Cosmological tests of models for the accelerating universe in terms of inhomogeneities

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Abstract

We study cosmological tests of models that can explain the apparent accelerated expansion of the present universe. In this thesis, we provide methods of testing these models by particularly focusing on inhomogeneities of the universe, because, practically, our universe is inhomogeneous.

First, we consider the effective gravitational stress-energy tensor for short-wavelength perturbations in modified gravity theories in the cosmological context. We address this problem in a simple class of $f(R)$ gravity theories on the assumptions that (i) the background, or coarse-grained metric averaged over several wavelengths, has the Friedmann-Lemaître-Robertson-Walker symmetry and that (ii) when our $f(R)$ theory reduces to Einstein gravity, the field equations of Einstein gravity should be reproduced. We show by explicit computation that the effective gravitational stress-energy tensor for a cosmological model in our $f(R)$ theories, as well as that obtained in the corresponding scalar-tensor theory, takes a similar form to that in general relativity and is in fact traceless, hence acting again like a radiation fluid as in the case of general relativity. If the assumption (ii) above is dropped, then an undetermined integration constant appears and the resultant effective stress-energy tensor acquires a term that is in proportion to the background metric, hence being, in principle, able to describe a cosmological constant. Whether this effective cosmological constant term is positive and whether it has the right magnitude as dark energy depends upon the value of the integration constant.

Second, we discuss temperature anisotropies of cosmic microwave background (CMB) in local void models. We derive analytic formulae for the dipole and quadrupole moments of the CMB temperature anisotropy that hold for any spherically symmetric universe model and can be used to compare consequences of this model with observations of the CMB temperature anisotropy rigorously. We check that our formulae are consistent with the numerical studies previously made for the CMB temperature anisotropy in the void model. We also update the constraints concerning the location of the observers in the void model by applying our analytic dipole formula with the latest WMAP data.

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Chapter 1

Introduction

In modern cosmology, it is commonly assumed that our universe be isotropic and homogeneous on sufficiently large scales under the cosmological principle, and accordingly be described by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric as a first approximation. Then, together with consequences of various cosmological observations such as the power spectrum of Cosmic Microwave Background (CMB) temperature anisotropy, the distance-redshift relation of SuperNova of type Ia (SN Ia) indicates that the expansion of the present universe is apparently accelerated as first observed in 1998 [1, 2].

In general relativity, if the universe is filled with ordinary non-relativistic matter and radiation that are the two known constituents of the universe, then gravity should lead to a slowing of the expansion. Since the expansion is apparently accelerated, there are three possibilities, any of which would have deep implications for our understanding of the universe and theory of gravity. The first is that about 75% of the energy density of the universe exists in a new form, known as "dark energy" which has negative pressure. However, there does not appear to be any satisfactory theory that can naturally explain the origin of dark energy and its magnitude required by observations. The second is that general relativity is broken on cosmological scales and must be replaced with a more appropriate theory of gravity, i.e., "modified gravity theory." The third is that the assumption that the universe is homogeneous breaks down, having instead an under-dense local void in the surrounding overdense universe, so-called "local void model."

We can regard the so-called cosmological constant Λ whose equation of state is $w_{\rm DE} =$

-1 as the simplest candidate for dark energy. If the cosmological constant originates from a vacuum energy of particle physics, its energy scale is significantly larger than the observed value of the present dark energy density [3]. We need to find a mechanism to obtain the tiny value of Λ consistent with observations.

To understand the property of dark energy, we should first clarify whether it is a simple constant or it dynamically changes in time. The dynamical dark energy models can be distinguished from the cosmological constant by considering the time evolution of $w_{\rm DE}$. A wide variety of variations of $w_{\rm DE}$ is predicted by the scalar field models, such as quintessence [4, 5] and k-essence [6–8]. Still, the current observational data are not sufficient to provide some preference of such models over the Λ-Cold Dark Matter (ΛCDM) model. Furthermore, the field needs to have sufficiently flat potentials such that the field evolves slowly enough to drive the accelerated expansion of the present universe. This demands that the field mass is extremely small relative to typical mass scales appearing in particle physics [9, 10]. It seems to be an extremely difficut task to construct viable scalar field dark energy models within the framework of particle physics.

There exists another class of dynamical dark energy models based on the large-distance modification of gravity, for example, $f(R)$ gravity [11], scalar-tensor theories, Gauss-Bonnet gravity [12, 13], Dvali-Gabadadze-Porrati (DGP) braneworld model [14], and so on. An attractive feature of these models is that the cosmic acceleration can be realized without introducing dark energy for matter contents of the universe. There are tight constraints coming from local gravity tests by modifying general relativity. Hence, in general, the restriction on modified gravity theories is stringent compared to modified matter models.

Among many, one of the simplest of modified theories so far proposed is the so-called $f(R)$ gravity, whose action is a generalization of the Einstein-Hilbert action to an arbitrary function, $f(R)$, of the scalar curvature R. For example, an $f(R)$ model of the form $f(R) = R - \mu^{2(n+1)}/R^n$ $(n > 0)$ was proposed to explain the late-time cosmic acceleration [15, 16]. However, this model suffers from a number of problems such as the incompatibility with local gravity constraints [17, 18], the instability of density perturbations [19, 20], and the absence of a matter-dominated epoch [21, 22]. As we will see in this thesis there are a number of conditions required for the viability of $f(R)$ dark energy models [23–25], which stimulated to propose viable models [26–28]. It is well known that $f(R)$ gravity is equivalent to a scalar-tensor theory which contains the coupling of the scalar curvature R to a scalar field ϕ in a certain way [17]. Brans-Dicke theory [29] is one of the simplest examples.

In addition to the scalar curvature R, we can construct other scalar quantities such as $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}$ from the Ricci tensor $R_{\mu\nu}$ and Riemann tensor $R_{\mu\nu\lambda\sigma}$ [30]. For the Gauss-Bonnet curvature invariant defined by $\mathcal{G} \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}$, it is known that we can avoid the appearance of spurious spin-2 ghosts [31]. In order to give rise to some contribution of the Gauss-Bonnet term to the Friedmann equation, we require that (i) the Gauss-Bonnet term couples to a scalar field ϕ, i.e., $F(\phi)\mathcal{G}$ or (ii) the Lagrangian density f is a function of $\mathcal{G}$, i.e., $f(\mathcal{G})$. We shall review such theories and observational constraints on them.

In DGP braneworld model, we consider a 3-dimensional brane embedded in the 5-dimensional Minkowski bulk spacetime [14]. The gravitational leakage to the extra dimension leads to a self-acceleration of the universe on the 3-dimensional brane. A longitudinal graviton (i.e. a branebending mode ϕ) gives rise to a nonlinear self-interaction of the form $(r_c^2/m_{\rm pl})\Box\phi\partial^\mu\phi\partial_\mu\phi$ through the mixing with a transverse graviton, where r_c is a cross-over scale (of the order of the Hubble radius H_0^{-1} today) and $m_{\rm pl}$ is the Planck mass [32]. In the local region where the energy density ρ is much larger than $r_c^{-2}m_{\rm pl}^2$, the nonlinear self-interaction can lead to the decoupling of the field from matter through the so-called Vainshtein mechanism [33], which allows a possibility for the consistency with local gravity constraints. However, the DGP braneworld model suffers from a ghost problem [34], in addition to the difficulty for satisfying the combined observational constraints.

In addition to the above mentioned models, there are attempts to explain the cosmic acceleration without dark energy. One of such attempts is the local void model proposed by Tomita [35, 36], also independently by Celerier [37] and by Goodwin *et al.* [38]. In this model, our universe is no longer assumed to be homogeneous, having instead an under-dense local void in the surrounding overdense universe. The isotropic nature of cosmological observations on large scales is realized by assuming the spherical symmetry and demanding that we live close to the center of the void. Furthermore, the model is supposed to contain only ordinary dust like cosmic matter, describing, say, CDM component. Such a spacetime can be described by the Lemaître-Tolman-Bondi (LTB) spacetime [39–41]. Since the rate of expansion in the void region is larger than that in the outer overdense region, this model can account for the observed dimming of SN Ia luminosity. In fact, recent numerical analyses [42–52] have shown that the LTB model can accurately reproduce the SN Ia distance-redshift relation. For this reason, despite the relinquishment of the widely accepted Copernican/cosmological principle, the local void model has recently attracted considerable attention.

Another example is the so-called backreaction model in which the backreaction of spatial inhomogeneities on the FLRW background is responsible for the real acceleration. Our observable universe appears to be homogeneous and isotropic on large scales, but highly inhomogeneous on small scales. It is therefore interesting to consider whether the local inhomogeneities can have any effects on the global dynamics of our universe, in particular, any effect that corresponds to a positive cosmological constant or dark energy. A number of authors have explored this possibility of explaining the present cosmic accelerating expansion by some backreaction effects of the local inhomogeneities [53–61]. Such a backreaction effect may be described in terms of an effective stress-energy tensor arising from metric as well as matter perturbations.

In this thesis, we discuss cosmological tests of modified gravity theories and the local void model in terms of inhomogeneities of the universe. We first consider backreaction of inhomogeneities and effective stress energy tensor in $f(R)$ gravity (Chapter 5) and then consider off-center CMB anisotropies in local void models (Chapter 6).

In Chapter 5, the high frequency limit in $f(R)$ gravity and scalar-tensor theory is studied [62]. In general relativity, a consistent expansion scheme for short-wavelength perturbations and the corresponding effective stress-energy tensor were largely developed by Isaacson [63, 64], in which the small parameter, say ϵ, corresponds to the amplitude and at the same time the wavelength of perturbations. Isaacson's expansion scheme is called the high frequency limit or the short-wavelength approximation. If the effective stress-energy tensor had a term proportional to the background spacetime metric, then it would correspond to adding a cosmological constant to the effective Einstein equations for the background metric, thereby explaining possible origin of dark energy from local inhomogeneities. It has been shown, however, that this effective gravitational stress-energy tensor is traceless and satisfies the weak energy condition, i.e. acts like radiation [65, 66], and thus cannot provide any effects that imitate dark energy in general relativity. However, it is far from obvious if this traceless property of the effective gravitational stress-energy tensor is a nature specific only to the general relativity or is rather a generic property that can also hold in other types of gravity theories.

The purpose of Chapter 5 is to address this question in a simple, concrete model in the cosmological context. Since $f(R)$ gravity contains higher-order derivative terms, one can anticipate the effective gravitational stress-energy tensor to be generally modified in the high frequency limit. Our analysis can be performed, in principle, either (i) by first translating a given $f(R)$ gravity into the corresponding scalar-tensor theory and then inspecting the stress-energy tensor for the scalar field ϕ, or (ii) by directly dealing with metric perturbations of $f(R)$ gravity. We may expect that the former approach is sufficient for our present purpose and much easier than the latter metric approach, as we have to deal with metric perturbations of complicated combinations of the curvature tensors in the latter case. Nevertheless, we will take both approaches. In fact, in the metric approach, by directly taking up perturbations of the scalar curvature R, the Ricci tensor $R_{\mu\nu}$ and the Riemann tensor $R^{\mu}{}_{\nu\lambda\sigma}$ involved in a given $f(R)$ theory, we can learn how to generalize our present analysis of a specific class of $f(R)$ gravity to analyses of other, different, types of modified gravity theory that cannot even be translated into a scalar-tensor theory, such as the Gauss-Bonnet gravity. Then, we will make sure that the effective stress-energy tensor in Brans-Dicke theory is consistent with that in our $f(R)$ gravity.

In Chapter 6, we turn to the local void model. In order to justify the local void model as a viable alternative to the standard ΛCDM model, we have to test this model by various observations other than the SN Ia distance-redshift relation. Most of previous analyses were performed for various types of local void models by using numerical methods, and it does not seem to be straightforward to compare analyses for each different model so as to have a coherent understanding of the results. In order to have general consequences of the local void model and systematically examine its viability, it is desirable to develop some general, analytic methods that can apply, independently of the details of each specific model.

The purpose of Chapter 6 is to derive the analytic formulae for the dipole and quadrupole of the CMB anisotropy in general spherically symmetric spacetimes, including the Λ-LTB spacetime, and to give constraints on the local void model. We will exploit the key requirement of the local void model that we, observers, are restricted to be around very near the center of the spherical symmetry: Namely, we first note that the small distance between the symmetry center and an off-center observer gives rise to a corresponding deviation in the photon distribution function. Then, by taking 'Taylor-expansions' of the photon distribution function at the center with respect to the deviation, we can read off the CMB temperature anisotropy caused by the deviation in the photon distribution function. By doing so, we can, in principle, construct the l-th order multiple moment of the CMB temperature anisotropy from the (up to) l-th order expansion coefficients, with the help of the background null geodesic equations and the Boltzmann equation. We will do so for the first and second-order expansions to find the CMB dipole and quadrupole moments. We also provide the concrete expression of the corresponding formulae for the local void model. Our formulae are then checked to be consistent with the numerical analyses of the CMB temperature anisotropy in the local void model, previously made by Alnes and Amarzguioui [47]. We apply our formulae to place the constraint on the distance between an observer and the symmetry center of the void, by using the latest Wilkinson Microwave Anisotropy Probe (WMAP) data, thereby updating the results of the previous analyses.

This thesis is organized as follows. In Chapter 2, we briefly review the FLRW cosmology and provide recent observational constraints on dark energy obtained from SN Ia, CMB and Baryon Acoustic Oscillations (BAO) data. Moreover, we introduce the cosmological constant Λ and the cosmological constant problem. In Chapter 3, we summarize current theoretical approaches to accelerated expansion and dark energy, including modified matter models, modified gravity theories, the local void model. In Chapter 4, we provide a number of ways to distinguish modified gravity theories and the local void model observationally from the ΛCDM model, and some constraints on these models. Chapter 5 and Chapter 6 are the main parts of this thesis. Chapter 7 is devoted to a summary and discussion.

Our signature convention for $g_{\mu\nu}$ is $(-, +, +, +)$. We define the Riemann tensor by

$R_{\mu\nu\lambda}{}^{\sigma}\omega_{\sigma} = 2\nabla_{[\mu}\nabla_{\nu]}\omega_{\lambda}$ and the Ricci tensor by $R_{\mu\nu} = R_{\mu\lambda\nu}{}^{\lambda}$ as in Wald's book [68]. We use units such that $c = \hbar = 1$, where c is the speed of light and $\hbar$ is reduced Planck's constant. The gravitational constant G is related to the Planck mass $m_{\rm pl} = 1.2211 \times 10^{19}$ GeV via $G = 1/m_{\rm pl}^2$ and the reduced Planck mass $M_{\rm pl} = 2.4357 \times 10^{18}$ GeV via $\kappa^2 = 8\pi G = 1/M_{\rm pl}^2$, respectively.

Chapter 2

Apparent accelerated expansion of the present universe

2.1 FLRW cosmology

2.1.1 Isotropic and homogeneous universe

FLRW spacetime

Our observable universe appears to be isotropic and homogeneous on large scales and accordingly be described by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric:

$$\begin{aligned} g_{\mu\nu}dx^\mu dx^\nu &= -dt^2 + a^2(t)\gamma_{ij}dx^i dx^j \\ &= -dt^2 + a^2(t)\left\{\frac{dr^2}{1-Kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right\}, \end{aligned} \tag{2.1.1}$$

where $a(t)$ is cosmic scale factor with cosmic time t. The coordinates r, θ and ϕ are known as comoving coordinates. The spatial curvature constant K describes the geometry of the spatial section. It may be convenient to write the metric (2.1.1) in the following form:

$$g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)\left\{d\chi^2 + f_K^2(\chi)\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right\}, \tag{2.1.2}$$

where

$$f_K(\chi) = \begin{cases} \sin\chi & (K>0); \quad \text{closed} \\ \chi & (K=0); \quad \text{flat} \\ \sinh\chi & (K<0); \quad \text{open.} \end{cases} \tag{2.1.3}$$

Note that (2.1.1) is invariant under the scaling $a(t) \to \xi a(t)$, $r \to r/\xi$, $K \to \xi^2 K$, where ξ is an arbitrary constant. This allows us to normalize k or $a(t)$ arbitrarily. We can set

$$a(t_0) \equiv 1, \tag{2.1.4}$$

where a subscript 0 means the present value.

The stress-energy tensor of the isotropic and homogeneous universe should take the same form as for a perfect fluid [69]:

$$T_{\mu\nu} = (\rho + P)U_\mu U_\nu + P g_{\mu\nu}, \tag{2.1.5}$$

where $\rho = \rho(t)$ and $P = P(t)$ are the energy density and the pressure density of the fluid, respectively, and $U^\mu \equiv dx^\mu/dt$ is the four-velocity with $U^\mu U_\mu = -1$. With one index raised, the stress-energy tensor takes the convenient form:

$$T^\mu{}_\nu = \mathrm{diag}(-\rho, P, P, P). \tag{2.1.6}$$

The Einstein field equation reads

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2.1.7}$$

where $G_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ is the Ricci tensor and $R \equiv R^\mu{}_\mu$ is the scalar curvature. In the FLRW background (2.1.1), the Einstein equation gives the Friedmann equations

$$H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}, \tag{2.1.8}$$

$$\frac{\ddot a}{a} = -\frac{4\pi G}{3}\left(\rho + 3P\right), \tag{2.1.9}$$

where *dot* denotes a derivative with respect to t, and

$$H(t) \equiv \frac{\dot{a}(t)}{a(t)} \tag{2.1.10}$$

is the Hubble parameter which characterizes the rate of cosmic expansion. The value of the Hubble parameter at the present epoch is the Hubble constant, H_0. Current observations lead us to believe that the Hubble constant is 73.8 ± 2.4 km/sec/Mpc [70]. (Mpc stands for megaparsec, which is 3.09×10^{24} cm.) Since there is still some uncertainty in this value, we often parametrize the Hubble constant as

$$H_0 = 100h \text{ km/sec/Mpc}, \tag{2.1.11}$$

so that $h \approx 0.7$.

Equation (2.1.8) can be rewritten

$$\Omega - 1 = \frac{K}{H^2 a^2}, \tag{2.1.12}$$

where $\Omega \equiv \rho_{\rm m}/\rho_c$ is the dimensionless density parameter and $\rho_{\rm c}$ is the critical density defined by

$$\rho_{\rm c} \equiv \frac{3H^2}{8\pi G}. \tag{2.1.13}$$

The density parameter tell us which of the three spatial geometries describes our universe, i.e.,

$$\begin{array}{lllllll} \rho > \rho_{\rm c} & \leftrightarrow & \Omega > 1 & \leftrightarrow & K > 0 & \leftrightarrow & \text{closed} \\ \rho = \rho_{\rm c} & \leftrightarrow & \Omega = 1 & \leftrightarrow & K = 0 & \leftrightarrow & \text{flat} \\ \rho < \rho_{\rm c} & \leftrightarrow & \Omega < 1 & \leftrightarrow & K < 0 & \leftrightarrow & \text{open.} \end{array} \tag{2.1.14}$$

Recent observations of the cosmic microwave background (CMB) anisotropy have shown that the current universe is very close to a spatially flat geometry ($\Omega \simeq 1$) [71]. This is actually a natural result from inflation in the early universe [72]. Hence, we will consider a flat universe ($K = 0$) as necessary.

The stress-energy tensor is conserved by virtue of the Bianchi identities, leading to the continuity equation

$$\dot{\rho} = -3H(\rho + P). \tag{2.1.15}$$

This equation can be derived from (2.1.8) and (2.1.9), which means that two of equations (2.1.8), (2.1.9) and (2.1.15) are independent.

Evolution of the scale factor

Let us consider the evolution of the universe filled with a barotropic perfect fluid with an equation of state

$$P = w\rho. \tag{2.1.16}$$

By solving (2.1.15), we obtain

$$\rho \propto \exp\left[-3\int^{a} \frac{da_1}{a_1}(1 + w(a_1))\right]. \tag{2.1.17}$$

When w is assumed to be constant, this can be integrated to derive

$$\rho \propto a^{-3(1+w)}. \tag{2.1.18}$$

The two most popular examples of cosmological fluids are known as non-relativistic matter ($w = 0$) and radiation ($w = 1/3$). The energy densities in the matter or radiation dominated universe are

$$\rho_{\rm m} \propto a^{-3} : \text{ matter}, \tag{2.1.19}$$

$$\rho_{\rm r} \propto a^{-4} : \text{ radiation}. \tag{2.1.20}$$

Solving the Friedmann equation (2.1.8) with $K = 0$ and (2.1.18) yields

$$a \propto (t - t_0)^{\frac{2}{3(1+w)}}. \tag{2.1.21}$$

The scale factors in the matter or radiation dominated universe behave

$$a \propto (t-t_0)^{\frac{2}{3}} : \text{matter}, \tag{2.1.22}$$

$$a \propto (t-t_0)^{\frac{1}{2}} : \text{radiation}. \tag{2.1.23}$$

Both cases correspond to a decelerated expansion of the universe. From (2.1.9), an accelerated expansion ($\ddot{a}(t) > 0$) occurs for the equation of state given by

$$w < -\frac{1}{3}. \tag{2.1.24}$$

In order to explain the current acceleration of the universe, we require an exotic energy dubbed "dark energy" with equation of state satisfying (2.1.24). From (2.1.18), the energy density is constant for $w = -1$. In this case, the Hubble parameter is also constant from (2.1.8), given the evolution of the scale factor:

$$a \propto e^{Ht}, \tag{2.1.25}$$

which is the de-Sitter universe.

Cosmological constant

The exponential expansion also arises by including a cosmological constant, Λ, in the Einstein equation. The Einstein tensor $G_{\mu\nu}$ and the stress-energy tensor $T_{\mu\nu}$ satisfy the Bianchi identity $\nabla_\mu G^\mu{}_\nu = 0$ and energy conservation $\nabla_\mu T^\mu{}_\nu = 0$. There is a freedom to add a term $\Lambda g_{\mu\nu}$ in the Einstein equation because of $\nabla_\mu g_{\nu\lambda} = 0$. Then, the Einstein equation is modified from (2.1.7) to

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{2.1.26}$$

In the FLRW background (2.1.1), the Friedmann equations are modified from (2.1.8) and (2.1.9) to

$$H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} + \frac{\Lambda}{3}, \tag{2.1.27}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) + \frac{\Lambda}{3}. \tag{2.1.28}$$

The cosmological constant term corresponds to the energy density term for $w = -1$. This clearly demonstrates that the cosmological constant contributes negatively to the pressure term and exhibits a repulsive effect. Equation (2.1.27) can be rewritten

$$\Omega + \Omega_K + \Omega_\Lambda = 1, \tag{2.1.29}$$

where $\Omega_K \equiv -K/\left(H^2a^2\right)$ and $\Omega_\Lambda \equiv \Lambda/\left(3H^2\right)$.

2.1.2 Kinematic properties of the expanding universe

Cosmological redshift

Suppose that a wavecrest of light is emitted from a source located at comoving coordinate r at time t, and arrives at the origin ($r = 0$) at time t_0. As a ray of light travels along a null geodesic ($ds^2 = 0$):

$$\frac{dt}{dr} = -\frac{a(t)}{\sqrt{1 - Kr^2}}, \tag{2.1.30}$$

the comoving distance is defined by

$$\chi(t) \equiv \int_t^{t_0} \frac{dt_1}{a(t_1)} = -\int_r^0 \frac{dr_1}{\sqrt{1 - Kr_1^2}}, \tag{2.1.31}$$

where the FLRW metric (2.1.1) is used. The next wavecrest emitted at time $t + \delta t$ will arrive at the origin at time $t_0 + \delta t_0$. Since the source position and the origin are both fixed in the comoving coordinate system, the right-hand-side of (2.1.31) is constant and

$$\int_t^{t_0} \frac{dt_1}{a(t_1)} = \int_{t+\delta t(t)}^{t_0+\delta t(t_0)} \frac{dt_1}{a(t_1)}. \tag{2.1.32}$$

We can select δt as a period of light, and then, $\delta t \ll t$. Tis equation can be Taylor expanded as

$$\frac{\delta t(t)}{a(t)} = \frac{\delta t(t_0)}{a(t_0)}, \tag{2.1.33}$$

The emitted and observed frequencies ν are related by

$$\frac{a(t_0)}{a(t)} = \frac{\delta t(t_0)}{\delta t(t)} = \frac{\nu(t)}{\nu(t_0)}. \tag{2.1.34}$$

The cosmological redshift is now expressed as

$$z \equiv \frac{\nu(t)}{\nu(t_0)} - 1 = \frac{a(t_0)}{a(t)} - 1. \tag{2.1.35}$$

If the universe is expanding, then $a(t_0) > a(t)$ and the light is red-shifted to give positive z. On the other hand, if the universe is contracting, then $a(t_0) < a(t)$ and the light is blue-shifted to negative z. We can also find the relationship to the Hubble parameter:

$$\begin{aligned} H &= (1+z)\frac{d}{dt}\left(\frac{1}{1+z}\right) \\ &= -\frac{1}{1+z}\frac{dz}{dt}. \end{aligned} \tag{2.1.36}$$

Definitions of distances

It is desirable to measure the angular size of an object at a cosmological redshift or the apparent luminosity of that object. Given a class of objects of the same size (standard rulers), we find that the corresponding angular size changes with redshift in a specific way that depends on the values of cosmological parameters. The same is also true for the apparent luminosities of objects with the same total brightness (standard candles). Therefore, if we know the appropriate dependencies for particular classes of standard rulers or standard candles, we can determine the cosmological parameters.

First, we introduce the angular diameter distance. Consider an object of proper diameter D located at coordinate r that emits light at time t. Using the FLRW metric (2.1.1), it follows that the angular diameter of the source θ observed at the origin $r = 0$ is related to D by

$$\theta = \frac{D}{a(t)r}. \tag{2.1.37}$$

The angular diameter distance is defined as

$$\begin{aligned} d_{\mathrm{A}} &\equiv \frac{D}{\theta} \\ &= a(t)r \\ &= \frac{r}{1+z}, \end{aligned} \tag{2.1.38}$$

where we use (2.1.4) and (2.1.35). Solving (2.1.31) yields

$$d_{\mathrm{A}}(z) = \frac{1}{(1+z)} \begin{cases} \dfrac{\sin(\sqrt{K}\chi)}{\sqrt{K}} & (K>0) \\ \chi & (K=0) \\ \dfrac{\sinh(\sqrt{-K}\chi)}{\sqrt{-K}} & (K<0) \end{cases}, \tag{2.1.39}$$

where

$$\begin{aligned} \chi &= \int_0^z \frac{dz_1}{H(z_1)} \\ &= \frac{1}{H_0}\int_0^z \frac{dz_1}{\sqrt{\Omega_{\mathrm{m}0}(1+z_1)^3 + \Omega_{\mathrm{r}0}(1+z_1)^4 + \Omega_{K0}(1+z_1)^2 + \Omega_{\Lambda 0}}} \end{aligned} \tag{2.1.40}$$

using (2.1.31)、(2.1.8) and (2.1.29).

Second, we introduce the luminosity distance. If source at coordinate r emits δN photons with frequency range from ν to $\nu+\delta\nu$ at time t, the luminosity (energy per unit time) in this frequency range is written as

$$\delta L = h\nu\frac{\delta N}{\delta t}, \tag{2.1.41}$$

where h is the Planck constant. Since the number of photon is conserved, the detector at the origin $r=0$ will receive at time t_0 the flux (energy per unit area per unit time) given by

$$\begin{aligned}\delta f &= h\nu_0 \frac{\delta N}{\delta t_0}\frac{1}{4\pi a^2(t_0)r^2} \\ &= \frac{\delta L}{4\pi a^2(t_0)r^2(1+z)^2},\end{aligned} \tag{2.1.42}$$

where (2.1.34), (2.1.35) and (2.1.41) are used. Since this equation is independent of the emitted frequency, the luminosity distance is defined by

$$\begin{aligned}d_{\mathrm{L}} &\equiv \sqrt{\frac{\delta L}{4\pi\delta f}} \\ &= (1+z)r,\end{aligned} \tag{2.1.43}$$

where (2.1.4) is used. Thus, the luminosity distance (2.1.43) and the angular diameter distance (2.1.38) are related as

$$d_{\mathrm{L}} = (1+z)^2 d_{\mathrm{A}}. \tag{2.1.44}$$

Instead of the flux f, astronomers often use the apparent (bolometric) magnitude m defined as

$$m_1 - m_2 = -2.5\log_{10}\frac{f_1}{f_2}. \tag{2.1.45}$$

A difference of five magnitudes should correspond to a ratio of a factor 100 in fluxes. The distance modulus is defined by

$$\begin{aligned}\mu &\equiv m - M \\ &= -2.5\log_{10}\left\{\frac{L}{4\pi d_{\mathrm{L}}^2}\frac{4\pi(10\ \mathrm{pc})^2}{L}\right\} \\ &= 5\log_{10}\frac{d_L}{10\ \mathrm{pc}},\end{aligned} \tag{2.1.46}$$

where the absolute magnitude M is the apparent magnitude which an object would have at a distance of 10 pc.

2.2 Observational constraints on dark energy

2.2.1 Constraints from SN Ia

SuperNovae of type Ia (SNe Ia) are believed to occur when a carbon-oxygen white dwarf star in a binary system accretes sufficient matter from its companion star to push its mass close to the Chandrasekhar mass, which is the maximum possible mass and can be supported by electron degeneracy pressure. The white dwarf becomes unstable as this happens. Then, the increase of temperature and density causes the conversion of carbon and oxygen into ^{56}Ni, and bring about a thermonuclear explosion. Light curves for SN Ia are powered by the radioactive decays of ^{56}Ni at early times, and ^{56}Co after a few weeks. The peak luminosity is determined by the mass of ^{56}Ni produced in the explosion. If the white dwarf is fully burned, we expect $\sim 0.6 M_\odot$ of ^{56}Ni to be produced. As a result, although the detailed mechanism of SN Ia explosions remains uncertain, SNe Ia are expected to have similar peak luminosities. Since they are about as bright as a typical galaxy when they peak, SNe Ia can be observed to distances of several thousand megaparsecs, recommending their utility as standard candles for cosmology.

The SN Ia data released by Riess *et al.* [1] and Perlmutter *et al.* [2] in the redshift regime $0.2 < z < 0.8$ showed that the luminosity distances of observed SN Ia tend to be larger than those predicted in the flat universe without dark energy. Perlmutter *et al.* [2] found that the cosmological constant is present at the 99% confidence level if we assume a flat universe with a dark energy equation of state $w_{\rm DE} = -1$ (i.e. the cosmological constant). According to their analysis, the density parameter of non-relativistic matter today was constrained to be $\Omega_{\rm m0} = 0.28^{+0.09}_{-0.08}$ (68% confidence level) in the flat universe with the cosmological constant.

Over the past decade, more SN Ia data have been collected by a number of high-redshift surveys, such as "SuperNova Legacy Survey" (SNLS) [73], "Hubble Space Telescope" (HST) [74,75] and "Equation of State: SupErNovae trace Cosmic Expansion" (ESSENCE) [76, 77] survey. These data also confirmed that the universe entered the late-time cosmic acceleration epoch after the matter-dominated epoch. If we allow the case in which dark energy is different from the cosmological constant (i.e. $w_{\rm DE} \neq -1$), then observational constraints on $w_{\rm DE}$ and $\Omega_{\rm DE0}$ (or $\Omega_{\rm m0}$) are not so stringent.

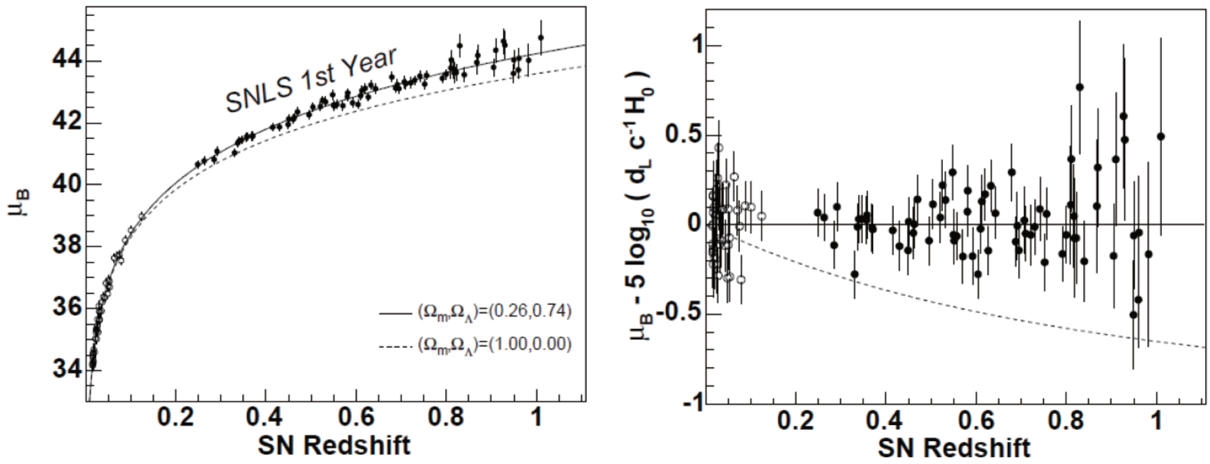

Figure 2.1: Hubble diagrams of SuperNova Legacy Survey (SNLS) and nearby SNe Ia, with various cosmologies superimposed [73]. The upper and bottom panels show the distance moduli plotted against redshift for a sample of SNe Ia. Solid and dashed curves give the distance moduli for cosmological models with $(\Omega_{\rm m0}, \Omega_{\Lambda 0}) = (0.26, 0.74)$ and $(\Omega_{\rm m0}, \Omega_{\Lambda 0}) = (1, 0)$, respectively.

Hubble diagrams of SNLS [73] and nearby data are shown in Fig. 2.1, together with the best fit ΛCDM cosmology for a flat universe. They make use of 44 nearby objects and 71 SNLS objects. In Fig. 2.2, we show the observational contours on $(\Omega_{\Lambda 0}, \Omega_{\rm m0})$ and $(w_{\rm DE}, \Omega_{\rm m0})$ for constant $w_{\rm DE}$ obtained from the "Union08" SN Ia data by Kowalski *et al.* [78]. Clearly the SN Ia data alone are not yet sufficient to place tight bounds on $w_{\rm DE}$.

2.2.2 Constraints from CMB

The presence of dark energy affect the temperature anisotropies in cosmic microwave background (CMB). The position of acoustic peaks in CMB anisotropies depends on the expansion history from the decoupling epoch to the present. Therefore, dark energy leads to the shift for positions of acoustic peaks. There is also another effect called the Integrated-Sachs-Wolfe (ISW) effect [79] induced by the variation of the gravitational potential during the cosmic acceleration epoch. The former effect is typically more important than the latter effect because the ISW effect is limited to large-scale perturbations.

The cosmic inflation in the early universe [80, 81] predicts nearly scale-invariant spectra of density perturbations through the quantum fluctuation of a scalar field. This is consistent with the CMB temperature anisotropies observed by "COsmic Background Ex-

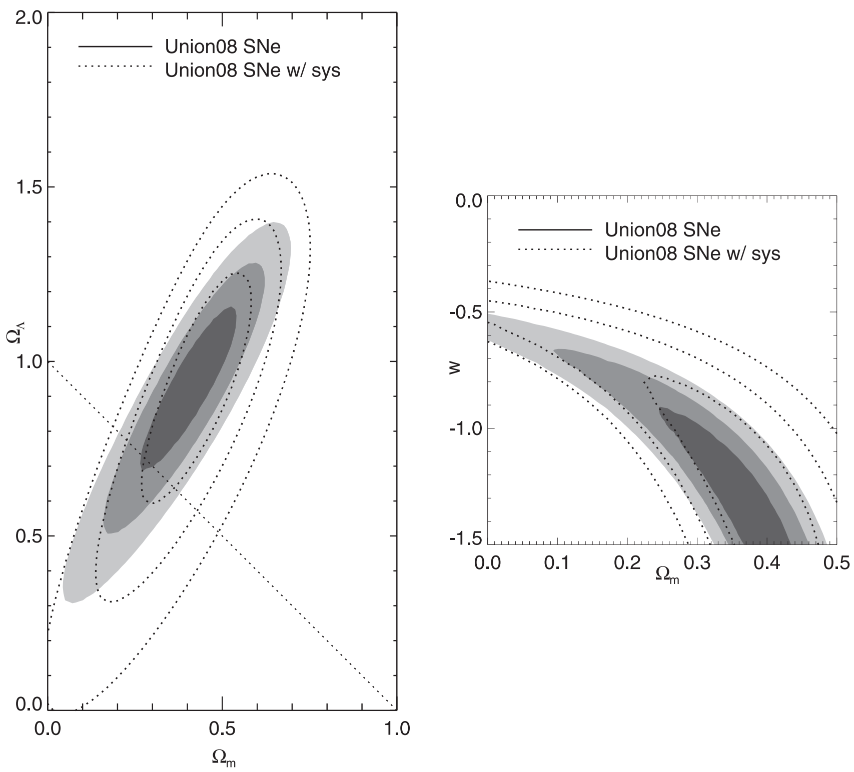

Figure 2.2: The left plot shows contours at 68.3%, 95.4% and 99.7% confidence level on $\Omega_{\Lambda 0}$ and $\Omega_{\rm m0}$ obtained with the Union08 [78] set, without (filled contours) and with (open contours) inclusion of systematic errors. The right plot shows the corresponding confidence level contours on the equation of state parameter w and $\Omega_{\rm m0}$, assuming a constant w.

plorer" (COBE) [82] and "Wilkinson Microwave Anisotropy Probe" (WMAP) [83]. After the scale $\lambda = 2\pi a/k$ (k is a comoving wavenumber) leaves the Hubble radius H^{-1} during inflation ($\lambda > H^{-1}$), perturbations are "frozen" [72]. After inflation, the perturbations cross inside the Hubble radius again ($\lambda < H^{-1}$) and they begin to oscillate as sound waves. This second horizon crossing happens earlier for larger k, i.e., for smaller scale perturbations.

The sound horizon is defined by

$$r_{\rm s} \equiv \int_0^\eta d\eta_1 c_{\rm s}(\eta_1), \tag{2.2.1}$$

where $c_{\rm s}$ is the sound speed and $d\eta \equiv dt/a$. The sound speed is given by

$$
\begin{aligned}
c_{\rm s} &\equiv \sqrt{\frac{\partial P}{\partial \rho}} \\
&= \sqrt{\frac{1}{3\,(1+R_{\rm s})}},
\end{aligned}
\tag{2.2.2}
$$

where

$$
R_{\rm s} \equiv \frac{3\rho_{\rm b}}{4\rho_{\gamma}} = \frac{3\Omega_{\rm b0}}{4\Omega_{\gamma 0}}a, \tag{2.2.3}
$$

and subscripts b and γ mean baryons and photons, respectively. The characteristic angular scale of the CMB acoustic peaks is set by [84]

$$
\theta_{\rm A} \equiv \frac{r_{\rm s}(z_{\rm dec})}{d_{\rm A}^{(\rm c)}(z_{\rm dec})}, \tag{2.2.4}
$$

where $d_{\rm A}^{(\rm c)}$ is the comoving angular diameter distance related with the proper angular diameter distance $d_{\rm A}$ via the relation $d_{\rm A}^{(\rm c)} \equiv d_{\rm A}/a = d_{\rm A}(1+z)$, and $z_{\rm dec} \simeq 1090$ is the redshift at the decoupling epoch. The CMB multipole $\ell_{\rm A}$ that corresponds to the angle (2.2.4) is given by

$$
\ell_{\rm A} = \frac{\pi}{\theta_{\rm A}} = \pi \frac{d_{\rm A}^{(\rm c)}(z_{\rm dec})}{r_{\rm s}(z_{\rm dec})}. \tag{2.2.5}
$$

Using the Friedmann equation (2.1.8), (2.1.39) and (2.1.40) for the redshift $z > z_{\rm dec}$, we obtain [85, 86]

$$
\ell_{\rm A} = \frac{3\pi}{4}\sqrt{\frac{\Omega_{\rm b0}}{\Omega_{\gamma 0}}}\left\{\ln\left(\frac{\sqrt{R_{\rm s}(a_{\rm dec})+R_{\rm s}(a_{\rm eq})}+\sqrt{1+R_{\rm s}(a_{\rm dec})}}{1+\sqrt{R_{\rm s}(a_{\rm eq})}}\right)\right\}^{-1}\mathcal{R}, \tag{2.2.6}
$$

where $a_{\rm dec}$ and $a_{\rm eq}$ correspond to the scale factor at the decoupling epoch and at the radiation-matter equality, respectively, and $\mathcal{R}$ is the so-called CMB shift parameter defined by [87]

$$
\mathcal{R} \equiv \sqrt{\frac{\Omega_{\rm m0}}{\Omega_{\rm K0}}}\sinh\left(\sqrt{\Omega_{K0}}\int_0^{z_{\rm dec}} dz \frac{H_0}{H(z)}\right). \tag{2.2.7}
$$

The CMB shift parameter $\mathcal{R}$ is affected by the cosmic expansion history from the decoupling epoch to the present. This leads to the shift for the multipole $\ell_{\rm A}$. The general relation for all peaks and troughs of observed CMB anisotropies is given by [88, 89]

$$\ell_m = \ell_{\rm A} \left(m - \phi_m \right), \tag{2.2.8}$$

where m represents peak numbers ($m = 1$ for the first peak, $m = 3/2$ for the first trough,...) and ϕ_m is the shift of multipoles. The quantity ϕ_m depends weakly on $\Omega_{\rm b0}$ and $\Omega_{\rm m0}$ for a given cosmic curvature Ω_{K0}. The shift of the first peak can be fitted as $\phi_1 = 0.265$ [89]. The WMAP 7-year bound on the CMB shift parameter is given by [71]

$$\mathcal{R} = 1.725 \pm 0.018, \tag{2.2.9}$$

at the 68% confidence level. Taking $\mathcal{R} = 1.725$ together with other values $\Omega_{\rm b0}h^2 = 0.02249$, $\Omega_{\rm m0}h^2 = 0.1345$ and $\Omega_{\gamma 0}h^2 = 2.469 \times 10^{-5}$ constrained by the WMAP 7-year data, we obtain $\ell_{\rm A} \simeq 300$ from (2.2.6). Using the relation (2.2.8) with $\phi_1 = 0.265$, we find that the first acoustic peak corresponds to $\ell_1 \simeq 220$, as observed in CMB anisotropies.

In the flat universe ($K = 0$), the CMB shift parameter is simply given by

$$\mathcal{R} = \sqrt{\Omega_{\rm m0}} \int_0^{z_{\rm dec}} dz \frac{H_0}{H(z)}. \tag{2.2.10}$$

For smaller $\Omega_{\rm m0}$ (i.e. for larger $\Omega_{\rm DE0}$), $\mathcal{R}$ tends to be smaller. For the cosmological constant ($w_{\rm DE} = -1$), the Hubble parameter is given by

$$H(z) = H_0 \sqrt{\Omega_{\rm m0}(1+z)^3 + \Omega_{\rm DE0}}. \tag{2.2.11}$$

Under the bound (2.2.9), the density parameter is constrained to be $0.70 < \Omega_{\rm DE0} < 0.76$. This is consistent with the bound coming from the SN Ia data. We can also show that, for increasing $w_{\rm DE}$, the observationally allowed values of $\Omega_{\rm m0}$ gets larger. However, $\mathcal{R}$ depends weakly on the $w_{\rm DE}$. Hence, the CMB data alone do not provide a tight constraint on $w_{\rm DE}$. In Fig. 2.3, we show the joint observational constraints on $w_{\rm DE}$ and $\Omega_{\rm m0}$ (for constant $w_{\rm DE}$) obtained from the WMAP 7-year data and the Union2 SN Ia data [90]. The joint observational constraints provide much tighter bounds compared to the individual constraint from CMB and SN Ia. For the flat universe, Amanullah

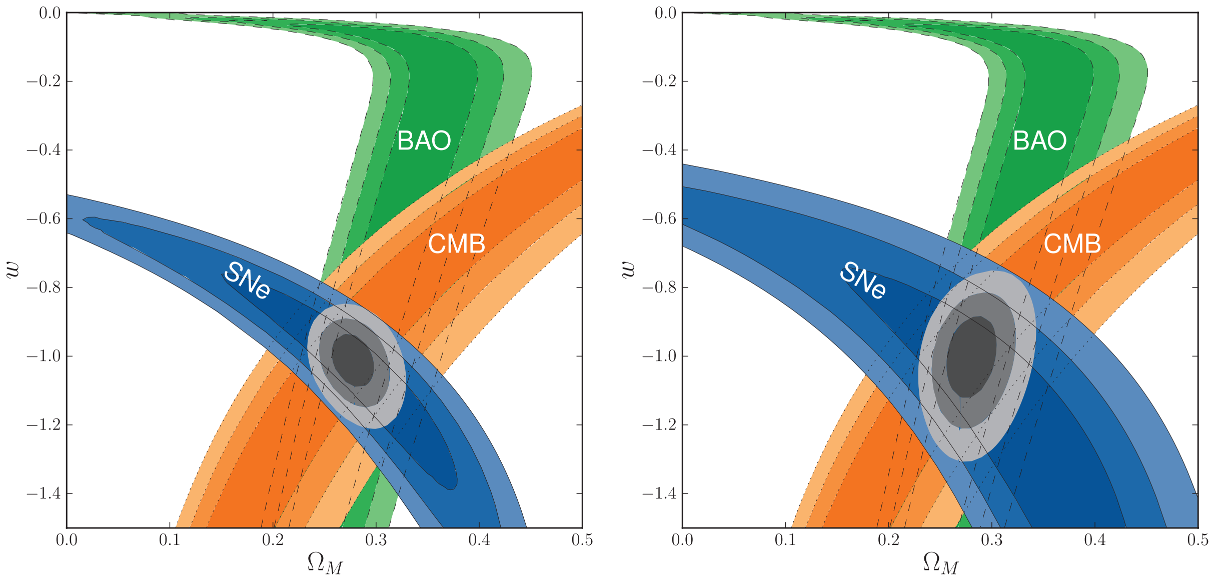

Figure 2.3: 68.3%, 95.4% and 99.7% confidence regions of the $(\Omega_{\rm M0}, w)$ plane from SN Ia combined with the constraints from BAO and CMB both without (left panel) and with (right panel) systematic errors. The flat universe ($K = 0$) and constant w have been assumed [90].

et al. [90] obtained the bounds $w_{\rm DE} = -0.997^{+0.050}_{-0.054}(\text{stat})^{+0.077}_{-0.082}(\text{stat} + \text{sys together})$ and $\Omega_{\rm m0} = 0.269^{+0.019}_{-0.017}(\text{stat})^{+0.023}_{-0.022}(\text{stat} + \text{sys together})$ from the combined data analysis of CMB and SN Ia.

2.2.3 Constraints from BAO

Eisenstein *et al.* [91] first reported the detection of Baryon Acoustic Oscillations (BAO) in a spectroscopic sample of 46, 748 luminous red galaxies observed by the "Sloan Digital Sky Survey" (SDSS). This has provided another test for probing the property of dark energy. Peaks and troughs in the angular power spectrum of CMB temperature anisotropies arise from gravity-driven acoustic oscillations of the coupled photon-baryon fluid in the early universe. After decoupling epoch, photons and baryons decouple and the sound speed of the baryons due to the loss of photon pressure. Sound waves remain imprinted in the baryon distribution and through gravitational interactions in the dark matter distribution as well. Since the sound horizon scale provides a standard ruler calibrated by CMB anisotropies, measurement of the BAO scale in the galaxy distribution provides a geometric probe of the expansion history. Since the impact of baryons on the far larger dark matter component is small, in the galaxy power spectrum, this scale appears as a series of oscillations with amplitude of order 10%. This is more subtle than the acoustic oscillations in CMB.

The location of BAO is determined by the sound horizon at which baryons were released from the Compton drag of photons. This epoch (, called the drag epoch,) occurs at the redshift $z_{\rm d}$. The sound horizon at $z = z_{\rm d}$ is given by $r_{\rm s}(z_{\rm d}) = \int_0^{\eta_{\rm d}} d\eta c_{\rm s}(\eta)$. According to the fitting formula of $z_{\rm d}$ by Eisenstein and Hu [92], $z_{\rm d}$ and $r_{\rm s}(z_{\rm d})$ are constrained to be around $z_{\rm d} \simeq 1020$ and $r_{\rm s}(z_{\rm d}) \simeq 150$ Mpc.

We observe the angular and redshift distributions of galaxies as a power spectrum $P(k_\perp, k_{//})$ in the redshift space, where $k_\perp$ and $k_{//}$ are the wavenumbers perpendicular and parallel to the direction of light respectively. In principle, we can measure the following two ratios [93]

$$\begin{aligned} \theta_{\rm s}(z) &= \frac{r_{\rm s}(z_{\rm d})}{d_{\rm A}^{\rm (c)}(z)}, && (2.2.12) \\ \delta z_{\rm s}(z) &= z_{\rm s}(z_{\rm d}) H(z), && (2.2.13) \end{aligned}$$

where the quantity $\theta_{\rm s}(z)$ characterizes the angle orthogonal to the line of sight, whereas the quantity $\delta z_{\rm s}$ corresponds to the oscillations along the line of sight.

The current BAO observations are not sufficient to measure both $\theta_{\rm s}(z)$ and $\delta z_{\rm s}(z)$ independently. From the spherically averaged spectrum, we can find a combined distance scale ratio given by [93]

$$\left(\theta_{\rm s}^2(z)\delta z_{\rm z}(z)\right)^{1/3} \equiv \frac{r_{\rm s}(z_{\rm d})}{\{(1+z)^2 d_{\rm A}^2(z)/H(z)\}^{1/3}}, \qquad (2.2.14)$$

or, alternatively, the effective distance ratio [91]

$$D_{\rm V}(z) \equiv \left\{\frac{(1+z)^2 d_{\rm A}^2(z) z}{H(z)}\right\}^{1/3}. \qquad (2.2.15)$$

In 2005, Eisenstein *et al.* [91] obtained the constraint $D_{\rm V}(z) = 1370 \pm 64$ Mpc at the redshift $z = 0.35$. In 2010, Percival *et al.* [94] measured the effective distance ratio defined by

$$r_{\rm BAO}(z) \equiv \frac{r_{\rm s}(z_{\rm d})}{D_{\rm V}(z)}, \tag{2.2.16}$$

at the three redshifts: $r_{\rm BAO}(z = 0.2) = 0.1905 \pm 0.0061$, $r_{\rm BAO}(z = 0.275) = 0.1390 \pm 0.0037$ and $r_{\rm BAO}(z = 0.35) = 0.1097 \pm 0.0036$. This is based on the data from SDSS Data Release 7 (DR7) and the 2-degree Field (2dF) Galaxy Redshift Survey. These data provide the observational contour of BAO plotted in Fig. 2.3. Amanullah *et al.* [90] placed the constraints $w_{\rm DE} = -1.009^{+0.050}_{-0.054}({\rm stat})^{+0.077}_{-0.082}({\rm sys + stat\ together})$ and $\Omega_{\rm m0} = 0.277^{+0.014}_{-0.014}({\rm stat})^{+0.017}_{-0.016}({\rm sys + stat\ together})$ for the constant equation of state of dark energy in the flat universe from the joint data analysis of SN Ia [90], WMAP 7-year [71] and BAO data [94]. Hence, the ΛCDM model is well consistent with a number of independent observational data.

Finally, we should mention that there are other constraints coming from the cosmic age [95], large-scale clustering [96], gamma ray bursts [97] and weak lensing [98]. So far we have not found strong evidence for supporting dynamical dark energy models over the ΛCDM model, but future high-precision observations may break this degeneracy.

2.3 Cosmological constant

In 1917, Einstein originally introduced the cosmological constant Λ to achieve a static universe. After Hubble discovered the expansion of the universe in 1929, Λ was dropped by Einstein as it was no longer required. From the point of view of particle physics, however, the cosmological constant naturally arises as an energy density of the vacuum. If Λ originates from the vacuum energy density, the energy scale of it should be much larger than that of the present Hubble constant H_0. Before the accelerated expansion of the universe is discovered [1,2], this "cosmological constant problem" [3] was well known to exist long. There have been a number of efforts to solve this problem.

Attempts to evaluate the value of the energy density of the quantum vacuum lead to very large or divergent results. There is a zero-point energy $\omega/2$ for each mode of a quantum field with mass m, so that the vacuum energy density is given by

$$\rho_{\rm vac} = \frac{1}{2} \sum_{\rm fields} {\rm g}_i \int_0^\infty \frac{d^3k}{(2\pi)^3} \sqrt{k^2 + m^2}, \tag{2.3.1}$$

where ${\rm g}_i$ means the degrees of freedom of the field (the sign of ${\rm g}_i$ is + for bosons and − for fermions) and the sum runs over all quantum fields, e.g., quarks, leptons, gauge fields, etc. This exhibits an ultraviolet divergence: $\rho_{\rm vac} \propto k^4$. However, we expect that quantum field theory is valid up to some cut-off scale $k_{\rm max}$ in which case the integral (2.3.1) is finite:

$$\rho_{\rm vac} \simeq \sum_{\rm fields} \frac{{\rm g}_i k_{\rm max}^4}{16\pi^2}. \tag{2.3.2}$$

On the one hand, Λ is known as of order the present value of the Hubble constant H_0 by observations, that is

$$\begin{aligned} \Lambda &\approx H_0^2 \\ &= \left(2.13h \times 10^{-42} \text{ GeV}\right)^2. \end{aligned} \tag{2.3.3}$$

This corresponds to the density of Λ

$$\begin{aligned} \rho_\Lambda &= \frac{\Lambda}{8\pi G} \\ &\approx 10^{-47} \text{ GeV}^4. \end{aligned} \tag{2.3.4}$$

On the other hand, if we take the cutoff to be the Planck scale ($k_{\rm max} = m_{\rm pl} = 1.22 \times 10^{19}$ GeV) where we expect the quantum field theory in a classical spacetime metric to break down, we find that the vacuum energy density is estimated as

$$\rho_{\rm vac} \approx 10^{74} \text{ GeV}^4, \tag{2.3.5}$$

which is about 10^{121} orders of magnitude larger than the observed value (2.3.4). It is very unlikely that a classical contribution to the vacuum energy density would cancel this quantum contribution to such high precision. Even if we take an energy scale of Quantum ChromoDynamics (QCD) for $k_{\rm max}$, we obtain $\rho_{\rm vac} \approx 10^{-3}$ GeV^4 which is still much larger than ρ_Λ. This very large discrepancy is known as the cosmological constant problem [3].

Supersymmetry (SUSY) which is the hypothetical symmetry between bosons and fermions, appears to provide a resolution of the zero-point energy. In SUSY theories, every fermion in the standard model of particle physics (SM) has an equal-mass SUSY bosonic partner that contributes to the zero point energy with an opposite sign compared to the fermionic degree of freedom, and vice versa. Therefore, fermionic and bosonic zero-point contributions to $\rho_{\rm vac}$ would exactly cancel. However, none of the SUSY particles has yet been observed in collider experiments, so they must be sufficiently heavier than their SM partners. If SUSY is spontaneously broken at a mass scale M, we would expect the imperfect cancellations to generate a finite vacuum energy density $\rho_{\rm vac} \sim M^4$. For a viable SUSY scenario, the SUSY breaking scale should be around $M_{\rm SUSY} \sim 1$ TeV because we want to ensure that no new scales are introduced between the electroweak scale (~ 246 GeV) and the Planck scale. This also leads to a very large discrepancy from the observed value (2.3.4). We do not know how the Planck scale or SUSY breaking scales are really related to the observed vacuum scale.

Chapter 3

Alternative models to ΛCDM model

3.1 Modified matter models

In this section, we shall briefly describe "modified matter models," such as quintessence, K-essence, etc. In these models, the energy-momentum tensor $T_{\mu\nu}$ on the right hand side of the Einstein equation contains an exotic matter source with a negative pressure. Scalar fields naturally arise in particle physics including string theory and these can act as candidates for dark energy. So far a wide variety of scalar-field dark energy models have been proposed. We have to keep in mind that the contribution of the dark matter component needs to be taken into account for a complete analysis. In this section, we shall study a flat FLRW universe ($K = 0$) unless otherwise specified.

3.1.1 Quintessence

A quintessence field [4, 5] is a scalar field with standard kinetic term, minimally coupled to gravity. The scalar field part action takes the form

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{2}\nabla^{\mu}\phi\nabla_{\mu}\phi - V(\phi)\right], \tag{3.1.1}$$

where $V(\phi)$ is the potential of the field. In the flat FLRW background, the variation of the action (3.1.1) with respect to ϕ gives

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0\,. \tag{3.1.2}$$

Taking variation of $g^{\mu\nu}$, we obtain the stress-energy tensor:

$$\begin{aligned} T_{\mu\nu} &= -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} \\ &= \nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}\nabla^\lambda\phi\nabla_\lambda\phi + V(\phi)\right]. \end{aligned} \tag{3.1.3}$$

The energy density and pressure can be derived from the stress-energy tensor as

$$\begin{aligned} \rho &= -T^0{}_0 \\ &= \frac{1}{2}\dot{\phi}^2 + V(\phi), \end{aligned} \tag{3.1.4}$$

$$\begin{aligned} P &= T^i{}_i \\ &= \frac{1}{2}\dot{\phi}^2 - V(\phi)\,. \end{aligned} \tag{3.1.5}$$

Then, the Einstein equations becomes

$$H^2 = \frac{8\pi G}{3}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right], \tag{3.1.6}$$

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3}\left[\dot{\phi}^2 - V(\phi)\right]. \tag{3.1.7}$$

The equation of state for the field ϕ is given by

$$\begin{aligned} w_\phi &= \frac{P_\phi}{\rho_\phi} \\ &= \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)}\,. \end{aligned} \tag{3.1.8}$$

In order to answer what kind of potentials can give rise to acceleration of the present universe, we consider the power-law expansion

$$a(t) \propto t^p, \tag{3.1.9}$$

where we have kept p general, keeping in mind that $p = 1$ corresponds to no acceleration and $p > 1$ corresponds to acceleration. From (3.1.6) and (3.1.7), we obtain the relation $\dot{H} = -4\pi G\dot{\phi}^2$. The field ϕ and the potential $V(\phi)$ can be solved from this relation as

$$\phi = \int \mathrm{d}t \left[-\frac{\dot{H}}{4\pi G}\right]^{1/2}, \tag{3.1.10}$$

$$V = \frac{3H^2}{8\pi G}\left(1 + \frac{\dot{H}}{3H^2}\right), \tag{3.1.11}$$

where we chose the positive sign of $\dot{\phi}$. Hence, The field and the potential driving the power-law expansion correspond to

$$\phi \propto \ln t, \tag{3.1.12}$$

$$V(\phi) \propto \exp\left(-\sqrt{\frac{16\pi}{p}}\frac{\phi}{m_{\mathrm{pl}}}\right). \tag{3.1.13}$$

Although the energy density for radiation (or matter) is much larger than that for the field ϕ during radiation-dominated (or matter-dominanted) era, the field energy density ρ_ϕ needs to dominate at present to be responsible for dark energy. From (2.1.24), the condition for the present cosmic acceleration corresponds to $w_\phi < -1/3$, i.e. $\dot{\phi}^2 < V(\phi)$ from (3.1.8). This means that the potential for the scalar field needs to be flat enough for the field to evolve slowly. If the slowly rolling scalar field satisfying the condition

$$\dot{\phi}^2 \ll V(\phi) \tag{3.1.14}$$

gives the dominant contribution to the energy density of the universe, we obtain the approximate relations

$$3H\dot{\phi} + V_{,\phi} \simeq 0 \tag{3.1.15}$$

and

$$3H^2 \simeq \kappa^2 V(\phi) \tag{3.1.16}$$

from (3.1.2) and (3.1.6), respectively. Then, the equation of state for field in (3.1.8) is approximately given by

$$w_\phi \simeq -1 + \frac{2\epsilon_s}{3}\,, \tag{3.1.17}$$

where $\epsilon_s \equiv \{(dV/d\phi)/V\}^2/(2\kappa^2)$ is the so-called slow-roll parameter [72]. Since the potential is sufficiently flat, $\epsilon_{\rm s}$ is much smaller than unity during the accelerated expansion of the universe. Note that the field equation of state deviates from -1 ($w_\phi > -1$) unlike the cosmological constant.

3.1.2 K-essence

There are often scalar fields with non-canonical kinetic terms in particle physics. The scalar-field action for such theories is generally given by

$$S = \int \mathrm{d}^4x\sqrt{-g}P(\phi, X)\,, \tag{3.1.18}$$

where $P(\phi, X)$ is a function of a scalar field ϕ and its kinetic energy

$$X \equiv -\frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi. \tag{3.1.19}$$

Quintessence relies on the field potential $V(\phi)$ to lead to the late-time accelerated expansion of the universe. Even in the absence of the potential energy of scalar fields, it is possible to realize the cosmic acceleration due to the kinetic energy X. Armendariz-Picon *et al.* [99] originally proposed kinetic energy driven inflation, called "k-inflation," to explain inflation in the early universe. The application of these theories to dark energy was first carried out by Chiba *et al.* [6]. The analysis was extended to a more general Lagrangian by Armendariz-Picon *et al.* [7,8] and this scenario based on the action (3.1.18) was called "k-essence." In these theories with the action (3.1.18), the pressure P_ϕ and the energy density ρ_ϕ of the field are

$$P_\phi = P, \tag{3.1.20}$$

$$\rho_\phi = 2X\frac{\partial P}{\partial X} - P. \tag{3.1.21}$$

Then, the equation of state of k-essence is given by

$$
\begin{aligned}
w_\phi &= \frac{P_\phi}{\rho_\phi} \\
&= \frac{P}{2X\partial P/\partial X - P} \,.
\end{aligned}
\tag{3.1.22}
$$

As long as the condition $|2XP_{,X}| \ll |P|$ is satisfied, w_ϕ can be close to -1.

The action (3.1.18) includes a wide variety of theories, for example, the 'ghost condensate model' [100]. The theories with a negative kinetic energy $-X$ generally suffer from the vacuum instability, but the presence of the quadratic term X^2 can avoid this problem. The model constructed in such context is the ghost condensate model whose Lagrangian is

$$
P = -X + \frac{X^2}{M^4} \,, \tag{3.1.23}
$$

where M is a constant. In this model, we have

$$
w_\phi = \frac{1 - X/M^4}{1 - 3X/M^4} \,, \tag{3.1.24}
$$

which gives $-1 < w_\phi < -1/3$ for $1/2 < X/M^4 < 2/3$. The de Sitter solution ($w_\phi = -1$), in particular, appears when $X/M^4 = 1/2$. We can explain the present cosmic acceleration for $M \simeq 10^{-3}$ eV because the energy density for the field is $\rho_\phi = M^4/4$ at the de Sitter point.

In order to discuss stability conditions of k-essence, we decompose the field into the homogeneous parts and perturbed parts as $\phi(t, \mathbf{x}) = \phi_0(t) + \delta\phi(t, \mathbf{x})$ in the Minkowski background [101]. Then, the second-order Hamiltonian reads

$$
\delta H = \left(\frac{dP}{dX} + 2X \frac{d^2P}{dX^2} \right) \frac{(\delta\dot{\phi})^2}{2} + \frac{dP}{dX} \frac{(\nabla\delta\phi)^2}{2} - \frac{d^2P}{d\phi^2} \frac{(\delta\phi)^2}{2} \,. \tag{3.1.25}
$$

The term $d^2P/d\phi^2$ is related with the effective mass of the field. From the positivity of the first two terms in (3.1.25), we can derive the stability conditions:

$$\frac{dP}{dX} + 2X\frac{d^2P}{dX^2} \geq 0, \tag{3.1.26}$$

$$\frac{dP}{dX} \geq 0. \tag{3.1.27}$$

The propagation speed $c_{\rm s}$ of the field is given by

$$\begin{aligned} c_{\rm s}^2 &= \frac{dP_\phi/dX}{d\rho_\phi/dX} \\ &= \frac{dP/dX}{dP/dX + 2Xd^2P/dX^2}, \end{aligned} \tag{3.1.28}$$

which is positive under the conditions (3.1.27). Furthermore, we also require the sound speed $c_{\rm s} \leq 1$. This is satisfied when

$$\frac{d^2P}{dX^2} > 0\,. \tag{3.1.29}$$

3.2 Modified gravity theories

There is another class of dark energy models in which gravity is modified from general relativity. This class contains $f(R)$ gravity, scalar-tensor theories, Gauss-Bonnet gravity, DGP braneworld model, etc. In this section, we will review “modified gravity theories.”

3.2.1 $f(R)$ gravity

The simplest modification to general relativity is $f(R)$ gravity with the action

$$S = \frac{1}{2\kappa^2}\int {\rm d}^4x\sqrt{-g}f(R) + \int {\rm d}^4x\,\mathcal{L}_{\rm M}, \tag{3.2.1}$$

where f is a function of the Ricci scalar R and $\mathcal{L}_{\rm M}$ is a matter Lagrangian for perfect fluids. The field equation can be derived by varying the action (3.2.1) with respect to $g_{\mu\nu}$:

$$F(R)R_{\mu\nu}(g) - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu F(R) + g_{\mu\nu}\Box F(R) = \kappa^2 T_{\mu\nu}\,, \tag{3.2.2}$$

where $F(R) \equiv df/dR$, and

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_M}{\delta g^{\mu\nu}} \tag{3.2.3}$$

is the energy-momentum tensor of matter. The trace of (3.2.2) is given by

$$3\,\Box F(R) + F(R)R - 2f(R) = \kappa^2 T\,, \tag{3.2.4}$$

where $T \equiv T^{\mu}{}_{\mu}$.

General relativity (without the cosmological constant) corresponds to $f(R) = R$, so that $\Box F(R) = 0$ and (3.2.4) becomes $R = -\kappa^2 T$. Then, the Ricci scalar R is directly determined by the matter T. In $f(R)$ gravity, the term $\Box F(R)$ does not vanish in (3.2.4) generally, which means that there is a propagating scalar degree of freedom, $\phi \equiv F(R)$. The trace equation (3.2.4) determines the dynamics of the scalar field ϕ (dubbed "scalaron" [102]).

The de Sitter point corresponds to a vacuum solution ($T = 0$) at which the Ricci scalar is constant. Since $\Box F(R) = 0$ at this point, we obtain

$$F(R)R - 2f(R) = 0\,. \tag{3.2.5}$$

The model $f(R) = cR^2$ satisfies this condition, and hence it gives rise to an exact de Sitter solution. In fact, the first model of inflation proposed by Starobinsky [102] corresponds to $f(R) = R + cR^2$, in which the inflationary expansion ends when the term cR^2 becomes smaller than the linear term R. This is followed by a reheating stage in which the oscillation of R leads to the gravitational particle production. Dark energy models based on $f(R)$ theories can be also constructed to realize the late-time de Sitter solution satisfying the condition (3.2.5).

In 2002, Capozziello [15] first suggested the possibility of the late-time cosmic acceleration in $f(R)$ gravity. The model takes the form

$$f(R) = R - \frac{\mu^{2(n+1)}}{R^n} \qquad (n > 0). \tag{3.2.6}$$

However, it became clear that this model suffers from a number of problems, such as inability to satisfy local gravity constraints [17, 18], the matter instability [19, 20] and absence of the matter era [21, 22]. The main reason why this model does not work is that the quantity $F_R \equiv d^2 f/dR^2$ is negative.

We can see why the models with negative values of F_R are excluded by considering perturbations. Decomposing quantities, such as $R = R^{(0)} + \delta R$, $F = F^{(0)}(1 + \delta_F)$, $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$ and $T_{\mu\nu} = T^{(0)}_{\mu\nu} + \delta T_{\mu\nu}$, we can expand the trace equation (3.2.4) [18]:

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\delta_F + M^2\,\delta_F = -\frac{\kappa^2}{3F^{(0)}}\delta T\,, \tag{3.2.7}$$

where $\delta T \equiv g^{(0)\mu\nu}\delta T_{\mu\nu}$, and

$$M^2 \equiv \frac{R^{(0)}}{3}\left\{\frac{F(R^{(0)})}{R^{(0)}F_R(R^{(0)})} - 1\right\}. \tag{3.2.8}$$

In the isotropic and homogeneous universe, δ_F is a function of the cosmic time t only and (3.2.7) reduces to

$$\ddot{\delta}_F + M^2\,\delta_F = -\frac{\kappa^2}{3F^{(0)}}\delta T. \tag{3.2.9}$$

Since we consider the models where the deviation from the ΛCDM model is small, we have $F_R(R^{(0)}) \ll 1$ so that $|M^2| \gg R^{(0)}$. If $M^2 < 0$, the perturbation δ_F exhibits a violent instability. Then, the condition $M^2 \simeq F(R^{(0)})/(3F_R(R^{(0)})) > 0$ is needed for the stability of cosmological perturbations. As we will see below, we also require that $F(R^{(0)}) > 0$ to avoid anti-gravity, i.e., to avoid a ghost. Hence, the condition

$$F_R > 0 \tag{3.2.10}$$

needs to hold for avoiding a tachyonic instability associated with the negative mass squared [19].

In order to derive conditions for the avoidance of ghosts, we expand the action (3.2.1) up to the second-order by considering the perturbed metric about the FLRW background:

$$\mathrm{d}s^2 = -(1+2\alpha)\mathrm{d}t^2 - 2a(t)\partial_i\beta\mathrm{d}t\mathrm{d}x^i + a^2(t)\left(\delta_{ij} + 2\psi\delta_{ij} + 2\partial_i\partial_j\gamma\right)\mathrm{d}x^i\mathrm{d}x^j\,, \tag{3.2.11}$$

where α, β, ψ and γ are scalar metric perturbations [103, 104]. We can construct the gauge-invariant curvature perturbation

$$\mathcal{R} \equiv \psi - \frac{H}{\dot{F}}\delta F\,. \tag{3.2.12}$$

Expanding the action (3.2.1) without the matter source, we obtain the second-order action for the curvature perturbation [11]

$$\delta S^{(2)} = \int d^4x \frac{3a^3\dot{F}^2}{2\kappa^2 F[H+\dot{F}/(2F)]^2}\left[\frac{1}{2}\dot{\mathcal{R}}^2 - \frac{1}{2}\frac{1}{a^2}(\nabla\mathcal{R})^2\right]. \tag{3.2.13}$$

If $F < 0$, there is a ghost field because of the negative kinetic energy. Hence, the condition for the avoidance of ghosts is given by

$$F > 0\,. \tag{3.2.14}$$

From local gravity constraints in solar system, $f(R)$ needs to be close to that in the ΛCDM model in the region of high density, i.e., $R \gg R_0$, where R_0 means the cosmological Ricci scalar today. We also require the existence of a stable late-time de Sitter point given in (3.2.5), and we can show that it is stable [11] for

$$0 < \frac{RF_R}{F} < 1. \tag{3.2.15}$$

Then, we can summarize the conditions for the viability of $f(R)$ dark energy models:

(i) $F > 0$ for $R \geq R_0$.

(ii) $F_R > 0$ for $R \geq R_0$.

(iii) $f(R) \to R - 2\Lambda$ for $R \gg R_0$.

(iv) $0 < RF_R/F < 1$ at the de Sitter point satisfying $RF = 2f$.

There are examples of viable models that satisfy all these requirements:

$$\text{(A)}\ f(R) = R - \mu R_c \frac{(R/R_c)^{2n}}{(R/R_c)^{2n}+1}, \tag{3.2.16}$$

$$\text{(B)}\ f(R) = R - \mu R_c \left[1 - \frac{1}{(1+R^2/R_c^2)^n}\right], \tag{3.2.17}$$

$$\text{(C)}\ f(R) = R - \mu R_c \tanh(R/R_c), \tag{3.2.18}$$

where μ, R_c, and n are positive constants, and R_c is roughly of the order of R_0 for $\mu = \mathrm{O}(1)$. The models (A), (B) and (C) have been proposed by Hu and Sawicki [26], Starobinsky [27] and Tsujikawa [28], respectively. If $R \gg R_c$, the models are close to the ΛCDM model ($f(R) \simeq R - \mu R_c$), so that general relativity is recovered in the region of high density. The models (A) and (B) have the asymptotic behavior

$$f(R) \simeq R - \mu R_c \left[1 - (R/R_c)^{-2n}\right] \qquad (R \gg R_c)\,, \tag{3.2.19}$$

which rapidly approaches the ΛCDM model for $n \gtrsim 1$. The model (C) shows an even faster increase of M^2 in the region $R \gg R_c$.

For example, let us consider the model (A). At the de Sitter point, we have the relation:

$$\mu = \frac{(1+x_d^{2n})^2}{x_d^{2n-1}(2+2x_d^{2n}-2n)}\,, \tag{3.2.20}$$

where $x_d \equiv R_1/R_c$ and R_1 is the Ricci scalar at the de Sitter point. The stability condition (3.2.15) gives [11]

$$2x_d^{4n} - (2n-1)(2n+4)x_d^{2n} + (2n-1)(2n-2) \geq 0\,. \tag{3.2.21}$$

The parameter μ has a lower bound determined by this condition. For example, when $n = 1$, we have $x_d \geq \sqrt{3}$ and $\mu \geq 8\sqrt{3}/9$. Under (3.2.21), the conditions for the viability of $f(R)$ dark energy models are also satisfied.

3.2.2 Scalar-tensor theories

In this section, we will see that $f(R)$ gravities are equivalent to Brans-Dicke theory [29] which is one of the simplest examples of scalar-tensor theory. Scalar-tensor theories are some of the most established and well studied alternative theories of gravity. Used as the prototypical way, they are of particular interest since the simple structure of field equations allow exact analytic solutions in physically interesting situations. Scalar-tensor theories arise naturally as the dimensionally reduced effective theories of higher dimensional theories, such as Kaluza-Klein and string models. We can construct viable dark energy models based on Brans-Dicke theory with a constant parameter $\omega_{\rm BD}$.

In the Jordan frame, the 4-dimensional action in Brans-Dicke theory is described by

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}\left[\phi R - \frac{\omega_{\rm BD}}{\phi}\nabla^\mu\phi\nabla_\mu\phi - \frac{1}{2}V(\phi)\right] + S_{\rm M}, \qquad (3.2.22)$$

where $\omega_{\rm BD}$ is the Brans-Dicke parameter which is a constant, $V(\phi)$ is a potential of the scalar field ϕ and $S_{\rm M}$ is an action of matter fields. The original Brans-Dicke theory [29] does not possess the field potential $V(\phi)$. Taking the variation of (3.2.22) with respect to ϕ and $g_{\mu\nu}$, we obtain the equations of motion for ϕ and $g_{\mu\nu}$ are, respectively, obtained as

$$\begin{aligned}
\Box\phi + \frac{\phi}{2\omega_{\rm BD}}\left(-\frac{\omega_{\rm BD}}{\phi^2}\nabla^\mu\phi\nabla_\mu\phi + R - 2\partial_\phi V(\phi)\right) &= \kappa^2 T^{(0)}_{\phi\ \mu\nu}\,, && (3.2.23)\\
\phi G_{\mu\nu} - \frac{\omega_{\rm BD}}{\phi}\left(\nabla_\mu\phi\nabla_\nu\phi - \frac{g_{\mu\nu}}{2}\nabla^\lambda\phi\nabla_\lambda\phi\right) - \nabla_\mu\nabla_\nu\phi & \\
+ g_{\mu\nu}\left(g^{\lambda\sigma}\nabla_\lambda\nabla_\sigma\phi + V(\phi)\right) &= \kappa^2 T^{(0)}_{\mu\nu}\,, && (3.2.24)
\end{aligned}$$

where $T^{(0)}_{\phi\ \mu\nu}$ and $T^{(0)}_{\mu\nu}$ are the stress-energy tensor for matter fields obtained by taking variations of ϕ and $g_{\mu\nu}$, respectively. The $f(R)$ gravity of the metric formalism (3.2.1) can be cast into the form of the above Brans-Dicke theory by setting

$$\phi = F(R), \quad \omega_{\rm BD} = 0, \quad V = \frac{F(R)R - f(R)}{2}. \qquad (3.2.25)$$

More general theories called scalar-tensor theory can be considered, in which R is coupled to a scalar field ϕ. The general 4-dimensional action for scalar-tensor theory can be written as

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}h(\phi)R - \frac{1}{2}\omega(\phi)\nabla^\mu\phi\nabla_\mu\phi - U(\phi)\right] + S_{\mathrm{M}}, \tag{3.2.26}$$

where $h(\phi)$ and $U(\phi)$ are functions of ϕ. Under the conformal transformation $\tilde{g}_{\mu\nu} = hg_{\mu\nu}$, we obtain the action in the Einstein frame [105]

$$S_{\mathrm{E}} = \int \mathrm{d}^4x\sqrt{-\tilde{g}}\left[\frac{1}{2}\tilde{R} - \frac{1}{2}\tilde{\nabla}^\mu\tilde{\phi}\tilde{\nabla}_\mu\tilde{\phi} - V(\tilde{\phi})\right] + \tilde{S}_{\mathrm{M}}, \tag{3.2.27}$$

where $V = U/h^2$. A new scalar field $\tilde{\phi}$ to make the kinetic term canonical is

$$\tilde{\phi} \equiv \int \mathrm{d}\phi\sqrt{\frac{3}{2}\left(\frac{dh/d\phi}{h}\right)^2 + \frac{\omega}{h}}\,. \tag{3.2.28}$$

These theories have the very useful property of being 'conformally equivalent' to general relativity. Under a transformation of the metric that changes scales but not angles, we can find a new metric that obeys the Einstein equation, with the scalar contributing as an ordinary matter field. However, scalar-tensor theories are not the same as general relativity since the metric that couples to matter fields must also transform. In the theory recovered after conformal transformation, the metric obeys field equations similar to these in general relativity, but with an unusual matter content that does not follow geodesics of the new metric with the exception of radiation fields, or null geodesics, which are themselves conformally invariant. This property can sometimes allow field equations to be manipulated into more familiar forms, which allow solutions to be found more easily.

3.2.3 Gauss-Bonnet models

In construction of $f(R)$ gravity, we consider modification to the Einstein-Hilbert action by introducing a general function of the Ricci scalar. This is a very special case. We can extend $f(R)$ gravity to more general theories in which the Lagrangian density f is an arbitrary function of all the infinite and possible scalars made out of the Riemann tensor and its derivatives [30], such as $R^{\mu\nu}R_{\mu\nu}$, $R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma}$, $\nabla^\mu R^\nu{}_\mu\nabla^\lambda R_{\nu\lambda}$, etc. $R_{\mu\nu}$ and $R_{\mu\nu\lambda\sigma}$ are Ricci tensor and Riemann tensor, respectively. However, we usually encounter serious problems of such theories. For example, besides the graviton, there typically appears another spin-2 ghost which has a kinetic term with an opposite sign [31].

In order to remove these spin-2 ghosts, we first introduce the Lovelock scalars [106] which are particular combinations and contractions of the Riemann tensor. The theory with Lovelock scalars does not include higher than third-order derivatives of $g_{\mu\nu}$ in the equations of motion. For example, in 1917, Kretschmann pointed out that the form of the Lagrangian cannot be determined from only general covariance. Instead of the Ricci scalar, he introduced the so-called Kretschmann scalar $R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma}$. The Riemann tensor $R_{\mu\nu\lambda\sigma}$ is a fundamental tensor in gravity theories, we think that this action is well motivated. Moreover, in this theory, Bianchi identities hold, as both sides of the equations of motion are covariantly conserved. However, there are third-order derivatives in the equations of motion.

The appearance of these terms in the equation of motion can be avoided by taking a Gauss-Bonnet combination [12, 13]

$$\mathcal{G} \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma} \,. \tag{3.2.29}$$

Then, the action for this invariant takes the form

$$S = \int d^4x\sqrt{-g}\,\mathcal{G}. \tag{3.2.30}$$

There are only the terms up to second-order derivatives of the metric in the equations of motion coming from this action. We can consider another class of general Gauss-Bonnet theories with a self-coupling of the form [13]

$$S = \int d^4x\sqrt{-g}\,\left[\frac{1}{2\kappa^2}R + f(\mathcal{G})\right]\,, \tag{3.2.31}$$

where $f(\mathcal{G})$ is a function of the Gauss-Bonnet term.

At first glance, the number of the Lovelock scalars looks infinite. In four dimensions, however, the only non-zero Lovelock scalars are the Ricci scalar R and the Gauss-Bonnet term $\mathcal{G}$ because of topological reasons. Therefore, in four dimensions, we can study the theories with the action

$$S = \int d^4x\sqrt{-g}\, f(R, \mathcal{G})\,. \tag{3.2.32}$$

This theory will not introduce spin-2 ghosts.

3.2.4 DGP braneworld model

The so-called Dvali-Gabadadze-Porrati (DGP) braneworld model [14] is considered as a model which realize the "self accelerating universe" even without dark energy because of the existence of the extra-dimensions. In the DGP braneworld model, the 3-dimensional brane is embedded in the 5-dimensional Minkowski bulk spacetime with large extra-dimensions. Newton gravity can be recovered by adding a 4-dimensional Einstein-Hilbert action sourced by the brane curvature to the 5-dimensional action. While the standard 4-dimensional gravity is recovered for small distances, the 5-dimensional gravity appears for large distances. The accelerated expansion of the present universe can be realized without introducing dark energy.

The action for the DGP braneworld model is given by

$$S = \frac{1}{2\kappa^2_{(5)}} \int d^5X\sqrt{-\tilde{g}}\,\tilde{R} + \frac{1}{2\kappa^2_{(4)}} \int d^4X\sqrt{-g}R - \int d^5X\sqrt{-\tilde{g}}\,\mathcal{L}_{\mathrm{M}}\,, \tag{3.2.33}$$

where $\tilde{g}_{AB}$ is the metric in the 5-dimensional bulk, $g_{\mu\nu} = \partial_\mu X^A \partial_\nu X^B \tilde{g}_{AB}$ is the induced metric on the brane, $X^A(x^c)$ are the coordinates of an event on the brane labelled by x^c. $\kappa^2_{(5)}$ and $\kappa^2_{(4)}$ are the 5-dimensional and 4-dimensional gravitational constants, respectively. The first and second terms in (3.2.33) correspond to Einstein-Hilbert actions in the 5-dimensional bulk and on the brane. The matter action is composed of matter localized on the 3-dimesional brane. The action for it is given by $\int d^4x\sqrt{-g}\left(\sigma + \mathcal{L}^{\mathrm{brane}}_{\mathrm{M}}\right)$, where σ is the 3-dimensional brane tension and $\mathcal{L}^{\mathrm{brane}}_{M}$ is the Lagrangian density on the brane.

The equation of motion in the 5-dimensional bulk is

$$G^{(5)}_{AB} = 0, \tag{3.2.34}$$

where $G^{(5)}_{AB}$ is the 5-dimensional Einstein tensor. The 4-dimensional Einstein equation is given by

$$G_{\mu\nu} - \frac{1}{r_c}(K_{\mu\nu} - Kg_{\mu\nu}) = \kappa^2_{(4)} T_{\mu\nu}\,, \tag{3.2.35}$$

where $K_{\mu\nu}$ is the extrinsic curvature on the brane and $T_{\mu\nu}$ is the stress-energy tensor for localized matter. The continuity equation $\nabla^\mu T_{\mu\nu} = 0$ can be derived because $\nabla^\mu(K_{\mu\nu} - g_{\mu\nu}K) = 0$. The cross-over scale r_c is defined by

$$r_\mathrm{c} \equiv \frac{\kappa^2_{(5)}}{2\kappa^2_{(4)}}. \tag{3.2.36}$$

The modified Friedmann equation on the flat FLRW brane ($K = 0$) takes the form

$$H^2 - \frac{\epsilon}{r_c}H = \frac{\kappa^2_{(4)}}{3}\rho_\mathrm{M}\,, \tag{3.2.37}$$

where $\epsilon = \pm 1$, and ρ_M is the energy density of matter on the brane. When r_c is much larger than the Hubble radius H^{-1}, the first term in (3.2.37) dominates over the second one. In this case, the standard Friedmann equation is recovered.

The two branches of the solution of (3.2.37) are often called the *normal* branch ($\epsilon = -1$) and the *self-accelerating* branch ($\epsilon = +1$). In the *self-accelerating* branch, if the universe is dominated by non-relativistic matter ($\rho_\mathrm{M} \propto a^{-3}$), the universe approaches a de Sitter solution

$$H_\mathrm{dS} = \frac{1}{r_\mathrm{c}}. \tag{3.2.38}$$

We can think that the present cosmic acceleration occurs when r_c is of the order of the present Hubble radius H_0^{-1}. This self acceleration is the result of gravitational leakage into extra-dimensions at large distances. In the *normal* branch, such cosmic acceleration is not realized.

3.3 Local void models

There are works to explain the apparent accelerated expansion of the present universe by inhomogeneities in the distribution of matter without the need for dark energy. One of such approaches is the "local void model." In this model, there are the underdense void which realizes the faster cosmic expansion compared to the outer overdense region. That is, we live near the center of the void and can interpret the evolution of the void as an apparent cosmic acceleration. These models can be represented by the Lemaître-Tolman-Bondi (LTB) spacetime [39–41].

3.3.1 LTB spacetime

A spherically symmetric spacetime with only non-relativistic matter, or dust, is described by the Lemaître-Tolman-Bondi (LTB) metric [39–41]:

$$ds^2 = -dt^2 + \frac{(R'(t,r))^2}{1-k(r)r^2}dr^2 + R^2(t,r)\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{3.3.1}$$

where $k(r)$ is an arbitrary function of r. Then, the Einstein equation (2.1.7) reduce to

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{2GM(r)}{R^3} - \frac{k(r)r^2}{R^2}, \tag{3.3.2}$$

$$4\pi\rho(t,r) = \frac{M'(r)}{R^2R'}, \tag{3.3.3}$$

where $M(r)$ is an arbitrary function of only r, and $\rho(t,r)$ is the energy density of the dust fluid. The solutions to (3.3.2) depend on the sign of $k(r)$ and can be expressed in parametric form: For $k(r) > 0$, we have

$$R(t,r) = \frac{M(r)}{k(r)r^2}(1-\cos\eta), \tag{3.3.4}$$

$$t - t_s(r) = \frac{M(r)}{\{k(r)r^2\}^{\frac{3}{2}}}(\eta - \sin\eta), \tag{3.3.5}$$

where $t_s(r)$ is an arbitrary function of only r. For $k(r) = 0$, we have

$$R(t,r) = \left(\frac{9}{2}\right)^{\frac{1}{3}} M^{\frac{1}{3}}(r)\{t - t_s(r)\}^{\frac{2}{3}}. \tag{3.3.6}$$

For $k(r) < 0$, we have

$$R(t,r) = \frac{M(r)}{-k(r)r^2}(\cosh\eta - 1), \tag{3.3.7}$$

$$t - t_s(r) = \frac{M(r)}{\{-k(r)r^2\}^{\frac{3}{2}}}(\sinh\eta - \eta). \tag{3.3.8}$$

The area radius $R(t,r)$ vanishes at $t = t_s(r)$, so that $t_s(r)$ is called the big-bang time. The solutions admit three arbitrary functions $k(r)$, $M(r)$ and $t_s(r)$, but due to one degree of freedom in rescaling r, only two of them are independent. By appropriately choosing the profile of these two arbitrary functions, we can construct LTB cosmological models that can reproduce the observed SN Ia distance-redshift relation.

3.3.2 Local void models

In the model introduced by Tomita [35, 36] soon after the first discovery of the apparent cosmic acceleration of the present universe, there are a local homogeneous void and the outer overdense region, which are described by homogeneous FLRW spacetimes with a singular mass shell. The former is described by an open FLRW spacetime, and the latter tends to the Einstein-de Sitter (EdS) spacetime in the limit of radial infinity. In this model, by adjusting the Hubble constants H_0 and the density parameters Ω_{M0} in the two regions, we can find that the theoretical distance-redshift relation of SN Ia is consistent with the observed relation. At that time, the boundary is at the distance of $z \approx 0.07$ from the center of the void.

Afterwards, general inhomogeneous models using LTB spacetimes are presented by Iguchi, Nakamura and Nakao [42]. The distance-redshift relation of SN Ia can be reproduced in the concordant model. Then, they showed that a critical point appears at an intermediate radius ($z < 1.7$) when they assume the big-bang time and asymptotic vanishing spatial curvature. By improving the model introduced by Iguchi *et al.*, when we assume only the uniform big-bang time, Yoo, Kai and Nakao [51] showed that we can derive the LTB void models which reproduce the distance-redshift relation in the concordant model and have no critical point.

On the other hand, Kasai [50] found that SN Ia data can be divided into the low-

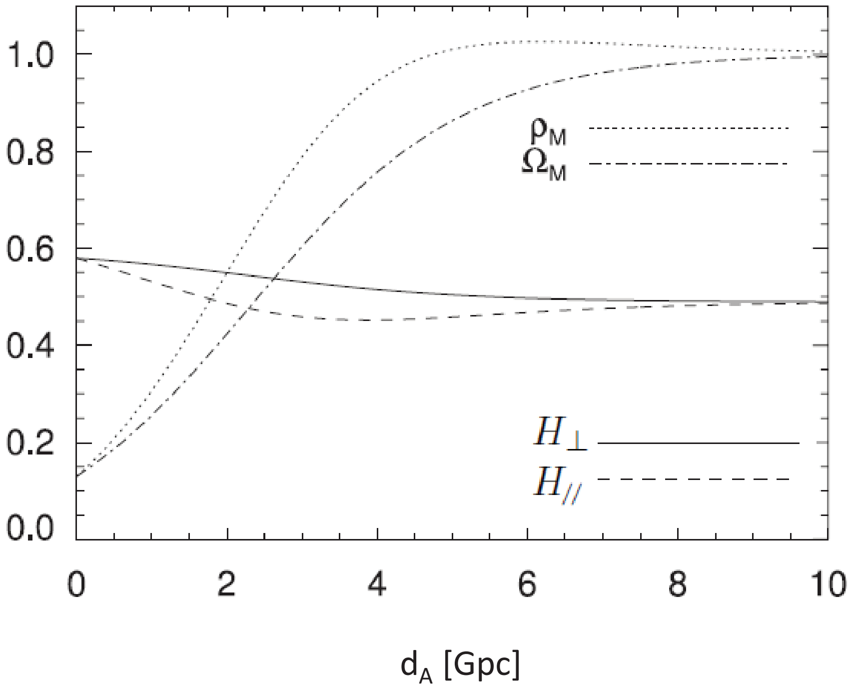

Figure 3.1: The radial dependence of the matter density in units of the critical density ($\rho_{\rm m}$), of density parameter (Ω_m) and of the transverse and longitudinal expansion rates ($H_\perp$ and $H_{//}$) in units of 100 km/s/Mpc [52]. The radial axis is the angular diameter distance ($d_{\rm A}$), and everything is taken at the present time for the observer of the center.

redshift group ($z < 0.2$) and the high-redshift group ($z > 0.3$), which correspond to higher and lower Hubble constants. He also suggested that the different trend of the data about redshifts shows the inhomogeneity of the universe.

We will show the example of local void models which can successively reproduce the observed distance-redshift relation below. From (3.3.2), the effective density parameters of matter and the spatial curvature today can be shown, respectively, as

$$\Omega_{\rm m0}(r) \equiv \frac{2GM}{R_0^3 H_{\perp 0}^2}, \tag{3.3.9}$$

$$\begin{aligned}\Omega_{K0}(r) &\equiv 1-\Omega_{\rm m0}(r) \\ &= -\frac{kr^2}{R_0^2 H_{\perp 0}^2},\end{aligned} \tag{3.3.10}$$

where

$$H_\perp(t,r) \equiv \frac{\dot{R}}{R} \tag{3.3.11}$$

denotes the Hubble parameter in the transverse direction. For example, local void models can be written by choosing $\Omega_{\rm m0}(r)$ and $H_{\perp 0}$ in the following form [52] (see also [46] for another choice)

$$\Omega_{\rm m0}(r) \equiv \Omega_{\rm out} + (\Omega_{\rm in} - \Omega_{\rm out}) \frac{1 - \tanh((r - r_0)/2\Delta)}{1 + \tanh(r_0/2\Delta)}, \tag{3.3.12}$$

$$H_{\perp 0}(r) \equiv H_{\perp 0,{\rm out}} + (H_{\perp 0,{\rm in}} - H_{\perp 0,{\rm out}}) \frac{1 - \tanh((r - r_0)/2\Delta)}{1 + \tanh(r_0/2\Delta)}, \tag{3.3.13}$$

where r_0 and Δ describe a size of void and thickness, and "in" and "out" represent quantities inside and outside the void, respectively.

Figure 3.1 show a plot of $\rho_{\rm M}$, $\Omega_{\rm M}$, $H_\perp$ and $H_{/\!/}$ for this model, as a function of the angular diameter distance today, $d_{\rm A}$, where

$$H_{/\!/}(t, r) \equiv \frac{\dot{R}'}{R'} \tag{3.3.14}$$

denotes the Hubble parameter in the longitudinal direction.

Chapter 4

Previous attempts to test alternative models to ΛCDM model

4.1 Previous tests of modified gravity theories

4.1.1 Parameterized post-Newtonian framework

In this section, we will introduce the "Parameterised Post-Newtonian" (PPN) framework [107] that contains a wide class of different gravitational theories, and that contains parameters which can be constrained by observations. Observers can apply observational results to constrain a wide class of theories without considering the details of the individual theories themselves. Theorists can straightforwardly constrain new theories by comparing to the already established bounds on the PPN parameters without recalculating individual gravitational phenomena. Then, this approach has been highly successful.

In the weak gravitational system, such as the solar system, where gravitation is weak enough for Newton's theory of gravity, there are the relation between the velocity $\tilde{v}$ and the Newtonian gravitational potential $\tilde{U}$

$$\begin{aligned} v^2 \sim U &= \frac{GM}{c^2 R} \\ &\sim 2 \times 10^{-6} \left(\frac{M}{M_\odot}\right) \left(\frac{R}{R_\odot}\right)^{-1}, \end{aligned} \tag{4.1.1}$$

where $v \equiv \tilde{v}/c$ and $U \equiv \tilde{U}/c^2$ are dimensionless in geometrized units ($c = G = 1$), and

$G = 6.7 \times 10^{-8}$ cm^{3}g^{-1}s^{-2}, $M_\odot = 2.0 \times 10^{33}$ g and $R_\odot = 7.0 \times 10^{10}$ cm are the gravitational constant, the solar mass and the solar radius, respectively. Then, we can find that

$$U_\odot \sim 10^{-5}, \quad U_{\mathrm{earth}} \sim 10^{-11}, \quad U_{\mathrm{NS}} \sim 10^{-1}, \quad U_{\mathrm{BH}} \sim 10^{0}, \tag{4.1.2}$$

in the solar system, the system of earth, neutron star and black hole, respectively. When we begin to demand accuracies greater than a part in 10^5, the Newtonian limit no longer suffices. For example, it cannot account for Mercury's additional perihelion shift of $\sim 5 \times 10^{-7}$ radians per orbit. Then, we need a more accurate approximation to the metric that goes "post"-Newtonian theory.

The PPN framework is a perturbative treatment of weak-field gravity, and therefore requires a small parameter. For this purpose, we define an order of smallness:

$$U \sim v^2 \sim \frac{P}{\rho} \sim \Pi \sim \mathrm{O}\left(\epsilon^2\right), \tag{4.1.3}$$

where P is the pressure of the fluid, ρ is its rest-mass density, Π is the ratio of energy density to rest-mass density and ϵ meas the small parameter. The single powers of velocity v are $\mathrm{O}(\epsilon)$, U^2 is $\mathrm{O}(\epsilon^4)$, Uv is $\mathrm{O}(\epsilon^3)$, $U\Pi$ is $\mathrm{O}(\epsilon^4)$, and so on. Derivatives with respect to time also have an order of smallness associated with them, relative to spatial derivatives:

$$\frac{|\partial/\partial t|}{|\partial/\partial x|} \sim \mathrm{O}(\epsilon). \tag{4.1.4}$$

The PPN framework proceeds as an expansion in this order of smallness.

Equations of motion for time-like particles show that the approximation required to recover the Newtonian limit is g_{00} to $\mathrm{O}(\epsilon^2)$. The post-Newtonian limit for time-like particles requires

$$g_{00} \text{ to } \mathrm{O}\left(\epsilon^4\right), \quad g_{0i} \text{ to } \mathrm{O}\left(\epsilon^3\right), \quad g_{ij} \text{ to } \mathrm{O}\left(\epsilon^2\right). \tag{4.1.5}$$

In order to obtain the Newtonian limit of null particles, we have to know the metric to background order. Light follows straight lines, to Newtonian accuracy. Then, it requires g_{00} and g_{ij} both to $\mathrm{O}(\epsilon^2)$.

Dynamical fields should be perturbed from background values, and perturbations should be assigned an appropriate order of smallness each. The appropriate expansion is usually

$$g_{00} = -1 + h_{00}^{(2)} + h_{00}^{(4)} + \mathrm{O}\left(\epsilon^{6}\right), \tag{4.1.6}$$

$$g_{0i} = h_{0i}^{(3)} + \mathrm{O}\left(\epsilon^{5}\right), \tag{4.1.7}$$

$$g_{ij} = \delta_{ij} + h_{ij}^{(2)} + \mathrm{O}\left(\epsilon^{4}\right), \tag{4.1.8}$$

where superscripts in brackets denote the order of smallness. When the theory contains an additional scalar field, then the usual expansion is

$$\phi = \phi_0 + \phi^{(2)} + \phi^{(4)} + \mathrm{O}\left(\epsilon^{6}\right), \tag{4.1.9}$$

where ϕ_0 is the background value of ϕ. We can also specify additional vector and tensor gravitational fields in a corresponding way.

The stress-energy tensor in the PPN framework is taken to be that of a perfect fluid. In a similar manner, we can derive the components of this:

$$T_{00} = \rho(1 + \Pi + v^2 - h_{00}) + \mathrm{O}\left(\epsilon^{6}\right), \tag{4.1.10}$$

$$T_{0i} = -\rho v_i + \mathrm{O}\left(\epsilon^{5}\right), \tag{4.1.11}$$

$$T_{ij} = \rho v_i v_j + P\delta_{ij} + \mathrm{O}\left(\epsilon^{6}\right). \tag{4.1.12}$$

Taking these expressions, and substituting in the perturbed expressions for the fields, the field equations can be solved for order by order in the smallness parameter.

The first step is to solve for $h_{00}^{(2)}$. We then proceeds to solve for $h_{ij}^{(2)}$ and $h_{0i}^{(3)}$ simultaneously with this solution, and finally $h_{00}^{(4)}$ can be solved for. To find $h_{ij}^{(2)}$, $h_{0i}^{(3)}$ and $h_{00}^{(4)}$, we need to specify a gauge. We have the freedom to make gauge transformations of the form $x^\mu \to x^\mu + \xi^\mu$, where $\xi^\mu \lesssim \mathrm{O}\left(\epsilon^2\right)$. This freedom should be used to obtained into the "standard post-Newtonian gauge". In this gauge, the spatial part of the metric is diagonal, and terms containing time derivatives are removed.

In order to derive the appropriate form of the metric, we need to follow that couples to matter fields in the weak-field limit. The result can be compared to the PPN metric below:

$$
\begin{aligned}
g_{00} &= -1 + 2GU - 2\beta G^2U^2 - 2\xi G^2\Phi_W + (2\gamma + 2 + \alpha_3 + \beta_1 - 2\xi)G\Phi_1 \\
&\quad +2(1 + 3\gamma - 2\beta + \beta_2 + \xi)G^2\Phi_2 + 2(1 + \beta_3)G\Phi_3 - (\beta_1 - 2\xi)G\mathcal{A} \\
&\quad +2(3\gamma + 3\beta_4 - 2\xi)G\Phi_4, \qquad (4.1.13) \\
g_{0i} &= -\frac{1}{2}(3 + 4\gamma + \alpha_1 - \alpha_2 + \beta_1 - 2\xi)GV_i - \frac{1}{2}(1 + \alpha_2 - \beta_1 + 2\xi)GW_i, \qquad (4.1.14) \\
g_{ij} &= (1 + 2\gamma GU)\delta_{ij}, \qquad (4.1.15)
\end{aligned}
$$

where β, γ, ξ, β_1, β_2, β_3, β_4, α_1, α_2 and α_3 are the 'post-Newtonian parameters', and Φ_W, Φ_1, Φ_2, Φ_3, Φ_4, $\mathcal{A}$, V_i and W_i are the 'post-Newtonian gravitational potentials' [107]. They have particular physical significance.

Comparison of the metric in the theory with the PPN metric allows us to find values for the PPN parameters. The great utility of the PPN formalism is that observers can take the PPN metric and can constrain the parameters without having a particular theory in mind. These constraints can be applied directly to a number of gravitational theories, without working out how complicated gravitational phenomena work in each theory. In general relativity, for example, we have

$$
\begin{aligned}
&\beta = \gamma = 1, \\
&\xi = \beta_1 = \beta_2 = \beta_3 = \beta_4 = \alpha_1 = \alpha_2 = \alpha_3 = 0. \qquad (4.1.16)
\end{aligned}
$$

Other theories predict other values for these parameters.

Observations that involve only null geodesics are sensitive to the Newtonian part of the metric $g_{00}^{(2)}$ and $g_{ij}^{(2)}$ that involve the PPN parameter γ only. In order to constrain on γ, we can use constraints on the bending of light by the Sun. Using the PPN metric, the predicted bending of light is [107]

$$
\begin{aligned}
\theta &= 2(1+\gamma)\frac{M_\odot}{R_\odot} \\
&= \frac{(1+\gamma)}{2}\theta_{\mathrm{GR}}, \qquad (4.1.17)
\end{aligned}
$$

where $\theta_{\rm GR}$ is the prediction in general relativity. Using the observed value of θ gives [107]

$$\gamma - 1 = (-1.7 \pm 4.5) \times 10^{-4}, \tag{4.1.18}$$

which is consistent with the value of $\gamma = 1$ in general relativisty. Similarly, we can use the PPN metric to find that the Shapiro time delay effect is given by [107]

$$\Delta t = \frac{(1+\gamma)}{2} \Delta t_{\rm GR}. \tag{4.1.19}$$

Taking the observed value of Δt gives the even tighter constraint [107]

$$\gamma - 1 = (2.1 \pm 2.3) \times 10^{-5}, \tag{4.1.20}$$

again consistent with $\gamma = 1$. In fact, the bending of light by the Sun and the Shapiro time delay effect constrain the same aspect of space-time geometry.

4.1.2 Parameterized post-Friedmann framework

Similar approaches to the PPN framework have been developed, which describe the deviations from the standard perturbed universe in general relativity on cosmological scale, such as [108]. Since there is freedom to modify evolution over the whole of cosmic time and over a range of scales on cosmological scales, such parameterizations are not unique unlike the PPN framework. These approaches have been dubbed the "Parameterized Post-Friedmann" (PPF) framework as we will show below.

On the one hand, they introduce two free functions directly into the modified Poisson equation and the 'slip relation', i.e., the transverse and traceless component of field equations, as these are the expressions relevant to observables, such as weak lensing of galaxies and measures of structure growth. One of these free functions acts as an effective rescaling of gravitational constant G, while the other is defined as the ratio of the two potentials that describe the perturbed metric in the conformal Newtonian gauge. We will refer to this approach as 'phenomenology-based.' In this approach, the free functions should be considered as indication of non-general relativistic behavior. A disadvantage of phenomenological-based parameterizations is their tendency to obscure which regions they correspond to, because there is no direct mapping between the free functions and the parameters of a specific theory. Therefore, it is difficult to translate constraints on the two free functions into those on a given modified gravity theory.

On the other hand, they consider directly specifying quantities describing a 4×4 tensor of scalar modifications to the linearly perturbed Einstein equations [109]. Then, we can derive the corresponding Poisson equation and slip relation, which will contain components of this new tensor. We call this approach 'mathematics-based.'

Mathematics-based PPF framework

In the mathematics-based approach, a parameterization was proposed by writing modifications as an additional tensor to the Einstein equations in general relativity:

$$G_{\mu\nu} = 8\pi G_0 a^2 T^M_{\mu\nu} + a^2 U_{\mu\nu}, \tag{4.1.21}$$

where $T^M_{\mu\nu}$ is the stress-energy tensor for all standard fluids and the tensor $U_{\mu\nu}$ may contain metric, matter and additional field degrees of freedom coming from a modified gravity theory. We assume that all known matter fields which are part of $T^M_{\mu\nu}$ couple to the same metric $g_{\mu\nu}$. Then, the stress-energy tensor of matter obeys its usual conservation equations, and hence $U_{\mu\nu}$ is separately conserved.

We parameterize around the linearly perturbed equation of (4.1.21). In order to compare with the phenomenology-based PPF framework below directly, we will specialize to the case of purely metric theories, that is, those for which the action is constructed from curvature invariants, e.g., $f(R)$ gravity, or non-local gravities. The most general perturbations of $U_{\mu\nu}$ in a second-order purely metric theory are as follows [109]:

$$\begin{aligned}
-a^2\delta U^0{}_0 &= k^2 A_0 \hat{\Phi}, \\
-a^2\delta U^0{}_i &= k B_0 \hat{\Phi}, \\
a^2\delta U^i{}_i &= k^2 C_0 \hat{\Phi} + k C_1 \dot{\hat{\Phi}}, \\
a^2\delta U^i{}_j &= D_0 \hat{\Phi} + \frac{D_1}{k} \dot{\hat{\Phi}},
\end{aligned} \tag{4.1.22}$$

where *dots* denote derivatives with respect to conformal time η and k is the Fourier wavenumber. The gauge-invariant metric perturbation $\hat{\Phi}$ reduces to the curvature perturbation Φ in the conformal Newtonian gauge which is defined by

$$ds^2 = -a^2(1+2\Psi)d\eta^2 + a^2(1+2\Phi)d\vec{x}^2. \tag{4.1.23}$$

Further below we introduce a second gauge-invariant metric perturbation, $\hat{\Psi}$ which reduces to Ψ in this gauge.

The coefficients A_0, B_0, C_0, C_1, D_0 and D_1 are functions of background quantities, $A_0 = A_0(k,\eta)$ etc., and the factors of k are chosen such that these coefficients are dimensionless. However, these functions are not all independent. Perturbations of the Bianchi identity $\nabla_\mu U^\mu{}_\nu = 0$ yield additional constraint equations which can be used to reduce the six free functions in (4.1.22) down to just two. One of these is defined by

$$\frac{D_1}{k} = \frac{\tilde{g}}{\mathcal{H}}, \tag{4.1.24}$$

where $\mathcal{H} \equiv \dot{a}/a$ and

$$\tilde{g} \equiv -\frac{1}{2}\left(A_0 + 3\frac{\mathcal{H}}{k}B_0\right). \tag{4.1.25}$$

Then, in Fourier space, the modified Poisson equation can be written by

$$k^2\hat{\Phi} = 4\pi G_{\rm eff}a^2\rho\Delta. \tag{4.1.26}$$

where the gauge-invariant comoving density perturbation $\rho\Delta$ is a summation over all conventional cosmologically-relevant fluids, and

$$G_{\rm eff} \equiv \frac{G}{1-\tilde{g}} \tag{4.1.27}$$

is an effective (modified) gravitational constant.

We choose the second free function to be D_0, which we will hereafter relabel as $\zeta = \zeta(k,\eta)$ to distinguish it from its appearance in the more general format of (4.1.22). The function ζ appears in the 'slip relation' between the potentials $\hat{\Phi}$ and $\hat{\Psi}$ which are equal to one another in general relativity:

$$\hat{\Phi} + \hat{\Psi} = -8\pi G_0 a^2(\rho + P)\Sigma + \zeta\hat{\Phi} + \frac{\tilde{g}}{\mathcal{H}}\dot{\hat{\Phi}}, \tag{4.1.28}$$

where the anisotropic stress of matter Σ is negligible after the radiation-dominated era.

Note that the slip relation and the modified gravitational constant are not independent, as the function $\tilde{g}$ appears in both. This special case of the mathematics-based PPF framework does not capture the behavior of many modified gravity theories, because we have not allowed for additional degrees of freedom to appear [109]. However, it is directly comparable to a phenomenological-based PPF framework. Hence, we will regard (4.1.26) and (4.1.28) simply as possible alternatives to (4.1.31) and (4.1.32).

Phenomenology-based PPF framework

We can find that it is enough to work with only two of the four Einstein field equations, i.e., the Poisson equation and the slip equation. The only observable modifications to gravity at least at the perturbative level will be modifications to the these equations:

$$k^2\Phi - 4\pi G_0 a^2\rho\Delta = F_1, \tag{4.1.29}$$

$$(\Psi + \Phi) + 8\pi G_0 a^2(\rho + P)\Sigma = F_2, \tag{4.1.30}$$

where F_1 and F_2 are arbitrary functions of time and space. We take a simple ansatz that $F_1 = -\alpha k^2\Phi$ and $F_2 = \zeta\Phi$. Then, we can find

$$k^2\Phi = 4\pi G_{\text{eff}} a^2\rho\Delta, \tag{4.1.31}$$

$$\Psi = -8\pi G_0 a^2(\rho + P)\Sigma - (1 - \zeta)\Phi. \tag{4.1.32}$$

The phenomenology-based PPF framework is defined in the conformal-Newtonian gauge, so that Φ and Ψ replace $\hat{\Phi}$ and $\hat{\Psi}$ in (4.1.26) and (4.1.28). This parameterization suggests that, once the anisotropic stress has become negligible, a modified gravity theory could potentially modify one of equations (4.1.31) or (4.1.32) while leaving the other unchanged. This behavior does not arise analytically from theories below third order [109], but given enough freedom in the functional ansatz the numerical behavior that occurs in any theory can be realized with this parameterization.

4.2 Previous tests of local void models

In order to justify the local void model as a viable alternative model to the standard ΛCDM model, we have to test this model by various observations other than the distance-redshift relation of SN Ia. A number of papers for this purpose have appeared recently, studying constraints from observations, such as BAO [43], the kinematic Sunyaev-Zeldovich (kSZ) effect [44, 45], the CMB temperature anisotropy [43, 46, 47], the redshift drift [110], cosmological perturbations [111, 112], etc. Many local void models with a small, a few hundreds Mpc size void are already ruled out, but at present, the models with a huge, Gpc size void still remain to be tested.

Apart from the quest of alternatives to dark energy, the LTB metric may also serve as a toy model for getting some insights into the dynamics and possible observational effects of non-linear perturbations in the standard ΛCDM cosmology. The LTB metric can incorporate a cosmological constant in a straightforward manner, hence being able to describe, as an exact solution, highly non-linear inhomogeneities —almost arbitrary in magnitude, as long as being spherically symmetric— in the FLRW universe with dark energy. In view of this, it is also worth attempting to derive some analytic formulae that can be used to make theoretical predictions of the Λ-LTB spacetimes and rigorous comparison with cosmological observations.

4.2.1 Constraints from BAO

When we calculate the BAO scale in local void models, we have to consider that the transverse and the longitudinal expansion rates are different. In order to compare with observations, we need to make two assumptions. The first assumption is to neglect the effects of the growth of perturbations because the evolution of perturbations in the LTB model is not clearly understood [111, 112]. The second assumption is to consider that the sound horizon r_{s} is given by recombination physics in the FLRW cosmology (see also Sec.

2.2.3) because it is true that the BAO scale is imprinted in the sky at early times when the universe can be regarded as homogeneous enough.

As we showed in Sec. 2.2.3, the observed BAO scales brought from the galaxy samples in the surveys, such as SDSS DR7 and the 2dF Galaxy Redshift Survey, are used to constrain the values of the effective distance ratio (2.2.15) which is the average of the transverse and logitudinal scales. Alexander, Biswas, Notari and Vaid [43] constrained the lower limit of the size of the void from observations of the effective distance ratio defined by (2.2.16) at $z = 0.2$ and $z = 0.35$ [113]:

$$\begin{aligned} r_{\mathrm{BAO}}(z = 0.2) &= 0.1980 \pm 0.0058, \\ r_{\mathrm{BAO}}(z = 0.35) &= 0.1094 \pm 0.0033. \end{aligned} \tag{4.2.1}$$

In the ΛCDM model, $r_{\mathrm{BAO}}(z = 0.2)$ and $r_{\mathrm{BAO}}(z = 0.35)$ can be reproduced well. However, in the models with about Mpc void size, these observed effective distance ratios (4.2.1) cannot be reproduced. This is because, in such models, the domains $z = 0.2$ and $z = 0.35$ belong to the outer region with the EdS model.

In order to make the local void model be consistent with the observations of the BAO scale, the epoch $z = 0.35$ must belong to the inner underdense region, so that the size of the void must be Gpc size. Thus, the models whose void size is smaller than about Gpc size are ruled out.

4.2.2 Constraints from kSZ effect

If we live at the center of the large (about Gpc size) void, observers in other clusters that is not at the center will see a large dipole in the CMB temperature anisotropy. We can regard such a dipole as the kinematic Sunyaev-Zeldovich (kSZ) effect.

The (first-order/thermal) Sunyaev-Zeldovich (SZ) effect is a small spectral distortion of the CMB radiation caused by the interaction of CMB photons with electrons that have high energies because of their temperature. CMB photons that pass through the hot center of massive clusters collide with electrons there, and take a small distortion in the CMB spectrum due to the inverse-Compton scattering, in which the low energy CMB photons receive energy from the high energy cluster electrons.

The (second-order) kSZ effect is an additional spectral distortion because of the Doppler effect of the cluster velocities. If the clusters are moving with respect to the CMB rest frame, CMB photons gain energy from electrons with high energies because clusters where electrons belong to have bulk velocity. The kSZ effect can be ued to map the cosmic peculiar velocity field inside our own light cone.

García-Bellido and Haugbølle [44] constrained the upper limit of the size of the void from available obserbations of the kSZ effect. In local void models, the CMB photon received by an observer of the center of the void is emitted at the outer overdense region. The Hubble constant outside the void ($H_{\rm out}$) is smaller than that in inner underdense region ($H_{\rm in}$). Then, a cluster in the distance r from the center of the void has a velocity

$$v_{\rm pec} = (H_{\rm in} - H_{\rm out})\, r, \tag{4.2.2}$$

relative to the CMB rest frame, and $v_{\rm pec}$ can be observed as a peculiar velocity of the cluster in the kSZ effect. They showed that a strong constraint is given to the Gpc void models. using the observed kSZ effect from 9 clusters. The size of the void is smaller than about 1.5 Gpc.

On the other hand, Yoo, Nakao and Sasaki [45] studied the amount of inhomogeneity in the bang time $t_{\rm s}(r)$ that is need to fit the obserbations of the kSZ effect. They tested the same dataset considered in [44] and found their void model ruled out because of the high peculiar velocities. They considered the case of an inhomogeneous decoupling hypersurface. A large kSZ effect is owing to a large cluster velocity, but this can be relieved if there are radial inhomogeneities in the non-relativistic matter on the decoupling hypersurface.

4.2.3 Constraints from CMB temperature anisotropies

From the anisotropic relation between the luminosity distance and the redshift of the observation of SN Ia, Alnes and Amarzguioui [114] have constrained the observer's position to about 200 Mpc from the center of the void. However, Alnes and Amarzguioui [47] have presented much tighter constraints by numerically using the observation of the CMB dipole. An observer near the center of the void and comoving with the LTB metric has a peculiar velocity with respect to the void model. This can be represented b y $\Delta v = \Delta H \cdot d_{\rm obs}$ and is due to the inhomogeneous Hubble expansion rate. This gives rise to a Doppler shift in the CMB temperature and a consequent dipole [47]:

$$\begin{aligned} a_{10} &= \sqrt{\frac{4\pi}{3}}\frac{\Delta v}{c} \\ &= \sqrt{\frac{4\pi}{3}}\frac{h_{\rm in}-h_{\rm out}}{3000\,{\rm Mpc}}\,d_{\rm obs}\,. \end{aligned} \tag{4.2.3}$$

Typical void models have $h_{\rm in} - h_{\rm out} \sim 0.2$, so that the distance from an observer to the center is about 20 Mpc by using the observed CMB dipole.

In Chapter 6, we will discuss analytical results for off-center CMB anisotropies in general spherically-symmetric spacetimes and update the constraints concerning the location of the observers in the void model by applying our analytic dipole formula with the latest WMAP data [67].

4.2.4 Tests by using redshift drift

Yoo, Kai and Nakao [110] studied the redshift drift, i.e., the time derivative of the cosmological redshift in LTB void models. They showed that the redshift drift is one of decisive differences between LTB void models and the standard ΛCDM model. Although the redshift drift in the ΛCDM model is positive, that in LTB void models is negative. Their results suggest that we can determine whether the local void model is a viable model or not by observing the redshift drift.

In the same way as we showed in Sec. 2.1.2, we can define the cosmological redshift in the general LTB spacetime. The null geodesic equation in the LTB spacetime

$$\frac{dt(r)}{dr} = -\frac{R'}{\sqrt{1-kr^2}} \tag{4.2.4}$$

leads to

$$\frac{d\,\delta t(r)}{dr} = -\frac{\dot{R}'}{\sqrt{1-kr^2}}\delta t. \tag{4.2.5}$$

Then, we have the equation for the redshift z as

$$\frac{dz}{dr} = (1+z)\frac{\dot{R}'}{\sqrt{1-kr^2}}. \tag{4.2.6}$$

Equations (4.2.4) and (4.2.6) can be rewritten in the following form

$$\begin{aligned} \frac{d}{dz}\delta z &= \frac{1}{1+z}\delta z + \frac{\ddot{R}'}{\dot{R}'}\delta t, \qquad (4.2.7) \\ \frac{d}{dz}\delta t &= -\frac{1}{1+z}\delta t. \qquad (4.2.8) \end{aligned}$$

We can easily integrate (4.2.8) to derive

$$\delta t = \frac{1}{1+z}\delta t_0. \tag{4.2.9}$$

By using above results, (4.2.7) is rewritten with the equation of the redshift drift

$$\frac{d}{dz}\left(\frac{\delta z}{1+z}\right) = \frac{1}{(1+z)^2}\frac{\ddot{R}'}{\dot{R}'}\delta t_0. \tag{4.2.10}$$

Thus, if we know the signs of $\dot{R}'$ and $\ddot{R}'$, we can find the sign of δz, i.e., the redshift drift $\delta z/\delta t$ because we know $\delta z = 0$ at $z = 0$.

Near the center of the general spherically symmetric LTB spacetime, we have

$$\begin{aligned} \dot{R}'\Big|_{t=t_0,r=0} &= H_0, \qquad (4.2.11) \\ \ddot{R}'\Big|_{t=t_0,r=0} &= -\frac{1}{2}\Omega_{\mathrm{m}0}H_0^2, \qquad (4.2.12) \end{aligned}$$

because of the regularity of the metric at $r = 0$, i.e., $R(t,r) \sim a(t)r$ near the center. Then, in the neighborhood of the symmetric center, (4.2.10) becomes

$$\begin{aligned} \frac{d}{dz}\delta z\Big|_{t=t_0,r=0} &= \frac{\ddot{R}'}{\dot{R}'}\Big|_{t=t_0,r=0}\delta t_0 + \mathrm{O}(z) \\ &= -\frac{1}{2}\Omega_{\mathrm{m}0}\delta t_0 + \mathrm{O}(z). \qquad (4.2.13) \end{aligned}$$

From this equation, we have

$$\begin{aligned} \frac{\delta z}{\delta t_0} &= -\frac{1}{2}\Omega_{\mathrm{m0}} z + \mathrm{O}\left(z^2\right) \\ &\leq 0, \end{aligned} \tag{4.2.14}$$

and then, we can find that the redshift drift is non-positive near the center in the general LTB spacetime.

In the LTB void model, we have

$$\dot{R}' > 0 \tag{4.2.15}$$

because of (4.2.6), and

$$\begin{aligned} \ddot{R}' &= \frac{4\pi G R'}{R^3}\left(-\rho R^3 + 2\int_0^r dr_1 \rho(t, r_1) R^2(t, r_1) R'(t, r_1)\right) \\ &= -4\pi G \frac{R'}{R^3}\int_0^R dR_1 \left(\frac{d\rho}{dR_1} R_1^3 + \rho R_1^2\right) \\ &< 0 \end{aligned} \tag{4.2.16}$$

because $R' > 0$ from (3.3.3) by the definition of LTB void models, i.e., $M' > 0$, and $d\rho/dR = \rho'/R' > 0$. Then, the eqaution of the redshift drift (4.2.10) in the LTB void models becomes

$$\frac{d}{dz}\left(\frac{\delta z}{1+z}\right) < 0. \tag{4.2.17}$$

From this inequality, it can be seen that the redshift drift is negative:

$$\frac{\delta z}{\delta t} < 0 \tag{4.2.18}$$

for $z > 0$ because $\delta z = 0$ at $z = 0$.

On the other hand, in the ΛCDM model, we find that, for $z < 2$,

$$\begin{aligned} \frac{\ddot{R}'}{\dot{R}'} &= \frac{\ddot{a}}{\dot{a}} \\ &> 0, \end{aligned} \tag{4.2.19}$$

and then, the redshift drift is positive:

$$\frac{\delta z}{\delta t} > 0. \tag{4.2.20}$$

Yoo *et al.* [110] showed that, assuming the mass density of the dust is positive, the redshift drift of an off-center source is negative unlike that in the ΛCDM model. Thus, when we observe the redshift drift, we can get a strong constraint on LTB void models. If the redshift drift turns out to be positive at some redshift, we can reject LTB void models. From, for example, the observation of compact binary stars by DECIGO [119] or BBO [120], the redshift drift at $z \simeq 1$ can be measured [118].

Chapter 5

High frequency limit in modified gravity theories

In general relativity, a consistent expansion scheme for short-wavelength perturbations and the corresponding effective stress-energy tensor were largely developed by Isaacson [63, 64], in which the small parameter, say ϵ, corresponds to the amplitude and at the same time the wavelength of perturbations. Isaacson's expansion scheme is called the "high frequency limit" or the "short-wavelength approximation." In this expansion, the dominant order of the Einstein equation with respect to this parameter ϵ corresponds to the equations of motion for linearized gravitational waves in the ordinary perturbation theory, and is in fact divergent as ϵ^{-1}. The next order of the expansion of the Einstein equation provides the Einstein equation for the background metric with an effective stress-energy tensor, which is essentially given as minus the second-order Einstein tensor averaged over a spacetime region of several wavelengths of metric perturbations. Since taking a derivative of perturbations corresponds, roughly speaking, to multiplying the inverse of the smallness parameter (or the inverse of the wavelength of perturbations), the effective stress-energy tensor consisting of the square of derivatives of the first-order metric perturbations can have large effects on the background dynamics. Furthermore it can be shown that the effective stress-energy tensor thus constructed is gauge-invariant, hence has a physical meaning. If the effective stress-energy tensor had a term proportional to the background spacetime metric, then it would correspond to adding a cosmological constant to the effective Einstein equations for the background metric, thereby explaining possible origin of dark energy from local inhomogeneities. It has been shown, however, that this effective gravitational stress-energy tensor is traceless and satisfies the weak energy condition, i.e. acts like radiation [65, 66], and thus cannot provide any effects that imitate dark energy in general relativity.

However, it is far from obvious if this traceless property of the effective gravitational stress-energy tensor is a nature specific only to Einstein gravity or is rather a generic property that can also hold in other types of gravity theory. The purpose of this chapter is to address this question in a simple, concrete model in the cosmological context. Since $f(R)$ gravity contains higher-order derivative terms, one can anticipate the effective gravitational stress-energy tensor to be generally modified in the high frequency limit.

Over the past decade, cosmological implications of $f(R)$ gravity theories have been extensively studied especially in the quest of finding an alternative cosmology to Λ-CDM model. A viable class of $f(R)$ theories is summarized in Sec. 3.2.1. Although it is desirable to examine all these cosmologically favored models, in this chapter, we will restrict our attention to the simplest model $f(R) = R + cR^2$ with an eye to applications to analyses of more generic cases, which are left for future study. This model itself is not considered as a cosmologically favored modified gravity theory for describing the present accelerating universe, but has rather been introduced as a prototype of an inflationary universe model by Starobinsky [102]. However, this simple model can be viewed as the leading term truncation of a more generic class of $f(R)$ theories that take an analytic form with respect to R around the vacuum solution $R = 0$ and therefore provides, as the first step toward this line of research, a good starting point for our analysis. It would also be interesting to check whether or not a once-excluded model can possibly revive as a cosmologically favored model, due to the inclusion of the backreaction effects of local inhomogeneities.

As we have showed in Sec. 3.2.2, it is well known that $f(R)$ gravity is equivalent to a scalar-tensor theory which contains the coupling of the scalar curvature R to a scalar field ϕ in a certain way [11, 17]. The Brans-Dicke theory [29] is one of the simplest examples. Our analysis can therefore be performed, in principle, either (i) by first translating a given $f(R)$ theory into the corresponding scalar-tensor theory and then inspecting the stress-energy tensor for the scalar field ϕ, or (ii) by directly dealing with metric perturbations of the $f(R)$ theory. We may expect that the former approach is much easier than the latter metric approach, as one has to deal with metric perturbations of complicated combinations of the curvature tensors in the latter case. Nevertheless, we will take both approaches. In fact, in the metric approach, by directly taking up perturbations of the scalar curvature R, the Ricci tensor $R_{\mu\nu}$ and the Riemann tensor $R^{\mu}{}_{\nu\lambda\sigma}$ involved in a given $f(R)$ theory, we can learn how to generalize our present analysis of a specific class of $f(R)$ gravity to analyses of other, different, types of modified gravity theory that cannot even be translated into a scalar-tensor theory, such as the Gauss-Bonnet gravity.

The theory we consider in this chapter contains higher-order derivative terms. As in the case of general relativity, we can consider short-wavelength metric perturbations with a small parameter ϵ and expand the field equations with respect to ϵ. In contrast to Einstein gravity, the dominant part of the field equations for this theory is of order $O(\epsilon^{-3})$. In $\mathrm{O}(\epsilon^{-1})$, we have equations of motion for linearized gravitational waves. In $\mathrm{O}(1)$ we obtain equations for the background metric with a source term arising from the short-wavelength perturbations. This source term contains a number of higher-order derivatives of metric perturbations. However, as one cannot have any meaningful notion of stress energy for gravitational waves in a local sense (at least within a wavelength), we have to take a suitable spacetime average over several wavelengths. Also, since we are interested in backreaction effects on the cosmological dynamics, we assume that our background metric takes the form of the FLRW metric. We also assume that in the limit to Einstein gravity, i.e., $f(R) \to R$ (as $c \to 0$), the field equations for our $f(R)$ theory reduce to those for Einstein gravity in the corresponding order of the expansion parameter. At this point, a number of terms that involve higher-order derivatives of metric perturbations vanish by the spacetime averaging procedure and the assumption of background FLRW symmetry. Eventually, besides the terms corresponding to Isaacson's formula in Einstein gravity, only a few terms that contain higher-order derivatives of metric perturbations can remain in the effective stress-energy tensor for short-wavelength perturbations in our modified gravity. Furthermore, in this case, the resultant effective stress-energy tensor is shown to be traceless as in Einstein gravity case.

It is interesting to note that if we drop one of our assumptions that our $f(R)$ theory reproduces the field equations for Einstein gravity in the limit $c \to 0$, then the resultant effective stress-energy tensor is no longer traceless and in fact acquires a term proportional to the background metric and therefore can have effects that mimic a cosmological constant. However, as we will see later, the cosmological constant term depends on an undetermined constant and there does not seem to be a way to determine the value of the constant within the framework of the theory considered.

We briefly comment on previous work along the similar line. Since the effective stress-energy tensor can be used to measure energy flux carried out by gravitational radiation from astrophysical sources, such as inspiral binary systems, it can be used by near-future gravitational wave detectors to test various modified gravity theories [115–123]. For this purpose, the effective stress-energy tensor for gravitational radiation has been derived by Sopuerta and Yunes [124] by applying Isaacson's scheme to the field equations of dynamical Chern-Simons theory. A more general formalism to compute the effective stress-energy tensor from the effective action, which can apply to a wide class of modified gravity theories, has been proposed by Stein and Yunes [125]. As a concrete example, the formula has been applied to dynamical Chern-Simons gravity as well as theories with dynamical scalar fields coupled to higher-order curvature invariants. It has been shown that in these modified theories the stress-energy tensor for gravitational radiations reduces, at future null infinity, to that in Einstein gravity. Also, Berry and Gair [126] have derived the effective stress-energy tensor for gravitational waves in $f(R)$ gravity which is analytic around the vacuum $R = 0$. However, since the main concern in these studies is to test alternative theories in the astrophysical context by using gravitational wave detectors, the formulas mentioned above have been formulated for asymptotically flat spacetimes (as the energy flux of gravitational waves needs to be evaluated at future null infinity) and therefore do not appear to apply to cosmological models. In contrast, our analysis will proceed by exploiting the cosmological setup that our background metric possesses FLRW symmetry.

In the next section, before going into the effective stress-energy tensor in modified gravity theories, we will first briefly summarize the high frequency limit in general relativity. In Sec. 5.2, we consider the high frequency limit in $f(R)$ gravity theory. Based on Isaacson's scheme, we expand the field equations for $f(R) = R + cR^2$ theory and first derive the general expression of the effective stress-energy tensor for gravitational perturbations in our $f(R)$ gravity. Then, assuming that our background metric has FLRW symmetry and also that the resulting equations reduce to the corresponding equations for Einstein gravity in the limit $c \to 0$, we see that the effective stress-energy tensor, whose expression is significantly simplified, is in fact traceless as in Einstein gravity case. As briefly mentioned above, when a given $f(R)$ gravity is translated into the corresponding scalar-tensor theory, the scalar field ϕ, which expresses an extra degree of freedom in the $f(R)$ theory, possess a non-trivial potential term. [Compare with the earlier work by Lee [127] on a computation of the effective stress-energy tensor in a scalar-tensor theory with vanishing potential term.] In Sec. 5.3, we will make sure that the effective stress-energy tensor in Brans-Dicke theory is consistent with that in our $f(R)$ gravity. We will also see that, in the Einstein frame, the traceless property of the effective stress-energy tensor is shown to hold in more generic circumstances.

5.1 High frequency limit in general relativity

In this section, we introduce our notation by recapitulating Isaacson's expansion scheme for short-wavelength gravitational perturbations in general relativity.

Let $g_{\mu\nu}$ be the metric with linear perturbation $h_{\mu\nu}$; it is described by $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$ with $g^{(0)}_{\mu\nu}$ being the background metric including the backreaction from perturbations. The amplitude of $h_{\mu\nu}$ is of order $h_{\mu\nu} \sim \mathrm{O}(\epsilon)$ with ϵ being the small parameter, which also corresponds to the wavelength λ of perturbations compared with the background characteristic curvature radius, L. The order of derivatives of $h_{\mu\nu}$ are

$$\nabla_{\mu_1}\nabla_{\mu_2}\cdots\nabla_{\mu_m} h_{\nu\lambda} \sim \mathrm{O}\left(\frac{\epsilon}{(\lambda/L)^m}\right) \sim \mathrm{O}\left(\epsilon^{1-m}\right), \tag{5.1.1}$$

where ∇_μ denotes the covariant derivative with respect to $g^{(0)}_{\mu\nu}$, so that $\nabla_\mu g^{(0)}_{\nu\lambda} = 0$. We may bear in mind perturbations of the form $h \sim \epsilon \sin(x/\lambda)$ and $\lambda/L \sim \mathrm{O}(\epsilon)$. In what follows we normalize $L \sim 1$. The inverse metric takes the form $g^{\mu\nu} = g^{(0)\mu\nu} - h^{\mu\nu} + h^{\mu}{}_{\lambda} h^{\lambda\nu} + \cdots$, where $h^{\mu\nu} \equiv g^{(0)\mu\lambda} g^{(0)\nu\sigma} h_{\lambda\sigma}$.

There is the general relationship between the Ricci tensor of $g_{\mu\nu}$ and that of $g^{(0)}_{\mu\nu}$, namely

$$R_{\mu\nu} = R_{\mu\nu}[g^{(0)}] + 2\nabla_{[\lambda}C^{\lambda}_{\nu]\mu} + 2C^{\lambda}_{\sigma[\lambda}C^{\sigma}_{\nu]\mu}\,, \tag{5.1.2}$$

where $C^{\mu}_{\nu\lambda} \equiv \Gamma^{\mu}_{\nu\lambda} - \Gamma^{\mu}_{\nu\lambda}[g^{(0)}] = g^{\mu\sigma}\left(\nabla_\nu g_{\sigma\lambda} + \nabla_\lambda g_{\nu\sigma} - \nabla_\sigma g_{\nu\lambda}\right)/2$. Since $R_{\mu\nu}$ contains terms such as those schematically expressed as $g^{-1}\nabla\nabla g$, $\nabla g^{-1}\nabla g$ and $g^{-1}g^{-1}\nabla g\nabla g$, we can find

$$R^{(n)}_{\mu\nu}[h] \sim \mathrm{O}\left(\epsilon^{n-2}\right)\,, \tag{5.1.3}$$

where n is the number of $h_{\mu\nu}$ included in $R_{\mu\nu}$. We also find

$$R^{(n)}[h] \sim G^{(n)}_{\mu\nu}[h] \sim \mathrm{O}\left(\epsilon^{n-2}\right)\,, \tag{5.1.4}$$

where $R \equiv g^{\mu\nu}R_{\mu\nu}$, $G_{\mu\nu} \equiv R_{\mu\nu} - g_{\mu\nu}R/2$ is the Einstein tensor, and $R^{(n)}[h]$ and $G^{(n)}_{\mu\nu}[h]$ do not contain $R_{\mu\nu}[g^{(0)}]$.

The Einstein equation is

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa^2 T^{(0)}_{\mu\nu}\,, \tag{5.1.5}$$

where $\kappa^2 = 8\pi G$ and $T^{(0)}_{\mu\nu}$ is the stress-energy tensor for the background matter fields. In the following, for simplicity, we focus on metric perturbations and ignore perturbations of the matter fields in $T^{(0)}_{\mu\nu}$. We can find the dominant terms, $\mathrm{O}\left(\epsilon^{-1}\right)$, of the Einstein equation as

$$G^{(1)}_{\mu\nu}[h] = R^{(1)}_{\mu\nu}[h] - \frac{1}{2}g^{(0)}_{\mu\nu}R^{(1)}[h] = 0\,,$$

or simply

$$R^{(1)}_{\mu\nu}[h] = 0\,. \tag{5.1.6}$$

This is equivalent to the equation for linearized gravitational waves in the ordinary perturbation theory. Next, in the order of $\mathrm{O}\left(1\right)$, we find

$$G_{\mu\nu}[g^{(0)}] \;=\; \kappa^2 T^{(0)}_{\mu\nu} + \kappa^2 T^{\mathrm{eff}}_{\mu\nu}\,, \tag{5.1.7}$$

where the effective gravitational stress-energy tensor, $T^{\rm eff}_{\mu\nu}$, is given by

$$
\begin{aligned}
\kappa^2 T^{\rm eff}_{\mu\nu} &\equiv -\left\langle G^{(2)}_{\mu\nu}[h]\right\rangle \\
&= -\left\langle R^{(2)}_{\mu\nu}[h] - \frac{1}{2} g^{(0)}_{\mu\nu} g^{(0)\lambda\sigma} R^{(2)}_{\lambda\sigma}[h]\right\rangle \\
&= \left\langle \frac{1}{4}\nabla_\mu h^{{\rm TT}\lambda\sigma}\nabla_\nu h^{\rm TT}_{\lambda\sigma}\right\rangle \,. \qquad (5.1.8)
\end{aligned}
$$

Here, and in the following, $\langle\cdots\rangle$ denotes taking a spacetime average over several wavelengths of perturbations. Here the indices are raised and lowered with $g^{(0)\mu\nu}$ and $g^{(0)}_{\mu\nu}$. For the expression of the third line, the transverse-traceless gauge, $\nabla_\mu h^\mu{}_\nu = 0 = h^\mu{}_\mu$, denoted by h^{TT}, and (A.6) have been used (see also Appendix. A). We can check that $T^{\rm eff}_{\mu\nu}$ is traceless, i.e., it acts like radiation:

$$
\kappa^2 T^{{\rm eff}\mu}{}_\mu = 0\,, \qquad (5.1.9)
$$

from (5.1.6) and (A.2). Thus, in particular, it cannot provide any effects that mimic dark energy in general relativity. More general treatment of the effective stress-energy tensor for gravitational waves is given in Appendix. B. For more mathematically rigorous treatments of short-wavelength perturbations and the effective stress-energy tensor, see [65, 66].

The effective stress-energy tensor (5.1.8) can be shown to be gauge-invariant [65]. In fact, the expression of the right-hand side of (5.1.8) is given by manifestly gauge-invariant part of $h^{\rm TT}_{\mu\nu}$. For this purpose, one can introduce the polarization tensors $\epsilon^{(+,\times)}_{\mu\nu}$, as usual, and decompose the metric perturbation accordingly $h^{\rm TT}_{\mu\nu} = \epsilon^{(+)}_{\mu\nu} h^{(+)} + \epsilon^{(\times)}_{\mu\nu} h^{(\times)}$. In the cosmological context, one is concerned with the FLRW metric,

$$
ds^2 = -dt^2 + a(t)^2 \gamma_{ij} dx^i dx^j \qquad (5.1.10)
$$

with $d\sigma^2 = \gamma_{ij} dx^i dx^j$ being the metric of 3-dimensional constant curvature space. So, it may be more convenient to impose the transverse-traceless condition with respect to this FLRW time-slicing, i.e., $h^{\rm TT}_{00} = 0, h^{\rm TT}_{0i} = 0, \nabla_\mu h^{{\rm TT}\mu}{}_i = 0, h^{{\rm TT}i}{}_i = 0$. This condition completely fixes the gauge freedom and (5.1.8) is written by the gauge invariant variable $h^{\rm TT}_{ij}$ as $\langle \frac{1}{4}\nabla_\mu h^{{\rm TT}ij}\nabla_\nu h^{\rm TT}_{ij}\rangle$.

5.2 High frequency limit in $f(R)$ gravity

The general action for $f(R)$ gravity is given by

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}f(R) + \int d^4x\mathcal{L}_{\mathrm{M}}\,, \tag{5.2.1}$$

where $\mathcal{L}_{\mathrm{M}}$ is the Lagrangian for matter fields, such as perfect fluid in the cosmological context. Varying this action with respect to the metric, we have the field equations

$$G^{f(R)}_{\mu\nu} \equiv G_{\mu\nu} + \hat{F}R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\hat{f} - \nabla_\mu\nabla_\nu\hat{F} + g_{\mu\nu}g^{\lambda\sigma}\nabla_\lambda\nabla_\sigma\hat{F} = \kappa^2 T^{(0)}_{\mu\nu}\,, \tag{5.2.2}$$

where $\hat{f} \equiv f - R$, $\hat{F} \equiv d\hat{f}/dR$, and $T^{(0)}_{\mu\nu}$ denotes the matter stress-energy tensor.

The field equations in $f(R)$ gravity have terms consisting of higher-order derivatives of R, and the orders of those derivatives are higher than that of R:

$$\nabla_{\mu_1}\nabla_{\mu_2}\cdots\nabla_{\mu_m}R^{(n)}[h] \sim \mathrm{O}\left(\epsilon^{n-2-m}\right)\,. \tag{5.2.3}$$

Therefore it is expected that the effect of the short-wavelength approximation would be enhanced. In order to see whether this is the case, from now on we restrict our attention to the concrete model

$$f(R) = R + cR^2\,, \tag{5.2.4}$$

where c is a constant. This model has been considered for the first time in the context of inflationary universe [102]. The field equations are

$$G^{f(R)}_{\mu\nu} \equiv G_{\mu\nu} + 2c\left(RR_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R^2 - \nabla_\mu\nabla_\nu R + g_{\mu\nu}g^{\lambda\sigma}\nabla_\lambda\nabla_\sigma R\right) = \kappa^2 T^{(0)}_{\mu\nu}\,. \tag{5.2.5}$$

As in Isaacson's formula reviewed in the previous section, we expand the above equations with respect to the small parameter ϵ. Then, the dominant part is of order $\mathrm{O}\left(\epsilon^{-3}\right)$, in which we have the following equations

$$\nabla_\mu\nabla_\nu R^{(1)}[h] - g^{(0)}_{\mu\nu}\Box R^{(1)}[h] = 0\,. \tag{5.2.6}$$

By contracting with $g^{(0)\mu\nu}$, we immediately have

$$\nabla_\mu \nabla_\nu R^{(1)}[h] = 0 \,. \tag{5.2.7}$$

Next, for order $\mathrm{O}\left(\epsilon^{-2}\right)$, we have

$$R^{(1)}[h] R^{(1)}_{\mu\nu}[h] - \frac{1}{4} g^{(0)}_{\mu\nu} \left(R^{(1)}[h]\right)^2 - \nabla_\mu \nabla_\nu R^{(2)}[h] + g^{(0)}_{\mu\nu} \Box R^{(2)}[h] = 0 \,. \tag{5.2.8}$$

By dotting with $g^{(0)ab}$, we have

$$\begin{aligned} \Box R^{(2)}[h] &= 0 \,, && (5.2.9) \\ \nabla_\mu \nabla_\nu R^{(2)}[h] &= R^{(1)}[h] R^{(1)}_{\mu\nu}[h] - \frac{1}{4} g^{(0)}_{\mu\nu} \left(R^{(1)}[h]\right)^2 \,. && (5.2.10) \end{aligned}$$

Note that since we are working in the short-wavelength approximation, we find that $g^{(0)\mu\nu} R^{(1)}_{\mu\nu}[h] = R^{(1)}[h]$ in $\mathrm{O}\left(\epsilon^{-1}\right)$, which is different from calculations in ordinary perturbation theory, where in general $g^{(0)\mu\nu} R^{(1)}_{\mu\nu}[h] \neq R^{(1)}[h]$. For $\mathrm{O}\left(\epsilon^{-1}\right)$, we have

$$\begin{aligned} & R^{(1)}_{\mu\nu}[h] - \frac{1}{2} g^{(0)}_{\mu\nu} R^{(1)}[h] + 2c \left\{ \left(R[g^{(0)}] + R^{(2)}[h]\right) R^{(1)}_{\mu\nu}[h] + R^{(1)}[h] \left(R_{\mu\nu}[g^{(0)}] + R^{(2)}_{\mu\nu}[h]\right) \right\} \\ & - \frac{c}{2} \left\{ 2 g^{(0)}_{\mu\nu} \left(R[g^{(0)}] + R^{(2)}[h]\right) R^{(1)}[h] + h_{\mu\nu} \left(R^{(1)}[h]\right)^2 \right\} \\ & - 2c \left(\nabla_\mu \nabla_\nu R^{(3)}[h] - R_{\lambda\sigma}[g^{(0)}] \nabla_\mu \nabla_\nu h^{\lambda\sigma}\right) \\ & + 2c \left\{ g^{(0)}_{\mu\nu} \left(\Box R^{(3)}[h] - R_{\lambda\sigma}[g^{(0)}] \Box h^{\lambda\sigma}\right) - g^{(0)}_{\mu\nu} h^{\lambda\sigma} \nabla_\lambda \nabla_\sigma R^{(2)}[h] \right\} = 0 \,. \end{aligned} \tag{5.2.11}$$

Again, by dotting with $g^{(0)ab}$, we have

$$R^{(1)}[h] + \frac{c}{2} h^\mu{}_\mu \left(R^{(1)}[h]\right)^2 - 6c \left(\Box R^{(3)}[h] - R_{\lambda\sigma}[g^{(0)}] \Box h^{\lambda\sigma}\right) + 8c h^{\mu\nu} \nabla_\mu \nabla_\nu R^{(2)}[h] = 0 \,. \tag{5.2.12}$$

In order $\mathrm{O}(1)$, as in Isaacson's formula in general relativity, we have the field equations for the background metric with the backreaction source term: $G^{f(R)}_{\mu\nu}[g^{(0)}] = \kappa^2 T^{(0)}_{\mu\nu} + \kappa^2 T^{\mathrm{eff}}_{\mu\nu}$, where

$$
\begin{aligned}
\kappa^2 T^{\mathrm{eff}}_{\mu\nu} \equiv -\Bigg\langle & R^{(2)}_{\mu\nu}[h] - \frac{1}{2}\left(g^{(0)}_{\mu\nu} R^{(2)}[h] + h_{\mu\nu} R^{(1)}[h]\right) \\
& + 2c \Big\{ R^{(1)}[h] R^{(3)}_{\mu\nu}[h] + R[g^{(0)}] R^{(2)}_{\mu\nu}[h] \\
& \qquad + R^{(2)}[h]\left(R_{\mu\nu}[g^{(0)}] + R^{(2)}_{\mu\nu}[h]\right) + \left(R^{(3)}[h] - h^{\lambda\sigma} R_{\lambda\sigma}[g^{(0)}]\right) R^{(1)}_{\mu\nu}[h] \Big\} \\
& - \frac{c}{2}\Big[g^{(0)}_{\mu\nu}\Big\{ 2R^{(1)}[h] R^{(3)}[h] - 2R^{(1)}[h] h^{\lambda\sigma} R_{\lambda\sigma}[g^{(0)}] + 2R[g^{(0)}] R^{(2)}[h] + \left(R^{(2)}[h]\right)^2 \Big\} \\
& \qquad + 2h_{\mu\nu}\left(R[g^{(0)}] + R^{(2)}[h]\right) R^{(1)}[h] \Big] \\
& + 2c \Big\{ \left(-g^{(0)}_{\mu\nu} h^{\lambda\sigma} + h_{\mu\nu} g^{(0)\lambda\sigma}\right)\left(\nabla_\lambda \nabla_\sigma R^{(3)}[h] - R_{\alpha\beta}[g^{(0)}] \nabla_\lambda \nabla_\sigma h^{\alpha\beta}\right) \\
& \qquad + \left(g^{(0)}_{\mu\nu} h^{\lambda}{}_{\alpha} h^{\alpha\sigma} - h_{\mu\nu} h^{\lambda\sigma}\right) \nabla_\lambda \nabla_\sigma R^{(2)}[h] \Big\} \Bigg\rangle . \qquad (5.2.13)
\end{aligned}
$$

This is the expression of the stress-energy tensor for short-wavelength metric perturbations on the generic background metric $g^{(0)}_{\mu\nu}$ in our $f(R)$ gravity.

From now on, we consider the cosmological context. We assume that our background is spatially homogeneous and isotropic, that is, our background metric possesses FLRW symmetry and therefore takes the form of (5.1.10). Then, thanks to this background symmetry we can explicitly solve equations of the form $\nabla_\mu \nabla_\nu S(t, \vec{x}) = 0$, such as (5.2.7) (see Appendix. C). Equation (5.2.7) (the equations of motion of $\mathrm{O}\left(\epsilon^{-3}\right)$) is solved to yield

$$
R^{(1)}[h] = const\,. \qquad (5.2.14)
$$

Taking the average, we find

$$
R^{(1)}[h] = const. = \langle const. \rangle = \left\langle R^{(1)}[h] \right\rangle = 0\,. \qquad (5.2.15)
$$

Then, the equations (5.2.10) (those of $\mathrm{O}\left(\epsilon^{-2}\right)$) become

$$
\nabla_\mu \nabla_\nu R^{(2)}[h] = 0\,. \qquad (5.2.16)
$$

Again using the result in Appendix. C, we find

$$R^{(2)}[h] \equiv S_1 = const\,. \tag{5.2.17}$$

By using (5.2.15) and (5.2.16), the equation (5.2.12) (of $\mathrm{O}\left(\epsilon^{-1}\right)$) immediately yields

$$\Box R^{(3)}[h] - R_{\mu\nu}[g^{(0)}]\Box h^{\mu\nu} = 0\,, \tag{5.2.18}$$

and the equations (5.2.11) (of $\mathrm{O}\left(\epsilon^{-1}\right)$) reduce to

$$\left(1 + 2cR[g^{(0)}] + 2cS_1\right) R^{(1)}_{\mu\nu}[h] = 2c\left(\nabla_\mu\nabla_\nu R^{(3)}[h] - R_{\lambda\sigma}[g^{(0)}]\nabla_\mu\nabla_\nu h^{\lambda\sigma}\right)\,. \tag{5.2.19}$$

The effective stress-energy tensor (5.2.13) is then expressed as

$$\begin{aligned} \kappa^2 T^{\mathrm{eff}}_{\mu\nu} \;=\; -\Bigg\langle & \left(1 + 2cR[g^{(0)}]\right) R^{(2)}_{\mu\nu}[h] - \frac{1}{2} g^{(0)}_{\mu\nu} S_1 \\ & + 2c\left\{\left(R^{(3)}[h] - h^{\lambda\sigma}R_{\lambda\sigma}[g^{(0)}]\right) R^{(1)}_{\mu\nu}[h] + S_1\left(R_{\mu\nu}[g^{(0)}] + R^{(2)}_{\mu\nu}\right)\right\} \\ & - \frac{c}{2} g^{(0)}_{\mu\nu}\left(2R[g^{(0)}] + S_1\right) S_1 \\ & + 2c\left(-g^{(0)}_{\mu\nu}h^{\lambda\sigma}\right)\left(\nabla_\lambda\nabla_\sigma R^{(3)}[h] - R_{\alpha\beta}[g^{(0)}]\nabla_\lambda\nabla_\sigma h^{\alpha\beta}\right)\Bigg\rangle\,. \end{aligned} \tag{5.2.20}$$

This is the most general expression of our effective stress-energy tensor.

We immediately notice that our expression (5.2.20) contains the integration constant S_1. There does not seem to be a definite way to determine S_1 within the framework of the present $f(R)$ theory itself. As a sensible way to specify S_1, let us assume in the following that the effective stress-energy tensor (5.2.20) in the R^2 model should reduce to that in general relativity when $c = 0$, and accordingly choose S_1 $(= R^{(2)}[h])$ to be 0.

Then, (5.2.20) becomes

$$\begin{aligned}
\kappa^2 T^{\rm eff}_{\mu\nu} &= -\Bigg\langle \left(1+2cR[g^{(0)}]\right) R^{(2)}_{\mu\nu}[h] + 2c\left(R^{(3)}[h] - h^{\lambda\sigma}R_{\lambda\sigma}[g^{(0)}]\right) R^{(1)}_{\mu\nu}[h] \\
&\qquad -g^{(0)}_{\mu\nu}\left(1+2cR[g^{(0)}]\right) h^{\lambda\sigma}R^{(1)}_{\lambda\sigma}[h] \Bigg\rangle \\
&= -\Bigg\langle \left(1+2cR[g^{(0)}]\right) R^{(2)}_{\mu\nu}[h] + 2c\left(R^{(3)}[h] - h^{\lambda\sigma}R_{\lambda\sigma}[g^{(0)}]\right) R^{(1)}_{\mu\nu}[h] \Bigg\rangle, \qquad (5.2.21)
\end{aligned}$$

where we have used (5.2.19) in the first equality above, and

$$\left\langle h^{\lambda\sigma}R^{(1)}_{\lambda\sigma}[h]\right\rangle = 0 \qquad (5.2.22)$$

in the second equality so as to make the above expression compatible with that of general relativity in the $c = 0$ case. The expression, (5.2.21), is our main result of this section. From $R^{(2)}[h] = 0$ and (5.2.22), we see

$$\begin{aligned}
\left\langle g^{(0)\mu\nu}R^{(2)}_{\mu\nu}[h]\right\rangle &= \left\langle R^{(2)}[h] + h^{\mu\nu}R^{(1)}_{\mu\nu}[h]\right\rangle \\
&= 0\,. \qquad (5.2.23)
\end{aligned}$$

Then using this and $R^{(1)}[h] = 0$, we can find that $\kappa^2 T^{\rm eff}_{\mu\nu}$ is in fact traceless:

$$\kappa^2 T^{{\rm eff}\mu}{}_{\mu} = 0\,. \qquad (5.2.24)$$

It should be stressed that as mentioned above, there is no a priori way to determine S_1 by the theory itself. If we choose S_1 to be, instead, a non-zero constant, then the effective stress-energy tensor, (5.2.20), has a term proportional to the background metric, that is, a cosmological-constant-like term, even in the limit to the Einstein gravity.

5.3 High frequency limit in scalar-tensor theory

In the previous section, the scalar curvature R and the Ricci tensor $R_{\mu\nu}$ are taken up directly in the metric formalism of the $f(R)$ gravity. As we have shown in Sec. 3.2.2, it is well known that any $f(R)$ gravity theory is included in Brans-Dicke theory, which is one of the simplest examples of scalar-tensor theory [11, 17]. In this section, we will see that the results obtained in the previous section are indeed consistent with those obtained within the corresponding scalar-tensor theory.

In the Jordan frame, the action of Brans-Dicke theory [29] is

$$S = \frac{1}{\kappa^2}\int d^4x\sqrt{-g}\left\{\frac{1}{2}\phi R - \frac{\omega_{\rm BD}}{2\phi}\nabla^\mu\phi\nabla_\mu\phi - V(\phi)\right\} + \int d^4x\mathcal{L}_{\rm M}\,, \tag{5.3.1}$$

where $\omega_{\rm BD}$ is a constant called the Brans-Dicke parameter and ϕ is a dimensionless scalar field, and $\mathcal{L}_{\rm M}$ denotes the Lagrangian for matter fields, which can in general couple to the metric $g_{\mu\nu}$ as well as the scalar field ϕ. Then the equations of motion for ϕ and $g_{\mu\nu}$ are, respectively, obtained as

$$\begin{aligned} \Box\phi + \frac{\phi}{2\omega_{\rm BD}}\left(-\frac{\omega_{\rm BD}}{\phi^2}\nabla^\mu\phi\nabla_\mu\phi + R - 2\partial_\phi V(\phi)\right) &= \kappa^2 T^{(0)}_{\phi\ \mu\nu}\,, & (5.3.2)\\ \phi G_{\mu\nu} - \frac{\omega_{\rm BD}}{\phi}\left(\nabla_\mu\phi\nabla_\nu\phi - \frac{g_{\mu\nu}}{2}\nabla^\lambda\phi\nabla_\lambda\phi\right) - \nabla_\mu\nabla_\nu\phi & \\ + g_{\mu\nu}\left(g^{\lambda\sigma}\nabla_\lambda\nabla_\sigma\phi + V(\phi)\right) &= \kappa^2 T^{(0)}_{\mu\nu}\,, & (5.3.3)\end{aligned}$$

where $T^{(0)}_{\phi\ \mu\nu}$ and $T^{(0)}_{\mu\nu}$ are the stress-energy tensor for matter fields obtained by taking variations of ϕ and $g_{\mu\nu}$, respectively.

The $f(R)$ gravity of the metric formalism, (5.2.1), can be cast into the form of the above Brans-Dicke theory by setting

$$\phi = F(R) \equiv \frac{df(R)}{dR}\,, \quad \omega_{\rm BD} = 0\,, \quad V = \frac{F(R)R - f(R)}{2}\,. \tag{5.3.4}$$

In this case, as one can find $R - 2\partial_\phi V = 0$, the equations of motion for ϕ and $g_{\mu\nu}$ given above become, respectively

$$\begin{aligned} \Box\phi - \frac{1}{2\phi}\nabla^\mu\phi\nabla_\mu\phi &= \kappa^2 T^{(0)}_{\phi\ \mu\nu}, & (5.3.5)\\ G^{\rm ST}_{\mu\nu} \equiv \phi\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) - \nabla_\mu\nabla_\nu\phi + g_{\mu\nu}\left(g^{\lambda\sigma}\nabla_\lambda\nabla_\sigma\phi + V(\phi)\right) &= \kappa^2 T^{(0)}_{\mu\nu}. & (5.3.6)\end{aligned}$$

From now we consider short-wavelength perturbations for ϕ: $\phi = \phi_0 + \delta\phi$. We also assume that there is no coupling of matter fields with the second-order derivatives of ϕ, so that there are no non-vanishing terms of order $O(\epsilon^{-1})$ in the stress-energy tensor for matter fields. Then, the equation of motion for ϕ of $\mathrm{O}\left(\epsilon^{-1}\right)$ is

$$\Box\delta\phi = 0\,, \tag{5.3.7}$$

and the equations of motion for $g_{\mu\nu}$ of $\mathrm{O}\left(\epsilon^{-1}\right)$ are

$$\phi_0\left(R^{(1)}_{\mu\nu}[h] - \frac{1}{2}g^{(0)}_{\mu\nu}R^{(1)}[h]\right) = \nabla_\mu\nabla_\nu\delta\phi - g^{(0)}_{\mu\nu}\Box\delta\phi\,. \tag{5.3.8}$$

Contracting with $g^{(0)\mu\nu}$, we have

$$R^{(1)}[h] = \frac{3}{\phi_0}\Box\delta\phi = 0\,, \tag{5.3.9}$$

where we have used (5.3.7). From this equation, we can immediately find

$$R^{(1)}_{\mu\nu}[h] = \frac{1}{\phi_0}\nabla_\mu\nabla_\nu\delta\phi\,. \tag{5.3.10}$$

The equations of motion of $\mathrm{O}\left(1\right)$ are given by $G^{\mathrm{ST}}_{\mu\nu}[g^{(0)},\phi_0] = \kappa^2 T^{(0)}_{\mu\nu} + \kappa^2 T^{\mathrm{eff}}_{\mu\nu}$, where

$$\kappa^2 T^{\mathrm{eff}}_{\mu\nu} \equiv -\left\langle \phi_0 R^{(2)}_{\mu\nu}[h] + \delta\phi R^{(1)}_{\mu\nu}[h] - g^{(0)}_{\mu\nu}h^{\lambda\sigma}\nabla_\lambda\nabla_\sigma\delta\phi\right\rangle\,. \tag{5.3.11}$$

Here we would like to emphasise that so far we have made no assumptions concerning the form of $f(R)$ or the symmetry of our background metric $g^{(0)}_{\mu\nu}$; the above expression, (5.3.11), applies to the generic $f(R)$ theory with an arbitrary background metric.

If we restrict the form of $f(R)$ to be (5.2.4), then by inspecting the expansions $\phi = \phi_0 + \delta\phi + \cdots$ and $F(R) = 1 + 2cR = \left(1 + 2cR[g^{(0)}]\right) + 2c\left(R^{(3)}[h] - h^{\lambda\sigma}R_{\lambda\sigma}[g^{(0)}]\right) + \cdots$, we find

$$\phi_0 = 1 + 2cR[g^{(0)}]\,,\quad \delta\phi = 2c\left(R^{(3)}[h] - h^{\lambda\sigma}R_{\lambda\sigma}[g^{(0)}]\right)\,. \tag{5.3.12}$$

Using these and (5.3.10), we have

$$\begin{aligned}\kappa^2 T^{\rm eff}_{\mu\nu} &= -\Big\langle \left(1+2cR[g^{(0)}]\right) R^{(2)}_{\mu\nu}[h] \\ &\qquad +2c\left(R^{(3)}[h]-h^{\lambda\sigma}R_{\lambda\sigma}[g^{(0)}]\right) R^{(1)}_{\mu\nu}[h] - g^{(0)}_{\mu\nu}\phi_0 h^{\lambda\sigma}R^{(1)}_{\lambda\sigma}[h]\Big\rangle . \quad (5.3.13)\end{aligned}$$

Now we consider the cosmological situation so that the background metric has FLRW symmetry. Provided that the limit $c \to 0$ should reproduce the same results as in the Einstein gravity case, we finally obtain

$$\kappa^2 T^{\rm eff}_{\mu\nu} = -\left\langle \left(1+2cR[g^{(0)}]\right) R^{(2)}_{\mu\nu}[h] + 2c\left(R^{(3)}[h]-h^{\lambda\sigma}R_{\lambda\sigma}[g^{(0)}]\right) R^{(1)}_{\mu\nu}[h]\right\rangle , \quad (5.3.14)$$

where we have used (5.2.22), derived under FLRW symmetry in Sec. 5.2. We see that expression (5.3.14) is precisely the same as (5.2.21) derived within the metric formalism of the $f(R)$ gravity. This verifies our methods for dealing with short-wavelength perturbations of the $f(R)$ gravity within the metric formalism.

In the Einstein frame, the action becomes [105]

$$S_{\rm E} = \frac{1}{\kappa^2}\int d^4x\sqrt{-\tilde{g}}\left\{\frac{1}{2}\tilde{R} - \frac{1}{2}\left(\tilde{\nabla}\tilde{\phi}\right)^2 - \tilde{V}(\tilde{\phi})\right\} , \quad (5.3.15)$$

where $\tilde{g}_{\mu\nu} \equiv F g_{\mu\nu}$, $\tilde{\phi} \propto \ln\phi$ and $\tilde{V} \equiv V/F^2$. From this action, one can find that the equation of motion of $\mathrm{O}\left(\epsilon^{-1}\right)$ is $\Box\delta\tilde{\phi} = 0$ and the effective stress-energy tensor is $\kappa^2 T^{\rm eff}_{\mu\nu} = \left\langle \tilde{\nabla}_\mu\delta\tilde{\phi}\tilde{\nabla}_\nu\delta\tilde{\phi}\right\rangle$. It then immediately follows that the effective stress-energy tensor must be traceless, i.e., ${T^{\rm eff}}^{\mu}{}_{\mu} = 0$. This can be shown only on the assumption of FLRW symmetry.

Chapter 6

Off-center CMB anisotropies in local void models

As we show in Sec. 4.2, there have been a number of tests of local void models from observations other than the SN Ia distance-redshift relation. The purpose of this chapter is to derive analytic formulae for the dipole and quadrupole moments of the CMB temperature anisotropy in general spherically symmetric inhomogeneous spacetimes, including the Λ-LTB spacetime as a particular case, and give constraints on the local void model. In the standard cosmology, basic properties of the CMB temperature anisotropy are derived by inspecting perturbations of the Einstein and Boltzmann equations in the FLRW background universe. Ideally, it is desirable to do the same thing in the LTB background. However, perturbations in a LTB spacetime, let alone those in a general spherically symmetric spacetime, have not been very well studied mainly because the perturbation equations are much more involved to solve in the spherical but inhomogeneous background, though the linear perturbation formulae themselves have long been available [128, 129]. There are some recent papers relevant to the LTB spacetime [111, 112].

In this chapter, we are not going to deal with perturbations of the LTB metric. Instead, we will exploit the key requirement of the local void model that we, observers, are restricted to be around very near the center of the spherical symmetry: Namely, we first note that the small distance between the symmetry center and an off-center observer gives rise to a corresponding deviation in the photon distribution function. Then, by taking

'Taylor-expansions' of the photon distribution function at the center with respect to the deviation, we can read off the CMB temperature anisotropy caused by the deviation in the photon distribution function. By doing so, we can, in principle, construct the l-th order multiple moment of the CMB temperature anisotropy from the (up to) l-th order expansion coefficients, with the help of the background null geodesic equations and the Boltzmann equation. We will do so for the first and second-order expansions to find the CMB dipole and quadrupole moments. We also provide the concrete expression of the corresponding formulae for the LTB cosmological model. Our formulae are then checked to be consistent with the numerical analyses of the CMB temperature anisotropy in the LTB model, previously made by Alnes and Amarzguioui [47]. We then apply our formulae to place the constraint on the distance between an observer and the symmetry center of the void, by using the latest WMAP data, thereby updating the results of the previous analyses.

In the next section, we derive analytic formulae for the CMB temperature anisotropy in the most general spherically symmetric spacetime. In Sec. 6.2, we obtain analytic formulae for the CMB temperature anisotropy in the LTB model, and some constraints concerning the position of the observer.

6.1 CMB anisotropies in general spherically symmetric spacetime

In this section, we will derive analytic formulae for the dipole and quadrupole moments of the CMB temperature anisotropy. We first briefly discuss the null geodesic equations, the photon distribution function and its relation to the CMB photon temperature in the most general spherically symmetric spacetime. Next, inspecting the first-order derivatives of the photon distribution function at the center of the spherical symmetry and using the null geodesic equations, we derive the dipole formula for the CMB. Then, taking further derivatives, we derive the quadrupole formula of the CMB temperature anisotropy in general spherically symmetric spacetimes.

6.1.1 Photon distribution function in a general spherically symmetric spacetime

The most general spherically symmetric metric can be written in the following form

$$ds^2 = -N^2(t,r)dt^2 + S^2(t,r)dr^2 + R^2(t,r)(d\theta^2 + \sin^2\theta d\phi^2). \tag{6.1.1}$$

In this background, the relevant geodesic equations are given by $dp^\mu/d\lambda = -\Gamma^\mu_{\alpha\beta}p^\alpha p^\beta$ which can read

$$\begin{aligned}\frac{dp^t}{d\lambda} &= -\frac{\dot{N}}{N}(p^t)^2 - \frac{2N'}{N}p^t p^r - \frac{S\dot{S}}{N^2}(p^r)^2 - \frac{\dot{R}}{N^2 R}p_\perp^2, && (6.1.2)\\ \frac{dp^r}{d\lambda} &= -\frac{NN'}{S^2}(p^t)^2 - \frac{2\dot{S}}{S}p^t p^r - \frac{S'}{S}(p^r)^2 + \frac{R'}{S^2 R}p_\perp^2, && (6.1.3)\end{aligned}$$

where $p^\mu = dx^\mu/d\lambda$, $p_\mu p^\mu = 0$ with λ being an affine parameter, and $p_\perp$ is given by $p_\perp^2 \equiv R^2\{(p^\theta)^2 + (p^\phi)^2\sin^2\theta\}$. Here and in the rest of this chapter, *prime* and *dot* denote the derivatives with respect to r and t, respectively.

We are concerned with the distribution function, $F(x,p)$, for the CMB photons that leave the "last scattering surface" appropriately defined, say, $t = t_i$ hypersurface, in the universe modelled by the above metric and that eventually reach an "observer" very near the symmetry center. We assume that the photon distribution function $F(x,p)$ is spherically symmetric, respecting the symmetry of the background geometry, so that

$$F(x,p) = F_0(t,r,\omega,\mu), \tag{6.1.4}$$

where $\omega \equiv p^t$ and $\mu \equiv Sp^r/(N\omega)$. Note that from the above geodesic equations, we have

$$\begin{aligned}\dot{r} &= \mu\frac{N}{S}, && (6.1.5)\\ \dot{\omega} &= -\omega\left(\frac{\dot{N}}{N} + 2\mu\frac{N'}{S} + \mu^2\frac{\dot{S}}{S} + (1-\mu^2)\frac{\dot{R}}{R}\right), && (6.1.6)\\ \dot{\mu} &= (1-\mu^2)\left\{\frac{N}{S}\left(\frac{R'}{R} - \frac{N'}{N}\right) + \mu\left(\frac{\dot{R}}{R} - \frac{\dot{S}}{S}\right)\right\}. && (6.1.7)\end{aligned}$$

In particular, it follows from the above equations

$$\partial_\omega \dot{r} = \partial_\omega \dot{\mu} = 0, \quad \partial_r \dot{\mu} = 0, \tag{6.1.8}$$

where the last one holds for the radial null geodesics, for which $\mu = \pm 1$.

Now, suppose the universe is locally in thermal equilibrium, that is, F is given as the Planck distribution function, Φ, at the last scattering surface, as required in most of the known LTB models. Then, given a photon geodesic γ, the ratio of the temperature T of the photon and its energy ω is preserved along the trajectory γ. Therefore, ω comes in F in the form

$$F = \Phi(\omega/T). \tag{6.1.9}$$

Then, the CMB temperature anisotropy $\delta T/T$ is generally given by

$$(\delta F)^{(1)} + (\delta F)^{(2)} + \cdots = \left\{ -\frac{\delta T}{T}\omega\partial_\omega + \frac{1}{2}\left(\frac{\delta T}{T}\right)^2 (\omega\partial_\omega)^2 + \cdots \right\} \Phi. \tag{6.1.10}$$

Now, suppose that an observer lives at a distance of δx^i from the symmetry center. Then, the left-side are written as

$$(\delta F)^{(1)} = \delta x^i (\partial_i F)_0, \qquad (\delta F)^{(2)} = \frac{1}{2}\delta x^i \delta x^j (\partial_i \partial_j F)_0, \tag{6.1.11}$$

where here and in the following, the subscript '0' implies the value evaluated at the center ($r = 0$) at the present time ($t = t_0$). The CMB temperature anisotropy dipole $(\delta T/T)^{(1)}$ and quadrupole $(\delta T/T)^{(2)}$ are therefore given by

$$\left(\frac{\delta T}{T}\right)^{(1)} = -\frac{\delta x^i (\partial_i F)_0}{\omega \partial_\omega F_0}, \tag{6.1.12}$$

$$\left(\frac{\delta T}{T}\right)^{(2)} = -\frac{1}{2}\frac{\delta x^i \delta x^j (\partial_i \partial_j F)_0}{\omega \partial_\omega F_0} + \frac{1}{2}\left\{\left(\frac{\delta T}{T}\right)^{(1)}\right\}^2 \frac{(\omega\partial_\omega)^2 F_0}{\omega \partial_\omega F_0}. \tag{6.1.13}$$

Since (6.1.4) implies $\partial_i F = (\partial_i r)\partial_r F_0 + (\partial_i \omega)\partial_\omega F_0 + (\partial_i \mu)\partial_\mu F_0$, $(\partial_i F)_0$, $(\partial_i \partial_j F)_0$ in the right-hand side of the above equations are given by $\partial_\alpha F_0$ and $\partial_\alpha \partial_\beta F_0$ $(\alpha, \beta = r, \omega, \mu)$. Therefore, our task is to find the concrete expressions of $\partial_\alpha F_0$ and $\partial_\alpha \partial_\beta F_0$ in terms of relevant geometric quantities.

6.1.2 Analytic formula for CMB dipole

First, we will obtain the CMB temperature dipole formula. For this purpose, we derive the expression for $\partial_\alpha F_0$. Our stating point is the Boltzmann equation for $F_0(t, r, \omega, \mu)$,

$$\frac{d}{dt}F_0 = \partial_t F_0 + \dot{r}\partial_r F_0 + \dot{\omega}\partial_\omega F_0 + \dot{\mu}\partial_\mu F_0 = 0, \tag{6.1.14}$$

where $\dot{r}, \dot{\omega}, \dot{\mu}$ are defined for a given null geodesic curve γ. By differentiating this equation by $\alpha = r, \omega, \mu$, and using the formula (6.1.8), we obtain the first order differential equations for $\partial_\alpha F$,

$$\frac{d}{dt}\begin{pmatrix} \partial_\omega F_0 \\ \partial_r F_0 \\ \partial_\mu F_0 \end{pmatrix} = -\begin{pmatrix} \partial_\omega \dot{\omega} & 0 & 0 \\ \partial_r \dot{\omega} & \partial_r \dot{r} & 0 \\ \partial_\mu \dot{\omega} & \partial_\mu \dot{r} & \partial_\mu \dot{\mu} \end{pmatrix}\begin{pmatrix} \partial_\omega F_0 \\ \partial_r F_0 \\ \partial_\mu F_0 \end{pmatrix}. \tag{6.1.15}$$

The set of these equations can easily be integrated along the given photon trajectory γ to yield the solutions

$$\begin{aligned}
\omega\partial_\omega F_0 &= (\omega\partial_\omega F_0)_i, && (6.1.16)\\
\partial_r F_0 &= e^{-P(t,t_i)}(\partial_r F_0)_i + (\omega\partial_\omega F_0)_i \int_{t_i}^{t} dt_1 e^{-P(t,t_1)} A(t_1), && (6.1.17)\\
\partial_\mu F_0 &= e^{-Q(t,t_i)}(\partial_\mu F_0)_i + (\partial_r F_0)_i \int_{t_i}^{t} dt_1 e^{-Q(t,t_1)-P(t_1,t_i)} \left(-\frac{N}{S}\right)_{t_1} \\
&\quad + (\omega\partial_\omega F_0)_i \int_{t_i}^{t} dt_1 e^{-Q(t,t_1)} \left\{ B(t_1) - \left(\frac{N}{S}\right)_{t_1} \int_{t_i}^{t_1} dt_2 e^{-P(t_1,t_2)} A(t_2) \right\} && (6.1.18)
\end{aligned}$$

where

$$A \equiv \left(\frac{\dot{N}}{N} \pm 2\frac{N'}{S} + \frac{\dot{S}}{S}\right)', \quad B \equiv 2\left\{\frac{N'}{S} \pm \left(\frac{\dot{R}}{R} - \frac{\dot{S}}{S}\right)\right\}, \tag{6.1.19}$$

and

$$
\begin{aligned}
P(t_1, t_2) &\equiv \mp \int_{t_1}^{t_2} dt \left(\frac{N}{S} \right)' , &(6.1.20)\\
Q(t_1, t_2) &\equiv 2 \int_{t_1}^{t_2} dt \left\{ \pm \frac{N}{S} \left(\frac{R'}{R} - \frac{N'}{N} \right) + \frac{\dot{R}}{R} - \frac{\dot{S}}{S} \right\} \\
&= 2 \left[\ln \frac{R}{N} \right]_{t_1}^{t_2} + 2 \int_{t_1}^{t_2} dt \left(\frac{\dot{N}}{N} - \frac{\dot{S}}{S} \right), &(6.1.21)
\end{aligned}
$$

and where the subscript 'i' denotes the value evaluated at the last scattering surface.

Note that $\omega \partial_\omega F_0$ is constant for any null geodesics, as in (6.1.16). Furthermore, as shown in Appendix, by inspecting the regularity of F at the center, as well as the behavior of some relevant geometric quantities near the center, we can observe that $\partial_r F_0$ does not contribute to the leading behavior of $\partial_i F$ in the limit $r \to 0$. Therefore, we only need to find the expression of $\partial_\mu F$.

Using the null geodesic equation for a radial geodesic, $\mu = \pm 1$, $dt/dr = \pm S/N$, we find that (6.1.20) and (6.1.21) become

$$
P(t_1, t_2) = \left[\ln \frac{S}{N} \right]_{t_1}^{t_2} + U(t_1, t_2), \qquad Q(t_1, t_2) = 2 \left[\ln \frac{R}{N} \right]_{t_1}^{t_2} + 2U(t_1, t_2), \tag{6.1.22}
$$

where

$$
U(t_1, t_2) \equiv \int_{t_1}^{t_2} dt \left(\frac{\dot{N}}{N} - \frac{\dot{S}}{S} \right). \tag{6.1.23}
$$

Substituting these into (6.1.18), we have

$$
\begin{aligned}
\partial_\mu F_0 \;=\;& \frac{R^2}{N^2} \frac{N_i^2}{R_i^2} e^{-2U(t,t_i)} (\partial_\mu F_0)_i - \frac{R^2}{N^2} \frac{N_i}{S_i} (\partial_r F_0)_i \int_{t_i}^{t} dt_1 \left(\frac{NS}{R^2} \right)_{t_1} e^{-2U(t,t_1) - U(t_1,t_i)} \\
&+ \frac{R^2}{N^2} (\omega \partial_\omega F_0)_i \int_{t_i}^{t} dt_1 \left(\frac{N^2}{R^2} \right)_{t_1} e^{-2U(t,t_1)} \left\{ B(t_1) - \int_{t_i}^{t_1} dt_2 e^{-U(t_1,t_2)} A(t_2) \right\}.
\end{aligned}
\tag{6.1.24}
$$

Here we note that the second and third terms in the right-side of the above equation have the form

$$
I(r) \equiv \int_{t_i}^{t} dt_1 \frac{V(t, t_1)}{R^2(t_1)} = \pm \int_{r_i}^{r} dr_1 \left(\frac{S}{N} \right)_{r_1} \frac{V(r, r_1)}{R^2(r_1)}, \tag{6.1.25}
$$

where $V(r, r_1)$ is a function of r and r_1 that is regular at $0 \leq r \leq r_1$. For the radial geodesic that reaches $r = 0$ at $t = t_0$, $I(r)$ behaves as

$$I(r) \simeq -\frac{1}{r}\frac{S_0}{N_0}\frac{V(0,0)}{(R_0')^2}. \qquad (6.1.26)$$

Substituting the above expression of $\partial_\mu F_0$ into

$$\partial_i F \simeq S_0 \frac{p^i}{p}(f_\nu)_0 = S_0 \frac{p^i}{p}\left(\frac{\partial_\mu F_0}{r}\right)_0, \quad \text{as } r \to 0, \qquad (6.1.27)$$

which is derived in Appendix (see (D.16)), we have

$$(\partial_i F)_0 \simeq \mp \frac{S_0^2}{N_0}\frac{p^i}{p}\left[\frac{N_i}{S_i}e^{-U(t_0,t_i)}(\partial_r F_0)_i + (\omega\partial_\omega F_0)_i\left\{\int_0^{r_i} dr e^{-U(t_0,t)}A(t) - B_0\right\}\right]. \qquad (6.1.28)$$

Thus, using this expression, we can write (6.1.12) as

$$\left(\frac{\delta T}{T}\right)^{(1)} = \mp\frac{\delta L\boldsymbol{n}\cdot\Omega}{N_0}\left[\frac{N_i}{S_i}e^{-U(t_0,t_i)}\left(\frac{\partial_r F_0}{\omega\partial_\omega F_0}\right)_i + \int_0^{r_i} dr e^{-U(t_0,t)}A(t) - B_0\right], \qquad (6.1.29)$$

where $\delta L\boldsymbol{n}$ is the position vector of the observer. This is our dipole formula for the CMB temperature anisotropy in the most general spherically symmetric spacetime. The regularity of the metric, (6.1.1) at the symmetry center implies that $N^2 \simeq C_1 + \mathrm{O}(r^2)$, $R/r \simeq C_2 + \mathrm{O}(r^2)$, and $S^2 \simeq (R/r)^2 + \mathrm{O}(r^2)$ near the center, with C_1, C_2 being some constants with respect to r. Using these estimations, we can check that the right-hand side of (6.1.29) is convergent, hence well-defined.

6.1.3 Analytic formula for CMB quadrupole

Next, we will derive the CMB quadrupole formula by inspecting the second-order derivatives, $\partial_\alpha\partial_\beta F_0$, $(\alpha, \beta = r, \omega, \mu)$. By differentiating the first row of (6.1.15) with respect to

$\ln\omega$, r, and μ, we obtain

$$
\begin{aligned}
\frac{d}{dt}\{(\omega\partial_\omega)^2F_0\} &= 0, & (6.1.30)\\
\frac{d}{dt}(\omega\partial_\omega\partial_rF_0) &= \mp\left(\frac{N}{S}\right)'\omega\partial_\omega\partial_rF_0 + A(\omega\partial_\omega)^2F_0, & (6.1.31)\\
\frac{d}{dt}(\omega\partial_\omega\partial_\mu F_0) &= 2\left\{\pm\frac{N}{S}\left(\frac{R'}{R}-\frac{N'}{N}\right)+\frac{\dot{R}}{R}-\frac{\dot{S}}{S}\right\}\omega\partial_\omega\partial_\mu F_0 & \\
&\quad -\frac{N}{S}\omega\partial_\omega\partial_rF_0 + B(\omega\partial_\omega)^2F_0, & (6.1.32)
\end{aligned}
$$

for a radial geodesic for which $\mu = \pm 1$. We can easily integrate this set of equations, and get the solutions

$$
\begin{aligned}
(\omega\partial_\omega)^2F_0 &= \{(\omega\partial_\omega)^2F_0\}_i, & (6.1.33)\\
\omega\partial_\omega\partial_rF_0 &= e^{-P(t,t_i)}(\omega\partial_\omega\partial_rF_0)_i + \{(\omega\partial_\omega)^2F_0\}_i\int_{t_i}^t dt_1 e^{-P(t,t_1)}A(t_1), & (6.1.34)\\
\omega\partial_\omega\partial_\mu F_0 &= e^{-Q(t,t_i)}(\omega\partial_\omega\partial_\mu F_0)_i - \{\omega\partial_\omega\partial_rF_0\}_i\int_{t_i}^t dt_1 e^{-Q(t,t_1)-P(t_1,t_i)}\left(\frac{N}{S}\right)_{t_1} & \\
&\quad +\{(\omega\partial_\omega)^2F_0\}_i\int_{t_i}^t dt_1 e^{-Q(t,t_1)}\left\{B(t_1)-\left(\frac{N}{S}\right)_{t_1}\int_{t_i}^{t_1}dt_2 e^{-P(t_1,t_2)}A(t_2)\right\}. & (6.1.35)
\end{aligned}
$$

Similarly, we can get the ordinary differential equations

$$
\begin{aligned}
\frac{d}{dt}(\partial_r^2F_0) &= \mp 2\left(\frac{N}{S}\right)'\partial_r^2F_0 \mp \left(\frac{N}{S}\right)''\partial_rF_0 + A'(\omega\partial_\omega F_0)_i + 2A\omega\partial_\omega\partial_rF_0, & (6.1.36)\\
\frac{d}{dt}(\partial_r\partial_\mu F_0) &= \left[\mp\left(\frac{N}{S}\right)' + 2\left\{\pm\frac{N}{S}\left(\frac{R'}{R}-\frac{N'}{N}\right)+\frac{\dot{R}}{R}-\frac{\dot{S}}{S}\right\}\right]\partial_r\partial_\mu F_0 - \frac{N}{S}\partial_r^2F_0 & \\
&\quad + B\omega\partial_\omega\partial_rF_0 + A\omega\partial_\omega\partial_\mu F_0 - \left(\frac{N}{S}\right)'\partial_rF_0 + C\partial_\mu F_0 + B'(\omega\partial_\omega F_0)_i, & (6.1.37)\\
\frac{d}{dt}(\partial_\mu^2F_0) &= 4\left\{\pm\frac{N}{S}\left(\frac{R'}{R}-\frac{N'}{N}\right)+\frac{\dot{R}}{R}-\frac{\dot{S}}{S}\right\}\partial_\mu^2F_0 - 2\frac{N}{S}\partial_r\partial_\mu F_0 + 2B\omega\partial_\omega\partial_\mu F_0 & \\
&\quad + D\partial_\mu F_0 + 2\left(\frac{\dot{S}}{S}-\frac{\dot{R}}{R}\right)(\omega\partial_\omega F_0)_i, & (6.1.38)
\end{aligned}
$$

for a radial geodesic, $\mu = \pm 1$. We can also integrate this set of equations, and obtain the solutions

$$\begin{aligned}
\partial_r^2 F_0 &= e^{-2P(t,t_i)}(\partial_r^2 F_0)_i \\
&\quad + \int_{t_i}^{t} dt_1 e^{-2P(t,t_1)} \left\{ \mp \left(\frac{N}{S}\right)'' \partial_r F_0 + A'(\omega\partial_\omega F_0)_i + 2A\omega\partial_\omega\partial_r F_0 \right\}_{t_1}, \quad (6.1.39) \\
\partial_r\partial_\mu F_0 &= e^{-P(t,t_i)-Q(t,t_i)}(\partial_r\partial_\mu F_0)_i \\
&\quad + \int_{t_i}^{t} dt_1 e^{-P(t,t_1)-Q(t,t_1)} \left\{ -\frac{N}{S}\partial_r^2 F_0 - \left(\frac{N}{S}\right)' \partial_r F_0 + C\partial_\mu F_0 \right. \\
&\qquad \left. + B'(\omega\partial_\omega F_0)_i + B\omega\partial_\omega\partial_r F_0 + A\omega\partial_\omega\partial_\mu F_0 \right\}_{t_1}, \quad (6.1.40) \\
\partial_\mu^2 F_0 &= e^{-2Q(t,t_i)}(\partial_\mu^2 F_0)_i + \int_{t_i}^{t} dt_1 e^{-2Q(t,t_1)} \left\{ -2\frac{N}{S}\partial_r\partial_\mu F_0 + D\partial_\mu F_0 \right. \\
&\qquad \left. + 2\left(\frac{\dot{S}}{S} - \frac{\dot{R}}{R}\right)(\omega\partial_\omega F_0)_i + 2B\omega\partial_\omega\partial_\mu F_0 \right\}_{t_1}, \\
&\qquad\qquad (6.1.41)
\end{aligned}$$

where

$$C \equiv 2\left\{ \pm\frac{N}{S}\left(\frac{R'}{R} - \frac{N'}{N}\right) + \frac{\dot{R}}{R} - \frac{\dot{S}}{S} \right\}', \quad D \equiv 2\frac{N}{S}\left(\frac{R'}{R} - \frac{N'}{N}\right) \pm 6\left(\frac{\dot{R}}{R} - \frac{\dot{S}}{S}\right). \quad (6.1.42)$$

Thus, we have the six solutions, $(\omega\partial_\omega)^2 F_0$, $\omega\partial_\omega\partial_r F_0$, etc. However, we note that $(\omega\partial_\omega)^2 F_0$ is just constant. Furthermore, by inspecting the regularity of F_0 at the center, as well as the behavior of some geometric quantities near the center, we can find that only $\partial_r^2 F_0$ becomes relevant to the evaluation of $\partial_i\partial_j F$ in the limit $r \to 0$. In fact, as we show in Appendix (see (D.30)),

$$\begin{aligned}
\partial_i\partial_j F \;\to\; & 2\left(\delta_{ij} - S_0^2\frac{p^i p^j}{p^2}\right)(f_2)_0 + S_0^2\frac{p^i p^j}{p^2}(\partial_r^2 F_0)_0 \\
& + \left\{ \left(\frac{a_\perp''}{a_\perp} - \frac{N''}{N}\right)_0 \delta_{ij} + C_0\frac{p^i p^j}{p^2} \right\}(\omega\partial_\omega F_0)_i, \quad (6.1.43)
\end{aligned}$$

where $f_2 = \partial_{r^2} F$ (see (D.7) in Appendix), and $a_\perp \equiv R/r$, which corresponds to the 'scale factor' perpendicular to the radial direction. Then, in terms of f_2 and $\partial_r^2 F_0$, the CMB quadrupole formula (6.1.13) is written as

$$
\begin{aligned}
\left(\frac{\delta T}{T}\right)^{(2)} &= -\frac{1}{2}\frac{\delta x^i \delta x^j}{(\omega\partial_\omega F_0)_i}\Bigg[2(\delta_{ij}-\Omega_i\Omega_j)(f_2)_0+\Omega_i\Omega_j(\partial_r^2 F_0)_0 \\
&\qquad + \left\{\frac{a''_\perp}{a_\perp}\delta_{ij}+a_\perp\left(S''-a''_\perp\right)\frac{\Omega_i\Omega_j}{S^2}\right\}_0(\omega\partial_\omega F_0)_i\Bigg] \\
&\quad +\frac{1}{2}\left\{\left(\frac{\delta T}{T}\right)^{(1)}\right\}^2\frac{\{(\omega\partial_\omega)^2F_0\}_i}{(\omega\partial_\omega F_0)_i},
\end{aligned}
\tag{6.1.44}
$$

where $\Omega_i \equiv \delta_{ij}x^j/r$. So, the remaining task is to find the expressions of the leading behavior of f_2 and $\partial_r^2 F_0$ at the center $r\to 0$.

First, we note from (D.28) that $(f_2)_0$ is given by

$$
(f_2)_0=\frac{1}{2}\left(\frac{\partial_r F_0}{r}\mp\frac{\partial_\mu F_0}{r^2}\right)_0 . \tag{6.1.45}
$$

In the limit $r\to 0$, from (6.1.18), the second term of this equation can be written as

$$
\begin{aligned}
\frac{\partial_\mu F_0}{r^2} &\to \frac{a_{\perp 0}^{\ 2}}{N_0^2}\frac{N_i^2}{R_i^2}e^{-2U(t_0,t_i)}(\partial_\mu F_0)_i \mp \frac{a_{\perp 0}^{\ 2}}{N_0^2}\int_0^{r_i}dr\frac{N^2}{r^2}\frac{e^{-2U(t_0,t)}}{a_\perp^2}\left(-\partial_r F_0+\frac{S}{N}B\omega\partial_\omega F_0\right) \\
&= \frac{a_{\perp 0}^{\ 2}}{N_0^2}\frac{N_i^2}{R_i^2}e^{-2U(t_0,t_i)}(\partial_\mu F_0)_i \pm \frac{a_{\perp 0}^{\ 2}}{a_{\perp i}^{\ 2}}\frac{e^{-2U(t_0,t_i)}}{r_i}\left(-\partial_r F_0+\frac{S}{N}B\omega\partial_\omega F_0\right)_i \\
&\quad \mp\left\{\frac{1}{r}\left(-\partial_r F_0+\frac{S}{N}B\omega\partial_\omega F_0\right)\right\}_0 \\
&\quad \pm\frac{a_{\perp 0}^{\ 2}}{N_0^2}\int_{t_i}^{t_0}dt\frac{N^2}{r}\frac{d}{dt}\left\{\frac{e^{-2U(t_0,t)}}{a_\perp^2}\left(-\partial_r F_0+\frac{S}{N}B\omega\partial_\omega F_0\right)\right\}.
\end{aligned}
\tag{6.1.46}
$$

From the second row of (6.1.15), we have

$$
\begin{aligned}
-\frac{N^2}{r}\frac{d}{dt}\left(\frac{\partial_r F_0}{a_\perp^2}e^{-2U(t_0,t)}\right) &= -\frac{N^2}{r}\Bigg[\left\{2\left(\frac{\dot S}{S}-\frac{\dot R}{R}-\frac{\dot N}{N}\mp\frac{N}{S}\frac{a'_\perp}{a_\perp}\right)\mp\left(\frac{N}{S}\right)'\right\}\frac{\partial_r F_0}{a_\perp^2}e^{-2U(t_0,t)} \\
&\qquad +\frac{A}{a_\perp^2}e^{-2U(t_0,t)}(\omega\partial_\omega F_0)_i\Bigg].
\end{aligned}
\tag{6.1.47}
$$

Thus, $(f_2)_0$ becomes

$$
\begin{aligned}
(f_2)_0 = &\mp \frac{1}{2}\frac{{a_\perp}_0^2}{N_0^2}\frac{N_i^2}{R_i^2}e^{-2U(t_0,t_i)}(\partial_\mu F_0)_i - \frac{{a_\perp}_0^2}{{a_\perp}_i^2}\frac{e^{-2U(t_0,t_i)}}{2r_i}\left(-\partial_r F_0 + \frac{S}{N}B\omega\partial_\omega F_0\right)_i \\
&+ \left(\frac{1}{r}\frac{S}{N}B\omega\partial_\omega F_0\right)_0 \\
&- \frac{1}{2}\frac{{a_\perp}_0^2}{N_0^2}\int_{t_i}^{t_0} dt \frac{N^2}{r}\left[-\left\{2\left(\frac{\dot{S}}{S} - \frac{\dot{R}}{R} - \frac{\dot{N}}{N} \mp \frac{N}{S}\frac{a'_\perp}{a_\perp}\right) + \left(\frac{N}{S}\right)'\right\}\partial_r F_0\right. \\
&\left. + \left\{\frac{\dot{N}}{N} - \left(\frac{\dot{S}}{S}\right)' - 2\left(\frac{\dot{N}}{N} - \frac{\dot{S}}{S}\right)\frac{S}{N}B + a_\perp^2\frac{d}{dt}\left(\frac{SB}{Na_\perp^2}\right)\right\}(\omega\partial_\omega F_0)_i\right]\frac{e^{-2U(t_0,t)}}{a_\perp^2}.
\end{aligned}
\tag{6.1.48}
$$

Next, from (6.1.39), we obtain

$$
\begin{aligned}
(\partial_r^2 F_0)_0 \;=\;& e^{-2P(t_0,t_i)}(\partial_r^2 F_0)_i \\
&+ \int_{t_i}^{t_0} dt e^{-2P(t_0,t)}\left\{2A\omega\partial_\omega\partial_r F_0 + A'(\omega\partial_\omega F_0)_i \pm \left(\frac{N}{S}\right)''\partial_r F_0\right\}.
\end{aligned}
\tag{6.1.49}
$$

Thus, substituting (6.1.48) and (6.1.49) with (6.1.17) and (6.1.34) into (6.1.44), we finally obtain the quadrupole formula for the CMB temperature anisotropy. As in the dipole formula case, under the assumption that our metric (6.1.1) is regular at the symmetry center, we can check that the above quadrupole formula is well-defined.

6.2 CMB anisotropies in local void models

In this section, the CMB temperature anisotropy formulae obtained in the previous section will be given concrete expressions for the LTB cosmological models, and then will be applied in some specific LTB models considered in the numerical analyses of Alnes and Amarzguioui [47], to check the consistency of the formulae. Further, the constraint on the location of off-center observers will be derived by our analytic formulae with the latest WMAP data, thereby updating the previous results numerically obtained. But before doing so, we will first briefly recapitulate the LTB metric.

6.2.1 LTB spacetime

A spherically symmetric spacetime with only non-relativistic matter, or dust, is described by the LTB metric, which may be given by setting in (6.1.1),

$$N^2 = 1, \quad S = \frac{R'(t,r)}{1-k(r)r^2}, \tag{6.2.1}$$

with $k(r)$ begin an arbitrary function of r. Then, the Einstein equations reduce to

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{2GM(r)}{R^3} - \frac{k(r)r^2}{R^2}, \qquad 4\pi\rho(t,r) = \frac{M'(r)}{R^2R'}, \tag{6.2.2}$$

where $M(r)$ is an arbitrary function of only r, and $\rho(t,r)$ is the energy density of the dust fluid. The solutions to (3.3.2) depend on the sign of $k(r)$ and can be expressed in parametric form: For $k(r) > 0$, we have

$$R(t,r) = \frac{M(r)}{k(r)r^2}(1-\cos\eta), \qquad t - t_s(r) = \frac{M(r)}{\{k(r)r^2\}^{\frac{3}{2}}}(\eta - \sin\eta), \tag{6.2.3}$$

where $t_s(r)$ is an arbitrary function of only r. For $k(r) = 0$, we have

$$R(t,r) = \left(\frac{9}{2}\right)^{\frac{1}{3}} M^{\frac{1}{3}}(r)\{t - t_s(r)\}^{\frac{2}{3}}. \tag{6.2.4}$$

For $k(r) < 0$, we have

$$R(t,r) = \frac{M(r)}{-k(r)r^2}(\cosh\eta - 1), \qquad t - t_s(r) = \frac{M(r)}{\{-k(r)r^2\}^{\frac{3}{2}}}(\sinh\eta - \eta). \tag{6.2.5}$$

The area radius $R(t,r)$ vanishes at $t = t_s(r)$, so that $t_s(r)$ is called the big-bang time. The solutions admit three arbitrary functions $k(r)$, $M(r)$ and $t_s(r)$, but due to one degree of freedom in rescaling r, only two of them are independent. By appropriately choosing the profile of these two arbitrary functions, one can construct LTB cosmological models that can reproduce the observed SN Ia distance-redshift relation.

6.2.2 CMB dipole in local void models

In the LTB cosmological model, in addition to $a_\perp = R/r$ defined previously, we also introduce the 'scale factor along the radial direction' by $a_{//}(t,r) \equiv R'(t,r)$. Accordingly, we also define two Hubble expansion rates in the radial and azimuthal direction, respectively, by

$$H_{//} \equiv \frac{\dot{S}}{S} = \frac{\dot{a}_{//}}{a_{//}}, \quad H_\perp \equiv \frac{\dot{R}}{R} = \frac{\dot{a}_\perp}{a_\perp}. \tag{6.2.6}$$

From (6.1.19) and (6.1.23), we obtain

$$U(t_1,t_2) = -\int_{t_1}^{t_2} dt H_{//}, \qquad A = H'_{//}, \qquad B = \pm 2(H_{//} - H_\perp). \tag{6.2.7}$$

Then, the analytic formula for the CMB temperature anisotropy dipole (6.1.29) takes the form

$$\left(\frac{\delta T}{T}\right)^{(1)} = \mp \delta L \boldsymbol{n} \cdot \Omega \left\{ \frac{e^{-U(t_0,t_i)}}{S_i} \left(\frac{\partial_r F_0}{\omega \partial_\omega F_0}\right)_i + \int_0^{r_i} dr H'_{//} e^{-U(t_0,t)} \right\}. \tag{6.2.8}$$

Now, using this formula, we derive some constraints concerning the position of off-center observers. In general, the CMB temperature anisotropy can be decomposed in terms of the spherical harmonics Y_{lm} by $\delta T/T = \sum_{l,m} a_{lm} Y_{lm}$. We are interested in a_{10} as the dipole moment. From (6.2.8), we obtain

$$a_{10} = \sqrt{\frac{4\pi}{3}} \delta L \left\{ \frac{e^{-U(t_0,t_i)}}{S_i} \left(\frac{\partial_r F_0}{\omega \partial_\omega F_0}\right)_i + \int_0^{r_i} dr H'_{//} e^{-U(t_0,t)} \right\}. \tag{6.2.9}$$

Assuming that the universe is locally in thermal equilibrium at the last scattering surface, the second term in the bracket is of the order of $(\partial_r T/T)_i$ because F_0 is isotropic and depends only on ω/T_i. In the models we consider in this chapter, this term can be neglected because the void size is sufficiently smaller than the horizon size and therefore, the observed region of the LTB universe is homogeneous with good accuracy on the last scattering surface. Furthermore, in order to estimate this term correctly, we have to specify the universe model before the last scattering. This is beyond the scope of this chapter. Therefore, we have only estimated the contribution of the second term numerically. For the LTB model considered in [47], we have found that the induced a_{10} is about 1.23×10^{-3} or less, according to WMAP data [130], which implies that the distance from the observer to the center, δL, has to be, $\delta L \lesssim 16$Mpc. This is consistent with the numerical result of [47]. We have also applied this formula to various LTB models, and found, for example, $\delta L \lesssim 14$Mpc in the Garfinkle model [48], and $\delta L \lesssim 12$Mpc in the GBH model [52].

6.2.3 CMB quadrupole in local void models

As for the quadrupole moment in the LTB model, from (6.1.48) and (6.1.49), we obtain

$$\begin{aligned}
(f_2)_0 &= \mp\frac{1}{2}\frac{a_0^2}{R_i^2}e^{-2U(t_0,t_i)}(\partial_\mu F_0)_i - \frac{a_0^2}{{a_\perp}_i^2}\frac{e^{-2U(t_0,t_i)}}{2r_i}\left(-\partial_r F_0 + \frac{a_{//}}{\sqrt{1-k(r)r^2}}B\omega\partial_\omega F_0\right)_i \\
&\quad -\frac{a_0^2}{2}\int_{t_i}^{t_0} dt\frac{1}{r}\left[-\left\{2\left(H_{//} - H_\perp \mp \frac{\xi}{a_{//}}\frac{a_\perp'}{a_\perp}\right) + \left(\frac{\xi}{a_{//}}\right)'\right\}\partial_r F_0\right. \\
&\qquad \left. + \left\{-H_{//}' + 2H_{//}\frac{a_{//}}{\xi}B + a_\perp^2\frac{d}{dt}\left(\frac{SB}{Na_\perp^2}\right)\right\}(\omega\partial_\omega F_0)_i\right]\frac{e^{-2U(t_0,t)}}{a_\perp^2},
\end{aligned} \tag{6.2.10}$$

$$\begin{aligned}
(\partial_r^2 F_0)_0 &= e^{-2P(t_0,t_i)}(\partial_r^2 F_0)_i \\
&\quad + \int_{t_i}^{t_0} dt e^{-2P(t_0,t)}\left\{2H_{//}'\omega\partial_\omega\partial_r F_0 + H_{//}''(\omega\partial_\omega F_0)_i + \left(\frac{\xi}{a_{//}}\right)''\partial_r F_0\right\},
\end{aligned} \tag{6.2.11}$$

where $a_0 \equiv S_0 = a_{//0} = a_{\perp 0}$, and $\xi \equiv \sqrt{1-k(r)r^2}$, just for notational simplicity, and where

$$P(t_1,t_2) = \mp\int_{t_1}^{t_2} dt\left(\frac{\xi}{a_{//}}\right)', \tag{6.2.12}$$

which is obtained from (6.1.20). Thus, we now have the analytic formula for quadrupole moment of the CMB temperature anisotropy in the LTB model: (6.1.44) together with (6.2.10) and (6.2.11).

From (6.1.44), we derive

$$a_{20} = -\sqrt{\frac{16\pi}{45}}\frac{(\delta L)^2}{2a_0^2}\left\{-\frac{2(f_2)_0}{(\omega\partial_\omega F_0)_i} + \frac{(\partial_r^2 F_0)_0}{(\omega\partial_\omega F_0)_i} + \frac{(S'' - a_\perp'')_0}{a_0}\right\} + \frac{(a_{10})^2}{2\sqrt{5\pi}}\frac{((\omega\partial_\omega)^2 F_0)_i}{(\omega\partial_\omega F_0)_i}. \tag{6.2.13}$$

If the universe is locally in thermal equilibrium at the beginning, we can set $(\partial_\alpha \partial_\mu F_0)_i$ to be zero. Further, for the same reason as we explained for the dipole formula, we neglect the term $(\partial_\alpha \partial_r F_0)_i$ in this chapter. Under these assumption, we have estimated the quadrupole moment using this formula numerically for the model in [47], and found that $a_{20} \simeq 8.61 \times 10^{-7}$. This is consistent with the numerical result of [47]. For other models, for example, $a_{20} \simeq 5.51 \times 10^{-6}$ in the Garfinkle model [48], and $a_{20} \simeq -9.27 \times 10^{-7}$ in the GBH model [52].

Chapter 7

Summary and discussion

In this thesis, we have provided formalisms for cosmological tests of models that can explain the apparent accelerated expansion of the present universe, such as modified gravity models and local void models, and also give constraints on local void models. Although the universe is isotropic and homogeneous on sufficiently large scales under the cosmological principle, the universe has inhomogeneities in practice. Therefore, we particularly focus on inhomogeneities of the universe in this thesis.

In Chapter 5, we have addressed the effective gravitational stress-energy tensor for short-wavelength perturbations in the simple class of $f(R)$ gravity of R^2 type in the cosmological context. As in Isaacson's formula for Einstein gravity reviewed in Sec. 5.1, we have obtained the field equations for the background metric with a backreaction source term $T^{\rm eff}_{\mu\nu}$ of order $O(1)$ of the small parameter ϵ. Reflecting the fact that our $f(R)$ theory contains higher-order derivative terms, the source term or the effective stress-energy tensor $T^{\rm eff}_{\mu\nu}$ takes, as given in (5.2.13), quite a complex form that contains, in principle, terms of fourth-order derivatives, schematically expressed as $\langle \nabla h \nabla h \nabla h \nabla h \rangle$. The resultant expression, (5.2.13), of the effective stress-energy tensor in fact applies to any background metric $g^{(0)}_{\mu\nu}$; until this point, no symmetry assumption on the background metric has been used. Then, by imposing that our background has FLRW symmetry, we have derived our effective stress-energy tensor for short-wavelength metric perturbations in cosmological models. At this point, thanks to the background FLRW symmetry, the spacetime averaging over several wavelengths and our choice of the constant $S_1 = 0$, the expression of our effective stress-energy tensor has been significantly reduced to have the simple form, (5.2.21). We have also shown that the effective stress-energy tensor obtained is traceless, so that it acts like a radiation fluid as in the Einstein gravity case and thus, in particular, cannot mimic dark energy.

We would like to stress that in order to obtain the traceless feature of our effective stress-energy tensor, we have set $S_1 = 0$. However, the field equations for Einstein gravity need not be, a priori, reproduced in the limit to Einstein gravity: $c \to 0$. In that case S_1 could take a non-vanishing value and give rise to a cosmological-constant-like term in our effective stress-energy (5.2.20). It would be interesting to consider the question of whether there exists any sensible way to provide the right sign and magnitude for S_1 so that (5.2.20) can mimic dark energy within the framework of our modified gravity theory.

Since any $f(R)$ gravity theory is known to be equivalent to a scalar-tensor theory, we have cast our $f(R)$ theory into the corresponding scalar-tensor theory. Then, within the scalar-tensor theory, we have derived the effective stress-energy tensor for short-wavelength perturbations of the scalar field and checked consistency with the stress-energy tensor obtained within the metric perturbations of the original $f(R)$ theory.

Although we have focused on the R^2 model, especially concerning the FLRW background, we have pushed forward our calculations with a general $f(R)$ gravity about an arbitrary background as far as possible, and have not used the property of the R^2 model about the FLRW background, up to (5.3.11) in Sec. 5.3. We can immediately note that (5.3.11) does not involve any terms of fourth-order derivatives but has only terms of the square of first-order derivatives of perturbations $h_{\mu\nu}$ and $\delta\phi$. We have mainly worked in the Jordan frame in Sec. 5.2. When working in the Einstein frame, we have shown the traceless nature of the effective stress-energy tensor by merely using the background FLRW symmetry; we did not have to set S_1 to any particular value. This is in contrast to the case of the Jordan frame. This result obtained within the framework of scalar-tensor theory indicates that the higher-order derivatives could also vanish in the metric framework of general $f(R)$ gravity theory for a generic background. However, to see whether this is indeed the case needs further involved calculation, and is beyond the scope of this chapter. This is left open for future study.

Our formulas derived in Sec. 5.2 deal directly with the scalar curvature R and the

Ricci tensor $R_{\mu\nu}$, and therefore should apply to similar analyses of other modified gravity theories which contain higher-order curvature terms composed of R, $R_{\mu\nu}$, and $R^a{}_{bcd}$ and which cannot even be cast in the form of a scalar-tensor theory. It would be interesting to consider an extension of our present work to a wide class of modified gravity theories with high-rank curvatures.

In Chapter 6, we derived the analytic formulae for the dipole (6.1.29) and quadrupole (6.1.44) moments of the CMB temperature anisotropy in general spherically symmetric spacetimes, including the LTB cosmological model as a special case. The formulae can be used to compare consequences of the LTB/local void models with observations of the CMB temperature anisotropy rigorously. The formulae also enable us to identify physical origins of the CMB temperature anisotropy in the LTB models. For example, in the CMB dipole formula (6.1.29), the first term comes from the initial (spherical) inhomogeneity at the last scattering surface, while the second term represents the integrated Sachs-Wolfe effect. Note that the first term also contains a contribution that reduces to the second-order ISW effect in the spatially homogeneous case.

We have checked the consistency of our formulae for both dipole and quadrupole, with the widely-used recent numerical results for the special LTB model by H. Alnes and M. Amarzguioui [47]. Furthermore, we applied our formulae to other LTB models, such as those in [48, 52] and in particular, for the dipole moment, we found the constraints on the distance between the void center and an off-center observer, by using the latest WMAP data.

We can also utilise our analytic quadrupole formula to discuss the relevance of the LTB model to observed anomalies. For example, the observed magnitude of the quadrupole is known to be significantly lower than the ΛCDM model predicts. This is usually understood as a cosmic variance, i.e., to be produced by a special feature of our Universe, one particular realisation of the statistical ensemble. Because the local void model is one of such realisation with a very low probalibity in the standard ΛCDM model, it is tempting to see whether the quadrupole anomaly of the CMB anisotropy can be explained by a local void model. Unfortunately, however, the above analysis of the constraint on the oberver offset by the dipole moment implies that the observed anomaly cannot be explained soly by the induced quadrupole moment in LTB models such as those in [47, 48, 52]. Nevetheless, this result is not conclusive. For example, we have implicitly assumed that the off-center observer stays at a fixed comoving position. If the observer has a peculiar velocity pointed toward the center of the void, however, the value of δL could be chosen to be much larger than the case with no peculiar velocity. If it is the case, then the observed anomaly of the quadrupole could be explained within the LTB models of [47, 48, 52]. Therefore, it would also be worth attempting to develop other analytic formulae concerning CMB polarizations, lensing effects, etc. (cf. [131]) that can be used to distinguish the LTB and FLRW cosmologies.

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Appendix A

Perturbation formulas in transverse-traceless gauge

The conditions of transverse-traceless waves are $\nabla_\mu h^{\mathrm{TT}\mu}{}_\nu = h^{\mathrm{TT}\mu}{}_\mu = 0$. The Ricci tensors for $h^{\mathrm{TT}}_{\mu\nu}$ are

$$
\begin{aligned}
C^{\mu(1)}_{\mu\nu}[h^{\mathrm{TT}}] &= 0\,, && \text{(A.1)}\\
R^{(1)}_{\mu\nu}[h^{\mathrm{TT}}] &= \nabla_\lambda C^{\lambda(1)}_{\nu\mu}[h^{\mathrm{TT}}]\,, \\
&= -\frac{1}{2}\Box h^{\mathrm{TT}}_{\mu\nu} && \text{(A.2)}\\
R^{(2)}_{\mu\nu}[h^{\mathrm{TT}}] &= 2\nabla_{[\lambda}C^{\lambda(2)}_{\nu]\mu}[h^{\mathrm{TT}}] - C^{\lambda(1)}_{\sigma\nu}[h^{\mathrm{TT}}]C^{\sigma(1)}_{\lambda\mu}[h^{\mathrm{TT}}] \\
&= -\frac{1}{2}h^{\mathrm{TT}\lambda\sigma}\nabla_\lambda\Big(\nabla_\nu h^{\mathrm{TT}}_{\sigma\mu} + \nabla_\mu h^{\mathrm{TT}}_{\nu\sigma} - \nabla_\sigma h^{\mathrm{TT}}_{\nu\mu}\Big) \\
&\quad +\frac{1}{2}\nabla_\nu\left\{h^{\mathrm{TT}\lambda\sigma}\Big(\nabla_\lambda h^{\mathrm{TT}}_{\sigma\mu} + \nabla_\mu h^{\mathrm{TT}}_{\lambda\sigma} - \nabla_\sigma h^{\mathrm{TT}}_{\lambda\mu}\Big)\right\} \\
&\quad -\frac{1}{4}\Big(\nabla_\sigma h^{\mathrm{TT}\lambda}{}_\nu + \nabla_\nu h^{\mathrm{TT}}{}_\sigma{}^\lambda - \nabla^\lambda h^{\mathrm{TT}}_{\sigma\nu}\Big)\Big(\nabla_\lambda h^{\mathrm{TT}\sigma}{}_\mu + \nabla_\mu h^{\mathrm{TT}}{}_\lambda{}^\sigma - \nabla^\sigma h^{\mathrm{TT}}_{\lambda\mu}\Big)\,, \\
&&& \text{(A.3)}
\end{aligned}
$$

and the scalar curvatures for $h^{\mathrm{TT}}_{\mu\nu}$ are

$$\begin{aligned} R^{(1)}[h^{\mathrm{TT}}] &= g^{(0)\mu\nu} R^{(1)}_{\mu\nu}[h^{\mathrm{TT}}] \\ &= 0\,, \qquad \text{(A.4)} \\ R^{(2)}[h^{\mathrm{TT}}] &= g^{(0)\mu\nu} R^{(2)}_{\mu\nu}[h^{\mathrm{TT}}] - h^{\mathrm{TT}\mu\nu} R^{(1)}_{\mu\nu}[h^{\mathrm{TT}}] \\ &= \frac{3}{4}\nabla^{\mu} h^{\mathrm{TT}\nu\lambda}\nabla_{\mu} h^{\mathrm{TT}}_{\nu\lambda} - \frac{1}{2}\nabla^{\mu} h^{\mathrm{TT}\nu\lambda}\nabla_{\lambda} h^{\mathrm{TT}}_{\mu\nu} + h^{\mathrm{TT}\mu\nu}\Box h^{\mathrm{TT}}_{\mu\nu}\,, \qquad \text{(A.5)} \end{aligned}$$

where we have used $[\nabla_\mu, \nabla_\nu] h^{\mathrm{TT}}_{\lambda\sigma} = R_{\lambda}{}^{\alpha}{}_{\mu\nu}[g^{(0)}] h^{\mathrm{TT}}_{\alpha\sigma} + R_{\sigma}{}^{\alpha}{}_{\mu\nu}[g^{(0)}] h^{\mathrm{TT}}_{\lambda\alpha} = \mathrm{O}(\epsilon)$. From (5.1.6) and (A.2), we find $\Box h^{\mathrm{TT}}_{\mu\nu} = 0$. Using these we find

$$\left\langle R^{(2)}_{\mu\nu}[h^{\mathrm{TT}}] \right\rangle = -\frac{1}{4}\nabla_\mu h^{\mathrm{TT}\lambda\sigma}\nabla_\nu h^{\mathrm{TT}}_{\lambda\sigma}, \qquad \text{(A.6)}$$

$$\left\langle R^{(2)}[h^{\mathrm{TT}}] \right\rangle = 0\,. \qquad \text{(A.7)}$$

Appendix B

General treatment of high frequency limit in general relativity

B.1 Characterization of high frequency limit in general relativity

The metric with the linear perturbation h_{ab} is described by

$$g_{ab} = \gamma_{ab} + h_{ab}, \tag{B.1}$$

where γ_{ab} is the background metric including the backreaction from perturbations, and is normalized: $\gamma_{ab} \sim \mathrm{O}(1)$. h_{ab} and its derivatives are of order $h_{ab} \sim \mathrm{O}(\epsilon)$ and

$$\begin{aligned} \nabla_{a_1}\nabla_{a_2}\cdots\nabla_{a_m}h_{bc} &\sim \mathrm{O}\left(\frac{\epsilon}{(\lambda/L)^m}\right) \\ &\sim \mathrm{O}\left(\epsilon^{1-m}\right), \end{aligned} \tag{B.2}$$

where ∇_a denotes the covariant derivative with respect to γ_{ab}, so that $\nabla_a\gamma_{bc} = 0$, and ϵ is the smallness parameter. λ is the wavelength of perturbations, such as $h \sim \epsilon\sin(x/\lambda)$ and $\lambda/L \sim \mathrm{O}(\epsilon)$. L is a characteristic length over the background, and normalized: $L \sim 1$. The inverse metric takes the form

$$g^{ab} = \gamma^{ab} - h^{ab} + h^a{}_c h^{cb} + \cdots, \tag{B.3}$$

where $h^{ab} \equiv \gamma^{ac}\gamma^{bd}h_{cd}$.

We can not have $\nabla_a h_{bc} \to 0$ pointwise $\lambda \to 0$, but suitable spacetime averages of $\nabla_a h_{bc}$ will go to 0. More precisely, if f^{abc} is any smooth tensor field of compact support, we have

$$\begin{aligned} \int f^{abc}\nabla_a h_{bc} &= -\int \left(\nabla_a f^{abc}\right) h_{bc} \\ &\underset{\lambda\to 0}{\longrightarrow} 0. \end{aligned} \tag{B.4}$$

If this equation holds for all test tensor fields f^{abc}, we say that $\nabla_a h_{bc} \to 0$ *weakly*.

There is the general relationship between the Ricci tensor of g_{ab} and of γ_{ab}, namely

$$R_{ab} = R_{ab}[\gamma] + 2\nabla_{[c}C^c_{b]a} + 2C^c_{d[c}C^d_{b]a}, \tag{B.5}$$

where

$$\begin{aligned} C^a_{bc} &\equiv \Gamma^a_{bc} - \Gamma^a_{bc}[\gamma] \\ &= \frac{g^{ad}}{2}\left(\nabla_b g_{dc} + \nabla_c g_{bd} - \nabla_d g_{bc}\right). \end{aligned} \tag{B.6}$$

Because R_{ab} includes $g^{-1}\nabla\nabla g$, $\nabla g^{-1}\nabla g$ and $g^{-1}g^{-1}\nabla g\nabla g$, we can find

$$R^{(n)}_{ab}[h] \sim \mathrm{O}\left(\epsilon^{n-2}\right), \tag{B.7}$$

where n is the number of h_{ab} included in R_{ab} and a natural number. We also find

$$R^{(n)}[h] \sim G^{(n)}_{ab}[h] \sim \mathrm{O}\left(\epsilon^{n-2}\right), \quad \nabla_{a_1}\nabla_{a_2}\cdots\nabla_{a_m}R^{(n)}[h] \sim \mathrm{O}\left(\epsilon^{n-2-m}\right), \tag{B.8}$$

where $R = g^{ab}R_{ab}$, $G_{ab} = R_{ab} - g_{ab}R/2$ is the Einstein tensor, and $R^{(n)}[h]$ and $G^{(n)}_{ab}[h]$ do not include $R_{ab}[\gamma]$.

We define a smooth tensor field μ_{abcdef} as

$$\mu_{abcdef} \equiv \underset{\lambda\to 0}{\text{w-lim}}\left(\nabla_a h_{cd}\nabla_b h_{ef}\right), \tag{B.9}$$

where "w-lim" means the weak limit. It follows immediately that $\mu_{abcdef} = \mu_{ab(cd)(ef)} = \mu_{ba(ef)(cd)}$. The tensor field μ_{abcdef} also has the symmetry $\mu_{abcdef} = \mu_{(ab)cdef}$ because

$$\begin{aligned}
\mu_{[ab]cdef} &= \operatorname*{w\text{-}lim}_{\lambda\to 0}\left(\nabla_{[a|}h_{cd}\nabla_{|b]}h_{ef}\right) \\
&= -\operatorname*{w\text{-}lim}_{\lambda\to 0}\left(h_{cd}\nabla_{[a}\nabla_{b]}h_{ef}\right) \\
&= -\operatorname*{w\text{-}lim}_{\lambda\to 0}\left(h_{cd}R_{ab(e}{}^{g}[\gamma]h_{f)g}\right) \\
&= 0,
\end{aligned} \tag{B.10}$$

where $R_{abc}{}^{d}[\gamma]$ is the Riemann tensor. Using the Leibniz rule, we derive the relation

$$\begin{aligned}
\operatorname*{w\text{-}lim}_{\lambda\to 0}\left(h_{cd}\nabla_a\nabla_b h_{ef}\right) &= -\operatorname*{w\text{-}lim}_{\lambda\to 0}\left(\nabla_a h_{cd}\nabla_b h_{ef}\right) \\
&= -\mu_{abcdef}.
\end{aligned} \tag{B.11}$$

B.2 Effective gravitational stress-energy tensor

The Einstein equation is

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R = \kappa^2 T_{ab}, \tag{B.1}$$

where $\kappa^2 = 8\pi G$. We can derive the vacuum Einstein equation of $\mathrm{O}\left(\epsilon^{-1}\right)$:

$$G^{(1)}_{ab}[h] = R^{(1)}_{ab}[h] - \frac{1}{2}\gamma_{ab}R^{(1)}[h] = 0, \tag{B.2}$$

and that of $\mathrm{O}\,(1)$:

$$G_{ab}[\gamma] = -\operatorname*{w\text{-}lim}_{\lambda\to 0} G^{(2)}_{ab}[h] \equiv \kappa^2 T^{\mathrm{eff}}_{ab}, \tag{B.3}$$

where T^{eff}_{ab} is the effective gravitational stress-energy tensor. From (B.2), we find

$$R^{(1)}_{ab}[h] = 0. \tag{B.4}$$

Then, $T^{\rm eff}_{ab}$ becomes

$$\begin{aligned}
\kappa^2 T^{\rm eff}_{ab} &= -\operatorname*{w-lim}_{\lambda\to 0}\left(R^{(2)}_{ab}[h]-\frac{1}{2}\gamma_{ab}R^{(2)}[h]\right)\\
&= -\operatorname*{w-lim}_{\lambda\to 0}\left(R^{(2)}_{ab}[h]-\frac{1}{2}\gamma_{ab}\gamma^{cd}R^{(2)}_{cd}[h]\right)\\
&= \frac{\gamma_{ab}}{8}\left(2\mu^{cd}{}_{cd}{}^{e}{}_{e}+\mu^{c}{}_{c}{}^{de}{}_{de}-2\mu^{cd}{}_{c}{}^{e}{}_{de}-\mu^{c}{}_{c}{}^{d}{}_{d}{}^{e}{}_{e}\right)\\
&\quad -\frac{1}{4}\left(4\mu^{c}{}_{[cd]a}{}^{d}{}_{b}+2\mu^{c}{}_{(ab)c}{}^{d}{}_{d}-\mu^{c}{}_{c}{}^{d}{}_{dab}-\mu_{ab}{}^{cd}{}_{cd}\right),
\end{aligned} \tag{B.5}$$

because

$$\begin{aligned}
\operatorname*{w-lim}_{\lambda\to 0} R^{(2)}_{ab}[h] &= \operatorname*{w-lim}_{\lambda\to 0}\left(2C^{c\,(1)}_{d[c}[h]C^{d\,(1)}_{b]a}[h]\right)\\
&= \frac{1}{4}\left(4\mu^{c}{}_{[cd]a}{}^{d}{}_{b}+2\mu^{c}{}_{(ab)c}{}^{d}{}_{d}-\mu^{c}{}_{c}{}^{d}{}_{dab}-\mu_{ab}{}^{cd}{}_{cd}\right),
\end{aligned} \tag{B.6}$$

where we use the fact that $\nabla_d C^{a(2)}_{bc}[h]\to 0$ weakly since $C^{a(2)}_{bc}[h]\to 0$ weakly, and indices are raised and lowered with γ^{ab} and γ_{ab}.

μ_{abcdef} is completely determined by the combination $\alpha_{abcdef}\equiv\mu_{[c|[ab]|d]ef}$ and $\beta_{abcdef}\equiv\mu_{(abcd)ef}$, so that

$$\mu_{abcdef} = -\frac{4}{3}\left(\alpha_{c(ab)def}+\alpha_{e(ab)fcd}-\alpha_{e(cd)fab}\right)+\beta_{abcdef}+\beta_{abefcd}-\beta_{cdefab}, \tag{B.7}$$

since

$$\begin{aligned}
-\frac{4}{3}\alpha_{c(ab)def}+\beta_{abcdef} &= -\frac{2}{3}\left(\mu_{[b|[ca]|d]ef}+\mu_{[a|[cb]|d]ef}\right)+\mu_{(abcd)ef}\\
&= \frac{1}{2}\left(\mu_{abcdef}+\mu_{cdabef}\right).
\end{aligned} \tag{B.8}$$

Note that $\alpha_{abcdef}=\alpha_{[ab][cd](ef)}=\alpha_{[cd][ab](ef)}$ and $\beta_{abcdef}=\beta_{(abcd)(ef)}$ from their definitions.

We can also find

$$\begin{aligned}
\alpha_{[abc]def} &= \frac{1}{3}\left(\alpha_{[ab]cdef} + \alpha_{[bc]adef} + \alpha_{[ca]bdef}\right) \\
&= \frac{1}{3}\left(\alpha_{abcdef} + \alpha_{bcadef} + \alpha_{cabdef}\right) \\
&= \frac{1}{3}\left(\mu_{[c|[ab]|d]ef} + \mu_{[a|[bc]|d]ef} + \mu_{[b|[ca]|d]ef}\right) \\
&= 0 \qquad \text{(B.9)}
\end{aligned}$$

and

$$\begin{aligned}
\alpha_{abc\ \ ef}^{\ \ \ \ b} &= \mu_{[c|[ab]|d]ef}\gamma^{bd} \\
&= -\operatorname*{w-lim}_{\lambda\to 0}\left(h_{ef}\nabla_{[a|}\nabla_{[c}h_{d]|b]}\gamma^{bd}\right) \\
&= -\operatorname*{w-lim}_{\lambda\to 0}\left(h_{ef}\nabla_{[a}C^{b(1)}_{b]c}[h]\right) \\
&= -\operatorname*{w-lim}_{\lambda\to 0}\left(h_{ef}\frac{1}{2}R^{(1)}_{ac}[h]\right) \\
&= 0, \qquad \text{(B.10)}
\end{aligned}$$

because $[\nabla_a, \nabla_b]h_{cd} = \mathrm{O}(\epsilon)$ and (B.4).

Thus, (B.5) becomes

$$\begin{aligned}
\kappa^2 T^{\mathrm{eff}}_{ab} &= \frac{\gamma_{ab}}{8}\cdot 0 - \frac{1}{4}\left\{-\frac{4}{3}\left(2\alpha^{d}{}_{(ca)b}{}^{c}{}_{d} - \alpha^{c}{}_{(ab)}{}^{d}{}_{cd} - \alpha_{c(ab)d}{}^{cd}\right)\right\} \\
&= \alpha_{a}{}^{c}{}_{b}{}^{d}{}_{cd}, \qquad \text{(B.11)}
\end{aligned}$$

and then

$$T^{\mathrm{eff}\,a}{}_{a} = 0, \qquad \text{(B.12)}$$

because α_{abcdef} is trace-free on its first four indices. Note that the right hand side is independent of β_{abcdef}.

B.3 Gauge

A infinitesimal gauge transformation is expressed as $x^a \to \tilde{x}^a = x^a + \xi^a$, where ξ^a is regarded as a quantity of the same order as the perturbations. The change of the perturbed metric tensor under this gauge transformation is given by

$$\begin{aligned} \tilde{g}_{ab}(x) &\simeq g_{ab}(x) - 2\gamma_{c(a}(x)\partial_{b)}\xi^c - \xi^c\partial_c\gamma_{ab}(x) \\ &= \gamma_{ab}(x) + h_{ab}(x) - 2\gamma_{c(a}(x)\partial_{b)}\xi^c - \xi^c\partial_c\gamma_{ab}(x) \end{aligned} \tag{B.1}$$

because

$$\begin{aligned} \tilde{g}_{ab}(\tilde{x}) &= \frac{\partial x^c}{\partial \tilde{x}^a}\frac{\partial x^d}{\partial \tilde{x}^b} g_{cd}(x) \\ &\simeq \left(\delta^c{}_a - \frac{\partial \xi^c}{\partial \tilde{x}^a}\right)\left(\delta^d{}_b - \frac{\partial \xi^d}{\partial \tilde{x}^b}\right)\left(g_{cd}(\tilde{x}) - \xi^l\tilde{\partial}_l g_{cd}(\tilde{x})\right). \end{aligned} \tag{B.2}$$

Thus, we have

$$\tilde{\gamma}_{ab}(x) = \gamma_{ab}(x), \tag{B.3}$$

$$\epsilon\tilde{h}_{ab}(x) = h_{ab}(x) - 2\gamma_{c(a}(x)\partial_{b)}\xi^c - \xi^c\partial_c\gamma_{ab}(x). \tag{B.4}$$

We define some tensor fields $\tau_{abc}{}^{d}{}_{e}{}^{f}$ and $\sigma_{abc}{}^{d}{}_{ef}$ as

$$\underset{\lambda\to 0}{\text{w-lim}}\left\{\nabla_a\left(\tilde{X}_c - X_c\right)\nabla_b\left(\tilde{Y}_e - Y_e\right)\right\} \equiv \tau_{abc}{}^{d}{}_{e}{}^{f}X_dY_f, \tag{B.5}$$

$$\underset{\lambda\to 0}{\text{w-lim}}\left\{\nabla_a\left(\tilde{X}_c - X_c\right)\nabla_b\left(\epsilon h_{ef}\right)\right\} \equiv \sigma_{abc}{}^{d}{}_{ef}X_d \tag{B.6}$$

for any smooth fields X^a and Y_b. It follows that $\tau_{abc}{}^{d}{}_{e}{}^{f}$ and $\sigma_{abc}{}^{d}{}_{ef}$ have symmetries $\sigma_{abc}{}^{d}{}_{ef} = \sigma_{(abc)}{}^{d}{}_{(ef)}$ and $\tau_{abc}{}^{d}{}_{e}{}^{f} = \tau_{(abc}{}^{(d}{}_{e)}{}^{f)}$. The symmetries $\sigma_{[ab]c}{}^{d}{}_{ef} = \sigma_{abc}{}^{d}{}_{[ef]} = 0$ are clear. Note that

$$\begin{aligned} 2\nabla_{[a}\left(\tilde{X}_{b]} - X_{b]}\right) &= (d\tilde{X})_{ab} - (dX)_{ab} \\ &= \widetilde{(dX)}_{ab} - (dX)_{ab}, \end{aligned} \tag{B.7}$$

where d is the exterior derivative and we use the fact that $(d\tilde{X})_{ab} = \widetilde{(dX)}_{ab}$. Since $\tilde{X}_a \to X_a$ uniformly as $\epsilon \to 0$, the symmetry $\sigma_{a[bc]}{}^{d}{}_{ef} = 0$ follows. The symmetries of $\tau_{abc}{}^{d}{}_{e}{}^{f}$ follow similarly. In addition, we can find the symmetry $\sigma_{[c|[a|ef|b]|d]}\gamma^{bd} = 0$ similarly to (B.10). The change of μ_{abcdef} under this gauge transformation is given by

$$\begin{aligned}
\tilde{\mu}_{abcdef} &= \operatorname*{w-lim}_{\lambda\to 0}\left(\nabla_a\tilde{h}_{cd}\nabla_b\tilde{h}_{ef}\right) \\
&= \operatorname*{w-lim}_{\lambda\to 0}\Big[\nabla_a\left(\tilde{h}_{cd}-h_{cd}\right)\nabla_b\left(\tilde{h}_{ef}-h_{ef}\right) \\
&\qquad +\nabla_a\left(\tilde{h}_{cd}-h_{cd}\right)\nabla_b h_{ef}+\nabla_a h_{cd}\nabla_b\left(\tilde{h}_{ef}-h_{ef}\right)+\nabla_a h_{cd}\nabla_b h_{ef}\Big] \\
&= \tau_{abc}{}^{m}{}_{e}{}^{n}g_{md}g_{nf}+\tau_{abc}{}^{m}{}_{f}{}^{n}g_{md}g_{en}+\tau_{abd}{}^{m}{}_{e}{}^{n}g_{cm}g_{nf}+\tau_{abd}{}^{m}{}_{f}{}^{n}g_{cm}g_{en} \\
&\quad +\sigma_{abc}{}^{m}{}_{ef}g_{md}+\sigma_{abd}{}^{m}{}_{ef}g_{cm}+\sigma_{abc}{}^{m}{}_{ef}g_{id}+\sigma_{abc}{}^{m}{}_{ef}g_{md} \\
&\quad +\mu_{abcdef} \\
&= \mu_{abcdef}+2\left(\sigma_{ab(cd)ef}+\sigma_{ab(ef)cd}\right)+4\tau_{ab(cd)(ef)},
\end{aligned} \tag{B.8}$$

where, in the third step, we use the fact that $\tilde{h}_{ab}-h_{ab}=\tilde{g}_{ab}-g_{ab}$ from (B.3) and (B.4). The fourth step results by taking the terms of order $\mathrm{O}(1)$. From this relation, we can find that

$$\begin{aligned}
\tilde{\alpha}_{abcdef} &= \tilde{\mu}_{[c|[ab]|d]ef} \\
&= \alpha_{abcdef}+2\sigma_{[c|[a|(ef)|b]|d]}.
\end{aligned} \tag{B.9}$$

We define two tensor fields $T_{ab}\equiv\alpha_{acbd}{}^{cd}$ and $S_{ab}\equiv{}^{*}\alpha_{acbd}{}^{cd}$ where ${}^{*}\alpha_{abcdef}\equiv\epsilon_{ab}{}^{gh}\alpha_{ghcdef}$. We can show that, if $\tilde{\alpha}_{abcdef}$ and α_{abcdef} give rise to the same T_{ab} and S_{ab}, then $\tilde{\alpha}_{abcdef}$ and α_{abcdef} are related as in (B.9) for some $\sigma_{efa}{}^{b}{}_{cd}$ satisfying the symmetries and traces given above. That is, T_{ab} and S_{ab} are the only gauge invariant parts of μ_{abcdef}. The tensor field T_{ab} is just the effective stress energy associated with the high frequency gravitational waves that this field is gauge invariant.

Appendix C

Solution of $\nabla_\mu \nabla_\nu S(t, \vec{x}) = 0$

We solve the equation

$$\nabla_\mu \nabla_\nu S(t, \vec{x}) = 0\,, \tag{C.1}$$

such as (5.2.7), in the FLRW background spacetime:

$$ds^2 = -dt^2 + a^2(t)\gamma_{ij}dx^i dx^j\,. \tag{C.2}$$

Since we are interested in the expanding universe, in what follows we assume that the scale factor $a(t)$ is dynamical, i.e., $a(t) \neq$ const. The $(0,0)$, $(0,i)$ and (i,j) components of (C.1) are

$$\ddot{S}(t, \vec{x}) = 0, \tag{C.3}$$

$$\partial_i \dot{S}(t, \vec{x}) - \frac{\dot{a}(t)}{a(t)}\partial_i S(t, \vec{x}) = 0, \tag{C.4}$$

$$\partial_i \partial_j S(t, \vec{x}) - a(t)\dot{a}(t)\gamma_{ij}\dot{S}(t, \vec{x}) - \Gamma^{k(3)}_{ij}\partial_k S = 0, \tag{C.5}$$

where the *dot* denotes the derivative with respect to the cosmic time. The solution of (C.3) is

$$S(t, \vec{x}) = c_1(\vec{x})t + c_2(\vec{x})\,, \tag{C.6}$$

where $c_1(\vec{x})$ and $c_2(\vec{x})$ are arbitrary functions of $\vec{x}$. Since $\dot{a} \neq 0$, Equation (C.4) becomes

$$\left(\frac{a(t)}{\dot{a}(t)} - t\right)\partial_i c_1(\vec{x}) = \partial_i c_2(\vec{x})\,. \tag{C.7}$$

Therefore, either

$$c_1, c_2 = const. \tag{C.8}$$

or $\partial_i c_2/\partial_i c_1 =$ const. In the latter case, by shifting $t \to t-$ const., we have $\dot{a}/a = 1/t$, and therefore have $a(t) \propto t$; the behavior of the background FLRW universe is determined. This is, in our present context, too restrictive, and for this reason, we should take the former case; $c_1, c_2 =$ const. Then, equation (C.5) for $i = j$ becomes

$$a(t)\dot{a}(t)\gamma_{ij}c_1 = 0\,, \tag{C.9}$$

which immediately implies

$$c_1 = 0\,. \tag{C.10}$$

Therefore we find the solution of (C.1) to be

$$S = const. \tag{C.11}$$

Appendix D

The regularity and derivatives of F_0 near the center

In this appendix, we discuss the regularity of the distribution function F at the symmetry center, and find which of $\partial_\alpha F$ (resp. $\partial_\alpha \partial_\beta F$), $\alpha, \beta = r, \omega, \mu$, become relevant in the leading behavior of the first- (resp. second-) order derivatives of F in the limit $r \to 0$.

Mathematically, the distribution function can become singular at some radius including at the center ($r = 0$). Furthermore, some authors concluded that a C^{2-}-class singularity of the metric should be allowed at the center in order to construct a model with accelerated expansion exactly at the center [49]. However, such a model can be easily made smooth by appropriate smoothing and the original singular model can be recovered as a limit such that the smoothing length approaches zero. Hence, the final formulae for the dipole and quadrupole anisotropies of CMB given in this appendix can be applied also to models with singularity in the metric or the distribution function simply if the results are finite. Hence, in this appendix, we assume that the metric and the distribution function are regular and smooth everywhere in Cartesian-type space coordinates on which the rotational symmetry group acts as on the standard Cartesian coordinates of the Euclidean space.

In such a coordinate system $\boldsymbol{x} = (x^i)$, the spatial part of the metric (6.1.1), denoted here by g_{ij}, can be written as

$$g_{ij} = S^2 \Omega_i \Omega_j + a_\perp^2 (\delta_{ij} - \Omega_i \Omega_j), \tag{D.1}$$

where $\Omega^i \equiv x^i/r$ and $a_\perp \equiv R/r$. For any smooth and rotationally invariant function h in this coordinate system, when it is Taylor expanded as

$$h = h_0 + h_i x^i + h_{ij} x^i x^j + \cdots , \tag{D.2}$$

where each coefficient $h_{ij\cdots}$ must be a rotationally invariant constant tensor. Hence, these coefficient tensors vanish for odd ranks and can be written as the sum of products of the Kronecker delta δ_{ij} for even ranks. This implies that h should be a smooth function of $r^2 = \delta_{ij} x^i x^j$.

Similarly, a smooth distribution function in this coordinate system can be Taylor expanded as

$$F(t, \boldsymbol{x}, p) = b_0(t, \boldsymbol{p}) + b_i(t, \boldsymbol{p}) x^i + b_{ij}(t, \boldsymbol{p}) x^i x^j + \cdots , \tag{D.3}$$

where $\boldsymbol{p} = (p^i)$. When F is rotationally invariant, each term on the right-hand side is rotationally invariant seperately. This implies that each $b_{i_1 \cdots i_l}$ is an SO(3) tensor depending only on the non-trivial vector p^i and therefore, can be written as the sum of products of the Kronecker delta δ_{ij} and the vector $\boldsymbol{p} = (p^i)$. This implies that F can be written

$$F(t, \boldsymbol{x}, \boldsymbol{p}) = \tilde{f}(t, x_i x^i, |p|, x_i p^i), \tag{D.4}$$

where $x_i x^i = r^2$ and $|p| = (\delta_{ij} p^i p^j)^{1/2}$. Here, from

$$p^2 \equiv g_{ij} p^i p^j = C (x_i p^i)^2 + a_\perp^2 \delta_{ij} p^i p^j, \tag{D.5}$$

where C is a smooth function defined by

$$C(t, r) \equiv \frac{S^2 - a_\perp^2}{r^2}, \tag{D.6}$$

it follows that $|p|$ is a smooth function of t, $\omega = p$, $x_i p^i$ and r^2. Therefore, the regularity of F at the center is equivalent to the condition that the corresponding function $F_0(t, r, \omega, \mu)$ can be written in terms of a smooth function f with the four arguments $t, r^2, y \equiv \ln \omega, \nu \equiv r\mu = x_i p^i / p$

$$F_0(t, r, \omega, \mu) = f(t, r^2, y, \nu). \tag{D.7}$$

Then,

$$\partial_i F = \partial_i f = \{\partial_i(r^2)\} f_2 + (\partial_i y) f_y + (\partial_i \nu) f_\nu, \tag{D.8}$$

where $f_2 \equiv \partial_{r^2} f$, $f_y \equiv \partial_y f$, and $f_\nu \equiv \partial_\nu f$. Here, because the spatial derivative of p^2 can be written

$$\partial_i(p^2) = \left\{ \frac{C'}{r}(rp^r)^2 + \frac{(a_\perp^2)'}{r}\delta_{jk}p^j p^k \right\} x^i + 2C(rp^r)p^i, \tag{D.9}$$

we have

$$\begin{aligned}
\partial_i(r^2) &= 2x^i, & \text{(D.10)}\\
\partial_i(\ln\omega) &= \left\{ -\frac{N'}{rN} + \frac{a'_\perp}{ra_\perp} + \left(\frac{C'}{2r} - \frac{Ca'_\perp}{ra_\perp}\right)\left(\frac{\mu r}{S}\right)^2 \right\} x^i + C\frac{\mu r}{S}\frac{p^i}{p}, & \text{(D.11)}\\
\partial_i \nu &= S\frac{p^i}{p} + rS'\frac{x^i}{r}\frac{p^r}{p} - rSp^r\frac{\partial_i(p^2)}{2p^3}. & \text{(D.12)}
\end{aligned}$$

In particular, in the limit $r \to 0$, we see

$$\begin{aligned}
C &\to \{a_\perp(S'' - a''_\perp)\}_0, & \text{(D.13)}\\
\partial_i(p^2),\ \partial_i(r^2),\ \partial_i(\ln\omega) &\to \mathrm{O}(r), & \text{(D.14)}\\
\partial_i\nu &\to S_0\frac{p^i}{p}. & \text{(D.15)}
\end{aligned}$$

Therefore, from (D.8), we find that the first derivative of F_0 behaves at the center as

$$\partial_i F \ \to\ S_0\frac{p^i}{p}(f_\nu)_0 = S_0\frac{p^i}{p}\left(\frac{\partial_\mu F_0}{r}\right)_0. \tag{D.16}$$

Next we study the second order derivatives of F with respect to x^i, which are written as

$$\begin{aligned}
\partial_i\partial_j F = {} & \{\partial_i\partial_j(r^2)\}f_2 + \{\partial_j(r^2)\}\left[\{\partial_i(r^2)\}f_{22} + (\partial_i y)f_{2y} + (\partial_i\nu)f_{2\nu}\right]\\
& + (\partial_i\partial_j y)f_y + (\partial_j y)\left[\{\partial_i(r^2)\}f_{y2} + (\partial_i y)f_{yy} + (\partial_i\nu)f_{y\nu}\right]\\
& + (\partial_i\partial_j\nu)f_\nu + (\partial_i\nu)\left[\{\partial_i(r^2)\}f_{\nu 2} + (\partial_i y)f_{\nu y} + (\partial_i\nu)f_{\nu\nu}\right]. \qquad \text{(D.17)}
\end{aligned}$$

We find that

$$
\begin{aligned}
\partial_i\partial_j(r^2) &= 2\delta_{ij}, && \text{(D.18)}\\
\partial_i\partial_j(p^2) &= \left[\left(\frac{C'}{r}\right)'(rp^r)^2+\left\{\frac{(a_\perp^2)'}{r}\right\}'\delta_{kl}p^kp^l\right]x^i\frac{x^j}{r}+2\frac{C'}{r}(rp^r)(p^ix^j+p^jx^i)\\
&\quad+\left\{\frac{C'}{r}(rp^r)^2+\frac{(a_\perp^2)'}{r}\delta_{kl}p^kp^l\right\}\delta_{ij}+2Cp^ip^j, && \text{(D.19)}\\
\partial_i\partial_j(\ln\omega) &= \frac{\partial_i\partial_j(p^2)}{2p^2}-\frac{\partial_i(p^2)\partial_j(p^2)}{2(p^2)^2}-\left(\frac{N'}{rN}\right)'\frac{x^i}{r}x^j-\frac{N'}{rN}\delta_{ij}, && \text{(D.20)}\\
\partial_i\partial_j\nu &= \frac{S'}{r}\left[\frac{x^ip^j+2x^jp^i}{p}-\frac{rp^r}{2p^3}\{x^j\partial_i(p^2)+x^i\partial_j(p^2)\}\right]+\left(\frac{S'}{r}\right)'\frac{rp^r}{p}\frac{x^ix^j}{r}\\
&\quad-\frac{S}{2p^3}\{p^i\partial_j(p^2)+p^j\partial_i(p^2)\}-S\frac{rp^r}{2p^3}\left\{\partial_i\partial_j(p^2)-\frac{3}{2p^2}\partial_i(p^2)\partial_j(p^2)\right\}. && \text{(D.21)}
\end{aligned}
$$

In particular, in the limit $r\to 0$,

$$
\begin{aligned}
\partial_i\partial_j(r^2) &\to 2\delta_{ij}, && \text{(D.22)}\\
\partial_i\partial_j(p^2) &\to 2\left(\frac{a_\perp''}{a_\perp}\right)_0 p^2\delta_{ij}+2C_0p^ip^j, && \text{(D.23)}\\
\partial_i\partial_j(\ln\omega) &\to \left(\frac{a_\perp''}{a_\perp}-\frac{N''}{N}\right)_0\delta_{ij}+C_0\frac{p^ip^j}{p^2}, && \text{(D.24)}\\
\partial_i\partial_j(\nu) &\to \mathrm{O}\left(r\right). && \text{(D.25)}
\end{aligned}
$$

From these we find that in the limit $r\to 0$,

$$
\partial_i\partial_jF\to 2\delta_{ij}(f_2)_0+\left\{\left(\frac{a_\perp''}{a_\perp}-\frac{N''}{N}\right)_0\delta_{ij}+C_0\frac{p^ip^j}{p^2}\right\}(f_y)_0+S_0^2\frac{p^ip^j}{p^2}(f_{\nu\nu})_0. \qquad \text{(D.26)}
$$

Now, from (D.7), we find that $\partial_rF_0=2rf_2+\mu f_\nu$, $\partial_\mu F_0=rf_\nu$, $\omega\partial_\omega F_0=f_y$, and $\partial_r^2F_0=$

$2f_2 + 4r^2 f_{22} + 4r\mu f_{2\nu} + \mu^2 f_{\nu\nu}$, and hence in the limit $r \to 0$,

$$\begin{aligned} f_\nu &\to \left(\frac{\partial_\mu F_0}{r}\right)_0, &\text{(D.27)}\\ f_2 &\to \frac{1}{2}\left(\frac{\partial_r F_0}{r} - \mu\frac{\partial_\mu F_0}{r^2}\right)_0, &\text{(D.28)}\\ f_{\nu\nu} &\to \frac{1}{\mu^2}\left(\partial_r^2 F_0 - \frac{\partial_r F_0}{r} + \frac{\mu}{r^2}\partial_\mu F_0\right)_0 \\ &= \frac{1}{\mu^2}\left(\partial_r^2 F_0 - 2f_2\right)_0. &\text{(D.29)} \end{aligned}$$

Thus, we finally obtain

$$\begin{aligned} \partial_i\partial_j F \;\to\; & 2\left(\delta_{ij} - S_0^2\frac{p^i p^j}{p^2}\right)(f_2)_0 + S_0^2\frac{p^i p^j}{p^2}(\partial_r^2 F_0)_0 \\ & + \left\{\left(\frac{a_\perp''}{a_\perp} - \frac{N''}{N}\right)_0 \delta_{ij} + C_0\frac{p^i p^j}{p^2}\right\}(\omega\partial_\omega F_0)_i. \qquad \text{(D.30)} \end{aligned}$$

Bibliography

[1] A. G. Riess *et al.*, Astron. J. **116**, 1009 (1998).

[2] S. Perlmutter *et al.*, Astrophys. J. **517**, 565 (1999).

[3] S. Weinberg, Rev. Mod. Phys. **61**, 1 (1989).

[4] R. R. Caldwell, R. Dave and P. J. Steinhardt, Phys. Rev. Lett. **80**, 1582 (1998).

[5] I. Zlatev, L. M. Wang and P. J. Steinhardt, Phys. Rev. Lett. **82**, 896 (1999).

[6] T. Chiba, T. Okabe and M. Yamaguchi, Phys. Rev. D **62**, 023511 (2000).

[7] C. Armendariz-Picon, V. F. Mukhanov and P. J. Steinhardt, Phys. Rev. Lett. **85**, 4438 (2000).

[8] C. Armendariz-Picon, V. F. Mukhanov and P. J. Steinhardt, Phys. Rev. D **63**, 103510 (2001).

[9] S. M. Carroll, Phys. Rev. Lett. **81**, 3067 (1998).

[10] C. F. Kolda and D. H. Lyth, Phys. Lett. B **458**, 197 (1999).

[11] A. D. Felice and S. Tsujikawa, Living Rev. Rel. **13**, 3 (2010).

[12] S. Nojiri, S. D. Odintsov and M. Sasaki, Phys. Rev. D **71**, 123509 (2005).

[13] S. Nojiri and S. D. Odintsov, Phys. Lett. B **631**, 1 (2005).

[14] G. R. Dvali, G. Gabadadze and M. Porrati, Phys. Lett. B **485**, 208 (2000).

[15] S. Capozziello, Int. J. Mod. Phys. D **11**, 483, (2002).

[16] S. M. Carroll, V. Duvvuri, M. Trodden and M. S. Turner, Phys. Rev. D **70**, 043528 (2004).

[17] T. Chiba, Phys. Lett. B **575**, 1 (2003).

[18] I. Navarro and K. V. Acoleyen, JCAP **0702**, 022 (2007).

[19] S. M. Carroll, I. Sawicki, A. Silvestri and M. Trodden, New J. Phys. **8**, 323 (2006).

[20] I. Sawicki and W. Hu, Phys. Rev. D **75**, 127502 (2007).

[21] L. Amendola, D. Polarski and S. Tsujikawa, Phys. Rev. Lett. **98**, 131302 (2007).

[22] L. Amendola, D. Polarski and S. Tsujikawa, Int. J. Mod. Phys. D **16**, 1555 (2007).

[23] L. Amendola and S. Tsujikawa, Phys. Lett. B **660**, 125 (2008).

[24] L. Pogosian and A. Silvestri, Phys. Rev. D **77**, 023503 (2008).

[25] N. Deruelle, M. Sasaki and Y. Sendouda, Phys. Rev. D **77**, 124024 (2008).

[26] W. Hu and I. Sawicki, Phys. Rev. D **76**, 064004 (2007).

[27] A. A. Starobinsky, JETP Lett. **86**, 157 (2007).

[28] S. Tsujikawa, Phys. Rev. D **77**, 023507 (2008).

[29] C. Brans and R. H. Dicke, Phys. Rev. **124**, 925 (1961).

[30] S. M. Carroll, A. D. Felice, V. Duvvuri, D. A. Easson, M. Trodden and M. S. Turner, Phys. Rev. D, **71**, 063513, (2005).

[31] K. S. Stelle, Gen. Rel. Grav. **9**, 353 (1978).

[32] C. Deffayet, G. R. Dvali, G. Gabadadze and A. I. Vainshtein, Phys. Rev. D **65**, 044026 (2002).

[33] A. I. Vainshtein, Phys. Lett. B 39, 393 (1972).

[34] A. Nicolis and R. Rattazzi, JHEP **0406**, 059 (2004).

[35] K. Tomita, Astrophys. J. **529**, 26 (2000).

[36] K. Tomita, Astrophys. J. **529**, 38 (2000).

[37] M. N. Célérier, Astron. Astrophys. **353**, 63 (2000).

[38] S. P. Goodwin, A. J. Barber, J. Griffin and L. I. Onuora, arXiv: astro-ph/9906187.

[39] G. Lemaître, Ann. Soc. Sci. Brux **53**, 51 (1933).

[40] R. C. Tolman, Proc. Nat. Acad. Sci. USA. **20**, 169 (1934).

[41] H. Bondi, Mon. Not. Roy. Astron. Soc. **107**, 410 (1947).

[42] H. Iguchi, T. Nakamura and K. Nakao, Prog. Theor. Phys. **108**, 809 (2002).

[43] S. Alexander, T. Biswas, A. Notari and D. Vaid, JCAP **0909**, 025 (2009).

[44] J. García-Bellido and T. Haugbølle, JCAP **0809**, 016 (2008).

[45] C.-M. Yoo, K. Nakao and M. Sasaki, JCAP **10**, 011 (2010).

[46] H. Alnes, M. Amarzguioui and Ø. Grøn, Phys. Rev. D **73**, 083519 (2006).

[47] H. Alnes and M. Amarzguioui, Phys. Rev. D **74**, 103520 (2006).

[48] D. Garfinkle, Class. Quant. Grav. **23**, 4811 (2006).

[49] R. A. Vanderveld, E. E. Flanagan and I. Wasserman, Phys. Rev. D **74**, 023506 (2006)

[50] M. Kasai, Prog. Theor. Phys. **117**, 1067 (2007).

[51] C.-M. Yoo, T. Kai and K. Nakao, Prog. Theor. Phys. **120**, 937 (2008).

[52] J. García-Bellido and T. Haugbølle, JCAP **0804**, 003 (2008).

[53] T. Buchert, Gen. Rel. Grav. **32**, 105 (2000).

[54] S. Rasanen, JCAP **0402**, 003 (2004).

[55] E. W. Kolb, S. Matarrese, A. Notari and A. Riotto, Phys. Rev. D **71**, 023524 (2005).

[56] C. M. Hirata and U. Seljak, Phys. Rev. D **72**, 083501 (2005).

[57] A. Ishibashi and R. M. Wald, Class. Quant. Grav. **23**, 235 (2006).

[58] E. W. Kolb, S. Matarrese and A. Riotto, New J. Phys. **8**, 322 (2006).

[59] M. Kasai, H. Asada and T. Futamase, Prog. Theor. Phys. **115**, 827 (2006).

[60] V. Marra, E. W. Kolb, S. Matarrese and A. Riotto, Phys. Rev. D **76**, 123004 (2007).

[61] E. W. Kolb, V. Marra and S. Matarrese, Gen. Rel. Grav. **42**, 1399 (2010).

[62] K. Saito and A. Ishibashi, Prog. Theor. Exp. Phys. 013E04 (2013).

[63] R. A. Isaacson, Phys. Rev. **166**, 1263 (1968).

[64] R. A. Isaacson, Phys. Rev. **166**, 1272 (1968).

[65] G. A. Burnett, J. Math. Phys. **30**, 90 (1989).

[66] S. R. Green and R. M. Wald, Phys. Rev. D **83**, 084020 (2011).

[67] H. Kodama, K. Saito and A. Ishibashi, Prog. Theor. Phys. **124**, 163 (2010).

[68] R. M. Wald, *General Relativity*, University of Chicago Press (1984).

[69] S. Weinberg, *Cosmology*, Oxford Univ. Pr. (2008).

[70] A. G. Riess *et al.*, Astrophys. J. **730**, 119 (2011).

[71] E. Komatsu *et al*, Astrophys. J. Suppl. **192**, 18 (2011).

[72] A. R. Liddle and D. H. Lyth, *Cosmological inflation and large-scale structure*, Cambridge University Press (2000).

[73] P. Astier *et al.* [The SNLS Collaboration], Astron. Astrophys. **447**, 31-48 (2006).

[74] A. G. Riess *et al.* [Supernova Search Team Collaboration], Astrophys. J. **607**, 665 (2004).

[75] A. G. Riess *et al.*, Astrophys. J. **659**, 98 (2007).

[76] W. M. Wood-Vasey *et al.* [ESSENCE Collaboration], Astrophys. J. **666**, 694 (2007).

[77] T. M. Davis *et al.*, Astrophys. J. **666**, 716 (2007).

[78] M. Kowalski *et al.* [Supernova Cosmology Project Collaboration], Astrophys. J. **686**, 749 (2008).

[79] R. K. Sachs and A. M. Wolfe, Astrophys. J. **147**, 73 (1967).

[80] K. Sato, Mon. Not. R. Astron. Soc. **195**, 467 (1981).

[81] A. H. Guth, Phys. Rev. D **23**, 347 (1981).

[82] G. F. Smoot *et al.*, Astrophys. J. **396**, L1 (1992).

[83] D. N. Spergel *et al.* [WMAP Collaboration], Astrophys. J. Suppl. **148**, 175 (2003).

[84] L. Page *et al.* [WMAP Collaboration], Astrophys. J. Suppl. **148**, 233 (2003).

[85] W. Hu and N. Sugiyama, Astrophys. J. **444**, 489 (1995).

[86] W. Hu and N. Sugiyama, Astrophys. J. **471**, 542 (1996).

[87] G. Efstathiou and J. R. Bond, Mon. Not. Roy. Astron. Soc. **304**, 75 (1999).

[88] W. Hu, M. Fukugita, M. Zaldarriaga and M. Tegmark, Astrophys. J. **549**, 669 (2001).

[89] M. Doran and M. Lilley, Mon. Not. Roy. Astron. Soc. **330**, 965 (2002).

[90] R. Amanullah *at al.*, Astrophys. J. **716**, 712 (2010).

[91] D. J. Eisenstein *et al.* [SDSS Collaboration], Astrophys. J. **633**, 560 (2005).

[92] D. J. Eisenstein and W. Hu, Astrophys. J. **496**, 605 (1998).

[93] M. Shoji, D. Jeong and E. Komatsu, Astrophys. J. **693**, 1404 (2009).

[94] W. J. Percival *et al.*, Mon. Not. Roy. Astron. Soc. **401**, 2148 (2010).

[95] B. Feng, X. L. Wang and X. M. Zhang, Phys. Lett. B **607**, 35 (2005).

[96] B. A. Reid *et al*, Mon. Not. Roy. Astron. Soc. **404**, 60 (2010).

[97] R. Tsutsui, T. Nakamura, D. Yonetoku, T. Murakami, S. Tanabe, Y. Kodama and K. Takahashi, Mon. Not. Roy. Astron. Soc. **394**, L31 (2009).

[98] L. Hollenstein, D. Sapone, R. Crittenden and B. M. Schaefer, JCAP **0904**, 012 (2009).

[99] C. Armendariz-Picon, T. Damour and V. F. Mukhanov, Phys. Lett. B **458**, 209 (1999).

[100] N. Arkani-Hamed, H. C. Cheng, M. A. Luty and S. Mukohyama, JHEP **0405**, 074 (2004).

[101] F. Piazza and S. Tsujikawa, JCAP **0407**, 004 (2004).

[102] A. A. Starobinsky, Phys. Lett. B **91**, 99 (1980).

[103] J. M. Bardeen, Phys. Rev. D **22**, 1882 (1980).

[104] H. Kodama and M. Sasaki, Prog. Theor. Phys. Suppl. **78**, 1 (1984).

[105] K. i. Maeda, Phys. Rev. D **39**, 3159 (1989).

[106] D. Lovelock, J. Math. Phys., **12**, 498 (1971).

[107] C. M. Will, *Theory and experiment in gravitational physics*, Cambridge University Press (1993).

[108] W. Hu and I. Sawicki, Phys. Rev. D **76**, 104043 (2007).

[109] T. Baker, P. G. Ferreira and C. Skordis, arXiv:1209.2117.

[110] C.-M. Yoo, T. Kai and K. Nakao, Phys. Rev. D **83**, 043527 (2011).

[111] C. Clarkson, T. Clifton and S. February, JCAP **0906**, 025 (2009).

[112] R. Nshikawa, C.-M. Yoo and K. Nakao, Phys. Rev. D **85**, 103511 (2012).

[113] W.J. Percival *et al.*, MNRAS **381**, 1053 (2007).

[114] H. Alnes and M. Amarzguioui, Phys. Rev. D **75**, 023506 (2007).

[115] *LIGO*, www.ligo.caltech.edu.

[116] *VIRGO*, www.virgo.infn.it.

[117] *Lcgt*, gw.icrr.u-tokyo.ac.jp/lcgt/.

[118] N. Seto, S. Kawamura and T. Nakamura, Phys. Rev. Lett. **87**, 221103 (2001).

[119] S. Kawamura *et al.*, Class. Quant. Grav. **28**, 094011 (2011).

[120] E. S. Phinney *et al.*, Big Bang Obsever Mission Concept Study (NASA) (2003).

[121] B. F. Schutz, Class. Quant. Grav. **28**, 125023 (2011).

[122] K. Danzmann and A. Rudiger, Class. Quant. Grav. **20**, S1 (2003).

[123] T. Prince, American Astronomical Society Meeting **202**, 3701 (2003).

[124] C. F. Sopuerta and N. Yunes, Phys. Rev. D **80**, 064006 (2009).

[125] L. C. Stein and N. Yunes, Phys. Rev. D **83** 064038 (2011).

[126] C. P. L. Berry and J. R. Gair, Phys. Rev. D **83**, 104022 (2011).

[127] D. L. Lee, Phys. Rev. D **10**, 2374 (1974).

[128] U. Gerlach and U. Sengupta, Phys. Rev. D **19**, 2268 (1979).

[129] U. Gerlach and U. Sengupta, Phys. Rev. D **20**, 3009 (1979).

[130] G. Hinshaw et al, Astrophys. J. Suppl. **170**, 288 (2007).

[131] H. Goto and H. Kodama, Prog. Theor. Phys. **125**, 815 (2011).