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THE EXISTENCE OF A STATIC SOLUTION FOR THE HAISSINSKI EQUATION WITH PURELY INDUCTIVE WAKE FORCE

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The equilibrium longitudinal distribution of electrons in circular accelerators is discussed for the case of the δ' wake function. Contrary to the well known fact that the solution does not exist in this case beyond a threshold, it is strongly suggested that the solution actually exists when we regularize the singularity of the δ' wake function in a physical way, therefore the non-existence of the solution has no physical consequence.

Keywords: Electron accelerators; Instabilities; Impedance; Storage rings

1 INTRODUCTION

The Haissinski equation[1] describes the equilibrium longitudinal distribution of electrons in circular accelerators. The analysis of the stability of this solution against a small perturbation is the basis of the theory of longitudinal instabilities.[2] Nonetheless, the existence (and the uniqueness) of the solution of this equation is not assured. It is difficult to solve this problem in complete generality.

Electrons in an accelerator interact with their environment because they are enclosed in metals (vacuum pipe, the RF cavities, etc.). This effect is represented by the wake function.[3] The wake field acting on an electron is determined by the distribution of electrons ahead of it.

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At the same time, the distribution of electrons is influenced by the wake field. Hence, to determine the distribution function, one should solve coupled non-linear equations. The single particle equations of motion are as follows:

$$\frac{\mathrm{d}}{\mathrm{d}s}\xi = -\frac{\omega_{\mathrm{s}}}{c\sigma_\epsilon}\epsilon, \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\epsilon = \frac{\omega_{\mathrm{s}}\sigma_\epsilon}{c}\xi - \frac{e^2LN}{T_0E_0c}\int_\xi^\infty \mathrm{d}\xi'\rho(\xi')W(\xi'-\xi). \tag{2}$$

Here, we have used the dimensionless parameter ξ as

$$\xi \equiv \frac{\omega_{\mathrm{s}}}{\alpha\sigma_\epsilon}\tau, \tag{3}$$

where σ_ϵ is the nominal rms relative energy spread, c is the velocity of light, e is the electric charge of the electron, L is the total length of the pipe structure in which the wake field is generated, E_0 is the reference energy of the beam, N is the number of electrons in a bunch, T_0 is the revolution period of the beam, τ is the time displacement between an electron and the reference synchronous particle, ϵ is the relative energy $(E-E_0)/E_0$ with E being the electron energy, ω_{s} is the synchrotron oscillation frequency, α is the momentum compaction factor and s is the longitudinal coordinate along the ring. The second term of Eq. (2) is the retarding force seen by a particle at ξ due to the longitudinal wake force which is produced by all particles infront of it; $\rho(\xi)$ is the particle density at location ξ.

In the presence of radiation, two more parameters are necessary: b is the damping coefficient, and $D = b\sigma_\epsilon^2$ is the diffusion coefficient representing the amount of quantum excitation due to photon emission. The dynamics with radiation can be described by the Fokker–Planck equation for the phase space particle distribution $\psi(\epsilon, \xi, s)$

$$\begin{aligned}\frac{\partial\psi}{\partial s} = &-\frac{\omega_{\mathrm{s}}\epsilon}{\alpha\sigma_\epsilon}\frac{\partial}{\partial\xi}\psi + b\frac{\partial}{\partial\epsilon}\epsilon\psi \\ &+ \left(\frac{\omega_{\mathrm{s}}\sigma_\epsilon}{\alpha}\xi - \frac{e^2LN}{T_0E_0\alpha}\int_\xi^\infty \mathrm{d}\xi'\rho(\xi')W(\xi'-\xi)\right)\frac{\partial}{\partial\epsilon}\psi + D\frac{\partial^2}{\partial\epsilon^2}\psi.\end{aligned} \tag{4}$$

This equation has a static solution given by

$$\psi(\epsilon, \xi) = \exp\left(-\frac{\epsilon^2}{2\sigma_\epsilon^2}\right)\rho(\xi), \tag{5}$$

$$\rho(\xi) = A\exp\left(-\frac{\xi^2}{2} + \int_\xi^\infty d\xi' V(\xi')\right), \tag{6}$$

$$V(\xi) = \int_\xi^\infty d\xi' \rho(\xi') w(\xi' - \xi), \tag{7}$$

$$w(\xi' - \xi) = -\frac{e^2 LN}{\omega_s \sigma_\epsilon T_0 E_0} W(\xi' - \xi). \tag{8}$$

Equation (6) is the Haissinski equation (A is the normalization constant where $\int \rho\, d\xi = 1$). Since $\rho(\xi)$ depends only on $\rho(\xi')$ for $\xi < \xi'$, and we know

$$\rho(\xi) \sim A\exp\left(-\frac{\xi^2}{2}\right), \quad \xi \to \infty, \tag{9}$$

Eq. (6) can be integrated from the head of the bunch to the tail for a given value A. Let us call the result of such an integration as $\rho(\xi, A)$ and define the "charge" Q as

$$Q = Q(A) = \int_{-\infty}^{\infty} \rho(\xi, A)\, d\xi. \tag{10}$$

If a value A exists such that $Q(A) = 1$, it gives the solution of the Haissinski equation. Usually, we find the solution of Eq. (6) numerically.

Previous independent studies seem to indicate that there is always a unique solution,[4] except for a single well known case, the δ' wake function[1,3]

$$w(\xi) = S\delta'(\xi). \tag{11}$$

This case is called the purely inductive wake function[5] and corresponds to a beam induced voltage $V(\xi) = -S\rho'(\xi)$ proportional to the derivative of the particle density. This case is of practical importance: machines with many small discontinuities, bellows, masks, etc. will tend to be more inductive.[3,6] For the damping ring of the Stanford Linear Collider, the observation is quite consistent with the assumption of the purely inductive case.[6–10] For example, S is evaluated as -7.5 for the SLAC damping ring[6] and -0.94 for KEKB.[10] Here we have to notice that the sign of S depends on that of α, because the sign of ω_s depends on that of α (see Eq. (8)).

This paper is organized as follows: In Section 2 we review the solution of the Haissinski equation with purely inductive wake force. Then in Section 3, we show that the solution actually depends on how to construct the derivative of a delta function when a parameter is over a threshold. Conclusions and discussions follow under Section 4.

2 PURELY INDUCTIVE IMPEDANCE

For Eq. (11), Eq. (6) is rewritten as

$$\frac{\rho'}{\rho} = -\xi + S\rho', \tag{12}$$

where the prime denotes differentiation with respect to ξ. By putting

$$\rho' = \frac{-\xi\rho}{1 - S\rho}, \tag{13}$$

we get the solution

$$\log\rho - S\rho = -\tfrac{1}{2}\xi^2 + \log A. \tag{14}$$

If S is negative, Eq. (13) has no singularity in its denominator and there is always a continuous unique solution that is normalizable. From now on, we consider the case that S is positive. Given A, the solution is shown in Figure 1. The manifold of the solution changes suddenly at

$$A = A_{\max} = \frac{1}{Se}, \quad e = 2.7182\ldots \tag{15}$$

When $A > A_{\max}$, the solution $\rho(\xi, A)$ defined for $-\infty < \xi < \infty$ does not exist. Clearly, $Q(A)$ increases when A increases. For ρ to be

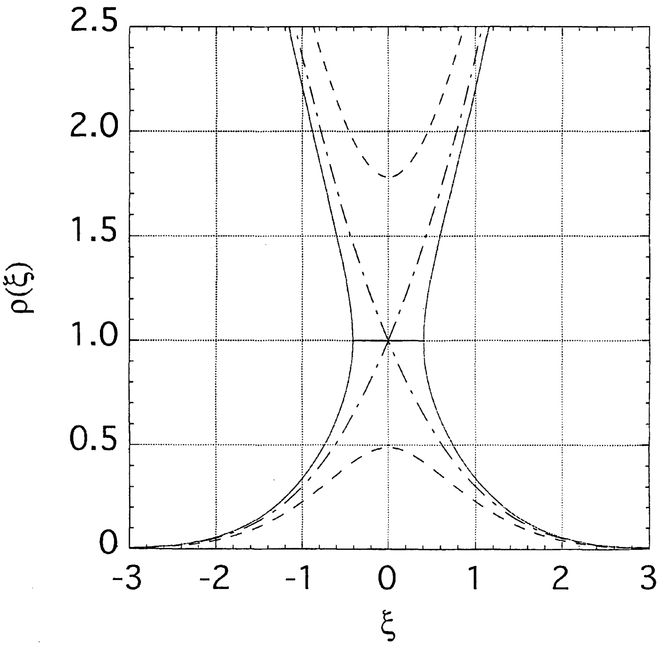

FIGURE 1 The solution manifolds of Eq. (13) for $A < A_{\max}$ (– – –), $A = A_{\max}$ (-----) and $A > A_{\max}$ (—).

normalizable, $Q(A_{\max})$ must be greater than 1. Using Eq. (14) to express ξ in terms of ρ, this gives

$$S \leq S_{\max} = \sqrt{2} \int_0^1 \mathrm{d}x \frac{1-x}{\sqrt{x - \log x - 1}} \simeq 1.55061. \tag{16}$$

When $S > S_{\max}$, no value of A can give $Q = 1$.[3] Previous numerical study told $S_{\max}$ was 1.53 instead of 1.55061.[3] This value is slightly wrong.

3 A PROPERTY OF SINGULARITY POINT

Let us consider more carefully why there is a threshold beyond which no solution exists. The δ' function is an approximation. As shown in Eq. (12), the δ' function induces the derivative of ρ in the r.h.s. In order to define the derivative, we need information around a neighborhood. It means that we cannot determine $\rho(\xi')$ from information for $\xi > \xi'$ only. In this sense, it breaks causality.[3] Further, the case with $A = A_{\max}$ plays an essential role: when $A < A_{\max}$, $\rho(A)$ is flat at $\xi = 0$, as expected from Eq. (13). But at $A = A_{\max}$, $\rho(0)$ jumps to $1/S$, which causes a bifurcation

$$\rho'(0) = \pm 1/S. \tag{17}$$

This is possible only because the r.h.s. of Eq. (12) contains ρ'.

From the above discussions, it seems reasonable to replace δ' by

$$\delta'(\xi) \rightarrow \frac{\delta(\xi) - \delta(\xi - a)}{a}. \tag{18}$$

Note that a must be positive in order to satisfy the causality condition. Instead of Eq. (12), we get

$$\frac{\rho'}{\rho} = -\xi + \frac{S}{a}\big(\rho(\xi + a) - \rho(\xi)\big). \tag{19}$$

The δ' wake function is regained when $a \rightarrow 0$. With a finite a, we have the possibility to avoid the bifurcation which occurred for the δ' case with $A = A_{\max}$. Let us study this possibility and show that Eq. (19) has a solution for arbitrary a and S.

When a is sufficiently large, it is easily found that Eq. (19) reduces to the case of the δ-wake where the existence of the solution is well known:[3,11] the term $\rho(\xi + a)$ of Eq. (19) can be neglected compared with the $\rho(\xi)$ term

$$\frac{\rho'}{\rho} \simeq -\xi - \frac{S}{a}\rho(\xi). \tag{20}$$

It is obvious that a continuous and normalizable ρ exists for arbitrary S. A solution that satisfies the normalization condition can be written as[3,11]

$$\rho(\xi) = \frac{e^{-(\xi^2/2)}}{\sqrt{(\pi/2)}(S/a)\big(\coth(S/2a) + \mathrm{erf}(\xi/\sqrt{2})\big)}. \tag{21}$$

Numerically we confirm that the solution of Eq. (19) can be approximated by that of Eq. (20) (see Figure 2). When a is small and S is

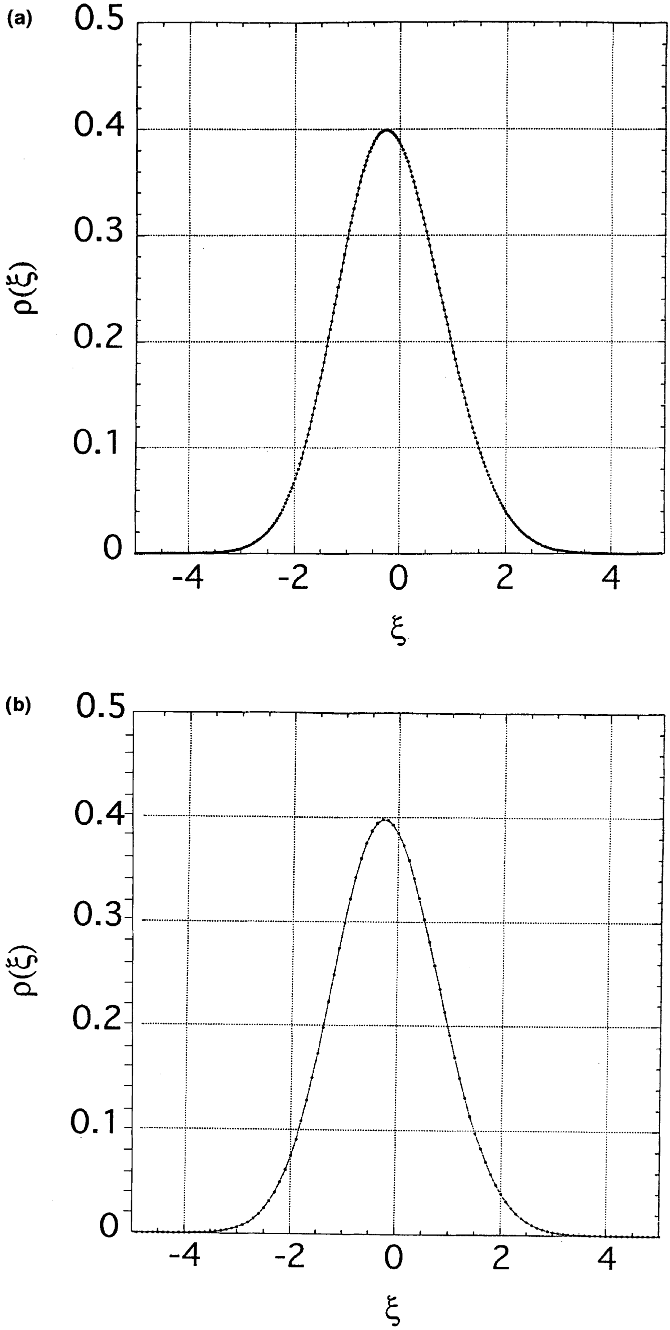

FIGURE 2 (a) The solution of Eq. (19) at $a=3$ and $S=2$; (b) Its approximate solution (Eq. (21)).

smaller than S_{max}, the solution of Eq. (19) can be well approximated by that of Eq. (13).

Let us numerically study the case where a is small and S is larger than S_{max}. Numerical study shows that Eq. (19) has a solution for arbitrary small a. Let us see the feature of this solution and show the reason why Eq. (19) can have solution. In Figure 3, we show the solution manifold for the case with $a = 0.1$. The most remarkable change is that the solution is not symmetric with respect to ξ. It is obvious, because Eq. (19) is not symmetric when a is finite. Further, ρ is not symmetric even when $a \to 0$. It is no surprise that a solution is not symmetric even when its original equation is symmetric. This kind of symmetry breaking really occurs. Actually, ξ_0 that satisfies $\rho'(\xi_0) = 0$ is leaving away from 0 as $a \to 0$ (see Figure 4). We expect that remains true for much smaller values of a.

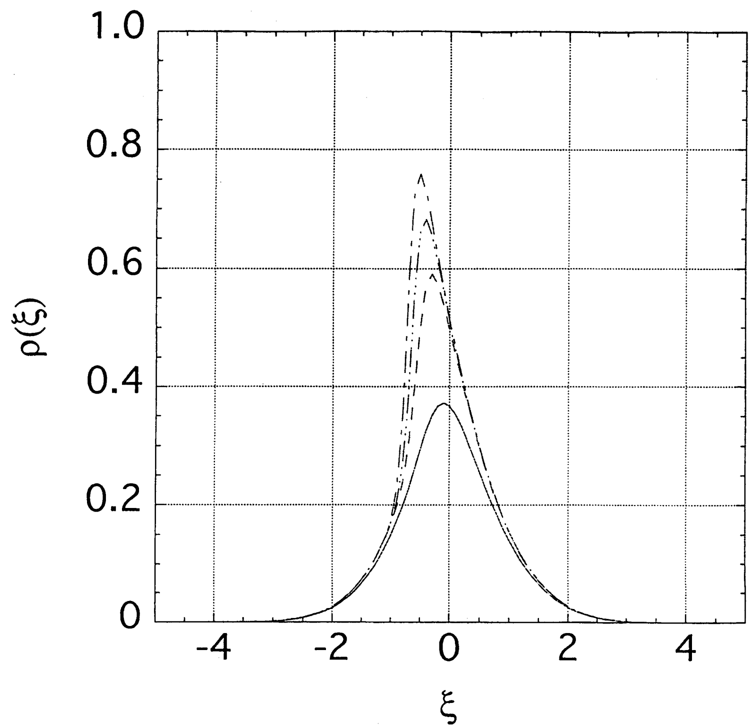

FIGURE 3 The solution of Eq. (19) at $a = 0.1$ for $A = 0.17785$ (—), $A = 0.19119$ (– – –), $A = 0.19208$ (–· · ·–· · ·–) and $A = 0.19230$ (–·–·–).

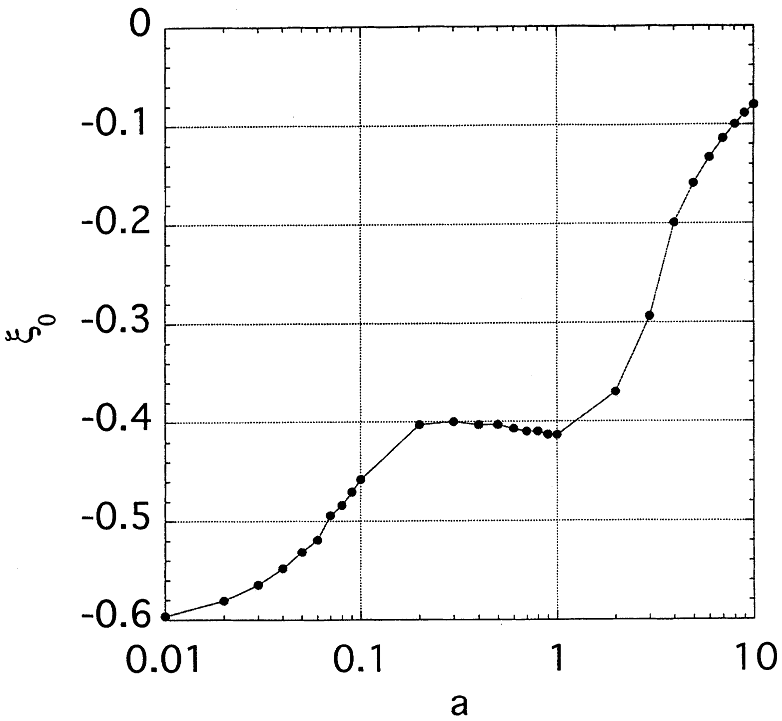

FIGURE 4 The $\xi_0 - a$ relation.

This symmetry breaking can also be seen by $Q(A)$ point of view. We saw in the $a = 0$ case that $Q(A)$ had an upper bound because $\rho(\xi)$ must be continuous and symmetric. However, if symmetry breaking occurs for $A = A_{\max}$, the upper bound of $Q(A)$ due to its bifurcation is removed. It strongly suggests that, with finite a, a continuous solution exists for all S and that the restriction of Eq. (16) does not apply. Actually, $Q(A)$ is a smooth function and monotonically increasing. There exists always one and only one value of A satisfying $Q(A) = 1$ (see Figure 5).

In the $a = 0$ case, the bifurcation occurs and there is a threshold beyond which no solution exists. In the $a \neq 0$ case, the bifurcation does not occur and a solution always exists. Further, the solution obtained by $a \to +0$ is not smoothly connected to the solution obtained by $a \to -0$ when S is larger than $S_{\max}$. Thus, $a = 0$ is mathematically a branch point and the solution manifolds are

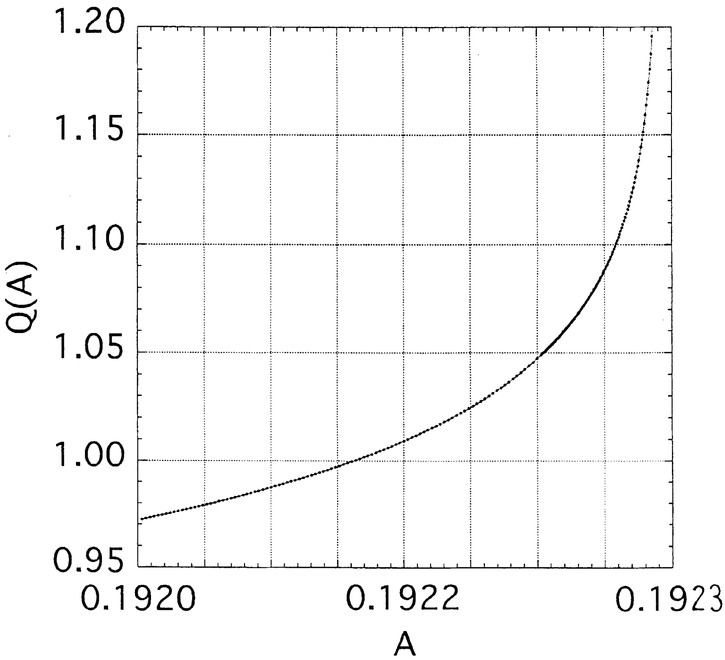

FIGURE 5 The function $Q(A)$ for $a=0.1$.

topologically different from those of the $a \neq 0$ case. In the a–S plane, the line $a=0, S>S_{\max}$ forms a "cut" and only the positive side of a is physical as we have mentioned before.

In the following way, we can also make the approximate formula where a is so small and S is larger than $S_{\max}$. First, we choose Eq. (14) in which $A=A_{\max}$ (see Figure 6). Then, we find ξ_0: ξ_0 can be decided by integrating this function in order to satisfy the normalization condition.

In the above discussions, we used Eq. (18) in order to regularize the $\delta'(\xi)$. There can be other regularization of $\delta'(\xi)$. It is interesting that our discussion is valid also for other choice of the regularization. Let us see briefly the case that we use a resonator wake[3] as a regularization function because it is sometimes used to parametrize the wake function. For simplicity, we consider the case with $Q=1/2$. In this case a resonator wake is written by the following form:[3]

$$\begin{aligned} W(\xi) &= L\omega_R^2 e^{-\omega_R \xi}(1-\omega_R \xi) \quad && \text{for } \xi \geq 0, \\ W(\xi) &= 0 && \text{for } \xi \leq 0. \end{aligned} \tag{22}$$

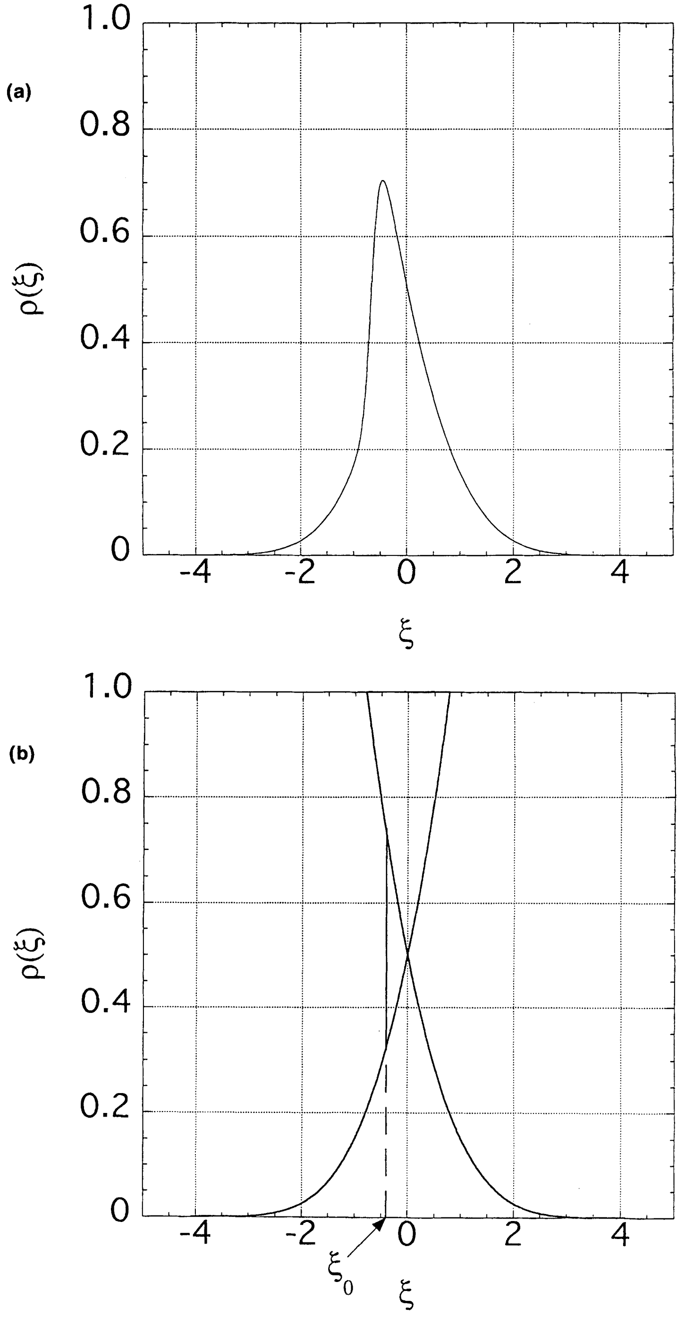

FIGURE 6 (a) The solution of Eq. (19) at $a = 0.1$ and $S = 2$; (b) Its approximate solution (Eq. (14) where $A = A_{max}$ and $S = 2$).

Here L and ω_R are parameters. Impedance of this wake is written as

$$Z(\omega) = -\mathrm{i}\omega L \frac{\omega_R^2}{(\omega_R - \mathrm{i}\omega)^2}. \tag{23}$$

Since $Z(\omega) \to -\mathrm{i}\omega L$ for $\omega_R \to \infty$, ω_R can be seen as a regularization parameter. Given A, the solution is shown in Figure 7. For small A, the solution is almost symmetric. As A becomes larger, ρ becomes asymmetric. Further, there is no upper bound for A. These properties are identical with what we saw in the case of Eq. (18). Thus, we expect that this regularization is equivalent to the previous one when $\omega_R \to \infty$. It may imply that our conclusion does not depend on the choice of regularization function.

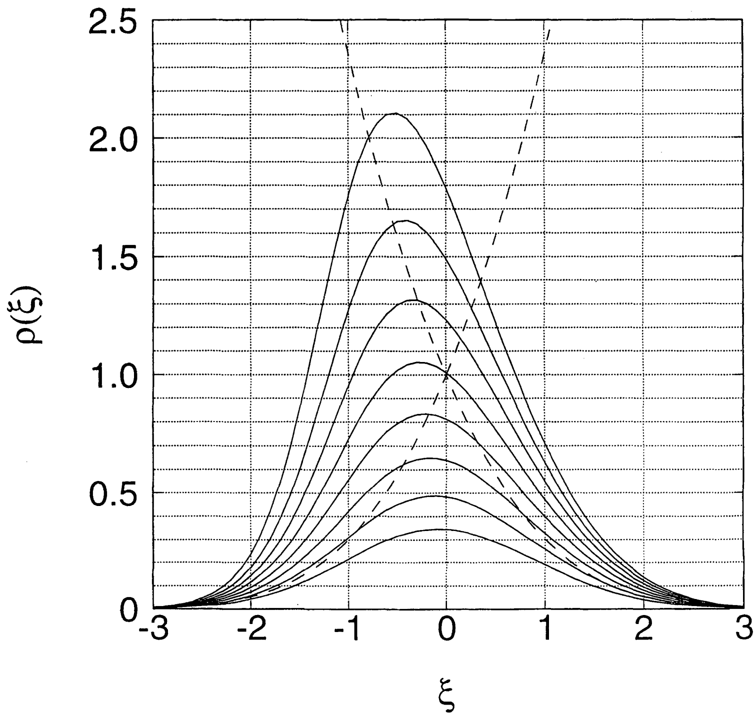

FIGURE 7 The solution manifolds for several values of A for Eq. (22) at $\omega_R = 2$ (–) and the solution manifold of Eq. (13) for $A = A_{\max}$ (– – –).

4 CONCLUSIONS AND DISCUSSIONS

The purely inductive wake function ($S\delta'$) was the only known example where the solution of the Haissinski equation may not exist. With physical and appropriate regularization of this wake function, we have shown that the solution still exists. It is of great interest to see whether the Haissinski equation has at least one solution for any physical wake function. Although it seems true intuitively, we are far from its proof. The present work is a first step towards a solution of this problem. More extensive work will be published elsewhere.

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