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LINEAR BEAM–BEAM RESONANCES DUE TO COHERENT DIPOLE MOTION

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We study the transverse motion of the barycentres of bunches in two beams that circulate in opposite direction in a storage ring, and are coupled by the beam-beam effect. The motion is described by a linear system of oscillators, and represented by a matrix. We distinguish between perfect machines in which the bunch parameters and the arc and interaction point parameters are all equal, and machines with errors in which at least one of these conditions is not satisfied. We determine the regions in tune space where the motion is unstable, analytically for perfect machines, and numerically for machines with errors. We identify these regions as resonances are excited under given conditions. We find that more resonances occur in machines with errors than in perfect machines. By multi-particle tracking, we study the instability at finite amplitudes.

Keywords: Colliding beams; instabilities; resonances; storage rings.

1 INTRODUCTION

The beam-beam interaction is one of the major performance limitations of circular colliders and has been studied theoretically, experimentally and by computer simulation for a long time. Several effects of this interaction can be studied by assuming that the bunches behave as rigid bodies: in the Rigid Gaussian Model (RGM)^{1,2}, we assume that the density distributions are a Gaussian with fixed beam radii, and apply the beam-beam forces between the barycentres.

Carrying the simplification one step further, and limiting ourselves to the study of the small amplitude oscillation of the bunches around the closed orbit, we can linearize the beam-beam force and reduce the study of the bunch motion to the study of the symplectic one-turn matrix. This Linear RGM (LRGM)³ addresses the following main issues:

- *Beam–Beam Modes:* When the motion of the bunches is stable, the eigenvalues of the one-turn matrix yield the visible betatron tunes, i.e. betatron frequencies divided by the revolution frequency, each corresponding to a coherent dipole oscillation mode of the system of oscillating bunches. They are used in setting up colliders such that the luminosity reaches a maximum.^{4–7}
- *Linear Beam–Beam Resonances:* Whenever an eigenvalue exceeds unity in absolute value, the motion becomes unstable. The LRGM yields the conditions when the instability takes place. It explained why space charge compensation with four beams in the 'Dispositif à Collisions dans le Igloo' (DCI)⁸ did not work as originally planned.^{9,10} The LRGM also gave the argument for abandoning the idea of building high-luminosity B (and other) factories composed of two rings with different circumferences.^{1,11}

Piwinski¹² was the first to use the eigenvalue technique. He treated a whole bunch as a point particle, found an instability when integral and half-integral tunes are approached from below (above) for attractive (repulsive) beam-beam forces, and gave a closed expression for the threshold of the instability. Our work can be thought of as a refinement and a generalization of Piwinski's approach.

Chao and Keil¹³ demonstrated that half-integral resonances are not excited in the case of N bunches in each beam colliding at 2N interaction points (IPs) in a symmetric machine, and also found more resonances in machines with phase errors between the interaction points. They identified these resonances as complex resonances: a pair of complex eigenvalues become larger than unity in absolute value. Keil continued the linear analysis and found that resonances growing out of the half-integral tunes are a generic feature of machines with phase advance errors between the collision points.¹⁴ He computed the visible tunes, and demonstrated that resonances arise when two visible tunes meet.¹⁵ In the study of coherent beam–beam effects in the SSC, Chao and Furman^{16,17} found half-integral resonances, which they called sail-shaped objects.

Keil used a multi-particle tracking code¹⁸ to compute the relation between the beam-beam strength parameter ξ and the visible frequencies of the barycentre modes (σ and π). He found that the frequency difference was about a factor of two smaller than expected from Piwinski's theory.¹² Hirata³ showed that in the RGM the slope of the force between two bunches is a factor of two smaller than that between a bunch and a test particle, when the collisions are head-on. Hofmann and Myers¹⁹ obtained the same result. It was confirmed experimentally in the SLC.²⁰ Our LRGM includes this effect.

Based on the LRGM, we constructed a computer code, BBMODE²¹, which finds the eigenvalues and allows all possible errors. When we apply BBMODE to LEP with various errors, we find many narrow resonances in surprisingly complicated patterns.²² A systematic survey of all these resonances is one of the purposes of the present paper.

We assume throughout that the N equidistant bunches of the e⁺ beam and the N equidistant bunches of the e⁻ beam circulate in opposite directions in a single storage ring or in two rings of the same circumference. In a single ring, 2N equidistant interaction points are possible at most. When all of them except $N_{\rm IP}$ are made inactive by separating the two beams, we denote this case as $N \oplus N = N_{\rm IP}$. We distinguish between perfect machines in which the bunch parameters and the arc and interaction point parameters are all equal, and machines with errors in which at least one of these conditions is not satisfied. Whenever we can relax this most restrictive definition of a perfect machine, we shall explicitly state it. However, the phase advances of the two beams in the arcs need not be the same. When we calculate growth rates of the beam-beam modes, we do not include any damping mechanism, e.g. synchrotron radiation damping, feedback systems, and Landau damping.⁶

Our paper is organized as follows. In Section 2, we summarize the Linear Rigid Gaussian Model and its simplest application to the $1 \oplus 1 = 1$ case. In Section 3, we study perfect machines. Section 4 is devoted to machines with errors. Section 5 contains the discussion of the complex resonances, and Section 6 our conclusions. Appendix A contains mathematical lemmas which allow the reduction of the $N \oplus N = N_{\text{IP}}$ case to the $N' \oplus N' = N_{\text{IP}}$ case, where $N' \leq N$. Appendix B compares the results of the LRGM and multi-particle tracking.

2 MACHINES WITH ONE INTERACTION POINT

Here, we introduce the notation, and describe the simplest case $(1 \oplus 1 = 1)$ in detail, including all possible differences between bunches and the arcs. We denote the tune of the e^{\pm} beam without the beam-beam effect as Q_z^{\pm} (z stands for either x, horizontal, or y, vertical). The revolution matrix for the z coordinate is given by

$$M_z = \hat{U}(Q_z^+, Q_z^-) R(\Xi_z^+, \Xi_z^-) .$$
 (1)

The transformations \hat{U} through the arcs are block-diagonal 4 \times 4 matrices:

$$\hat{U}(Q^+, Q^-) = \begin{pmatrix} U(Q^+) & O \\ O & U(Q^-) \end{pmatrix},$$

$$U(Q) = \begin{pmatrix} \cos 2\pi Q & \sin 2\pi Q \\ -\sin 2\pi Q & \cos 2\pi Q \end{pmatrix}.$$
(2)

Here O is the null matrix of suitable dimension whose components are 0. The revolution matrix M_z operates on the dynamical variables

$$\begin{pmatrix} \mathbf{Z}_{+} \\ \mathbf{Z}_{-} \end{pmatrix} \text{ where } \mathbf{Z}_{\pm} = \sqrt{N_{\pm}\gamma_{\pm}} \begin{pmatrix} z_{\pm} \\ \alpha_{z}^{\pm} z_{\pm}/\sqrt{\beta_{z}^{\pm}} + \sqrt{\beta_{z}^{\pm}} z_{\pm}' \end{pmatrix}.$$
(3)

Here, N_{\pm} is the number of particles, γ_{\pm} is the relativistic Lorentz factor, z_{\pm} is the *z* coordinate of the barycentre, z'_{\pm} is its slope, and α_z^{\pm} and β_z^{\pm} are nominal Twiss parameters of the e^{\pm} beam at the IP. The beam-beam kick matrix *R* is defined by:

$$R(\Xi_{z}^{+}, \Xi_{z}^{-}) = \begin{pmatrix} I - A(\Xi_{z}^{+}) & A(\sqrt{\Xi_{z}^{+}\Xi_{z}^{-}}) \\ A(\sqrt{\Xi_{z}^{+}\Xi_{z}^{-}}) & I - A(\Xi_{z}^{-}) \end{pmatrix}, \quad A(\Xi) = \begin{pmatrix} 0 & 0 \\ 4\pi \Xi & 0 \end{pmatrix},$$
(4)

with the coherent beam-beam parameter, Ξ_z^{\pm} ,

$$\Xi_{z}^{\pm} = \frac{r_{e} N_{\mp} \beta_{z}^{\pm} Y_{z}}{2\pi \gamma_{\pm} \Sigma_{z} (\Sigma_{x} + \Sigma_{y})} , \qquad \Sigma_{z} = \sqrt{(\sigma_{z}^{+})^{2} + (\sigma_{z}^{-})^{2}} .$$
(5)

Here and in the following, I is the unit matrix of suitable dimension, r_e is the classical electron radius, and Y_z is the Yokoya factor. This factor describes the change of the visible tunes caused by the distortion of the beam distribution by the beam-beam collision.^{*a*} We have assumed that the force is attractive.

^{*a*} It should be noted that (5) is only phenomenologically correct for head-on collision. Meller and Siemann²³ and Yokoya and Koiso²⁴ calculated the beam-beam effect on the coherent tunes for head-on beam-beam collisions and found that the visible tune difference is larger than what we expect from the LRGM. We include this effect by multiplying the focusing force with a phenomenological factor which we call Yokoya factor Y_z . Typical values for flat beams with $\sigma_y \ll \sigma_x$ are $Y_x \approx 1.33$ and $Y_y \approx 1.24$. The factor is not known for cases with an offset between the axes of the two beams of the order of the beam sizes or larger, and this simple treatment is no longer accurate. Beam-beam collisions with an offset also change the closed orbits of the bunches.² To treat such cases, we should go beyond the linear analysis. In most of this paper, we shall confine ourselves to the case where the collision is head-on.

For a repulsive force, the sign of Ξ should be changed. When $\sigma^+ = \sigma^-$, we have $\Xi_z = Y_z \xi_z/2$, where ξ_z is the usual beam-beam parameter

$$\xi_z^{\pm} = \frac{r_e N_{\mp} \beta_z^{\pm}}{2\pi \gamma_{\pm} \sigma_z^{\mp} (\sigma_x^{\mp} + \sigma_y^{\mp})} . \tag{6}$$

As is clear from (4), the beam-beam collision is represented by Ξ_z^{\pm} only: any difference in N^{\pm} , β_z^{\pm} , α_z^{\pm} , γ^{\pm} , σ_x^{\pm} , and σ_y^{\pm} is taken into account. As long as the collision is head-on and the directions of the betatron modes are the same in the e⁺ and e⁻ beams, the linear motions in the x and y degrees of freedom are independent. We assume this to be the case and drop the subscripts z hereafter.

Since *M* is a symplectic 4×4 matrix, we can use a well-known technique²⁵ to obtain the average of the eigenvalue λ and its reciprocal $1/\lambda$:

$$\cos \bar{\mu}^{\pm} = \frac{\lambda + 1/\lambda}{2} = \frac{\cos \mu^{+} + \cos \mu^{-}}{2} - \pi \Xi^{+} \sin \mu^{+} - \pi \Xi^{-} \sin \mu^{-} \pm \frac{1}{2} \sqrt{D} , \qquad (7)$$

$$D = [\cos \mu^{+} - \cos \mu^{-} - 2\pi \Xi^{+} \sin \mu^{+} + 2\pi \Xi^{-} \sin \mu^{-}]^{2} + 16\pi^{2} \Xi^{-} \Xi^{+} \sin \mu^{+} \sin \mu^{-}.$$
(8)

The mapping M is stable if and only if both $\cos \bar{\mu}^+$ and $\cos \bar{\mu}^-$ are real and fall into the region between -1 and +1. The mapping is unstable, in terms of Q^{\pm} and D, if:

- D > 0 and Q⁺ ≤ integer or Q⁻ ≤ integer, then one λ is real and λ > 1. We call this a positive or integral resonance.
- D > 0 and Q⁺ ≤ half-integer or Q⁻ ≤ half-integer, then one λ is real and λ < -1. We call this a negative or half-integral resonance.
- D < 0 and Q⁺ + Q⁻ ≤ integer, then there is a complex conjugate pair of λ's with |λ| > 1. We call this a complex or sum resonance.

Here, $Q^+ \lesssim$ integer, for example, indicates that the instability occurs when Q^+ approaches the integer from below. The instability does not necessarily persist until Q^+ reaches the integer. More than one type of resonance can occur at the same tunes.

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FIGURE 1 Unstable region in tune space for the $1 \oplus 1=1$ case with $\Xi^+ = 0.1$ and $\Xi^- = 0.01$. The abscissa is $0 \le Q^+ \le 1$, the ordinate is $0 \le Q^- \le 1$. The type of eigenvalue with the largest growth rate is indicated: H, I and C stand for half-integral, integral and complex sum resonances, respectively. The pattern repeats itself with period 1 for Q^+ and Q^- .

The unstable region in the (Q^+, Q^-) -plane is shown in Figure 1, which displays a case with $\Xi^+ \neq \Xi^-$. All unstable regions listed above can be seen clearly. As is clear from (1), under a replacement $(Q^+, Q^-) \rightarrow$ $(Q^+ + 1/2, Q^- \pm 1/2)$, M becomes -M, so that λ remains the same but with opposite sign. This is the reason why, in Figure 1, $(0 \le Q^+ \le 1/2, 0 \le Q^- \le 1/2)$ and $(1/2 \le Q^+ \le 1, 1/2 \le Q^- \le 1)$ and also $(0 \le Q^+ \le 1/2, 1/2 \le Q^- \le 1)$ and $(1/2 \le Q^+ \le 1, 0 \le Q^- \le 1/2)$ are identical apart from the replacement H \leftrightarrow I, forming a chessboard pattern. For $\Xi^+ = \Xi^-$, the graph would be symmetric under the reflection with respect to the line $Q^+ = Q^-$.

3 PERFECT MACHINES

Here, we consider cases of machines with equally spaced bunches and equally spaced IPs. All bunches and IPs are equal, i.e. $\Xi^+ = \Xi^- = \Xi$. The phase advances $2\pi\nu_+$ for the e⁺ beam in the arcs connecting the IPs are all the same, and so are the phase advances $2\pi\nu_-$ of the e⁻ beam. However, ν_+ and ν_- may be different. The eigenvalues can be found analytically.

In some cases, the problem is reducible: The $2 \oplus 2 = 2$ case splits into two mutually independent and identical $1 \oplus 1 = 2$ cases with the same eigenvalues. We have to examine only the irreducible cases. The number of such cases is not large. We prove in Appendix A that $N \oplus N = N$ (with odd N) and $N \oplus N = 2N$ are the only irreducible cases with equally spaced IPs. We begin with the $3 \oplus 3 = 3$ and $3 \oplus 3 = 6$ cases, because N = 3is the smallest N which contains all essential features of cases. Finally, we derive a closed expression for the threshold of the instability for the general $N \oplus N = N_{\rm IP}$ case.

3.1 The 3 \oplus 3 = 3 Case

We use the Eulerian view²⁶; instead of labelling the bunches and following them around the ring, we label the IPs, and call them IP₁, IP₂, and IP₃ in a clockwise manner, and we give the bunches, which collide there at a particular instant in time, the label of the IP. We define the state vector $\mathbf{Y} = (\mathbf{Y}_1, \mathbf{Y}_2, \cdots \mathbf{Y}_6)^t$, where,

$\mathbf{Y}_1: \mathbf{e}^+ \text{ at } \mathbf{IP}_1$	\mathbf{Y}_4 : e ⁻ at IP ₁
\mathbf{Y}_2 : e ⁺ at IP ₂	\mathbf{Y}_5 : e ⁻ at IP ₂
\mathbf{Y}_3 : e ⁺ at IP ₃	\mathbf{Y}_6 : \mathbf{e}^- at IP ₃ .

Let \mathbf{Y}_1 represent \mathbf{Z}^+ for the first e^+ bunch e_1^+ at a particular instant of time. One-third of a turn later, this bunch has moved to \mathbf{IP}_2 and is represented by \mathbf{Y}_2 , while the third e^+ bunch e_3^+ has moved from \mathbf{IP}_3 to \mathbf{IP}_1 and is now represented by \mathbf{Y}_1 . The e^+ bunches pass through \mathbf{IP}_1 in the order 1, 3, 2, through \mathbf{IP}_2 in the order 2, 1, 3, and through \mathbf{IP}_3 in the order 3, 2, 1. Similarly, the e^- bunches pass through \mathbf{IP}_1 in the order 1, 2, 3, through \mathbf{IP}_2 in the order 2, 3, 1, and through \mathbf{IP}_3 in the order 3, 1, 2. The correspondence between state vectors \mathbf{Y}_i and e^\pm bunch numbers for three successive collisions is shown in Table I.

The cyclic permutation of the e^- bunches from (1, 2, 3) to (2, 3, 1) is described by the 6×6 matrix *P*:

$$P = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ I & 0 & 0 \end{pmatrix}.$$
 (9)

				·		
i	\mathbf{Y}_1	\mathbf{Y}_2	\mathbf{Y}_3	\mathbf{Y}_4	\mathbf{Y}_5	Y ₆
1	e ₁ ⁺	e ₂ ⁺	e ₃ ⁺	e ₁	e	e_3
2	e_3^+	e_1^+	e_2^+	e_2^-	e ₃ -	e_1^-
3	e_2^+	e_3^+	e_1^+	e ₃ -	e_1^-	e_2^-

TABLE I Correspondence between state vectors \mathbf{Y}_i and \mathbf{e}^{\pm} bunches for the 3 \oplus 3 = 3 case

with $P^3 = I$. Because of the Eulerian view, the 6×6 one-turn matrix M_1 can be written as the cube of a matrix $M_{1/3}$ for 1/3 of a turn:

$$M_1 = [M_{1/3}]^3, (10)$$

$$M_{1/3} = \begin{pmatrix} P^2 & \\ & P \end{pmatrix} \hat{U}R , \qquad (11)$$

$$\hat{U} = \text{diag}(U_+, U_+, U_+, U_-, U_-, U_-),$$
 (12)

$$U_{\pm} = \begin{pmatrix} \cos 2\pi \nu_{\pm} & \sin 2\pi \nu_{\pm} \\ -\sin 2\pi \nu_{\pm} & \cos 2\pi \nu_{\pm} \end{pmatrix}, \qquad (13)$$

$$R = \begin{pmatrix} I - A & A & \\ & I - A & & A \\ & & I - A & & A \\ A & & I - A & & \\ & A & & I - A & \\ & & A & & I - A \\ & & & A & & I - A \end{pmatrix}.$$
 (14)

Here, diag (A, B, \dots) is a matrix which has the submatrices A, B, \dots along the main diagonal, $A = A(\Xi)$ and $\nu_{\pm} = Q^{\pm}/3$. To bring $M_{1/3}$ into block diagonal form, we introduce $\chi = \exp(2\pi i/3)$ and the matrix V

$$V = \frac{1}{\sqrt{3}} \begin{pmatrix} I & I & I \\ I & \chi I & \chi^* I \\ I & \chi^* I & \chi I \end{pmatrix} .$$
(15)

It is easily found that $V^{\dagger}PV = \text{diag}(I, \chi I, \chi^*I)$, where V^{\dagger} is the Hermitian conjugate of V, and $VV^{\dagger} = I$ holds. The first step towards block diagonalizing $M_{1/3}$ is multiplying with diag (V, V^{\dagger}) from the left and with diag (V^{\dagger}, V) from the right:

$$M_{1/3} \sim \operatorname{diag}\left(I, \chi I, \chi^* I, I, \chi I, \chi^* I\right) U R', \qquad (16)$$

$$R' = \begin{pmatrix} I - A & & A & & \\ & I - A & & & A \\ & & I - A & & A \\ A & & I - A & & \\ & A & & I - A & \\ & A & & & I - A \end{pmatrix}.$$
 (17)

When all eigenvalues of any two matrices A and B are identical, we say that A and B are equivalent, and write $A \sim B$. By reordering of the basis vectors, we can bring $M_{1/3}$ into block diagonal form:

$$M_{1/3} \sim \text{diag}(M_{\sigma\pi}, M_{\chi}, M_{\chi^*}).$$
 (18)

Here

$$M_{\sigma\pi} = \begin{pmatrix} U_+ & 0\\ 0 & U_- \end{pmatrix} \begin{pmatrix} I - A & A\\ A & I - A \end{pmatrix},$$
 (19)

$$M_{\chi} = \begin{pmatrix} \chi U_{+} & 0\\ 0 & \chi^{*} U_{-} \end{pmatrix} \begin{pmatrix} I - A & A\\ A & I - A \end{pmatrix},$$
(20)

and M_{χ^*} is the complex conjugate of M_{χ} . Since the eigenvalues of a block diagonal matrix are equal to the set of eigenvalues of the blocks, we can discuss the blocks one by one.

As its name implies, $M_{\sigma\pi}$ is identical with the one-turn matrix for the $1 \oplus 1 = 1$ case, (1). It is symplectic. Although we have given the exact eigenvalues earlier in (8), we start, for later convenience, the discussion of $M_{\sigma\pi}$ with $\Xi \ll 1$ where the eigenvalues of $M_{\sigma\pi}$ are:

$$\lambda_{+} \approx \exp[2\pi i(+\nu_{+} + \Xi)], \quad \lambda_{+}^{*} \approx \exp[2\pi i(-\nu_{+} - \Xi)],$$

$$\lambda_{-} \approx \exp[2\pi i(+\nu_{-} + \Xi)], \quad \lambda_{-}^{*} \approx \exp[2\pi i(-\nu_{-} - \Xi)].$$
(21)

Encounter	Unstable region	Туре
λ_+ vs. λ_+^*	$\nu_+ \lesssim 1/2$	
	$\nu_+ \lesssim 1$	+
λ_{-} vs. λ_{-}^{*}	$ u_{-} \lesssim 1/2 $	-
	$\nu_{-} \lesssim 1$	+
λ_+ vs. λ^*	$\nu_+ + \nu \stackrel{<}{_\sim}$ integer /2	с
λ vs. λ_+^*		с

TABLE II Mode coupling pattern of $M_{\sigma\pi}$ for the 3 \oplus 3 = 3 case. The type -, + and c refers to negative, positive and complex eigenvalues

As Ξ grows, the eigenvalues move on the unit circle until two of them meet. We have observed that then one of the eigenvalues λ of the matrix $M_{\sigma\pi}$ becomes larger than unity in absolute value. We summarize the mode coupling pattern in Table II.

When λ is an eigenvalue of M_{χ} , λ^* is an eigenvalue of M_{χ^*} , and vice versa. Thus only M_{χ} or M_{χ^*} needs to be studied. Also for the discussion of M_{χ} , we start with $\Xi \ll 1$ where the eigenvalues of M_{χ} are:

$$\lambda_{1} \approx \exp[2\pi i(+1/3 + \nu_{+} + \Xi)],$$

$$\lambda_{2} \approx \exp[2\pi i(+1/3 - \nu_{+} - \Xi)],$$

$$\lambda_{3} \approx \exp[2\pi i(-1/3 + \nu_{-} + \Xi)],$$

$$\lambda_{4} \approx \exp[2\pi i(-1/3 - \nu_{-} - \Xi)].$$
(22)

As Ξ grows, the eigenvalues move on the unit circle until two of them meet. We have observed numerically that then one of the eigenvalues λ of the matrix M_{χ} becomes larger than unity in absolute value. All possible encounters of the eigenvalues and corresponding instabilities can thus be understood and are shown in Table III. The unstable regions of M_{χ}^3 , $M_{\sigma\pi}^3$ and M_1 are compared in Figure 2. We can limit the ranges of ν_+ and ν_- because (13) shows that U_{\pm} is invariant to a change of ν_{\pm} by one unit. Furthermore, a simultaneous change $\nu_+ \rightarrow \nu_+ \pm 1/2$ and $\nu_- \rightarrow \nu_- \pm 1/2$ changes only the sign of \hat{U} but not the absolute values of the eigenvalues, resulting in the chessboard pattern. Figure 2 and Tables II and III summarize the $3 \oplus 3 = 3$ case.



FIGURE 2 The largest absolute value of the eigenvalues of M_{χ}^3 (A), $M_{\sigma\pi}^3$ (B) and M_1 (C) as functions of $(\nu_+, \nu_-) = (Q^+/3, Q^-/3)$ for the $3 \oplus 3 = 3$ case with $\Xi = 0.025$.

Encounter	Unstable region	Туре
λ_1 vs. λ_2	$\nu_+ \lesssim 1/2$	_
	$\nu_+ \lesssim 1$	+
λ3 vs. λ4	$\nu_{-} \lesssim 1/2$	-
	$\nu_{-} \gtrsim 1$	+
λ_1 vs. λ_4	$\nu_+ + \nu \lesssim 1/3$	c
	$\nu_+ + \nu \lesssim 4/3$	с
λ_2 vs. λ_3	$\nu_+ + \nu \lesssim 2/3$	c
	$\nu_+ + \nu \lesssim 5/3$	с

TABLE III Mode coupling pattern of M_{χ} . The type -, + and c refers to negative, positive and complex eigenvalues for M_{χ}^3

3.2 The 3 \oplus 3 = 6 Case

This case is the simplest non-trivial $N \oplus N = 2N$ case. We add three primed IPs, labelled IP'_1 , IP'_2 and IP'_3 to the IPs of the $3 \oplus 3 = 3$ case, such that primed and unprimed IPs alternate. Let all bunches collide at the unprimed IPs at a particular instant of time. They will then collide at the primed IPs one collision later. We still use the state vectors **Y** for the unprimed IPs. For the primed IPs, we define new state vectors $\mathbf{W} = (\mathbf{W}_1, \mathbf{W}_2, \cdots, \mathbf{W}_6)^t$, where,

\mathbf{W}_1 : e ⁺ at \mathbf{IP}'_1	\mathbf{W}_4 : e ⁻ at IP' ₁
\mathbf{W}_2 : e ⁺ at \mathbf{IP}_2^{\prime}	\mathbf{W}_5 : e ⁻ at \mathbf{IP}_2^{i}
\mathbf{W}_3 : e ⁺ at $\mathbf{IP}_3^{\overline{7}}$	$\mathbf{W}_6: e^- \text{ at } \mathrm{IP}_3^7$

The correspondence between state vectors \mathbf{W}_i and \mathbf{Y}_i and \mathbf{e}^{\pm} bunch numbers for six successive collisions is shown in Table IV.

TABLE IV Correspondence between state vectors \mathbf{W}_i and \mathbf{Y}_i and \mathbf{e}^{\pm} bunches for six successive instants *i* during a turn in the $3 \oplus 3 = 6$ case

i	Y ₁	Y ₂	Y ₃	Y ₄	Y ₅	Y ₆	\mathbf{W}_1	W ₂	W ₃	\mathbf{W}_4	W ₅	\mathbf{W}_6
1	e ₁ ⁺	e_2^+	e ₃ ⁺	e_1	e	e_3						
2	a ⁺	° ⁺	°+	a	o	o	e_1^+	e_2^+	e_3^+	e_1^-	e ₂	e ₃
3 4	e ₃	e ₁	e ₂	e ₂	e ₃	\mathbf{e}_1	e_2^+	e_1^+	e_2^+	e_	e_2^-	e_1^-
5	e_2^+	e_3^+	e_1^+	e ₃ -	e_1^-	e_2^-	3	1	2	2	3	1
6							e ₂ ⁺	e ₃ ⁺	e ₁ ⁺	e ₃ ⁻	e_1	e ₂ ⁻

The W is related to the Y at the previous collision by

$$\mathbf{W} = \begin{pmatrix} I & O \\ O & P \end{pmatrix} \hat{U} R \mathbf{Y}_{\text{previous}} .$$
 (23)

This W is related to the next Y by

$$\mathbf{Y}_{\text{new}} = \begin{pmatrix} P^2 & O \\ O & I \end{pmatrix} \hat{U} R \mathbf{W}.$$
 (24)

Thus, we have

$$M_1 = M_{1/3}^3, \qquad M_{1/3} = \begin{pmatrix} P^2 & O \\ O & I \end{pmatrix} \hat{U} R \begin{pmatrix} I & O \\ O & P \end{pmatrix} \hat{U} R.$$
(25)

As before, by multiplying with diag (V, V^{\dagger}) from the left and with diag (V^{\dagger}, V) from the right and by reordering the basis vectors, we have three mutually decoupled systems:

$$M_{1/3} \sim \text{diag}\left[M_{\sigma\pi}^2, M_{\chi^{1/2}}^2, M_{\chi^{1/2*}}^2\right].$$
 (26)

Here $M_{\sigma\pi}$ is defined by (19), and $M_{\chi^{1/2}}$ is the same as that defined by (20) but χ is replaced by $\chi^{1/2} = \exp(\pi i/3)$. We thus arrive at

$$M_1 \sim \text{diag}\left[M_{\sigma\pi}^6, M_{\chi^{1/2}}^6, M_{\chi^{1/2*}}^6\right].$$
 (27)

For $\Xi \ll 1$, the eigenvalues of $M_{\chi^{1/2}}$ can be predicted as shown in (22) with 1/3 replaced by 1/6. Table III also applies to the 3 \oplus 3 = 6 case if we interchange " λ_1 vs. λ_4 " and " λ_2 vs. λ_3 ".

Thus we conclude that, in the (ν_+, ν_-) -plane, the $3 \oplus 3 = 6$ case has exactly the same instability pattern as the $3 \oplus 3 = 3$ case in Tables II and III. We will show in Section 3.5 that this is not an accident. In the (Q^+, Q^-) -plane, the resonances are twice as far apart, compared to the $3 \oplus 3 = 3$ case.

3.3 The $N \oplus N = N$ Case

For the discussion of the $N \oplus N = N$ case, we can restrict ourselves to the case where N is odd. It follows from the family theorem in Appendix A that the $N \oplus N = N$ case is irreducible when N is odd. When N is even, the $N \oplus N = N$ case splits into two mutually independent and identical $N/2 \oplus N/2 = N$ cases with the same eigenvalues. The latter case will be studied later when we discuss the $N \oplus N = 2N$ case. As before, we define the state vector $\mathbf{Y} = (\mathbf{Y}_1, \cdots \mathbf{Y}_{2N})^t$, where \mathbf{Y}_i denotes the state vector of the e⁺ bunch at the *i*-th IP and \mathbf{Y}_{N+i} that of the e⁻ bunch at the *i*-th IP ($i = 1, 2, \cdots, N$). With $\chi_N \equiv \exp(2\pi i/N)$, the matrices P in (9) and V in (15) are replaced by

$$P = \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & I \\ I & 0 & \cdots & \cdots & 0 \end{pmatrix},$$
(28)

$$V = \frac{1}{\sqrt{N}} \begin{pmatrix} I & I & I & \cdots & I \\ I & \chi_N I & \chi_N^2 I & \cdots & \chi_N^{N-1} I \\ I & \chi_N^2 I & \chi_N^4 I & \cdots & \chi_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ I & \chi_N^{N-1} I & \chi_N^{2(N-1)} & \cdots & \chi_N^{(N-1)^2} \end{pmatrix}.$$
 (29)

Then we have $M_1 \sim M_{1/N}^N$, where

$$M_{1/N} \sim \text{diag} \ [m(0), m(1/N), m(-1/N), \cdots, m(n/N), m(-n/N), \dots, m((N-1)/(2N)), m(-(N-1)/(2N))].$$
(30)

Here m(n/N) is the same as M_{χ} (20), with $\chi = \exp(2\pi i n/N)$, and m(0) is identical to $M_{\sigma\pi}$ (19). Clearly, m(n/N) and m(-n/N) have the same instability property. The generalization of (22) to m(n/N) for $\Xi \ll 1$ yields:

$$\lambda_{1} \approx \exp[2\pi i(+n/N + \nu_{+} + \Xi)],$$

$$\lambda_{2} \approx \exp[2\pi i(+n/N - \nu_{+} - \Xi)],$$

$$\lambda_{3} \approx \exp[2\pi i(-n/N + \nu_{-} + \Xi)],$$

$$\lambda_{4} \approx \exp[2\pi i(-n/N - \nu_{-} - \Xi)].$$
(31)

Note that n/N = 1/2 never happens because N is odd. Also note that we can assume $0 \le n < N/2$ without losing generality. All mode-coupling patterns for the $N \oplus N = N$ case are listed in Table V.

TABLE V Mode coupling pattern of m(n/N) for the $N \oplus N = N$ case. Note that $\nu_{\pm} = Q^{\pm}/N \pmod{1}$. The type -, + and c refers to the negative, positive and complex eigenvalues for $m(n/N)^N$. We have $0 \le n < N/2$. The case n = 0 corresponds to $M_{\sigma\pi}$

Encounter	Unstable region	Туре
λ_1 vs. λ_2	$\nu_+ \lesssim 1/2$	_
	$\nu_+ \lesssim 1$	+
λ_3 vs. λ_4	$\nu_{-} \lesssim 1/2$	_
	$\nu_{-}\lesssim 1$	+
λ_1 vs. λ_4	$\nu_+ + \nu \lesssim (N-2n)/N$	с
	$\nu_+ + \nu \lesssim 2(N-n)/N$	с
λ_2 vs. λ_3	$ u_+ + u \lesssim 2n/N $	с
	$\nu_+ + \nu \stackrel{<}{_\sim} (N+2n)/N$	с

3.4 The $N \oplus N = 2N$ Case

In the general $N \oplus N = 2N$ case, N can either be odd or even. Both cases are irreducible. By the same transformations as before, using diag (V, V^{\dagger}) with V defined by (29) and $\chi = \exp(2\pi i/N)$ as before, we get

$$M_1 = M_{1/(2N)}^{2N}, (32)$$

where

$$M_{1/(2N)} \sim \text{ diag } [m(0), m(1/(2N)), m(-1/(2N)),$$

..., m(n/(2N)), m(-n/(2N)), (33)
..., m((N-1)/(4N)), m(-(N-1)/(4N))] (N \text{ odd })

$$M_{1/(2N)} \sim \text{diag } [m(0), m(1/(2N)), m(-1/(2N)),$$

..., m(n/(2N)), m(-n/(2N)), (34)
..., m((N-2)/(4N)), m(-(N-2)/(4N)), m(1/4)] (N even)

Encounter	Unstable region	Туре
λ_1 vs. λ_2	$\nu_+ \lesssim 1/2$	+
	$\nu_+ \lesssim 1$	+
λ_3 vs. λ_4	$ u \lesssim 1/2 $	+
	$\nu_{-} \lesssim 1$	+
λ_1 vs. λ_4	$\nu_+ + \nu \lesssim (N-n)/N$	с
	$\nu_+ + \nu \stackrel{<}{\sim} (2N - n)/N$	c
λ_2 vs. λ_3	$ u_+ + \nu \lesssim n/N $	с
	$ u_+ + u \lesssim (N+n)/N $	с

TABLE VI Mode-coupling pattern of m(n/(2N)). Note that $v_{\pm} = Q^{\pm}/(2N)$ (mod 1). For the $N \oplus N = 2N$ case, the type -, + and c refers to the negative, positive and complex eigenvalues for $m(n/(2N))^{2N}$. We have $0 \le n \le N/2$

For $\Xi \ll 1$, the eigenvalues of m(n/(2N)) are

$$\lambda_{1} \approx \exp[2\pi i (+n/(2N) + \nu_{+} + \Xi)]$$

$$\lambda_{2} \approx \exp[2\pi i (+n/(2N) - \nu_{+} - \Xi)]$$

$$\lambda_{3} \approx \exp[2\pi i (-n/(2N) + \nu_{-} + \Xi)]$$

$$\lambda_{4} \approx \exp[2\pi i (-n/(2N) - \nu_{-} - \Xi)]$$
(35)

Here $0 \le n \le N/2$ is assumed and the n = 0 case corresponds to $M_{\sigma\pi}$. The case n = N/2 occurs only when N is even. Table VI shows the types of the unstable regions, and Figure 3 the schematic instability lines.

3.5 Equivalence of $N \oplus N = N$ and $N \oplus N = 2N$ Cases for Odd N

So far, we have examined the cases $N \oplus N = N$ for odd N and $N \oplus N = 2N$ for arbitrary N, and we show in the Appendix A that they are the only irreducible cases for equally spaced bunches and interaction points. We now show that the $N \oplus N = N$ and $N \oplus N = 2N$ cases are equivalent when N is odd. We observe that there are identical terms for even n in Equations (30) and (33). For odd n, we use $m(\omega) = -m(\omega \pm 1/2)$, and change the typical term in (33) as follows:

$$m\left(\frac{n}{2N}\right) = -m\left(\frac{n}{2N} + \frac{1}{2}\right) = -m\left(-\frac{N-n}{2N}\right).$$
 (36)



FIGURE 3 The edges of the unstable region in (ν_+, ν_-) plane for $0 \le \nu_{\pm} \le 1$. For the $N \oplus N = 2N$ case, these graphs can be taken as graphs in the (Q^+, Q^-) -plane for $0 \le Q^{\pm}$ (mod 2N) $\le 2N$. For the $N \oplus N = N$ case with odd N, they can be used in the (Q^+, Q^-) -plane for $0 \le Q^{\pm}$ (mod $N) \le N$. N is indicated in each graph.

We then notice that the modified terms in (33) are identical to the remaining terms in (30), apart from the sign which is irrelevant for the stability. Hence, we have demonstrated that, apart from the sign, the matrices

$$M_{1/N}(N \oplus N = N) \sim M_{1/(2N)}(N \oplus N = 2N),$$
 (37)

are equivalent, i.e. have the same absolute eigenvalues when N is odd. Therefore, the graphs in Figure 3 can be used both for $N \oplus N = N$ cases with odd N and $N \oplus N = 2N$ cases.

3.6 The $N \oplus N = N_{\text{IP}}$ Case for Arbitrary N_{IP}

We now consider the case of equally spaced bunches and IPs for arbitrary $N \oplus N = N_{\text{IP}}$, such that N_{IP} is an integral fraction of 2*N*. The algorithm for reducing *N* is as follows:

- 1. calculate the number of families $N_{\rm F} = \gcd(N, 2N/N_{\rm IP})$, (see Appendix A),
- 2. reduce the $N \oplus N = N_{\rm IP}$ case to the $N/N_{\rm F} \oplus N/N_{\rm F} = N_{\rm IP}$ case, arriving at either $N' \oplus N' = N' (N' \text{ odd})$ or at $N' \oplus N' = 2N'$ for $N' = N/N_{\rm F}$, which are both irreducible,
- 3. find a graph in Figure 3 with index N', which shows the unstable region in the (Q^+, Q^-) -plane for $0 \le Q^{\pm} \pmod{N_{\text{IP}}} \le N_{\text{IP}}$.

Let us consider cases with N = 6 and all possible values of $N_{\rm IP}$, starting at the highest value $N_{\rm IP} = 12$, and taking all values which are divisors of 12, as shown in Table VII. For example, we find the instability pattern of the $6 \oplus 6 = 3$ case by looking at the index 3 in Figure 3 where both axes are from 0 to 3 in Q^{\pm} .

$N \oplus N = N_{\rm IP}$	S	N _F	N'	reduced case
$6 \oplus 6 = 12$	1	1	6	irreducible
$6 \oplus 6 = 6$	2	2	3	$3 \oplus 3 = 6$
$6 \oplus 6 = 4$	3	3	2	$2 \oplus 2 = 4$
$6 \oplus 6 = 3$	4	2	3	$3 \oplus 3 = 3$
$6 \oplus 6 = 2$	6	6	1	$1 \oplus 1 = 2$
$6 \oplus 6 = 1$	12	6	1	$1 \oplus 1 = 1$

TABLE VII Table of all possible cases for 6 bunches in each beam

3.7 Instability Threshold for Equal Tunes

In all $N \oplus N = N_{\text{IP}}$ cases, where $\nu_+ = \nu_- \equiv \nu$ holds in addition to $\Xi^+ = \Xi^- \equiv \Xi$, we can analytically calculate the threshold of the instability, i.e. the minimum value of Ξ that gives the instability.

The instability threshold $\hat{\Xi}_{\sigma\pi}$ for the $M_{\sigma\pi}$ block, (19), is well known. Applying a similarity transformation, we can reduce $M_{\sigma\pi}$ to blockwise diagonal form:

$$M_{\sigma\pi} \sim T\hat{U}(\nu)TTR(\Xi, \Xi)T \sim \hat{U}(\nu) \begin{pmatrix} I & O \\ O & I-2A \end{pmatrix}$$

$$\sim \begin{pmatrix} U(\nu) & O \\ O & U(\nu)(I-2A) \end{pmatrix}$$
(38)

where U is defined by (2) and T is the symplectic 4×4 matrix

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ I & -I \end{pmatrix}, \tag{39}$$

which satisfies $T^2 = I$. The upper half of (38) corresponds to the so-called σ mode whose tune is not shifted by the beam-beam interaction. The lower half is the π mode. Its perturbed tune ν_{π} is

$$\cos 2\pi \nu_{\pi} = \cos 2\pi \nu - 4\pi \Xi \sin 2\pi \nu, \tag{40}$$

which can also be derived from (8). For $\Xi \ll 1$, $\nu_{\pi} \approx \nu + 2\Xi$. The π mode becomes unstable if and only if $\cos 2\pi \nu_{\pi}$ becomes ± 1 . Solving (40) with this condition yields for the instability threshold

$$\hat{\Xi}_{\sigma\pi} = \frac{\cos 2\pi \nu \mp 1}{4\pi \sin 2\pi \nu}.$$
(41)

The $\hat{\Xi}_{\sigma\pi}$ is shown in the graph in Figure 4, labelled 1.

To find the threshold $\hat{\Xi}_{\chi}$ of the M_{χ} block, (20), for arbitrary χ , explicit expressions for the eigenvalues are not needed, because we know from Tables V and VI that the instability develops if and only if an eigenvalue λ becomes ± 1 . The eigenvalue equation for M_{χ} yields:

$$\Xi = \frac{(1 - 2\lambda\chi\cos 2\pi\nu + \lambda^2\chi^2)(\chi^2 + \lambda^2 - 2\lambda\chi\cos 2\pi\nu)}{(4\pi\lambda\chi\sin 2\pi\nu)[4\lambda\chi\cos 2\pi\nu - (\lambda^2 + 1)(\chi^2 + 1)]}, \qquad (42)$$

Hence, we find the threshold for arbitrary χ by substituting $\lambda = \pm 1$:

$$\hat{\Xi}_{\chi} = \frac{\cos 2\pi \nu \mp \cos 2\pi \omega}{4\pi \sin 2\pi \nu} , \qquad (43)$$

Here $\chi = \exp(2\pi i\omega)$. When $\omega = 0$, this equation is identical to (41).

In order to obtain the instability threshold $\hat{\Xi}$ for the one-turn map M_1 of the $N \oplus N = N_{\rm IP}$ case we first find the irreducible $N' \oplus N' = N_{\rm IP}$ case, cf. Section 3.6. Then we evaluate (43) for all values of ω appearing in Equations (33) and (34), and obtain the smallest positive $\hat{\Xi}_{\chi}$ which is $\hat{\Xi}$. Since we know the branch which leads to the lowest value of $\hat{\Xi}$, we can avoid looking for the minimum $\hat{\Xi}_{\chi}$, and find:



FIGURE 4 The threshold value of $\hat{\Xi}$ for $N \oplus N = 2N$ cases as a function of ν . The N is indicated in the graph. Each graph gives the $\hat{\Xi}$ for the case given in Figure 3 with the same index. It is thus also usable for general cases if N in the graph is interpreted as $N' = N/N_{\rm F}$ and the horizontal axis as Q (modulo $N_{\rm IP}$) from 0 to $N_{\rm IP}$.

$$\hat{\Xi} = \frac{\cos\{2\pi Q/N_{\rm IP}\} - \cos\{2\pi ([Q+1])/(N_{\rm IP})\}}{4\pi \sin\{2\pi Q/N_{\rm IP}\}}.$$
(44)

Here we use the tune $Q \equiv N_{\rm IP} \nu$, and [a] is the largest positive integer that does not exceed a. The value of $\hat{\Xi}$ given by (44) is shown as a function of ν in Figure 4 with $1 \leq N \leq 6$ as a parameter. The graphs are valid both for $N' \oplus N' = N'$ with odd N' and $N' \oplus N' = 2N'$ cases. The horizontal axes can also be taken as $0 \leq Q \pmod{N_{\rm IP}} \leq N_{\rm IP}$.

Piwinski's result¹² applies to the $N \oplus N = 2N$ case and looks similar but is different even in this case in two important respects: (i) Piwinski finds resonances at half-integral values of Q which do not exist, (ii) a factor of two is caused by the ratio of Ξ and ξ . Our result applies to all $N \oplus N = N_{\rm IP}$ cases.

It is interesting to note that the threshold $\hat{\Xi}$ can be obtained by requiring that the π mode eigenvalues of M_1 become ± 1 , although it is not always the π mode which causes the instability. (The eigenvalue can pass ± 1 along the unit circle without causing instability.) This is so because (42) is symmetric in χ and λ , and therefore the eigenvalue λ_{π} of $M_{\sigma\pi}$ becomes $\pm \exp(2\pi i\omega)$ when the eigenvalue λ_{ω} of M_{χ} becomes ± 1 . Thus, (43) can also be derived by putting $\cos 2\pi \nu_{\pi} = \pm \cos \omega$ into (40).

3.8 Summary for Perfect Machines

We have studied the coherent beam-beam effects in the framework of the LRGM where we consider only the linear focusing force between the barycentres of two bunches colliding head-on. The dominant effect in this case is a change of focusing, parametrized by the beam-beam strength parameter Ξ . We study the stability of the motion of the barycentres using the eigenvalues of the one-turn map. We call the case of N electron bunches colliding with N positron bunches in $N_{\rm IP}$ interaction points $N \oplus N = N_{\rm IP}$, and give closed analytical solutions for it.

In Section 3.6, we give the algorithm for finding the irreducible $N' \oplus N' = N_{\rm IP}$ case for the arbitrary $N \oplus N = N_{\rm IP}$ case. All irreducible cases are either $N \oplus N = N$ with odd N or $N \oplus N = 2N$. Furthermore, these two cases show exactly the same instability patterns in the (Q^+, Q^-) -plane. The period of the resonances in Q^{\pm} is N for the $N \oplus N = N$ case with odd N, and 2N for the $N \oplus N = 2N$ case with odd N, and the (Q^+, Q^-) -plane is filled in a chess board pattern. In all cases with even N, the period in Q^{\pm} is N, and the resonance pattern repeats itself in the (Q^+, Q^-) -plane every N units of tune. For all cases, the edges of the resonances in tune space can be found in Figure 3. The case with equal tunes, $Q^+ = Q^-$, corresponds to the diagonal from lower left to upper right, the threshold is given by (44) and is found in Figure 4.

4 MACHINES WITH ERRORS

We have neglected up to now the errors in real machines which occur there for several reasons:

- 1. The currents of the bunches in the two beams are not exactly equal.
- 2. The betatron functions α and β differ from IP to IP and between the two beams.
- 3. The phase advances μ between IPs differ from arc to arc and between the two beams.
- 4. Electrons and positrons have different energies at all IPs because of asymmetries of the RF accelerations between the IPs, either by design or by errors in the RF system.²⁷
- 5. The emittances of the two beams are different.

In this section, we present the results of numerical computations of the consequences of these errors, which cause $N \oplus N = 2N$ machines with errors to behave to a large extent like $1 \oplus 1 = 1$ machines. In particular, the half-integral resonances appear, and the complex sum resonances occur when the sum of the tunes in the two beams is just below an integer. We have already discussed errors in $1 \oplus 1 = 1$ machines in Section 2.

4.1 $1 \oplus 1 = 2$ Machines with Errors

We simultaneously put errors on the phase advances $v_i^{\pm} = Q^{\pm}/2 + \delta v_{\pm}r$, bunch currents $I_i^{\pm} = I + \delta I^{\pm}r$, and β -functions $\beta_{xi}^{\pm} = \beta_{xi}^{\pm} + \delta \beta_x^{\pm}r$, $\beta_{yi}^{\pm} = \beta_{yi}^{\pm} + \delta \beta_y^{\pm}r$, where the r's are all independent random Gaussian variables with zero average and unit variance. In Figure 5 we show the average growth rates in the (Q^+, Q^-) -plane for ten sets of random errors. The excitation of the error-driven resonances depends strongly on the random errors, while the resonances already present in the perfect machine remain about the same. In contrast to the perfect $1 \oplus 1 = 2$ machine, the errors cause the following error-driven resonances: (i) half-integral resonances when either Q^+ or Q^- is equal to an integer and one-half, (ii) complex sum resonances when the sum of the tunes $(Q^+ + Q^-)$ is just below an integer. These resonances are already present in perfect $1 \oplus 1 = 1$ machines. Figure 5 shows that the histogram channels just below the integral and half-integral tunes are filled with the integral and half-integral resonances. Hence, the upper edge of these resonances coincides with the exact integral or half-integral tunes within the accuracy of the histograms. However, the upper edge of the error-driven sum resonances is one channel or more below the integral value of the sum $(Q^+ + Q^-)$. We will come across other examples of this shifting of error-driven sum resonances later, and believe that it is

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FIGURE 5 Vertical growth rate for the $1 \oplus 1 = 2$ case with random errors on phase advances δv_i^{\pm} , bunch currents $\delta I^{\pm}/I^{\pm}$, β -functions at the IP $\delta \beta_x^{\pm}/\beta_x^{\pm}$ and $\delta \beta_y^{\pm}/\beta_y^{\pm}$, averaged over 10 sets of random errors. The standard deviations are $\delta v_i^{\pm} = 0.16$, $\delta I_i^{\pm}/I_i^{\pm} = 0.2$, and $\delta \beta_x^{\pm}/\beta_x^{\pm} = \delta \beta_y^{\pm}/\beta_y^{\pm} = 0.2$. The abscissa and ordinate are the tunes of the two beams in the intervals $0 < Q^+ < 2$ and $0 < Q^- < 1$, respectively. The nominal beam-beam parameter is $\Xi = 0.015$. The resonances have period 2 in Q^{\pm} . The left and right half fill the (Q^+, Q^-) -plane in a chessboard pattern (cf. Sections 2 and 3.5). The value of the growth rate is indicated by the code at the bottom.

a generic feature. Energy differences of the two beams at the IPs do not produce qualitatively new features.

4.2 $2 \oplus 2 = 4$ Machines with Errors

In practice, $2 \oplus 2 = 4$ machines with errors are important since they represent the case of TRISTAN. The $4 \oplus 4 = 4$ and $8 \oplus 8 = 4$ cases of LEP can be reduced to it when we neglect the beam-beam interactions in every second IP where the beams are vertically separated and at the centres of all

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FIGURE 6 Vertical growth rate for the $2 \oplus 2 = 4$ case with random errors on phase advances δv_i^{\pm} , bunch currents $\delta I_i^{\pm}/I_i^{\pm}$, β -functions at the IP $\delta \beta_x^{\pm}/\beta_x^{\pm}$ and $\delta \beta_y^{\pm}/\beta_y^{\pm}$, averaged over 10 sets of random errors. The standard deviations are $\delta v_i^{\pm} = 0.16$, $\delta I_i^{\pm}/I_i^{\pm} = 0.2$, and $\delta \beta_x^{\pm}/\beta_x^{\pm} = \delta \beta_y^{\pm}/\beta_y^{\pm} = 0.2$. The abscissa and ordinate are the tunes of the two beams in the intervals $0 < Q^+ < 2$ and $0 < Q^- < 2$, respectively. The nominal beam-beam parameter is $\Xi = 0.015$. The resonances have period 2 in Q^{\pm} . The value of the growth rate is indicated by the code at the bottom.

eight arcs where the beams are horizontally separated. We have studied this case numerically, including random errors on the phases $v_i^{\pm} = Q^{\pm}/4 + \delta v_{\pm}r$, on the bunch currents $I_i^{\pm} = I^{\pm} + \delta I^{\pm}r$, and on the β -functions at the IPs $\beta_{xi}^{\pm} = \beta_x^{\pm} + \delta \beta_x^{\pm}r$ and $\beta_{yi}^{\pm} = \beta_y^{\pm} + \delta \beta_y^{\pm}r$, where the *r*'s are all independent random Gaussian variables with zero average and unit variance. In Figure 6 we show the average growth rate over ten different seeds of the random numbers. We observe instability when either Q^+ or Q^- are just below a half-integer or an integer, and when the sum $(Q^+ + Q^-)$ is below an integer. We should compare Figure 6 with the lower half of the N = 2 sketch in Figure 3 for the perfect $2 \oplus 2 = 4$ case. The number of sum resonances is now twice as large, and the number of resonances in Q^+ or Q^- is four times as large. The increased number of resonances divides the (Q^+, Q^-) -plane into much smaller pieces, and reduces the fraction not covered by resonances from 0.696 to 0.582 \pm 0.045.

4.3 Machines with $N \ge 3$

Further numerical studies with random errors on the bunch currents, phase advances and β -functions at the IP, in both $N \oplus N = N$ for N = 3 and $N \oplus N = 2N$ cases with N = 3, 4, show that resonances occur when $Q^{\pm} \pmod{1} \lesssim 1/2, Q^{\pm} \pmod{1} \lesssim 1$, and $(Q^{+} + Q^{-}) \pmod{1} \lesssim 1$, i.e. at the same tunes where resonances occur in the $1 \oplus 1 = 1$ case. Figure 7 shows the $3 \oplus 3 = 3$ case as an example. Compared to the perfect $N \oplus N = N$ case with odd N, the number of resonances in Q^{\pm} increases by a factor of N, and the number of sum resonances remains the same. Compared to the perfect $N \oplus N = 2N$ case with any N, the number of resonances in Q^{\pm} increases by a factor of 2N, and the number of sum resonances doubles. The periodicity of the resonances in the (Q^+, Q^-) -plane is the same as in perfect machines, dicussed in Section 3.8.

5 DISCUSSION OF COMPLEX RESONANCES

In this section, we first calculate analytically the complex resonances for the $1 \oplus 1 = 2$ case which are excited by the differences of the phase advances in the arcs. Next we discuss LEP, where the separation in half of the interaction points is not so large that the beam-beam tune shifts can be neglected completely.

5.1 Complex Resonances in Asymmetric $1 \oplus 1 = 2$ Machines

Chao and Keil¹³ studied the case of a conventional single-ring machine where the phase advances of the e⁺ and the e⁻ beams are identical in the same arc, $v_1^+ = v_1^- \equiv Q/2 + \delta$ and $v_2^+ = v_2^- \equiv Q/2 - \delta$, with $\delta \neq 0$, and where all $\Xi_{1,2}^{\pm} = \Xi$ are identical. We use this case for a demonstration of the behaviour of the complex resonances. The one-turn matrix M_1 is:

$$M_{1} = \hat{U}(Q/2 + \delta, Q/2 - \delta)R(\Xi, \Xi)\hat{U}(Q/2 - \delta, Q/2 + \delta)R(\Xi, \Xi) .$$
(45)

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FIGURE 7 Vertical growth rate for the $3 \oplus 3 = 3$ case with random errors on phase advances δv_i^{\pm} , bunch currents $\delta I_i^{\pm}/I_i^{\pm}$, β -functions at the IP $\delta \beta_x^{\pm}/\beta_x^{\pm}$ and $\delta \beta_y^{\pm}/\beta_y^{\pm}$, averaged over 10 sets of random errors. The standard deviations are $\delta v_i^{\pm} = 0.16$, $\delta I_i^{\pm}/I_i^{\pm} = 0.2$, and $\delta \beta_x^{\pm}/\beta_x^{\pm} = \delta \beta_y^{\pm}/\beta_y^{\pm} = 0.2$. The abscissa and ordinate are the tunes of the two beams in the intervals $0 < Q^+ < 3$ and $0 < Q^- < 1.5$, respectively. The nominal beam-beam parameter is $\Xi = 0.015$. The resonances have period 3 in Q^{\pm} . The left and right half fill the (Q^+, Q^-) -plane in a chessboard pattern (cf. Sections 2 and 3.5). The value of the growth rate is indicated by the code at the bottom.

Here \hat{U} is defined in (2) and R in (4). The eigenvalues can be obtained from

$$\cos \bar{\mu} = \frac{\lambda + \lambda^{-1}}{2} = \cos 2\pi Q - a \sin 2\pi Q + (a \sin \pi Q \cos \pi \delta)^2 \pm \sqrt{D} ,$$
(46)

$$D = (2a\cos\pi\delta\sin\pi Q)^2 \left[(2\cos\pi Q - a\sin\pi Q)^2 - (a\sin\pi Q\sin\pi\delta)^2 \right],$$
(47)



FIGURE 8 Half-integral stopband in a $1 \oplus 1 = 2$ machine with phase advances $2\pi(Q/2+0.1)$ and $2\pi(Q/2-0.1)$. The abscissa is the tune Q, the ordinate is Ξ .

where $a = 4\pi \Xi$. The complex resonance with D < 0 can only occur when the phases in the two arcs are different, i.e. $\delta \neq 0$ holds. It is centred where the first term in the square bracket of D vanishes, at $\Xi = (1/2\pi) \cot \pi Q$, i.e. at Q < 1/2 for attractive beam-beam forces with a > 0. Its width in the Ξ direction is determined by the second term in the square bracket for D:

$$\frac{\cot \pi Q}{2\pi (1+|\sin \pi \delta|)} < \Xi < \frac{\cot \pi Q}{2\pi (1-|\sin \pi \delta|)}.$$
(48)

The region of instability in the (Q, Ξ) -plane is shown in Figure 8. For small Ξ , the instability occupies a small range of Q just below Q = 0.5. With increasing Ξ , the instability becomes wider and shifts towards smaller values of Q. For a fixed value of Q < 1/2, the instability occupies a range of Ξ values, with stability above and below this range.

Another simple case is that of DA Φ NE^{28,29} which has two rings, each consisting of two rather different arcs with phase advances $2\pi (Q/2 + \delta)$ and $2\pi (Q/2 - \delta)$, respectively, and collisions in two interaction points. The revolution matrix is different from (45):

$$M = \hat{U}(Q/2 + \delta, Q/2 + \delta)R(\Xi, \Xi)\hat{U}(Q/2 - \delta, Q/2 - \delta)R(\Xi, \Xi) .$$
(49)

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(B)

FIGURE 9 (A) Tune space (0 < Q^+ < 2, 0 < Q^- < 1). Parameters: $\Xi = 0.05$, $v_1^{\pm} = Q^{\pm}/2 + 0.1, v_2^{\pm} = Q^{\pm}/2 - 0.1$, corresponding to the conventional 1 \oplus 1 = 2 type. (B) Tune space (0 < Q^+ < 2, 0 < Q^- < 1). Parameters: $\Xi = 0.05, v_1^+ = v_2^- = Q^+/2 + 0.1, v_1^- = v_2^+ = Q^-/2 - 0.1$, corresponding to the DA Φ NE 1 \oplus 1 = 2 type.

The real parts of the eigenvalues are:

$$\cos \bar{\mu}_{\sigma} = \cos 2\pi Q ,$$

$$\cos \bar{\mu}_{\pi} = \cos 2\pi Q - 2a \sin 2\pi Q + a^2 (\cos 4\pi \delta - \cos 2\pi Q) .$$
(50)

Hence, the tune of the σ -mode does not move, $\cos \bar{\mu}_{\pi}$ is real, and the complex resonance does not occur.

In Figure 9 we show the instability region in the tune space for both cases. When moving along the diagonal line $(\nu_+ = \nu_-)$ in the left halves of Figures 9A and 9B, instability occurs at half-integers. It is of complex type for the conventional case (45), and of half-integral type for DA Φ NE (49). If the phase advances in the arcs are different in the $1 \oplus 1 = 2$ case, we can predict the existence but not the type of the instability originating at half-integral tunes.

Chao and Furman^{16,17} used a tracking method for the beam-beam modes in the SSC where the interaction points are arranged in two different 'clusters'. They found half-integral resonances of a characteristic sail shape very similar to our Figure 8. This result is not surprising, given the fact that the two halves of the SSC are different.

5.2 The LEP Case

In LEP, the two beams are vertically separated at the odd-numbered pits by about 3 mm. The beam radii there are $\sigma_x \approx 0.57$ mm and $\sigma_y \approx 50 \ \mu$ m. This separation is not quite enough to make the beam-beam focusing completely negligible. In both even- and odd-numbered pits in LEP, the ratio $\beta_x/\beta_y \gg 1$. Therefore, the ratio $\Xi_x/\Xi_y \gg 1$ in the odd-numbered pits where the beams are separated, while in the even-numbered pits where the beams collide head-on, the ratio $\Xi_x/\Xi_y \approx 1$. Hence, the effects of the residual beam-beam tune shifts in the odd-numbered pits are stronger in the horizontal plane. When we apply our formalism to LEP, we are confronted with two difficulties:

 The beam-beam force in the odd-numbered pits modifies the closed orbits of all bunches in both beams around the whole circumference of LEP.² We overcome this difficulty by assuming that the closed orbit in the even-numbered pits, where the head-on beam-beam collisions occur, is corrected locally by fine adjustments of the vertical electrostatic separators there, as is actually the case to optimize the luminosity.

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94.29	٠			26	EBG84	+46CJ	POOPHE842	2	•	5
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FIGURE 10 Horizontal growth rate for the $4 \oplus 4 = 8$ LEP case with vertical separation in every second IP and random errors on the phase advances $\delta v_i^{\pm} = 0.01$, averaged over 10 sets of random errors. The abscissa and ordinate are the horizontal tunes of the two beams in the intervals $94.25 < Q_x^{\pm} < 94.5$. The nominal beam-beam parameter is $\Xi = 0.015$. The value of the growth rate is indicated by the code at the bottom of the graph, in steps of 0.002.

2. The Yokoya factor²⁴ is not known for collisions with offsets. Since it is typically 4/3 for head-on collisions, and may tend towards unity for separated beams, we leave it at the typical value. The difference is small enough to be ignored.

We studied this case numerically with phase advance errors with standard deviations $\delta v_x = \delta v_y = 0.01$ in the arcs, and observed stopbands. The average growth rate obtained for ten seeds of the phase errors are shown in Figure 10 for the horizontal tune range $94.25 < Q_x^{\pm} < 94.5$ in the two beams which was used in actual LEP operation in 1992. Two systems of complex instabilities cross over when the two horizontal tunes are equal. Between seeds there are large variations in the widths and growth rates of these stopbands. However, their positions in the tune plane are always the same, which makes it possible to display a meaningful average.

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FIGURE 11 Horizontal growth rate for the $4 \oplus 4 = 8$ LEP case with vertical separation in every second IP and random errors on the phase advances $\delta v_i^{\pm} = 0.01$, averaged over 10 sets of random errors. The abscissa is the horizontal tune of the two beams in the interval $94.25 < Q_x^+ = Q_x^- < 94.5$. The ordinate is the bunch current in the interval $0 \le I \le 0.0005$ A. The beam-beam parameter is $\Xi = 0.015$ at the maximum *I*. The value of the growth rate is indicated by the code at the bottom.

Figure 11 shows these stopbands as a function of the bunch current I for the case with equal horizontal tunes in the two beams. The stopbands start at the half-integral tune $Q_x^{\pm} = 94.5$ at small bunch current I, and move towards smaller fractional tunes as I increases. This behaviour is very similar to Figure 8 and the SSC results.^{16,17} It is very plausible that coherent horizontal oscillations, which are often observed in LEP when the two beams are colliding in the even-numbered pits, are indeed related to these stopbands.³⁰

6 CONCLUSIONS

We studied the coherent beam-beam effects in storage rings in the framework of the linear rigid Gaussian model where we consider only the dipole forces between the barycentres of two bunches, assumed to be rigid bodies colliding head-on. The dominant effect in this case is a change of focusing, parametrized by the beam-beam strength parameter Ξ . We study the stability of the motion of the barycentres in linear approximation, using the eigenvalues of the one-turn map. We call the case of N equidistant electron bunches colliding with N equidistant positron bunches at $N_{\rm IP}$ equidistant interaction points $N \oplus N = N_{\rm IP}$. In Section 3.6, we give the procedure which reduces the general case to its associated irreducible case. Figure 3 shows schematic diagrams of the resonances in perfect machines up to N = 6, while Figure 4 shows and (44) gives a closed expression for the threshold in terms of the coherent beam-beam parameter Ξ for the case of equal tunes in the two beams.

We find that errors in the machine, in particular in the bunch currents, the β -functions at the IPs and the phase advances through the arcs, which may be caused by the design or by inevitable errors in the construction, increase the number of beam-beam resonances. In machines with errors, integral resonances occur when one of the tunes $Q^{\pm} \pmod{1} < 1$, half-integral resonances occur when one of the tunes $Q^{\pm} \pmod{1} < 1/2$, and complex resonances occur when the sum of the tunes $(Q^{+} + Q^{-}) \pmod{1} < 1$. The schematic resonance diagram for N = 1 in Figure 3 shows the resonances in machines with errors for all N. Figures 1, 5–7, show examples of these resonances in the cases $1 \oplus 1 = 1$, $1 \oplus 1 = 2$, $2 \oplus 2 = 4$, and $3 \oplus 3 = 3$, respectively. The growth rates displayed correspond to *e*-folding times of a few turns, making it rather unlikely that the resonances can be cured by a feedback system at present or in the near future.

APPENDIX A BEAM-BEAM FAMILIES

In a machine of type $N \oplus N = N_{\rm IP}$, $N_{\rm IP}$ is at most 2N. When some of these IPs are inactivated by separators, for example, it might happen that the one-turn matrix M_1 can be blockwise diagonalized by reordering the basis vectors. When this happens, we call this case reducible. If the one-turn matrix cannot be blockwise diagonalized by any permutation of basis vectors, we call this case irreducible. The reducible one-turn matrix can be reduced to $N_{\rm F}$, the number of families, irreducible diagonal blocks. When $N_{\rm F} = 1$, the case is irreducible.

Each block of bunches, corresponding to an irreducible diagonal block, is called a (beam-beam) family. Bunches belonging to different families do

not interact with each other even indirectly: if we kick a bunch, the bunches belonging to the same family of the kicked bunch are eventually affected by the kick but those belonging to other families are not affected. Conversely, bunches belonging to different families do not collide with each other in any active IP. In this appendix, we show how to calculate N_F . A family is composed of one or more e⁺ bunches and the same number of e⁻ bunches. We consider the e⁺ bunches only because e⁻ bunches can be treated in the same manner.

We label IPs and bunches in a different way from the main text. Each beam is composed of N equidistant bunches. The e⁺ beam rotates clockwise and the e⁻ counterclockwise in the ring. Possible IPs are labelled clockwise as IP_i, $0 \le i \le 2N - 1$. Bunches are called clockwise $e_j^{\pm} = 0 \le j \le N - 1$. The chronological step is numbered as t_k , $k = \ldots, -2, 0, 2, \ldots$. At t_0 , e_j^+ and e_j^- sit at IP_{2j}. It is convenient to define these numbers in a cyclic manner: e_j^{\pm} is identical with e_{j+nN}^{\pm} , IP_i with IP_{i+2nN} and t_k with t_{k+2nN} , where *n* is any integer. The e_j^{\pm} bunch comes to IP_i at t_k as

$$i(\text{ IP}) = 2j(\text{ bunch }) \pm k(\text{ time }) \mod 2N.$$
(51)

Two bunches e_i^{\pm} and e_j^{\pm} are in the same family when e_i^{\pm} can influence e_j^{\pm} through a beam-beam interaction. We write

$$e_i^{\pm} \sim e_j^{\pm}$$
.

If only one IP, IP₀, is active, each e^+ bunch forms a family so that N_F is N.

Let us first consider the case where two IPs (IP₀, IP_s) are active. We can assume that $1 \le s \le N$ without loss of generality. From (51), e_0^- collides with e_0^+ at IP_s at t_{-s} and with e_0^+ at IP₀ at t_0 . Repetition of this process yields

$$\mathrm{e}^+_{-s}\sim\,\mathrm{e}^+_0\sim\,\mathrm{e}^+_s.$$

By repeating the same argument, we finally arrive at

$$e_0^+ \sim e_{ns}^+$$

for any value of the integer *n*. Thus, the collisions at IP₀ and IP_s cause all e_j^+ with j = 0, s, 2s, ... to be in one family. It follows that the number of families is

$$N_{\rm F} = \gcd(N, s), \tag{52}$$

where gcd stands for the greatest common divisor. Each family contains $N/N_{\rm F}$ bunches in each beam. In particular, $N_{\rm F} = 1$ when s = 1 or when N is a prime number and $N_{\rm F} = N$ when s = N (two diametrically opposite IPs). The family due to two diametrically opposite IPs (i.e. IP₀ and IP_N) is identical with the family due to IP₀ (or IP_N) alone.

A simple consequence is that the family due to IPs of (IP₀, IP_s, IP_N) is identical with the family due to (IP₀, IP_s). The proof is as follows: Consider a family due to (IP₀, IP_s). This family might interact with other families (so that families will merge) by introducing a new IP. By IP_N, however, this merging of families does not happen because IP_N defines the same family as IP₀. Hence, we need not consider the diametrically opposite IP. Another simple consequence is that the family due to (IP₀, IP_s, IP_{2s}, ..., IP_K) is the same as the family due to (IP₀, IP_s). The equidistant (K + 1) IPs define the same family so that the number of families N_F is also N_F = gcd(N, s).

Let us now discuss machines with equidistant periodic IPs, in which $N_{\rm IP}$ must be a divisor of 2N. We can apply the discussion above using $s = 2N/N_{\rm IP}$. All possibilities for $N_{\rm F} = \gcd(N, s)$ are listed in Table VIII.

For LEP with complete separation at the odd pits, the IPs are (IP₀, IP₂, IP₄, IP₆) and N = 4. Thus $N_F = \text{gcd}(4, 2) = 2$. For Furman's case^{16,17} N = 26 and active IPs are (IP₀, IP₄, IP₂₆, IP₃₀). The IP₀ and IP₂₆ are a diametrically opposite pair and so are IP₄ and IP₃₀. Thus the family structure is the same as that due to (IP₀, IP₄), and we have $N_F = \text{gcd}(26, 4) = 2$. More general and complicated cases can be discussed in the same manner.

The discussion given here applies to machines with errors as well as without errors because it depends only on the geometrical configuration of the collisions.

TABLE VIII Number of families N_F for perfectly periodic $N \oplus N = N_{IP}$ cases, where *m* is a divisor of *N* (if any) but not 1, 2 or *N*

N IP	1	2	m	m	N	N	2 <i>N</i>
N _F	N	N	N/m	2N/m	1	2	1
when	always	always	N odd	N even	N odd	N even	always

CHADDELS	10		0	1		2		3		4	5	0	
	1	1	123456	789012	34567	789012:	34567	890123	456789	01234	567890	¥	
			*****	*****	****	*****	****	*****	*****	*****	*****		
OVE	٠											•	OVE
4.95	٠											•	20
4.9	٠		+2	2222+								•	19
4.85	٠			23344	44433	32+						•	18
4.8	٠			+34	45566	5666555	5432					٠	17
4.75	٠				24566	5777786	88777	766543	+			٠	16
4.7	٠				+3	5677889	99999	999999	98876	543		•	15
4.65	٠					24678	89444	BBBBBB	88888	844998	87642	٠	14
4.6	٠					35	78994	BBCCCD	DDDDDI	DDDDDC	DCCBBA	•	13
4.55							+4689	ABBCCD	DEEEFI	FFFFF	FFFFFF	•	12
4.5	•						35	79ABCC	DEEFF	CCERT	XXXXXII	٠	11
4.45	٠							4689B	CODEFI	FGRRRI	113333	٠	10
4.4	٠							+57	9ACDE	EFCHNI	IJJKKK	٠	9
4.35	•								368AB	DEFECI	IJJKKL		8
4.3	٠								47	BCDFG	HHIJKK	٠	7
4.25										58ACE	EFNIIJ	•	6
4.2	٠									269	BCEFGN	•	5
4.15	٠										37ABDF	٠	4
4.1	٠										484	•	3
4.05	٠											•	2
4	•											٠	1
WED	•											•	UND
+10++ 1		***	*****	*****	****	*****	****	*****	*****	*****	*****	••••	
LOV-EDGE1	000			11	1111:	111122:	22222	222333	333333	334444	444444		
*10**- 8	100		12345	678901	2345	678901:	23456	789012	345671	890123	456789		
	10		00000	000000	0000	000000	00000	000000	00000	00000	000000		
	1	•	00000	000000	0000	000000	00000	000000	00000	00000	000000		
• SCALE	.,.	+,2	,3,.,.	, A,B,									
• STEP =	.10	00E	-01 +	HINING	H=0.(000E+0	0						

FIGURE 12 Vertical growth rate from the eigenvalue analysis. The abscissa is the bunch population in the interval $0 < N < 5 \times 10^{11}$. The maximum N corresponds to $\Xi = 0.027$. The ordinate is the tune in the interval $0.4 < Q_y^{\pm} < 0.5$. The phase advance error is $\delta = \pm 0.16$. The value of the growth rate per turn is indicated by the code at the bottom.

APPENDIX B COMPARISON WITH MULTI-PARTICLE TRACKING

The eigenvalue analysis in the LRGM does not allow us to state what happens when the instability has grown to amplitudes comparable to the r.m.s. beam radius or larger, where the beam-beam tune shifts drop in proportion to the square of the separation. We have therefore compared the results of the eigenvalue analysis of BBMODE for the half-integral resonance in the $1 \oplus 1 = 2$ case with the results of the multi-particle tracking program BB.¹⁸ Figure 12 shows the growth rate obtained as a function of the tune Q and the bunch current I, and the characteristic sail shape of the stopband. When the tune is somewhat below the half-integer and the bunch current varies, the instability occurs for a range of the bunch current, with stability above and below that range.

CHABBELS	10		0	1	2	3	4	5 0)	
	1	5	12345678	90123456	578901234	567890123	156789012	34567890 1	I .	
	٠	••	*******	*******	*******	********	********	•••••	**	
OVE	٠								٠	BAO
4.925	٠		.+234467	899ABCCI	DEEFGGGNI	JIJKLLHHN	OSOPPLQL	RSSRT	٠	19
4.875	٠		++232	+3445667	78899441	ABBBCDDDE	EEEFFFGGG	KKKKII JJ	٠	18
4.825	•		+2	22+2222	344445666	5788899999	AAABBBBBB	CBCCCDDD	٠	17
4.775	٠			++2222+	++22223+3	3355356777	878888899	9444448	٠	16
4.725	٠			.++22222	2++++2+2:	222222+233	235546677	78888889	٠	15
4.675	•			+++2	2222++++	2222+23+2	2++ 32+	636.3567	٠	14
4.625	٠			++	+22222+++	+++++2++2	223242 +	+42+2+	٠	13
4.575	•			· · · · · · · · · •	+++++22+3	2+2+++++2	+22222223	3++++2+ .	٠	12
4.525	•				. * * * * * * * *	++222+ . +++	+++++22++	++232323	٠	11
4.475	•				++ ++++	+2+2222.++	*******	+222222+	٠	10
4.425	•		• • • • • • • •		• • • • • • • • •	+++++2++2	++++ . ++++	+++++2+2	٠	9
4.375	٠					. + + + + + + + 2 +	++++ ++	*******	٠	8
4.325	•		• • • • • • • •			***** *	++++ . +2++	. + + + + +	٠	7
4.275	•					** . **	+++2+.+++	****.	•	6
4.225	•					* * . *	*******	++2++.+.	•	5
4.175	•		• • • • • • • •			 •	. * * * * * * * * *	******	•	4
4.125	•			• • • • • • • •	· · · · · · · · ·		++.++++	++2++++.	•	3
4.075	•		• • • • • • •	• • • • • • • •	• • • • • • • •	• • • • • • • • • •		.+++++++	•	2
4.025	•	•	• • • • • • •		• • • • • • • •	• • • • • • • • • •		*****.**	٠	1
W D D	•	•							•	
•10 • • 1	•	••	*******	*******	*******	*********	********	********	***	
LOW-EDGE1	000)		11111	11111222	2222222333	333333344	44444444		
♦10♦♦ - 8	100)	123456	78901234	56789012	3456789012	345678901	23456789		
	10)	5555555	55555555	55555555	5555555555	555555555	55555555		
	1	۱.	0000000	00000000	00000000	0000000000	000000000	00000000		
			2 1	A R						
+ STEP :		1	• H	TETHEN=0						

FIGURE 13 Absolute value of the vertical beam separation in units of σ_y , averaged over 1000 turns, from multi-particle simulation. The abscissa is the bunch population in the interval $0 < N < 5 \times 10^{11}$. The ordinate is the vertical tune in the interval $0.4 < Q_y^{\pm} < 0.5$. The horizontal tune is $Q_x^{\pm} = 0.3$. The maximum N corresponds to $\Xi = 0.027$. The phase advance error is $\delta = \pm 0.16$. The value of the separation is indicated by the code at the bottom.

Figure 13 shows the most dramatic effect seen in the multi-particle tracking. The two bunches separate spontaneously in the vertical plane. An increase in the vertical r.m.s. beam radius is also observed. Figure 13 also shows that the vertical separation at a given tune below the half-integer does not stop above a certain bunch current as was the case for the growth rate shown in Figure 12. However, the agreement between the lower edge of the instability stopband in Figure 12 and the onset of the spontaneous separation in Figure 13 is better than about $\pm 10\%$.

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