-Yang-Mills systems [49]. In this case, the Yang-Mills field breaks the $SO(d-1)$ rotational invariance on the boundary theory, which has two implications. First, the hydrodynamic limit is not described by a single shear viscosity.[1] Second, for $d = 3$, one does not have a tensor mode which decouples from the Yang-Mills field. (Namely, item 1 of the above list fails.) As a result, the existing technique is not applicable. However, for $d \geq 4$, one has the $SO(d-2)$ invariance. In this case, a tensor mode exists, and the universality holds for the shear viscosity associated with the tensor mode.

(iii) The $(p+ip)$-wave holographic superfluid is described by the same system as the p-wave holographic superfluid (with $d = 3$), but the symmetry breaking pattern is different [50]. Although the metric keeps the $SO(2)$ invariance, the Yang-Mills field breaks the $SO(2)$ invariance. As a result, there does not exist the tensor mode which decouples from Yang-Mills perturbations and the existing technique is not applicable.

[1] In the context of the AdS/CFT duality, anisotropic shear viscosities have been computed for the non-commutative $\mathcal{N} = 4$ plasma [57].

Our results indicate that the shear viscosity has no singular behavior across the phase transition for holographic superfluids (See Sec. 2.3.2).

The plan of this chapter is as follows. In Sec. 2.1, we consider η/s for the s-wave holographic superfluids, explaining basic prescriptions of AdS/CFT. In Sec. 2.2, we consider anisotropic holographic superfluids, the p-wave and $(p+ip)$-wave holographic superfluids. For the $(p+ip)$-wave case, we identify the Yang-Mills perturbations which couple to the would-be tensor mode of metric perturbations.

2.1 The s-wave Superfluids

2.1.1 Background

The s-wave holographic superfluids are described by Einstein-Maxwell-complex scalar system:

$$S_s = \frac{1}{16\pi G_{d+1}} \int d^{d+1}x\sqrt{-g}\Big\{ R - 2\Lambda - \frac{1}{4}K_1\left(|\Psi|^2\right)F^{MN}F_{MN} - K_2\left(|\Psi|^2\right)|\nabla_M\Psi - iqA_M\Psi|^2 - V\left(|\Psi|^2\right)\Big\} \tag{2.1.1}$$

with the ansatz

$$ds^2_{d+1} = -g_{tt}(r)dt^2 + g_{xx}(r)\sum_{i=1}^{d-1}dx_i^2 + g_{rr}(r)dr^2 \ , \tag{2.1.2}$$

$$A = A_t(r)dt \ , \tag{2.1.3}$$

$$\Psi = \Psi(r) \ . \tag{2.1.4}$$

Here, capital Latin indices $M, N, \dots$ run through bulk spacetime coordinates (t, x_i, r), where (t, x_i) are the boundary coordinates and r is the AdS radial coordinate. Greek indices $\mu, \nu, \dots$ run though only the boundary coordinates. K_1, K_2 and V are arbitrary real functions of the scalar field. This action includes not only the conventional s-wave holographic superfluids [47, 48] but also generalized models [58, 59, 60]. We impose the regularity condition on the metric at the horizon $r = r_h$:

$$g_{tt} \to c_t(r - r_h) \ , \quad g_{xx} \to c_x \ , \quad g_{rr} \to c_r(r - r_h)^{-1} \ . \tag{2.1.5}$$

These conditions fix the Hawking temperature and the entropy density of the bulk geometry:

$$T = \frac{1}{4\pi}\sqrt{\frac{c_t}{c_r}}\ , \quad s = \frac{c_x^{(d-1)/2}}{4G_{d+1}}\ . \tag{2.1.6}$$

The model exhibits a second-order phase transition. At high temperatures, the scalar field Ψ vanishes and one obtains the standard Reissner-Nordström-AdS black hole. But at low temperatures, the Reissner-Nordström-AdS black hole becomes unstable and is replaced by a charged black hole with a scalar "hair."

This system is supposed to be dual to some kind of superconductors/superfluids. In fact, the low temperature phase shows the expected behavior for superconductors/superfluids. As superconductors, one can see the divergence of the DC conductivity, an energy gap proportional to the size of the condensate, and the holographic London equation [48, 51, 61, 62]. As superfluids, one can see the existence of the second and fourth sounds [63, 64]. Also, vortex solutions have been constructed [65, 66, 67, 68, 69].

2.1.2 η/s

Since we are interested in the viscosity, the main object to study is the boundary energy-momentum tensor. According to the standard AdS/CFT dictionary [3, 4, 5, 6], the bulk gravitational perturbations act as the source for the boundary energy-momentum tensor. Thus, our task amounts to solve the bulk gravitational perturbations.

Consider the fluctuations of the energy-momentum tensor $T_{\mu\nu}$ which behaves as $e^{-i\omega t}$. The fluctuations are decomposed by the little group $SO(d-1)$ acting on x_i ($i = 1, \cdots, d-1$). The fluctuations are decomposed as the tensor mode, the vector mode ("shear mode"), and the scalar mode ("sound mode").

One can use various methods to compute the shear viscosity. Among them, the most powerful one is the Kubo formula method , which uses the tensor mode (1.2.20) :

$$\eta = -\lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}\, G_R^{1212}(\omega,\vec{0})\ , \tag{2.1.7}$$

where $G_R^{1212}(\omega,\vec{0})$ is the retarded Green function for the tensor mode T^{12} at zero spatial momentum (1.2.19):

$$G_R^{1212}(\omega,\vec{0}) = -i\int_{-\infty}^{\infty} d^dx\, e^{i\omega t}\theta(t)\left\langle [T^{12}(t,\vec{x}), T^{12}(0,\vec{0})]\right\rangle\ . \tag{2.1.8}$$

To obtain the retarded Green function, we consider homogeneous gravitational perturbations which take the form

$$g_{MN} = \bar{g}_{MN} + h_{MN} \ , \tag{2.1.9}$$

where $\bar{g}_{MN}$ is the background metric (2.1.2). In the Lorentzian prescription of the AdS/CFT duality [44], the retarded Green function (2.1.8) can be calculated from the tensor mode h_{12}. We expand the action in terms of $\phi(t,r) := h^1{}_2(t,r)$ up to quadratic order and use the Fourier transformation

$$\phi(t,r) = \int \frac{d^d k}{(2\pi)^d} e^{-i\omega t + i\vec{k}\cdot\vec{x}} f_k(r)\tilde{\phi}_0(k). \tag{2.1.10}$$

The retarded Green function is obtained as follows:

1. Solve the classical equation of motion for the field $f_k(r)$ with the ingoing-wave boundary condition at the horizon and $f_k(r) \to 1$ at the boundary.

2. Substitute the classical solution into the action and represent the action in terms of the boundary value $\tilde{\phi}_0$. Only surface terms remain, and drop the contribution from the horizon.

3. The retarded Green's function is given by the kernel of the on-shell action:

$$S_{\text{on-shell}} = -\frac{1}{2}\int \frac{d^d k}{(2\pi)^d} \tilde{\phi}_0(-k) G_R(k) \tilde{\phi}_0(k) \tag{2.1.11}$$

   where the on-shell action is defined as $S_{\text{on-shell}} = (S + S_{\text{GH}} + S_{\text{c.t.}})|_{\text{on-shell}}$. S_{GH} is the Gibbons-Hawking term to provide a correct variational problem for the background geometry. $S_{\text{c.t.}}$ is the counterterm to renormalize divergences in the classical action.

Thus, the problem is to solve the equation of motion for the field ϕ under the appropriate boundary conditions.

From Eq. (2.1.1), the action which is quadratic in ϕ is

$$\begin{aligned} {}^{(2)}S_s = \frac{1}{16\pi G_{d+1}} \int d^{d+1}x \bigg[& -\frac{1}{2}\sqrt{\bar{g}}(\nabla_M \phi)^2 \\ & + \partial_r \left\{ \sqrt{\bar{g}} \left(2g^{rr}\phi\partial_r\phi + \frac{1}{2}\frac{g'_{xx}}{g_{xx}} g^{rr}\phi^2 \right) \right\} \bigg] \ , \end{aligned} \tag{2.1.12}$$

with the help of background equation of motions (See Appendix A.2). Because of the little group $SO(d-1)$ acting on x_i, the tensor mode of the metric perturbations decouples from the rest of perturbations: the other modes of the metric perturbations h_{MN}, the gauge field perturbations δA_M and the scalar field perturbation $\delta\Psi$. Thus, they can be set to zero consistently. Since the background geometry must satisfy the stationary condition, we add the Gibbons-Hawking term

$$S_{\rm GH} = \frac{1}{16\pi G_{d+1}} \int_{r\to\infty} d^d x \sqrt{-\gamma} 2K \ , \tag{2.1.13}$$

where $\gamma^{\mu\nu}$ is the boundary induced metric, n_M is the normal vector to the boundary and $K = \gamma^{\mu\nu}\nabla_\mu n_\nu$ is the trace of the extrinsic curvature of the boundary. This provides surface terms

$$^{(2)}S_{\rm GH} = \frac{1}{16\pi G_{d+1}} \int_{r\to\infty} d^d x \left(-2\sqrt{\bar{g}} g^{rr} \phi \partial_r \phi - \frac{1}{\sqrt{g_{rr}}} \partial_r \left(\sqrt{-\bar{\gamma}}\right) \phi^2 \right) \ . \tag{2.1.14}$$

Therefore, the bare action is

$$\begin{aligned} ^{(2)}(S_s + S_{\rm GH}) =& \frac{1}{16\pi G_{d+1}} \int d^{d+1}x \sqrt{\bar{g}} \left[-\frac{1}{2} \left(\nabla_M \phi\right)^2 \right] \\ &+ \frac{1}{16\pi G_{d+1}} \int_{r\to\infty} d^d x \left(\frac{g'_{xx}}{2g_{xx}} \frac{\sqrt{-\bar{\gamma}}}{\sqrt{g_{rr}}} - \frac{1}{\sqrt{g_{rr}}} \partial_r \left(\sqrt{-\bar{\gamma}}\right) \right) \phi^2 \ . \end{aligned} \tag{2.1.15}$$

The action diverges as $r \to \infty$, so the counterterms at the boundary must be added. We need only the gravitational counterterm in order to evaluate the retarded Green's function for the energy-momentum tensor. According to the holographic renormalization procedure, the counterterm is

$$\begin{aligned} S_{\rm c.t.} = -\frac{1}{16\pi G_{d+1}} \int_{r\to\infty} d^d x \sqrt{-\gamma} \Bigg[& \frac{2(d-1)}{L} + \frac{L}{d-2} \mathcal{R}[\gamma] \\ &+ \frac{L^3}{(d-4)(d-2)^2} \left(\mathcal{R}_{\mu\nu}[\gamma] \mathcal{R}^{\mu\nu}[\gamma] - \frac{d}{4(d-1)} \mathcal{R}[\gamma]^2 \right) + \cdots \Bigg] \ , \end{aligned} \tag{2.1.16}$$

where L is the AdS radius and $\mathcal{R}_{\mu\nu}[\gamma]$ is the Ricci tensor made from the induced metric $\gamma_{\mu\nu}$. These terms largely depend on the spacetime dimensions[2]. However, in order to evaluate the shear viscosity, we need boundary terms only up to first order in ω: $\mathcal{O}(\omega^2)$ terms in the action do not contribute to the Kubo formula because of the $\omega \to 0$ limit. So, only the first

[2]One has to be careful when the number of the boundary spacetime dimensions d is an even number. See Ref. [70] for details.

term in Eq. (2.1.16) is important and it becomes

$$ {}^{(2)}S_{\text{c.t.}} = \frac{1}{16\pi G_{d+1}} \int_{r\to\infty} d^d x \sqrt{-\bar{\gamma}} (d-1)\phi^2 \ , \tag{2.1.17} $$

for the tensor mode perturbation. This term removes the divergences from the second term of Eq. (2.1.15). As a result, the renormalized action is

$$ \begin{aligned} &16\pi G_{d+1}{}^{(2)}(S_s + S_{\text{GH}} + S_{\text{c.t.}}) \\ &\quad = \int \frac{d^d k}{(2\pi)^d} \tilde{\phi}_0(-k) \left(-\frac{1}{2} \frac{\sqrt{g}}{g_{rr}} f_{-k}(r) \partial_r f_k(r) \right) \tilde{\phi}_0(k) \Bigg|_{r\to\infty} \\ &\qquad + (\text{terms which are proportional to the EOM}) \\ &\qquad + (\text{contact terms}) \ , \end{aligned} \tag{2.1.18} $$

with (2.1.10). Here, we neglected the second derivative respect to t because it provide only $\mathcal{O}(\omega^2)$ terms. "(contact terms)" provide contact terms in the Green function and have the form $f_{-k} f_k$. They will not affect the shear viscosity since they do not give an imaginary part of retarded Green's function. The terms which give the imaginary part take the form like $f_{-k}\partial_r f_k$ [3]. We will see this at the end of this section.

In order to find the on-shell action, we need to solve the equation of motion for $f_k(r)$:

$$ f_k'' + \frac{g_{rr}}{g_{tt}} \omega^2 f_k + \frac{(g^{rr}\sqrt{g})'}{g^{rr}\sqrt{g}} f_k' = 0 \ , \tag{2.1.19} $$

where the long wavelength limit $\vec{k} \to 0$ is taken since $\mathcal{O}(|\vec{k}|)$ terms in the action don't contribute to the Kubo formula. The equations of motion can be solved as follows. First, solve this equation of motion near the horizon and impose the ingoing-wave boundary condition. Second, find the solution over the whole region in the bulk up to first order in $\mathfrak{w}$. Finally, match these solutions.

First, consider the near-horizon limit of Eq. (2.1.19). With asymptotics of the metric (2.1.5)

$$ f_k(r) \sim (r/r_h - 1)^{\pm i\mathfrak{w}} = \exp\Big[\pm i\mathfrak{w} \ln[r/r_H - 1] \Big] \ . \tag{2.1.20} $$

where $\mathfrak{w} := \omega/4\pi T$ is the rescaled dimensionless frequency. The ingoing-wave solution is given by $f_k(r) = \exp[-i\mathfrak{w} \ln(r/r_h - 1)]$. We expand this solution in terms of $\mathfrak{w} \ln(r/r_h - 1)$

[3] So, the second term of Eq. (2.1.15) and the counterterm (2.1.17) do not affect the shear viscosity.

near the horizon since we take the $\mathfrak{w} \to 0$ limit at the end of the analysis. So,

$$f_k(r) \sim 1 - i\mathfrak{w} \ln\left[r/r_h - 1\right] \ . \tag{2.1.21}$$

is the boundary condition as $r \to r_h$. The overall factor will be determined by the boundary condition at $r \to \infty$.

Next, get the solution of Eq. (2.1.19) for all r. In order to evaluate the Kubo formula, it is enough to obtain $f_k(r)$ up to first order in $\mathfrak{w}$. Thus, expand $f_k(r)$ in power of $\mathfrak{w}$:

$$f_k(r) = f^{(0)}(r) + \mathfrak{w} f^{(1)}(r) + \mathcal{O}(\mathfrak{w}^2) \ . \tag{2.1.22}$$

Inserting this into the equation of motion, $f^{(0)}$ and $f^{(1)}$ satisfy

$$f^{(i)\prime\prime} + \frac{(g^{rr}\sqrt{\bar{g}})'}{g^{rr}\sqrt{\bar{g}}} f^{(i)\prime} = 0 \ , \tag{2.1.23}$$

where i runs $i = 0, 1$. Solutions are given by

$$f^{(i)}(r) = C_1^{(i)} + C_2^{(i)} \int_r^\infty dr' \frac{g_{rr}(r')}{\sqrt{-\bar{g}(r')}} \ , \tag{2.1.24}$$

where $C_j^{(i)}$'s are integration constants. From the boundary condition at $r \to \infty$,

$$C_1^{(0)} = 1 \ , \quad C_1^{(1)} = 0 \ . \tag{2.1.25}$$

The rest of constants are determined by the boundary condition at the horizon. Since the integrand in Eq. (2.1.24) has a simple pole at the horizon,

$$\int_{r\sim r_h}^\infty dr' \frac{g_{rr}(r')}{\sqrt{-\bar{g}(r')}} \sim \sqrt{\frac{c_r}{c_t c_x^{d-1}}} \int_{r/r_h\sim 1}^\infty \frac{d(r'/r_h)}{(r'/r_h - 1)} = -\frac{1}{16\pi G_{d+1} sT} \ln\left[r/r_h - 1\right] \ . \tag{2.1.26}$$

Comparing this with the boundary condition (2.1.21), one gets

$$C_2^{(0)} = 0, \quad \mathfrak{w} C_2^{(1)} = 4G_{d+1} \cdot i\omega s \ . \tag{2.1.27}$$

Therefore, the solution of Eq. (2.1.19) with the appropriate boundary conditions is

$$f_k(r) = \left(1 + 4G_{d+1} \cdot i\omega s \int_r^\infty dr' \frac{g_{rr}(r')}{\sqrt{-\bar{g}(r')}} + \mathcal{O}\left(\mathfrak{w}^2\right)\right) \ , \tag{2.1.28}$$

which becomes

$$f_k(r) \to 1, \quad \partial_r f_k(r) \to -4G_{d+1} \cdot i\omega s \frac{g_{rr}(r)}{\sqrt{-\bar{g}(r)}} \ , \tag{2.1.29}$$

as $r \to \infty$. Thus, the terms $f_{-k}f_k$ and $f_{-k}\partial_r f_k$ in the action provide real and imaginary parts, respectively. So, the contact terms, which have the form $f_{-k}f_k$, do not contribute to the shear viscosity.

Now, we are ready to extract the Green function from the on-shell action. Substituting the solution (2.1.28) into the on-shell action, one gets

$$\begin{aligned} S_{\text{on-shell}} &= \frac{1}{16\pi G_{d+1}} \int \frac{d^d k}{(2\pi)^d} \tilde{\phi}_0(-k) \left(-\frac{1}{2} \frac{\sqrt{g}}{g_{rr}} f_{-k}(r) \partial_r f_k(r) \right) \tilde{\phi}_0(k) \Bigg|_{r\to\infty} \\ &= -\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \tilde{\phi}_0(-k) \left(-\frac{i\omega s}{4\pi} \right) \tilde{\phi}_0(k) \end{aligned} \tag{2.1.30}$$

This leads to the retarded Green's function

$$G_R^{1212}(\omega, 0) = -\frac{i\omega s}{4\pi} + \mathcal{O}\left((\omega/T)^2\right) \ , \tag{2.1.31}$$

from the prescription (2.1.11). Finally, the Kubo formula (2.1.7) derives the shear viscosity,

$$\eta = -\lim_{\omega \to 0} \frac{1}{\omega} G_R^{1212}(\omega, \vec{0}) = \frac{s}{4\pi} \ . \tag{2.1.32}$$

Thus, the ratio of the shear viscosity to the entropy density is

$$\frac{\eta}{s} = \frac{1}{4\pi} \ . \tag{2.1.33}$$

Therefore, the universality of η/s holds in this system irrespective of whether the complex scalar condenses or not.

2.2 Anisotropic superfluids

The p-wave or the $(p+ip)$-wave holographic superfluids are described by the Einstein-Yang-Mills system:

$$S_{\text{EYM}} = \frac{1}{16\pi G_{d+1}} \int d^{d+1}x \sqrt{-g} \left\{ R - 2\Lambda - \frac{1}{4} (F^a_{MN})^2 \right\} \ , \tag{2.2.1}$$

where $F^a_{MN} = \partial_M A^a_N - \partial_N A^a_M + g_{\rm YM}\epsilon^{abc}A^b_M A^c_N$ is the field strength of $SU(2)$ gauge fields, $g_{\rm YM}$ is the Yang-Mills gauge coupling and ϵ^{abc} is the totally antisymmetric tensor with $\epsilon^{123} = 1$. The gauge field is written as a matrix-valued form:

$$A = A^a_M \tau^a dx^M \ , \tag{2.2.2}$$

where $\tau^a = \sigma^a/(2i)$ using the Pauli matrices, so $[\tau^a, \tau^b] = \epsilon^{abc}\tau^c$.

2.2.1 The p-wave superfluids

The p-wave case is described by the ansatz

$$\begin{aligned} ds^2_{d+1} &= -g_{tt}(r)dt^2 + g_{x_1x_1}(r)dx_1^2 + g_{x_2x_2}(r)\sum_{i=2}^{d-1} dx_i^2 + g_{rr}(r)dr^2 \ , & (2.2.3) \\ A &= \Phi(r)\tau^3 dt + w(r)\tau^1 dx_1 \ . & (2.2.4) \end{aligned}$$

The function $\Phi(r)$ gives the background static electric potential whereas the function $w(r)$ represents the condensate. We impose the regularity condition at the horizon $r = r_h$:

$$g_{tt} \to c_t(r - r_h) \ , \quad g_{x_1x_1} \to c_{x_1} \ , \quad g_{x_2x_2} \to c_{x_2} \ , \quad g_{rr} \to c_r(r - r_h)^{-1} \ . \tag{2.2.5}$$

Then, the temperature and the entropy density are given by

$$T = \frac{1}{4\pi}\sqrt{\frac{c_t}{c_r}} \ , \quad s = \frac{\sqrt{c_{x_1}c_{x_2}^{d-2}}}{4G_{d+1}} \ , \tag{2.2.6}$$

respectively.

As is clear from the metric (2.2.3), the boundary spacetime is anisotropic. In such a case, the shear viscosity is no longer given by a single coefficient η. Rather, one is interested in

$$\begin{aligned} \eta^{ijkl} &= -\lim_{\omega\to 0}\frac{1}{\omega}{\rm Im}\, G_R^{ijkl}(\omega, \vec{0}) \ , & (2.2.7) \\ G_R^{ijkl}(\omega, \vec{0}) &= -i\int_{-\infty}^{\infty} d^dx\, e^{i\omega t}\theta(t)\left\langle [T^{ij}(t,\vec{x}), T^{kl}(0,\vec{0})]\right\rangle \ . & (2.2.8) \end{aligned}$$

From the symmetric nature of the metric and the $SO(d-2)$ invariance acting on $x_2, \cdots, x_{d-1}$, there are only two nontrivial independent coefficients, *e.g.*, η^{1212} and η^{2323}. We will examine these coefficients below.

The shear viscosities of anisotropic fluids have been widely discussed in the context of liquid crystal [75]. There are various parametrizations known in the literature. Among them, the most well-studied parametrization is the Miesowicz viscosity coefficients. The coefficients η^{1212} and η^{2323} are related to the Miesowicz coefficients [76]. However, various conventions are found in the literature for the Miesowicz coefficients. To avoid the confusion, we keep using η^{ijkl}.

η^{2323} (for $d \geq 4$)

First, let us consider η^{2323}. The coefficient exists for $d \geq 4$, and the metric has the $SO(d-2)$ invariance, so the perturbation h_{23} transforms as a tensor mode. Then, the discussion is similar to the s-wave superfluid case.

The action (2.2.1) with appropriate boundary terms reduces to ($h^2{}_3 =: \phi(r,t) = \int \frac{d^dk}{(2\pi)^d} e^{ikx} f_k(r) \tilde{\phi}_0(k)$)

$$\begin{aligned} &16\pi G_{d+1}{}^{(2)}(S_p + S_{\mathrm{GH}} + S_{\mathrm{c.t.}}) \\ &\quad = \int \frac{d^dk}{(2\pi)^d} \tilde{\phi}_0(-k) \left(-\frac{1}{2} \frac{\sqrt{g}}{g_{rr}} f_{-k}(r) \partial_r f_k(r) \right) \tilde{\phi}_0(k) \Bigg|_{r\to\infty} \\ &\qquad + (\text{terms which are proportional to the EOM}) \\ &\qquad + (\text{contact terms}) \ , \end{aligned} \tag{2.2.9}$$

using the ansatz (2.2.3) and the equation of motion for the background geometry (See Appendix A.3.1). The equation of motion is given by

$$f_k'' + \frac{g_{rr}}{g_{tt}} \omega^2 f_k + \frac{(g^{rr}\sqrt{g})'}{g^{rr}\sqrt{g}} f_k' = 0 \ . \tag{2.2.10}$$

This takes the same form as the s-wave case (2.1.19), so the solution under the appropriate boundary conditions is given by

$$f_k(r) = \left(1 + 4G_{d+1} \cdot i\omega s \int_r^\infty dr' \frac{g_{rr}(r')}{\sqrt{-\bar{g}(r')}} + \mathcal{O}\left((\omega/T)^2\right) \right) \ , \tag{2.2.11}$$

and the retarded Green function has the same form as the s-wave one (2.1.31). From the Kubo formula, the shear viscosity to the entropy density ratio is

$$\frac{\eta^{2323}}{s} = \frac{1}{4\pi} \ . \tag{2.2.12}$$

The universality holds for this case as well.

η^{1212}

Next, let us consider η^{1212}. For $d = 3$, this is the only shear viscosity coefficient. The perturbation h_{12} transforms as a vector under $SO(d-2)$. Thus, it couples to the vector mode of the Yang-Mills perturbations. As a result, the existing technique is not applicable.

It should be straightforward to obtain the action for the relevant vector mode perturbations since they are standard vector perturbations for which one can rely on the symmetry. However, it does not seem straightforward to solve them analytically. Actually, η^{1212} was calculated numerically [78, 79] and take non-universal value $\eta^{1212}/s \geq 1/(4\pi)$ in the low temperature phase. We will discuss it in Sec. 2.3.1.

2.2.2 The $(p+ip)$-wave superfluids

For completeness, let us consider the $(p+ip)$-wave holographic superfluid. The $(p+ip)$-wave case is described by the ansatz

$$\begin{aligned} ds^2 &= -g_{tt}(r)dt^2 + g_{xx}(r)(dx_1^2 + dx_2^2) + g_{rr}(r)dr^2 \ , \qquad (2.2.13) \\ A &= \Phi(r)\tau^3 dt + w(r)(\tau^1 dx_1 + \tau^2 dx_2) \ . \qquad (2.2.14) \end{aligned}$$

As in previous models, the regularity condition at the horizon $r = r_h$ implies

$$g_{tt} \to c_t(r - r_h) \ , \quad g_{xx} \to c_x \ , \quad g_{tt} \to c_t(r - r_h)^{-1} \ , \qquad (2.2.15)$$

which leads to the temperature and the entropy density as

$$T = \frac{1}{4\pi}\sqrt{\frac{c_t}{c_r}} \ , \quad s = \frac{c_x}{4G_4} \ , \qquad (2.2.16)$$

respectively. It is argued that the $(p+ip)$-wave background is unstable and it turns into the p-wave background [50]. But the analysis was carried out only in the probe limit and the full analysis including the backreaction has not been done.

Unlike the p-wave case, the metric is isotropic. Thus, the shear viscosity is described by a single coefficient. The anisotropy in the (x_1, x_2)-plane is caused by the Yang-Mills field. The condensation breaks the $SO(2)$ rotational symmetry in the (x_1, x_2)-plane as well as the $U(1)$ gauge symmetry. But, as is clear from Eq. (2.2.14), it preserves a diagonal $U(1)$ which is a combination of the two. Thus, there does not exist the tensor mode which decouples from Yang-Mills perturbations. As a result, the existing technique is not applicable.

Since the whole system does not have the $SO(2)$ symmetry, it is worthwhile to see explicitly how Yang-Mills perturbations couple to the "tensor mode" perturbations. In Appendix A.3, we derived the interaction of the Yang-Mills perturbations and the metric perturbations [Eq. (A.3.8)]. For the tensor mode metric perturbations $h^2{}_2 = -h^1{}_1 (:= \phi_d)$ and $h^1{}_2 = h^2{}_1 (:= \phi_{od})$[4], the interaction reduces to

$$\frac{1}{2} h_{ij} Q_a^{ijMN} f^a_{MN} = h^i{}_j \left(F \cdot f\right)^j{}_i \ . \tag{2.2.17}$$

Here, we defined

$$\begin{aligned} f^a_{MN} &:= D_M a^a_N - D_N a^a_M \ , & (2.2.18)\\ D^{ab}_M &:= \nabla_M \delta^{ab} + g_{\rm YM} \epsilon^{acb} A^c_M \ , & (2.2.19)\\ (F \cdot f)^i{}_j &:= F^{aiN} f^a_{jN} \ . & (2.2.20) \end{aligned}$$

As is clear from Eq. (2.2.17), the tensor mode $h^i{}_j$ couples to $(F \cdot f)^i{}_j$, which transforms as a tensor under the diagonal $U(1)$ symmetry. ϕ_d and ϕ_{od} couple to

$$(F \cdot f)_d := (F \cdot f)^2{}_2 = -(F \cdot f)^1{}_1 \ , \qquad (F \cdot f)_{od} := (F \cdot f)^1{}_2 = (F \cdot f)^2{}_1 \ , \tag{2.2.21}$$

respectively. They contain the following components of $\delta A^a{}_M$:

$$a_d := \delta A^1{}_1 = -\delta A^2{}_2 \ , \qquad a_{od} := \delta A^1{}_2 = \delta A^2{}_1 \ , \qquad \text{(the other modes)} = 0 \ . \tag{2.2.22}$$

These perturbations ϕ_{od}, ϕ_d, a_{od} and a_d are all coupled. The explicit form of $(F \cdot f)$ is given by

$$\begin{aligned} (F \cdot f)_{od} &= g^{rr} g^{xx} (\partial_r w)(\partial_r a_{od}) + g^{tt} g^{xx} g_{\rm YM} \Phi w (D_t a_d), & (2.2.23)\\ (F \cdot f)_d &= g^{rr} g^{xx} (\partial_r w)(\partial_r a_d) - g^{tt} g^{xx} g_{\rm YM} \Phi w (D_t a_{od}), & (2.2.24) \end{aligned}$$

which include the covariant derivatives of a_i:

$$D_t a_{od} = \partial_t a_{od} + g_{\rm YM} \Phi a_d \ , \qquad D_t a_d = \partial_t a_d - g_{\rm YM} \Phi a_{od} \ . \tag{2.2.25}$$

[4]In the s-wave and p-wave cases, the diagonal perturbation ϕ_{od} and the off-diagonal perturbation ϕ_d are completely decoupled. So, we have set $\phi_d = 0$. But this does not hold for the $(p + ip)$-wave case as we will see below.

Therefore, these forms mix a_{od} and a_d.

Let us summarize how the tensor mode metric perturbations couple with Yang-Mills perturbations:

- The tensor mode metric perturbations ϕ_{od} and ϕ_d couple to the tensor $(F \cdot f)_{od}$ and $(F \cdot f)_d$, respectively, where $(F \cdot f)$ is made from the Yang-Mills perturbations. So, the action for $\phi_{od/d}$ no longer takes the minimally-coupled scalar form.

- The Yang-Mills perturbations $a_{od/d}$ couple to ϕ_{od} and ϕ_d. As a result, ϕ_{od} couples to ϕ_d through a_i.

The complete actions in terms of these "tensor mode" fluctuations are given by

$$
\begin{aligned}
{}^{(2)}(S_{(p+ip)} \quad &+ \quad S_{\rm GH}) = \frac{1}{16\pi G_4}\int d^4x^{(2)}(\mathcal{L}_{\rm grav} + \mathcal{L}_{\rm gauge} + \mathcal{L}_{\rm int}) \ ; && (2.2.26) \\
{}^{(2)}\mathcal{L}_{\rm grav} \quad &= \quad \sqrt{\bar{g}}\sum_{i=1}^{2}\Bigg[-\frac{1}{2}\left\{-g^{tt}(\partial_t\phi_i)^2 + g^{rr}(\partial_r\phi_i)^2\right\} - \frac{1}{2}M(r)^2\phi_i^2 \\
&\qquad\qquad + (\text{surface term})\Bigg] \ , && (2.2.27) \\
{}^{(2)}\mathcal{L}_{\rm gauge} \quad &= \quad \sqrt{\bar{g}}g^{xx}\sum_{i=1}^{2}\left[-g^{rr}(\partial_r a_i)^2 + g_{\rm YM}^2 w^2 a_i^2 + g^{tt}(D_t a_i)^2\right] \ , && (2.2.28) \\
{}^{(2)}\mathcal{L}_{\rm int} \quad &= \quad \sqrt{\bar{g}}\sum_{i=1}^{2}\phi_i(F\cdot f)_i \ , && (2.2.29)
\end{aligned}
$$

where we defined two-component vectors as $\phi_i = (\phi_{od}, \phi_d)$, $a_i = (a_{od}, a_d)$, $(F \cdot f)_i = ((F \cdot f)_1, (F \cdot f)_2)$ and i runs isotropic components $i = 1, 2$. ${}^{(2)}S_{\rm int}$ is the interaction term we have discussed in Eq. (2.2.17). The mass-like function $M(r)$ is defined by

$$
M(r)^2 := g^{rr}g^{xx}(\partial_r w)^2 + g^{xx}g_{\rm YM}^2\left(g^{xx}w^2 - g^{tt}\Phi^2\right)w^2 \ . \qquad (2.2.30)
$$

The action leads to coupled equations of motion for ϕ_{od}, ϕ_d, a_{od} and a_d. It is difficult to solve them analytically, and it does not seem straightforward to obtain the shear viscosity to the entropy ratio.

2.3 Implications of the results

We study η/s for s-wave, $(p + ip)$-wave, and p-wave holographic superfluids. The shear viscosity for the s-wave superfluids satisfies the universality. The p-wave superfluids are anisotropic, and there are two nontrivial independent shear viscosities, η^{2323} and η^{1212}. We show that one of the coefficients η^{2323} satisfies the universality.

On the other hand, for another coefficient η^{1212} of the p-wave superfluids and for the shear viscosity of the $(p+ip)$-wave superfluid, the gravitational perturbations in question couple to the Yang-Mills perturbations even in Kubo-formula method, and the existing technique is not applicable. We extract the modes which couple to the gravitational perturbations. For the $(p+ip)$-wave case, we write down the perturbed action for those modes.

There is another technique to derive η/s in the membrane paradigm context [24]. According to the method, transport coefficients in the boundary field theory can be determined by (i) the flow equation for r-dependent transport coefficients, *e.g.*, the shear viscosity $\eta(r)$ and by (ii) their values at the horizon. If the tensor mode metric perturbation is written as a free scalar, the flow equation becomes trivial in the hydrodynamic limit: $\partial_r \eta(r) = 0$. In this case, $(\eta/s)|_{\text{boundary}} = (\eta/s)|_{\text{horizon}} = 1/4\pi$. This method works for η in the s-wave case and for η^{2323} in the p-wave case. But it does not work for η^{1212} in the p-wave case and for η in the $(p+ip)$-wave case. This is because the interactions of the metric and Yang-Mills perturbations provide a non-trivial flow equation $\partial_r \eta(r) \neq 0$.

We now discuss the shear viscosity itself of holographic superfluids below.

2.3.1 Viscosity of superfluids

The holographic superfluid shows a nonzero viscosity. To interpret the result, note the following points.

First, a superfluid has a nonzero viscosity. For example, for ^{4}He no viscous resistance is observed when it goes through a narrow pipe, but a viscous drag is observed when a test body is moved in the liquid.

According to the two-fluid model, a superfluid consists of the superfluid component and the normal component. The normal component has a nonzero viscosity, so a superfluid has a nonzero viscosity as a whole. The normal component represents the effect of thermal fluctuation, and it always exists at finite temperatures. And the quasi-particle description is valid for the normal component. Since we do not separate the normal and superfluid components, one cannot observe the zero viscosity for the superfluid component.

Second, currently the boundary theory description is not clear for holographic superfluids, but the boundary theory presumably contains the fields which may not play an important role in the superfluid behavior. Among the other things, the boundary theory should include the $SU(N)$ non-Abelian gauge field, which is unlikely to play an important role. The computation of η/s includes the dissipation not only from the normal component but also from these fields.

Isotropic Components

Obviously, for the universal components, (*i.e.*, η in the s-wave case and η^{2323} in the p-wave case,) while η/s is the same in both phases, η itself can have different functional forms. This requires the knowledge of s, and it would be interesting to compute it. In the probe limit, both phases are described by the same bulk geometry since the backreaction of matter fields onto the geometry is ignored. Thus, one needs the fully backreacted metric to find a nontrivial behavior. In particular, it would be interesting to see if η in the superfluid state is lower than η in the (unstable) normal state. Again one needs a fully backreacted metric, but analysis near the critical point or a numerical computation would suffice for the purpose.

In fact, the entropy density s has been obtained for a limited class of holographic superfluids. Especially, Ref. [77] obtained s for the $(4+1)$-dimensional p-wave superfluid in the grand canonical ensemble.[5] They use the parameter $\alpha := \kappa_5/\hat{g}$. In our notation, $\alpha \propto 1/g_{\rm YM}$, and $\alpha \to 0$ corresponds to the probe limit. Once the backreaction is taken into account, the p-wave superfluid undergoes the second-order phase transition only when $\alpha < \alpha_c$, where $\alpha_c \sim 0.365$, and it undergoes the first-order phase transition when $\alpha > \alpha_c$. Namely, the phase transition becomes first-order when the backreaction becomes large.

According to their computation, s in the superfluid state is lower than s in the unstable normal state below $T < T_c$ at fixed chemical potential $\mu = A_t^3$. See Fig. 3(b) of Ref. [77]. Since η^{2323} satisfies the universality, η^{2323} in the superfluid state is lower than the normal state one. This may be due to the zero viscosity of the superfluid component. Needless to say, this statement is only for one coefficient of shear viscosities of one p-wave superfluid. At this moment, it is not clear if the same holds in general.

Anisotropic Components

For the anisotropic components, (*i.e.*, η^{1212} in the p-wave case and η in the $(p+ip)$-wave case,) it seems to be difficult to obtain them analytically. The former, η^{1212} in the p-wave, was calculated numerically [78, 79] and takes non-universal value $\eta^{1212}/s \geq 1/(4\pi)$ in the low temperature phase. As noted above, since s in the superfluid state is lower than one in the normal state, the decrement of η^{1212} is lower than that of s. The behavior of η^{1212} itself is not, however, unclear.

[5] For a s-wave superfluid, s has been obtained in the microcanonical ensemble [80].

Moreover, there is another difficulty to define the shear viscosity in the anisotropic fluids. In the anisotropic fluids, additional viscosity coefficients appear since director fields, which cause anisotropy [6], provide additional terms for the stress tensor. Therefore, η^{1212} may be made of the shear viscosity η from the isotropic part in Eq. (1.2.11) and other viscosities η_{director} come from the directors. We can expect two possibilities:

1. Although the contribution from η_{director} makes $\eta^{1212}/s \geq 1/(4\pi)$, the viscosity from the isotropic part preserves $\eta/s = 1/(4\pi)$.

2. The viscosity from the isotropic part even breaks the universality: $\eta/s = 1/(4\pi)$.

It is not necessary that η preserves the universality since h_{12} couples with other components. If it turns out that it also satisfy the universality, these shear viscosities will give highly nontrivial tests for the universality.

2.3.2 Implication to dynamic critical phenomena

We found that the universality of η/s holds both for high temperature phase and for low temperature phase. In the second-order phase transition, critical phenomena occur and one has singular behaviors in physical quantities. In the dynamic case, one has singular behaviors in various transport coefficients [72]. But our results indicate that there is no divergence in the shear viscosity. (Since the entropy density is the first derivative of the free energy, it is continuous across the phase transition. Thus, the universality of η/s implies that the shear viscosity is also continuous across the transition.)

More precisely, in the dynamic critical phenomena, the relaxation time of the order parameter diverges, which is known as the critical slowing down. In fact, for s-wave holographic superfluids, the relaxation time of the order parameter diverges near the critical point [73]. In general, when a system has a conserved charge, the associated transport coefficient diverges as well. For example, for $T_{\mu\nu}$, one has a (mild) singularity in η. But our results indicate that this does not happen in the holographic superfluids. The fact that singular behavior does not occur in η has been observed in the critical phenomena of R-charged black holes [74].

[6] *e.g.*, the boundary field corresponding to $w(r)$ in Eq. (2.2.4).

Chapter 3

Another Realization of the Relationship between Gravity and Hydrodynamics

3.1 Linearized hydrodynamics by Metric Perturbations

3.1.1 Homogeneous Perturbations

The basic hydrodynamic equation is the conservation equation

$$\nabla_\mu T^{\mu\nu} = 0 \tag{3.1.1}$$

(or the continuity equation and the Navier-Stokes equation). In $(3+1)$-dimensions, there are 4 equations whereas the stress tensor has 10 components. Since the equation of motion is not closed, one introduces the constitutive equation[1]:

$$T^{\mu\nu} = (\varepsilon + P)u^\mu u^\nu + P g^{\mu\nu} - \tau^{\mu\nu} \ , \tag{3.1.2}$$

$$\tau^{\mu\nu} := P^{\mu\alpha}P^{\nu\beta}\left[\eta\left(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - \frac{2}{p} g_{\alpha\beta}\nabla_\lambda u^\lambda\right) + \zeta g_{\alpha\beta}\nabla_\lambda u^\lambda\right] \ , \tag{3.1.3}$$

where $P^{\mu\nu} := g^{\mu\nu} + u^\mu u^\nu$ is the projection tensor, η is the shear viscosity, and ζ is the bulk viscosity. In other words, one chooses the velocity field u^μ and the pressure P as the

[1] We use $\mu, \nu, \dots$ for the $(p+1)$-dimensional boundary coordinate indices. The boundary spatial coordinates x_i are also denoted as $x_i = (x, y, z)$ for $p = 3$. We use indices $a, b, \dots$ for the spatial coordinates transverse to z.

basic hydrodynamic variables. Note that ε and P are not independent; rather they are determined by an equation of state. We choose P as the independent variable. We assume $\varepsilon = \varepsilon(P)$ and use $c_s^2 = \partial P/\partial\varepsilon$, where c_s is the speed of sound. Then, there are 4 degrees of freedom in $(3+1)$-dimensions (three from u^μ because of $u^2 = -1$ and one from P), and the equation of motion is closed.

In equilibrium, there is no spatial flow, so one can take the rest frame $u^i = 0$. Then, one has

$$T^t{}_t = -\bar{\varepsilon}\ , \quad T^i{}_j = \bar{P}\delta^i{}_j\ , \tag{3.1.4}$$

where " $^-$ " denotes an equilibrium value[2].

When one adds external gravitational perturbations $h^\mu{}_\nu$, the hydrodynamic variables P and u^i have responses following the conservation equation. By solving the conservation equation, one can determine the responses. For hydrodynamic computations, we always use the Minkowski background $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$. We consider the metric perturbations of the form

$$h^\mu{}_\nu(t,z) = h^\mu{}_\nu e^{-i\omega t + iqz}\ . \tag{3.1.5}$$

Then, the metric perturbations are decomposed as the tensor, shear and sound modes. We consider the sound mode, which consists of

$$h^t{}_t\ , \quad h^a{}_a = h^x{}_x\ , \quad h^z{}_z\ , \quad h^z{}_t\ . \tag{3.1.6}$$

The metric becomes

$$ds^2 = -(1 + h^t{}_t)dt^2 + \sum_i (1 + h^i{}_i)dx_i^2 + 2h^z{}_t dtdz\ . \tag{3.1.7}$$

We write the responses as

$$P(t,z) = \bar{P} + \delta P e^{-i\omega t + iqz}\ , \tag{3.1.8}$$

$$u^i(t,z) = \delta u^i e^{-i\omega t + iqz}\ , \tag{3.1.9}$$

and $\varepsilon(t,z) = \bar{\varepsilon} + \delta\varepsilon e^{-i\omega t + iqz}$. Accordingly, the stress tensor has the response

$$T^\mu{}_\nu(t,z) = \bar{T}^\mu_\nu + \delta T^\mu{}_\nu e^{-i\omega t + iqz}\ . \tag{3.1.10}$$

[2]In this chapter, we consider $T^\mu{}_\nu$, the stress tensor with one upper and one lower indices, which is convenient to compare with the Brown-York tensor (Sec. 3.2.1).

We first consider homogeneous perturbations $q = 0$. Since $u^2 = -1$, $u^t = 1 - h^t{}_t/2$ (one can set $u^a = 0$). From the conservation equation $\nabla_\mu T^{\mu\nu} = 0$, one gets

$$i\omega \left\{ \delta\varepsilon + \frac{\bar{\varepsilon} + \bar{P}}{2} h_s \right\} = 0 \ , \tag{3.1.11}$$

$$i\omega(\bar{\varepsilon} + \bar{P})(h^z{}_t + \delta u^z) = 0 \ , \tag{3.1.12}$$

where $h_s := \sum_k h^k{}_k$ is the *spatial* trace. Then, $\delta T^\mu{}_\nu$ becomes

$$\delta T^t{}_t = -\delta\varepsilon = \frac{\bar{\varepsilon} + \bar{P}}{2} h_s \ , \tag{3.1.13a}$$

$$\delta T^z{}_t = (\bar{\varepsilon} + \bar{P}) h^z{}_t \ , \tag{3.1.13b}$$

$$\delta T^i{}_j = \delta P(h) \delta^i{}_j + i\eta\omega h^i{}_j - i \left(\frac{\eta}{p} - \frac{\zeta}{2} \right) \omega h_s \delta^i{}_j \ , \tag{3.1.13c}$$

where

$$\delta P(h) = c_s^2 \delta\varepsilon = -\frac{\bar{\varepsilon} + \bar{P}}{2} c_s^2 h_s \ . \tag{3.1.14}$$

These expressions may be familiar to readers. For instance, see App. A of Ref. [82] for $\delta T^i{}_j$. However, inhomogeneous perturbation case ($q \neq 0$) in the next subsection is more involved and deserves a close inspection.

Anticipating the bulk results in the following sections, let us consider the $c_s \to \infty$ limit. In the $c_s \to \infty$ limit, the continuity equation (3.1.11) becomes

$$i\omega \bar{P} h_s = 0 \ . \tag{3.1.15}$$

The $c_s \to \infty$ limit is rather special. In this limit, the conservation equation gives a condition for the perturbations instead of a response. In order that time-dependent perturbations are allowed, the spatial perturbations must be traceless. Or the fluid must be compressible for generic time-dependent homogeneous perturbations. Then, one obtains

$$\delta T^t{}_t \to 0 \ , \tag{3.1.16a}$$

$$\delta T^z{}_t \to (\bar{\varepsilon} + \bar{P}) h^z{}_t \ , \tag{3.1.16b}$$

$$\delta T^i{}_j \to i\eta\omega h^i{}_j \ . \tag{3.1.16c}$$

3.1.2 Inhomogeneous Perturbations

We turn to inhomogeneous perturbations $q \neq 0$. Again take $u^t = 1 - h^t{}_t/2$. For $q \neq 0$, the continuity equation becomes

$$- i\omega \left(\frac{\delta\varepsilon}{\bar{\varepsilon} + \bar{P}} + \frac{1}{2} h_s \right) + iq\delta u^z = 0 \ . \tag{3.1.17}$$

Combining this with the Navier-Stokes equation gives

$$\begin{aligned}
\delta u^z &= \frac{\omega}{q} \frac{c_s^2 q^2}{c_s^2 q^2 - \omega^2 - i\Gamma_s \omega q^2} \left[\frac{1}{2} h_s + \frac{\omega}{c_s^2 q} h^z{}_t - \frac{h^t{}_t}{2c_s^2} \right. \\
&\quad \left. - \frac{i}{c_s^2} \left\{ \frac{1}{2} \left(\hat{\zeta} - \frac{2}{3} \hat{\eta} \right) \omega (h^x{}_x + h^y{}_y) + \frac{\Gamma_s}{2} \omega h^z{}_z \right\} \right] \ , && (3.1.18a) \\
\delta P &= (\bar{\varepsilon} + \bar{P}) \frac{c_s^2 q^2}{c_s^2 q^2 - \omega^2 - i\Gamma_s \omega q^2} \left[-\frac{1}{2} h^t{}_t + \frac{\omega}{q} h^z{}_t + \frac{\omega^2}{2q^2} h_s + i\hat{\eta}\omega \left(h^x{}_x + h^y{}_y \right) \right] \ , && (3.1.18b) \\
\hat{\eta} &:= \frac{\eta}{\bar{\varepsilon} + \bar{P}} \ , \quad \hat{\zeta} := \frac{\zeta}{\bar{\varepsilon} + \bar{P}} \ , && (3.1.18c) \\
\Gamma_s &:= \frac{1}{\bar{\varepsilon} + \bar{P}} \left(\frac{4}{3} \eta + \zeta \right) \ , && (3.1.18d)
\end{aligned}$$

where Γ_s is the sound attenuation constant. Also, these are momentum-space expressions so are complex; in real-space, they are real.

Having written down all hydrodynamic variables via metric perturbations, we are ready to express the hydrodynamic stress tensor via metric perturbations only. The full expression is rather cumbersome, so we give the expressions only in the $c_s \to \infty$ limit, which is relevant to the Rindler case. In the $c_s \to \infty$ limit,

$$\delta u^z \to \frac{\omega}{2q} h_s \ , \tag{3.1.19}$$

$$\delta P \to (\bar{\varepsilon} + \bar{P}) \left[-\frac{1}{2} h^t{}_t + \frac{\omega}{q} h^z{}_t + \frac{\omega^2}{2q^2} h_s \right] + i\eta\omega \left(h^x{}_x + h^y{}_y \right) \ . \tag{3.1.20}$$

Note that the sound pole $(c_s^2 q^2 - \omega^2 + i\Gamma_s \omega q^2)^{-1}$ in Eqs. (3.1.18) disappears in this limit. At the same time, the dependence on the bulk viscosity disappears. As a check, substituting Eq. (3.1.19) into the continuity equation gives $\delta\varepsilon = 0$ as expected. Also, some components of covariant derivatives are

$$\nabla_x u^x = -\frac{1}{2} i\omega h^x{}_x \ , \tag{3.1.21a}$$
$$\nabla_y u^y = -\frac{1}{2} i\omega h^y{}_y \ , \tag{3.1.21b}$$
$$\nabla_z u^z = \frac{1}{2} i\omega (h^x{}_x + h^y{}_y) \ , \tag{3.1.21c}$$

so u^μ obeys

$$\nabla_\mu u^\mu = 0 \ . \tag{3.1.22}$$

Then, $\delta T^\mu{}_\nu$ becomes

$$\delta T^t{}_t \to 0 \ , \tag{3.1.23a}$$
$$\delta T^z_t \to -(\bar{\varepsilon} + \bar{P}) \frac{\omega}{2q} h_s \ , \tag{3.1.23b}$$
$$\delta T^x_x \to \delta P(h) + i\eta\omega h^x{}_x \ , \tag{3.1.23c}$$
$$\delta T^y_y \to \delta P(h) + i\eta\omega h^y{}_y \ , \tag{3.1.23d}$$
$$\delta T^z_z \to \delta P(h) - i\eta\omega (h^x{}_x + h^y{}_y) \ , \tag{3.1.23e}$$

in the $c_s \to \infty$ limit. Several points of Eqs. (3.1.23) deserve comments. (i) T^z_t is not proportional to $h^z{}_t$ [*cf.*, Eq. (3.1.13b)]. (ii) The $O(i\omega)$ terms of T^z_z is not proportional to $h^z{}_z$. (iii) While Eqs. (3.1.18) themselves have the well-defined $q \to 0$ limit, the $c_s \to \infty$ case does not have the limit. So, we consider the $q = 0$ and $q \neq 0$ cases separately.

We now compare hydrodynamic expressions obtained in this section with the Brown-York tensor in Rindler space and in the SAdS_5 black hole.

3.2 Sound Mode in Rindler Space

3.2.1 Thermodynamic Quantities

The $(p+2)$-dimensional Rindler space is given by

$$ds^2_{p+2} = -r dt^2 + \frac{dr^2}{r} + \sum_i dx_i^2 \ . \tag{3.2.1}$$

The Rindler horizon is located at $r = 0$ and the Hawking temperature is given by $T = 1/(4\pi)$.

We consider the timelike surface $r = r_c$. The Brown-York tensor is given by

$$\mathcal{T}^{\mu}{}_{\nu} = \frac{1}{8\pi G} \left(\delta^{\mu}{}_{\nu} K - K^{\mu}{}_{\nu} \right) \ , \tag{3.2.2}$$

where $K^{\mu}{}_{\nu}$ is the extrinsic curvature of the surface, and K is its trace. In this thesis, we denote the Brown-York tensor as $\mathcal{T}^{\mu}{}_{\nu}$ to avoid confusion with the hydrodynamic stress tensor $T^{\mu}{}_{\nu}$. For a diagonal metric, the extrinsic curvature takes a simple form:

$$K^{\mu}{}_{\nu} = \frac{1}{2} n^r g^{\mu\rho} \partial_r g_{\rho\nu} \ , \tag{3.2.3}$$

where $g_{\mu\nu}$ is the induced metric on the surface. For Rindler space, $g_{\mu\nu} = \mathrm{diag}(-r_c, \mathbf{1})$. Also, n^r is the unit normal to the $r = r_c$ surface: $n^r = 1/\sqrt{g_{rr}}$.

We consider the Brown-York tensor with one upper and one lower indices from the following reasons:

1. The counterterm takes the form $\mathcal{T}^{\mu}{}_{\nu}{}^{(\mathrm{CT})} \propto \delta^{\mu}{}_{\nu}$ (see below), so the counter term dependence is absent upon metric perturbations.

2. We chose the Minkowski background $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ for the hydrodynamic computations. But this differs from the induced metric used for the Brown-York tensor by r-rescaling, *e.g.*, $g_{\mu\nu} = \mathrm{diag}(-r_c, \mathbf{1})$ for Rindler space. One way is to transform the Brown-York tensor from the original coordinates x^{μ} to the proper coordinates $x^{\tilde{\mu}}$:

$$\tilde{t} = \sqrt{-\bar{g}_{tt}}\, t \ , \quad x^{\tilde{i}} = \sqrt{\bar{g}_{ii}}\, x^{i} \ . \tag{3.2.4}$$

   However, it is not necessary to distinguish x^{μ} and $x^{\tilde{\mu}}$ for $\mathcal{T}^{t}{}_{t}$ and $\mathcal{T}^{i}{}_{j}$ since the upper and lower indices receive the opposite scaling. (This does not apply to $\mathcal{T}^{t}{}_{z}$, so a care is necessary.)

One can add "counterterms" to the Brown-York tensor. From the AdS/CFT point of view, the counterterms regularize divergences in physical quantities [83]. They are given by

$$\mathcal{T}^{\mu}{}_{\nu}{}^{(\mathrm{CT})} = -\frac{1}{16\pi G} \left(c_1 \delta^{\mu}{}_{\nu} + c_2 G^{\mu}{}_{\nu}{}^{(p+1)} + \cdots \right) \ . \tag{3.2.5}$$

The coefficient $c_1 = 2p/L$, where L is the AdS radius. $G^{\mu}{}_{\nu}{}^{(p+1)}$ is the Einstein tensor built from the induced metric $g_{\mu\nu}$. For Rindler space, $G^{\mu}{}_{\nu}{}^{(p+1)}$ vanishes since the surface is flat[3].

[3]This will not be the case when one adds metric perturbations. But our primary concern is thermodynamic quantities and transport coefficients. The transport coefficients of first-order hydrodynamics appear

We include the boundary cosmological constant term in order to compare with the SAdS_5 result (Sec. 3.3).

For Rindler space, $K^\mu{}_\nu = \text{diag}(1/(2r_c^{1/2}), \mathbf{0})$. Then, one gets

$$\mathcal{T}^t{}_t = -\frac{c_1}{16\pi G} \ , \tag{3.2.6}$$

$$\mathcal{T}^i{}_j = \frac{1}{16\pi G}\left(\frac{1}{\sqrt{r_c}} - c_1\right)\delta^i{}_j \ , \tag{3.2.7}$$

which gives

$$\bar{\varepsilon} = \frac{c_1}{16\pi G} \ , \tag{3.2.8a}$$

$$\bar{P} = \frac{1}{16\pi G}\left(\frac{1}{\sqrt{r_c}} - c_1\right) = \frac{\tilde{T}}{4G} - \frac{c_1}{16\pi G} \ . \tag{3.2.8b}$$

Here, $\tilde{T}$ is the proper temperature not the Hawking temperature T:

$$\tilde{T}(r_c) = \frac{1}{\sqrt{-\bar{g}_{tt}(r_c)}} T \ . \tag{3.2.9}$$

When $c_1 = 0$, $\bar{P}$ agrees with the membrane paradigm result [1, 2]. The thermodynamic relation $\tilde{T}\bar{s} = \bar{\varepsilon} + \bar{P}$ gives the entropy density $\bar{s} = 1/(4G)$. Since the energy density is constant, the stress tensor describes an incompressible fluid, and the speed of sound $c_s^2 = \partial\bar{P}/\partial\bar{\varepsilon}$ diverges.

3.2.2 Sound Mode Perturbations

We consider sound mode perturbations in Rindler space. We take the gauge where $h_{*r} = 0$ for all $_*$, and the metric is given by

$$ds_{p+2}^2 = -r(1 + h^t{}_t)dt^2 + \sum_i (1 + h^i{}_i)dx_i^2 + 2h^z{}_t dtdz + \frac{dr^2}{r} \ . \tag{3.2.10}$$

We consider the perturbations of the form

$$h^\mu{}_\nu(t, z, r) = h^\mu{}_\nu(r)\, e^{-i\omega t + iqz} \ . \tag{3.2.11}$$

In the gauge $h_{*r} = 0$, the extrinsic curvature takes the simple form (3.2.3). The response

only in $O(\omega)$ terms in the stress tensor, while $G_{\mu\nu}^{(p+1)}$ gives $O(\omega^2, q^2)$ terms, so we can safely ignore the Einstein tensor.

of the Brown-York tensor is given by ($' = \partial_r$)

$$\delta \mathcal{T}^t{}_t = (\bar{\varepsilon} + \bar{P}) r_c h_s{}' \ , \tag{3.2.12a}$$

$$\delta \mathcal{T}^z{}_t = (\bar{\varepsilon} + \bar{P}) (h^z{}_t - r_c h^z{}_t{}') \ , \tag{3.2.12b}$$

$$\delta \mathcal{T}^i{}_j = (\bar{\varepsilon} + \bar{P}) r_c \left[-h^i{}_j{}' + \delta^i{}_j (h^t{}_t{}' + h_s{}') \right] \ , \tag{3.2.12c}$$

where we used $\bar{\varepsilon} + \bar{P} = 1/(16\pi G r_c^{1/2})$. In order to compare the Brown-York tensor with the hydrodynamic tensor, one needs to rewrite $h^\mu{}_\nu{}'$. This requires the Einstein equation.

We first consider homogeneous perturbations $q = 0$. The Einstein equation with $q = 0$ gives

$$h^i{}_i{}'' + \frac{1}{r} h^i{}_i{}' + \frac{\omega^2}{r^2} h^i{}_i = 0 \ . \tag{3.2.13a}$$

$$(r^{3/2} h^t{}_t{}')' = 0 \ , \tag{3.2.13b}$$

$$-2\omega h^z{}_t{}' = 0 \ , \tag{3.2.13c}$$

$$r h_s{}' + O(\omega^2) = 0 \ . \tag{3.2.13d}$$

Equations (3.2.13c) and (3.2.13d) are two of the "constraint equations" which are first-order differential equations[4]. The solution of Eq. (3.2.13a) is given by

$$h^i{}_i(r) = h^i{}_i(r_c) \left(\frac{r}{r_c} \right)^{-i\omega} \ , \tag{3.2.14}$$

where we imposed the "incoming wave" boundary condition at the horizon. The remaining integration constant is fixed by the Dirichlet boundary condition $h^i{}_i(r_c)$. Equation (3.2.14) is the exact solution for all r. For $h^t{}_t$, imposing the regularity condition at the horizon, one gets $h^t{}_t = h^t{}_t(r_c)$.

From Eq. (3.2.13d), we obtain $\delta \mathcal{T}^t{}_t = 0$, so the Brown-York tensor describes an incompressible fluid. From Eq. (3.2.13d) and $h^t{}_t{}' = 0$, the terms proportional to $\delta^i{}_j$ vanish, which implies an incompressible fluid as well. Finally, from Eq. (3.2.13c), a non hydrodynamic term in $\delta \mathcal{T}^z{}_t$ [the second term of Eq. (3.2.12b)] vanishes.

[4] In this thesis, we use the word "constraint equations" in the sense of the radial foliation, not the time foliation.

Thus, the Brown-York tensor becomes

$$\delta \mathcal{T}^t{}_t = 0 \ , \tag{3.2.15a}$$

$$\delta \mathcal{T}^z{}_t = (\bar{\varepsilon} + \bar{P}) h^z{}_t \ , \tag{3.2.15b}$$

$$\delta \mathcal{T}^i{}_j = \frac{i\tilde{\omega}}{16\pi G} h^i{}_j \ , \tag{3.2.15c}$$

where we used Eq. (3.2.14) and $\tilde{\omega}$ is the proper frequency. In order to compare the Brown-York tensor with the Minkowski hydrodynamic stress tensor (3.1.13), one needs to rewrite the Brown-York tensor in proper coordinates $\tilde{t} = \sqrt{-\bar{g}_{tt}}\, t$ and $x^{\tilde{i}} = \sqrt{\bar{g}_{zz}}\, x^i$. In this thesis, " ˜ " denotes proper coordinates and proper quantities. Proper frequencies and wave number are given by

$$\tilde{\omega} = \frac{\omega}{\sqrt{-\bar{g}_{tt}}} = \frac{\omega}{r_c^{1/2}} \ , \quad \tilde{q} = \frac{q}{\sqrt{\bar{g}_{zz}}} = q \ . \tag{3.2.16}$$

However, as discussed previously, it is not necessary to distinguish x^μ and $x^{\tilde{\mu}}$ for $\delta \mathcal{T}^t{}_t$ and $\delta \mathcal{T}^i{}_j$ except the replacement (3.2.16). For the off-diagonal component, $\delta \mathcal{T}^z{}_t \propto h^z{}_t$, so the expression does not change under the coordinate transformation.

Equations (3.2.15) take the same form as the hydrodynamic stress tensor in the $c_s \to \infty$ limit (3.1.23) with

$$\eta = \frac{1}{16\pi G} \ . \tag{3.2.17}$$

This agrees with the membrane paradigm result and the BKLS result [1, 2, 27, 28]. On the other hand, the result of an incompressible fluid differs from the membrane paradigm.

From Eqs. (3.2.12), the term $h_s{}'$ gives $\delta\varepsilon$ and the bulk viscosity, but $h_s{}' = 0$ up to first order in (ω, q), so one immediately has an incompressible fluid. This is true even for $q \neq 0$ [Eq. (3.2.21c)], thus one expects that the fluid remains incompressible even for $q \neq 0$. However, it is not obvious that the Brown-York tensor takes the same form as the hydrodynamic tensor when $q \neq 0$. Thus, we turn to the $q \neq 0$ case in Sec. 3.2.4.

3.2.3 Possible Connection with the Membrane Paradigm?

Our result shows that the Brown-York tensor gives an incompressible fluid, which differs from the membrane paradigm. However, there is an interesting "coincidence" with the membrane paradigm if one ignores part of the Einstein equation.

Let us ignore the constraint equation (3.2.13d) for a moment, which gives the incompressible condition. In hydrodynamic analysis, the incompressible condition comes from the continuity equation (Sec. 3.1), so ignoring the constraint equation (3.2.13d) corresponds to ignore the continuity equation. From Eq. (3.2.14), $h_s{}' = -i\omega h_s/r$. Substituting this into Eq. (3.2.12c) gives

$$\delta\mathcal{T}^i{}_j = \frac{i\tilde{\omega}}{16\pi G}\left[h^i{}_j - \delta^i{}_j h_s\right] \ . \tag{3.2.18}$$

If one compares this with the hydrodynamic stress tensor (3.1.13), one would get

$$\zeta = -\frac{p-1}{p}\frac{1}{8\pi G} \ , \tag{3.2.19}$$

which coincides with the membrane paradigm [1, 2, 84]. The original membrane paradigm focuses on the $(3+1)$-dimensional case, but the extension into the generic dimensions exists [84]. Note that $\zeta < 0$.

This is an interesting coincidence, and the result may have some relevance with the membrane paradigm. On the other hand, we should stress that this result itself does not give a consistent hydrodynamic interpretation completely. For example, Eq. (3.2.18) seems to lack δP term in Eq. (3.1.13c). Also, $\delta\mathcal{T}^t{}_t$ is nonvanishing, but

$$\delta\mathcal{T}^t{}_t = -(\bar{\varepsilon} + \bar{P})i\omega h_s \ . \tag{3.2.20}$$

With Eqs. (3.1.13), this is consistent only if $i\omega = -1/2$. But this brings us another issue. First, we consider the hydrodynamic limit $|\omega| \to 0$, so it is not clear if such an interpretation is possible. Second, when $|\omega|$ is not small, it is not clear if $O(h_s)$ term in Eq. (3.2.18) is really the viscosity term: the first term and the third term of Eq. (3.1.13c) are not distinguishable.

Thus, the only consistent interpretation is the incompressible fluid by taking Eq. (3.2.13d) into account. But the coincidence (3.2.19) is suggestive. This might indicate that the membrane paradigm is not fully consistent.

3.2.4 Inhomogeneous Perturbations

The Einstein equation consists of second-order differential equations which are dynamical equations and first-order differential equations which are constraint equations. The dynamics of the field obeying the constraints is determined by one dynamical equation. They are referred as the master field and the master equation, respectively. This counting goes as follows:

- For the sound mode in 5-dimensional spacetime, 4 components of metric perturbations are relevant.

- The Einstein equation gives 4 dynamical equations and 3 constraint equations. Thus, one obtains 1 master equation, which gives the solution for a combination of 4 metric components.

- The solution of the master equation has two integration constants. One is fixed by imposing the incoming wave boundary condition at the horizon. Thus, one obtains the solution for a combination of 4 metric components with one integration constant.
- The other 3 components are calculated using 3 constraint equations which give one integration constant for each component.
- In summary, we obtain 4 solutions with 4 integration constants. These integration constants are fixed by imposing boundary conditions at the boundary for each components, $h^\mu{}_\nu(r_c)$.

In reality, in order to compute the Brown-York tensor, one only needs $h^\mu{}_\nu{}'(r_c)$. They can be determined from the constraint equations and the master equation. So, one does not have to solve the constraint equations.

In Rindler space, the constraint equations are given by[5]

$$\omega\left(2rh_s{}' - h_s\right) + 2q\left(rh^z{}_t{}' - h^z{}_t\right) = 0 \ , \tag{3.2.21a}$$

$$-2\omega h^z{}_t{}' + q\left(2rh^t{}_t{}' + h^t{}_t + 4rh^x{}_x{}'\right) = 0 \ , \tag{3.2.21b}$$

$$rh_s{}' + O(\omega^2, \omega q, q^2) = 0 \ . \tag{3.2.21c}$$

The master field is $h^x{}_x$ which obeys

$$h^x{}_x{}'' + \frac{1}{r}h^x{}_x{}' + \frac{\omega^2 - rq^2}{r^2}h^x{}_x = 0 \ . \tag{3.2.22}$$

We solve the master equation by imposing (i) the incoming wave boundary condition at the horizon and (ii) the Dirichlet boundary condition at $r = r_c$, $h^x{}_x(r_c)$. After imposing the former boundary condition, the solution takes the form

$$h^x{}_x(r) = \left.\frac{h^x{}_x}{F}\right|_{r_c} F(r) \ , \tag{3.2.23}$$

where we fixed the remaining overall integration constant by the Dirichlet boundary condition $h^x{}_x(r_c)$. The solution of the master equation in the near-horizon limit $r \to 0$ is given

[5] These equations correspond to (t, r), (z, r), and (r, r)-components of the Einstein equation, respectively.

by Eq. (3.2.14). When $q = 0$, it is the exact solution for all r. This is not the case when $q \neq 0$, but the master equation takes the form

$$h^x{}_x{}'' + \frac{1}{r} h^x{}_x{}' + O(\omega^2, q^2) = 0 \ . \tag{3.2.24}$$

Thus, Eq. (3.2.14) still gives the solution to first order in (ω, q):

$$F(r) = 1 - i\omega \log r + O(\omega^2, q^2) \ . \tag{3.2.25}$$

For the sound mode, $h^a{}_a = h^x{}_x$, but it is convenient to keep each components separately. Also, we focus on the five-dimensional case ($p = 3$) for simplicity. Substitute Eqs. (3.2.21) and (3.2.23) into Eqs. (3.2.12). To first order in (ω, q), the Brown-York tensor becomes

$$\delta \mathcal{T}^t{}_t = 0 \ , \tag{3.2.26a}$$

$$\delta \mathcal{T}^{\tilde{z}}{}_{\tilde{t}} = \frac{1}{r_c^{1/2}} \delta \mathcal{T}^z{}_t = -(\bar{\varepsilon} + \bar{P}) \frac{\tilde{\omega}}{2\tilde{q}} h_s \ , \tag{3.2.26b}$$

$$\delta \mathcal{T}^x{}_x = \delta P(h) + \frac{i\tilde{\omega}}{16\pi G} h^x{}_x \ , \tag{3.2.26c}$$

$$\delta \mathcal{T}^y{}_y = \delta P(h) + \frac{i\tilde{\omega}}{16\pi G} h^y{}_y \ , \tag{3.2.26d}$$

$$\delta \mathcal{T}^z{}_z = \delta P(h) - \frac{i\tilde{\omega}}{16\pi G} \left(h^x{}_x + h^y{}_y \right) \ , \tag{3.2.26e}$$

where

$$\delta P = (\bar{\varepsilon} + \bar{P}) \left[-\frac{1}{2} h^t{}_t + \frac{\omega}{q r_c} h^z{}_t + \frac{\omega^2}{2q^2 r_c} h_s \right] + \frac{i\omega}{16\pi G r_c^{1/2}} \left(h^x{}_x + h^y{}_y \right) \ , \tag{3.2.27}$$

$$= (\bar{\varepsilon} + \bar{P}) \left[-\frac{1}{2} h^t{}_t + \frac{\tilde{\omega}}{\tilde{q}} h^{\tilde{z}}{}_{\tilde{t}} + \frac{\tilde{\omega}^2}{2\tilde{q}^2} h_s \right] + \frac{i\tilde{\omega}}{16\pi G} \left(h^x{}_x + h^y{}_y \right) \ . \tag{3.2.28}$$

Again, the Brown-York tensor in proper coordinates $x^{\tilde{\mu}}$ takes the same form except $\delta \mathcal{T}^z{}_t$. So, we have rewritten $\delta \mathcal{T}^z{}_t$ (and δP) in proper coordinates. The Brown-York tensor takes the same form as the hydrodynamic stress tensor in the $c_s \to \infty$ limit (3.1.23) with $\eta = 1/(16\pi G)$.

3.3 Sound Mode in Schwarzschild-AdS Black Hole

In this section, we study the bulk viscosity for the $SAdS_5$ black hole. In the AdS/CFT duality, the bulk viscosity for the $SAdS_5$ black hole vanishes in the limit $r_c \to \infty$ because of the scale invariance of the geometry. However, when $r_c \neq \infty$, the stress tensor for the $SAdS_5$ black hole is no longer traceless (3.3.2), so one must examine the bulk viscosity in this case.

The near-horizon limit of the $SAdS_5$ black hole is Rindler space. So, one expects that the bulk viscosity for the $SAdS_5$ black hole agrees with the Rindler result in the limit $r_c \to r_0$. We find that the bulk viscosity vanishes even for arbitrary r_c. One might wonder how this result is compatible with the Rindler result. The answer is that the hydrodynamic regime used in the SAdS computation differs from the hydrodynamic regime used in the Rindler computation (when expressed in terms of the SAdS variables). We also compute a second-order hydrodynamic coefficient τ_π for arbitrary r_c.

3.3.1 Thermodynamic Quantities

The $SAdS_5$ metric is given by

$$ds_5^2 = \left(\frac{r}{L}\right)^2 [-f(r)dt^2 + dx_i^2] + \frac{dr^2}{\left(\frac{r}{L}\right)^2 f(r)} \ , \tag{3.3.1a}$$

$$= \frac{1}{u}\left[-f(u)dt^2 + dx_i^2\right] + \frac{du^2}{4u^2 f(u)} \ , \tag{3.3.1b}$$

$$f(r) = 1 - \left(\frac{r_0}{r}\right)^4 \ , \quad f(u) = 1 - u^2 \ , \tag{3.3.1c}$$

where $u = (r_0/r)^2$. The Hawking temperature is given by $T = r_0/(\pi L^2)$. We take the horizon radius $r_0 = 1$ by rescaling t and x_i, and we set the AdS radius $L = 1$. The boundary position will be denoted as $u = u_c$.

The Brown-York tensor and thermodynamic relations give the following thermodynamic quantities:

$$\tilde{T} = \frac{1}{\pi}\sqrt{\frac{u_c}{1-u_c^2}} \ , \tag{3.3.2a}$$

$$\bar{\varepsilon} = \frac{3}{8\pi G}\left(\frac{c_1}{6} - \sqrt{1-u_c^2}\right) \ , \tag{3.3.2b}$$

$$\bar{P} = \frac{1}{8\pi G}\left(\frac{3-u_c^2}{\sqrt{1-u_c^2}} - \frac{c_1}{2}\right) \ , \tag{3.3.2c}$$

$$\bar{s} = \frac{\bar{\varepsilon}+\bar{P}}{\tilde{T}} = \frac{u_c^{3/2}}{4G} \ , \tag{3.3.2d}$$

$$c_s^2 = \frac{\partial \bar{P}}{\partial \bar{\varepsilon}} = \frac{1+u_c^2}{3(1-u_c^2)} \ , \tag{3.3.2e}$$

where c_1 is the counterterm dependence (3.2.5) ($c_1 = 6$ for asymptotically AdS_5 spacetime). In the above expressions, one can eliminate u_c by proper temperature $\tilde{T}$, but the result is not very illuminating.

Note that the stress tensor is no longer traceless. Also, one always has $\bar{\varepsilon} < 0$ for $c_1 = 0$, which may be troublesome, but $\bar{\varepsilon} > 0$ for $c_1 = 6$. On the other hand, $\bar{P} > 0$ for both values of c_1.

The computation of thermodynamic quantities has some differences in the AdS/CFT duality. The Brown-York tensor is the stress tensor with respect to the intrinsic metric on the surface, and it is natural to use the proper temperature. In the AdS/CFT duality, one identifies the gauge theory metric $\gamma_{\mu\nu}^{(\mathrm{FT})}$ as $g_{\mu\nu} = (r/L)^2\gamma_{\mu\nu}^{(\mathrm{FT})}$. As a result, it is natural to use the Hawking temperature in the AdS/CFT duality. The field theory stress tensor is defined with respect to $\gamma_{\mu\nu}^{(\mathrm{FT})}$[6]. Then, the AdS/CFT stress tensor $\mathcal{T}_{\mu\nu}^{(\mathrm{GKPW})}$ is related to the Brown-York tensor as

$$\mathcal{T}_{\mu\nu}^{(\mathrm{GKPW})} = -\frac{2}{\sqrt{-\gamma^{(\mathrm{FT})}}}\frac{\delta S}{\delta \gamma^{(\mathrm{FT})\mu\nu}} \sim \left(\frac{r}{L}\right)^2 \mathcal{T}_{\mu\nu}^{(\mathrm{BY})} \tag{3.3.3}$$

for $p = 3$. However, physical quantities from the Brown-York tensor in terms of the proper temperature takes the same from as the standard AdS/CFT expressions in the limit $r_c \to \infty$ (see below).

Consider two interesting limits, the $u_c \to 0$ limit and the $u_c \to 1$ limit. They correspond to the low-$\tilde{T}$ limit and the high-$\tilde{T}$ limit, respectively.

1. In the AdS/CFT limit ($u_c \to 0$), thermodynamic quantities take the same form as the

[6] The induced metric $\gamma_{\mu\nu}$ in chapter 2 is not the same as this $\gamma_{\mu\nu}^{(\mathrm{FT})}$ but rather is the same as the induced metric $g_{\mu\nu}$ in this chapter.

standard AdS/CFT result:

$$\bar{\varepsilon} = \frac{3}{16\pi G}(\pi\tilde{T})^4 \ , \tag{3.3.4a}$$
$$\bar{P} = \frac{1}{16\pi G}(\pi\tilde{T})^4 \ , \tag{3.3.4b}$$
$$\bar{s} = \frac{1}{4G}(\pi\tilde{T})^3 \ , \tag{3.3.4c}$$
$$c_s^2 = \frac{1}{3} \ . \tag{3.3.4d}$$

2. In the Rindler limit ($u_c \to 1$), they reduce to Eqs. (3.2.8):

$$\bar{\varepsilon} = \frac{c_1}{16\pi G} \ , \tag{3.3.5a}$$
$$\bar{P} = \frac{\tilde{T}}{4G} - \frac{c_1}{16\pi G} \ , \tag{3.3.5b}$$
$$\bar{s} = \frac{1}{4G} \ , \tag{3.3.5c}$$
$$c_s^2 \to \infty \ . \tag{3.3.5d}$$

3.3.2 Sound Mode Perturbations

We consider sound mode perturbations in the SAdS$_5$ black hole. Again we take the gauge $h_{*u} = 0$, and the metric is given by

$$ds_5^2 = \frac{1}{u}\left[-f(1+h^t{}_t)dt^2 + \sum_i (1+h^i{}_i)dx_i^2 + 2h^z{}_t dtdz\right] + \frac{du^2}{4u^2 f} \ . \tag{3.3.6}$$

Like the Rindler analysis, one can obtain the master equation for the master field after some algebra. The definition of the master field is not unique, but different definitions are related to each other describing the same physics. We take the following combination for the master field:

$$\Phi(u) = h^x{}_x + f\frac{4h^x{}_x{}' + 2h^z{}_z{}'}{4q^2 - 3f'} \ , \tag{3.3.7}$$

which obeys the following master equation:

$$\left[u^{-1}f(u)\Phi(u)'\right]' + V(u)\Phi(u) = 0 \ , \tag{3.3.8}$$

where

$$V(u) = \frac{1}{u^3 f \left(4\mathfrak{q}^2 - 3f'\right)^2} \times \left[u\mathfrak{w}^2 \left(4\mathfrak{q}^2 - 3f'\right)^2 + \mathfrak{q}^2 f \left(15uf'^2 - 36ff'\right) + \mathfrak{q}^4 f \left(16f - 8uf'\right) - 16\mathfrak{q}^6 uf\right] , \tag{3.3.9}$$

$\mathfrak{w} := \omega/(2\pi T) = \omega/2$, and $\mathfrak{q} := q/(2\pi T) = q/2$.

We again solve the master equation by imposing (i) the incoming wave boundary condition at the horizon $u = 1$ and (ii) the Dirichlet boundary condition at $u = u_c$, $h^\mu{}_\nu(u_c)$. After imposing the former boundary condition, the solution takes the form

$$\Phi(u) = CF(u) . \tag{3.3.10}$$

The remaining integration constant C is fixed by the boundary condition $h^\mu{}_\nu(u_c)$. We expand the solution in $\mathfrak{w}$ and $\mathfrak{q}$:

$$F(u) = (1-u)^{-i\mathfrak{w}/2} \left[F_{00}(u) + (\mathfrak{w}F_{10}(u) + \mathfrak{q}F_{01}(u)) + \left(\mathfrak{w}^2 F_{20}(u) + \mathfrak{w}\mathfrak{q}F_{11}(u) + \mathfrak{q}^2 F_{02}(u)\right) + \cdots\right] . \tag{3.3.11}$$

Here, we factorized $(1-u)^{-i\mathfrak{w}/2}$ to implement the incoming wave boundary condition at the horizon. Then, the incoming wave boundary condition becomes the regularity condition for $F_{ij}(u)$ at the horizon. One can easily check $F_{00} = 1$. The master equation has no terms with odd powers in q, so one can set $F_{01} = F_{11} = 0$ without loss of generality. The solutions are

$$F_{10} = -\frac{i}{2} \ln(1+u) , \tag{3.3.12a}$$

$$F_{02} = -\frac{2}{3u} + \frac{1}{3} \ln(1+u) , \tag{3.3.12b}$$

$$F_{20} = \frac{1}{2}\mathrm{Li}_2\left(\frac{u+1}{2}\right) - \frac{1}{2}\{\ln 2 - \ln(1+u)\} \ln(1-u) + \ln(1+u) \left\{\frac{1}{8} \ln(1+u) + 1 - \ln 2\right\} . \tag{3.3.12c}$$

The integration constant C is fixed by the boundary condition $h^\mu{}_\nu(u_c)$. Using the definition of the master field (3.3.7) and the Einstein equation, we obtain $C = C_{\mathrm{num}}/C_{\mathrm{den}}$. [See Eqs. (B.2.1) for the detailed form of C_{num} and C_{den}.]

The response of the Brown-York tensor is given by

$$\delta \mathcal{T}^t{}_t = -\frac{\bar{\varepsilon}+\bar{P}}{2}\frac{f}{u_c}h_s' \ , \tag{3.3.13a}$$

$$\delta \mathcal{T}^z{}_t = (\bar{\varepsilon}+\bar{P})\left(h^z{}_t + \frac{f}{2u_c}h^z{}_t{}'\right) \ , \tag{3.3.13b}$$

$$\delta \mathcal{T}^x{}_x = -\frac{\bar{\varepsilon}+\bar{P}}{2}\frac{f}{u_c}(-h^x{}_x{}' + h^t{}_t{}' + h_s') \ , \tag{3.3.13c}$$

$$\delta \mathcal{T}^z{}_z = -\frac{\bar{\varepsilon}+\bar{P}}{2}\frac{f}{u_c}(-h^z{}_z{}' + h^t{}_t{}' + h_s') \ , \tag{3.3.13d}$$

where we used $\bar{\varepsilon}+\bar{P} = u_c^2/(4\pi G f^{1/2})$. In order to compare the Brown-York tensor with the hydrodynamic tensor, one needs to rewrite $h^\mu{}_\nu{}'$. Using Eq. (3.3.7) together with three constraint equations, one can write $h^\mu{}_\nu{}'$ in terms of $h^\mu{}_\nu$ and Φ, schematically in the form of

$$h^\mu{}_\nu{}' = A^\mu{}_{\nu\alpha}{}^\beta h^\alpha{}_\beta + B^\mu{}_\nu \Phi \ . \tag{3.3.14}$$

[See Eqs. (B.2.2) for the detailed form of A and B.]

3.3.3 Homogeneous Perturbations

One can carry out the same analysis as the $q \neq 0$ Rindler case. However, the full form of the Brown-York tensor is rather complicated, so we focus on the following two cases. First, we consider the $q \to 0$ limit and compare the Brown-York tensor with the hydrodynamic stress tensor (3.1.13). Second, we consider the $q \neq 0$ case and extract the sound pole.

We first consider homogeneous perturbations. Take the $\mathfrak{q} \to 0$ limit in Eqs. (B.2.2) and then expand it in $\mathfrak{w}$. One obtains

$$h^t{}_t{}' = \frac{u\,(3-u^2)}{3f^2}h_s + \mathcal{O}\left(\mathfrak{w}^2\right) \ , \tag{3.3.15a}$$

$$h^z{}_t{}' = 0 \ , \tag{3.3.15b}$$

$$h^x{}_x{}' = -\frac{u}{3f}h_s + \frac{i\mathfrak{w}u(h^x{}_x - h^z{}_z)}{3f} + \mathcal{O}\left(\mathfrak{w}^2\right) \ , \tag{3.3.15c}$$

$$h^z{}_z{}' = -\frac{u}{3f}h_s - \frac{2i\mathfrak{w}u(h^x{}_x - h^z{}_z)}{3f} + \mathcal{O}\left(\mathfrak{w}^2\right) \ . \tag{3.3.15d}$$

Substituting them into the Brown-York tensor (3.3.13), one obtains

$$\delta\mathcal{T}^{t}{}_{t} = \frac{\bar{\varepsilon}+\bar{P}}{2} h_s \ , \tag{3.3.16a}$$

$$\delta\mathcal{T}^{z}{}_{t} = (\bar{\varepsilon}+\bar{P}) h^{z}{}_{t} \ , \tag{3.3.16b}$$

$$\delta\mathcal{T}^{x}{}_{x} = -\frac{\bar{\varepsilon}+\bar{P}}{2} c_s^2 h_s + \frac{i\tilde{\omega} u_c^{3/2}}{16\pi G}\left(h^{x}{}_{x} - \frac{1}{3} h_s \right) \ , \tag{3.3.16c}$$

$$\delta\mathcal{T}^{z}{}_{z} = -\frac{\bar{\varepsilon}+\bar{P}}{2} c_s^2 h_s + \frac{i\tilde{\omega} u_c^{3/2}}{16\pi G}\left(h^{z}{}_{z} - \frac{1}{3} h_s \right) \ . \tag{3.3.16d}$$

Again it is not necessary to distinguish x^μ and $x^{\tilde{\mu}}$ for $\delta\mathcal{T}^{t}{}_{t}$ and $\delta\mathcal{T}^{i}{}_{j}$. Since $\delta\mathcal{T}^{z}{}_{t} \propto h^{z}{}_{t}$, $\delta\mathcal{T}^{z}{}_{t}$ takes the same form in proper coordinates. Equations (3.3.16) take the same form as the hydrodynamic stress tensor (3.1.13) with

$$\eta = \frac{u_c^{3/2}}{16\pi G} \ , \qquad \zeta = 0 \ , \tag{3.3.17}$$

which satisfies $\eta/s = 1/(4\pi)$.

3.3.4 Inhomogeneous Perturbations and Sound Pole

We now consider the $q \neq 0$ case and the sound pole in the Brown-York tensor. The coefficients of $h^{\mu}{}_{\nu}$ and $h^{\mu}{}_{\nu}{}'$ of the Brown-York tensor do not have non-trivial singularities. Thus, the pole can appear in $h^{\mu}{}_{\nu}{}'$. But $h^{\mu}{}_{\nu}{}'$ can be written by Eq. (3.3.14), so the pole can appear in the integration constant C in Φ. Namely, the pole is given by $C_{\text{den}} = 0$. From Eqs. (B.2.1), the hydrodynamic pole is located at

$$\mathfrak{w} = d_1 \mathfrak{q} + d_2 \mathfrak{q}^2 + d_3 \mathfrak{q}^3 + \cdots \ , \tag{3.3.18}$$

with

$$d_1 = \sqrt{\frac{1+u_c^2}{3}} \ , \tag{3.3.19a}$$

$$d_2 = -i\frac{1}{3}(1-u_c^2) \ , \tag{3.3.19b}$$

$$d_3 = (1-u_c^2)\frac{(1+u_c^2)[3 - 2\ln 2 + 2\ln(1+u_c)] - 2u_c}{6\sqrt{3(1+u_c^2)}} \ . \tag{3.3.19c}$$

In terms of proper quantities,

$$\tilde{\omega} = \frac{d_1}{f^{1/2}}\tilde{q} + \frac{d_2}{f}\frac{\tilde{q}^2}{2\pi\tilde{T}} + \frac{d_3}{f^{3/2}}\frac{\tilde{q}^3}{(2\pi\tilde{T})^2} + \cdots \ . \tag{3.3.20}$$

We compare this pole with the dispersion relation of hydrodynamic sound mode (Sec. B.1):

$$\begin{aligned}\omega &= c_s q - i\left(\frac{p-1}{p}\hat{\eta} + \frac{1}{2}\hat{\zeta}\right) q^2 \\ &+ \frac{1}{2c_s}\left[\frac{p-1}{p}\hat{\eta}\left(2c_s^2\tau_\pi - \frac{p-1}{p}\hat{\eta}\right) + \zeta\left(c_s^2\tau_\Pi - \frac{p-1}{p}\hat{\eta} - \frac{1}{4}\hat{\zeta}\right)\right] q^3 + \cdots \ .\end{aligned} \tag{3.3.21}$$

The $O(q^3)$ terms are the modification by the second-order hydrodynamics. The coefficients τ_π and τ_Π are two coefficients appeared in the second-order hydrodynamics. The coefficient τ_π gives the relaxation time of the shear stress.

Comparing Eq. (3.3.21) and Eq. (3.3.20) and using $\eta/s = 1/(4\pi)$, we obtain[7]

$$c_s^2 = \frac{1+u_c^2}{3(1-u_c^2)} \ , \tag{3.3.22a}$$

$$\zeta = 0 \ , \tag{3.3.22b}$$

$$\tau_\pi = \frac{(1+u_c)[1-\ln 2 + \ln(1+u_c)] + 1 - u_c}{2\pi\tilde{T}(1+u_c)^2} \ . \tag{3.3.22c}$$

The speed of sound c_s agrees with the thermodynamic result (3.3.2). The second-order coefficient τ_π behaves as follows:

$$\tau_\pi = \frac{2-\ln 2}{2\pi\tilde{T}} \quad (u_c \to 0) \ , \tag{3.3.23}$$

$$= \frac{1}{2\pi\tilde{T}} \quad (u_c \to 1) \ . \tag{3.3.24}$$

The $u_c \to 0$ limit takes the same form as the standard AdS/CFT result.

3.4 Relation between Rindler and SAdS Results

In Sec. 3.2, the Rindler result gives an incompressible fluid. On the other hand, the SAdS result gives $\zeta = 0$ even in the near-horizon limit $u_c \to 1$. An incompressible fluid is different from a fluid with $\zeta = 0$. To answer to the question, let us study the relation

[7] In second-order hydrodynamics, τ_Π is defined as $\tau_\Pi \propto \zeta$, so τ_π vanishes automatically.

between the SAdS black hole and Rindler space.

The Rindler limit of the $SAdS_5$ black hole is given by

$$u = 1 - 8\epsilon^2 r \ , \qquad t = \frac{1}{4}\, t^{\rm NH} \ , \qquad x_i = \epsilon\, x_i^{\rm NH} \ , \qquad \text{with } \epsilon \to 0 \ . \tag{3.4.1}$$

Note that x_i is ϵ-rescaled, but t is not. The coefficient of 1/4 in the definition of $t^{\rm NH}$ is necessary to match the $SAdS_5$ Hawking temperature $T_{\rm SAdS} = 1/\pi$ with the Rindler temperature $T_{\rm Rindler} = 1/(4\pi)$. Under the rescaling, the $SAdS_5$ metric becomes the Rindler metric up to an overall rescaling:

$$ds^2_{\rm SAdS} = \epsilon^2 ds^2_{\rm Rindler} \ . \tag{3.4.2}$$

Consider the perturbations under the rescaling. The momentum is rescaled as

$$\omega = 4\omega_{\rm NH} \ , \quad q = \frac{1}{\epsilon} q_{\rm NH} \ . \tag{3.4.3}$$

Since x_i is rescaled but t is not, $h^z{}_t$ must be rescaled as

$$h^z{}_t = 4\epsilon\, h^z{}_t{}^{\rm NH} \ . \tag{3.4.4}$$

The other components are not ϵ-rescaled since the upper and lower indices receive the opposite rescaling. In the Rindler limit, the SAdS master equation (3.3.8) becomes

$$[r\Phi']' + \left(\frac{\omega_{\rm NH}^2}{r} - q_{\rm NH}^2 \right) \Phi = 0 \ , \tag{3.4.5}$$

which is identical to the Rindler master equation (3.2.22). The Dirichlet boundary condition for the master field is also identical to the Rindler case. Using Eqs. (B.2.1),

$$C = \frac{h^x{}_x}{F} + \mathcal{O}(\epsilon^2) \bigg|_{u_c} \ , \tag{3.4.6}$$

which reduces to Eq. (3.2.23). Thus, the SAdS master field is completely identical to the Rindler master field.

Then, why does the SAdS result differ from the Rindler result? In the sound mode computation in Sec. 3.3.4, we looked at the hydrodynamic regime $\omega \sim O(q)$. But this does not mean $\omega_{\rm NH} \sim O(q_{\rm NH})$ in the Rindler limit because of the scaling (3.4.3). In order to have the hydrodynamic regime $\omega_{\rm NH} \sim O(q_{\rm NH})$, one must look at $\omega \sim O(\epsilon q)$ in the original

SAdS variables. Namely, the hydrodynamic regime used in the SAdS computation differs from the hydrodynamic regime used in the Rindler computation (when expressed in terms of the SAdS variables). In fact, one can show that the full SAdS Brown-York tensor reduces to the Rindler Brown-York tensor (3.2.26) in the $\epsilon \to 0$ limit. *The full SAdS Brown-York tensor contains not only the SAdS $\zeta = 0$ hydrodynamics but also the Rindler incompressible hydrodynamics.*

For the $q = 0$ case, the reason is slightly different. If one takes the $q \to 0$ limit in the SAdS boundary condition (B.2.1),

$$C = \left. \frac{h^x{}_x - h^z{}_z}{3F} \right|_{u_c} , \tag{3.4.7}$$

which does not reduce to Eq. (3.4.6). Thus, in the boundary condition, the $q \to 0$ limit and the $\epsilon \to 0$ limit do not commute.

Conclusion

We have studies two approaches for the realization of the relation between gravity and hydrodynamics: the AdS/CFT duality and the BKLS approach. The results are summarized as follows.

We have studied the shear viscosities for s-wave, $(p+ip)$-wave, and p-wave holographic superfluids. The shear viscosity for the s-wave superfluids satisfies the universality of η/s. The p-wave superfluids are anisotropic, and there are two nontrivial independent shear viscosities. We have showed that one of the coefficients satisfies the universality.

On the other hand, for the other coefficient of the p-wave superfluids and for the $(p+ip)$-wave superfluid, the gravitational perturbations in question couple to the Yang-Mills perturbations even in Kubo-formula method, and the existing technique is not applicable. We have extracted the modes which couple to the gravitational perturbations. For the $(p+ip)$-wave case, we have written down the perturbed action for those modes. In fact, the non-universality of p-wave holographic superfluids has been showed numerically [78, 79]. So our proposal is the first example of the non-universal shear viscosity. In addition, inspired by our work, an application to a strongly coupled anisotropic plasma, which is a model for the pre-equilibrium stage of quark-gluon plasma in heavy-ion collisions, has been studied [86, 87, 88, 89].

We have studied yet another realization, the BKLS approach. Although one often starts to identify the velocity field of the fluid in the bulk spacetime in the BKLS approach, we have not taken such a path. We have written down the velocity field in terms of the metric perturbations, and compared with the Brown-York tensor.

In particular, we have studied the issue of the bulk viscosity ζ, which is non-negative in the AdS/CFT duality, negative in the membrane paradigm, and is irrelevant in the BKLS approach (because of an incompressible fluid). At first glance, it might seem inconsistent since the near horizon limit of the SAdS black hole includes Rindler space. The result is summarized schematically in Fig.3.1.

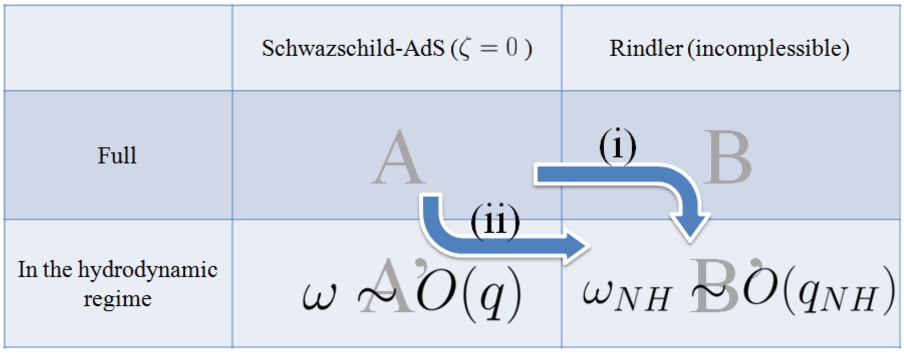

Fig. 3.1: The matrix of the geometries

- Path (i): The region A is the SAdS without the hydrodynamic limit, so one can take the near horizon limit (from A to B) and then the hydrodynamic limit (from B to B') in terms of ω_{NH} and q_{NH} consistently. That is to say, the path (i) is allowed.
- Path (ii): Once the hydrodynamic limit of SAdS (from A to A') is taken, the near horizon limit (from A' to B') can *not* be taken consistently. This is because the momentum diverges $q \to \infty$ (in SAdS) with the near horizon limit $\epsilon \to 0$. (See Eq. (3.4.3).) Therefore, there are no contradictions between two results since the hydrodynamic regime used for the SAdS black hole "differs" from the hydrodynamic regime used for Rindler space (when expressed in terms of the SAdS variables).

We have also found an interesting "coincidence" with the membrane paradigm in the Rindler analysis. If one does not take into account a constraint equation of the Einstein equation (in hydrodynamics, this corresponds not to take into account the continuity equation), one would get the negative bulk viscosity in accordance with the membrane paradigm. The precise relation to the membrane paradigm is not clear and is left to a future work.

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Appendix A

Quadratic forms of perturbations for Einstein-Matter actions

In this Appendix, we derive the quadratic forms of the tensor mode perturbation for the s-wave, p-wave and $(p + ip)$-wave holographic superfluids. First, we derive the quadratic form of the Einstein-Hilbert action. Then, we derive quadratic forms of the matter action for these three models. Since we will not consider scalar perturbations, we will focus on the metric perturbations and the gauge field perturbations.

A.1 The Quadratic Form of the Einstein-Hilbert Action

Consider the general perturbation h_{MN} to the background metric $\bar{g}_{MN}$:

$$g_{MN} = \bar{g}_{MN} + h_{MN} \ . \tag{A.1.1}$$

Under the perturbation,

$$\sqrt{-g} = \sqrt{-\bar{g}} \left[1 + \frac{1}{2} h + \frac{1}{2} \left(\frac{1}{4} h^2 - \frac{1}{2} h^{MN} h_{MN} \right) \right] \ , \tag{A.1.2}$$

and

$$\begin{aligned} {}^{(2)}R =& \nabla_M \left(h^{IJ} \nabla^M h_{IJ} + h^{MN} \nabla_N h - h^{MN} \nabla_I h^I{}_N - h^{IJ} \nabla_I h_J{}^M \right) \\ &- \frac{1}{4} (\nabla^N h^{IJ}) \nabla_N h_{IJ} + \frac{1}{2} (\nabla^N h^{IJ}) \nabla_I h_{JN} - \frac{1}{4} \nabla_N h \nabla^N h + \bar{R}_{MI} h^I{}_N h^{MN}. \end{aligned} \tag{A.1.3}$$

Therefore, the quadratic form of h_{MN} in the Einstein-Hilbert action is

$$\frac{{}^{(2)}(\sqrt{-g}R)}{\sqrt{-\bar{g}}} = \left(\frac{1}{8}h^2 - \frac{1}{4}h^{MN}h_{MN}\right)\bar{R} + \left(h^{MI}h_I{}^N - \frac{1}{2}hh^{MN}\right)\bar{R}_{MN}$$
$$- \frac{1}{4}(\nabla^M h^{IJ})(\nabla_M h_{IJ}) + \frac{1}{2}(\nabla^M h^{IJ})(\nabla_I h_{JM}) + \frac{1}{4}(\nabla_M h)(\nabla^M h) - \frac{1}{2}(\nabla_N h^{MN})(\nabla_M h)$$
$$+ \nabla_M \left(h^{IJ}\nabla^M h_{IJ} + h^{MN}\nabla_N h - h^{MN}\nabla_I h^I{}_N - h^{IJ}\nabla_I h_J{}^M - \frac{1}{2}h\nabla^M h + \frac{1}{2}h\nabla_N h^{MN}\right) . \tag{A.1.4}$$

If we restrict the perturbation h_{MN} to the tensor mode perturbation h_{12}, this quadratic form reduces to ($\phi := h^1{}_2$)

$${}^{(2)}(\sqrt{-g}R) = \sqrt{-\bar{g}}\left[{}^{(2)}R - \frac{1}{2}\bar{R}\phi^2\right] , \tag{A.1.5}$$

where ${}^{(2)}R$ is the quadratic form of the Ricci scalar with the tensor mode perturbation:

$${}^{(2)}R = \sqrt{-\bar{g}}\left[-\frac{1}{2}(\nabla_M\phi)^2 + \bar{R}^2{}_2\phi^2\right]$$
$$+ \partial_r\left(2\frac{\sqrt{-\bar{g}}}{g_{rr}}\phi\partial_r\phi + \frac{\sqrt{-\bar{g}}}{g_{rr}}\frac{g'_{x_2x_2}}{2g_{x_2x_2}}\phi^2\right) - \partial_t\left(2\frac{\sqrt{-\bar{g}}}{g_{tt}}\phi\partial_t\phi\right) . \tag{A.1.6}$$

Here, we take the $(d+1)$-dimensional p-wave metric (2.2.3) to preserve generality. Except the free scalar part, all these terms will be removed in the end.

A.2 The Quadratic Form of the s-wave Holographic Superfluid Action

The s-wave holographic superfluid is described by Eq. (2.1.1):

$$S_s = \frac{1}{16\pi G_{d+1}}\int d^{d+1}x(\sqrt{-g}R + \mathcal{L}_{s\text{-matter}}) ; \tag{A.2.1}$$

$$\mathcal{L}_{s\text{-matter}} := \sqrt{-g}\left[-\frac{1}{4}K_1(|\Psi|)F_{MN}F^{MN} - K_2(|\Psi|)|(D_M\Psi)|^2 - V(|\Psi|)\right] , \tag{A.2.2}$$

where we defined a covariant derivative as $D_M := \nabla_M - iqA_M$. Under the general gravitational perturbation

$$g_{MN} = \bar{g}_{MN} + h_{MN} , \tag{A.2.3}$$

one can easily find

$$ {}^{(2)}\mathcal{L}_{s\text{-matter}} = \frac{1}{2}\sqrt{-\bar{g}}h_{MN}\left[K_1X^{MNIJ} + K_2Y^{MNIJ} + VP^{MNIJ}\right]h_{IJ} \ , \tag{A.2.4} $$

where

$$ P^{MNIJ} = \frac{1}{4}\left(\bar{g}^{MI}\bar{g}^{NJ} + \bar{g}^{MJ}\bar{g}^{NI} - \bar{g}^{MN}\bar{g}^{IJ}\right) \ , \tag{A.2.5} $$

$$ X^{MNIJ} = \frac{1}{4}F_{AB}F^{AB}P^{MNIJ} + \frac{1}{2}F_A{}^MF^{AN}\bar{g}^{IJ} - F_A{}^MF^{AJ}\bar{g}^{NI} - \frac{1}{2}F^{MI}F^{NJ}, \tag{A.2.6} $$

$$ Y^{MNIJ} = |D_A\Psi|^2P^{MNIJ} + (D^M\Psi)(D^N\Psi)^*\bar{g}^{IJ} - 2(D^M\Psi)(D^J\Psi)^*\bar{g}^{NI} \ . \tag{A.2.7} $$

Note that some modes of the metric perturbations couple with the gauge field perturbations δA_I and the complex scalar perturbation $\delta\Psi$ in general. However, we drop these perturbations since these decouple from the tensor mode metric perturbations.

The equation of motion for the background field is

$$ \left(\frac{1}{2}\bar{R}\bar{g}^{MN} - \bar{R}^{MN}\right) + \bar{T}^{MN} = 0 \ . \tag{A.2.8} $$

The background energy-momentum tensor $\bar{T}_{MN}$ is defined as [1]

$$ \begin{aligned} \bar{T}^{MN} &= \frac{1}{\sqrt{-\bar{g}}}\frac{\partial\mathcal{L}_{s\text{-matter}}}{\partial g_{MN}} \\ &= \frac{1}{2}K_1\left(-\frac{1}{4}\bar{g}^{MN}F_{AB}F^{AB} + F^{(M}{}_AF^{N)A}\right) \\ &\quad + K_2\left(-\frac{1}{2}\bar{g}^{MN}(D_A\Psi)(D^A\Psi)^* + (D^{(M}\Psi)(D^{N)}\Psi)^*\right) - \frac{1}{2}\bar{g}^{MN}V \ . \end{aligned} \tag{A.2.9} $$

This equation of motion leads to a relation between the Ricci scalar and the matter fields

$$ \bar{R} = \frac{d-3}{4(d-1)}K_1F_{MN}F^{MN} + K_2|D_M\Psi|^2 + \frac{d+1}{d-1}V \ , \tag{A.2.10} $$

and an isotropic component leads to

$$ R^2{}_2 = \frac{1}{d-1}\left(V - \frac{1}{4}K_1F_{MN}F^{MN}\right) \ . \tag{A.2.11} $$

So far our discussion does not assume an explicit background nor perturbations. Now,

[1]Here, we defined the symmetric symbol as $F^{(M}{}_AF^{N)A} = \frac{1}{2}\left(F^M{}_AF^{NA} + F^N{}_AF^{MA}\right)$.

we take the s-wave background ansatz (2.1.2)-(2.1.4) and the tensor mode $h^1{}_2 = \phi(t,r)$. The quadratic form (A.2.4) reduces to

$$
{}^{(2)}\mathcal{L}_{s\text{-matter}} = \frac{1}{2}\sqrt{-\bar{g}}\left(\frac{1}{4}K_1 F_{MN}F^{MN} + K_2|D_M\Psi|^2 + V\right)\phi^2 \ , \tag{A.2.12}
$$

where we set the gauge field perturbations a_M to zero since these decouple from ϕ. Using the trace of the equations of motion (A.2.10) and an isotropic component of Eq. (A.2.11), one gets

$$
{}^{(2)}\mathcal{L}_{s\text{-matter}} = \sqrt{-\bar{g}}\left(\frac{1}{2}\bar{R} - \bar{R}^2{}_2\right)\phi^2 \ . \tag{A.2.13}
$$

Combining the Einstein-Hilbert term (A.1.4) and the matter term (A.2.13), we obtain the quadratic form of the s-wave holographic superfluid action (2.1.12):

$$
\begin{aligned}
{}^{(2)}S_s = \frac{1}{16\pi G_{d+1}}\int d^{d+1}x\Bigg[& -\frac{1}{2}\sqrt{-g}(\nabla_M\phi)^2 - \partial_t\left(2\sqrt{-\bar{g}}g^{tt}\phi\partial_t\phi\right) \\
& + \partial_r\left\{\sqrt{-\bar{g}}\left(2g^{rr}\phi\partial_r\phi + \frac{1}{2}\frac{g'_{xx}}{g_{xx}}g^{rr}\phi^2\right)\right\}\Bigg] \ ,
\end{aligned} \tag{A.2.14}
$$

with $g_{x_1x_1} = g_{x_2x_2} = g_{xx}$. The second term (the total derivative with respect to t) does not affect the correlator, so we ignored the term in Eq. (2.1.12).

A.3 The Quadratic Form of the Einstein-Yang-Mills Action

The Einstein-Yang-Mills action is

$$
S_{\rm EYM} = \frac{1}{16\pi G_{d+1}}\int d^{d+1}x(\sqrt{-g}R + \mathcal{L}_{\text{EYM-matter}}) \ , \tag{A.3.1}
$$

$$
\mathcal{L}_{\text{EYM-matter}} := \sqrt{-g}\left[-\frac{1}{4}F^a_{MN}F^{aMN} - 2\Lambda\right] \ , \tag{A.3.2}
$$

where A^a_M is $SU(2)$ gauge field and the field strength is defined as

$$
F^a_{MN} = \partial_M A^a_N - \partial_N A^a_M + g_{\rm YM}\epsilon^{abc}A^b_M A^c_N \ . \tag{A.3.3}
$$

Under the metric and gauge field perturbations

$$g_{MN} = \bar{g}_{MN} + h_{MN} \ , \quad A^a_M = \bar{A}^a_M + a^a_M \ , \tag{A.3.4}$$

one can find[2]

$$\begin{aligned}
{}^{(2)}\mathcal{L}_{\rm EYM} &= {}^{(2)}\mathcal{L}_{\rm grav} + {}^{(2)}\mathcal{L}_{\rm gauge} + {}^{(2)}\mathcal{L}_{\rm int} \ ; && \text{(A.3.5)}\\
{}^{(2)}\mathcal{L}_{\rm grav} &= {}^{(2)}(\sqrt{-g}R) + \sqrt{-\bar{g}}\left[\frac{1}{2}h_{MN}\left(\bar{X}^{MNIJ} + 2\Lambda P^{MNIJ}\right)h_{IJ}\right] \ , && \text{(A.3.6)}\\
{}^{(2)}\mathcal{L}_{\rm gauge} &= \sqrt{-\bar{g}}\left[-\frac{1}{4}f^a_{MN}f^{aMN} + \frac{1}{2}a^a_M Z^{MN}_{ab}a^b_N\right] \ , && \text{(A.3.7)}\\
{}^{(2)}\mathcal{L}_{\rm int} &= \frac{1}{2}\sqrt{-\bar{g}}h_{MN}Q^{MNIJ}_a f^a_{IJ} \ , && \text{(A.3.8)}
\end{aligned}$$

where

$$\begin{aligned}
f^a_{MN} &= D_M a^a_N - D_N a^a_M && \text{(A.3.9)}\\
\bar{X}^{MNIJ} &= \frac{1}{4}F^a_{AB}F^{aAB}P^{MNIJ} + \frac{1}{2}F^a{}_A{}^M F^{aAN}\bar{g}^{IJ} && \text{(A.3.10)}\\
&\quad -F^a{}_A{}^M F^{aAJ}\bar{g}^{NI} - \frac{1}{2}F^{aMN}F^{aIJ} \ , &&\\
Q^{MNIJ}_a &= 2\bar{g}^{MI}F^{aNJ} - \frac{1}{2}\bar{g}^{MN}F^{aIJ} \ , && \text{(A.3.11)}\\
Z^{MN}_{ab} &= -g_{\rm YM}\epsilon^{abc}F^{cMN} \ . && \text{(A.3.12)}
\end{aligned}$$

The background satisfies the Einstein equation (A.2.8) with the energy-momentum tensor given by [3]

$$\bar{T}^{MN} = \frac{1}{2}\left(-\frac{1}{4}\bar{g}^{MN}F^a_{IJ}F^{aIJ} - F^{a(M}{}_A F^{|a|N)A} - \bar{g}^{MN}2\Lambda\right) \ . \tag{A.3.13}$$

This equation of motion leads to a relation between the Ricci scalar and the matter fields

$$\bar{R} = \frac{d-3}{4(d-1)}F^a_{MN}F^{aMN} + \frac{d+1}{d-1}2\Lambda \ , \tag{A.3.14}$$

[2] The quadratic form of the Einstein-Yang-Mills action was obtained in Ref. [81] in order to calculate the one-loop divergence, but they omitted surface terms.

[3] Here, the vertical bars indicate that we do not symmetrize over a: $F^{a(M}{}_A F^{|a|N)A} = \frac{1}{2}\left(F^{aM}{}_A F^{aNA} + F^{aN}{}_A F^{aMA}\right)$

and an isotropic component leads to

$$R^2{}_2 = \frac{1}{d-1}\left(2\Lambda - \frac{1}{4}F^a_{MN}F^{aMN} + \frac{d-1}{2}F^a_{Mx_2}F^{aMx_2}\right) \ . \tag{A.3.15}$$

A.3.1 The p-wave Holographic Superfluid Action (Tensor Mode)

Here, we derive the effective action of the tensor mode metric perturbation for the $(d+1)$-dimensional p-wave system. If we set metric perturbation (A.3.4) to the tensor mode $h^2{}_3 = \phi(t,r)$, all the other perturbations are decoupled, so these perturbations can be ignored consistently. The quadratic action (A.3.5) reduces to

$${}^{(2)}\mathcal{L}_{\text{EYM-matter}} = \frac{1}{2}\sqrt{-\bar{g}}\left[\frac{1}{4}F^a_{MN}F^{aMN} + 2\Lambda\right]\phi^2 = \sqrt{-\bar{g}}\left(\frac{1}{2}\bar{R} - \bar{R}^2{}_2\right)\phi^2 \ , \tag{A.3.16}$$

using Eqs. (A.3.14) and (A.3.15). Then, one obtains the quadratic form of the p-wave action for the tensor mode metric perturbation:

$$\begin{aligned}{}^{(2)}S_p = \frac{1}{16\pi G_{d+1}}\int d^{d+1}x\sqrt{-\bar{g}}\Bigg[&-\frac{1}{2}(\nabla_M\phi)^2 - \partial_t\left(2\sqrt{-\bar{g}}g^{tt}\phi\partial_t\phi\right) \\ &+ \partial_r\left\{\sqrt{-\bar{g}}\left(2g^{rr}\phi\partial_r\phi + \frac{1}{2}\frac{g'_{x_2x_2}}{g_{x_2x_2}}g^{rr}\phi^2\right)\right\}\Bigg] \ .\end{aligned} \tag{A.3.17}$$

A.3.2 The $(p+ip)$-wave Holographic Superfluid Action

Let us derive the effective action of the "tensor mode" metric perturbations for the 4-dimensional $(p+ip)$-wave system. In this case, we must turn on four perturbations $\phi_{od} = h^1{}_2 = h^2{}_1$, $\phi_d = h^1{}_1 = -h^2{}_2$, $a_{od} = a^1{}_2 = a^2{}_1$ and $a_d = a^1{}_1 = -a^2{}_2$. The perturbed action is obtained by substituting these perturbations into Eq. (A.3.5):

$${}^{(2)}S_{(p+ip)} = \frac{1}{16\pi G_4}\int d^4x\sqrt{-\bar{g}}\left({}^{(2)}\mathcal{L}_{\text{grav}} + {}^{(2)}\mathcal{L}_{\text{gauge}} + {}^{(2)}\mathcal{L}_{\text{int}}\right) \ ; \tag{A.3.18}$$

$${}^{(2)}\mathcal{L}_{\text{grav}} = \sqrt{-\bar{g}}\sum_{i=1}^{2}\left[-\frac{1}{2}\left\{-g^{tt}(\partial_t\phi_i)^2 + g^{rr}(\partial_r\phi_i)^2\right\} - \frac{1}{2}M(r)^2\phi_i^2\right] \ , \tag{A.3.19}$$

$${}^{(2)}\mathcal{L}_{\text{gauge}} = \sqrt{-\bar{g}}g^{xx}\sum_{i=1}^{2}\left[-g^{rr}(\partial_r a_i)^2 + g_{\text{YM}}^2 w^2 a_i^2 + g^{tt}(D_t a_i)^2\right] \ , \tag{A.3.20}$$

$${}^{(2)}\mathcal{L}_{\text{int}} = \sqrt{-\bar{g}}\sum_{i=1}^{2}\phi_i(F\cdot f)_i \ , \tag{A.3.21}$$

where we defined two-component vectors $\phi_i = (\phi_{od}, \phi_d)$, $a_i = (a_{od}, a_d)$, $(F \cdot f)_i = ((F \cdot f)_1, (F \cdot f)_2)$ and i runs isotropic components $i = 1, 2$. Here, we have omitted the surface term in Eq. (A.3.19). The mass-like function $M(r)$ is defined by

$$M(r)^2 := g^{rr} g^{xx} (\partial_r w)^2 + g^{xx} g_{\rm YM}^2 \left(g^{xx} w^2 - g^{tt} \Phi^2 \right) w^2 \ , \tag{A.3.22}$$

and the explicit form of $(F \cdot f)_i$ is

$$(F \cdot f)_{od} = g^{rr} g^{xx} (\partial_r w)(\partial_r a_{od}) + g^{tt} g^{xx} g_{\rm YM} \Phi w (D_t a_d), \tag{A.3.23}$$

$$(F \cdot f)_d = g^{rr} g^{xx} (\partial_r w)(\partial_r a_d) - g^{tt} g^{xx} g_{\rm YM} \Phi w (D_t a_{od}). \tag{A.3.24}$$

Note that the covariant derivatives of a_i have forms

$$D_t a_{od} = \partial_t a_{od} + g_{\rm YM} \Phi a_d \ , \qquad D_t a_d = \partial_t a_d - g_{\rm YM} \Phi a_{od} \ . \tag{A.3.25}$$

Therefore, these forms mix a_{od} and a_d.

Appendix B

A Note on Chapter 3

B.1 The Dispersion relation of Second Order Hydrodyanmics

In this section, we derive the sound mode dispersion relation of the second order hydrodynamics (3.3.21). For hydrodynamic computations, we use the Minkowski back ground $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ and the rest fluid $\bar{u}^\mu = (1, 0, 0, 0)$. Therefore, coefficients κ_*, which are the coefficients of Riemann tensor or Ricci tensor (1.2.15), are absent in the dispersion relations. Without loss of generality we choose momentum direction to be the z-direction as in Eqs. (3.1.9, 3.1.10). Using the constitutive equation of the second order fluid (1.2.15), one can find the linearized stress tensor,

$$\begin{aligned} \delta T^{tt} &= \delta\epsilon \ , && \text{(B.1.1)} \\ \delta T^{tz} &= (\bar{\varepsilon} + \bar{P})\delta u_z \ , && \text{(B.1.2)} \\ \delta T^{zz} &= c_s^2 \delta\epsilon \left[-iq \left(2\frac{p-1}{p}\hat{\eta} + \hat{\zeta} \right) + \omega q \left(2\frac{p-1}{p}\hat{\eta}\tau_\pi + \hat{\zeta}\tau_\Pi \right) \right] \delta u_z \ . && \text{(B.1.3)} \end{aligned}$$

Hydrodynamic variables δu_z and $\delta\epsilon$ can be easily removed from the above expressions, and with the conservation equation,

$$-i\omega\delta T^{tt} + iq\delta T^{zt} = 0 \ , \quad -i\omega\delta T^{tz} + iq\delta T^{zz} = 0 \ , \qquad \text{(B.1.4)}$$

the full dispersion relation can be obtained

$$\omega^2 - c_s^2 q^2 - \left[-i\omega q^2 \left(2\frac{p-1}{p}\hat{\eta} + \hat{\zeta} \right) + (\omega q)^2 \left(2\frac{p-1}{p}\hat{\eta}\tau_\pi + \hat{\zeta}\tau_\Pi \right) \right] = 0 \ . \qquad \text{(B.1.5)}$$

Solving the equation for ω and expanding it in terms of q, one can find the dispersion relation:

$$\omega = \pm c_s q - i\left(\frac{p-1}{p}\hat{\eta} + \frac{1}{2}\hat{\zeta}\right) q^2 + \frac{1}{2c_s}\left[\frac{p-1}{p}\hat{\eta}\left(\pm 2c_s^2\tau_\pi - \frac{p-1}{p}\hat{\eta}\right) + \zeta\left(\pm c_s^2\tau_\Pi - \frac{p-1}{p}\hat{\eta} - \frac{1}{4}\hat{\zeta}\right)\right] q^3 + O(q^4)\ . \tag{B.1.6}$$

B.2 Several Expressions Used in Chapter 3

B.2.1 Integration constant

The integration constant C of the master field in terms of boundary values $h^\mu{}_\nu(u_c)$:

$$C = \frac{C_{\text{num}}}{C_{\text{den}}} \tag{B.2.1a}$$

$$C_{\text{num}} = u(4\mathfrak{q}^2 - 3f')^2\left[-2\mathfrak{q}\mathfrak{w}h^z{}_t + \mathfrak{q}^2 f h^t{}_t - \{\mathfrak{q}^2(1+u^2) - \mathfrak{w}^2\}h^x{}_x - \mathfrak{w}^2 h^z{}_z\right]\Big|_{u_c} \tag{B.2.1b}$$

$$C_{\text{den}} = 12\mathfrak{q}^2 u f^2(4\mathfrak{q}^2 - 3f')F'\Big|_{u_c} + 4[4\mathfrak{q}^6 u(-3+u^2) + 27u^3\mathfrak{w}^2 - 9\mathfrak{q}^2u^2(u + u^3 - 4\mathfrak{w}^2) - 12\mathfrak{q}^4(1 + u^4 - u\mathfrak{w}^2)]F\Big|_{u_c} \tag{B.2.1c}$$

B.2.2 Explicit expression of Eq. (3.3.14)

Explicit expressions of Eq. (3.3.14):

$$h^t{}_t{}' = \frac{1}{6f^2}\Big[-4\mathfrak{q}^2 h^t{}_t + [8\mathfrak{w}^2 + 4\mathfrak{q}^2(f - uf') - 3(3f - uf')f']h^x{}_x + 4\mathfrak{w}^2 h^z{}_z + 8\mathfrak{q}\mathfrak{w}h^z{}_t - (4\mathfrak{q}^2 - 3f')(3f - uf')\Phi\Big]\ , \tag{B.2.2a}$$

$$h^x{}_x{}' = \frac{1}{12\mathfrak{q}^2 f^2}\Big[\mathfrak{q}^2 f(4\mathfrak{q}^2 - 3f')h^t{}_t + [(4\mathfrak{q}^2 - 3f')(\mathfrak{w}^2 + \mathfrak{q}^2 uf') - (4\mathfrak{q}^4 - 9\mathfrak{q}^2 f')f]h^x{}_x - \mathfrak{w}^2(4\mathfrak{q}^2 - 3f')h^z{}_z - 2\mathfrak{q}\mathfrak{w}(4\mathfrak{q}\mathfrak{w} - 3f')h^z{}_t - (4\mathfrak{q}^2 - 3f')(3\mathfrak{w}^2 - 3\mathfrak{q}^2 f + \mathfrak{q}^2 uf')\Phi\Big]\ , \tag{B.2.2b}$$

$$h^z{}_z{}' = \frac{1}{6\mathfrak{q}^2 f^2}\Big[-\mathfrak{q}^2 f(4\mathfrak{q}^2 - 3f')h^t{}_t - [8\mathfrak{q}^4 f + (4\mathfrak{q}^2 - 3f')(\mathfrak{w}^2 + \mathfrak{q}^2 uf')]h^x{}_x + \mathfrak{w}^2(4\mathfrak{q}^2 - 3f')h^z{}_z + 2\mathfrak{q}\mathfrak{w}(4\mathfrak{q}^2 - 3f')h^z{}_t + (4\mathfrak{q}^2 - 3f')(3\mathfrak{w}^2 + \mathfrak{q}^2 uf')\Phi\Big]\ , \tag{B.2.2c}$$

$$h^z{}_t{}' = \frac{1}{2\mathfrak{q}f}\Big[\mathfrak{w}(4\mathfrak{q}^2 - f')h^x{}_x + \mathfrak{w}f'h^z{}_z + 2\mathfrak{q}f'h^z{}_t - \mathfrak{w}(4\mathfrak{q}^2 - 3f')\Phi\Big]\ . \tag{B.2.2d}$$

Bibliography

[1] R. H. Price and K. S. Thorne, "Membrane Viewpoint On Black Holes: Properties And Evolution Of The Stretched Horizon," Phys. Rev. D **33** (1986) 915.

[2] M. Parikh and F. Wilczek, "An Action for black hole membranes," Phys. Rev. D **58** (1998) 064011 [arXiv:gr-qc/9712077].

[3] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2** (1998) 231 [Int. J. Theor. Phys. **38** (1999) 1113] [arXiv:hep-th/9711200].

[4] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2** (1998) 253 [arXiv:hep-th/9802150].

[5] E. Witten, "Anti-de Sitter space, thermal phase transition, and confinement in gauge theories," Adv. Theor. Math. Phys. **2** (1998) 505 [arXiv:hep-th/9803131].

[6] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "Gauge theory correlators from non-critical string theory," Phys. Lett. B **428** (1998) 105 [arXiv:hep-th/9802109].

[7] J. Polchinski, Phys. Rev. Lett. **75**, 4724 (1995) [hep-th/9510017].

[8] M. Natsuume, "String theory and quark-gluon plasma," arXiv:hep-ph/0701201.

[9] D. T. Son and A. O. Starinets, "Viscosity, Black Holes, and Quantum Field Theory," Ann. Rev. Nucl. Part. Sci. **57** (2007) 95 [arXiv:0704.0240 [hep-th]].

[10] G. Policastro, D. T. Son and A. O. Starinets, Phys. Rev. Lett. **87**, 081601 (2001) [hep-th/0104066].

[11] C. P. Herzog, JHEP **0212**, 026 (2002) [hep-th/0210126].

[12] P. Kovtun, D. T. Son and A. O. Starinets, JHEP **0310**, 064 (2003) [hep-th/0309213].

[13] A. Buchel and J. T. Liu, Phys. Rev. Lett. **93**, 090602 (2004) [hep-th/0311175].

[14] A. Buchel, Nucl. Phys. B **708**, 451 (2005) [hep-th/0406200].

[15] A. Buchel, Phys. Lett. B **609**, 392 (2005) [hep-th/0408095].

[16] P. K. Kovtun and A. O. Starinets, Phys. Rev. D **72**, 086009 (2005) [hep-th/0506184].

[17] J. Mas, "Shear viscosity from R-charged AdS black holes," JHEP **0603** (2006) 016 [arXiv:hep-th/0601144].

[18] D. T. Son and A. O. Starinets, "Hydrodynamics of R-charged black holes," JHEP **0603** (2006) 052 [arXiv:hep-th/0601157].

[19] O. Saremi, "The viscosity bound conjecture and hydrodynamics of M2-brane theory at finite chemical potential," JHEP **0610**, 083 (2006) [arXiv:hep-th/0601159].

[20] K. Maeda, M. Natsuume and T. Okamura, "Viscosity of gauge theory plasma with a chemical potential from AdS/CFT correspondence," Phys. Rev. D **73** (2006) 066013 [arXiv:hep-th/0602010].

[21] D. Mateos, R. C. Myers and R. M. Thomson, Phys. Rev. Lett. **98**, 101601 (2007) [hep-th/0610184].

[22] R. A. Janik, Phys. Rev. Lett. **98**, 022302 (2007) [hep-th/0610144].

[23] P. Kovtun and D. Nickel, Phys. Rev. Lett. **102**, 011602 (2009) [arXiv:0809.2020 [hep-th]].

[24] N. Iqbal and H. Liu, "Universality of the hydrodynamic limit in AdS/CFT and the membrane paradigm," Phys. Rev. D **79** (2009) 025023 [arXiv:0809.3808 [hep-th]].

[25] P. Benincasa, A. Buchel and R. Naryshkin, "The shear viscosity of gauge theory plasma with chemical potentials," Phys. Lett. B **645** (2007) 309 [arXiv:hep-th/0610145].

[26] J. Casalderrey-Solana, H. Liu, D. Mateos, K. Rajagopal and U. A. Wiedemann, "Gauge/String Duality, Hot QCD and Heavy Ion Collisions," arXiv:1101.0618 [hep-th].

[27] I. Bredberg, C. Keeler, V. Lysov and A. Strominger, "Wilsonian Approach to Fluid/Gravity Duality," JHEP **1103** (2011) 141 [arXiv:1006.1902 [hep-th]].

[28] I. Bredberg, C. Keeler, V. Lysov and A. Strominger, "From Navier-Stokes To Einstein," arXiv:1101.2451 [hep-th].

[29] G. Compere, P. McFadden, K. Skenderis and M. Taylor, "The Holographic fluid dual to vacuum Einstein gravity," JHEP **1107** (2011) 050 [arXiv:1103.3022 [hep-th]].

[30] D. K. Brattan, J. Camps, R. Loganayagam and M. Rangamani, "CFT dual of the AdS Dirichlet problem : Fluid/Gravity on cut-off surfaces," JHEP **1112** (2011) 090 [arXiv:1106.2577 [hep-th]].

[31] J. W. York, Jr., "Black hole thermodynamics and the Euclidean Einstein action," Phys. Rev. **D33** (1986) 2092.

[32] J. D. Brown and J. W. York, "Quasilocal energy and conserved charges derived from the gravitational action," Phys. Rev. D **47** (1993) 1407 [arXiv:gr-qc/9209012].

[33] D. Marolf and M. Rangamani, "Causality and the AdS Dirichlet problem," arXiv:1201.1233 [hep-th].

[34] C. Eling, A. Meyer and Y. Oz, "The Relativistic Rindler Hydrodynamics," arXiv:1201.2705 [hep-th].

[35] C. Eling, A. Meyer and Y. Oz, "Local Entropy Current in Higher Curvature Gravity and Rindler Hydrodynamics," arXiv:1205.4249 [hep-th].

[36] M. Natsuume and M. Ohta, Prog. Theor. Phys. **124**, 931 (2010) [arXiv:1008.4142 [hep-th]].

[37] Y. Matsuo, M. Natsuume, M. Ohta and T. Okamura, arXiv:1206.6924 [hep-th].

[38] L. D. Landau, and E. M. Lifshitz, "Fluid Mechanics, Second Edition: Volume 6 (Course of Theoretical Physics)," Butterworth-Heinemann (1987) 552p

[39] R. Baier, P. Romatschke, D. T. Son, A. O. Starinets and M. A. Stephanov, JHEP **0804**, 100 (2008) [arXiv:0712.2451 [hep-th]].

[40] W. Israel, Annals Phys. **100**, 310 (1976).

[41] W. Israel and J. M. Stewart, Annals Phys. **118**, 341 (1979).

[42] M. Natsuume and T. Okamura, "Causal hydrodynamics of gauge theory plasmas from AdS/CFT duality," Phys. Rev. D **77** (2008) 066014 [Erratum-ibid. D **78** (2008) 089902] [arXiv:0712.2916 [hep-th]].

[43] S. Bhattacharyya, JHEP **1207**, 104 (2012) [arXiv:1201.4654 [hep-th]].

[44] D. T. Son and A. O. Starinets, "Minkowski-space correlators in AdS/CFT correspondence: Recipe and applications," JHEP **0209** (2002) 042 [arXiv:hep-th/0205051].

[45] G. Policastro, D. T. Son and A. O. Starinets, JHEP **0209**, 043 (2002) [hep-th/0205052].

[46] P. Kovtun, D. T. Son and A. O. Starinets, "Viscosity in strongly interacting quantum field theories from black hole physics," Phys. Rev. Lett. **94** (2005) 111601 [arXiv:hep-th/0405231].

[47] S. S. Gubser, "Breaking an Abelian gauge symmetry near a black hole horizon," Phys. Rev. D **78** (2008) 065034 [arXiv:0801.2977 [hep-th]].

[48] S. A. Hartnoll, C. P. Herzog and G. T. Horowitz, "Building a Holographic Superconductor," Phys. Rev. Lett. **101**, 031601 (2008) [arXiv:0803.3295 [hep-th]].

[49] S. S. Gubser, "Colorful horizons with charge in anti-de Sitter space," Phys. Rev. Lett. **101** (2008) 191601 [arXiv:0803.3483 [hep-th]].

[50] S. S. Gubser and S. S. Pufu, "The gravity dual of a p-wave superconductor," JHEP **0811** (2008) 033 [arXiv:0805.2960 [hep-th]].

[51] S. A. Hartnoll, C. P. Herzog and G. T. Horowitz, "Holographic Superconductors," JHEP **0812** (2008) 015 [arXiv:0810.1563 [hep-th]].

[52] C. P. Herzog, P. K. Kovtun and D. T. Son, "Holographic model of superfluidity," arXiv:0809.4870 [hep-th].

[53] P. Basu, A. Mukherjee and H. H. Shieh, "Supercurrent: Vector Hair for an AdS Black Hole," arXiv:0809.4494 [hep-th].

[54] S. A. Hartnoll, "Lectures on holographic methods for condensed matter physics," Class. Quant. Grav. **26** (2009) 224002 [arXiv:0903.3246 [hep-th]].

[55] C. P. Herzog, "Lectures on Holographic Superfluidity and Superconductivity," J. Phys. A **42** (2009) 343001 [arXiv:0904.1975 [hep-th]].

[56] G. T. Horowitz, "Introduction to Holographic Superconductors," arXiv:1002.1722 [hep-th].

[57] K. Landsteiner and J. Mas, "The shear viscosity of the non-commutative plasma," JHEP **0707** (2007) 088 [arXiv:0706.0411 [hep-th]].

[58] S. Franco, A. Garcia-Garcia and D. Rodriguez-Gomez, "A general class of holographic superconductors," JHEP **1004** (2010) 092 [arXiv:0906.1214 [hep-th]].

[59] S. Franco, A. M. Garcia-Garcia and D. Rodriguez-Gomez, "A holographic approach to phase transitions," Phys. Rev. D **81** (2010) 041901 [arXiv:0911.1354 [hep-th]].

[60] C. P. Herzog, "An Analytic Holographic Superconductor," arXiv:1003.3278 [hep-th].

[61] K. Maeda and T. Okamura, "Characteristic length of an AdS/CFT superconductor," Phys. Rev. D **78** (2008) 106006 [arXiv:0809.3079 [hep-th]].

[62] K. Maeda, M. Natsuume and T. Okamura, "On two pieces of folklore in the AdS/CFT duality," arXiv:1005.2431 [hep-th].

[63] C. P. Herzog and S. S. Pufu, "The Second Sound of SU(2)," JHEP **0904** (2009) 126 [arXiv:0902.0409 [hep-th]].

[64] A. Yarom, "Fourth sound of holographic superfluids," JHEP **0907** (2009) 070 [arXiv:0903.1353 [hep-th]].

[65] T. Albash and C. V. Johnson, "Phases of Holographic Superconductors in an External Magnetic Field," arXiv:0906.0519 [hep-th].

[66] T. Albash and C. V. Johnson, "Vortex and Droplet Engineering in Holographic Superconductors," Phys. Rev. D **80** (2009) 126009 [arXiv:0906.1795 [hep-th]].

[67] M. Montull, A. Pomarol and P. J. Silva, "The Holographic Superconductor Vortex," Phys. Rev. Lett. **103** (2009) 091601 [arXiv:0906.2396 [hep-th]].

[68] K. Maeda, M. Natsuume and T. Okamura, "Vortex lattice for a holographic superconductor," Phys. Rev. D **81** (2010) 026002 [arXiv:0910.4475 [hep-th]].

[69] V. Keranen, E. Keski-Vakkuri, S. Nowling and K. P. Yogendran, "Inhomogeneous Structures in Holographic Superfluids: II. Vortices," Phys. Rev. D **81**, 126012 (2010) [arXiv:0912.4280 [hep-th]].

[70] S. de Haro, S. N. Solodukhin and K. Skenderis, "Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence," Commun. Math. Phys. **217** (2001) 595 [arXiv:hep-th/0002230].

[71] A. Buchel and S. Cremonini, "Viscosity Bound and Causality in Superfluid Plasma," arXiv:1007.2963 [hep-th].

[72] P. C. Hohenberg and B. I. Halperin, "Theory Of Dynamic Critical Phenomena," Rev. Mod. Phys. **49** (1977) 435.

[73] K. Maeda, M. Natsuume and T. Okamura, "Universality class of holographic superconductors," Phys. Rev. D **79** (2009) 126004 [arXiv:0904.1914 [hep-th]].

[74] K. Maeda, M. Natsuume and T. Okamura, "Dynamic critical phenomena in the AdS/CFT duality," Phys. Rev. D **78** (2008) 106007 [arXiv:0809.4074 [hep-th]].

[75] P. G.de Gennes, *The physics of liquid crystals* (Clarendon Press, Oxford, 1974).

[76] S. Sarman and D. J. Evans, "Statistical mechanics of viscous flow in nematic fluids," J. Chem. Phys. **99** (1999) 9021.

[77] M. Ammon, J. Erdmenger, V. Grass, P. Kerner and A. O'Bannon, Phys. Lett. B **686** (2010) 192 [arXiv:0912.3515 [hep-th]].

[78] J. Erdmenger, P. Kerner and H. Zeller, "Non-universal shear viscosity from Einstein gravity," Phys. Lett. B **699**, 301 (2011) [arXiv:1011.5912 [hep-th]].

[79] J. Erdmenger, P. Kerner and H. Zeller, JHEP **1201**, 059 (2012) [arXiv:1110.0007 [hep-th]].

[80] K. Maeda, S. Fujii and J. -i. Koga, Phys. Rev. D **81**, 124020 (2010) [arXiv:1003.2689 [gr-qc]].

[81] S. Deser, H. S. Tsao and P. van Nieuwenhuizen, "One Loop Divergences Of The Einstein Yang-Mills System," Phys. Rev. D **10** (1974) 3337.

[82] S. S. Gubser, S. S. Pufu and F. D. Rocha, "Bulk viscosity of strongly coupled plasmas with holographic duals," JHEP **0808** (2008) 085 [arXiv:0806.0407 [hep-th]].

[83] V. Balasubramanian and P. Kraus, "A Stress tensor for Anti-de Sitter gravity," Commun. Math. Phys. **208** (1999) 413 [hep-th/9902121].

[84] C. Eling and Y. Oz, "Relativistic CFT Hydrodynamics from the Membrane Paradigm," JHEP **1002** (2010) 069 [arXiv:0906.4999 [hep-th]].

[85] C. P. Herzog and D. T. Son, JHEP **0303**, 046 (2003) [hep-th/0212072].

[86] D. Mateos and D. Trancanelli, Phys. Rev. Lett. **107**, 101601 (2011) [arXiv:1105.3472 [hep-th]].

[87] D. Mateos and D. Trancanelli, JHEP **1107**, 054 (2011) [arXiv:1106.1637 [hep-th]].

[88] A. Rebhan and D. Steineder, Phys. Rev. Lett. **108**, 021601 (2012) [arXiv:1110.6825 [hep-th]].

[89] K. A. Mamo, JHEP **1210**, 070 (2012) [arXiv:1205.1797 [hep-th]].

[90] S. Sarman, D. J. Evans, "Statistical mechanics of viscous flow in nematic fluids," J. Chem. Phys. **99**, 9021 (1993)

[91] P. G. deGennes, "The Physics of Liquid Crystals," Clarendon, Oxford, (1979)