

Revisiting chiral phase transition of two
flavor QCD with effective theory approach

Tomomi Sato

Doctor of Philosophy

Department of Particle and Nuclear Physics
School of High Energy Accelerator Science
SOKENDAI (The Graduate University for
Advanced Studies)

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Abstract

Chiral phase transition at finite temperature is one of the most important features of QCD. In this work, we focus on the phase transition of the two-flavor massless QCD using effective theory approach. It is well studied in two extremal cases, the infinitely large broken $U_A(1)$ case and the $U_A(1)$ restored case. Assuming that the breaking of the $U_A(1)$ symmetry is finite at the critical temperature, we investigate the $U(2) \times U(2)$ linear sigma model (LSM) with the $U_A(1)$ breaking term, the $U_A(1)$ broken model. We take a working hypothesis that the $U_A(1)$ broken model undergoes second order phase transition, and we examine the existence of an infrared fixed point as a consistency check by the ϵ expansion. In order to establish the IR nature of the model, the reduction of the $U_A(1)$ broken model to the $O(4)$ LSM is argued. In this argument, the decoupling of the massive fields has a significant role. We find that there is the attractive basin where the RG flow reaches to the infrared fixed point of the $O(4)$ LSM. We calculate the exponent ω which characterizes the sub-leading behavior of the critical phenomena, and show that there is the discrepancy of ω between the $O(4)$ LSM and the $U_A(1)$ broken model. This discrepancy would be interpreted as the remnant of the would-be decouple field.

Contents

| | | |
|----------|---|-----------|
| 1 | Introduction | 4 |
| 2 | Effective theory | 7 |
| 2.1 | Field theory with finite temperature | 7 |
| 2.2 | Effective theory without the $U_A(1)$ anomaly | 8 |
| 2.3 | $U_A(1)$ broken model | 12 |
| 2.4 | $O(4)$ limit | 14 |
| 3 | Renormalization flow of the $U_A(1)$ broken model | 15 |
| 3.1 | β functions | 15 |
| 3.2 | IR behavior of couplings | 18 |
| 3.3 | Attractive basin | 22 |
| 4 | IR nature in the attractive basin | 22 |
| 4.1 | Four-point functions with the RG improvement | 22 |
| 4.1.1 | $O(4)$ limit | 26 |
| 4.1.2 | The $U_A(1)$ broken model with the symmetric scheme | 27 |
| 4.1.3 | $U_A(1)$ broken model with \overline{MS} scheme | 27 |
| 4.2 | N -point vertex function with $N \geq 6$ | 30 |
| 5 | Critical exponents | 31 |
| 5.1 | $O(4)$ LSM | 32 |
| 5.2 | $U_A(1)$ broken model | 33 |
| 5.3 | Sub-leading exponent | 36 |
| 6 | Lattice calculation | 40 |
| 6.1 | $U_A(1)$ broken model | 40 |
| 6.2 | $O(4)$ LSM | 44 |
| 7 | Summary | 48 |
| A | Critical exponents and scaling law | 50 |
| A.1 | Critical exponents | 50 |
| A.2 | Scaling law and finite size scaling | 52 |
| A.2.1 | Scaling law in infinite volume | 52 |
| A.2.2 | Sub-leading term of the scaling law | 54 |

| | | |
|----------|---|-----------|
| A.2.3 | Finite size scaling | 54 |
| A.2.4 | Binder ratio and cumulant | 55 |
| A.2.5 | Next to leading term of the finite size scaling | 56 |
| B | Hessian matrix | 56 |
| C | Dimensional regularization | 58 |
| D | Correlation functions and renormalization | 60 |
| D.1 | Renormalization scheme | 63 |
| E | β functions | 67 |
| F | Mass renormalization | 69 |
| G | RG equation of external momentum | 71 |
| G.1 | Symmetric scheme | 71 |
| G.1.1 | Correlation function at asymmetric point | 72 |
| G.1.2 | Other channels | 73 |
| G.2 | \overline{MS} scheme | 74 |
| G.3 | $O(4)$ LSM | 76 |
| G.4 | Scheme independence | 77 |
| H | Anomalous dimensions with operator mixing | 82 |
| I | Feynman parameter integral | 84 |

1 Introduction

Quantum chromodynamics (QCD) is the gauge theory describing interaction of quarks and gluons. Because we are able to observe non-perturbative effects of QCD experimentally, this theory is studied for understanding not only the hadronic dynamics but also strong coupling natures. In the QCD with N_f massless flavors, there is the global symmetry of $SU_L(N_f) \times SU_R(N_f) \times U_V(1) \times U_A(1)$ classically, but the axial part of the $U(1)$ ($U_A(1)$) symmetry is broken by the quantum anomaly. This symmetry is called as chiral symmetry. One of the most important features of QCD is the spontaneous breaking of chiral symmetry in low temperature, and its restoration in high temperature. In low temperature, the quark and anti-quark pair ($\bar{q}q$) condenses in vacuum, and it breaks chiral symmetry as $SU_L(N_f) \times SU_R(N_f) \rightarrow SU_V(N_f)$. The $\bar{q}q$ condensate vanishes in high temperature, and axial $SU(N_f)$ ($SU_A(N_f)$) symmetry is restored. The transition between the $SU_A(N_f)$ broken phase and the restored phase at finite temperature is called as chiral phase transition. There are a number of studies of the chiral phase transition both analytically and numerically.¹

In this study, we focus on the two-flavor massless QCD with vanishing density. Since there are six flavors of massive quarks, the two-flavor massless QCD obviously differs from the real world. However, it is frequently considered as one of the extreme case of the realistic QCD. Regarding two flavors of the light quarks u and d as approximately massless, and ignoring other heavier quarks, the two-flavor massless system is obtained. Thus, the study of the phase transition in this system will provide a fundamental understanding of chiral phase transition not only in the realistic QCD in vanishing density but also in the QCD with various flavors, masses, and finite density.

Because of the strong coupling feature of QCD, the direct calculation of this theory is arduous. Thus, various effective theory approaches has been performed. In 1983, Pisarski and Wilczek classified the order of the chiral phase transition with arbitrary numbers of massless quarks by the examination of a renormalization group (RG) flow of the linear sigma model (LSM) in the ϵ expansion [1]. This model is regarded as the Landau-Ginzburg-Wilson (LGW) theory corresponding to chiral phase transition. Because there is infinitely long-range correlation at the critical point of second order phase transition, an infrared fixed point (IRFP) arises in a theory which undergoes second order phase transition. They showed that the fate of this flow depends on a presence of the $U_A(1)$ symmetry at the critical temperature in the two-flavor case.

With large effect of $U_A(1)$ breaking, the chiral phase transition of the two-flavor massless

¹The lattice results of chiral phase transition in the 2+1 flavor QCD are reported in , for example, Refs. [9, 10, 11, 12, 13, 14].

QCD is described by the $O(4)$ LSM. It is well-established that $O(4)$ LSM has an IRFP, or the Wilson-Fisher fixed point, by many studies both analytically and numerically (Refs. [17, 18, 19, 20, 21], for example). Therefore, the phase transition with the infinitely large $U_A(1)$ breaking will be classified into second order with the $O(4)$ universality class.

When the $U_A(1)$ symmetry is effectively restored at the critical temperature (T_c), the model corresponding to the chiral phase transition turns to the $U(2) \times U(2)$ LSM. Various approaches have been attempted to investigate the nature of this theory. Presence of an IRFP in the $U(2) \times U(2)$ LSM is still under debate [22, 23, 24, 25, 26, 27, 28, 29, 30]. When an IRFP exists, the chiral phase transition without $U_A(1)$ breaking is classified in second order with the $U(2) \times U(2)$ universality.

In 1996, Cohen says that the correlators of the bilinear operators of q and \bar{q} connected by the $U_A(1)$ (and chiral) transformation, as the isosinglet scalar $\bar{q}q$ and the isotriplet scalar $\bar{q}t^3q$, degenerate in the $SU_A(2)$ restored phase [32]. The degeneracy of the correlators is called as *the effective restoration of $U_A(1)$ symmetry*. In the same year, Lee and Hatsuda showed that the non-trivial topological sector breaks the degeneracy, even in the $SU_A(2)$ restored phase [33]. However, Aoki, Fukaya and Taniguchi showed that the correlators will degenerate with some assumptions, and claimed that the degeneracy breaking caused by the instanton sector is explained as a finite size effect [34]. Determination of the strength of the $U_A(1)$ breaking at the critical point has been addressed [13, 31, 35, 36, 37]. Splitting of the correlators of a scalar and a pseudoscalar operator connected by the $U_A(1)$ and chiral symmetry is often used as a parameter of the $U_A(1)$ breaking. A consensus about the restoration of the $U_A(1)$ symmetry has not been achieved yet, but it is established that the splitting of the correlators, *i.e.* the strength of the $U_A(1)$ breaking, is much smaller at T_c than that at zero temperature.²

The $U_A(1)$ broken model corresponds to the finite breaking of $U_A(1)$ symmetry case. This model is constructed by $U(2) \times U(2)$ symmetric terms and $U_A(1)$ breaking terms. The $U(2) \times U(2)$ symmetric terms construct the $U(2) \times U(2)$ LSM. The $U(2) \times U(2)$ LSM is constructed by eight degenerate real scalar fields. They become massless at T_c . In the $U_A(1)$ broken model, the $U_A(1)$ breaking term gives a mass splitting [2]. Thus, four of massless fields and four of massive exist at T_c .

An interesting point of the $U_A(1)$ broken model is that, one can achieve conflicting predictions to the phase transition in this model. Second order phase transition is a physics in infinitely long distance. Thus, one may naively expect that we can regard any fields having a finite mass as infinitely heavy, because any mass is sufficiently heavier than the scale of

²Recent results of the lattice calculation about chiral phase transition of two-flavor massless QCD are reported in Refs. [15, 38, 39, 40], for example.

the phase transition if it is second order. In this case, the $U_A(1)$ broken model is reduced into the $O(4)$ LSM having an IRFP. On the other hand, it is reported by the functional RG analysis that there is no IR stable fixed point in the $U_A(1)$ broken model unless the infinitely large $U_A(1)$ breaking limit are taken [43]. Therefore, they concluded that the $U_A(1)$ broken model with the finite breaking ends up with fluctuation induced first order phase transition.

In this work, we investigate the fate of the phase transition in the $U_A(1)$ broken theory. In the sec.2, we briefly review the effective theoretical analysis and introduce the model which we use. In the sec.3, we calculate the RG flow of the $U_A(1)$ broken model. In order to trace the effect of the mass of the would-be decouple fields accurately and examine the detail feature of the decoupling, we take the ϵ expansion and a mass-dependent scheme. The β functions are obtained in the leading order of the ϵ expansion. We show that there is no IR stable fixed point in the full space of the couplings. However, we find that we can classify the RG flow, and in the one of them, the RG flow projected onto a particular axis reaches to the IRFP of the $O(4)$ LSM. The decoupling of the massive fields is discussed in this case. And we search the region where the RG flow in the $U_A(1)$ broken model reaches to the IRFP of the $O(4)$ LSM. In the sec.4, we show the equivalence of the IR nature in the $U_A(1)$ LSM and that in the $O(4)$ LSM in terms of the correlation functions and the effective action. In the sec.5, we calculate the critical exponents which determines a universality class, and the sub-leading exponent ω in the $U_A(1)$ broken model. ω characterizes the sub-leading behavior of the critical phenomena [3, 4, 5, 6]. We point out that the exponent ω in the $U_A(1)$ broken model differs from that in the $O(4)$ LSM, even though all of the leading exponents are equivalent. The discrepancy of ω implies that there is the footprint of the massive fields in the IR nature of the $U_A(1)$ broken model. Finally, we carry out a lattice calculation of the $U_A(1)$ broken model in the sec.6 as a non-perturbative check of the decoupling, and we obtain the critical exponents. The calculation of the $O(4)$ LSM is also done for comparison. The appendices A and B are devoted to review general arguments of the scaling laws and the Hessian matrix. In the appendices C, D, E and G, H, I, we show the detail derivations of the equations we use in the discussions. And in the appendix F, we show the renormalization of the two point functions with the on-shell scheme.

This work is mainly based on³

- T. Sato and N. Yamada, "*Linking $U(2) \times U(2)$ to $O(4)$ model via decoupling*", Phys. Rev. D **91** (2015) 3, 034025 [arXiv:1412.8026 [hep-lat]] [46].

³See also Refs [45, 50].

2 Effective theory

In this section, we briefly review the effective theory analysis done by Pisarski and Wilczek [1]. They used the linear sigma model which has the same global symmetry with QCD as an effective theory in hadronic level, and as an expansion in the order parameter (LGW theory). They discussed the $U_A(1)$ restored case (the $U(2) \times U(2)$ LSM) and the infinitely strong $U_A(1)$ broken case (the $O(4)$ LSM).

2.1 Field theory with finite temperature

First, we show the formulation of the finite temperature field theory. A partition function Z at temperature T is obtained as

$$Z = \text{Tr} (e^{-H/T}) = \sum_{\alpha} \langle \alpha | e^{-H/T} | \alpha \rangle. \quad (2.1)$$

In the rightmost, we sum up all possible state $|\alpha\rangle$. Using a path integral of the field theory, we can calculate the kernel as

$$\langle \beta | e^{-itH} | \alpha \rangle = \int_{\alpha}^{\beta} \mathcal{D}\Phi e^{-iS[\Phi]}, \quad (2.2)$$

where the path is set from the initial state $|\alpha\rangle$ to the final state $\langle\beta|$, and there is a time separation t between the initial state and the final state. Then we can calculate a partition function by the path integral with Euclidean time $x_4 = -it$ and periodic boundary condition of period $\Delta x_4 = 1/T$ in (Euclidean) time direction.

We consider a scalar field $\bar{\Phi}(\mathbf{x}, x_4)$ in three spatial dimensions ($\mathbf{x} = (x_1, x_2, x_3)$) and one Euclidean time x_4 with periodic boundary condition of period $1/T$, and its Fourier transformation,

$$\bar{\Phi}(\mathbf{x}, x_4) = T \sum_{n=0}^{\infty} \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i(\mathbf{k}\cdot\mathbf{x} + \omega_n x_4)} \bar{\Phi}(\omega_n, \mathbf{k}). \quad (2.3)$$

$\bar{\Phi}$ can be decomposed by the zero mode Φ and the excited mode Φ' as

$$\bar{\Phi}(\mathbf{x}, x_4) = \Phi(\mathbf{x}) + \Phi'(\mathbf{x}, x_4), \quad (2.4)$$

where

$$\Phi(\mathbf{x}) = T \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \Phi(\mathbf{k}), \quad (2.5)$$

$$\Phi'(\mathbf{x}, x_4) = T \sum_{n \neq 0}^{\infty} \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i(\mathbf{k}\cdot\mathbf{x} + \omega_n x_4)} \Phi'(\omega_n, \mathbf{k}). \quad (2.6)$$

On the critical point, this system is governed by the zero mode Φ [16], and it is approximately described by three dimensional theory. A leading contribution of Φ' gives an additional

correction to a mass of Φ as

$$m_0^2 \rightarrow m^2(T) = m_0^2 + cT^2, \quad (2.7)$$

where m_0^2 is the mass which Φ (or $\bar{\Phi}$) has in four dimension, and c is a real constant.

2.2 Effective theory without the $U_A(1)$ anomaly

In this analysis, we make a working hypothesis that the system undergoes second order phase transition. Hence the order parameter of the transition near by the critical point is sufficiently small. In this case, we can use it as an expansion parameter to construct Landau-Ginzburg-Wilson (LGW) theory, *i.e.*, we use the model constructed by low-order terms of the order parameter as an effective theory of chiral phase transition. Because this system becomes to be scale invariant in the IR limit, the LGW model describing second order phase transition must have an infrared fixed point(IRFP). If we obtain an IRFP, we estimate the phase transition to be second order. While, if there is no IRFP, we conclude the transition to be first order phase transition (or crossover).

We take a 2×2 complex scalar matrix

$$\Phi = \sqrt{2}(\sigma_a + i\pi_a)t_a \quad (2.8)$$

as the order parameter. Where $t_0 = 1_{2 \times 2}/2$ and $t_i = \sigma_i/2$ ($i = 1, 2, 3$) is the generators of $SU(2)$, and σ_a and π_a ($a = 0, 1, 2, 3$) are real scalar fields. σ_0 and σ_i are usually called as σ and δ_i respectively. Similarly, π_0 is called as η' and π_i is the pion at zero temperature. It is transformed as

$$\Phi \rightarrow e^{2i\theta_A} L^\dagger \Phi R \quad (L \in SU_L(2), R \in SU_R(2), \theta_A \in \text{Re}), \quad (2.9)$$

under the chiral transformation. $U_V(1)$ symmetry is omitted because it corresponds to baryon number conservation, and thus is not broken.

First of all, we ignore the axial anomaly. The most general renormalizable $U(2) \times U(2)$ symmetric Lagrangian (the $U(2) \times U(2)$ LSM) is described by

$$\mathcal{L}_{U(2) \times U(2)} = \frac{1}{2} \text{Tr}[\partial_\mu \Phi \partial_\mu \Phi^\dagger] + \frac{1}{2} m^2 \text{Tr}[\Phi \Phi^\dagger] + \frac{\pi^2}{3} g_1 (\text{Tr}[\Phi \Phi^\dagger])^2 + \frac{\pi^2}{3} g_2 \text{Tr}[(\Phi \Phi^\dagger)^2], \quad (2.10)$$

where m^2 depends on temperature T . In terms of the component fields σ_a and π_a , each terms of eq. (2.10) can be written as

$$\text{Tr}[\Phi \Phi^\dagger] = (\sigma_a + i\pi_a)(\sigma_b - i\pi_b) \text{Tr}[t^a t^b] = (\sigma_a)^2 + (\pi_a)^2, \quad (2.11)$$

$$\begin{aligned}
\text{Tr} [(\Phi\Phi^\dagger)^2] &= (\sigma_a + i\pi_a)(\sigma_b - i\pi_b)(\sigma_c + i\pi_c)(\sigma_d - i\pi_d) \text{Tr}[t^a t^b t^c t^d] \\
&= (\sigma_a + i\pi_a)(\sigma_b - i\pi_b)(\sigma_c + i\pi_c)(\sigma_d - i\pi_d) \left(\frac{1}{2} \delta^{ab} \delta^{cd} - \frac{1}{2} \epsilon^{abe} \epsilon^{cde} \right) \\
&= \frac{1}{2} (\sigma_a^2 + \pi_a^2)^2 \\
&\quad + 2 \{ \sigma_0^2 \sigma_i^2 + \pi_0^2 \pi_i^2 + \sigma_i^2 \pi_i^2 + 2\sigma_0 \pi_0 \sigma_i \pi_i - (\sigma_i \pi_i)^2 \}.
\end{aligned} \tag{2.12}$$

Using $\{\phi_a\} \equiv \{\sigma_0, \pi_i\}$, $\{\chi_a\} \equiv \{-\pi_0, \sigma_i\}$, we obtain

$$\sigma_a^2 + \pi_a^2 = \phi_a^2 + \chi_a^2, \tag{2.13}$$

$$\begin{aligned}
&\sigma_0^2 \sigma_i^2 + \pi_0^2 \pi_i^2 + \sigma_i^2 \pi_i^2 + 2\sigma_0 \pi_0 \sigma_i \pi_i - (\sigma_i \pi_i)^2 \\
&= \phi_0^2 \chi_i^2 + \chi_0^2 \phi_i^2 + \chi_i^2 \phi_i^2 + \phi_0^2 \chi_0^2 - (\phi_0 \chi_0)^2 - 2\phi_0 \chi_0 \phi_i \chi_i - (\phi_i \chi_i)^2 \\
&= \phi_a^2 \chi_b^2 - (\phi_a \chi_a)^2.
\end{aligned} \tag{2.14}$$

Thus,

$$\begin{aligned}
\mathcal{L}_{U(2) \times U(2)} &= \frac{1}{2} (\partial_\mu \phi_a)^2 + \frac{1}{2} m^2 \phi_a^2 + \frac{1}{2} (\partial_\mu \chi_a)^2 + \frac{1}{2} m^2 \chi_a^2 \\
&\quad + \frac{\pi^2}{3} \left[\left(g_1 + \frac{g_2}{2} \right) \{ (\phi_a^2)^2 + (\chi_a^2)^2 \} + 2 \left(g_1 + \frac{3}{2} g_2 \right) \phi_a^2 \chi_b^2 - 2g_2 (\phi_a \chi_a)^2 \right] \\
&= \frac{1}{2} (\partial_\mu \phi_a)^2 + \frac{1}{2} m^2 \phi_a^2 + \frac{1}{2} (\partial_\mu \chi_a)^2 + \frac{1}{2} m^2 \chi_a^2 \\
&\quad + \frac{\pi^2}{3} \left[\lambda \{ (\phi_a^2)^2 + (\chi_a^2)^2 \} + 2(\lambda + g_2) \phi_a^2 \chi_b^2 - 2g_2 (\phi_a \chi_a)^2 \right],
\end{aligned} \tag{2.15}$$

where $\lambda = g_1 + g_2/2$.

RG flow and phase transition

In order to check a consistency with our assumption of second order phase transition, we investigate the RG flow of this model using the leading order ϵ expansion method. That is, we calculate the β function of the couplings in $d = 4 - \epsilon$ dimension with sufficiently small ϵ , where the couplings are taken as $\mathcal{O}(\epsilon)$. In the end of analysis, we set ϵ to unity. Obviously, this limit differs from that of calculated in three dimension quantitatively. However, we expect that we can extract a qualitatively reliable result. Since we are now assuming that this system ends up with second order phase transition, the critical temperature T_c is determined by vanishment of the mass $m^2(T_c) = 0$.

In the leading order of the ϵ expansion, we obtain

$$\beta_{\hat{\lambda}}^{U(2)\times U(2)} = \mu \frac{d\hat{\lambda}}{d\mu} = -\epsilon\hat{\lambda} + \frac{8}{3}\hat{\lambda}^2 + \hat{\lambda}\hat{g}_2 + \frac{1}{2}\hat{g}_2^2, \quad (2.16)$$

$$\beta_{\hat{g}_2}^{U(2)\times U(2)} = \mu \frac{d\hat{g}_2}{d\mu} = -\epsilon\hat{g}_2 + 2\hat{\lambda}\hat{g}_2 + \frac{1}{3}\hat{g}_2^2, \quad (2.17)$$

where μ is the renormalization scale, $\hat{\lambda}$ and \hat{g}_2 are dimensionless couplings normalized by μ as $\hat{\lambda} = \mu^{-\epsilon}\lambda$, $\hat{g}_2 = \mu^{-\epsilon}g_2$.

These β functions becomes to zero at i: $(\hat{\lambda}, \hat{g}_2) = (0, 0)$ and ii: $(\hat{\lambda}, \hat{g}_2) = (8\epsilon/3, 0)$. In order to estimate stability of the fixed points, we calculate the Hessian matrix ω ,

$$\omega_{ij} = \left(\begin{array}{cc} \frac{\partial\beta_{\hat{\lambda}}}{\partial\hat{\lambda}} & \frac{\partial\beta_{\hat{\lambda}}}{\partial\hat{g}_2} \\ \frac{\partial\beta_{\hat{g}_2}}{\partial\hat{\lambda}} & \frac{\partial\beta_{\hat{g}_2}}{\partial\hat{g}_2} \end{array} \right) \Big|_{\hat{\lambda}^*, \hat{g}_2^*}, \quad (2.18)$$

where $\hat{\lambda}^*$ and \hat{g}_2^* are the value of couplings at the fixed points i and ii. When all eigenvalues of Hessian matrix at a fixed point are positive, it is IR stable. On the other hand, UV fixed point has only negative eigenvalues. When there are both of positive and negative eigenvalues, it is a saddle point.⁴ The eigenvalues of each fixed points are calculated as $\{-\epsilon, -\epsilon\}$ at i, and $\{-\epsilon/4, \epsilon\}$ at ii. Thus, i is a UV fixed point and ii is a saddle point, no IRFP arises in this order. Though, existence of an IRFP in higher order is reported in Ref. [21], and possibility of second order phase transition is suggested by different approach [29].

Symmetry breaking pattern

With $T < T_c$, non-zero vev of Φ arises, and it breaks a part of the symmetry. Since the chiral symmetry breaking of two flavor QCD is $SU_L(2) \times SU_R(2) \times U_A(1) \rightarrow SU_V(2)$, we need a model that has the same breaking pattern. Thus there are some constraints on the parameters in the model.

There are two breaking patterns depending on coupling g_2 in this model. The classical potential of this model is

$$\begin{aligned} V(\Phi) &= -\frac{1}{2}\mu^2(\phi_a^2 + \chi_a^2) + \frac{\pi^2}{3} \{ \lambda(\phi_a^2 + \chi_a^2)^2 + 2g_2(\phi_a^2\chi_b^2 - (\phi_a\chi_a)^2) \} \\ &= \frac{\pi^2}{3} \left[\lambda \left(\phi_a^2 + \chi_a^2 - \frac{3}{4\pi^2\lambda}\mu^2 \right)^2 + 2g_2(\phi_a^2\chi_b^2 - (\phi_a\chi_a)^2) + \text{const.} \right], \end{aligned} \quad (2.19)$$

where $m^2(T < T_c) = -\mu^2 < 0$. Rewriting ϕ_a and χ_a by three parameters as

$$\phi^2 \equiv \phi_a^2, \quad \chi^2 \equiv \chi_a^2, \quad \cos\theta \equiv \frac{\phi_a\chi_a}{\sqrt{\phi\chi}} \quad (\theta \in [0, 2\pi)), \quad (2.20)$$

⁴See also appendix B.

the potential is described as

$$V(\Phi) = \frac{\pi^2}{3} \left[\lambda \left(\phi^2 + \chi^2 - \frac{3}{4\pi^2\lambda} \mu^2 \right)^2 + 2(1 - \cos^2 \theta) g_2 \phi^2 \chi^2 + \text{const.} \right]. \quad (2.21)$$

The first term becomes minimum at $\phi^2 + \chi^2 = v^2 \equiv \frac{3}{4\pi^2} \mu^2 / \lambda$, when $\lambda > 0$. Absolut value of second term becomes zero at $\phi = 0$ or $\chi = 0$, and maximum at $\phi = \chi$. Therefore, classical vacuum is determined as $\phi = v$, $\chi = 0$ with $g_2 > 0$ ⁵, and $\phi = \chi = v/\sqrt{2}$, $\cos \theta = 0$ with $g_2 < 0$. In the positive g_2 case, the symmetry is broken as $U(2) \times U(2) \rightarrow U_V(2)$, this is what we need. On the other hands, negative case has breaking pattern $U(2) \times U(2) \rightarrow U(1)$.

Classical stability bound

Next, we discuss the stability of the $U(2) \times U(2)$ LSM in the classical level at critical point (thus $m^2 = 0$). Parameterizing ϕ_a and χ_a as

$$\phi_a^2 + \chi_a^2 \equiv \bar{\Phi}^2, \quad \frac{\sqrt{\phi_a^2}}{\bar{\Phi}} \equiv \cos \varphi, \quad \frac{\sqrt{\chi_a^2}}{\bar{\Phi}} \equiv \sin \varphi, \quad \frac{\phi_a \chi_a}{\bar{\Phi}^2 \sin \varphi \cos \varphi} \equiv \cos \theta, \quad (2.22)$$

the classical potential can be described as

$$V(\Phi) = \frac{\pi^2}{3} \left[\lambda + 2(1 - \cos^2 \theta) g_2 \sin^2 \varphi \cos^2 \varphi \right] \bar{\Phi}^4, \quad (2.23)$$

where $\theta, \varphi \in [0, 2\pi)$. When the coefficient $\bar{\Phi}^4$ is positive for any φ and θ , this potential is bounded. Thus, we obtain a constraint to the couplings as

$$f(\xi) = 2(1 - \cos^2 \theta) g_2 \xi(1 - \xi) + \lambda > 0 \quad (2.24)$$

for any $\xi = \sin^2 \varphi \in [0, 1]$ and any $\theta \in [0, 2\pi)$. With $2(1 - \cos^2 \theta) g_2 > 0$, thus positive g_2 , $f(\xi) > f(0) = f(1) = \lambda$ with $0 < \xi < 1$. Hence, the stability bound is $\lambda > 0$. With negative g_2 , we need $f(1/2) > 0$. This is because $f(\xi)$ becomes the minimum at $\xi = 1/2$ in this case. Thus,

$$f\left(\frac{1}{2}\right) = \frac{1}{2}(1 - \cos^2 \theta) g_2 + \lambda > 0, \quad (2.25)$$

for any $\theta \in [0, 2\pi)$. The strongest constraint comes from $\theta = \pi/2$.

Eventually, we obtain the classical stability bound,

$$\lambda > 0 \quad (g_2 > 0), \quad (2.26)$$

$$\lambda + \frac{1}{2} g_2 > 0 \quad (g_2 < 0). \quad (2.27)$$

⁵In practice, we can choose the vacuum arbitrarily as $\phi^2 + \chi^2 = v^2$ and $\cos \theta = 0$. However, the same symmetry remains after the SSB.

2.3 $U_A(1)$ broken model

Next, we show how the anomaly affects to chiral phase transition of the two flavor massless QCD. $U_A(1)$ part of $U(2) \otimes U(2)$ symmetry is broken by the quantum anomaly. Thus, we add $U_A(1)$ breaking term to $U(2) \times U(2)$ LSM. The most general renormalizable $U_A(1)$ breaking operators are

$$\begin{aligned} \mathcal{L}_{\text{breaking}} = & -\frac{c_A}{4}(\det \Phi + \det \Phi^\dagger) + \frac{\pi^2}{3}x \text{Tr}[\Phi\Phi^\dagger](\det \Phi + \det \Phi^\dagger) + \frac{\pi^2}{3}y(\det \Phi + \det \Phi^\dagger)^2 \\ & + w(\text{Tr}[\partial_\mu \Phi t_2 \partial_\mu \Phi^T t_2] + \text{h.c.}), \end{aligned} \quad (2.28)$$

where c_A has mass dimension two, and x, y has dimension ϵ , w is a dimensionless parameter. These terms break the $U_A(1)$ symmetry and preserve the $SU(2) \times SU(2)$ symmetry. Using $\Phi_a = \sigma_a + i\pi_a$,

$$\begin{aligned} \det \Phi = \det \frac{1}{\sqrt{2}} \begin{pmatrix} \Phi_0 + \Phi_3 & \Phi_1 + i\Phi_2 \\ \Phi_1 - i\Phi_2 & \Phi_0 + \Phi_3 \end{pmatrix} &= \frac{1}{2}(\Phi_0^2 - \Phi_i^2) \\ &= \frac{1}{2}(\sigma_0^2 - \pi_0^2 - \sigma_i^2 + \pi_i^2 + 2i\sigma_0\pi_0 - 2i\sigma_i\pi_i), \end{aligned} \quad (2.29)$$

$$\det \Phi + \det \Phi^\dagger = \sigma_0^2 - \pi_0^2 - \sigma_0^2 + \pi_0^2 = \phi_a^2 - \chi_a^2. \quad (2.30)$$

Similarly,

$$\begin{aligned} \text{Tr}[\partial_\mu \Phi t_2 \partial_\mu \Phi^T t_2] &= \frac{1}{8} \text{Tr} \left[\partial_\mu \begin{pmatrix} \Phi_0 + \Phi_3 & \Phi_1 + i\Phi_2 \\ \Phi_1 - i\Phi_2 & \Phi_0 + \Phi_3 \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \right. \\ &\quad \left. \times \partial_\mu \begin{pmatrix} \Phi_0 + \Phi_3 & \Phi_1 - i\Phi_2 \\ \Phi_1 + i\Phi_2 & \Phi_0 + \Phi_3 \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \right] \\ &= \frac{1}{4} \{(\partial_\mu \Phi_0)^2 - (\partial_\mu \Phi_i)^2\}, \end{aligned} \quad (2.31)$$

$$\text{Tr}[\partial_\mu \Phi t_2 \partial_\mu \Phi^T t_2] + \text{h.c.} = \frac{1}{2}(\partial_\mu \phi_a)^2 - \frac{1}{2}(\partial_\mu \chi_a)^2. \quad (2.32)$$

Hence, we obtain

$$\begin{aligned} \mathcal{L}_{\text{breaking}} = & -\frac{c_A}{4}(\phi_a^2 - \chi_a^2) + \frac{\pi^2}{3}[(x+y)(\phi_a^2)^2 + (-x+y)(\chi_a^2)^2 - 2y\phi_a^2\chi_b^2] \\ & + \frac{w}{2}\{(\partial_\mu \phi_a)^2 + (\partial_\mu \chi_a)^2\}. \end{aligned} \quad (2.33)$$

Now therefore, the whole Lagrangian is described as

$$\begin{aligned}
\mathcal{L}_{U_A(1) \text{ br}} &= \mathcal{L}_{U(2) \times U(2)} + \mathcal{L}_{\text{breaking}} \\
&= \frac{1}{2}(1+w)(\partial_\mu \phi_a)^2 + \frac{1}{2} \left(m^2 - \frac{c_A}{2} \right) \phi_a^2 + \frac{1}{2}(1-w)(\partial_\mu \chi_a)^2 + \frac{1}{2} \left(m^2 + \frac{c_A}{2} \right) \chi_a^2 \\
&\quad + \frac{\pi^2}{3} \left[\left(g_1 + \frac{g_2}{2} + x + y \right) (\phi_a^2)^2 + \left(g_1 + \frac{g_2}{2} - x + y \right) (\chi_a^2)^2 \right. \\
&\quad \left. + 2 \left(g_1 + \frac{3}{2}g_2 - y \right) \phi_a^2 \chi_b^2 - 2g_2(\phi_a \chi_a)^2 \right]. \quad (2.34)
\end{aligned}$$

Using $\lambda \equiv g_1 + \frac{g_2}{2} + x + y$ and $z \equiv x + 2y$, the Lagrangian can be written as

$$\begin{aligned}
\mathcal{L}_{U_A(1) \text{ br}} &= \frac{1}{2}(1+w)(\partial_\mu \phi_a)^2 + \frac{1}{2} \left(m^2 - \frac{c_A}{2} \right) \phi_a^2 + \frac{1}{2}(1-w)(\partial_\mu \chi_a)^2 + \frac{1}{2} \left(m^2 + \frac{c_A}{2} \right) \chi_a^2 \\
&\quad + \frac{\pi^2}{3} \left[\lambda(\phi_a^2)^2 + (\lambda - 2x)(\chi_a^2)^2 + 2(\lambda + g_2 - z)\phi_a^2 \chi_b^2 - 2g_2(\phi_a \chi_a)^2 \right], \quad (2.35)
\end{aligned}$$

or,

$$\begin{aligned}
\mathcal{L}_{U_A(1) \text{ br}} &= \frac{1}{2}(1+w)(\partial_\mu \phi_a)^2 + \frac{1}{2} \left(m^2 - \frac{c_A}{2} \right) \phi_a^2 + \frac{1}{2}(1-w)(\partial_\mu \chi_a)^2 + \frac{1}{2} \left(m^2 + \frac{c_A}{2} \right) \chi_a^2 \\
&\quad + \lambda_1(\phi_a^2)^2 + \lambda_2(\chi_a^2)^2 + \lambda_3\phi_a^2 \chi_b^2 + \lambda_4(\phi_a \chi_a)^2, \quad (2.36)
\end{aligned}$$

where

$$\lambda_1 = \frac{\pi^2}{3}\lambda, \quad \lambda_2 = \frac{\pi^2}{3}(\lambda - 2x), \quad \lambda_3 = \frac{2}{3}\pi^2(\lambda + g_2 - z), \quad \lambda_4 = -\frac{2}{3}\pi^2 g_2.$$

It is important notice that the masses of ϕ_a and χ_a are split by the $U_A(1)$ breaking parameter c_A . In this case, critical temperature T_c is defined by the vanishment of lighter mass. We take $c_A > 0$ in this analysis, thus

$$m_\phi^2(T_c) \equiv m^2(T_c) - \frac{c_A}{2} = 0, \quad (2.37)$$

and

$$m_\chi^2(T_c) \equiv m^2(T_c) - \frac{c_A}{2} = c_A. \quad (2.38)$$

Hereafter, we carry out the calculations at critical temperature. Hence, there are four massless fields ϕ_a and four massive fields χ_a . The $U_A(1)$ breaking coupling w affects to the renormalization of the wave functions. We take $w = 0$, and it does not run at least in the leading order of the ϵ expansion.

Finally, note that there is another $U_A(1)$ breaking term $(\det \Phi)^2 + (\det \Phi^\dagger)^2$. However,

this term can be rewritten as

$$(\det \Phi)^2 + (\det \Phi^\dagger)^2 = (\det \Phi + \det \Phi^\dagger)^2 - 2 \det \Phi \det \Phi^\dagger. \quad (2.39)$$

The first term is $U_A(1)$ breaking term having coefficient y . Second term does not break $U_A(1)$, and we can decompose as

$$\begin{aligned} \det \Phi \det \Phi^\dagger &= \frac{1}{4}(\phi_a^2 - \chi_a^2 + 2i\phi_a\chi_a)(\phi_a^2 - \chi_a^2 + 2i\phi_a\chi_a) \\ &= \frac{1}{4}(\phi_a^2 + \chi_a^2)^2 - \{\phi_a^2\chi_a^2 - (\phi_a\chi_a)^2\}. \end{aligned} \quad (2.40)$$

Therefore, it can be absorbed by λ and g_2 .

The symmetry of the $U_A(1)$ broken model is $O(4)$. This symmetry is rotation in the four dimensional space of ϕ_a and χ_a with same angle. With some specific value of couplings, enhanced symmetry arises.

When g_2 is set to zero, all terms are constructed by products of ϕ_a^2 and χ_a^2 . Therefore, we can rotate ϕ_a and χ_a independently. In this case, the symmetry is enhanced to $O(4) \times O(4)$. Another case is $c_A = 0$ and $x = 0$. In the $U_A(1)$ transformation $\Phi \rightarrow e^{i\theta}\Phi$, $\det \Phi \rightarrow e^{2i\theta} \det \Phi$ in the two flavor case. The determinant is invariant in the rotation with $\theta = \pi$, and $\det \Phi \rightarrow -\det \Phi$ with $\theta = \pi/2$ rotation. So with $c_A = 0$ and $x = 0$, additional Z_2 symmetry of $\det \Phi \rightarrow -\det \Phi$ (or Z_4 of $\Phi \rightarrow e^{i\pi/4}\Phi$) arises.

2.4 $O(4)$ limit

Taking $c_A \rightarrow \infty$ *i.e.* $m_\chi^2 \rightarrow \infty$, the massive fields χ_a decouple from IR physics, and the model is reduced into the $O(4)$ LSM,

$$\mathcal{L}_{O(4)} = \frac{1}{2}(\partial_\mu\phi_a)^2 + \frac{1}{2}m_\phi^2\phi_a^2 + \frac{\pi^2}{3}\lambda(\phi_a^2)^2. \quad (2.41)$$

Note that the only remained coupling is λ .

The nature of the $O(4)$ LSM has been well studied (Ref. [17] and *et al.*), and it is well established that the model undergoes with second order phase transition with the $O(4)$ universality class. In the leading order of the ϵ expansion, for instance, the β function in this model is obtained as

$$\beta_{O(4)} = -\epsilon\hat{\lambda} + 2\hat{\lambda}^2. \quad (2.42)$$

There is an IRFP at $\hat{\lambda} = \epsilon/2$, thus $\hat{\lambda}$ reaches to $\epsilon/2$ as long as the initial value of $\hat{\lambda}$ positive. Because the existence of the IRFP agrees with our working hypothesis, we estimate the phase transition in this model is second order. Therefore, the chiral phase transition will be second order as long as the $c_A \rightarrow \infty$ is a good approximation.

3 Renormalization flow of the $U_A(1)$ broken model

In order to classify the critical phenomena of the $U_A(1)$ broken model, we calculate the renormalization group (RG) flow in this section. Calculations are done in $4 - \epsilon$ dimension and in the leading order of the ϵ expansion. In the end of the analysis, we take ϵ to unity.

3.1 β functions

In the \overline{MS} scheme, we obtain the β functions in the $U_A(1)$ broken model (eq. (2.34)) as⁶

$$\beta_{\hat{\lambda}}^{\overline{MS}} = -\epsilon\hat{\lambda} + \frac{8}{3}\hat{\lambda}^2 + \hat{\lambda}\hat{g}_2 + \frac{1}{2}\hat{g}_2^2 - \frac{4}{3}\hat{\lambda}\hat{z} - \hat{g}_2\hat{z} + \frac{2}{3}\hat{z}^2, \quad (3.1)$$

$$\beta_{\hat{g}_2}^{\overline{MS}} = -\epsilon\hat{g}_2 + 2\hat{\lambda}\hat{g}_2 + \frac{1}{3}\hat{g}_2^2 - \frac{2}{3}\hat{g}_2\hat{x} - \frac{4}{3}\hat{g}_2\hat{z}, \quad (3.2)$$

$$\beta_{\hat{x}}^{\overline{MS}} = -\epsilon\hat{x} + 4\hat{\lambda}\hat{x} - 4\hat{x}^2, \quad (3.3)$$

$$\beta_{\hat{z}}^{\overline{MS}} = -\epsilon\hat{z} + 2\hat{\lambda}\hat{x} + 2\hat{\lambda}\hat{z} + \hat{g}_2\hat{x} - \hat{g}_2\hat{z} - 2\hat{x}\hat{z}. \quad (3.4)$$

Each of the first term in eqs. (3.1-3.4) comes from the canonical dimension of the original coupling constant. Thus they behave as $\mu^{-\epsilon}$ in the tree level.

It is worthy of note that $\beta_{\hat{g}_2}$ vanishes at $\hat{g}_2 = 0$. As seen above, the symmetry of the $U_A(1)$ broken model is enhanced to $O(4) \times O(4)$ with vanishing \hat{g}_2 . So, the vanishment of \hat{g}_2 is guaranteed by this enhanced symmetry. Similarly, $\beta_{\hat{x}} = 0$ with vanishing \hat{x} , that is guaranteed by Z_4 symmetry. The coefficients of the couplings in the β functions calculated in the \overline{MS} scheme are μ independent. They coincides with those in Ref. [48] which is calculated in the limit of $c_A \rightarrow 0$ with MS scheme.

In order to trace the effect of the mass parameter $\sqrt{c_A}$, we take the mass-dependent renormalization conditions, the symmetric scheme, as

$$G^{(4,0)}(\phi_1(p_1), \phi_1(p_2)\phi_2(p_3), \phi_2(p_4))|_{amp, s=t=u=\mu^2} = -8\lambda_1, \quad (3.5)$$

$$G^{(0,4)}(\chi_1(p_1), \chi_1(p_2)\chi_2(p_3), \chi_2(p_4))|_{amp, s=t=u=\mu^2} = -8\lambda_2, \quad (3.6)$$

$$G^{(2,2)}(\phi_1(p_1), \chi_2(p_2)\phi_1(p_3), \chi_2(p_4))|_{amp, s=t=u=\mu^2} = -4\lambda_3, \quad (3.7)$$

$$G_4^{(2,2)}(\phi_1(p_1), \chi_1(p_2)\phi_2(p_3), \chi_2(p_4))|_{amp, s=t=u=\mu^2} = -2\lambda_4, \quad (3.8)$$

where $s = (p_1 + p_2)^2 = (p_3 + p_4)^2$, $t = (p_1 + p_3)^2 = (p_2 + p_4)^2$, and $u = (p_1 + p_4)^2 = (p_2 + p_3)^2$. $G^{(n,m)}$ is the correlation function of n points of ϕ and m points of χ . With the symmetric

⁶See also appendices C-E for the detail derivation.

scheme, we obtain

$$\beta_{\hat{\lambda}}^{sym} = -\epsilon\hat{\lambda} + 2\hat{\lambda}^2 + \frac{1}{6}f(\hat{\mu}) \left(4\hat{\lambda}^2 + 6\hat{\lambda}\hat{g}_2 + 3\hat{g}_2^2 - 8\hat{\lambda}\hat{z} - 6\hat{g}_2\hat{z} + 4\hat{z}^2 \right), \quad (3.9)$$

$$\beta_{\hat{g}_2}^{sym} = -\epsilon\hat{g}_2 + \frac{1}{3}\hat{\lambda}\hat{g}_2 + \frac{1}{3}f(\hat{\mu})\hat{g}_2 \left(\hat{\lambda} - 2\hat{x} \right) + \frac{1}{3}h(\hat{\mu})\hat{g}_2 \left(4\hat{\lambda} + \hat{g}_2 - 4\hat{z} \right), \quad (3.10)$$

$$\begin{aligned} \beta_{\hat{x}}^{sym} &= -\epsilon\hat{x} + 4f(\hat{\mu}) \left(\hat{\lambda}\hat{x} - \hat{x}^2 \right) \\ &+ \frac{1}{12} (1 - f(\hat{\mu})) \left(8\hat{\lambda}^2 - 6\hat{\lambda}\hat{g}_2 - 3\hat{g}_2^2 + 8\hat{\lambda}\hat{z} + 6\hat{g}_2\hat{z} - 4\hat{z}^2 \right), \end{aligned} \quad (3.11)$$

$$\begin{aligned} \beta_{\hat{z}}^{sym} &= -\epsilon\hat{z} + \frac{1}{2} \left(2\hat{\lambda}^2 - \hat{\lambda}\hat{g}_2 + 2\hat{\lambda}\hat{z} \right) - \frac{1}{6}h(\hat{\mu}) \left(4\hat{\lambda}^2 + 3\hat{g}_2^2 - 8\hat{\lambda}\hat{z} + 4\hat{z}^2 \right) \\ &+ \frac{1}{6}f(\hat{\mu}) \left(-2\hat{\lambda}^2 + 3\hat{\lambda}\hat{g}_2 + 3\hat{g}_2^2 - 2\hat{\lambda}\hat{z} - 6\hat{g}_2\hat{z} + 12\hat{\lambda}\hat{x} + 6\hat{g}_2\hat{x} - 12\hat{x}\hat{z} + 4\hat{z}^2 \right) \end{aligned} \quad (3.12)$$

where $\hat{\mu} = \mu/\sqrt{c_A}$ is the dimensionless renormalization scale, and f and h are functions of $\hat{\mu}$ as

$$\begin{aligned} f(\hat{\mu}) &= \mu \frac{\partial}{\partial \mu} \int_0^1 d\xi \frac{1}{2} \log[c_A + \xi(1-\xi)\mu^2] = \int_0^1 d\xi \frac{\xi(1-\xi)\mu^2}{c_A + \xi(1-\xi)\mu^2} \\ &= 1 - \frac{4}{\hat{\mu}\sqrt{4+\hat{\mu}^2}} \arctan \frac{\hat{\mu}}{\sqrt{4+\hat{\mu}^2}}, \end{aligned} \quad (3.13)$$

$$\begin{aligned} h(\hat{\mu}) &= \mu \frac{\partial}{\partial \mu} \int_0^1 d\xi \frac{1}{2} \log[\xi c_A + \xi(1-\xi)\mu^2] \\ &= \int_0^1 d\xi \frac{\xi(1-\xi)\mu^2}{\xi c_A + \xi(1-\xi)\mu^2} = \int_0^1 d\xi' \frac{\xi'}{\hat{\mu}^2 + \xi'} \quad (\xi' = 1-\xi) \\ &= 1 - \frac{1}{\hat{\mu}^2} \log[1 + \hat{\mu}^2]. \end{aligned} \quad (3.14)$$

In the $\hat{\mu} \rightarrow \infty$ and $\hat{\mu} \rightarrow 0$ limits, they behave

$$\lim_{\hat{\mu} \rightarrow \infty} f(\hat{\mu}) = \lim_{\hat{\mu} \rightarrow \infty} h(\hat{\mu}) = 1, \quad (3.15)$$

$$\lim_{\hat{\mu} \rightarrow 0} f(\hat{\mu}) \approx \frac{1}{3}\hat{\mu}^2 + \mathcal{O}(\hat{\mu}^4), \quad \lim_{\hat{\mu} \rightarrow 0} h(\hat{\mu}) \approx \frac{1}{2}\hat{\mu}^2 + \mathcal{O}(\hat{\mu}^4). \quad (3.16)$$

Note that the $c_A \rightarrow 0$ limit (thus $\hat{\mu} \rightarrow \infty$ limit) of eqs. (3.9-3.12) coincide with these in \overline{MS} scheme. On the other hand, in the limit of $c_A \rightarrow \infty$, $\beta_{\hat{\lambda}}^{sym}$ coincides with that of in the $O(4)$ LSM (eq. (2.42)) as naively expected. These factors arise in the contribution from the loop diagrams of the massive field χ_a . Thus, they are explained as the suppression factor of the mass.

With the dimensional regularization, the wave function renormalizations for ϕ_a and χ_a do not receive corrections at the one-loop. We take the on-shell scheme in the renormalization of two-point functions. Thus, $\sqrt{c_A}$ is defined as the pole mass of χ_a and does not depend on the renormalization scale. The renormalizations of the two point functions are argued in appendix F.

There is no IRFP both in the \overline{MS} and the symmetric scheme. Fig. 1 shows examples of the flow lines in the $U_A(1)$ broken model projected into $\hat{\lambda} - \hat{g}_2$ plane with $\epsilon = 1$. When

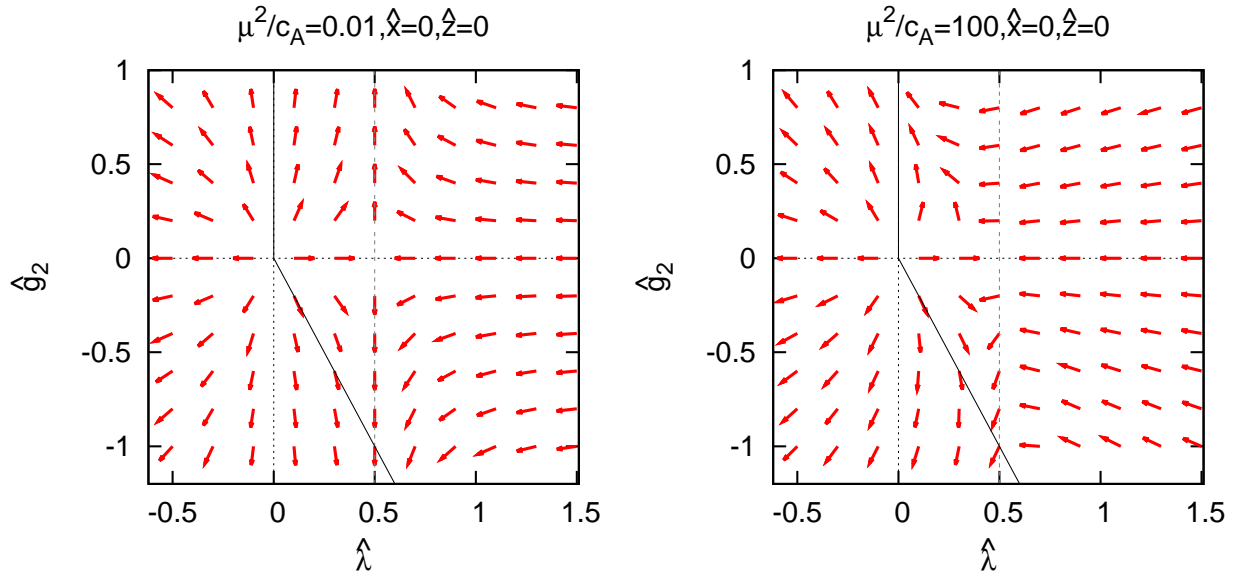


Figure 1: The RG flow projected into the $\hat{\lambda} - \hat{g}_2$ plane in the $U_A(1)$ broken model is shown. The flow grows in accordance with the arrows with a decreasing of μ . μ^2/c_A is fixed as 0.01 in the left panel and 100 in the right panel. ϵ is taken to unity. The arrows represent only the direction of the flow, does not represent the velocity. The solid lines show the classical stability bound in $\hat{x} = \hat{z} = 0$ in which the potential in the tree level is bounded. The dashed lines show $\hat{\lambda} = 0$ and $\hat{g}_2 = 0$ axes, and the dotted line shows $\hat{\lambda} = 1/2$. These lines are plotted for the guide to eyes.

μ decreases, the couplings grow in accordance with the arrows. The arrows are absorbed to an IRFP, when it arises, though there is no such a point.

Henceforth, we focus on symmetric scheme, and omit the superscript *sym* of the β functions eqs. (3.9-3.12).

In order to investigate more detail feature of the RG flow, we calculate the RG flows with initial conditions, $(\hat{\lambda}(\Lambda), \hat{g}_2(\Lambda), \hat{x}(\Lambda), \hat{z}(\Lambda)) = (0.25, 0.25, 0, 0)$ and $(0.75, 0.25, 0, 0)$ with varying c_A/Λ^2 . The mass parameter Λ is a initial value of the renormalization scale μ .

The results projected onto the $\hat{\lambda}-\hat{g}_2$ plane are shown in fig. 2. We can classify the flows in two types, blue dashed curves and the red solid curves. In the blue dashed curves, $\hat{\lambda}$ flows to $-\infty$, and all couplings, including \hat{x} and \hat{z} , diverge. No flow converges to anywhere, and then we expect the phase transition with this initial condition to be first order.

Because no IRFP arises in this order, the RG flow never reaches to any IRFP even in the red solid curves. However, projecting into $\hat{\lambda}$ -axis, the flow converges to the fixed value $\hat{\lambda} = \epsilon/2$ in the IR limit. $\hat{\lambda}$ is the coupling which remains in the decoupling limit of the massive fields χ_a , that is the $O(4)$ LSM limit, and the fixed value of $\hat{\lambda} = \epsilon/2$ is just the IRFP arising in the $O(4)$ LSM. Therefore, the $U_A(1)$ broken model looks like to be reduced into

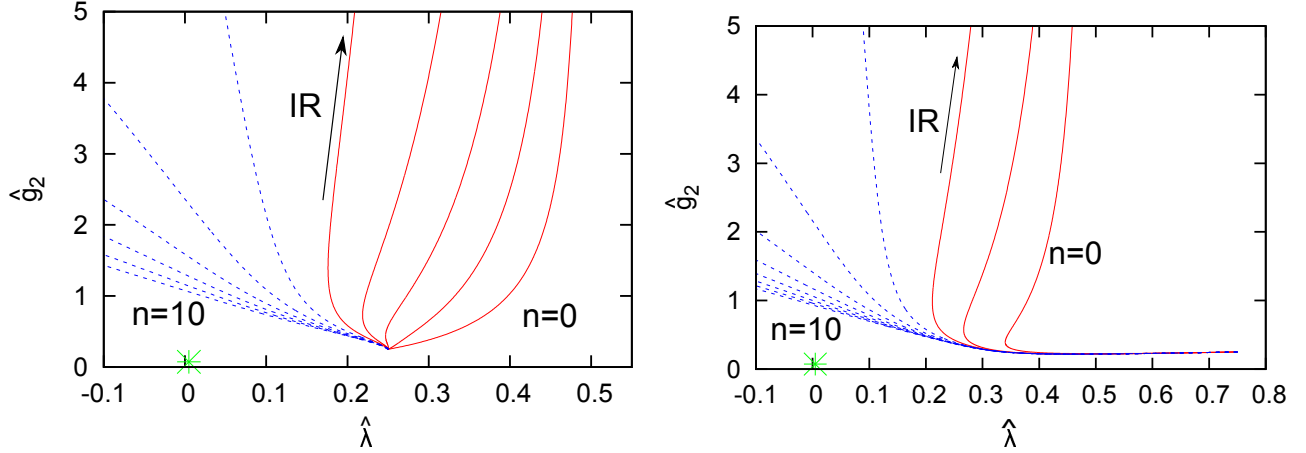


Figure 2: The RG flows projected onto $\hat{\lambda}$ - \hat{g}_2 plane in the $U_A(1)$ broken model starting from two of initial. In the left panel, the flows start from $(\hat{\lambda}(\Lambda), \hat{g}_2(\Lambda), \hat{x}(\Lambda), \hat{z}(\Lambda)) = (0.25, 0.25, 0, 0)$, and c_A/Λ^2 is varied as $c_A/\Lambda^2 = \left(\frac{1}{2n+1}\right)^2$. In the right panel, the flows start from $(0.75, 0.25, 0, 0)$, and $c_A/\Lambda^2 = \left(\frac{1}{10(2n+1)}\right)^2$. ϵ is taken to unity. These initial conditions are chosen just as example. The cross at $(\hat{\lambda}, \hat{g}_2) \sim (0.0048, 0.073)$ is the IRFP of the $U(2) \times U(2)$ LSM reported in Ref. [28], it is plotted as a reference.

the $O(4)$ LSM in this case. This reduction would be interpreted as consequence of that, the mass of χ_a becomes to relatively larger than the renormalization scale in the IR region, and the massive fields decouple from the IR nature. Actually, the contribution to the $\beta_{\hat{\lambda}}$ from χ_a includes the suppression factor $f(\hat{\mu})$ which decreases with $\hat{\mu} = \mu/\sqrt{c_A}$ as eq. (3.16). However, other couplings \hat{g}_2 , \hat{x} and \hat{z} still diverge. Stability of the suppression of χ_a contribution is checked below.

3.2 IR behavior of couplings

We found that ϕ 's self four-point coupling $\hat{\lambda}$ approaches to the IRFP of the $O(4)$ LSM. However, other couplings \hat{g}_2 , \hat{x} and \hat{z} diverge in the IR limit. These divergence increase the coupling between massless fields ϕ_a and the would-be decoupling field χ_a . If the increase is stronger than the mass suppression, the decoupling does not occur, and $\hat{\lambda}$ will flow away from the IRFP $\epsilon/2$. Thus, we next estimate the IR asymptotic behavior of the couplings, and argue the stability of the convergence of $\hat{\lambda}$.

$\beta_{\hat{\lambda}}$ in the $U_A(1)$ broken model has two parts, as

$$\beta_{\hat{\lambda}} = \beta_{O(4)} + \beta_{\hat{\lambda}}^X. \quad (3.17)$$

The first term is the β function in the $O(4)$ LSM, it is the contribution of the canonical dimension and loop diagrams of the massless fields ϕ_a . The second term comes from the

loops of massive fields χ_a , and it is described as,

$$\beta_\lambda^x = \frac{1}{6} f(\hat{\mu}) \left(4\hat{\lambda}^2 + 6\hat{\lambda}\hat{g}_2 + 3\hat{g}_2^2 - 8\hat{\lambda}\hat{z} - 6\hat{g}_2\hat{z} + 4\hat{z}^2 \right). \quad (3.18)$$

When β_λ^x is suppressed to zero in the IR limit, $\hat{\lambda}$ converges to the IRFP $\epsilon/2$. In terms of the decoupling theorem [41, 42], furthermore, it means the decoupling of the massive fields.

Frist, we assume the divergence of \hat{g}_2 and \hat{z} are slower than μ^{-1} , and that of \hat{x} is slower than μ^{-2} . It is that, $f(\hat{\mu})\hat{g}_2^2$, $f(\hat{\mu})\hat{z}^2$, $f(\hat{\mu})\hat{g}_2\hat{z}$ and $f(\hat{\mu})\hat{x}$ are suppressed in the IR limit, hence $\lim(\mu \rightarrow 0)\beta_\lambda^x = 0$. Substituting $\hat{\lambda} = \epsilon/2$ to $\beta_{\hat{g}_2}$ (eq. (3.10)), we obtain

$$\beta_{g_2} \rightarrow -\epsilon\hat{g}_2 + \frac{\epsilon}{6}\hat{g}_2 = -\frac{5}{6}\epsilon\hat{g}_2. \quad (3.19)$$

Assuming the leading term of \hat{g}_2 in the IR limit as $\hat{g}_2(\mu \rightarrow 0) = c\hat{\mu}^p$,

$$\mu \left. \frac{d\hat{g}_2}{d\mu} \right|_{\mu \rightarrow 0} = p c \hat{\mu}^p = -\frac{5}{6}\epsilon c \hat{\mu}^p. \quad (3.20)$$

Thus, we obtain the scaling dimension of \hat{g}_2 as $p = -\frac{5}{6}\epsilon$. The constant c is determined by initial condition. Similarly,

$$\beta_z = -\epsilon\hat{z} - \frac{\epsilon}{4}\hat{g}_2 + \frac{\epsilon}{2}\hat{z} + \frac{\epsilon^2}{4} = -\frac{\epsilon}{4}\hat{g}_2 - \frac{\epsilon}{2}\hat{z} + \frac{\epsilon^2}{4}. \quad (3.21)$$

When $\hat{z} \gg \hat{g}_2$ in the IR limit, we obtain $\beta_z = -\epsilon\hat{z}/2$. And thus, $\hat{z} \sim \mu^{-\epsilon/2}$ in this case. However, it is inconsistent because $\hat{g} \sim \mu^{-\frac{5}{6}\epsilon} \gg \mu^{-\epsilon/2}$ in the IR limit. Assuming $\hat{z}(\mu \rightarrow 0) = k\hat{g}_2(\mu \rightarrow 0)$, and substituting the asymptotic behavior of \hat{g}_2 obtained above to eq. (3.21), we obtain

$$\beta_z(\mu \rightarrow 0) = k\beta_{g_2}(\mu \rightarrow 0) = -\left(\frac{1}{4} + \frac{k}{2}\right)\epsilon\hat{g}_2. \quad (3.22)$$

Thus, the RG equation is solved in self-consistently with $k = 3/4$. And, the IR behavior of \hat{x} is described as

$$\beta_x(\mu \rightarrow 0) \rightarrow -\epsilon\hat{x} - \frac{1}{4}\hat{g}_2^2 + \frac{1}{2}\hat{g}_2\hat{z} - \frac{1}{3}\hat{z}^2 = -\epsilon\hat{x} - \frac{1}{16}c^2\hat{\mu}^{-\frac{5}{3}\epsilon}. \quad (3.23)$$

When the divergence of \hat{x} is stronger than $\mu^{-\frac{5}{3}\epsilon}$, we obtain $\beta_x = -\epsilon\hat{x}$. In this case, $\hat{x}(\mu \rightarrow 0) \sim \mu^{-\epsilon}$ and it is inconsistent. Then, \hat{x} has asymptotic behavior as $\hat{x}(\mu \rightarrow 0)f(\hat{\mu})\hat{x} = c_x\mu^{-\frac{5}{3}\epsilon}$. Hence,

$$\beta_x(\mu \rightarrow 0) = -\frac{5}{3}\epsilon c_x \hat{\mu}^{-\frac{5}{3}\epsilon} = -\epsilon c_x \hat{\mu}^{-\frac{5}{3}\epsilon} - \frac{c^2}{16}\hat{\mu}^{-\frac{5}{3}\epsilon}. \quad (3.24)$$

The equation satisfied with $c_x = \frac{3}{32}\epsilon^{-1}c^2$.

Eventually, we obtain the IR asymptotic behaviors of the couplings as

$$\hat{g}_{2,\text{asym}}(\mu) = \lim_{\mu \rightarrow 0} \hat{g}_2(\mu) = c\hat{\mu}^{-\frac{5}{6}\epsilon}, \quad (3.25)$$

$$\hat{x}_{\text{asym}}(\mu) = \lim_{\mu \rightarrow 0} \hat{x}(\mu) = \frac{3}{32}\epsilon^{-1} \hat{g}_{2,\text{asym}}^2(\mu) \sim \hat{\mu}^{-\frac{5}{3}\epsilon} \quad (3.26)$$

$$\hat{z}_{\text{asym}}(\mu) = \lim_{\mu \rightarrow 0} \hat{z}(\mu) = \frac{3}{4} \hat{g}_{2,\text{asym}}(\mu) \sim \hat{\mu}^{-\frac{5}{6}\epsilon}. \quad (3.27)$$

They are consistent with our assumption that $f(\hat{\mu})\hat{g}_2^2$, $f(\hat{\mu})\hat{z}^2$, $f(\hat{\mu})\hat{g}_2\hat{z}$ and $f(\hat{\mu})\hat{x}$ are suppressed in the IR limit. Therefore, $\hat{\lambda}$ converges to the IRFP $\hat{\lambda}_* = \epsilon/2$, and the massive fields will decouple from IR nature in this case. Of course, this analysis is carried out with the assumption that $\hat{\lambda}$ is close to $\epsilon/2$ with sufficiently small $\hat{\mu}$. Once $\hat{\lambda}$ becomes negative, $\hat{\lambda}$ does not converge in the IR limit even if β_λ^x vanishes. And this analysis does not work in this case. Finally, we comment that the quantum correction to the converging coupling $\hat{\lambda}$ is still small, even though other couplings diverge.

Approaching ratio

Next, we calculate the approaches ratio ω which characterizing IR behavior of $\hat{\lambda}(\mu)$ around the IRFP $\hat{\lambda}_* = \epsilon/2$, as

$$\hat{\lambda}(\mu) - \hat{\lambda}_* \sim \mu^\omega. \quad (3.28)$$

It is worthy of note that the approaching ratio differs from that in the ordinary $O(4)$ LSM.

To distinguish the coupling in the $O(4)$ LSM and that in the $U_A(1)$ broken model, we add subscripts $O(4)$ and $U_A(1)$ br to $\hat{\lambda}$ in each theory. First, we estimate the IR behavior of $\hat{\lambda}_{O(4)}$ in the $O(4)$ LSM. Substituting $\hat{\lambda}_{O(4)} = \epsilon/2 + \alpha_{O(4)}(\mu)$ to the β function in the $O(4)$ LSM (eq. (2.42)),

$$\mu \frac{d\hat{\lambda}_{O(4)}}{d\mu} = \mu \frac{d\hat{\alpha}_{O(4)}}{d\mu} = \epsilon\alpha_{O(4)} + \mathcal{O}(\alpha^2). \quad (3.29)$$

With sufficiently small μ , the sub-leading terms in α is negligible. Thus we obtain the approaching ratio in the $O(4)$ LSM as

$$\omega_{O(4)} = \epsilon. \quad (3.30)$$

In the $U_A(1)$ broken model, substituting $\hat{\lambda}_{U_A(1)\text{ br}} = \epsilon/2 + \alpha_{U_A(1)\text{ br}}(\mu)$ and the IR asymptotic behavior (eq. (3.25-3.27)) to the β function of eq. (3.9), we obtain

$$\begin{aligned} \mu \frac{d\hat{\lambda}_{U_A(1)\text{ br}}}{d\mu} &= \mu \frac{d\alpha_{U_A(1)\text{ br}}}{d\mu} = -(\epsilon - 2\hat{\lambda})\hat{\lambda} + \frac{1}{2}f(\hat{\mu})\hat{g}_2^2 - f(\hat{\mu})\hat{g}_2\hat{z} + \frac{2}{3}f(\hat{\mu})\hat{z}^2 + \dots \\ &= \epsilon\alpha_{U_A(1)\text{ br}} + \frac{c^2}{24}\hat{\mu}^{2-\frac{5}{3}\epsilon} + \dots \end{aligned} \quad (3.31)$$

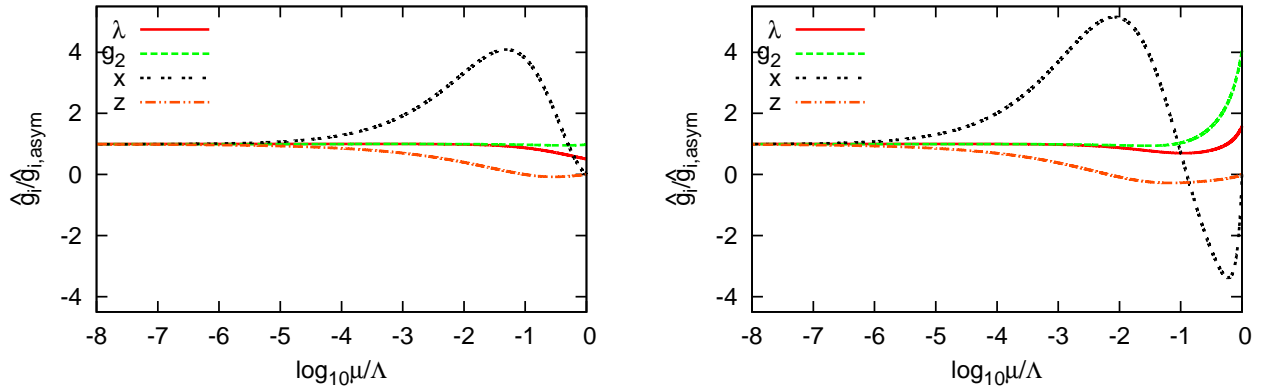


Figure 3: The ratios of the couplings divided by own asymptotic behavior eqs. (3.25-3.27) and eq. (3.33) are shown with two initial conditions. $\{\hat{g}_i\} = \{\hat{\lambda}, \hat{g}_2, \hat{x}, \hat{z}\}$ and $\{\hat{g}_{i,asym}\}$ are obtained in eqs. (3.25-3.27) and eq. (3.33). The initial conditions are set as $(\hat{\lambda}(\Lambda), \hat{g}_2(\Lambda), \hat{x}(\Lambda), \hat{z}(\Lambda)) = (0.25, 0.25, 0, 0)$ and $c_A/\Lambda^2 = 1$ in the left panel, and $(0.75, 0.25, 0, 0)$ and $c_A/\Lambda^2 = 0.01$ in the right panel. The RG flows starting from both points reach to the $O(4)$ IRFP. The coefficient c of $\hat{g}_{2,asym}$ (eq. (3.25)) is 0.2613774 in the left, and 0.4201792 in the right. ϵ is taken to unity.

Where we used $f(\hat{\mu} \rightarrow 0) \approx \hat{\mu}^2/3$. With $\epsilon < 3/4$, the second term becomes negligible. Thus, we obtain

$$\omega_{U_A(1) \text{ br}} = \epsilon \left(\epsilon < \frac{3}{4} \right). \quad (3.32)$$

On the other hand, the second term of eq. (3.31) governs the IR behavior of $\hat{\lambda}$ with $\epsilon > 3/4$. In this case, we obtain

$$\hat{\lambda}_{asym} = \lim_{\mu \rightarrow 0} \hat{\lambda}(\mu) = \frac{\epsilon}{2} + c_\alpha \hat{\mu}^{2 - \frac{5}{3}\epsilon}, \quad (3.33)$$

with the suitable constant c_α . The approaching ratio is obtained as

$$\omega_{U_A(1) \text{ br}} = 2 - \frac{5}{3}\epsilon \left(\epsilon > \frac{3}{4} \right). \quad (3.34)$$

Therefore, there is a discrepancy between the approaching ratio ω in the $O(4)$ LSM and that in the $U_A(1)$ broken model in the limite of $\epsilon \rightarrow 1$. Impact of this discrepancy is discussed in more detail in sec. 5.

The IR asymptotic behaviors obtained above (eqs. (3.25-3.27) and eq. (3.33)) are checked in numerically (fig. 3).

3.3 Attractive basin

There is a region in the space of the initial coupling where the RG flow reaches into the IRFP of the $O(4)$ LSM. We call that *the $O(4)$ attractive basin*. The slices of the $O(4)$ attractive basin with various values of $\hat{x}(\Lambda)$, $\hat{z}(\Lambda)$ and c_A/Λ^2 are shown in fig. 4 ($c_A/\Lambda^2 = 1$) and figs. 5 ($c_A/\Lambda^2 = 0.01$) by the hatched area. It can be seen that the attractive basin shrinks with decreasing of c_A/Λ^2 in each values of the $\hat{x}(\Lambda)$ and $\hat{z}(\Lambda)$. Therefore, regarding the initial scale Λ as the cutoff scale below which the $U_A(1)$ broken model well describes the nature of the two-flavor massless QCD, the reduction of the $U_A(1)$ broken model to the $O(4)$ LSM needs a fine tuning of the initial couplings, in the case that c_A becomes extremely smaller than the cutoff scale Λ . And thus, the phase transition tends to be first order in this case.

4 IR nature in the attractive basin

In the previous section, we showed that the convergence of $\hat{\lambda}$ to the IRFP of the $O(4)$ LSM. However, other couplings diverge in the IR limit, and no IRFP arises in the $U_A(1)$ broken model, to be exact. In this section, we estimate the IR nature of the $U_A(1)$ broken theory in the $O(4)$ attractive basin, and discuss the critical phenomena with the massive field χ_a in terms of the four-point functions and the effective action.

4.1 Four-point functions with the RG improvement

In order to estimate the IR nature of the $U_A(1)$ broken model, we calculate the four-point functions with RG improvement. The IR behavior of the four-point function in the $U_A(1)$ broken model is compared to that in the $O(4)$ LSM. Scheme dependence of the correlation function is also discussed. In this subsection, we set the external momentums at a symmetric point as $s = t = u = P^2$ for simplicity.

First, we review the RG improvement of the correlation function. A four point function of a scalar field ϕ , $G^{(4)}(\phi(p_1), \phi(p_2), \phi(p_3), \phi(p_4))$ with the typical external momentum P , that is $p_i^2 = c_i P^2$ is factorized as

$$G^{(4)}(P, \hat{\lambda}, \rho; \mu) = \prod_i \left(\frac{-1}{c_i P^2} \right)^4 \mu^\epsilon g^{(4)}(P/\mu, \hat{\lambda}, \rho), \quad (4.1)$$

where $\rho = \mu^{-2} m^2$ is a mass of ϕ , $\hat{\lambda}$ is a coupling constant, and $g^{(4)}(P/\mu, \hat{\lambda}, \rho)$ is a dimension-

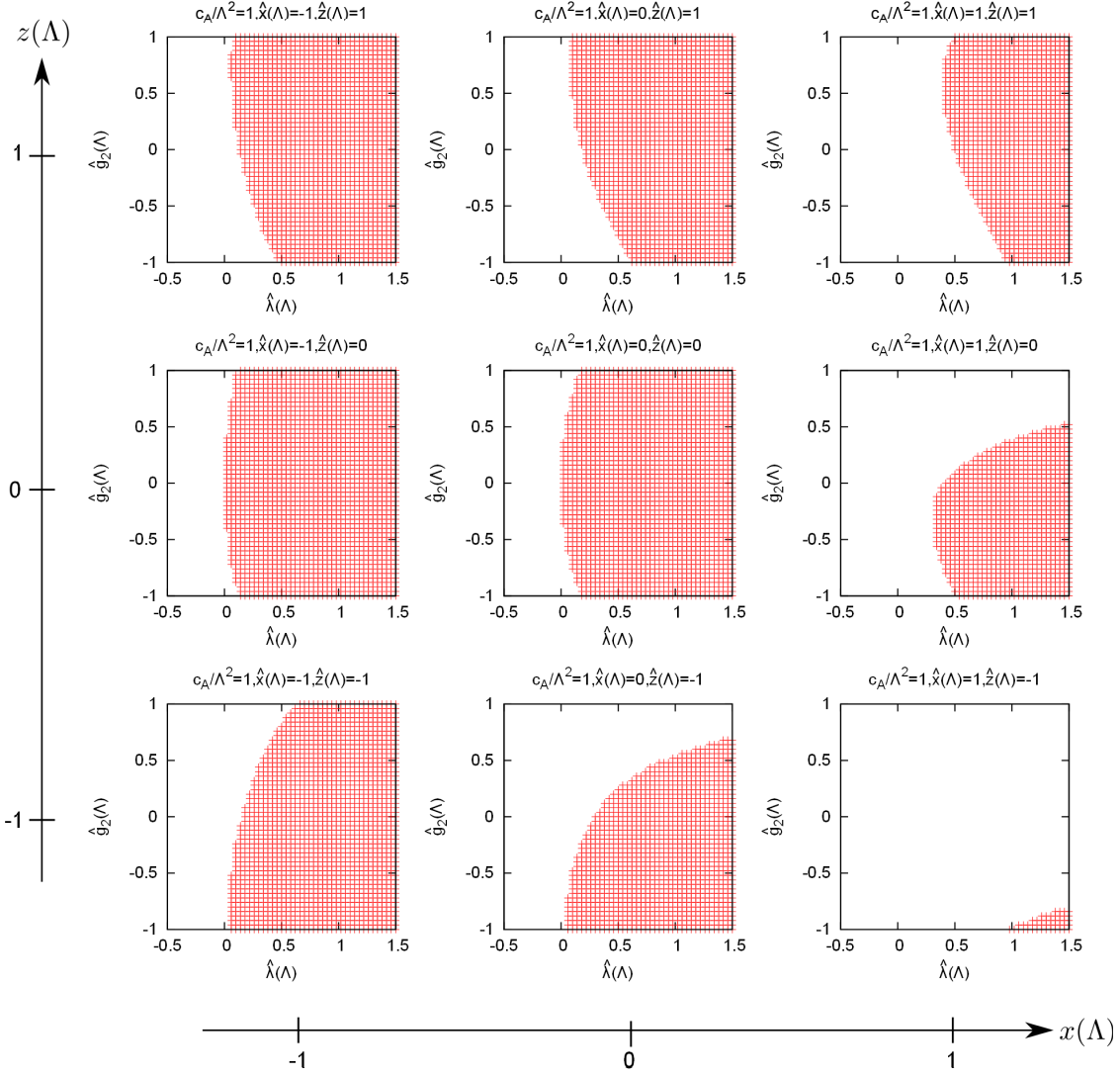


Figure 4: The hatched area shows the $O(4)$ attractive basin in the various \hat{x} and \hat{z} slices. $\hat{x}(\Lambda)$ and $\hat{z}(\Lambda)$ are varied to ± 1 and zero. $c_A/\Lambda^2 = 1$. ϵ is taken to unity.

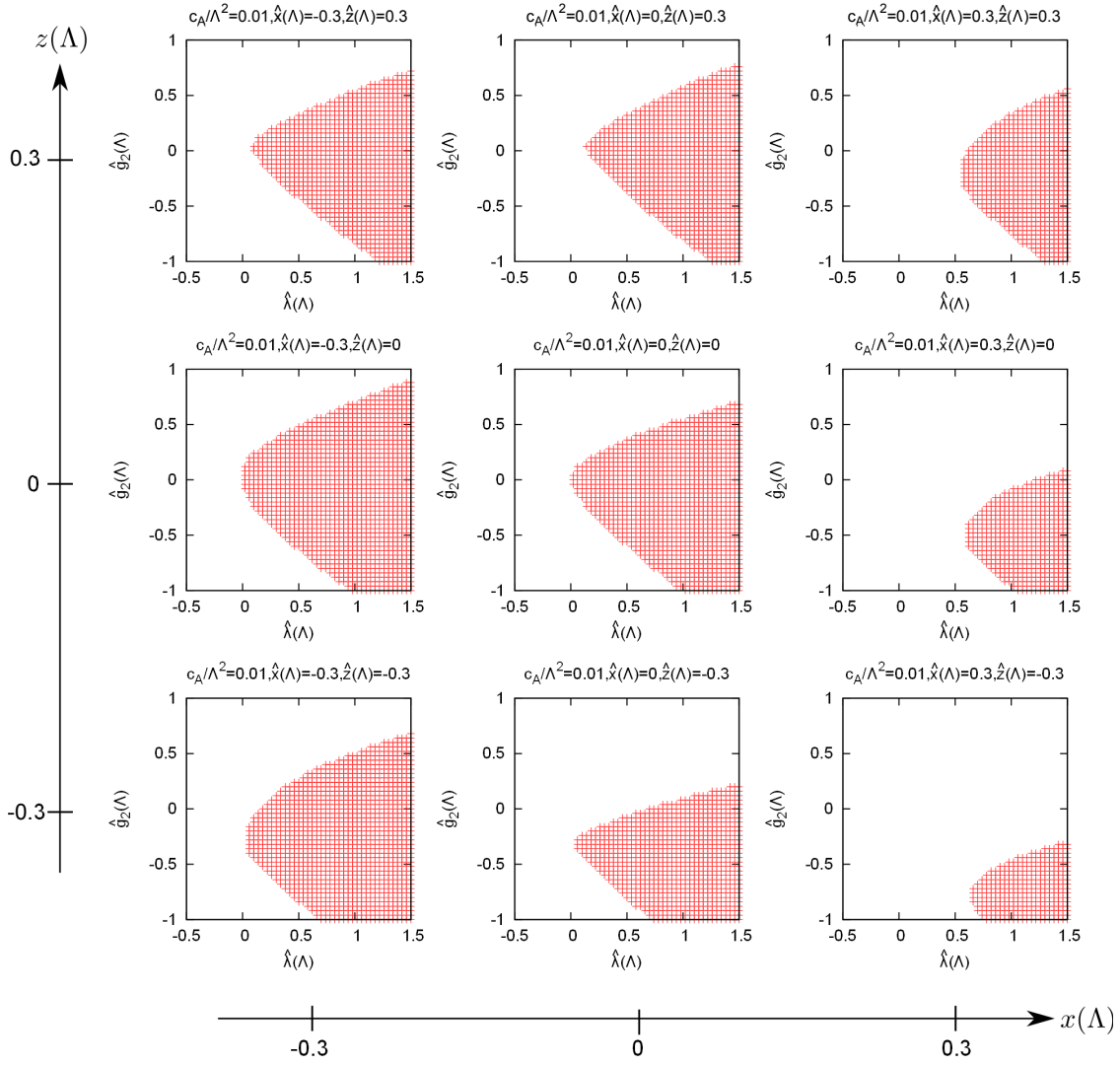


Figure 5: The similar plot as fig. 4 but different \hat{x} , \hat{z} and c_A . $\hat{x}(\Lambda)$ and $\hat{z}(\Lambda)$ are varied to ± 0.3 and zero. $c_A/\Lambda^2 = 0.01$. ϵ is taken to unity.

less function. The renormalization equation of $g^{(4)}(P/\mu, \hat{\lambda}, \rho)$ is obtained as

$$\begin{aligned}
& \left[\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial \hat{\lambda}} + \beta_\rho \frac{\partial}{\partial \rho} + 4\gamma \right] G^{(4)} \\
&= \left[\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial \hat{\lambda}} + \beta_\rho \frac{\partial}{\partial \rho} + 4\gamma \right] \left(\frac{-1}{P^2} \right)^4 \mu^\epsilon g^{(4)}(P/\mu, \hat{\lambda}, \rho) \\
&= \left(\frac{-1}{P^2} \right)^4 \left\{ \left(\mu \frac{\partial}{\partial \mu} \mu^\epsilon \right) g^{(4)}(P/\mu, \hat{\lambda}, \rho) \right. \\
&\quad \left. + \mu^\epsilon \left[\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial \hat{\lambda}_i} + \beta_\rho \frac{\partial}{\partial \rho} + 4\gamma \right] g^{(4)}(P/\mu, \hat{\lambda}, \rho) \right\} \\
&= \left(\frac{-1}{P^2} \right)^4 \mu^\epsilon \left[\epsilon + \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial \hat{\lambda}} + \beta_\rho \frac{\partial}{\partial \rho} + 4\gamma \right] g^{(4)}(P/\mu, \hat{\lambda}, \rho) = 0. \tag{4.2}
\end{aligned}$$

Thus, the RG equation of dimensionless function $g^{(4)}$ is given by

$$\left[\epsilon + \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial \hat{\lambda}} + \beta_\rho \frac{\partial}{\partial \rho} + 4\gamma \right] g^{(4)}(P/\mu, \hat{\lambda}, \rho) = 0. \tag{4.3}$$

Because $g^{(4)}$ is dimensionless, μ dependence of $g^{(4)}$ arises only in P/μ . Hence, we can rewrite the differentiation of the renormalization scale as

$$\begin{aligned}
\mu \frac{\partial}{\partial \mu} g^{(4)}(P/\mu, \hat{\lambda}, \rho) &= \left(\mu \frac{\partial \log[P/\mu]}{\partial \mu} \right) \frac{\partial}{\partial \log[P/\mu]} g^{(4)}(P/\mu, \hat{\lambda}, \rho) \\
&= - \frac{\partial}{\partial \log[P/\mu]} g^{(4)}(P/\mu, \hat{\lambda}, \rho). \tag{4.4}
\end{aligned}$$

Therefore, we obtain

$$\left[\frac{\partial}{\partial \log[P/\mu]} - \beta(\hat{\lambda}, \rho) \frac{\partial}{\partial \hat{\lambda}} - \beta_\rho(\hat{\lambda}, \rho) \frac{\partial}{\partial \rho} - n\gamma(\hat{\lambda}, \rho) - \epsilon \right] g^{(4)}(P/\mu, \hat{\lambda}, \rho) = 0. \tag{4.5}$$

When $\gamma = 0$, the solution of this equation is given as

$$\begin{aligned}
g^{(4)}(P/\mu, \hat{\lambda}, \rho) &= \mathcal{G}^{(4)} \left(\bar{\lambda}(P; \hat{\lambda}, \rho), \bar{\rho}(P; \hat{\lambda}, \rho) \right) \exp \left[\epsilon \int_0^{\log[P/\mu]} d \log[P'/\mu] \right] \\
&= \left(\frac{P}{\mu} \right)^\epsilon \mathcal{G}^{(4)} \left(\bar{\lambda}(P; \hat{\lambda}, \rho), \bar{\rho}(P; \hat{\lambda}, \rho) \right), \tag{4.6}
\end{aligned}$$

where $\bar{\lambda}_i$ and $\bar{\rho}$ grow in accordance with

$$\frac{d}{d \log[P/\mu]} \bar{\lambda} = \beta(\bar{\lambda}, \bar{\rho}), \quad \frac{d}{d \log[P/\mu]} \bar{\rho} = \beta_\rho(\bar{\lambda}, \bar{\rho}), \tag{4.7}$$

and,

$$\bar{\lambda}(P/\mu = 1) = \hat{\lambda}(\mu), \quad \bar{\rho}(P/\mu = 1) = \rho(\mu). \quad (4.8)$$

Substituting eq.(4.6) to eq.(4.1), we obtain

$$\begin{aligned} G^{(4)}(P) &= \left(\frac{-1}{P^2}\right)^4 \mu^\epsilon g^{(4)}(P/\mu, \hat{\lambda}, \rho) \\ &= \left(\frac{-1}{P^2}\right)^4 P^\epsilon \mathcal{G}^{(4)}\left(\bar{\lambda}(P; \hat{\lambda}, \rho), \bar{\rho}(P; \hat{\lambda}, \rho)\right). \end{aligned} \quad (4.9)$$

$\mathcal{G}^{(4)}(\bar{\lambda}, \bar{\rho})$ depends only on $\bar{\lambda}(P)$ and $\bar{\rho}(P)$, and it is determined by an initial condition at $P = \mu$.

4.1.1 $O(4)$ limit

The RG improved four point function in the $O(4)$ LSM is given as

$$G_{O(4)}^{(4)}(\phi_1(p_1), \phi_1(p_2), \phi_3(p_3), \phi_4(p_4)) = \left(\prod_{i=1}^4 \frac{-1}{p_i^2}\right) P^\epsilon \mathcal{G}_{O(4)}^{(4)}(\bar{\lambda}(P)), \quad (4.10)$$

where

$$\mathcal{G}_{O(4)}^{(4), sym}(\bar{\lambda}) = -\frac{8}{3}\pi^2\bar{\lambda}, \quad \mathcal{G}_{O(4)}^{(4), \overline{MS}}(\bar{\lambda}) = -\frac{8}{3}\pi^2(\bar{\lambda} + 2\bar{\lambda}^2), \quad (4.11)$$

in the symmetric scheme and the \overline{MS} scheme respectively, and $\bar{\lambda}(P)$ grows in the following equation,

$$\frac{d\bar{\lambda}(P)}{d\log[P/\mu]} = \beta_{O(4)}(\bar{\lambda}(P)) = -\epsilon\bar{\lambda} + 2\bar{\lambda}^2. \quad (4.12)$$

Using the IR behavior of the coupling, we obtain the IR behavior of the correlation function as

$$\mathcal{G}_{O(4)}^{(4), sym}(P \rightarrow 0) = -\frac{8}{3}\pi^2 \left\{ \frac{\epsilon}{2} + k \left(\frac{P}{\mu}\right)^\epsilon \right\}, \quad (4.13)$$

$$\mathcal{G}_{O(4)}^{(4), \overline{MS}}(P \rightarrow 0) = -\frac{8}{3}\pi^2 \left\{ \frac{\epsilon}{2} - \frac{\epsilon^2}{2} + k \left(\frac{P}{\mu}\right)^\epsilon \right\}, \quad (4.14)$$

where k is a real constant determined by an initial condition.

4.1.2 The $U_A(1)$ broken model with the symmetric scheme

In the $U_A(1)$ broken case, the four-point function $G_1^{(4,0)}$ is obtained as⁷

$$G_1^{(4,0)}(P) = \langle \phi_1(p_1) \phi_1(p_2) \phi_3(p_3) \phi_4(p_4) \rangle|_{s=t=u=P^2} = \left(\prod_{i=1}^4 \frac{-1}{p_i^2} \right) P^\epsilon \mathcal{G}_1^{(4,0)}, \quad (4.15)$$

$$\mathcal{G}_1^{(4,0), sym}(\bar{\lambda}, \bar{g}_2, \bar{x}, \bar{z}, \bar{\rho}) = -\frac{8}{3}\pi^2 \bar{\lambda}(P). \quad (4.16)$$

It coincide with that in the $O(4)$ LSM eq. 4.13, but the coupling does not follow same equation. $\bar{\lambda}$ flow with the differential equation

$$\frac{d\bar{\lambda}(P)}{d \log[P/\mu]} = \beta_{\hat{\lambda}}(\bar{\lambda}, \bar{g}_2, \bar{x}, \bar{z}, \bar{\rho}). \quad (4.17)$$

The right hand side is given in eq. (3.9), and it is substituted the couplings depending on P . Similarly, $\bar{g}_2(P)$ satisfies

$$\frac{d\bar{g}_2(P)}{d \log[P/\mu]} = \beta_{\hat{g}_2}(\bar{\lambda}, \bar{g}_2, \bar{x}, \bar{z}, \bar{\rho}), \quad (4.18)$$

and so on. $\bar{\rho} \propto P^2/\mu^2$ with the on-shell scheme. From the IR behavior of the couplings, we obtain

$$\mathcal{G}_1^{(4,0), sym}(P \rightarrow 0) = -\frac{8}{3}\pi^2 \left\{ \frac{\epsilon}{2} - k' \left(\frac{P}{\mu} \right)^{2-\frac{5}{3}\epsilon} \right\}, \quad (4.19)$$

with a constant k' . It is obvious that the four-point function in the $U_A(1)$ coincides with that in the $O(4)$ LSM in the IR limit. Even though, they approach with different ratio.

4.1.3 $U_A(1)$ broken model with \overline{MS} scheme

In order to check the scheme dependence of the IR behavior of the $U_A(1)$ broken theory, we next calculate the correlation function in \overline{MS} scheme. In the \overline{MS} scheme,

$$\begin{aligned} \mathcal{G}_1^{(4,0) \overline{MS}}(\bar{\lambda}, \bar{g}_2, \bar{x}, \bar{z}, \bar{\rho}) = & -\frac{8}{3}\pi^2 \left\{ \bar{\lambda} - 2\bar{\lambda}^2 + \frac{1}{6}(4\bar{\lambda}^2 + 6\bar{\lambda}\bar{g}_2 + 3\bar{g}_2^2 - 8\bar{\lambda}\bar{z} - 6\bar{g}_2\bar{z} + 4\bar{z}^2) \right. \\ & \left. \times \frac{1}{2} \int_0^1 d\xi \log[\bar{\rho} + \xi(1-\xi)] \right\}, \end{aligned} \quad (4.20)$$

⁷See also appendix G for the detail derivation.

where $\bar{\lambda}(P)$ glows as

$$\frac{d\bar{\lambda}(P)}{d\log[P/\mu]} = \beta_{\bar{\lambda}}^{\overline{MS}}(\bar{\lambda}, \bar{g}_2, \bar{x}, \bar{z}, \bar{\rho}), \quad (4.21)$$

and so on. To estimate dependence of the external momentum, we calculate a differential of $\mathcal{G}_1^{(4,0)\overline{MS}}$.

$$\begin{aligned} & \frac{d}{d\log[P/\mu]} \mathcal{G}_1^{(4,0)\overline{MS}}(P) \\ &= -\frac{8}{3}\pi^2 \left\{ \frac{d}{d\log[P/\mu]} \bar{\lambda} + \frac{1}{6}(4\bar{\lambda}^2 + 6\bar{\lambda}\bar{g}_2 + 3\bar{g}_2^2 - 8\bar{\lambda}\bar{z} - 6\bar{g}_2\bar{z} + 4\bar{z}^2) \right. \\ & \quad \left. \times \frac{1}{2} \frac{d\bar{\rho}}{d\log[P/\mu]} \frac{\partial}{\partial\bar{\rho}} \int_0^1 d\xi \log[\rho + \xi(1-\xi)] \right\} + \mathcal{O}(\lambda^2), \quad (4.22) \end{aligned}$$

where

$$\begin{aligned} \frac{\partial}{\partial\bar{\rho}} \int_0^1 d\xi \log[\rho + \xi(1-\xi)] &= \int_0^1 \frac{d\xi}{\bar{\rho} + \xi(1-\xi)} \\ &= \frac{1}{\bar{\rho}} \int_0^1 d\xi \left(1 - \frac{\xi(1-\xi)}{\bar{\rho} + \xi(1-\xi)} \right) = \frac{1}{\bar{\rho}} \left(1 - f(1/\sqrt{\bar{\rho}}) \right), \quad (4.23) \end{aligned}$$

and

$$\frac{d}{d\log[P/\mu]} \bar{\lambda} = -\epsilon\bar{\lambda} + \frac{8}{3}\bar{\lambda}^2 + \bar{\lambda}\bar{g}_2 + \frac{1}{2}\bar{g}_2^2 - \frac{4}{3}\bar{\lambda}\bar{z} - \bar{g}_2\bar{z} + \frac{2}{3}\bar{z}^2. \quad (4.24)$$

Hence, we obtain

$$\begin{aligned} & \frac{d}{d\log[P/\mu]} \mathcal{G}_1^{(4,0)\overline{MS}}(\bar{\lambda}, \bar{g}_2, \bar{x}, \bar{z}, \bar{\rho}) \\ &= -\frac{8}{3}\pi^2 \left\{ -\epsilon\bar{\lambda} + \frac{8}{3}\bar{\lambda}^2 + \bar{\lambda}\bar{g}_2 + \frac{1}{2}\bar{g}_2^2 - \frac{4}{3}\bar{\lambda}\bar{z} - \bar{g}_2\bar{z} + \frac{2}{3}\bar{z}^2 \right. \\ & \quad \left. - \frac{1}{6}(4\bar{\lambda}^2 + 6\bar{\lambda}\bar{g}_2 + 3\bar{g}_2^2 - 8\bar{\lambda}\bar{z} - 6\bar{g}_2\bar{z} + 4\bar{z}^2) \left(1 - f(1/\sqrt{\bar{\rho}}) \right) \right\} \\ &= -\frac{8}{3}\pi^2 \left\{ -\epsilon\bar{\lambda} + 2\bar{\lambda}^2 + \frac{1}{6}f(1/\bar{\rho}) (4\bar{\lambda}^2 + 6\bar{\lambda}\bar{g}_2 + 3\bar{g}_2^2 - 8\bar{\lambda}\bar{z} - 6\bar{g}_2\bar{z} + 4\bar{z}^2) \right\} \\ &= \frac{d}{d\log[P/\mu]} \mathcal{G}_1^{(4,0)\text{sym.}}(\bar{\lambda}, \bar{g}_2, \bar{x}, \bar{z}, \bar{\rho}). \quad (4.25) \end{aligned}$$

Therefore, the correlation functions calculated both in the \overline{MS} scheme and the symmetric scheme have same P dependence.

We can understand the IR behavior of the correlation functions in the \overline{MS} scheme in

another way. Introducing rewritten couplings

$$\bar{\lambda}' = \bar{\lambda} + \frac{1}{6}(4\bar{\lambda}^2 + 6\bar{\lambda}\bar{g}_2 + 3\bar{g}_2^2 - 8\bar{\lambda}\bar{z} - 6\bar{g}_2\bar{z} + 4\bar{z}^2)\frac{1}{2}\log\rho, \quad (4.26)$$

$$\bar{g}'_2 = \bar{g}_2 + \frac{1}{3}\bar{g}_2(\bar{\lambda} - 2\bar{x})\frac{1}{2}\log\rho + \frac{1}{3}\bar{g}_2(4\bar{\lambda} + \bar{g}_2 - 4\bar{z})\frac{1}{2}(\log\rho - 1), \quad (4.27)$$

$$\bar{x}' = \bar{x} + 4(\bar{\lambda} - \bar{x})\bar{x}\frac{1}{2}\log\rho - \frac{1}{6}(8\bar{\lambda}^2 - 6\bar{\lambda}\bar{g}_2 - 3\bar{g}_2^2 + 8\bar{\lambda}\bar{z} + 6\bar{g}_2\bar{z} - 4\bar{z}^2)\frac{1}{2}\log\rho, \quad (4.28)$$

$$\begin{aligned} \bar{z}' &= \bar{z} + \frac{1}{12}(4\bar{\lambda}^2 + 8\bar{\lambda}\bar{g}_2 + 5\bar{g}_2^2 - 8\bar{\lambda}\bar{z} - 8\bar{g}_2\bar{z} + 4\bar{z}^2) \\ &\quad + \frac{1}{6}(-6\bar{\lambda}^2 + 3\bar{\lambda}\bar{g}_2 + 12\bar{\lambda}\bar{x} + 6\bar{g}_2\bar{x} + 6\bar{\lambda}\bar{z} - \bar{g}_2\bar{z} - 12\bar{x}\bar{z})\frac{1}{2}\log\rho, \end{aligned} \quad (4.29)$$

the RG improved correlation function is described as

$$\begin{aligned} \mathcal{G}_1^{(4,0)\overline{MS}} &= -\frac{8}{3}\pi^2 \left\{ \bar{\lambda}' + 2\bar{\lambda}'^2 \frac{1}{2}(\log[P^2/\mu^2] - 2) \right. \\ &\quad \left. + \frac{1}{6}(4\bar{\lambda}'^2 + 6\bar{\lambda}'\bar{g}'_2 + 3\bar{g}'_2{}^2 - 8\bar{\lambda}'\bar{z}' - 6\bar{g}'_2\bar{z}' + 4\bar{z}'^2) \right. \\ &\quad \left. \times \frac{1}{2} \int_0^1 d\xi \log \left[1 + \xi(1-\xi) \frac{P^2}{\mu^2} \rho^{-1} \right] \right\} + \mathcal{O}(\epsilon^3). \end{aligned} \quad (4.30)$$

Using the β functions in the \overline{MS} scheme, we obtain

$$\begin{aligned} \frac{d\bar{\lambda}'}{d\log[P/\mu]} &= \frac{d\bar{\lambda}_{\overline{MS}}}{d\log[P/\mu]} + \frac{1}{6}(4\bar{\lambda}'^2 + 6\bar{\lambda}'\bar{g}'_2 + 3\bar{g}'_2{}^2 - 8\bar{\lambda}'\bar{z}' - 6\bar{g}'_2\bar{z}' + 4\bar{z}'^2)\frac{1}{2}\frac{d\bar{\rho}}{d\log[P/\mu]} + \mathcal{O}(\epsilon^3) \\ &= -\epsilon\bar{\lambda}' + 2\bar{\lambda}'^2 + \mathcal{O}(\epsilon^3). \end{aligned} \quad (4.31)$$

Thus, we obtain the IR behavior of $\bar{\lambda}'$ as

$$\bar{\lambda}'(P \rightarrow 0) = \frac{\epsilon}{2} + c' \left(\frac{P}{\mu} \right)^\epsilon, \quad (4.32)$$

with a constant c' . Similarly, we obtain

$$\bar{g}'_2(P \rightarrow 0) \sim \left(\frac{P}{\mu} \right)^{-\frac{5}{6}\epsilon}, \quad (4.33)$$

$$\bar{x}'(P \rightarrow 0) \sim \left(\frac{P}{\mu} \right)^{-\frac{5}{3}\epsilon}, \quad (4.34)$$

$$\bar{z}'(P \rightarrow 0) \sim \left(\frac{P}{\mu} \right)^{-\frac{5}{6}\epsilon}. \quad (4.35)$$

Substituting these, we obtain the IR behavior of the correlation function in \overline{MS} scheme as,

$$\mathcal{G}_1^{(4,0)\overline{MS}}(P \rightarrow 0) = -\frac{8}{3}\pi^2 \left\{ \frac{\epsilon}{2} - \frac{\epsilon^2}{2} + k \left(\frac{P}{\mu} \right)^{2-\frac{5}{3}\epsilon} \right\}, \quad (4.36)$$

with a constant k . Therefore, it converges to the same value with that in the $O(4)$ LSM with \overline{MS} scheme as in the case of the symmetric scheme.

Finally, we summarize this subsection as below. The RG improved four-point function calculated with the symmetric scheme and that calculated with \overline{MS} scheme have the same dependence of the external momentum P . They converge to that in the $O(4)$ LSM in each scheme.

4.2 N -point vertex function with $N \geq 6$

Next, we estimate an IR behavior of the vertex functions. When any N -point 1PI vertex functions $\Gamma^{(N)}$ (or $\Gamma^{(N,0)}$) coincides between two theories, it indicates the coincidence of the effective action. Thus, the nature in these theories will be equivalent as well. We have already shown that the four-point functions agree with in the IR limit. Because we use the on-shell scheme, the two-point functions have no correction in 1-loop order, two-point function in the $U_A(1)$ broken model and the $O(4)$ LSM agree with. Hence, we calculate the vertex function with $N \geq 6$.

In the $U_A(1)$ broken model, there are two types of diagram contributing to $\Gamma^{(N,0)}$. One is a contribution from ϕ_a internal loop $\Gamma_\phi^{(N,0)}$ (fig. 6a), and another is $\Gamma_\chi^{(N,0)}$ which is a contribution from χ_a (fig. 6b).



Figure 6: The diagrams contribute to the N -point 1PI vertex function are shown. The propagator of ϕ_a is represented by solid line, and that of χ_a is dashed line. There are $N/2$ vertices in each of the diagram (a) and (b).

On the other hand, only the diagram in fig. 6a contributes in the $O(4)$ LSM. The vertex function in the $O(4)$ LSM, $\Gamma_{O(4)}^{(N)}$ is roughly estimated as,

$$\begin{aligned} \Gamma_{O(4)}^{(N)}(P) &\sim \bar{\lambda}_{O(4)}^{N/2}(P) \int \frac{d^d k}{(2\pi)^d} \prod_{i=1}^{N/2} \frac{1}{(k + c_i P)^2}, \sim \bar{\lambda}_{O(4)}^{N/2}(P) \left(\frac{1}{P^2} \right)^{\frac{N-d}{2}} \\ &\xrightarrow{P \rightarrow 0} \left(\frac{\epsilon}{2} \right)^{N/2} P^{-N+d} \end{aligned} \quad (4.37)$$

where the coefficients $\{c_i\}$ are $\mathcal{O}(1)$ constants, and we use $\bar{\lambda}(P \rightarrow 0) = \epsilon/2$.⁸ Similarly, $\Gamma_\phi^{(N,0)}$ in the $U_A(1)$ broken model is

$$\Gamma_\phi^{(N,0)}(P) \sim \bar{\lambda}_{U_A(1)\text{ br}}^{N/2}(P) \left(\frac{1}{P^2}\right)^{\frac{N-d}{2}} \xrightarrow{P \rightarrow 0} \left(\frac{\epsilon}{2}\right)^{N/2} P^{-N+d}. \quad (4.38)$$

Thus, $\Gamma_{O(4)}^{(N)}$ and $\Gamma_\phi^{(N,0)}$ coincide in the IR limit. This is because $\bar{\lambda}$ reaches to the same IR fixed value in both models.

The contribution from the diagram in fig. 6b is estimated as

$$\Gamma_\chi^{(N,0)}(P) \sim \bar{g}_2^{N/2}(P) \int \frac{d^d k}{(2\pi)^d} \prod_{i=1}^{N/2} \left\{ \frac{1}{(k + c_i P)^2 + m^2} \right\} \sim \bar{g}_2^{N/2}(P) \left(\frac{1}{C' P^2 + c_A} \right)^{\frac{N-d}{2}}. \quad (4.39)$$

In the limit of $P \rightarrow 0$, $1/(P^2 + c_A) \sim c_A + \mathcal{O}(P^2/c_A)$. From the IR asymptotic behavior eq. (3.25), we obtain

$$\Gamma_\chi^{(N,0)}(P) \xrightarrow{P \rightarrow 0} P^{-\frac{5}{12}N\epsilon}, \quad (4.40)$$

with suitable coefficient. With sufficiently large N as

$$N > \frac{4 - \epsilon}{1 - 5\epsilon/12} \xrightarrow{\epsilon \rightarrow 1} \frac{36}{7}, \quad (4.41)$$

the contribution from massive field $\Gamma_\chi^{(N,0)}$ becomes negligible as compared with $\Gamma_\phi^{(N,0)}$ in the $P \rightarrow 0$ limit, because the power of the external momentum P of $\Gamma_\chi^{(N,0)}$ becomes larger than that of $\Gamma_\phi^{(N,0)}$. Therefore, the 1PI vertex function in the $U_A(1)$ broken model $\Gamma_{U_A(1)\text{ br}}^{(N,0)} = \Gamma_\phi^{(N,0)} + \Gamma_\chi^{(N,0)}$ approaches to that in the $O(4)$ LSM in the IR limit.

Because all of the vertex functions agree with, the $U_A(1)$ broken model has the same low energy effective action with the $O(4)$ LSM. Therefore, we conclude that the $U_A(1)$ broken model in the $O(4)$ attractive basin will end up with second order phase transition as well as the $O(4)$ LSM.

5 Critical exponents

In order to know the detail of the phase transition in each of the $O(4)$ LSM and the $U_A(1)$ broken model, we calculate the critical exponents⁹ and *the RG dimension of the leading irrelevant operator* called as ω .

The critical exponents are related to γ_ϕ^* which is the IR fixed value of the anomalous

⁸See also appendix I for the detail calculation.

⁹See also appendix A to the general argument of the critical exponents.

dimension of the wave function of ϕ_a , and that of the composite operator ϕ_a^2 , $\gamma_{\phi^2}^*$, as

$$\nu = \frac{1}{2 - \gamma_{\phi^2}^*}, \quad \eta = 2\gamma_{\phi^2}^*. \quad (5.1)$$

The exponent ν characterizes the divergence of the correlation length ξ as eq. (A.2), and η is the anomalous dimension of the correlator (eq. (A.3)).

Because of the wave function has no correction in the leading order of the ϵ expansion, $\eta = 0$ in both of the $O(4)$ LSM and the $U_A(1)$ LSM. Thus we calculate the mass anomalous dimension.

5.1 $O(4)$ LSM

First, we review the estimation of the anomalous dimension γ_{ϕ^2} and the critical exponent ν in the $O(4)$ LSM.

There is a divergence in the correlation function of bare composite operator $\phi^2 = \phi_a^2$ as,

$$\langle \phi_a(p)\phi_b(q)\phi^2(k) \rangle = 2\delta_{ab} \left\{ 1 - \frac{\hat{\lambda}}{2} \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \log[k^2/\mu^2] + 2 \right) \right\} \left(\frac{-1}{p^2} \right) \left(\frac{-1}{q^2} \right), \quad (5.2)$$

where $p+q+k=0$. In order to remove the divergence, we introduce the regularized operator $[\phi^2]$ as

$$[\phi^2] = Z_{\phi^2}^{-1}\phi^2 = (1 + A)\phi^2, \quad (5.3)$$

where A is determined by the renormalization scheme. We take \overline{MS} scheme,

$$A = \frac{\hat{\lambda}}{2} \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) \right). \quad (5.4)$$

Thus, we obtain the regularized correlation function as

$$\langle \phi_a(p)\phi_b(q)[\phi^2](k) \rangle = 2\delta_{ab} \left\{ 1 + \frac{\hat{\lambda}}{2} (\log[k^2/\mu^2] - 2) + \mathcal{O}(\epsilon^2) \right\} \left(\frac{-1}{p^2} \right) \left(\frac{-1}{q^2} \right). \quad (5.5)$$

And, the anomalous dimension is obtained by,

$$\left[\mu \frac{\partial}{\partial \mu} + \beta_{\hat{\lambda}} \frac{\partial}{\partial \hat{\lambda}} + 2\gamma_{\phi} + \gamma_{\phi^2} \right] \langle \phi_a(p)\phi_b(q)[\phi^2](k) \rangle = 0. \quad (5.6)$$

Using $\beta_{\hat{\lambda}} = \mathcal{O}(\epsilon^2)$ and $\gamma_{\phi} = \mathcal{O}(\epsilon^2)$,

$$\gamma_{\phi^2} = -\mu \frac{\partial}{\partial \mu} \left\{ 1 + \frac{\hat{\lambda}}{2} (\log[k^2/\mu^2] - 2) \right\} + \mathcal{O}(\epsilon^2) = \hat{\lambda} + \mathcal{O}(\epsilon^2). \quad (5.7)$$

Eventually, we obtain

$$\gamma_{\phi^2}^* = \lim_{\mu \rightarrow 0} \gamma_{\phi^2} = \frac{\epsilon}{2}, \quad (5.8)$$

and therefore,

$$\nu_{O(4)} = \frac{1}{2 - \gamma_{\phi^2}^*} = \frac{1}{2} + \frac{\epsilon}{8}, \quad (5.9)$$

in the leading order of ϵ .

5.2 $U_A(1)$ broken model

Because there is the mixing between $\phi^2 = \phi_a^2$ and $\chi^2 = \chi_a^2$, the calculation of the anomalous dimension of the $U_A(1)$ broken model is somewhat complicative. In the leading order of the ϵ expansion, we obtain

$$\langle \phi_a(p) \phi_b(q) \phi^2(k) \rangle|_{amp} = 2\delta_{ab} \left\{ 1 - \frac{1}{2} \hat{\lambda} \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \log[k^2/\mu^2] + 2 \right) \right\}, \quad (5.10)$$

$$\begin{aligned} \langle \chi_a(p) \chi_b(q) \phi^2(k) \rangle|_{amp} \\ = -2\delta_{ab} \left(\frac{\hat{\lambda}}{3} + \frac{g_2}{4} - \frac{z}{3} \right) \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \log[k^2/\mu^2] + 2 \right), \end{aligned} \quad (5.11)$$

$$\begin{aligned} \langle \chi_a(p) \chi_b(q) \chi^2(k) \rangle|_{amp} \\ = 2\delta_{ab} \left\{ 1 - \frac{1}{2} (\hat{\lambda} - 2\hat{x}) \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \int_0^1 d\xi \log[\{c_A + \xi(1-\xi)k^2\}/\mu^2] \right) \right\}, \end{aligned} \quad (5.12)$$

$$\begin{aligned} \langle \phi_a(p) \phi_b(q) \chi^2(k) \rangle|_{amp} \\ = -2\delta_{ab} \left(\frac{\hat{\lambda}}{3} + \frac{g_2}{4} - \frac{z}{3} \right) \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \int_0^1 d\xi \log[\{c_A + \xi(1-\xi)k^2\}/\mu^2] \right). \end{aligned} \quad (5.13)$$

These divergences are regularized by 2×2 matrix Z_{Φ^2} as

$$\begin{pmatrix} [\phi^2] \\ [\chi^2] \end{pmatrix} = Z_{\Phi^2}^{-1} \begin{pmatrix} \phi^2 \\ \chi^2 \end{pmatrix}, \quad Z_{\Phi^2}^{-1} \equiv \begin{pmatrix} 1+A & C_1 \\ C_2 & 1+B \end{pmatrix}. \quad (5.14)$$

These parameters A , B and C_1 , C_2 are determined by renormalization condition. Here, we take

$$\langle \phi_a(p)\phi_b(q)[\phi^2](k) \rangle \Big|_{k^2=\mu^2} = \frac{(-1)}{p^2} \frac{(-1)}{q^2} 2\delta_{ab}, \quad (5.15)$$

$$\langle \chi_a(p)\chi_b(q)[\phi^2](k) \rangle \Big|_{k^2=\mu^2} = 0, \quad (5.16)$$

$$\langle \chi_a(p)\chi_b(q)[\chi^2](k) \rangle \Big|_{k^2=\mu^2} = \frac{(-1)}{p^2 + m_\chi^2} \frac{(-1)}{q^2 + m_\chi^2} 2\delta_{ab}, \quad (5.17)$$

$$\langle \phi_a(p)\phi_b(q)[\chi^2](k) \rangle \Big|_{k^2=\mu^2} = 0. \quad (5.18)$$

with the renormalization scale μ . In this scheme, we obtain

$$A = \frac{1}{2} \hat{\lambda} \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) + 2 \right), \quad (5.19)$$

$$B = \frac{1}{2} (\hat{\lambda} - 2\hat{x}) \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \int_0^1 d\xi \log \left[\frac{c_A}{\mu^2} + \xi(1-\xi) \right] \right), \quad (5.20)$$

$$C_1 = \left(\frac{\hat{\lambda}}{3} + \frac{\hat{g}_2}{4} - \frac{\hat{z}}{3} \right) \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) + 2 \right), \quad (5.21)$$

$$C_2 = \left(\frac{\hat{\lambda}}{3} + \frac{\hat{g}_2}{4} - \frac{\hat{z}}{3} \right) \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \int_0^1 d\xi \log \left[\frac{c_A}{\mu^2} + \xi(1-\xi) \right] \right). \quad (5.22)$$

Hence, the correlation functions are regularized as,

$$\langle \phi_a(p)\phi_b(q)[\phi^2](k) \rangle \Big|_{amp} = 2\delta_{ab} \left\{ 1 - \frac{1}{2} \hat{\lambda} \log[k^2/\mu^2] \right\}, \quad (5.23)$$

$$\langle \chi_a(p)\chi_b(q)[\phi^2](k) \rangle \Big|_{amp} = -2\delta_{ab} \left(\frac{\hat{\lambda}}{3} + \frac{g_2}{4} - \frac{z}{3} \right) \log[k^2/\mu^2], \quad (5.24)$$

$$\langle \chi_a(p)\chi_b(q)[\chi^2](k) \rangle \Big|_{amp} = 2\delta_{ab} \left\{ 1 - \frac{1}{2} (\hat{\lambda} - 2\hat{x}) \int_0^1 d\xi \log \left[\frac{c_A + \xi(1-\xi)k^2}{c_A + \xi(1-\xi)\mu^2} \right] \right\}, \quad (5.25)$$

$$\langle \phi_a(p)\phi_b(q)[\chi^2](k) \rangle \Big|_{amp} = -2\delta_{ab} \left(\frac{\hat{\lambda}}{3} + \frac{g_2}{4} - \frac{z}{3} \right) \int_0^1 d\xi \log \left[\frac{c_A + \xi(1-\xi)k^2}{c_A + \xi(1-\xi)\mu^2} \right]. \quad (5.26)$$

For simplicity, we enforce the scale dependence to the Z factor as

$$A' = \frac{1}{2} \hat{\lambda} \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \log \mu^2 + 2 \right), \quad (5.27)$$

$$B' = \frac{1}{2} (\hat{\lambda} - 2\hat{x}) \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \int_0^1 d\xi \log [c_A + \xi(1-\xi)\mu^2] \right), \quad (5.28)$$

$$C'_1 = \left(\frac{\hat{\lambda}}{3} + \frac{\hat{g}_2}{4} - \frac{\hat{z}}{3} \right) \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \log \mu^2 + 2 \right), \quad (5.29)$$

$$C'_2 = \left(\frac{\hat{\lambda}}{3} + \frac{\hat{g}_2}{4} - \frac{\hat{z}}{3} \right) \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \int_0^1 d\xi \log [c_A + \xi(1-\xi)\mu^2] \right), \quad (5.30)$$

and ignoring the μ dependence of the bare correlation functions introduced for regularization of the logarithmical terms. Diagonalizing $Z_{\phi^2}^{-1}$ as

$$P^{-1} Z_{\phi^2}^{-1} P = \text{diag}\{Z_+^{-1}, Z_-^{-1}\}, \quad (5.31)$$

and using linear combinations of ϕ^2 and χ^2 , Φ_{\pm} , defined by

$$\begin{pmatrix} \Phi_+^2 \\ \Phi_-^2 \end{pmatrix} = P^{-1} \begin{pmatrix} \phi^2 \\ \chi^2 \end{pmatrix}, \quad (5.32)$$

we obtain the RG equation as

$$\left[\mu \frac{\partial}{\partial \mu} + \begin{pmatrix} \gamma_+ & 0 \\ 0 & \gamma_- \end{pmatrix} \right] \begin{pmatrix} \langle \phi_a \phi_b [\Phi_+^2] \rangle \\ \langle \phi_a \phi_b [\Phi_-^2] \rangle \end{pmatrix} = 0, \quad (5.33)$$

where γ_+ and γ_- are anomalous dimensions of Φ_+^2 and Φ_-^2 respectively, and $[\Phi_{\pm}] = Z_{\pm} \Phi_{\pm}$ are the regularized operator. We omitted sub-leading terms in ϵ . Hence,

$$\gamma_{\pm} = -\mu \frac{\partial}{\partial \mu} \log Z_{\pm}^{-1}. \quad (5.34)$$

In the $O(4)$ attractive basin, we obtain ¹⁰

$$\lim_{\mu \rightarrow 0} \gamma_+ = \lim_{\mu \rightarrow 0} \hat{\lambda}(\mu) = \frac{\epsilon}{2}, \quad \lim_{\mu \rightarrow 0} \gamma_- = -\lim_{\mu \rightarrow 0} 2f \left(\frac{\mu}{\sqrt{c_A}} \right) \hat{x}(\mu) = 0. \quad (5.35)$$

Thus, the linear combination of ϕ^2 and χ^2 , Φ_+ has the same anomalous dimension with that of ϕ^2 in the $O(4)$ LSM (eq. (5.8)) in the IR limit.

The matrix P is described as

$$P = \begin{pmatrix} 1 & P_{12} \\ P_{21} & 1 \end{pmatrix}, \quad (5.36)$$

¹⁰See also appendix H for the detail calculation.

where

$$P_{12} = \frac{A' - B'}{2C'_2} - \sqrt{\left(\frac{A' - B'}{2C'_2}\right)^2 + \frac{C'_1}{C'_2}} = -\frac{4\hat{\lambda} + 3\hat{g}_2 + 4\hat{z}}{12\hat{x}} (1 + \mathcal{O}(\epsilon)), \quad (5.37)$$

and

$$P_{21} = -\frac{A' - B'}{2C'_1} + \sqrt{\left(\frac{A' - B'}{2C'_1}\right)^2 + \frac{C'_2}{C'_1}} = \frac{4\hat{\lambda} + 3\hat{g}_2 + 4\hat{z}}{12\hat{x}} (1 + \mathcal{O}(\epsilon)). \quad (5.38)$$

Using the IR asymptotic behaviors of the couplings (eqs. (3.25-3.27)), we obtain the IR behaviors of P_{12} and P_{21} as

$$P_{12} \xrightarrow{\mu \rightarrow 0} 0, \quad P_{21} \xrightarrow{\mu \rightarrow 0} 0. \quad (5.39)$$

Therefore, P becomes the identity matrix in the IR limit, and

$$\lim_{\mu \rightarrow 0} \Phi_+^2 = \phi^2. \quad (5.40)$$

Thus, we conclude that

$$\nu_{U_A(1) \text{ br}} = \lim_{\mu \rightarrow 0} \frac{1}{2 - \gamma_+} = \frac{1}{2} + \frac{\epsilon}{8} + \mathcal{O}(\epsilon^2) = \nu_{O(4)}. \quad (5.41)$$

Eventually, we obtain

$$\nu_{U_A(1) \text{ br}} = \nu_{O(4)} = \frac{1}{2} + \frac{\epsilon}{8}, \quad \eta_{U_A(1) \text{ br}} = \eta_{O(4)} = 0.$$

Other critical exponents can be calculated by the scaling and the hyper scaling laws. Because the exponents η and ν in the $O(4)$ LSM and those in the $U_A(1)$ broken model agree with respectively, all of the critical exponents agree with. Therefore, there is no discrepancy in the leading behavior of the critical phenomena, the $U_A(1)$ broken model ends up with second order phase transition with the $O(4)$ universality class.

5.3 Sub-leading exponent

The corrections to the scaling behaviors in second order phase transition is characterized by *the RG dimension of the leading irrelevant operator* ω [3, 4, 5, 6].¹¹ For instance, the magnetic susceptibility χ near the critical temperature without an external field h is described as

$$\chi(t, h = 0) = \propto t^{-\nu(\eta-2)} (1 + k_\chi t^{\nu\omega} + \dots), \quad (5.42)$$

with reduced temperature $t = (T - T_c)/T_c$ and constant k_χ .

¹¹See also appendix A

With sufficiently small ϵ , the leading irrelevant coupling in the $O(4)$ LSM is $\hat{\lambda}_{O(4)} - \hat{\lambda}_*$. In the $U_A(1)$ broken model, the couplings \hat{g}_2 , \hat{x} and \hat{z} will not affect directly to the IR nature, scaling of the remaining coupling $\hat{\lambda}_{U_A(1)\text{br}} - \hat{\lambda}_*$ gives ω . The RG dimension of these couplings are calculated in eq. (3.30) and eq. (3.34) as

$$\omega_{O(4)} = \epsilon, \quad \omega_{U_A(1)\text{br}} = 2 - \frac{5}{3}\epsilon, \quad (5.43)$$

with $\epsilon > 3/4$. Because this exponent characterizes behavior of observables in second order phase transition, we can distinguish which model describes the transition even though the transition of both models undergoes with the same universality class.

Higher dimensional operator

There are two possibilities which would explain the discrepancy of the sub-leading exponent ω . There would be the higher dimensional operator consisting of ϕ_a and preserving the $O(4)$ symmetry with scaling dimension $2 - \frac{5}{3}\epsilon$. In this case, once the operator is switched in the $O(4)$ LSM, this operator becomes to be leading irrelevant, and it shifts the approaching ratio of $\hat{\lambda}$ in the $O(4)$ LSM. The discrepancy of ω is caused by a switching of the operator. If it is not in the case, the discrepancy of ω will be interpreted as the remnant of the massive fields χ_a . Focusing on the former possibility, we calculate the RG flow of the $O(4)$ LSM with higher dimensional operators.

Now, we add the higher dimensional terms

$$\mathcal{L}_{\text{high-dim}} = \lambda_6(\phi_a^2)^3 + \lambda_8(\phi_a^2)^4 + \lambda_A\phi_a^2(\partial_\mu\phi_b)^2 + \lambda_B(\phi_a\partial_\mu\phi_a)^2, \quad (5.44)$$

to $\mathcal{L}_O(4)$ for instance. λ_6 and λ_8 have dimension $2 - 2\epsilon$ and $4 - 3\epsilon$ respectively. Hence $(\phi_a^2)^3$ becomes a marginal operator at $\epsilon \rightarrow 1$. Dimensions of the derivative interactions are $2 - \epsilon$.

First, we estimate the effect of the derivative interactions. These terms have tree contribution,

$$\begin{aligned} & G^{(4)}(\phi_1(p_1), \phi_1(p_2), \phi_2(p_3), \phi_2(p_4)) \Big|_{amp} \\ &= -\frac{\pi^2}{3} 2^3 \lambda + (p_1 p_2 + p_3 p_4) 2^2 \lambda_A + (p_1 p_3 + p_1 p_4 + p_2 p_3 + p_2 p_4) 2 \lambda_B + \mathcal{O}(\epsilon^2). \end{aligned} \quad (5.45)$$

We decompose $G^{(4)}$ as

$$\begin{aligned} & G^{(4)}(\phi_1(p_1)\phi_1(p_2); \phi_2(p_3), \phi_2(p_4)) \\ &= G_\lambda^{(4)} + (p_1 \cdot p_2 + p_3 \cdot p_4) G_A^{(4)} + (p_1 \cdot p_3 + p_1 \cdot p_4 + p_2 \cdot p_3 + p_2 \cdot p_4) G_B^{(4)} + \dots \end{aligned} \quad (5.46)$$

Higher term in external momenta will arise from more higher derivative interactions. The counter term of λ is determined by a condition for $G^{(4)}$, for instance

$$G_\lambda^{(4)} \Big|_{s=t=u=\mu^2} = -\frac{\pi^2}{3} 2^3 \lambda(\mu). \quad (5.47)$$

Corrections from derivative interactions thus come from the loop diagrams which are not proportional to polynomial of the external momenta. In the leading order of the ϵ expansion, diagrams where there are contributions from the derivative interactions are enumerated in figs. 7. Upper diagrams of figs. 7 are proportional to $\lambda\lambda_A$ or $\lambda\lambda_B$, and lowers are λ_A^2 ,

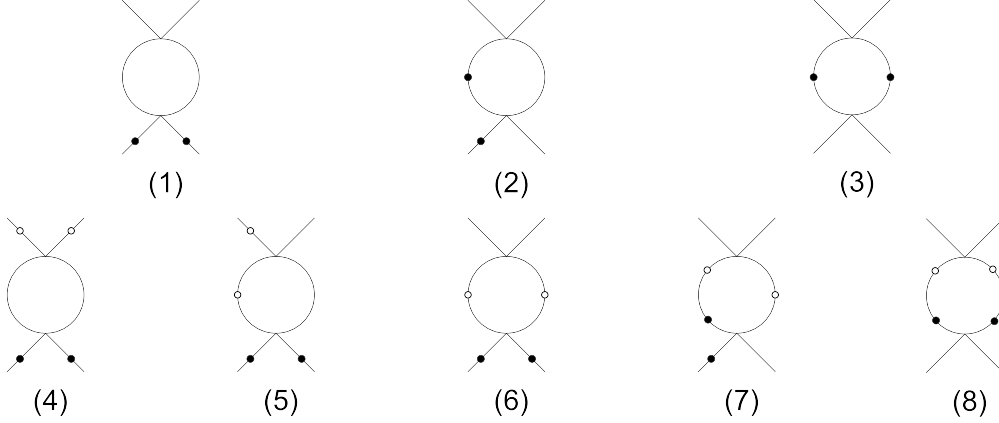


Figure 7: 1-loop contributions from the derivative couplings are shown. Derivatives act to dotted line, and colors of the dots indicate the contractions of derivatives.

λ_B^2 or $\lambda_A\lambda_B$. The diagram (1) is obviously a contribution to $G_A^{(4)}$ and $G_B^{(4)}$. (4) gives a corrections to more higher dimensional operators $(\partial_\mu\phi_a\partial_\mu\phi_a)^2$ and $(\partial_\mu\phi_a\partial_\nu\phi_a)(\partial_\mu\phi_b\partial_\nu\phi_b)$, and (6) contributes $G_A^{(4)}$, $G_B^{(4)}$ and more higher terms. The diagram (2) is proportional to

$$\begin{aligned} \int \frac{d^d k}{(2\pi)^d} \frac{p_i \cdot k}{k^2(k+P)^2} &= \int_0^1 dx \int \frac{d^d l}{(2\pi)^d} \frac{p_i \cdot (l-xP)}{[l^2 + x(1-x)P^2]^2} \quad (l \equiv k + xP) \\ &= p_i \cdot P \int_0^1 dx \int \frac{d^d l}{(2\pi)^d} \frac{-x}{[l^2 + x(1-x)P^2]^2} + \int_0^1 dx \int \frac{d^d l}{(2\pi)^2} \frac{p_i \cdot l}{[l^2 + x(1-x)P^2]^2}, \end{aligned} \quad (5.48)$$

where p_i is one of the external momenta, and P is a some linear combination of external momenta. The first term is a contribution to $G_A^{(4)}$ and $G_B^{(4)}$. The second term vanishes by the integration, it is a odd function of the loop momentum. Similarly, (5) and (7) contribute only $G_A^{(4)}$, $G_B^{(4)}$ and more higher terms. Eventually, diagrams which would contribute to the RG flow of λ are (3) and (8). The diagram (3) is proportional to

$$\begin{aligned} \int \frac{d^d k}{(2\pi)^d} \frac{k \cdot (k+P)}{k^2(k+P)^2} &= \int \frac{d^d k}{(2\pi)^d} \left(\frac{1}{(k+P)^2} + \frac{k \cdot P}{k^2(k+P)^2} \right) \\ &= \int \frac{d^d k'}{(2\pi)^d} \frac{1}{k'^2} + \int_0^1 dx \int \frac{d^d l}{(2\pi)^d} \frac{(l-xP) \cdot P}{[l^2 + x(1-x)P^2]^2} \\ &= \int \frac{d^d k'}{(2\pi)^d} \frac{1}{k'^2} + P^2 \int_0^1 dx \int \frac{d^d l}{(2\pi)^d} \frac{-x}{[l^2 + x(1-x)P^2]^2}, \end{aligned} \quad (5.49)$$

where we ignore the odd term of loop momentum. The first term contributes for the $G_\lambda^{(4)}$, and the second is a contribution for $G_A^{(4)}$, $G_B^{(4)}$. The first is quadratic divergence term, but it has no dependence of the external momenta. Thus it canceled by the counter term as a

constant in terms of the renormalization scale, and it cannot contribute to the RG flow of λ . The diagram (8) is proportional to

$$\begin{aligned}
\int \frac{d^d k}{(2\pi)^d} \frac{[k \cdot (k+P)]^2}{k^2(k+P)^2} &= \int \frac{d^d k}{(2\pi)^d} \left\{ 1 - \frac{P^2}{(k+P)^2} + \frac{k \cdot P}{k^2(k+P)^2} \right\} \\
&= \int \frac{d^d k}{(2\pi)^d} 1 - P^2 \int \frac{d^d k'}{(2\pi)^d} \frac{1}{k'^2} + P_\mu P_\nu \int_0^1 dx \int \frac{d^d l}{(2\pi)^d} \frac{l^\mu l^\nu}{[l^2 + x(1-x)P^2]^2} \\
&\quad + P^4 \int_0^1 dx \int \frac{d^d l}{(2\pi)^d} \frac{x^2}{[l^2 + x(1-x)P^2]^2}. \tag{5.50}
\end{aligned}$$

Only the first term contributes the $G_\lambda^{(4)}$, but it is a constant in external momenta. As a consequence, there is no contribution from derivative couplings.

Diagrams which arise in injection of the six and eight point interactions and contribute the four point functions at leading order of the ϵ expansion are enumerated in fig. 8. (a-i),

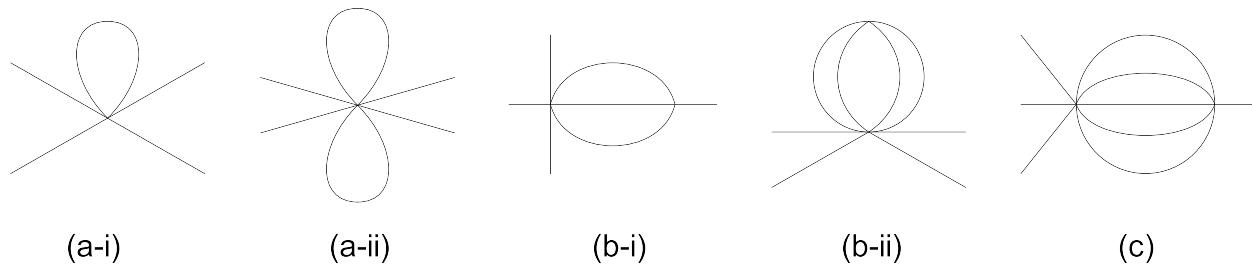


Figure 8: Leading contributions to the four-point correlation function from the six point and eight point interactions are shown.

(a-ii) are linear term of λ_6 and λ_8 . (b-i), (b-ii) are proportional to $\lambda\lambda_6$ and $\lambda\lambda_8$ respectively, and (c) are $\lambda_6\lambda_8$. Because the integrals of loop momenta in the diagram (a-i), (a-ii) and (b-ii) are independent of external momenta, they are canceled by the counter term of λ as a constant of the renormalization scale. Thus these diagrams does not contribute the RG flow of λ in this order. On the other hand, (b-i) and (c) have contribution to the RG flow as

$$\beta_{\hat{\lambda}}^{\phi^6, \phi^8} = \beta_{\hat{\lambda}} + c_1 \hat{\lambda} \hat{\lambda}_6 + c_2 \hat{\lambda}_6 \hat{\lambda}_8, \tag{5.51}$$

where $\beta_{\hat{\lambda}}$ is the β function of $\hat{\lambda}$ calculated in the theory without higher dimensional operators, and $\hat{\lambda}_6 = \mu^{-2+2\epsilon} \lambda_6$, $\hat{\lambda}_8 = \mu^{-4+3\epsilon} \lambda_8$, c_1 and c_2 are suitable constant. When $(\phi_a^2)^3$ or $(\phi_a^2)^4$ is relevant, it upset the stability of the IRFP. We ignore this possibility for a while, and assume that they are irrelevant. In this case, $\hat{\lambda}$ still converges to $\epsilon/2$ in the IR limit. Since we are assuming $(\phi_a^2)^3$ and $(\phi_a^2)^4$ are irrelevant, the third term converges to zero faster than the second. If the convergence of $\hat{\lambda}_6$ is slower than μ^ϵ , the approaching ratio of $\hat{\lambda}$ to $\epsilon/2$ becomes to the scaling dimension of $\hat{\lambda}_6$, and the leading irrelevant operator is $(\phi_a^2)^3$ in this

case. The β function of $\hat{\lambda}_6$ can be obtained as

$$\beta_{\hat{\lambda}_6} = (2 - 2\epsilon)\hat{\lambda}_6 + 7\lambda\lambda_6 + \mathcal{O}(\hat{\lambda}_6^2) \xrightarrow{\mu \rightarrow 0} \left(2 + \frac{3}{2}\epsilon\right) \hat{\lambda}_6. \quad (5.52)$$

This result is consistent with the assumption that $(\phi_a^2)^3$ is irrelevant, and moreover, means $(\phi_a^2)^3$ (and also $(\phi_a^2)^4$) do not change the IR approaching ratio of $\hat{\lambda}$ because the additional terms arising by insertion of the higher dimensional interactions decrease to zero faster than μ^ϵ .

As a consequence, we obtain that there is no operator which shifts the approaching ratio of $\hat{\lambda}$ in $\mathcal{L}_{\text{high-dim}}$. Needless to say, there are more higher dimensional operators, and perhaps, one of them might be able to change the approaching ratio of $\hat{\lambda}_{O(4)}$. However, it requires the large anomalous dimension. Therefore, we expect that the footprint of the massive fields will remain in the sub-leading behavior of second order phase transition.

6 Lattice calculation

Because of the diverging coupling, we need a non-perturbative check of the decoupling. In this section, we show our results of lattice calculation.

6.1 $U_A(1)$ broken model

For the $U_A(1)$ broken model, we use the discretized action described as

$$\begin{aligned} S_{U_A(1)\text{br}}^{\text{lat}} = a^{-3} \sum_x \left[& -\frac{1}{2} \sum_i \phi_a(x) (\phi_a(x + a\hat{i}) + \phi_a(x - a\hat{i}) - 2\phi_a(x)) + \frac{1}{2} \mu^2 \phi_a(x)^2 \right. \\ & - \frac{1}{2} \sum_i \chi_a(x) (\chi_a(x + a\hat{i}) + \chi_a(x - a\hat{i}) - 2\chi_a(x)) + \frac{1}{2} \mu^2 \phi_a(x)^2 \\ & + \frac{\pi^2}{3} \hat{\lambda} (\phi_a^2(x))^2 + \frac{\pi^2}{3} (\hat{\lambda} - 2\hat{x}) (\chi_a^2(x))^2 \\ & \left. + \frac{2}{3} \pi^2 (\hat{\lambda} + \hat{g}_2 - \hat{z}) \phi_a^2(x) \chi_b^2(x) - \frac{2}{3} \pi^2 \hat{g}_2 (\phi_a(x) \chi_a(x))^2 \right]. \quad (6.1) \end{aligned}$$

where a is a lattice spacing, and \hat{i} is the unit vector in the i th direction. We perform a lattice calculation in hybrid Monte-Carlo method in L^3 box with $L/a = 4, 8, 16, 32, 64$ and with periodic boundary condition. Because this model is an effective theory of low energy QCD, there is a cutoff scale. Therefore, the continuous limit is not taken, and we interpret the lattice spacing a as the cutoff scale $\Lambda = \pi/a$.

We perform the calculations with $L = 4, 8, 16, 32, 64$, and the results are shown in figs 9-12. Because $\langle M \rangle$ varies continuously, we suppose that the phase transition will be second order.

Fig. 9 shows μ^2 dependence of the expectation value of the effective magnetization M

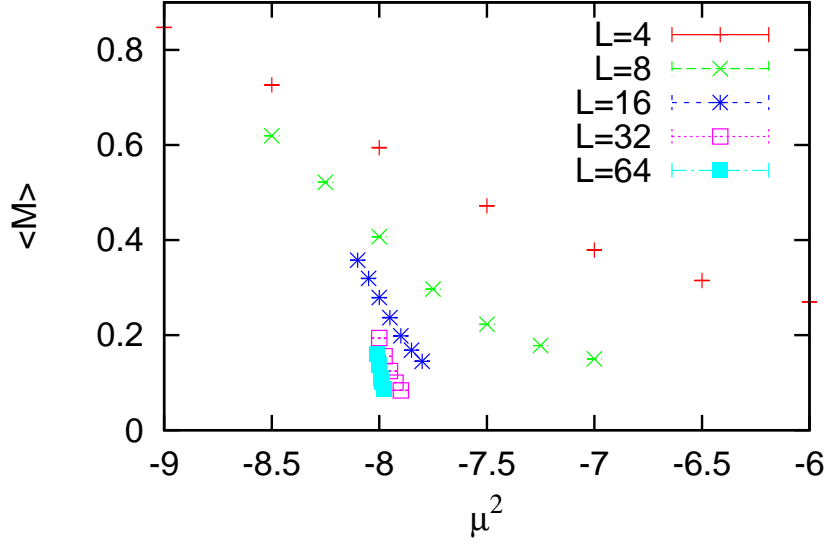


Figure 9: μ^2 dependence of the effective magnetizations $\langle M \rangle$ in the $U_A(1)$ broken model with $L = 4, 8, 16, 32, 64$ are shown. The input parameters is fixed to $a^2 c_A = 0.5$ and $(\hat{\lambda}, \hat{g}_2, \hat{x}, \hat{z}) = (3/\pi^2, 3/\pi^2, 0, 0)$.

defined as

$$M = \frac{1}{V} \sqrt{\sum_a \left(\int d^3x \phi_a(x) \right)^2}, \quad (6.2)$$

where $V = L^3$ is the volume of the box. Decomposing $\phi_a(x)$ by a vacuum expectation value φ_a and a fluctuation $\delta_a(x)$ as

$$\phi_a(x) = \varphi_a + \delta_a(x), \quad (6.3)$$

the effective magnetization is described as

$$\begin{aligned} M &= \frac{1}{V} \sqrt{\sum_a \int d^3x d^3y \phi_a(x) \phi_a(y)} \\ &= \sqrt{\sum_a \varphi_a^2 + V^{-1} \sum_a \int d^3x (\varphi_a \delta_a(x) + \delta_a(0) \delta_a(x)) + \dots} \end{aligned} \quad (6.4)$$

Thus, M becomes the correct vacuum expectation value in the infinitely volume limit. $\langle \mathcal{O} \rangle$ means a statistical average of an observable \mathcal{O} . The (negative) mass parameter μ^2 has an additive correction which is proportional to T^2 as $\mu^2 = cT^2 + \mu_0^2$. Because of the additive correction in the lattice regularization, $\mu(T_c)$ does not zero in this case. Near critical

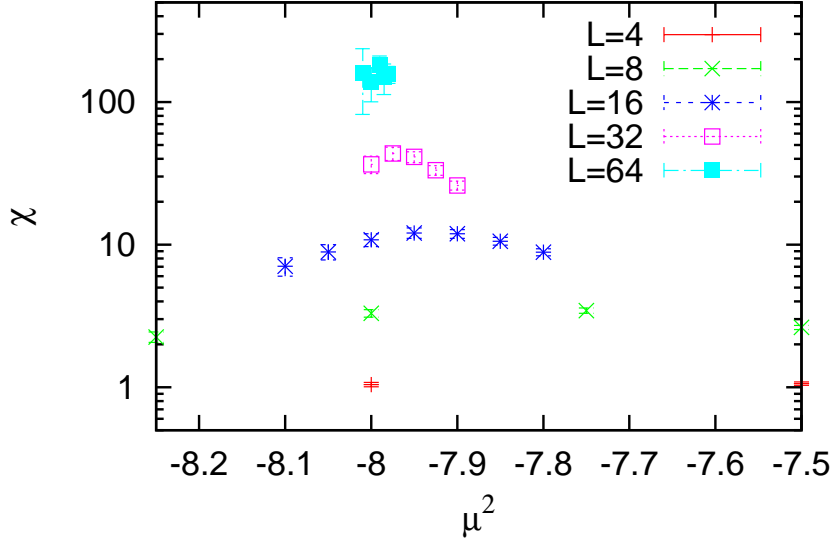


Figure 10: μ^2 dependence of the magnetic susceptibility χ in the $U_A(1)$ broken model with $L = 4, 8, 16, 32, 64$ are shown. The vertical axis is plotted in logarithmic scale. The input parameters is fixed to $a^2 c_A = 0.5$ and $(\hat{\lambda}, \hat{g}_2, \hat{x}, \hat{z}) = (3/\pi^2, 3/\pi^2, 0, 0)$.

temperature, we obtain

$$\mu^2(T) - \mu^2(T_c) = c(T^2 - T_c^2) \approx 2cT_c(T - T_c) + \mathcal{O}((T - T_c)^2). \quad (6.5)$$

Thus $\mu^2(T) - \mu^2(T_c) \propto t$ in $t = (T - T_c)/T_c \ll 1$, and we simply regard the μ^2 dependence as the temperature dependence near the critical point.

Fig. 9 shows that the magnetization has no gap even in large volume. It implicates that the phase transition will be second order.

The magnetic susceptibility χ is defined by

$$\chi = V (\langle M^2 \rangle - \langle M \rangle^2), \quad (6.6)$$

and it is shown in fig. 15. When the phase transition is second order phase transition, we can extract the critical exponent η and sub-leading exponent ω by the finite size scaling of the maximum value of the susceptibility χ_{max} (eq. (A.54)), as

$$\chi_{max} \propto L^{2-\eta}(1 + c_\chi L^{-\omega} + \dots).$$

On the other hand, $\chi_{max} \propto L^d$ in first order case. The fitting of the peaks in fig. 15 shows $\chi_{max} \sim L^2$, and it strongly suggests second order phase transition. Using the data in larger lattice ($L=16, 32, 64$), we obtain

$$\eta = 0.12 \pm 0.10. \quad (6.7)$$

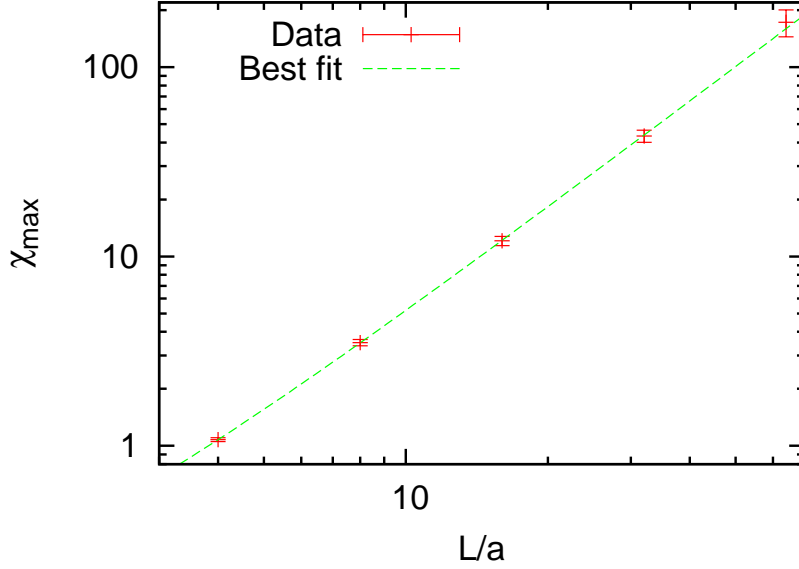


Figure 11: Lattice volume dependence of the χ_{max} in the $U_A(1)$ broken model is shown. The input parameters is fixed to $a^2 c_A = 0.5$ and $(\hat{\lambda}, \hat{g}_2, \hat{x}, \hat{z}) = (3/\pi^2, 3/\pi^2, 0, 0)$. The fitting curve is drawn with best fit value $\eta = 0.12$ and $\omega = 1.2$.

Fixing η to the best fit value, we obtain the sub-leading exponent as

$$\omega = 1.3 \pm 1.2 \quad (\text{with } \eta = 0.036). \quad (6.8)$$

The lattice data and the best fit curve of χ_{max} is plotted in fig. 11.

Assuming the $O(4)$ universality, we can use a referential value of the exponent $\eta = 0.036$ reported in Ref. [19] alternatively. With referential value, we obtain

$$\omega = 0.85 \pm 0.52. \quad (6.9)$$

The Binder cumulant [7, 8] U_L is shown in figs. 12. It is defined as

$$U_L = \frac{\langle M^4 \rangle}{(\langle M^2 \rangle)^2}. \quad (6.10)$$

We can see that the U_L is nearly volume independent at the critical point as expected by the finite size scaling (eq. (A.51)). One can extract the critical exponent ν by the slope of U_L (eq. (A.52)) as

$$\left. \frac{\partial}{\partial t} U_L \right|_{t=0} \propto L^{\frac{1}{\nu}}.$$

And we obtain

$$\nu = 0.70 \pm 0.10. \quad (6.11)$$

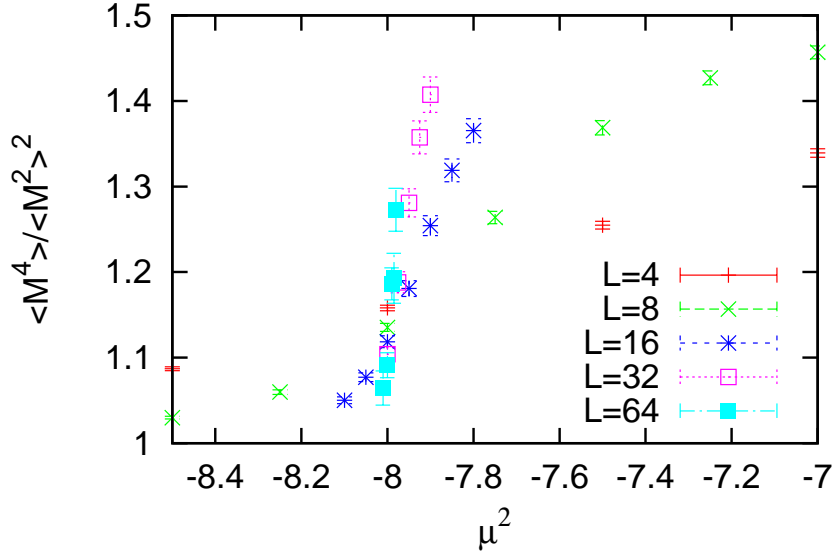


Figure 12: μ^2 dependence of the Binder cumulant U_L in the $U_A(1)$ broken model with $L = 4, 8, 16, 32, 64$ are shown. The input parameters is fixed to $a^2 c_A = 0.5$ and $(\hat{\lambda}, \hat{g}_2, \hat{x}, \hat{z}) = (3/\pi^2, 3/\pi^2, 0, 0)$.

The volume dependence of the slope is shown in fig. 13.

6.2 $O(4)$ LSM

In the aim of the comparison with the $U_A(1)$ broken model, we also carried out the lattice calculation on the $O(4)$ LSM. For $O(4)$ LSM, we use the discretized action as

$$S_{O(4)}^{\text{lat}} = a^{-3} \sum_x \left[-\frac{1}{2} \sum_{i=1}^3 \phi_a(x) (\phi_a(x + a\hat{i}) + \phi_a(x - a\hat{i}) - 2\phi_a(x)) + \frac{1}{2} \mu^2 \phi_a(x)^2 + \frac{\pi^2}{3} \lambda (\phi_a^2(x))^2 \right], \quad (6.12)$$

We perform the calculations with $L = 4, 8, 16, 32, 64$, and the results are shown in figs 14-16.

As similar as in the $U_A(1)$ broken model, we extract the critical exponents η , ν and ω as

$$\eta = 0.048 \pm 0.084, \quad (6.13)$$

$$\nu = 0.71 \pm 0.02, \quad (6.14)$$

$$\omega = 0.90 \pm 0.33. \quad (6.15)$$

The volume dependence of χ_{max} and slope of the Bindar cumulant U_L are shown in fig.17 and 18. We comment that, of course, the volume dependence of the susceptibility, $\chi_{max} \propto L^{(2-\eta)} \sim L^2$, strongly suggests second order phase transition.

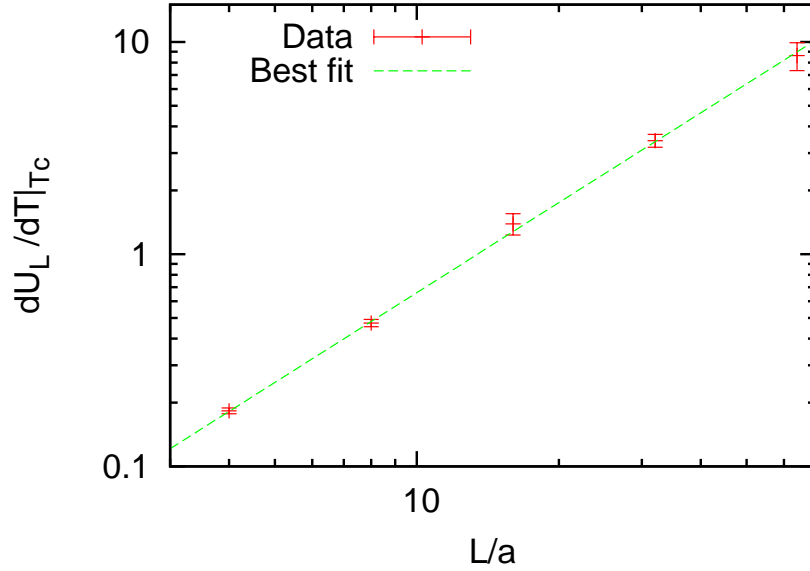


Figure 13: Volume dependence of the slope of U_L in the $U_A(1)$ broken model is shown. The input parameters is fixed to $a^2 c_A = 0.5$ and $(\hat{\lambda}, \hat{g}_2, \hat{x}, \hat{z}) = (3/\pi^2, 3/\pi^2, 0, 0)$. The fitting curve is drawn with best fit value $\eta = 0.12$ and $\omega = 1.2$.

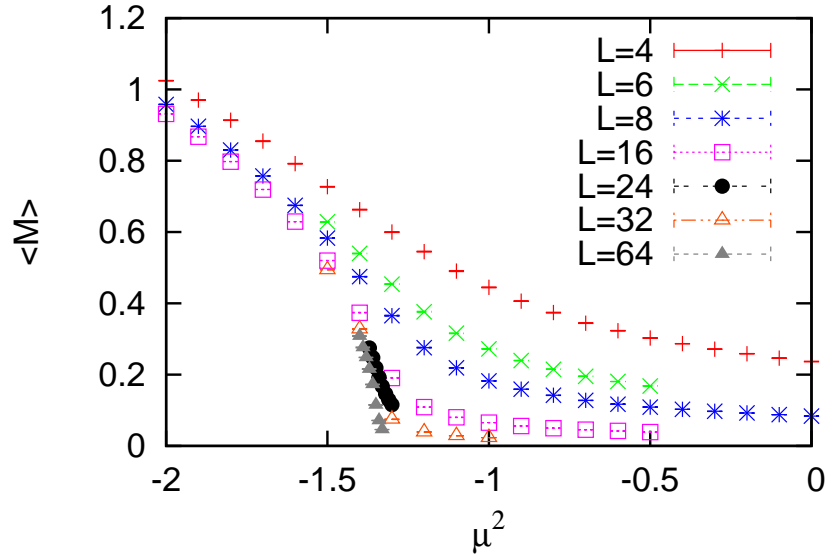


Figure 14: μ^2 dependence of the effective magnetizations $\langle M \rangle$ in $O(4)$ LSM with $L = 4, 8, 16, 32, 64$ are shown.

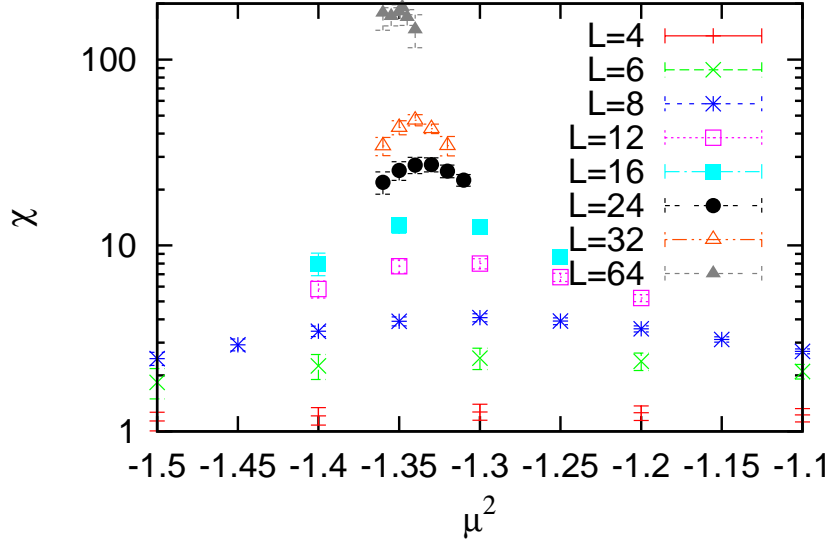


Figure 15: μ^2 dependence of the magnetic susceptibility χ of the $O(4)$ LSM in $L = 4, 6, 8, 12, 16, 24, 32, 64$ box is shown. The vertical axis is plotted in logarithmic scale. $\hat{\lambda}$ is fixed to $3/\pi^2$.

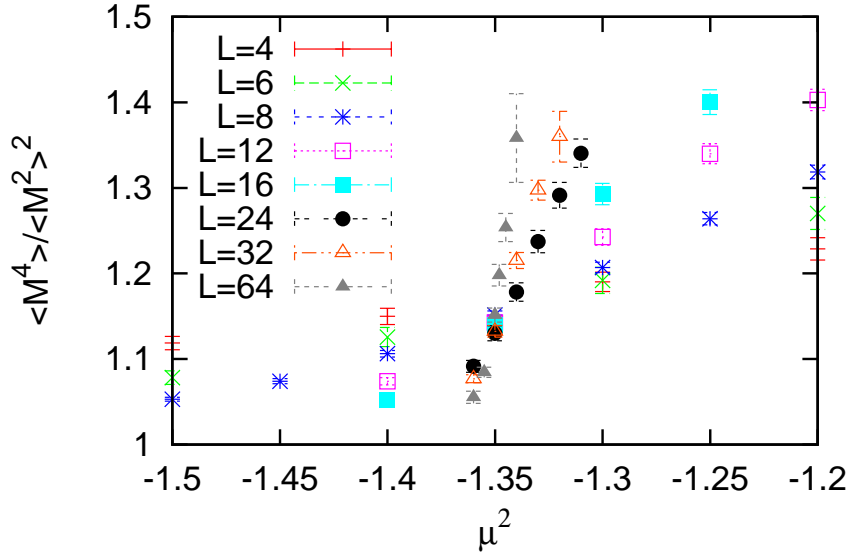


Figure 16: μ dependence of the Binder cumulants U_L in the $O(4)$ LSM with $L = 4, 6, 8, 12, 16, 24, 32, 64$ are shown. $\hat{\lambda}$ is fixed to $3/\pi^2$.

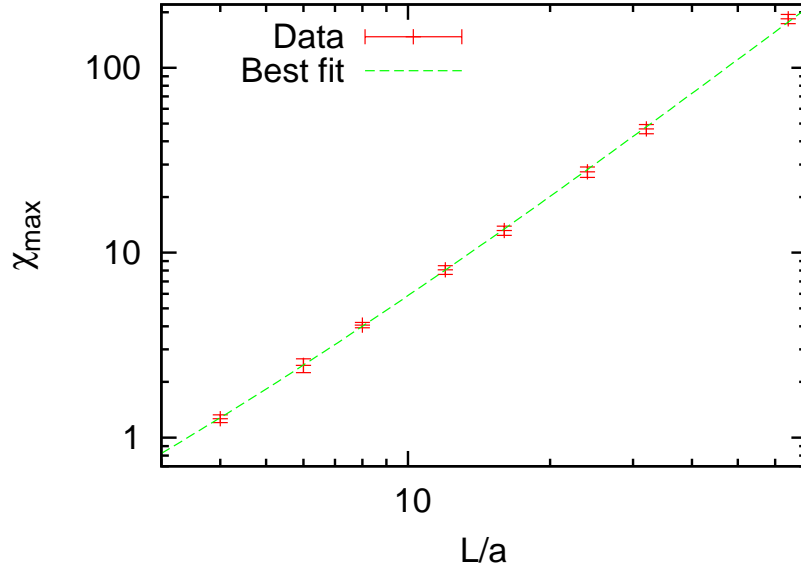


Figure 17: Volume dependence of the χ_{max} in the $O(4)$ LSM is shown. $\hat{\lambda}$ is fixed to $3/\pi^2$. The fitting curve is drawn with the best fit values $\eta = 0.048$ and $\omega = 0.90$.

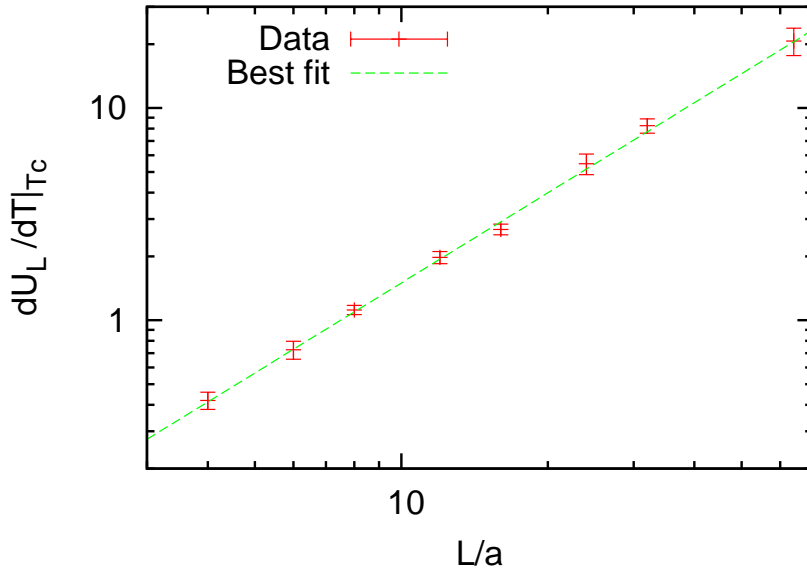


Figure 18: Volume dependence of the slope of U_L in the $O(4)$ LSM is shown. $\hat{\lambda}$ is fixed to $3/\pi^2$. The fitting curve are drawn with the best fit value $\nu = 0.71$.

| | Model | Method | η | ν | ω |
|-----------|------------------------|-------------------------------|------------|--------------------|------------------------|
| Ref. [17] | $O(4)$ Heisenberg | Lattice | 0.0254(38) | 0.7479(90) | |
| Ref. [19] | $O(4)$ LSM | Lattice | 0.0365(10) | 0.749(2) | 0.765(22) |
| Ref. [28] | $U(2) \times U(2)$ LSM | Perturbation (d=3) | 0.12(1) | 0.71(7) | |
| This work | $O(4)$ LSM | ϵ expansion | 0 | $(4 + \epsilon)/8$ | ϵ |
| | | Lattice [$\eta = 0.036$] | 0.048(84) | 0.71(2) | 0.90(33) [0.88(33)] |
| | $U_A(1)$ broken | ϵ expansion | 0 | $(4 + \epsilon)/8$ | $2 - 5\epsilon/3$ |
| | | Lattice [$\eta = 0.036$] | 0.12(10) | 0.70(1) | 1.3(1.2) [0.85(52)] |

Table 1: Our results of the ϵ expansion and the lattice calculation are summarized. The exponents reported in Refs. [17, 19, 28] are also shown as references. The values in the square brackets [] are the value of ω calculated with the referential value $\eta = 0.036$.

Using the referential value $\eta = 0.036$, we obtain

$$\omega = 0.88 \pm 0.33 \quad (\text{with } \eta = 0.036). \quad (6.16)$$

Finally, we summarize the exponents obtained by the ϵ expansion and the lattice calculation in table 1. These reported in Refs. [17, 19, 28] are also shown as a reference. Because of the large error, we cannot distinguish whether the discrepancy of the ω between the $O(4)$ LSM and the $U_A(1)$ broken model exists or not.

7 Summary

The two-flavor massless QCD is studied both analytically and numerically. The critical phenomena of this theory depends on strength of the $U_A(1)$ symmetry breaking at the critical point. The nature of the chiral phase transition with infinitely large breaking of the $U_A(1)$ symmetry and that with the effective restored $U_A(1)$ are well established by effective theory approaches. In this study, we investigated the critical phenomena of the $U(2) \times U(2)$ LSM with the $U_A(1)$ breaking term called the $U_A(1)$ broken model as an effective theory of the chiral phase transition of the two-flavor massless QCD with the finite $U_A(1)$ breaking. This model is constructed by two of four-component real scalar fields, the massless fields ϕ_a and the massive fields χ_a at the critical point. The strength of the $U_A(1)$ breaking is parameterized by c_A which is equivalent to the mass splitting of ϕ_a and χ_a in this model. With infinitely large c_A , the massive fields χ_a decouple from the IR nature, and the model is reduced into the $O(4)$ LSM.

The RG flow of the $U_A(1)$ broken model with finite c_A is investigated in the ϵ expansion.

In order to trace effects of the mass, we took a mass-dependent scheme to the regularization of the four-point functions, and the on-shell scheme to the two-point functions. No IR stable fixed point is obtained in the leading order of the expansion. However, depending on initial conditions, one of the couplings $\hat{\lambda}$ which is remained in the infinitely large $U_A(1)$ breaking limit converges to the IRFP of the $O(4)$ LSM. IR asymptotic behaviors of the couplings are obtained in the converging case. Contributions of the massive fields to the β function of $\hat{\lambda}$ vanish in the IR limit. It indicates the decoupling of the massive fields and the reduction of the $U_A(1)$ broken model to the $O(4)$ LSM. There is the $O(4)$ attractive basin where the remaining coupling $\hat{\lambda}$ reaches to the $O(4)$ IRFP in the initial coupling space. We found that the attractive basin shrinks as c_A decreases. Thus, if c_A turns out to be extremely small, the phase transition of massless two flavor QCD tends to be first order.

In order to establish the nature with the diverging couplings and the decoupling of the massive fields, we calculated the RG improved correlation functions. It was shown that the RG improved four-point functions in the $U_A(1)$ broken model with converging $\hat{\lambda}$ converges to these in the $O(4)$ LSM. We point out that the correlation functions calculated in the \overline{MS} scheme have same dependence on the external momenta with those calculated in the mass-dependent scheme. And, it was shown that the N -point functions with $N \geq 6$ converge to these in the $O(4)$ LSM. Therefore, the IR nature of the $U_A(1)$ broken model will approach to that of the $O(4)$ LSM, and we conclude that the $U_A(1)$ broken model undergoes second order phase transition in the $O(4)$ attractive basin.

We calculated the critical exponents of the $U_A(1)$ broken model. As a naive expectation, the $U_A(1)$ broken model shows the same exponent as the $O(4)$ LSM. On the other hand, it is worthy to note that the exponent ω which characterizes the sub-leading behaviors of the critical phenomena differs between the $U_A(1)$ broken model and the $O(4)$ LSM. This discrepancy implies that we can find the footprint of the massive fields from the sub-leading behavior of the phase transition.

Finally, we performed the lattice calculation on the $U_A(1)$ broken model. The scaling of the magnetic susceptibility at the critical point χ_{max} indicates that the model can end up with second order phase transition. It means that the decoupling of the massive fields χ_a occurs even in the non-perturbative formulation.

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Finally, I would like to thank my parents for the great supports.

A Critical exponents and scaling law

A.1 Critical exponents

When a theory is pointed away from the critical point, the two-point correlation function behaves as

$$G^{(2)}(x) \equiv \langle \Phi(x)\Phi(0) \rangle \sim \exp[-|x|/\xi], \quad (\text{A.1})$$

where ξ is the correlation length. Approaching to the critical point, that is the reduced temperature $t \equiv (T - T_c)/T_c$ decreases to zero, the correlation length ξ diverges with negative power of t ,

$$\xi \sim |t|^{-\nu}. \quad (\text{A.2})$$

As a consequence, the correlation function follows a power law of $|x|$ as

$$G^{(2)}(x) \sim \frac{1}{|x|^{d-2+\eta}}. \quad (\text{A.3})$$

Because of the RG equation of the correlation function,

$$\left[\mu \frac{\partial}{\partial \mu} + \sum_i \beta_{\rho_i} \frac{\partial}{\partial \rho_i} + 2\gamma_\phi \right] G^{(2)}(x; \mu, \{\rho_i\}) = 0, \quad (\text{A.4})$$

we obtain

$$G^{(2)}(x) = \frac{1}{|x|^{d-2}} h(\{\bar{\rho}_i\}) \cdot \exp \left[-2 \int_{1/\mu}^{|x|} d \log |x'| \gamma_\phi(\{\bar{\rho}_i(x')\}) \right]. \quad (\text{A.5})$$

Where, ρ_i is dimensionless couplings. $\bar{\rho}_i$ obeys the differential equation

$$\frac{d}{d \log[1/\mu|x]} \bar{\rho}_i = \beta_i(\{\bar{\rho}_i\}), \quad (\text{A.6})$$

and the initial condition,

$$\bar{\rho}_i|_{\mu|x|=1} = \rho_i(\mu). \quad (\text{A.7})$$

The dimensionless function $h(\{\bar{\rho}_i\})$ is determined from a initial condition.

When one set μ to sufficiently small, say, all parameters in the theory are sufficiently close to the IR fixed point, the exponential in eq. (A.5) can be written approximately as

$$\exp \left[-2 \int_{1/\mu}^{|x|} d \log |x'| \gamma_\phi(\{\bar{\rho}_i(x')\}) \right] \approx \exp \left[-2\gamma_\phi^* \int_{1/\mu}^{|x|} d \log |x'| \right] = \frac{1}{(\mu|x|)^{2\gamma_\phi^*}}, \quad (\text{A.8})$$

Where γ_ϕ^* is the value of γ_ϕ at the IR fixed point.

Considering the case that, there is only one relevant coupling ρ_m in the theory. In this case,

$$\bar{\rho}_m = \rho_m(\mu|x|)^{2-\gamma_\phi^*}, \quad (\text{A.9})$$

$$\bar{\rho}_{i \neq m} = \rho_i(\mu|x|)^{-A_i}. \quad (\text{A.10})$$

Where $A_i > 0$ at any $i \neq m$.¹² At large distances, all arguments in $h(\{\bar{\rho}_i\})$ expect for $\bar{\rho}_m$ become negligible. And, we can regard as the function h as univariate function of $\rho_m(\mu|x|)^{2-\gamma_\phi^*}$. To regularity of $G^{(2)}$, ρ_m should vanishes at the critical point. Typically, it proportional to t near $t = 0$. Hence, we obtain the asymptotic behavior of $G^{(2)}$ around the critical point as

$$G^{(2)}(x) \approx \frac{1}{|x|^{d-2}} \cdot \frac{1}{(\mu|x|)^{2\gamma_\phi^*}} \cdot h(t(\mu|x|)^{2-\gamma_\phi^*}). \quad (\text{A.11})$$

Therefore, the critical exponent η defined by eq. (A.3) is

$$\eta = 2\gamma_\phi^*. \quad (\text{A.12})$$

Because t dependence of $G^{(2)}$ comes only in a form of $h(t(\mu|x|)^{2-\gamma_\phi^*})$, the exponent ν defined by eq. (A.2) should be

$$\nu = \frac{1}{2 - \gamma_\phi^*}. \quad (\text{A.13})$$

There are other critical exponents. They characterize the behavior of a magnetic susceptibility χ , a specific heat C , a spontaneous magnetization (or the order parameter) m and

¹²There may be the couplings which does not vanish but converges to ρ_* . In this case, we can redefine $\rho' \equiv \rho - \rho_*$ which vanishes at critical point.

the response of that for an external field h as,

$$\chi(t, h = 0) \propto |t|^{-\gamma} \quad (t > 0), \quad (\text{A.14})$$

$$C(t, h = 0) \propto |t|^{-\alpha} \quad (t > 0), \quad (\text{A.15})$$

$$m(t, h = 0) \propto |t|^\beta \quad (t < 0), \quad (\text{A.16})$$

$$m(t = 0, h) \propto |h|^{\frac{1}{\delta}}. \quad (\text{A.17})$$

A.2 Scaling law and finite size scaling

In this section, we show a brief review of the scaling law and the finite size scaling. Finite size scaling is the most important technique to pick up information of a critical phenomenon from lattice calculation in a finite size box.

A.2.1 Scaling law in infinite volume

First of all, we consider a system which ends up to the second order phase transition in d dimensional infinite volume with temperature T , external field h and operators $\{g_i\}$. Near by the critical temperature T_c , free energy density of a system is transformed in the renormalization transformation $L \rightarrow b^{-1}L$ as

$$f(t, h, g_1, g_2, \dots) \rightarrow b^{-d} f(b^{y_t} t, b^{y_h} h, b^{y_1} g_1, b^{y_2} g_2, \dots), \quad (\text{A.18})$$

where L is some of a length scale, $t = (T - T_c)/T_c$ is the reduced temperature, and y_t , y_h and $\{y_i\}$ are scaling dimension of each operators. When all of operators $\{g_i\}$ are irrelevant, $y_i < 0$ for any i . Repeating the renormalization transformation, all couplings of irrelevant operator decrease to zero. Thus, we can write the free energy as the function depending only on temperature t and external field h as

$$f(t, h) = b^{-nd} f(b^{ny_t} t, b^{ny_h} h), \quad (\text{A.19})$$

with sufficiently large n . Choosing renormalization parameter b as $b^{ny_t} t = 1$, we obtain

$$f(t, h) = t^{d/y_t} f(1, ht^{-y_h/y_t}). \quad (\text{A.20})$$

Eventually, we can deal the free energy as a unary function practically. Using eq. (A.20),

$$C(t, 0) \propto \frac{\partial^2 f(t, 0)}{\partial t^2} \propto t^{\frac{d}{y_t} - 2}, \quad (\text{A.21})$$

$$\langle \phi \rangle(t, 0) \propto \left. \frac{\partial f(t, h)}{\partial h} \right|_{h=0} \propto t^{(d-y_h)/y_t}, \quad (\text{A.22})$$

$$\chi(t, 0) \propto \left. \frac{\partial^2 f(t, h)}{\partial h^2} \right|_{h=0} \propto t^{(d-2y_h)/y_t}. \quad (\text{A.23})$$

And from eq. (A.19),

$$\langle \phi \rangle(0, h) \propto \frac{\partial f(0, h)}{\partial h} \propto b^{-nd+ny_h} \left. \frac{\partial f(0, h')}{\partial h'} \right|_{h'=b^{ny_h}h}. \quad (\text{A.24})$$

Taking $b^{ny_h}h = 1$,

$$\langle \phi \rangle(0, h) \propto h^{(d-y_h)/y_h} \quad (\text{A.25})$$

Hence, the critical exponents can be written as

$$\alpha = 2 - \frac{d}{y_t}, \quad \beta = \frac{d - y_h}{y_t}, \quad \gamma = \frac{2y_h - d}{y_t}, \quad \delta = \frac{y_h}{d - y_h}. \quad (\text{A.26})$$

And therefore, we obtain the scaling relation,

$$\alpha + 2\beta + \gamma = 2, \quad \gamma = \beta(\delta - 1). \quad (\text{A.27})$$

Similarly, performing the renormalization transformation $L \rightarrow b^{-1}L$ to a correlation function,

$$G(r, t) = b^{-2d+2y_t} G(b^{-1}r, b^{y_t}t), \quad (\text{A.28})$$

where we use the scaling law of the spontaneous magnetization $m(t) = b^{-d+y_h}m(b^{y_h}t)$. Tking $b^{y_t}t = 1$,

$$G(r, t) = t^{2(d-y_h)/y_t} \Phi(rt^{1/y_t}). \quad (\text{A.29})$$

Where $\Phi(r) = G(r, 1)$. Using a correlation length ξ , the correlation function at $t \neq 0$ can be written as,

$$G(r, t) \propto e^{-r/\xi}. \quad (\text{A.30})$$

On the other hand, we can see that the r dependence of the correlation function only arises as rt^{1/y_t} from eq. (A.29). Thus,

$$\xi \propto t^{-1/y_t}. \quad (\text{A.31})$$

Taking $b^{-1}r = 1$ alternatively, and setting $t = 0$,

$$G(r, 0) \propto r^{-2d+2y_h}. \quad (\text{A.32})$$

Then

$$\nu = \frac{1}{y_h}, \quad \eta = d - 2y_h + 2. \quad (\text{A.33})$$

Using eq. (A.33) and eq. (A.26), we obtain the hyperscaling relations as

$$\alpha = 2 - d\nu, \quad \beta = \frac{\nu(d-2+\eta)}{2}, \quad \gamma = \nu(2-\eta), \quad \delta = \frac{d+2-\eta}{d-2+\eta}. \quad (\text{A.34})$$

A.2.2 Sub-leading term of the scaling law

Emphasizing the leading irrelevant operator $\delta\lambda$ that has largest scaling dimension $-\omega$ in eq. (A.20)

$$f(t, h, \delta\lambda) = b^{-nd} f(b^{ny_t} t, b^{ny_h} h, b^{-n\omega} \delta\lambda). \quad (\text{A.35})$$

Taking $b^{ny_t} t = 1$, we obtain

$$\begin{aligned} C(t, 0) &\sim t^{\frac{d}{y_t}-2} \Phi_C(\delta\lambda t^{\omega/y_t}) = t^{-\alpha} \left(\Phi_C(0) + t^{\omega\nu} \frac{d\Phi(\delta\lambda')}{d(\delta\lambda')} \Big|_{\delta\lambda'=0} \delta\lambda + \dots \right) \\ &\propto t^{-\alpha} (1 + k_C t^{\omega\nu} + \dots), \end{aligned} \quad (\text{A.36})$$

where Φ_X ($X = C, \chi, m, \dots$) is a some suitable function, and k_C is a constant. Similarly, we obtain

$$\langle \phi \rangle(t, 0) \propto t^\beta (1 + k_m t^{\omega\nu} + \dots), \quad (\text{A.37})$$

$$\chi(t, 0) \propto t^{-\gamma} (1 + k_\chi t^{\omega\nu} + \dots). \quad (\text{A.38})$$

Thus, the critical exponent ω characterizes the sub-leading behaviors of critical phenomena.

A.2.3 Finite size scaling

Next, we consider a system in a finite volume. In order to actualize the second order phase transition, we need to take L^{-1} to zero, as t and h . Hence L^{-1} is also a relevant operator with scaling dimension 1,

$$f(t, h, L^{-1}) = b^{-d} f(b^{y_t} t, b^{y_h} h, bL^{-1}). \quad (\text{A.39})$$

Taking $b = L$ (with suitable coefficient), we obtain the scaling law in a finite volume as

$$f(t, h, L^{-1}) = L^{-d} f(tL^{y_t}, hL^{y_h}, 1). \quad (\text{A.40})$$

Thus,

$$C(t, 0, L^{-1}) \propto \frac{\partial^2 f(t', 0, L^{-1})}{\partial t'^2} \Big|_{t'=t} = L^{2y_t-d} \Phi_C(tL^{y_t}) = L^{\alpha/\nu} \Phi_C(tL^{\frac{1}{\nu}}), \quad (\text{A.41})$$

$$\langle \phi \rangle(t, 0, L^{-1}) \propto \frac{\partial f(t, h, L^{-1})}{\partial h} \Big|_{h=0} = L^{-d+y_h} \Phi_m(tL^{y_t}) = L^{\beta/\nu} \Phi_m(tL^{\frac{1}{\nu}}), \quad (\text{A.42})$$

$$\chi(t, 0, L^{-1}) \propto \frac{\partial^2 f(t, h, L^{-1})}{\partial h^2} \Big|_{h=0} = L^{-d+2y_h} \Phi_\chi(tL^{y_t}) = L^{\gamma/\nu} \Phi_\chi(tL^{\frac{1}{\nu}}), \quad (\text{A.43})$$

where we use the scaling laws eq. (A.26) and eq. (A.33). Assuming that the finite size expressions $C(t, 0, L^{-1})$ and $\chi(t, 0, L^{-1})$ smoothly connect to that of in the infinite volume, $C(t, 0, L^{-1})$ and $\chi(t, 0, L^{-1})$ should have peaks at $t = 0$ ¹³. And the values of these at the peaks are

$$C_{max}(h = 0, L^{-1}) = C(0, 0, L^{-1}) \propto L^{\alpha/\nu}, \quad (\text{A.44})$$

$$\chi_{max}(h = 0, L^{-1}) = \chi(0, 0, L^{-1}) \propto L^{\gamma/\nu}. \quad (\text{A.45})$$

A.2.4 Binder ratio and cumulant

The Binder ratio B_L and cumulant U_L are usefull tools to classify the second order phase transition. They are defined as

$$B_L(t, h) = \frac{\langle \phi^2 \rangle(t, h, L^{-1})}{\langle \phi \rangle^2(t, h, L^{-1})}, \quad U_L(t, h) = \frac{\langle \phi^4 \rangle(t, h, L^{-1})}{\langle \phi^2 \rangle^2(t, h, L^{-1})}. \quad (\text{A.46})$$

$\langle \phi^n \rangle$ can be obtained by n th partial differentiation of the external field as

$$\begin{aligned} \langle \phi^2 \rangle &= V^{-2} Z^{-1} \int \mathcal{D}\phi \left(\int d^d x \phi(x) \right)^2 e^{-S+h \int d^d x \phi(x)} \\ &\propto V^{-2} \frac{\partial^2}{\partial h^2} \log Z + \left(V^{-1} \frac{\partial}{\partial h} \log Z \right)^2 \propto V^{-1} \chi(t, h) + \langle \phi \rangle^2. \end{aligned} \quad (\text{A.47})$$

Using the scaling laws eq. (A.42) and eq. (A.43),

$$\langle \phi^2 \rangle(t, 0) \propto L^{2(-d+y_n)} \Phi_{m^2}(tL^{\frac{1}{\nu}}) = L^{2\beta/\nu} \Phi_{m^2}(tL^{\frac{1}{\nu}}). \quad (\text{A.48})$$

Then, we obtain

$$B_L(t, 0) \propto \frac{\Phi_{m^2}(tL^{\frac{1}{\nu}})}{\Phi_m^2(tL^{\frac{1}{\nu}})}. \quad (\text{A.49})$$

Similarly,

$$\langle \phi^4 \rangle(t, 0) \propto L^{4\beta/\nu} \Phi_{m^4}(tL^{\frac{1}{\nu}}), \quad (\text{A.50})$$

and also

$$U_L(t, 0) \propto \frac{\Phi_{m^4}(tL^{\frac{1}{\nu}})}{\Phi_{m^2}^2(tL^{\frac{1}{\nu}})}. \quad (\text{A.51})$$

Therefore, the Binder ratio and cumulant are L independent at $t = 0$ ¹⁴. On the other hand,

¹³Acctually, because the critical temperature also shifts by the finite size effect, definition of t depends on the box size L .

¹⁴Acutually, L dependence is remained in the definition of t .

we can extract the critical exponent by the derivation of these as,

$$\left. \frac{\partial}{\partial t} B_L(t, 0) \right|_{t=0} \propto L^{\frac{1}{\nu}}, \quad \left. \frac{\partial}{\partial t} U_L(t, 0) \right|_{t=0} \propto L^{\frac{1}{\nu}}. \quad (\text{A.52})$$

A.2.5 Next to leading term of the finite size scaling

Emphasizing the leading irrelevant operator $\delta\lambda$ that has largest scaling dimension $-\omega$ in eq. (A.39), the scaling equation in finite volume is obtained as

$$f(t, h, L^{-1}, \delta\lambda) = b^{-d} f(b^{y_t} t, b^{y_h} h, bL^{-1}, b^{-\omega} \delta\lambda). \quad (\text{A.53})$$

Taking $bL^{-1} = 1$, we obtain

$$\begin{aligned} \chi(t, 0, L^{-1}) &= L^{\gamma/\nu} \Phi_\chi(tL^{\frac{1}{\nu}}, \delta\lambda L^{-\omega}) \\ &= L^{\gamma/\nu} \Phi_\chi(tL^{\frac{1}{\nu}}, 0) + L^{\gamma/\nu-\omega} \left. \frac{\partial}{\partial(\delta\lambda')} \Phi_\chi(tL^{\frac{1}{\nu}}, \delta\lambda') \right|_{\delta\lambda'=0} \delta\lambda + \dots \end{aligned} \quad (\text{A.54})$$

Thus the scaling dimension ω handles sub-leading term of the finite size scaling. Similarly,

$$\begin{aligned} C(t, 0, L^{-1}) &= L^{\alpha/\nu} \Phi_C(tL^{\frac{1}{\nu}}, \delta\lambda L^{-\omega}) \\ &= L^{\alpha/\nu} \Phi_C(tL^{\frac{1}{\nu}}, 0) + L^{\alpha/\nu-\omega} \left. \frac{\partial}{\partial(\delta\lambda')} \Phi_C(tL^{\frac{1}{\nu}}, \delta\lambda') \right|_{\delta\lambda'=0} \delta\lambda + \dots \end{aligned} \quad (\text{A.55})$$

$$\begin{aligned} \left. \frac{\partial}{\partial t} U_L(t, 0) \right|_{t=0} &= \left. \frac{\partial}{\partial t} \frac{\Phi_{m^4}(tL^{\frac{1}{\nu}}, \delta\lambda L^{-\omega})}{\Phi_{m^2}^2(tL^{\frac{1}{\nu}}, \delta\lambda L^{-\omega})} \right|_{t=0} \\ &\propto L^{\frac{1}{\nu}} \Phi_{B_2}(L^{\frac{1}{\nu}}, \delta\lambda L^{-\omega}) \\ &= L^{\frac{1}{\nu}} \Phi_{U_L}(L^{\frac{1}{\nu}}, 0) + L^{\frac{1}{\nu}-\omega} \left. \frac{\partial}{\partial(\delta\lambda')} \Phi_{U_L}(tL^{\frac{1}{\nu}}, \delta\lambda') \right|_{\delta\lambda'=0} \delta\lambda + \dots \end{aligned} \quad (\text{A.56})$$

B Hessian matrix

The RG flow near a fixed point of a theory $\{g_{i,*}\}$ grows in accordance with

$$\mu \frac{d}{d\mu} \begin{pmatrix} g_{1,*} + \delta_1 \\ g_{2,*} + \delta_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \end{pmatrix} \Big|_{\{g_{i,*}\}} + \mu \frac{d}{d\mu} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \end{pmatrix} = \mu \frac{d}{d\mu} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \end{pmatrix}. \quad (\text{B.1})$$

With sufficiently small $\{\delta_i\}$, the RG equation can be expanded as

$$\begin{aligned} \mu \frac{d}{d\mu} \begin{pmatrix} g_{1,*} + \delta_1 \\ g_{2,*} + \delta_2 \\ \vdots \end{pmatrix} &= \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \end{pmatrix} (\{g_{i,*} + \delta_i\}) \\ &\approx \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \end{pmatrix} \Big|_{\{g_{i,*}\}} + \begin{pmatrix} \frac{\partial \beta_1}{\partial \lambda_1} & \frac{\partial \beta_1}{\partial \lambda_2} & \cdots \\ \frac{\partial \beta_2}{\partial \lambda_1} & \ddots & \\ \vdots & & \ddots \end{pmatrix} \Big|_{\{g_{i,*}\}} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \end{pmatrix}. \end{aligned} \quad (\text{B.2})$$

From these,

$$\mu \frac{d}{d\mu} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \frac{\partial \beta_1}{\partial \lambda_1} & \frac{\partial \beta_1}{\partial \lambda_2} & \cdots \\ \frac{\partial \beta_2}{\partial \lambda_1} & \ddots & \\ \vdots & & \ddots \end{pmatrix} \Big|_{\{g_{i,*}\}} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \end{pmatrix}. \quad (\text{B.3})$$

The matrix $\omega_{ij} \equiv \frac{\partial \beta_i}{g_j}$ is called the Hessian matrix. Decomposing the vector $(\delta_1, \delta_2, \dots)$ using the eigenvectors $\omega v_i = \lambda_i v_i$ as

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \end{pmatrix} = \sum_i c_i v_i, \quad (\text{B.4})$$

we obtain

$$\mu \frac{d}{d\mu} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \end{pmatrix} = \sum_i \mu \frac{dc_i}{d\mu} v_i = \omega \sum_i c_i v_i = \sum_i c_i \lambda_i v_i, \quad (\text{B.5})$$

where we assume that the eigenvectors $\{v_i\}$ have no dependence of renormalization scale. Therefore,

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \end{pmatrix} = \sum_i c_i(\Lambda) \left(\frac{\mu}{\Lambda}\right)^{\lambda_i} v_i. \quad (\text{B.6})$$

When all eigenvalues are positive,

$$\lim_{\mu \rightarrow 0} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \end{pmatrix} = 0. \quad (\text{B.7})$$

In this case, the fixed point is stable in the IR limit. On the other hand, when there are only negative eigenvalues,

$$\lim_{\mu \rightarrow \infty} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \end{pmatrix} = 0. \quad (\text{B.8})$$

Thus, the fixed point is UV fixed point. If there are both of positive and negative eigenvalues, there are both of attractive and repulsive directions at $\{g_{i,*}\}$. Therefore it is a saddle point.

C Dimensional regularization

In this appendix, we calculate diverging function

$$V(p^2; m^2, M^2) = \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + m^2} \frac{1}{(k+p)^2 + M^2} \quad (\text{C.1})$$

in $4 - \epsilon$ dimension. Using integration formula

$$\int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2 + \Delta^2} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(2 - d/2)}{\Gamma(2)} \left(\frac{1}{\Delta} \right)^{2-d/2}. \quad (\text{C.2})$$

Taking the dimension to $d = 4 - \epsilon$, and using $\hat{\Delta} \equiv \Delta/\mu^2$

$$\begin{aligned} \frac{\Gamma(2 - d/2)}{(4\pi)^{d/2}} \left(\frac{1}{\Delta} \right)^{2-d/2} &= \frac{\Gamma(\epsilon/2)}{(4\pi)^{2-\epsilon/2}} \mu^{-\epsilon} \hat{\Delta}^{-\epsilon/2} \\ &= \frac{\mu^{-\epsilon}}{(4\pi)^2} \Gamma(\epsilon/2) \left(\frac{\hat{\Delta}}{4\pi} \right)^{-\epsilon/2} \\ &= \frac{\mu^{-\epsilon}}{(4\pi)^2} \left(\frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon) \right) \left(1 - \frac{\epsilon}{2} \log \hat{\Delta} + \frac{\epsilon}{2} \log 4\pi \right) \\ &= \frac{\mu^{-\epsilon}}{(4\pi)^2} \left(\frac{2}{\epsilon} - \gamma + \log 4\pi - \log \hat{\Delta} + \mathcal{O}(\epsilon) \right). \end{aligned} \quad (\text{C.3})$$

Feynman parameter integrals

Using Feynman parameter integral identity, we deform eq. (C.1) to the formula which we can use eq. (C.3).

Same mass case

At first, we consider the case of $M = m$.

$$V(p^2; m, m) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + m^2} \frac{1}{(k+p)^2 + m^2}.$$

Using the identity

$$\frac{1}{AB} = \int_0^1 \frac{d\xi}{[\xi A + (1-\xi)B]^2}, \quad (\text{C.4})$$

$$\begin{aligned} \frac{1}{k^2 + m^2} \frac{1}{(k+p)^2 + m^2} &= \int_0^1 \frac{d\xi}{[k^2 + 2\xi k \cdot p + \xi p^2 + m^2]^2} \\ &= \int_0^1 \frac{d\xi}{[l^2 + \xi(1-\xi)p^2 + m^2]^2} \quad (l = k + \xi p). \end{aligned} \quad (\text{C.5})$$

Using eq. (C.3)

$$\begin{aligned} V(p^2; m, m) &= \frac{1}{2} \int_0^1 d\xi \int \frac{d^d l}{(2\pi)^d} \frac{1}{[l^2 + \xi(1-\xi)p^2 + m^2]^2} \\ &\xrightarrow{d=4-\epsilon} -\frac{\mu^{-\epsilon}}{32\pi^2} \int_0^1 d\xi \left(\frac{2}{\epsilon} - \gamma + \log 4\pi - \log[\{m^2 + \xi(1-\xi)p^2\}/\mu^2] + \mathcal{O}(\epsilon) \right). \end{aligned} \quad (\text{C.6})$$

Taking $m \rightarrow 0$, we obtain

$$V(p^2; 0, 0) = -\frac{\mu^{-\epsilon}}{32\pi^2} \int_0^1 d\xi \left(\frac{2}{\epsilon} - \gamma + \log 4\pi - \log[\xi(1-\xi)p^2/\mu^2] + \mathcal{O}(\epsilon) \right). \quad (\text{C.7})$$

One massive and one massless case

Next, we consider the case of $M = 0$, $m \neq 0$.

$$V(p^2; 0, m) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + m^2} \frac{1}{(k+p)^2}.$$

In this case

$$\begin{aligned} \frac{1}{k^2 + m^2} \frac{1}{(k+p)^2} &= \int_0^1 \frac{d\xi}{[k^2 + 2(1-\xi)k \cdot p + (1-\xi)p^2 + \xi m^2]^2} \\ &= \int_0^1 \frac{d\xi}{[l^2 + \xi(1-\xi)p^2 + \xi m^2]^2} \quad (l = k + \xi p). \end{aligned} \quad (\text{C.8})$$

Using eq. (C.3)

$$\begin{aligned}
V(p^2; 0, m) &= \frac{1}{2} \int_0^1 d\xi \int \frac{d^d l}{(2\pi)^d} \frac{1}{[l^2 + \xi(1-\xi)p^2 + \xi m^2]^2} \\
&\xrightarrow{d=4-\epsilon} -\frac{\mu^{-\epsilon}}{32\pi^2} \int_0^1 d\xi \left(\frac{2}{\epsilon} - \gamma + \log 4\pi - \log[\{\xi m^2 + \xi(1-\xi)p^2\}/\mu^2] + \mathcal{O}(\epsilon) \right).
\end{aligned} \tag{C.9}$$

D Correlation functions and renormalization

In this appendix, we calculate four point correlation functions in the leading order of the ϵ expansion.

At first, we rewrite the bare parameters to renormalized one as

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}(1+w_0)(\partial_\mu \phi_{0a})^2 + \frac{1}{2}m_{\phi 0}^2 \phi_{0a}^2 + \frac{1}{2}(1-w_0)(\partial_\mu \chi_{0a})^2 + \frac{1}{2}m_{\chi 0}^2 \chi_{0a}^2 \\
&\quad + \lambda_{10}(\phi_{0a}^2)^2 + \lambda_{20}(\chi_{0a}^2)^2 + \lambda_{30}\phi_{0a}^2 \chi_{0b}^2 + \lambda_{40}(\phi_{0a}\chi_{0a})^2 \\
&= \frac{1}{2}(1+w_R)(\partial_\mu \phi_{Ra})^2 + \frac{1}{2}m_{\phi R}^2 \phi_{Ra}^2 + \frac{1}{2}(1-w_R)(\partial_\mu \chi_{Ra})^2 + \frac{1}{2}m_{\chi R}^2 \chi_{Ra}^2 \\
&\quad + \lambda_{1R}(\phi_{Ra}^2)^2 + \lambda_{2R}(\chi_{Ra}^2)^2 + \lambda_{3R}\phi_{Ra}^2 \chi_{Rb}^2 + \lambda_{4R}(\phi_{Ra}\chi_{Ra})^2 \\
&\quad + \frac{1}{2}\delta_\phi(\partial_\mu \phi_{Ra})^2 + \frac{1}{2}\delta_{m_\phi} \phi_{Ra}^2 + \frac{1}{2}\delta_\chi(\partial_\mu \chi_{Ra})^2 + \frac{1}{2}\delta_{m_\chi} \chi_{Ra}^2 \\
&\quad + \delta_1(\phi_{Ra}^2)^2 + \delta_2(\chi_{Ra}^2)^2 + \delta_3\phi_{Ra}^2 \chi_{Rb}^2 + \delta_4(\phi_{Ra}\chi_{Ra})^2,
\end{aligned} \tag{D.1}$$

where

$$\Phi_0 = Z_\Phi^{1/2} \Phi_R, \quad \delta_\Phi = Z_\Phi - 1, \quad w_0 = Z_\Phi^{-1}(w_R + \delta_w), \tag{D.2}$$

$$\delta_\phi = \delta_\Phi + \delta_w, \quad \delta_\chi = \delta_\Phi - \delta_w, \tag{D.3}$$

$$m_{\phi 0}^2 = Z_\Phi^{-1}(m_{\phi R}^2 + \delta_{m_\phi}), \quad m_{\chi 0}^2 = Z_\Phi^{-1}(m_{\chi R}^2 + \delta_{m_\chi}), \tag{D.4}$$

$$\lambda_{i0} = Z_\Phi^{-2}(\lambda_{iR} + \delta_i). \tag{D.5}$$

We add the subscript 0 to the bare parameters and R to the renormalized one. Here after, we drop the subscript R for simplicity.

The amputated ϕ 's four point functions $G^{(4,0)}$ is, for example

$$\begin{aligned}
G_1^{(4,0)}(\phi_1(p_1), \phi_1(p_2), \phi_2(p_3), \phi_2(p_4))|_{amp} \\
&= \langle \phi_1(p_1) \phi_1(p_2) \phi_2(p_3) \phi_2(p_4) \rangle|_{amp} \\
&= -8\lambda_1 - 8\delta_1 \\
&\quad + 2^9 \lambda_1^2 V(s; 0, 0) + (2^5 \lambda_3 \lambda_4 + 2^6 \lambda_3^2) V(s; c_A, c_A) \\
&\quad + 2^7 \lambda_1^2 \{V(t; 0, 0) + V(u; 0, 0)\} + 2^3 \lambda_4^2 \{V(t; c_A, c_A) + V(u; c_A, c_A)\},
\end{aligned} \tag{D.6}$$

where we take $m^2(T_c) = c_A/2$, thus $m_\phi^2 = 0$, $m_\chi^2 = c_A$, and $s = (p_1+p_2)^2$, $t = (p_1+p_3)^2$, $u = (p_1+p_4)^2$. $G^{(n,M)}$ is a correlation functions of n point ϕ_a and m point χ_a . The loop factor

V in eq. (D.6) is defined as

$$V(p^2; m^2, M^2) = \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \frac{(-1)}{k^2 - m^2} \frac{(-1)}{(k+p)^2 - M^2}. \quad (\text{D.7})$$

Using dimensional regularization in $d = 4 - \epsilon$ ¹⁵, we obtain

$$V(p^2; 0, 0) = \frac{\mu^{-\epsilon}}{32\pi^2} \int_0^1 d\xi \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \log[\xi(1-\xi)p^2/\mu^2] + \mathcal{O}(\epsilon) \right), \quad (\text{D.8})$$

$$V(p^2; m, m) = \frac{\mu^{-\epsilon}}{32\pi^2} \int_0^1 d\xi \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \log[\{m^2 + \xi(1-\xi)p^2\}/\mu^2] + \mathcal{O}(\epsilon) \right), \quad (\text{D.9})$$

$$V(p^2; 0, m) = \frac{\mu^{-\epsilon}}{32\pi^2} \int_0^1 d\xi \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \log[\{\xi m^2 + \xi(1-\xi)p^2\}/\mu^2] + \mathcal{O}(\epsilon) \right). \quad (\text{D.10})$$

Substituting these to eq. (D.6), we obtain

$$\begin{aligned} G_1^{(4,0)}(\phi_1(p_1), \phi_1(p_2), \phi_2(p_3), \phi_2(p_4))|_{amp} \\ = -8\lambda_1 - 8\delta_1 \\ + \frac{\mu^{-\epsilon}}{32\pi^2} \left[(2^8 \cdot 3\lambda_1^2 + 2^6\lambda_3^2 + 2^5\lambda_3\lambda_4 + 2^4\lambda_4^2) \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) + \mathcal{O}(\epsilon) \right) \right. \\ \left. - \int_0^1 d\xi \left\{ 2^9\lambda_1^2 \log[\xi(1-\xi)s/\mu^2] \right. \right. \\ \left. \left. + 2^7\lambda_1^2 (\log[\xi(1-\xi)t/\mu^2] + \log[\xi(1-\xi)u/\mu^2]) \right. \right. \\ \left. \left. + (2^5\lambda_3\lambda_4 + 2^6\lambda_3^2) \log[\{c_A + \xi(1-\xi)s\}/\mu^2] \right. \right. \\ \left. \left. + 2^3\lambda_4^2 (\log[\{c_A + \xi(1-\xi)t\}/\mu^2] + \log[\{c_A + \xi(1-\xi)u\}/\mu^2]) \right\} \right] \quad (\text{D.11}) \end{aligned}$$

¹⁵See also C.

Similarly, other four point functions can be written as

$$\begin{aligned}
G_2^{(0,4)}(\chi_1(p_1), \chi_1(p_2), \chi_2(p_3), \chi_2(p_4))|_{amp} &= \langle \chi_1(p_1) \chi_1(p_2) \chi_2(p_3) \chi_2(p_4) \rangle|_{amp} \\
&= -8\lambda_2 - 8\delta_2 \\
&\quad + 2^9 \lambda_2^2 V(s; c_A, c_A) + (2^5 \lambda_3 \lambda_4 + 2^6 \lambda_3^2) V(s; 0, 0) \\
&\quad + 2^7 \lambda_2^2 \{V(t; c_A, c_A) + V(u; c_A, c_A)\} + 2^3 \lambda_4^2 \{V(t; 0, 0) + V(u; 0, 0)\}, \\
&= -8\lambda_2 - 8\delta_2 \\
&\quad + \frac{\mu^{-\epsilon}}{32\pi^2} \left[(2^8 \cdot 3\lambda_2^2 + 2^6 \lambda_3^2 + 2^5 \lambda_3 \lambda_4 + 2^4 \lambda_4^2) \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) + \mathcal{O}(\epsilon) \right) \right. \\
&\quad \quad - \int_0^1 d\xi \left\{ 2^9 \lambda_2^2 \log[\{c_A + \xi(1 - \xi)s\}/\mu^2] \right. \\
&\quad \quad \quad 2^7 \lambda_2^2 (\log[\{c_A + \xi(1 - \xi)t\}/\mu^2] + \log[\{c_A + \xi(1 - \xi)u\}/\mu^2]) \\
&\quad \quad \quad + (2^5 \lambda_3 \lambda_4 + 2^6 \lambda_3^2) \log[\xi(1 - \xi)s/\mu^2] \\
&\quad \quad \quad \left. \left. + 2^3 \lambda_4^2 (\log[\xi(1 - \xi)t/\mu^2] + \log[\xi(1 - \xi)u/\mu^2]) \right\} \right], \quad (D.12)
\end{aligned}$$

$$\begin{aligned}
G_3^{(2,2)}(\phi_1(p_1), \chi_2(p_2), \phi_1(p_3), \chi_2(p_4))|_{amp} &= \langle \phi_1(p_1) \chi_2(p_2) \phi_1(p_3) \chi_2(p_4) \rangle|_{amp} \\
&= -4\lambda_3 - 4\delta_3 \\
&\quad + (2^5 \lambda_3^2 + 2^3 \lambda_4^2) \{V(s; 0, c_A) + V(u; 0, c_A)\} \\
&\quad + (2^6 \cdot 3\lambda_1 \lambda_3 + 2^5 \lambda_1 \lambda_4) V(t; 0, 0) + (2^6 \cdot 3\lambda_2 \lambda_3 + 2^5 \lambda_2 \lambda_4) V(t; c_A, c_A), \\
&= -4\lambda_3 - 4\delta_3 \\
&\quad + \frac{\mu^{-\epsilon}}{32\pi^2} \left[\{2^5(\lambda_1 + \lambda_2)(6\lambda_3 + \lambda_4) + 2^6 \lambda_3^2 + 2^4 \lambda_4^2\} \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) + \mathcal{O}(\epsilon) \right) \right. \\
&\quad \quad - \int_0^1 d\xi \left\{ (2^5 \lambda_3^2 + 2^3 \lambda_4^2) \right. \\
&\quad \quad \quad \times (\log[\{\xi c_A + \xi(1 - \xi)s\}/\mu^2] + \log[\{\xi c_A + \xi(1 - \xi)u\}/\mu^2]) \\
&\quad \quad \quad + (2^6 \cdot 3\lambda_1 \lambda_3 + 2^5 \lambda_1 \lambda_4) \log[\xi(1 - \xi)t/\mu^2] \\
&\quad \quad \quad \left. \left. + (2^6 \cdot 3\lambda_2 \lambda_3 + 2^5 \lambda_2 \lambda_4) \log[\{c_A + \xi(1 - \xi)t\}/\mu^2] \right\} \right], \quad (D.13)
\end{aligned}$$

$$\begin{aligned}
G_4^{(2,2)}(\phi_1(p_1), \chi_1(p_2), \phi_2(p_3), \chi_2(p_4))|_{amp} &= \langle \phi_1(p_1) \chi_1(p_2) \phi_2(p_3) \chi_2(p_4) \rangle|_{amp} \\
&= -2\lambda_4 - 2\delta_4 \\
&\quad + 2^5 \lambda_3 \lambda_4 V(s; 0, c_A) + 2^5 \lambda_1 \lambda_4 V(t; 0, 0) + 2^5 \lambda_2 \lambda_4 V(t; c_A, c_A) \\
&\quad + (2^5 \lambda_3 \lambda_4 + 2^4 \cdot 3\lambda_4^2) iV(u; 0, c_A) \\
&= -2\lambda_4 - 2\delta_4 \\
&\quad + \frac{\mu^{-\epsilon}}{32\pi^2} \left[\left\{ 2^5 (\lambda_1 + \lambda_2) \lambda_4 + 2^6 \lambda_3 \lambda_4 + 2^4 \cdot 3\lambda_4^2 \right\} \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) + \mathcal{O}(\epsilon) \right) \right. \\
&\quad \left. - \int_0^1 d\xi \left\{ 2^5 \lambda_3 \lambda_4 \log[\{\xi c_A + \xi(1-\xi)s\}/\mu^2] \right. \right. \\
&\quad \quad \left. \left. + 2^5 \lambda_1 \lambda_4 \log[\xi(1-\xi)t/\mu^2] + 2^5 \lambda_2 \lambda_4 \log[\{c_A + \xi(1-\xi)t\}/\mu^2] \right. \right. \\
&\quad \quad \left. \left. + (2^5 \lambda_3 \lambda_4 + 2^4 \cdot 3\lambda_4^2) \log[\{\xi c_A + \xi(1-\xi)u\}/\mu^2] \right\} \right]. \quad (\text{D.14})
\end{aligned}$$

D.1 Renormalization scheme

Because of the diversion of the correlation functions which we obtained above, we need to regularize these divergences by the counterterms. We introduce \overline{MS} scheme and symmetric scheme.

\overline{MS} scheme

All four point functions have the factor $\frac{2}{\epsilon} - \gamma + \log(4\pi)$. In \overline{MS} scheme, we choose counterterm δ_i to cancel this typical diverging term. We take

$$\delta_1 = \frac{\mu^{-\epsilon}}{\pi^2} (3\lambda_1^2 + 2^{-2}\lambda_3^2 + 2^{-3}\lambda_3\lambda_4 + 2^{-4}\lambda_4^2) \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) \right), \quad (\text{D.15})$$

$$\delta_2 = \frac{\mu^{-\epsilon}}{\pi^2} (3\lambda_2^2 + 2^{-2}\lambda_3^2 + 2^{-3}\lambda_3\lambda_4 + 2^{-4}\lambda_4^2) \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) \right), \quad (\text{D.16})$$

$$\delta_3 = \frac{\mu^{-\epsilon}}{\pi^2} \left\{ 2^{-3}(\lambda_1 + \lambda_2)(6\lambda_3 + \lambda_4) + 2^{-2}\lambda_3^2 + 2^{-4}\lambda_4^2 \right\} \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) \right), \quad (\text{D.17})$$

$$\delta_4 = \frac{\mu^{-\epsilon}}{\pi^2} \left\{ 2^{-3}(\lambda_1 + \lambda_2)\lambda_4 + 2^{-2}\lambda_3\lambda_4 + 2^{-4} \cdot 3\lambda_4^2 \right\} \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) \right). \quad (\text{D.18})$$

Substituting these,

$$\begin{aligned}
& G_1^{(4)}(\phi_1(p_1), \phi_1(p_2), \phi_2(p_3), \phi_2(p_4))|_{amp} \\
&= -8\lambda_1 \\
&\quad - \frac{\mu^{-\epsilon}}{32\pi^2} \int_0^1 d\xi \left\{ 2^9 \lambda_1^2 \log[\xi(1-\xi)s/\mu^2] + 2^7 \lambda_1^2 (\log[\xi(1-\xi)t/\mu^2] + \log[\xi(1-\xi)u/\mu^2]) \right. \\
&\quad\quad + (2^5 \lambda_3 \lambda_4 + 2^6 \lambda_3^2) \log[\{c_A + \xi(1-\xi)s\}/\mu^2] \\
&\quad\quad \left. + 2^3 \lambda_4^2 (\log[\{c_A + \xi(1-\xi)t\}/\mu^2] + \log[\{c_A + \xi(1-\xi)u\}/\mu^2]) \right\}, \tag{D.19}
\end{aligned}$$

$$\begin{aligned}
& G_2^{(4)}(\chi_1(p_1), \chi_1(p_2), \chi_2(p_3), \chi_2(p_4))|_{amp} \\
&= -8\lambda_2 \\
&\quad - \frac{\mu^{-\epsilon}}{32\pi^2} \int_0^1 d\xi \left\{ 2^9 \lambda_2^2 \log[\{c_A + \xi(1-\xi)s\}/\mu^2] \right. \\
&\quad\quad + 2^7 \lambda_2^2 (\log[\{c_A + \xi(1-\xi)t\}/\mu^2] + \log[\{c_A + \xi(1-\xi)u\}/\mu^2]) \\
&\quad\quad + (2^5 \lambda_3 \lambda_4 + 2^6 \lambda_3^2) \log[\xi(1-\xi)s/\mu^2] \\
&\quad\quad \left. + 2^3 \lambda_4^2 (\log[\xi(1-\xi)t/\mu^2] + \log[\xi(1-\xi)u/\mu^2]) \right\}, \tag{D.20}
\end{aligned}$$

$$\begin{aligned}
& G_3^{(4)}(\phi_1(p_1), \chi_2(p_2), \phi_1(p_3), \chi_2(p_4))|_{amp} \\
&= -4\lambda_3 \\
&\quad - \frac{\mu^{-\epsilon}}{32\pi^2} \int_0^1 d\xi \left\{ (2^5 \lambda_3^2 + 2^3 \lambda_4^2) \right. \\
&\quad\quad \times (\log[\{xc_A + \xi(1-\xi)s\}/\mu^2] + \log[\{xc_A + \xi(1-\xi)u\}/\mu^2]) \\
&\quad\quad + (2^6 \cdot 3\lambda_1 \lambda_3 + 2^5 \lambda_1 \lambda_4) \log[\xi(1-\xi)t/\mu^2] \\
&\quad\quad \left. + (2^6 \cdot 3\lambda_2 \lambda_3 + 2^5 \lambda_2 \lambda_4) \log[\{c_A + \xi(1-\xi)t\}/\mu^2] \right\}, \tag{D.21}
\end{aligned}$$

$$\begin{aligned}
& G_4^{(4)}(\phi_1(p_1), \chi_1(p_2), \phi_2(p_3), \chi_2(p_4))|_{amp} \\
&= -2\lambda_4 \\
&\quad - \frac{\mu^{-\epsilon}}{32\pi^2} \int_0^1 d\xi \left\{ 2^5 \lambda_3 \lambda_4 \log[\{\xi c_A + \xi(1-\xi)s\}/\mu^2] \right. \\
&\quad\quad + 2^5 \lambda_1 \lambda_4 \log[\xi(1-\xi)t/\mu^2] + 2^5 \lambda_2 \lambda_4 \log[\{c_A + \xi(1-\xi)t\}/\mu^2] \\
&\quad\quad \left. + (2^5 \lambda_3 \lambda_4 + 2^4 \cdot 3\lambda_4^2) \log[\{\xi c_A + \xi(1-\xi)u\}/\mu^2] \right\}. \tag{D.22}
\end{aligned}$$

Symmetric scheme

With the symmetric RG condition (eq. (3.5-3.8)), we obtain,

$$\begin{aligned}
\delta_1 &= -2^5 \cdot 3\lambda_1^2 V(\mu^2; 0, 0) - (2^3\lambda_3^2 + 2^2\lambda_3\lambda_4 + 2\lambda_4^2) V(\mu^2; c_A, c_A) \\
&= -\frac{\mu^{-\epsilon}}{\pi^2} \left[(3\lambda_1^2 + 2^{-2}\lambda_3^2 + 2^{-3}\lambda_3\lambda_4 + 2^{-4}\lambda_4^2) \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) + \mathcal{O}(\epsilon) \right) \right. \\
&\quad \left. - \int_0^1 d\xi \left\{ 3\lambda_1^2 \log[\xi(1-\xi)] \right. \right. \\
&\quad \left. \left. + 2^{-4}(2^2\lambda_3^2 + 2\lambda_3\lambda_4 + \lambda_4^2) \log \left[\frac{c_A}{\mu^2} + \xi(1-\xi) \right] \right\} \right], \quad (D.23)
\end{aligned}$$

$$\begin{aligned}
\delta_2 &= -2^5 \cdot 3\lambda_2^2 V(\mu^2; c_A, c_A) - (2^3\lambda_3^2 + 2^2\lambda_3\lambda_4 + 2\lambda_4^2) V(\mu^2; 0, 0) \\
&= -\frac{\mu^{-\epsilon}}{\pi^2} \left[(3\lambda_2^2 + 2^{-2}\lambda_3^2 + 2^{-3}\lambda_3\lambda_4 + 2^{-4}\lambda_4^2) \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) + \mathcal{O}(\epsilon) \right) \right. \\
&\quad \left. - \int_0^1 d\xi \left\{ 3\lambda_2^2 \log \left[\frac{c_A}{\mu^2} + \xi(1-\xi) \right] \right. \right. \\
&\quad \left. \left. + 2^{-4}(2^2\lambda_3^2 + 2\lambda_3\lambda_4 + \lambda_4^2) \log[\xi(1-\xi)] \right\} \right], \quad (D.24)
\end{aligned}$$

$$\begin{aligned}
\delta_3 &= -(2^4 \cdot 3\lambda_1\lambda_3 + 2^3\lambda_1\lambda_4) V(\mu^2; 0, 0) \\
&\quad - (2^4 \cdot 3\lambda_2\lambda_3 + 2^3\lambda_2\lambda_4) V(\mu^2; c_A, c_A) - (2^4\lambda_3^2 + 2^2\lambda_4^2) V(\mu^2; 0, c_A) \\
&= -\frac{\mu^{-\epsilon}}{\pi^2} \left[\{2^{-2}(\lambda_1 + \lambda_2)(6\lambda_3 + \lambda_4) + 2^{-1}\lambda_3^2 + 2^{-3}\lambda_4^2\} \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) + \mathcal{O}(\epsilon) \right) \right. \\
&\quad \left. - \int_0^1 d\xi \left\{ 2^{-2}\lambda_1(6\lambda_3 + \lambda_4) \log[\xi(1-\xi)] + 2^{-2}\lambda_2(6\lambda_3 + \lambda_4) \log \left[\frac{c_A}{\mu^2} + \xi(1-\xi) \right] \right. \right. \\
&\quad \left. \left. + 2^{-3}(2^2\lambda_3^2 + \lambda_4^2) \log \left[x \frac{c_A}{\mu^2} + \xi(1-\xi) \right] \right\} \right], \quad (D.25)
\end{aligned}$$

$$\begin{aligned}
\delta_4 &= -2^4 \lambda_1 \lambda_4 V(\mu^2; 0, 0) - 2^4 \lambda_2 \lambda_4 V(\mu^2; c_A, c_A) \\
&\quad - (2^5 \lambda_3 \lambda_4 + 2^3 \cdot 3 \lambda_4^2) V(\mu^2; 0, c_A) \\
&= -\frac{\mu^{-\epsilon}}{\pi^2} \left[\left\{ 2^{-1} (\lambda_1 + \lambda_2) \lambda_4 + \lambda_3 \lambda_4 + 2^{-2} \cdot 3 \lambda_4^2 \right\} \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) + \mathcal{O}(\epsilon) \right) \right. \\
&\quad \left. - \int_0^1 d\xi \left\{ 2^{-1} \lambda_1 \lambda_4 \log[\xi(1-\xi)] + 2^{-1} \lambda_2 \lambda_4 \log \left[\frac{c_A}{\mu^2} + \xi(1-\xi) \right] \right. \right. \\
&\quad \left. \left. + (\lambda_3 \lambda_4 + 2^{-2} \cdot 3 \lambda_4^2) \log \left[x \frac{c_A}{\mu^2} + \xi(1-\xi) \right] \right\} \right]. \tag{D.26}
\end{aligned}$$

Substituting these,

$$\begin{aligned}
&G_1^{(4)}(\phi_1(p_1), \phi_1(p_2), \phi_2(p_3), \phi_2(p_4))|_{amp} \\
&= -8\lambda_1 \\
&\quad - \frac{\mu^{-\epsilon}}{\pi^2} \int_0^1 d\xi \left\{ 2^4 \lambda_1^2 \log[s/\mu^2] + 2^2 \lambda_1^2 (\log[t/\mu^2] + \log[u/\mu^2]) \right. \\
&\quad \quad + (\lambda_3 \lambda_4 + 2\lambda_3^2) \log[\{c_A + \xi(1-\xi)s\}/\{c_A + \xi(1-\xi)\mu^2\}] \\
&\quad \quad + 2^{-2} \lambda_4^2 (\log[\{c_A + \xi(1-\xi)t\}/\{c_A + \xi(1-\xi)\mu^2\}] \\
&\quad \quad \left. \left. + \log[\{c_A + \xi(1-\xi)u\}/\{c_A + \xi(1-\xi)\mu^2\}]) \right\}, \tag{D.27}
\end{aligned}$$

$$\begin{aligned}
&G_2^{(4)}(\chi_1(p_1), \chi_1(p_2), \chi_2(p_3), \chi_2(p_4))|_{amp} \\
&= -8\lambda_2 \\
&\quad - \frac{\mu^{-\epsilon}}{\pi^2} \int_0^1 d\xi \left\{ 2^4 \lambda_2^2 \log[\{c_A + \xi(1-\xi)s\}/\{c_A + \xi(1-\xi)\mu^2\}] \right. \\
&\quad \quad + 2^3 \lambda_2^2 (\log[\{c_A + \xi(1-\xi)t\}/\{c_A + \xi(1-\xi)\mu^2\}] \\
&\quad \quad \quad \left. + \log[\{c_A + \xi(1-\xi)u\}/\{c_A + \xi(1-\xi)\mu^2\}]) \right. \\
&\quad \quad \left. + (\lambda_3 \lambda_4 + 2\lambda_3^2) \log[s/\mu^2] + 2^{-2} \lambda_4^2 (\log[t/\mu^2] + \log[u/\mu^2]) \right\}, \tag{D.28}
\end{aligned}$$

$$\begin{aligned}
& G_3^{(4)}(\phi_1(p_1), \chi_2(p_2), \phi_1(p_3), \chi_2(p_4))|_{amp} \\
&= -4\lambda_3 \\
&\quad - \frac{\mu^{-\epsilon}}{\pi^2} \int_0^1 d\xi \left\{ (\lambda_3^2 + 2^{-2}\lambda_4^2) (\log[\{\xi c_A + \xi(1-\xi)s\}/\{\xi c_A + \xi(1-\xi)\mu^2\}] \right. \\
&\quad\quad\quad + \log[\{\xi c_A + \xi(1-\xi)u\}/\{\xi c_A + \xi(1-\xi)\mu^2\}]) \\
&\quad\quad\quad + \lambda_1(6\lambda_3 + \lambda_4) \log[t/\mu^2] \\
&\quad\quad\quad \left. + \lambda_2(6\lambda_3 + \lambda_4) \log[\{c_A + \xi(1-\xi)t\}/\{c_A + \xi(1-\xi)\mu^2\}] \right\}, \tag{D.29}
\end{aligned}$$

$$\begin{aligned}
& G_4^{(4)}(\phi_1(p_1), \chi_1(p_2), \phi_2(p_3), \chi_2(p_4))|_{amp} \\
&= -2\lambda_4 \\
&\quad - \frac{\mu^{-\epsilon}}{\pi^2} \int_0^1 d\xi \left\{ \lambda_3\lambda_4 \log[\{\xi c_A + \xi(1-\xi)s\}/\{\xi c_A + \xi(1-\xi)\mu^2\}] + \lambda_1\lambda_4 \log[t/\mu^2] \right. \\
&\quad\quad\quad + \lambda_2\lambda_4 \log[\{c_A + \xi(1-\xi)t\}/\{c_A + \xi(1-\xi)\mu^2\}] \\
&\quad\quad\quad + (\lambda_3\lambda_4 + 2^{-1} \cdot 3\lambda_4^2) \\
&\quad\quad\quad \left. \times \log[\{\xi c_A + \xi(1-\xi)u\}/\{\xi c_A + \xi(1-\xi)\mu^2\}] \right\}. \tag{D.30}
\end{aligned}$$

E β functions

When one changes renormalization scale μ to $\mu + \delta\mu$, each parameter of Lagrangian becomes to¹⁶

$$\lambda_i \rightarrow \lambda_i + \delta\lambda_i, \quad \Phi \rightarrow (1 + \delta\Phi)\Phi. \tag{E.1}$$

Under this transformation, n-point function $G^{(n)} = \langle \Phi\Phi\dots\Phi \rangle$ becomes to

$$G^{(n)} \rightarrow (1 + n\delta\Phi)G^{(n)}. \tag{E.2}$$

So,

$$dG^{(n)} = \frac{\partial G^{(n)}}{\partial \mu} \delta\mu + \sum_i \frac{\partial G^{(n)}}{\partial \lambda_i} \delta\lambda_i = n\delta\Phi G^{(n)}. \tag{E.3}$$

Therefore

$$\left[\mu \frac{\partial}{\partial \mu} + \sum_i \beta'_i \frac{\partial}{\partial \lambda_i} + n\gamma \right] G^{(n)}(\mu, \lambda_i) = 0. \tag{E.4}$$

¹⁶We are now ignoring the running of masses. It will be argued later.

Where

$$\beta'_i \equiv \mu \frac{\delta \lambda_i}{\delta \mu}, \quad \gamma \equiv -\mu \frac{\delta \Phi}{\delta \mu}. \quad (\text{E.5})$$

Prime of β'_i means these β functions are derivation of couplings which has mass dimension ϵ in $d = 4 - \epsilon$. We use primeless β_i for derivation of dimensionless couplings $\hat{\lambda}_1 \equiv \lambda_i/\mu^\epsilon$ below.

The four point functions are described by eq. (D.19-D.22) for \overline{MS} scheme, and eq. (D.27-D.30) for symmetric scheme. Assuming that $\beta_i \sim \mathcal{O}(\epsilon^2)$, $\gamma \sim \mathcal{O}(\epsilon^2)$ ¹⁷, renormalization equation for $G_1^{(4,0)}$ is

$$-8\beta'_1 = -\mu \frac{\partial}{\partial \mu} G_1^{(4,0)} + \mathcal{O}(\epsilon^3). \quad (\text{E.6})$$

And,

$$\beta'_1 = \mu \frac{\partial}{\partial \mu} \mu^\epsilon \hat{\lambda}_1 = \epsilon \mu^\epsilon \hat{\lambda}_1 + \mu^\epsilon \beta_1. \quad (\text{E.7})$$

So,

$$8\mu^\epsilon (\epsilon \hat{\lambda}_1 + \beta_1) = \mu \frac{\partial}{\partial \mu} G_1^{(4,0)} + \mathcal{O}(\epsilon^3). \quad (\text{E.8})$$

Similarly, we get

$$c_i \mu^\epsilon (\epsilon \hat{\lambda}_i + \beta_i) = \mu \frac{\partial}{\partial \mu} G_i^{(n,4-n)} + \mathcal{O}(\epsilon^3), \quad (\text{E.9})$$

where $c_1 = c_2 = 8$, $c_3 = 4$, $c_4 = 2$, and $n = 4$ for $i = 1$, $n = 0$ for $i = 2$, $n = 2$ for $i = 3, 4$. Since the lowest order of a four-point function is $\mathcal{O}(\epsilon)$, next to leading contributions are $\mathcal{O}(\epsilon^3)$.

Scheme dependence

In this scheme, right hand side of eq. (E.9) (i=1) is

$$\begin{aligned} & \mu \frac{\partial}{\partial \mu} G_1^{(4,0)} \\ &= \frac{\mu^{-\epsilon}}{\pi^2} \mu \frac{\partial}{\partial \mu} \int_0^1 d\xi \left\{ 2^3 \cdot 3\lambda_1^2 \log \mu^2 + 2^{-1} (4\lambda_3^2 + 2\lambda_3\lambda_4 + \lambda_4^2) \log [c_A + \xi(1-\xi)\mu^2] \right\} + \mathcal{O}(\epsilon^2) \\ &= \mu^{-\epsilon} 8 \frac{3}{\pi^2} \left\{ 2\lambda_1^2 + \frac{1}{24} f(\hat{\mu}) (4\lambda_3^2 + 2\lambda_3\lambda_4 + \lambda_4^2) \right\} + \mathcal{O}(\epsilon^2). \end{aligned} \quad (\text{E.10})$$

¹⁷We set $\epsilon \sim \mathcal{O}(\lambda)$ in order counting.

Where $\hat{\mu} = \mu/\sqrt{c_A}$, and we drop $\mathcal{O}(\epsilon^3)$. So,

$$\beta_1^{sym} = -\epsilon\hat{\lambda}_1 + \frac{3}{\pi^2} \left\{ 2\hat{\lambda}_1^2 + \frac{1}{24}f(\hat{\mu})(4\hat{\lambda}_3^2 + 2\hat{\lambda}_3\hat{\lambda}_4 + \hat{\lambda}_4^2) \right\}. \quad (\text{E.11})$$

From this, we obtain β function of $\hat{\lambda} = \lambda/\mu^\epsilon$ as

$$\beta_\lambda^{sym} = \left(\frac{\pi}{3}\right)^{-1} \beta_1 = -\epsilon\hat{\lambda} + 2\hat{\lambda}^2 + \frac{1}{6}f(\hat{\mu}) \left(4\hat{\lambda}^2 + 6\hat{\lambda}\hat{g}_2 + 3\hat{g}_2^2 - 8\hat{\lambda}\hat{z} - 6\hat{g}_2\hat{z} + 4\hat{z}^2 \right). \quad (\text{E.12})$$

Similarly

$$\beta_{g_2}^{sym} = -\epsilon\hat{g}_2 + \frac{1}{3}\hat{\lambda}\hat{g}_2 + \frac{1}{3}f(\hat{\mu})\hat{g}_2 \left(\hat{\lambda} - 2\hat{x} \right) + \frac{1}{3}h(\hat{\mu})\hat{g}_2 \left(4\hat{\lambda} + \hat{g}_2 - 4\hat{z} \right), \quad (\text{E.13})$$

$$\begin{aligned} \beta_{\hat{x}}^{sym} &= -\epsilon\hat{x} + 4f(\hat{\mu}) \left(\hat{\lambda}\hat{x} - \hat{x}^2 \right) \\ &+ \frac{1}{12} (1 - f(\hat{\mu})) \left(8\hat{\lambda}^2 - 6\hat{\lambda}\hat{g}_2 - 3\hat{g}_2^2 + 8\hat{\lambda}\hat{z} + 6\hat{g}_2\hat{z} - 4\hat{z}^2 \right), \end{aligned} \quad (\text{E.14})$$

$$\begin{aligned} \beta_{\hat{z}}^{sym} &= -\epsilon\hat{z} + \frac{1}{2} \left(2\hat{\lambda}^2 - \hat{\lambda}\hat{g}_2 + 2\hat{\lambda}\hat{z} \right) - \frac{1}{6}h(\hat{\mu}) \left(4\hat{\lambda}^2 + 3\hat{g}_2^2 - 8\hat{\lambda}\hat{z} + 4\hat{z}^2 \right) \\ &+ \frac{1}{6}f(\hat{\mu}) \left(-2\hat{\lambda}^2 + 3\hat{\lambda}\hat{g}_2 + 3\hat{g}_2^2 - 2\hat{\lambda}\hat{z} - 6\hat{g}_2\hat{z} + 12\hat{\lambda}\hat{x} + 6\hat{g}_2\hat{x} - 12\hat{x}\hat{z} + 4\hat{z}^2 \right) \end{aligned} \quad (\text{E.15})$$

On the other hands, using the correlation functions in \overline{MS} scheme eq. (D.19-D.22)

$$\beta_\lambda^{\overline{MS}} = -\epsilon\hat{\lambda} + \frac{8}{3}\hat{\lambda}^2 + \hat{\lambda}\hat{g}_2 + \frac{1}{2}\hat{g}_2^2 - \frac{4}{3}\hat{\lambda}\hat{z} - \hat{g}_2\hat{z} + \frac{2}{3}\hat{z}^2, \quad (\text{E.16})$$

$$\beta_{g_2}^{\overline{MS}} = -\epsilon\hat{g}_2 + 2\hat{\lambda}\hat{g}_2 + \frac{1}{3}\hat{g}_2^2 - \frac{2}{3}\hat{g}_2\hat{x} - \frac{4}{3}\hat{g}_2\hat{z}, \quad (\text{E.17})$$

$$\beta_x^{\overline{MS}} = -\epsilon\hat{x} + 4\hat{\lambda}\hat{x} - 4\hat{x}^2, \quad (\text{E.18})$$

$$\beta_z^{\overline{MS}} = -\epsilon\hat{z} + 2\hat{\lambda}\hat{x} + 2\hat{\lambda}\hat{z} + \hat{g}_2\hat{x} - \hat{g}_2\hat{z} - 2\hat{x}\hat{z}. \quad (\text{E.19})$$

F Mass renormalization

In the main discussion, we dealt with the mass of the massive field $\sqrt{c_A}$ as a constant which is independent of the renormalization scale, because we take the on-shell scheme. In this appendix, we calculate the RG equation of the mass, and perform the independence of the mass in this scheme.

In the leading order of the ϵ expansion, we obtain

$$G^{(2,0)}(p) = \langle \phi_a(p)\phi_b(-p) \rangle = \frac{-\delta_{ab}}{p^2 + m_\phi^2 + \mathcal{M}_\phi^2(p^2)}, \quad (\text{F.1})$$

$$\begin{aligned} \mathcal{M}_\phi^2 = & 2^3 \cdot 3\lambda_1 \int \frac{d^d k}{(2\pi)^d} \frac{(-1)}{k^2 + m_\phi^2} \\ & + (2^3\lambda_3 + 2\lambda_4) \int \frac{d^d k}{(2\pi)^d} \frac{(-1)}{k^2 + m_\chi^2} + \delta_{m_\phi} + p^2\delta_\phi, \end{aligned} \quad (\text{F.2})$$

and

$$G^{(0,2)}(p) = \langle \chi_a(p)\chi_b(-p) \rangle = \frac{-\delta_{ab}}{p^2 + m_\chi^2 + \mathcal{M}_\chi^2(p^2)}, \quad (\text{F.3})$$

$$\begin{aligned} \mathcal{M}_\chi^2 = & (2^3\lambda_3 + 2\lambda_4) \int \frac{d^d k}{(2\pi)^d} \frac{(-1)}{k^2 + m_\phi^2} \\ & + 2^3 \cdot 3\lambda_2 \int \frac{d^d k}{(2\pi)^d} \frac{(-1)}{k^2 + m_\chi^2} + \delta_{m_\chi} + p^2\delta_\chi, \end{aligned} \quad (\text{F.4})$$

where δ_{m_ϕ} , δ_{m_χ} are the counter term of the each mass, and δ_ϕ , δ_χ are the counter term of the each wave function. In the end of this analysis, we take $m_\phi^2 = 0$ and $m_\chi^2 = c_A$.

In the on-shell scheme, we take the renormalization conditions as

$$\mathcal{M}_\phi^2(p^2 = m_\phi^2) = 0, \quad \mathcal{M}_\chi^2(p^2 = m_\chi^2) = 0, \quad (\text{F.5})$$

$$\left. \frac{d}{dp^2} \mathcal{M}_\phi^2 \right|_{p^2=m_\phi^2} = 0, \quad \left. \frac{d}{dp^2} \mathcal{M}_\chi^2 \right|_{p^2=m_\chi^2} = 0. \quad (\text{F.6})$$

Thus, we take the counter terms as

$$\delta_{m_\phi} = -2^3 \cdot 3\lambda_1 \int \frac{d^d k}{(2\pi)^d} \frac{(-1)}{k^2 + m_\phi^2} - (2^3\lambda_3 + 2\lambda_4) \int \frac{d^d k}{(2\pi)^d} \frac{(-1)}{k^2 + m_\chi^2}, \quad (\text{F.7})$$

$$\delta_{m_\chi} = - (2^3\lambda_3 + 2\lambda_4) \int \frac{d^d k}{(2\pi)^d} \frac{(-1)}{k^2 + m_\phi^2} - 2^3 \cdot 3\lambda_2 \int \frac{d^d k}{(2\pi)^d} \frac{(-1)}{k^2 + m_\chi^2}. \quad (\text{F.8})$$

Because the correction terms (eq. (F.2) and eq. (F.4)) have no dependence to the external momentum p , we obtain $\delta_\phi = \delta_\chi = 0$. Therefore, the wave functions do not affect the renormalization in the leading order of the ϵ expansion. Furthermore, once we take the RG condition eq. (F.5), $\mathcal{M}_{\phi(\chi)}^2(p^2) = 0$ for arbitrary p .

Taking the limit of $m_\phi \rightarrow 0$, and factorizing m_χ by renormalization scale μ and dimensionless constant ρ_χ as $m_\chi^2 = \rho_\chi \mu^2$, we obtain the mass-dependent renormalization equation as

$$\left[\mu \frac{\partial}{\partial \mu} + \sum_i \beta_i \frac{\partial}{\partial \hat{\lambda}_i} + \left(\mu \frac{d\rho_\chi}{d\mu} \right) \frac{\partial}{\partial \rho_\chi} + n\gamma_\phi + m\gamma_\chi \right] G^{(n,m)}(\mu, \lambda_i) = 0. \quad (\text{F.9})$$

Using eq.(F.1), we obtain

$$\begin{aligned}\mu \frac{\partial}{\partial \mu} G^{(0,2)} &= \frac{1}{(p^2 + \rho_\chi \mu^2 + \mathcal{M}_\chi^2(p^2))^2} \mu \frac{\partial}{\partial \mu} (\rho_\chi \mu^2 + \mathcal{M}_\chi^2(p^2)) \\ &= \frac{1}{(p^2 + \rho_\chi \mu^2 + \mathcal{M}_\chi^2(p^2))^2} 2\rho_\chi \mu^2,\end{aligned}\tag{F.10}$$

and

$$\begin{aligned}\frac{\partial}{\partial \rho_\chi} G^{(0,2)} &= \frac{1}{(p^2 + \rho_\chi \mu^2 + \mathcal{M}_\chi^2(p^2))^2} \frac{\partial}{\partial \rho_\chi} (\rho_\chi \mu^2 + \mathcal{M}_\chi^2(p^2)) \\ &= \frac{1}{(p^2 + \rho_\chi \mu^2 + \mathcal{M}_\chi^2(p^2))^2} \mu^2.\end{aligned}\tag{F.11}$$

Because the wave functions have no correction in this order, $\gamma_\phi, \gamma_\chi = \mathcal{O}(\epsilon^2)$, and $\beta_i = \mathcal{O}(\epsilon^2)$, we obtain

$$\beta_{\rho_\chi} \equiv \mu \frac{d\rho_\chi}{d\mu} = -2\rho_\chi + \mathcal{O}(\epsilon^2).\tag{F.12}$$

This equation is solved as

$$\rho_\chi(\mu) = \rho_\chi(\Lambda) \left(\frac{\Lambda}{\mu} \right)^2,\tag{F.13}$$

where Λ is initial value of the renormalization scale μ . Therefore,

$$m_\chi^2 = \rho_\chi(\mu) \mu^2 = \rho_\chi(\Lambda) \Lambda^2.\tag{F.14}$$

This means that $m_\chi^2 = c_A$ is independent of renormalization scale in the on-shell scheme.

G RG equation of external momentum

G.1 Symmetric scheme

From eq. (D.27), setting $s = t = u = P^2$,

$$\begin{aligned}g_1^{(4,0)}(P/\mu, \hat{\lambda}_i, \rho) &= -8\hat{\lambda}_1 \\ &\quad - \frac{1}{\pi^2} \int_0^1 d\xi \left\{ 2^4 \hat{\lambda}_1^2 \log[P^2/\mu^2] + 2^2 \hat{\lambda}_1^2 (\log[P^2/\mu^2] + \log[P^2/\mu^2]) \right. \\ &\quad \quad \quad + (\hat{\lambda}_3 \hat{\lambda}_4 + 2\hat{\lambda}_3^2) \log[\{\rho + \xi(1 - \xi)P^2/\mu^2\}/\{\rho + \xi(1 - \xi)\}] \\ &\quad \quad \quad + 2^{-2} \hat{\lambda}_4^2 (\log[\{\rho + \xi(1 - \xi)P^2/\mu^2\}/\{\rho + \xi(1 - \xi)\}] \\ &\quad \quad \quad \quad \quad \quad \left. + \log[\{\rho + \xi(1 - \xi)P^2/\mu^2\}/\{\rho + \xi(1 - \xi)\}]) \right\}.\end{aligned}\tag{G.1}$$

The initial condition for $\mathcal{G}_1^{(4,0)}$ is obtained as

$$\mathcal{G}_1^{(4,0)}(\bar{\lambda}_i = \hat{\lambda}_i, \bar{\rho} = \rho) = g_1^{(4,0)}(P/\mu = 1, \hat{\lambda}_i, \rho) = -\frac{8}{3}\pi^2 \hat{\lambda}. \quad (\text{G.2})$$

Thus we obtain

$$\mathcal{G}_1^{(4,0)}(\bar{\lambda}_i, \bar{\rho}) = -\frac{8}{3}\pi^2 \bar{\lambda}(P). \quad (\text{G.3})$$

In the limit of $P \rightarrow 0$,

$$\bar{\lambda}(P \rightarrow 0) = \frac{\epsilon}{2} + c \left(\frac{P}{\mu}\right)^{2-\frac{5}{3}\epsilon} + \mathcal{O}(P), \quad (\text{G.4})$$

with the constant c . Thus,

$$\mathcal{G}_1^{(4,0)}(P \rightarrow 0) = -\frac{8}{3}\pi^2 \left\{ \frac{\epsilon}{2} + c \left(\frac{P}{\mu}\right)^{2-\frac{5}{3}\epsilon} \right\} + \mathcal{O}(P). \quad (\text{G.5})$$

G.1.1 Correlation function at asymmetric point

In the analysis above, we fixed external momenta as the symmetric point $s = t = u = P^2$. Next, we calculate the difference appearing at asymmetric point. At $s = c_t^{-1}t = c_u^{-1}u = P^2$,

$$\begin{aligned} g_1^{(4,0)}(P) = & -8\hat{\lambda}_1 \\ & -\frac{1}{\pi^2} \int_0^1 d\xi \left\{ 2^3 \cdot 3\hat{\lambda}_1^2 \log[P^2/\mu^2] + 2^2\hat{\lambda}_1^2(\log c_t + \log c_u) \right. \\ & + (\hat{\lambda}_3\hat{\lambda}_4 + 2\hat{\lambda}_3^2) \log [\{\rho + \xi(1-\xi)P^2/\mu^2\}/\{\rho + \xi(1-\xi)\}] \\ & + 2^{-2}\hat{\lambda}_4^2 (\log [\{\rho + \xi(1-\xi)c_t P^2/\mu^2\}/\{\rho + \xi(1-\xi)\}]) \\ & \left. + \log [\{\rho + \xi(1-\xi)c_u P^2/\mu^2\}/\{\rho + \xi(1-\xi)\}] \right\}. \quad (\text{G.6}) \end{aligned}$$

$$\begin{aligned} \mathcal{G}_1^{(4,0)}(\{\hat{\lambda}_i\}, \rho) = & g_1^{(4,0)}(P = \mu) \\ = & -8\hat{\lambda}_1 - \frac{2^2}{\pi^2} \hat{\lambda}_1^2 (\log c_t + \log c_u) \\ & - \frac{2^{-2}}{\pi^2} \hat{\lambda}_4^2 \int_0^1 d\xi \left(\log [\{\rho + \xi(1-\xi)c_t\}/\{\rho + \xi(1-\xi)\}] \right. \\ & \left. + \log [\{\rho + \xi(1-\xi)c_u\}/\{\rho + \xi(1-\xi)\}] \right). \quad (\text{G.7}) \end{aligned}$$

Hence,

$$\begin{aligned} \mathcal{G}_1^{(4,0)}(\{\bar{\lambda}_i\}, \bar{\rho}) &= -8\bar{\lambda}_1 - \frac{2^2}{\pi^2} \bar{\lambda}_1^2 (\log c_t + \log c_u) \\ &\quad - \frac{2^{-2}}{\pi^2} \bar{\lambda}_4^2 \int_0^1 d\xi \left(\log \left[\frac{\{\rho + \xi(1-\xi)c_t\}}{\{\rho + \xi(1-\xi)\}} \right] \right. \\ &\quad \left. + \log \left[\frac{\{\rho + \xi(1-\xi)c_u\}}{\{\rho + \xi(1-\xi)\}} \right] \right). \end{aligned} \quad (\text{G.8})$$

In the $P \rightarrow 0$ limit, $\bar{\rho}$ diverges as $\bar{\rho} = \rho(P/\mu)^{-2}$. Therefore,

$$\lim_{P \rightarrow 0} \int_0^1 d\xi \log[\bar{\rho} + \xi(1-\xi)c] = \log \bar{\rho} + \int_0^1 d\xi \xi(1-\xi) \frac{c}{\bar{\rho}} = \log \bar{\rho} + \frac{1}{6} \frac{c}{\bar{\rho}}. \quad (\text{G.9})$$

And,

$$\lim_{P \rightarrow 0} \mathcal{G}_1^{(4,0)}(\{\bar{\lambda}_i\}, \bar{\rho}) = -8\bar{\lambda}_1 - \frac{2^2}{\pi^2} \bar{\lambda}_1^2 (\log c_t + \log c_u) - \frac{2^{-2}}{6\pi^2} \frac{\bar{\lambda}_4^2}{\bar{\rho}} (c_u + c_t - 2). \quad (\text{G.10})$$

Because $\bar{\lambda}_4 = -\frac{2}{3}\pi^2 g_2 \sim P^{-\frac{5}{6}\epsilon}$ and $\bar{\rho} \sim P^{-2}$, the last term decreases in $P^{2-\frac{5}{3}\epsilon}$ as the approaching of the other terms.

G.1.2 Other channels

Temporarily, we describe $\langle \phi_1(p_1) \phi_1(p_2) \phi_2(p_3) \phi_2(p_4) \rangle = G_s^{(4,0)}$. Similarly, $\langle \phi_1(p_1) \phi_2(p_2) \phi_1(p_3) \phi_2(p_4) \rangle = G_t^{(4,0)}$ and $\langle \phi_1(p_1) \phi_2(p_2) \phi_2(p_3) \phi_1(p_4) \rangle = G_u^{(4,0)}$. In the case of $c_s^{-1}s = c_t^{-1}t = c_u^{-1}u = P^2$,

$$\begin{aligned} \mathcal{G}_s^{(4,0)} &= -8\bar{\lambda}_1 - \frac{\bar{\lambda}_1^2}{\pi^2} (2^4 \log c_s + 2^2 \log c_t + 2^2 \log c_u) \\ &\quad - \frac{1}{\pi^2} \int_0^1 d\xi \left\{ (\bar{\lambda}_3 \bar{\lambda}_4 + 2\bar{\lambda}_3^2) \log \left[\frac{\{\rho + \xi(1-\xi)c_s\}}{\{\rho + \xi(1-\xi)\}} \right] \right. \\ &\quad \quad + 2^{-2} \bar{\lambda}_4^2 (\log \left[\frac{\{\rho + \xi(1-\xi)c_t\}}{\{\rho + \xi(1-\xi)\}} \right] \\ &\quad \quad \left. + \log \left[\frac{\{\rho + \xi(1-\xi)c_u\}}{\{\rho + \xi(1-\xi)\}} \right]) \right\}, \end{aligned} \quad (\text{G.11})$$

$$\begin{aligned} \mathcal{G}_t^{(4,0)} &= -8\bar{\lambda}_1 - \frac{\bar{\lambda}_1^2}{\pi^2} (2^4 \log c_t + 2^2 \log c_u + 2^2 \log c_s) \\ &\quad - \frac{1}{\pi^2} \int_0^1 d\xi \left\{ (\bar{\lambda}_3 \bar{\lambda}_4 + 2\bar{\lambda}_3^2) \log \left[\frac{\{\rho + \xi(1-\xi)c_t\}}{\{\rho + \xi(1-\xi)\}} \right] \right. \\ &\quad \quad + 2^{-2} \bar{\lambda}_4^2 (\log \left[\frac{\{\rho + \xi(1-\xi)c_u\}}{\{\rho + \xi(1-\xi)\}} \right] \\ &\quad \quad \left. + \log \left[\frac{\{\rho + \xi(1-\xi)c_s\}}{\{\rho + \xi(1-\xi)\}} \right]) \right\}, \end{aligned} \quad (\text{G.12})$$

$$\begin{aligned}
\mathcal{G}_u^{(4,0)} = & -8\bar{\lambda}_1 - \frac{\bar{\lambda}_1^2}{\pi^2} (2^4 \log c_u + 2^2 \log c_s + 2^2 \log c_t) \\
& - \frac{1}{\pi^2} \int_0^1 d\xi \left\{ (\bar{\lambda}_3 \bar{\lambda}_4 + 2\bar{\lambda}_3^2) \log[\{\rho + \xi(1 - \xi)c_u\}/\{\rho + \xi(1 - \xi)\}] \right. \\
& \quad + 2^{-2}\bar{\lambda}_4^2 (\log[\{\rho + \xi(1 - \xi)c_s\}/\{\rho + \xi(1 - \xi)\}] \\
& \quad \left. + \log[\{\rho + \xi(1 - \xi)c_t\}/\{\rho + \xi(1 - \xi)\}]) \right\}. \tag{G.13}
\end{aligned}$$

Another channel which has only ϕ external fields is $G_{sym}^{(4,0)} = \langle \phi_1(p_1) \phi_1(p_2) \phi_1(p_3) \phi_1(p_4) \rangle$. All diagrams contributing to the correlation functions $G_s^{(4,0)}$, $G_t^{(4,0)}$, $G_u^{(4,0)}$ contribute to $G_{1,sym}^{(4,0)}$ too, and there is no other diagram. Therefore,

$$\begin{aligned}
\mathcal{G}_{1,sym}^{(4,0)} = & \mathcal{G}_{1,s}^{(4,0)} + \mathcal{G}_{1,t}^{(4,0)} + \mathcal{G}_{1,u}^{(4,0)} \\
= & -24\bar{\lambda}_1 - \frac{2^3 \cdot 3}{\pi^2} \bar{\lambda}_1^2 (\log c_s + \log c_t + \log c_u) \\
& - \frac{1}{\pi^2} (\bar{\lambda}_3 \bar{\lambda}_4 + 2\bar{\lambda}_3^2 + 2^{-2}\bar{\lambda}_4^2) \\
& \times \int_0^1 d\xi \left\{ \log[\{\rho + \xi(1 - \xi)c_s\}/\{\rho + \xi(1 - \xi)\}] \right. \\
& \quad + \log[\{\rho + \xi(1 - \xi)c_t\}/\{\rho + \xi(1 - \xi)\}] \\
& \quad \left. + \log[\{\rho + \xi(1 - \xi)c_u\}/\{\rho + \xi(1 - \xi)\}] \right\}. \tag{G.14}
\end{aligned}$$

Especially, at the symmetric case $s = t = u = P^2$,

$$\mathcal{G}_{1,sym}^{(4,0)} = 3\mathcal{G}_{1,s}^{(4,0)} = -24\bar{\lambda}_1. \tag{G.15}$$

G.2 \overline{MS} scheme

In \overline{MS} scheme with $s = t = u = P^2$,

$$\begin{aligned}
g_1^{(4,0)\overline{MS}}(P) = & -8\hat{\lambda}_1 - \frac{1}{32\pi^2} \left\{ 2^8 \cdot 3\hat{\lambda}_1^2 (\log [P^2/\mu^2] - 2) \right. \\
& \left. + (2^6\hat{\lambda}_3^2 + 2^5\hat{\lambda}_3\hat{\lambda}_4 + 2^4\hat{\lambda}_4^2) \int_0^1 d\xi \log [\rho + \xi(1 - \xi)P^2/\mu^2] \right\}. \tag{G.16}
\end{aligned}$$

$$\begin{aligned}
g_1^{(4,0)\overline{MS}}(P = \mu) &= \mathcal{G}_1^{(4,0)\overline{MS}}(\hat{\lambda}_i, \rho) \\
&= -8\hat{\lambda}_1 \\
&\quad - \frac{1}{32\pi^2} \left\{ -2^9 \cdot 3\hat{\lambda}_1^2 + (2^6\hat{\lambda}_3^2 + 2^5\hat{\lambda}_3\hat{\lambda}_4 + 2^4\hat{\lambda}_4^2) \int_0^1 d\xi \log[\rho + \xi(1-\xi)] \right\} \\
&= -\frac{8}{3}\pi^2 \left\{ \hat{\lambda} - 2\hat{\lambda}^2 + \frac{1}{6}(4\hat{\lambda}^2 + 6\hat{\lambda}\hat{g}_2 + 3\hat{g}_2^2 - 8\hat{\lambda}\hat{z} - 6\hat{g}_2\hat{z} + 4\hat{z}^2) \right. \\
&\quad \left. \times \frac{1}{2} \int_0^1 d\xi \log[\rho + \xi(1-\xi)] \right\}. \tag{G.17}
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathcal{G}_1^{(4,0)\overline{MS}}(\bar{\lambda}_i, \bar{\rho}) &= -\frac{8}{3}\pi^2 \left\{ \bar{\lambda} - 2\bar{\lambda}^2 + \frac{1}{6}(4\bar{\lambda}^2 + 6\bar{\lambda}\bar{g}_2 + 3\bar{g}_2^2 - 8\bar{\lambda}\bar{z} - 6\bar{g}_2\bar{z} + 4\bar{z}^2) \right. \\
&\quad \left. \times \frac{1}{2} \int_0^1 d\xi \log[\rho + \xi(1-\xi)] \right\}. \tag{G.18}
\end{aligned}$$

Differentiating $\mathcal{G}_1^{(4,0)\overline{MS}}$ by P ,

$$\begin{aligned}
\frac{d}{d \log[P/\mu]} \mathcal{G}_1^{(4,0)\overline{MS}}(\bar{\lambda}_i, \bar{\rho}) \\
&= -\frac{8}{3}\pi^2 \left\{ \frac{d}{d \log[P/\mu]} \bar{\lambda} + \frac{1}{6}(4\bar{\lambda}^2 + 6\bar{\lambda}\bar{g}_2 + 3\bar{g}_2^2 - 8\bar{\lambda}\bar{z} - 6\bar{g}_2\bar{z} + 4\bar{z}^2) \right. \\
&\quad \left. \times \frac{1}{2} \frac{d\bar{\rho}}{d \log[P/\mu]} \frac{\partial}{\partial \bar{\rho}} \int_0^1 d\xi \log[\rho + \xi(1-\xi)] \right\} + \mathcal{O}(\lambda^2), \tag{G.19}
\end{aligned}$$

where

$$\begin{aligned}
\frac{\partial}{\partial \bar{\rho}} \int_0^1 d\xi \log[\rho + \xi(1-\xi)] &= \int_0^1 \frac{d\xi}{\bar{\rho} + \xi(1-\xi)} \\
&= \frac{1}{\bar{\rho}} \int_0^1 d\xi \left(1 - \frac{\xi(1-\xi)}{\bar{\rho} + \xi(1-\xi)} \right) \\
&= \frac{1}{\bar{\rho}} \left(1 - f(1/\sqrt{\bar{\rho}}) \right). \tag{G.20}
\end{aligned}$$

And,

$$\frac{d}{d \log[P/\mu]} \bar{\lambda} = -\epsilon \bar{\lambda} + \frac{8}{3}\bar{\lambda}^2 + \bar{\lambda}\bar{g}_2 + \frac{1}{2}\bar{g}_2^2 - \frac{4}{3}\bar{\lambda}\bar{z} - \bar{g}_2\bar{z} + \frac{2}{3}\bar{z}^2. \tag{G.21}$$

Eventually, we obtain

$$\begin{aligned}
& \frac{d}{d \log[P/\mu]} \mathcal{G}_1^{(4,0) \overline{MS}}(\bar{\lambda}_i, \bar{\rho}) \\
&= -\frac{8}{3} \pi^2 \left\{ -\epsilon \bar{\lambda} + \frac{8}{3} \bar{\lambda}^2 + \bar{\lambda} \bar{g}_2 + \frac{1}{2} \bar{g}_2^2 - \frac{4}{3} \bar{\lambda} \bar{z} - \bar{g}_2 \bar{z} + \frac{2}{3} \bar{z}^2 \right. \\
&\quad \left. - \frac{1}{6} (4 \bar{\lambda}^2 + 6 \bar{\lambda} \bar{g}_2 + 3 \bar{g}_2^2 - 8 \bar{\lambda} \bar{z} - 6 \bar{g}_2 \bar{z} + 4 \bar{z}^2) (1 - f(1/\sqrt{\bar{\rho}})) \right\} \\
&= -\frac{8}{3} \pi^2 \left\{ -\epsilon \bar{\lambda} + 2 \bar{\lambda}^2 + \frac{1}{6} f(1/\bar{\rho}) (4 \bar{\lambda}^2 + 6 \bar{\lambda} \bar{g}_2 + 3 \bar{g}_2^2 - 8 \bar{\lambda} \bar{z} - 6 \bar{g}_2 \bar{z} + 4 \bar{z}^2) \right\} \\
&= \frac{d}{d \log[P/\mu]} \mathcal{G}_1^{(4,0) \text{sym.}}(\bar{\lambda}_i, \bar{\rho}). \tag{G.22}
\end{aligned}$$

G.3 $O(4)$ LSM

Taking the $c_A \rightarrow \infty$ limit, the $U_A(1)$ broken model drops into the $O(4)$ LSM

$$\mathcal{L}_{O(4)} = \frac{1}{2} (\partial_\mu \phi_a)^2 + \frac{1}{2} m^2 (T) \phi_a^2 + \frac{\pi^2}{3} \lambda (\phi_a^2)^2. \tag{G.23}$$

In the 1-loop order, we obtain

$$\begin{aligned}
& G_{O(4)}^{(4) \overline{MS}}(\phi_1(p_1), \phi_1(p_2), \phi_2(p_3), \phi_2(p_4))|_{amp} \\
&= -\frac{8}{3} \pi^2 \lambda - \frac{i}{32} \pi^2 \mu^{-\epsilon} \lambda^2 \{ 2^4 (\log[s/\mu^2] - 2) + 2^2 (\log[t/\mu^2] + \log[u/\mu^2] - 4) \}, \tag{G.24}
\end{aligned}$$

for \overline{MS} scheme, and

$$\begin{aligned}
& G_{O(4)}^{(4) \text{sym.}}(\phi_1(p_1), \phi_1(p_2), \phi_2(p_3), \phi_2(p_4))|_{amp} \\
&= -\frac{8}{3} \pi^2 \lambda - \frac{i}{32} \pi^2 \mu^{-\epsilon} \lambda^2 \{ 2^4 \log[s/\mu^2] + 2^2 (\log[t/\mu^2] + \log[u/\mu^2]) \}, \tag{G.25}
\end{aligned}$$

for symmetric scheme. β function is described by¹⁸

$$\beta_{O(4)} = -\epsilon \hat{\lambda} + 2 \hat{\lambda}^2. \tag{G.26}$$

In the symmetric point $s = t = u = P^2$,

$$G_{O(4)}^{(4)}(P) = \left(\frac{-1}{P^2} \right)^4 P^\epsilon \mathcal{G}_{O(4)}^{(4)}(\bar{\lambda}(P)), \tag{G.27}$$

¹⁸The β function is scheme independent at least 1-loop order

where,

$$\mathcal{G}_{O(4)}^{(4)\overline{MS}}(\bar{\lambda}) = -\frac{8}{3}\pi^2 (\bar{\lambda} - 2\bar{\lambda}^2), \quad (\text{G.28})$$

for \overline{MS} scheme, and

$$\mathcal{G}_{O(4)}^{(4)sym}(\bar{\lambda}) = -\frac{8}{3}\pi^2 \bar{\lambda}, \quad (\text{G.29})$$

for symmetric scheme.

In the $O(4)$ LSM, the IR behavior of the coupling is described as

$$\bar{\lambda}(P \rightarrow 0) = \frac{\epsilon}{2} + c' \left(\frac{P}{\mu}\right)^\epsilon + \dots \quad (\text{G.30})$$

Therefore, we obtain

$$\mathcal{G}_{O(4)}^{(4)\overline{MS}}(P \rightarrow 0) = -\frac{8}{3}\pi^2 \left\{ \frac{\epsilon}{2} - \frac{\epsilon^2}{2} + c' \left(\frac{P}{\mu}\right)^\epsilon + \dots \right\}. \quad (\text{G.31})$$

Subleading term

When one calculate at 2-loop order, β function should be

$$\beta_{O(4)} = -\epsilon\hat{\lambda} + 2\hat{\lambda}^2 + A\epsilon^2\hat{\lambda} + B\epsilon\hat{\lambda}^2 + C\hat{\lambda}^3. \quad (\text{G.32})$$

The IR fixed point of the coupling $\hat{\lambda}$ is described as

$$\lambda^* = \frac{\epsilon}{2} - \left(\frac{A}{2} + \frac{B}{4} + \frac{C}{8}\right)\epsilon^2 + \mathcal{O}(\epsilon^3). \quad (\text{G.33})$$

Thus, $\mathcal{O}(\epsilon^2)$ term that we found in eq. (G.31) will be higher order contribution.

G.4 Scheme independence

The formulation of four-point functions eq. (D.19-D.22) has anomalous behavior in the limit of $\rho \rightarrow \infty$ ¹⁹. In this limit, contributions from χ 's loop must be vanish. However, they don't vanish but diverges in eq. (D.19-D.22). This is because of breakdown of the approximation as

$$x^\epsilon = e^{\epsilon \log x} \approx 1 + \epsilon \log x + \mathcal{O}(\epsilon^2), \quad (\text{G.34})$$

in C, in the limit of $x \rightarrow \infty$.

¹⁹Because the counter term has same divergence, this anomalous behavior does not exist in symmetric scheme.

In order to avoid this anomaly, we redefine the couplings as they absorb the divergence in $\rho \rightarrow \infty$. For example, four-point function in \overline{MS} schem eq. (D.19) can be rewritten as

$$\begin{aligned}
& g_1^{(4,0)}(P/\mu, \hat{\lambda}_i, \rho) \\
&= -\frac{8}{3}\pi^2 \left\{ \hat{\lambda} + 2\hat{\lambda}^2 \frac{1}{2}(\log[P^2/\mu^2] - 2) \right. \\
&\quad \left. + \frac{1}{6}(4\hat{\lambda}^2 + 6\hat{\lambda}g_2 + 3\hat{g}_2^2 - 8\hat{\lambda}\hat{z} - 6\hat{g}_2\hat{z} + 4\hat{z}^2) \right. \\
&\quad \left. \times \frac{1}{2} \int_0^1 d\xi \log \left[\rho + \xi(1-\xi) \frac{P^2}{\mu^2} \right] \right\} \\
&= -\frac{8}{3}\pi^2 \left\{ \hat{\lambda} + \frac{1}{6}(4\hat{\lambda}^2 + 6\hat{\lambda}g_2 + 3\hat{g}_2^2 - 8\hat{\lambda}\hat{z} - 6\hat{g}_2\hat{z} + 4\hat{z}^2) \frac{1}{2} \log \rho \right. \\
&\quad \left. + 2\hat{\lambda}^2 \frac{1}{2}(\log[P^2/\mu^2] - 2) \right. \\
&\quad \left. + \frac{1}{6}(4\hat{\lambda}^2 + 6\hat{\lambda}g_2 + 3\hat{g}_2^2 - 8\hat{\lambda}\hat{z} - 6\hat{g}_2\hat{z} + 4\hat{z}^2) \right. \\
&\quad \left. \times \frac{1}{2} \int_0^1 d\xi \log \left[1 + \xi(1-\xi) \frac{P^2}{\mu^2} \rho^{-1} \right] \right\} \\
&= -\frac{8}{3}\pi^2 \left\{ \hat{\lambda}' + 2\hat{\lambda}'^2 \frac{1}{2}(\log[P^2/\mu^2] - 2) \right. \\
&\quad \left. + \frac{1}{6}(4\hat{\lambda}'^2 + 6\hat{\lambda}'g_2 + 3\hat{g}_2^2 - 8\hat{\lambda}'\hat{z} - 6\hat{g}_2\hat{z} + 4\hat{z}^2) \right. \\
&\quad \left. \times \frac{1}{2} \int_0^1 d\xi \log \left[1 + \xi(1-\xi) \frac{P^2}{\mu^2} \rho^{-1} \right] \right\}. \tag{G.35}
\end{aligned}$$

There is no term which diverges in $\rho \rightarrow \infty$ in above formulation, and contributions from χ 's loop vanish in $\rho \rightarrow \infty$. Redefined coupling $\hat{\lambda}'$ is described as

$$\hat{\lambda}' \equiv \hat{\lambda} + \frac{1}{6}(4\hat{\lambda}^2 + 6\hat{\lambda}g_2 + 3\hat{g}_2^2 - 8\hat{\lambda}\hat{z} - 6\hat{g}_2\hat{z} + 4\hat{z}^2) \frac{1}{2} \log \rho. \tag{G.36}$$

It grows in accordance with

$$\begin{aligned}
\mu \frac{d\hat{\lambda}'}{d\mu} &= \mu \frac{d\hat{\lambda}}{d\mu} + \frac{1}{6}(4\hat{\lambda}^2 + 6\hat{\lambda}g_2 + 3\hat{g}_2^2 - 8\hat{\lambda}\hat{z} - 6\hat{g}_2\hat{z} + 4\hat{z}^2) \frac{1}{2} \mu \frac{d}{d\mu} \log \rho + \mathcal{O}(\hat{\lambda}_i^3) \\
&= -\epsilon \hat{\lambda} + 2\hat{\lambda}^2 + \mathcal{O}(\hat{\lambda}_i^3) \\
&= -\epsilon \hat{\lambda}' + 2\hat{\lambda}'^2 + \mathcal{O}(\hat{\lambda}_i^3), \tag{G.37}
\end{aligned}$$

where we use $\beta_\rho = -2\rho$. It coincides with β function in symmetric scheme with $\rho \rightarrow \infty$.

Similarly, from eq. (D.22),

$$\begin{aligned}
& g_4^{(2,2)}(P/\mu, \hat{\lambda}_i, \rho) \\
&= -\frac{4}{3}\pi^2 \left\{ \hat{g}_2 + \frac{1}{3}\hat{g}_2\hat{\lambda} \frac{1}{2}(\log[P^2/\mu^2] - 2) \right. \\
&\quad + \frac{1}{3}\hat{g}_2(\hat{\lambda} - 2\hat{x}) \frac{1}{2} \int_0^1 d\xi \log \left[\rho + \xi(1-\xi) \frac{P^2}{\mu^2} \right] \\
&\quad \left. + \frac{1}{3}\hat{g}_2(4\hat{\lambda} + \hat{g}_2 - 4\hat{z}) \frac{1}{2} \int_0^1 d\xi \log \left[x\rho + \xi(1-\xi) \frac{P^2}{\mu^2} \right] \right\} \\
&= -\frac{4}{3}\pi^2 \left\{ \hat{g}_2 + \frac{1}{3}\hat{g}_2(\hat{\lambda} - 2\hat{x}) \frac{1}{2} \log \rho + \frac{1}{3}\hat{g}_2(4\hat{\lambda} + \hat{g}_2 - 4\hat{z}) \frac{1}{2}(\log \rho - 1) \right. \\
&\quad + \frac{1}{3}\hat{g}_2\hat{\lambda} \frac{1}{2}(\log[P^2/\mu^2] - 2) \\
&\quad + \frac{1}{3}\hat{g}_2(\hat{\lambda} - 2\hat{x}) \frac{1}{2} \int_0^1 d\xi \log \left[1 + \xi(1-\xi) \frac{P^2}{\mu^2} \rho^{-1} \right] \\
&\quad \left. + \frac{1}{3}\hat{g}_2(4\hat{\lambda} + \hat{g}_2 - 4\hat{z}) \frac{1}{2} \int_0^1 d\xi \log \left[1 + x \frac{P^2}{\mu^2} \rho^{-1} \right] \right\} \\
&= -\frac{4}{3}\pi^2 \left\{ \hat{g}'_2 + \frac{1}{3}\hat{g}'_2\hat{\lambda}' \frac{1}{2}(\log[P^2/\mu^2] - 2) \right. \\
&\quad + \frac{1}{3}\hat{g}'_2(\hat{\lambda}' - 2\hat{x}') \frac{1}{2} \int_0^1 d\xi \log \left[1 + \xi(1-\xi) \frac{P^2}{\mu^2} \rho^{-1} \right] \\
&\quad \left. + \frac{1}{3}\hat{g}'_2(4\hat{\lambda}' + \hat{g}'_2 - 4\hat{z}') \frac{1}{2} \int_0^1 d\xi \log \left[1 + x \frac{P^2}{\mu^2} \rho^{-1} \right] \right\}. \tag{G.38}
\end{aligned}$$

\hat{g}' is defined as

$$\hat{g}'_2 = \hat{g}_2 + \frac{1}{3}\hat{g}_2(\hat{\lambda} - 2\hat{x}) \frac{1}{2} \log \rho + \frac{1}{3}\hat{g}_2(4\hat{\lambda} + \hat{g}_2 - 4\hat{z}) \frac{1}{2}(\log \rho - 1), \tag{G.39}$$

and

$$\mu \frac{d\hat{g}'_2}{d\mu} = -\epsilon\hat{g}'_2 + \frac{1}{3}\hat{\lambda}'\hat{g}'_2. \tag{G.40}$$

And, from eq. (D.20)

$$\begin{aligned}
& g_2^{(0,4)}(P/\mu, \hat{\lambda}_i, \rho) \\
&= -\frac{4}{3}\pi^2 \left\{ \hat{\lambda} - 2\hat{x} + 2(\hat{\lambda} - 2\hat{x})^2 \frac{1}{2} \int_0^1 d\xi \log \left[\rho + \xi(1-\xi) \frac{P^2}{\mu^2} \right] \right. \\
&\quad \left. + \frac{1}{6}(4\hat{\lambda}^2 + 6\hat{\lambda}\hat{g}_2 + 3\hat{g}_2^2 - 8\hat{\lambda}\hat{z} - 6\hat{g}_2\hat{z} + 4\hat{z}^2) \frac{1}{2} (\log[P^2/\mu^2] - 2) \right\} \\
&= -\frac{4}{3}\pi^2 \left\{ \hat{\lambda} - 2\hat{x} + 2(\hat{\lambda} - 2\hat{x})^2 \frac{1}{2} \log \rho \right. \\
&\quad \left. + 2(\hat{\lambda} - 2\hat{x})^2 \frac{1}{2} \int_0^1 d\xi \log \left[1 + \xi(1-\xi) \frac{P^2}{\mu^2} \rho^{-1} \right] \right. \\
&\quad \left. + \frac{1}{6}(4\hat{\lambda}^2 + 6\hat{\lambda}\hat{g}_2 + 3\hat{g}_2^2 - 8\hat{\lambda}\hat{z} - 6\hat{g}_2\hat{z} + 4\hat{z}^2) \frac{1}{2} (\log[P^2/\mu^2] - 2) \right\} \\
&= -\frac{4}{3}\pi^2 \left\{ \hat{\lambda}' - 2\hat{x}' + 2(\hat{\lambda}' - 2\hat{x}')^2 \frac{1}{2} \int_0^1 d\xi \log \left[1 + \xi(1-\xi) \frac{P^2}{\mu^2} \rho^{-1} \right] \right. \\
&\quad \left. + \frac{1}{6}(4\hat{\lambda}'^2 + 6\hat{\lambda}'\hat{g}'_2 + 3\hat{g}'_2{}^2 - 8\hat{\lambda}'\hat{z}' - 6\hat{g}'_2\hat{z}' + 4\hat{z}'^2) \frac{1}{2} (\log[P^2/\mu^2] - 2) \right\}, \quad (\text{G.41})
\end{aligned}$$

$$\begin{aligned}
\hat{x}' &= \hat{x} + \frac{1}{2}(\hat{\lambda}' - \hat{\lambda}) - (\hat{\lambda} - 2\hat{x})^2 \frac{1}{2} \log \rho \\
&= \hat{x} + 4(\hat{\lambda} - \hat{x})\hat{x} \frac{1}{2} \log \rho - \frac{1}{6}(8\hat{\lambda}^2 - 6\hat{\lambda}\hat{g}_2 - 3\hat{g}_2^2 + 8\hat{\lambda}\hat{z} + 6\hat{g}_2\hat{z} - 4\hat{z}^2) \frac{1}{2} \log \rho, \quad (\text{G.42})
\end{aligned}$$

$$\mu \frac{d\hat{x}'}{d\mu} = -\epsilon\hat{x}' + \frac{1}{6}(8\hat{\lambda}'^2 - 6\hat{\lambda}'\hat{g}'_2 - 3\hat{g}'_2{}^2 + 8\hat{\lambda}'\hat{z}' + 6\hat{g}'_2\hat{z}' - 4\hat{z}'^2). \quad (\text{G.43})$$

From eq. (D.22),

$$\begin{aligned}
& g_3^{(2,2)}(P/\mu, \hat{\lambda}_i, \rho) \\
&= -\frac{4}{3}\pi^2 \left\{ 2(\hat{\lambda} + \hat{g}_2 + \hat{z}) + \frac{1}{3}\hat{\lambda}(6\hat{\lambda} + 5\hat{g}_2 - 6\hat{z})\frac{1}{2}(\log(P^2/\mu^2)) \right. \\
&\quad + \frac{1}{3}(\hat{\lambda} - 2\hat{x})(6\hat{\lambda} + 5\hat{g}_2 - 6\hat{z})\frac{1}{2}\int_0^1 d\xi \log \left[\rho + \xi(1-\xi)\frac{P^2}{\mu^2} \right] \\
&\quad \left. + \frac{1}{3}(4\hat{\lambda}^2 + 8\hat{\lambda}\hat{g}_2 + 5\hat{g}_2^2 - 8\hat{\lambda}\hat{z} - 8\hat{g}_2\hat{z} + 4\hat{z}^2)\frac{1}{2}\int_0^1 d\xi \log \left[x\rho + \xi(1-\xi)\frac{P^2}{\mu^2} \right] \right\} \\
&= -\frac{4}{3}\pi^2 \left\{ 2(\hat{\lambda}' + \hat{g}'_2 + \hat{z}') + \frac{1}{3}\hat{\lambda}'(6\hat{\lambda}' + 5\hat{g}'_2 - 6\hat{z}')\frac{1}{2}(\log(P^2/\mu^2)) \right. \\
&\quad + \frac{1}{3}(\hat{\lambda}' - 2\hat{x}')(6\hat{\lambda}' + 5\hat{g}'_2 - 6\hat{z}')\frac{1}{2}\int_0^1 d\xi \log \left[1 + \xi(1-\xi)\frac{P^2}{\mu^2}\rho^{-1} \right] \\
&\quad + \frac{1}{3}(4\hat{\lambda}'^2 + 8\hat{\lambda}'\hat{g}'_2 + 5\hat{g}'_2{}^2 - 8\hat{\lambda}'\hat{z}' - 8\hat{g}'_2\hat{z}' + 4\hat{z}'^2) \\
&\quad \left. \times \frac{1}{2}\int_0^1 d\xi \log \left[1 + x\frac{P^2}{\mu^2}\rho^{-1} \right] \right\}, \tag{G.44}
\end{aligned}$$

$$\begin{aligned}
\hat{z}' &= \hat{z} + \hat{\lambda}' - \hat{\lambda} + \hat{g}'_2 - \hat{g}_2 \\
&= \hat{z} + \frac{1}{12}(4\hat{\lambda}^2 + 8\hat{\lambda}\hat{g}_2 + 5\hat{g}_2^2 - 8\hat{\lambda}\hat{z} - 8\hat{g}_2\hat{z} + 4\hat{z}^2) \\
&\quad + \frac{1}{6}(-6\hat{\lambda}^2 + 3\hat{\lambda}\hat{g}_2 + 12\hat{\lambda}\hat{x} + 6\hat{g}_2\hat{x} + 6\hat{\lambda}\hat{z} - \hat{g}_2\hat{z} - 12\hat{x}\hat{z})\frac{1}{2}\log \rho, \tag{G.45}
\end{aligned}$$

$$\mu \frac{d\hat{z}'}{d\mu} = -\epsilon \hat{z}' + \frac{1}{2}(2\hat{\lambda}'^2 - \hat{\lambda}\hat{g}_2 + 2\hat{\lambda}\hat{z}). \tag{G.46}$$

As discussed in section 3.2, IR behaviors of $\bar{\lambda}'$, \bar{g}'_2 , \bar{x}' and \bar{z}' is obtained as

$$\bar{\lambda}'(P \rightarrow 0) = \frac{\epsilon}{2} + c_\alpha \left(\frac{P}{\mu} \right), \quad \bar{g}'_2(P \rightarrow 0) = c' \left(\frac{P}{\mu} \right)^{-\frac{5}{6}\epsilon}, \tag{G.47}$$

$$\bar{x}'(P \rightarrow 0) = \frac{3}{32}c'^2\epsilon^{-1} \left(\frac{P}{\mu} \right)^{\frac{5}{3}\epsilon}, \quad \bar{z}'(P \rightarrow 0) = \frac{3}{4}c' \left(\frac{P}{\mu} \right)^{\frac{5}{6}\epsilon}. \tag{G.48}$$

Using these expressions and eq. (G.22),

$$\begin{aligned}
& \frac{d}{\log[P/\mu]} \mathcal{G}_1^{(4,0)\overline{MS}}(\bar{\lambda}_i, \bar{\rho}) \\
&= -\frac{8}{3}\pi^2 \left\{ -\epsilon\bar{\lambda} + 2\bar{\lambda}^2 + \frac{1}{6}f(1/\sqrt{\bar{\rho}})(4\bar{\lambda}^2 + 6\bar{\lambda}\bar{g}_2 + 3\bar{g}_2^2 - 8\bar{\lambda}\bar{z} - 6\bar{g}_2\bar{z} + 4\bar{z}^2) \right\} \\
&\rightarrow -\frac{8}{3}\pi^2 \frac{\bar{\rho}^{-1}}{24} c'^2 \left(\frac{P}{\mu} \right)^{-\frac{5}{6}\epsilon}, \tag{G.49}
\end{aligned}$$

in the $P \rightarrow 0$ limit. Assuming $\bar{\rho}(P \rightarrow 0) \sim P^2 + \mathcal{O}(\hat{\lambda}_i^2)$,

$$\mathcal{G}_1^{(4,0)\overline{MS}}(\bar{\lambda}_i, \bar{\rho}; P \rightarrow 0) \approx -\frac{8}{3}\pi^2 \left(\frac{\epsilon}{2} - c \left(\frac{P}{\mu} \right)^{2-\frac{5}{6}\epsilon} \right). \tag{G.50}$$

This is same as the case of symmetric scheme.²⁰

H Anomalous dimensions with operator mixing

In this section, we calculate the anomalous dimension of composit operators ϕ^2 and χ^2 with operator mixing which regularized as

$$\begin{pmatrix} [\phi^2] \\ [\chi^2] \end{pmatrix} = Z_{\Phi^2}^{-1} \begin{pmatrix} \phi^2 \\ \chi^2 \end{pmatrix}, \quad Z_{\Phi^2}^{-1} \equiv \begin{pmatrix} 1 + A' & C'_1 \\ C'_2 & 1 + B' \end{pmatrix}, \tag{H.1}$$

and,

$$A' = \frac{1}{2}\hat{\lambda} \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \log \mu^2 + 2 \right), \tag{H.2}$$

$$B' = \frac{1}{2}(\hat{\lambda} - 2\hat{x}) \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \int_0^1 d\xi \log [c_A + \xi(1-\xi)\mu^2] \right), \tag{H.3}$$

$$C'_1 = \left(\frac{\hat{\lambda}}{3} + \frac{\hat{g}_2}{4} - \frac{\hat{z}}{3} \right) \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \log \mu^2 + 2 \right), \tag{H.4}$$

$$C'_2 = \left(\frac{\hat{\lambda}}{3} + \frac{\hat{g}_2}{4} - \frac{\hat{z}}{3} \right) \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \int_0^1 d\xi \log [c_A + \xi(1-\xi)\mu^2] \right). \tag{H.5}$$

Taking a diagonalizing basis,

$$\begin{pmatrix} [\Phi_+^2] \\ [\Phi_-^2] \end{pmatrix} = \begin{pmatrix} Z_+^{-1} & 0 \\ 0 & Z_-^{-1} \end{pmatrix} \begin{pmatrix} \Phi_+^2 \\ \Phi_-^2 \end{pmatrix}, \tag{H.6}$$

²⁰Accurately, leading term of $\mathcal{G}_1^{(4)\overline{MS}}(P \rightarrow 0)$ is $(\epsilon - \epsilon^2)/2$. However, we regard quadric term as sub-leading term.

where

$$P^{-1}Z_{\Phi^2}^{-1}P = \text{diag}\{Z_+^{-1}, Z_-^{-1}\}, \quad Z_{\pm}^{-1} = 1 + \frac{A' + B'}{2} \pm \sqrt{\left(\frac{A' - B'}{2}\right)^2 + C'_1 C'_2} \quad (\text{H.7})$$

and

$$\begin{pmatrix} \Phi_+^2 \\ \Phi_-^2 \end{pmatrix} = P^{-1} \begin{pmatrix} \phi^2 \\ \chi^2 \end{pmatrix}. \quad (\text{H.8})$$

The diagonalizing matrix P is described as

$$\begin{aligned} P &= \begin{pmatrix} 1 & -\frac{1+B'-Z_-}{C'_2} \\ -\frac{1+A'-Z_+}{C'_1} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \frac{A'-B'}{2C'_2} - \sqrt{\left(\frac{A'-B'}{2C'_2}\right)^2 + \frac{C'_1}{C'_2}} \\ -\frac{A'-B'}{2C'_1} + \sqrt{\left(\frac{A'-B'}{2C'_1}\right)^2 + \frac{C'_2}{C'_1}} & 1 \end{pmatrix}. \end{aligned} \quad (\text{H.9})$$

The anomalous dimensions of Φ_+^2 and Φ_-^2 , γ_+ γ_- are calculated as

$$\gamma_{\pm} = -\mu \frac{\partial}{\partial \mu} \log Z_{\pm}^{-1} \approx \mu \frac{\partial}{\partial \mu} \left\{ \frac{A' + B'}{2} \pm \sqrt{\left(\frac{A' - B'}{2}\right)^2 + C'_1 C'_2} \right\}. \quad (\text{H.10})$$

Where

$$\begin{aligned} A' - B' &= \hat{x} \left(\frac{2}{\epsilon} - \gamma + \log 4\pi \right) - \frac{1}{2} \hat{\lambda} (\log \mu^2 - 2) + \frac{1}{2} (\hat{\lambda} - 2\hat{x}) \int_0^1 d\xi \log [c_A + \xi(1 - \xi)\mu^2] \\ &\approx \hat{x} \frac{2}{\epsilon} + \mathcal{O}(\epsilon), \end{aligned} \quad (\text{H.11})$$

and

$$\begin{aligned} C'_1 C'_2 &= \left(\frac{\hat{\lambda}}{3} + \frac{\hat{g}_2}{4} - \frac{\hat{z}}{3} \right)^2 \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \log \mu^2 + 2 \right) \\ &\quad \times \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \int_0^1 \xi \log [c_A + \xi(1 - \xi)\mu^2] \right) \\ &\approx \left(\frac{\hat{\lambda}}{3} + \frac{\hat{g}_2}{4} - \frac{\hat{z}}{3} \right)^2 \left(\frac{2}{\epsilon} \right)^2 + \mathcal{O}(\epsilon). \end{aligned} \quad (\text{H.12})$$

Assuming $(A' - B')^2 \gg C'_1 C'_2$,

$$\begin{aligned} \sqrt{\left(\frac{A' - B'}{2}\right)^2 + C'_1 C'_2} &= \frac{A' - B'}{2} \sqrt{1 + \frac{4C'_1 C'_2}{(A' - B')^2}} \\ &\approx \frac{A' - B'}{2} + \frac{C'_1 C'_2}{A' - B'} + \dots \end{aligned} \quad (\text{H.13})$$

Using the IR asymptotic behavior of the couplings, the second term is suppressed to zero in the IR limit. Therefore,

$$\gamma_+(\mu \rightarrow 0) = - \lim_{\mu \rightarrow 0} \mu \frac{\partial}{\partial \mu} A' = \lim_{\mu \rightarrow 0} \hat{\lambda}(\mu) \quad (\text{H.14})$$

$$\gamma_+(\mu \rightarrow 0) = - \lim_{\mu \rightarrow 0} \mu \frac{\partial}{\partial \mu} B' = - \lim_{\mu \rightarrow 0} 2f\left(\frac{\mu}{\sqrt{c_A}}\right) \hat{x}(\mu). \quad (\text{H.15})$$

I Feynman parameter integral

A loop factor of $2N$ point 1PI diagram is obtained as

$$\int \frac{d^d k}{(2\pi)^d} \prod_{i=1}^N \left\{ \frac{-1}{(k + \sum_{j=1}^i (p_j + q_j))^2 + m^2} \right\}, \quad (\text{I.1})$$

where p_i and q_i are external momenta that pour into i th vertex. The squared mass m^2 is zero for ϕ 's loop and c_A for χ 's loop. Using $P_i \equiv \sum_{j=1}^i (p_j + q_j)$ and Feynman parameter integral,

$$\frac{1}{A^\alpha B^\beta} = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dw \frac{w^{\alpha-1} (1-w)^{\beta-1}}{[wA + (1-w)B]^{\alpha+\beta}}, \quad (\text{I.2})$$

we obtain

$$\begin{aligned}
& \prod_{i=1}^N \left\{ \frac{-1}{(k + \sum_{j=1}^i (p_j + q_j))^2 + m^2} \right\} \\
&= (-1)^N \left\{ \prod_{i=1}^{N-2} \frac{1}{(k + P_i)^2 + m^2} \right\} \frac{1}{(k + P_{N-1})^2 + m^2} \frac{1}{k^2 + m^2} \\
&= (-1)^N \left\{ \prod_{i=1}^{N-2} \frac{1}{(k + P_i)^2 + m^2} \right\} \frac{\Gamma(2)}{\Gamma(1)^2} \int_0^1 dw_1 \frac{1}{[w_1 \{(k + P_{N-1})^2 + m^2\} + (1 - w_1)(k^2 + m^2)]^2} \\
&= (-1)^N \left\{ \prod_{i=1}^{N-2} \frac{1}{(k + P_i)^2 + m^2} \right\} \frac{\Gamma(2)}{\Gamma(1)^2} \int_0^1 dw_1 \frac{1}{[k^2 + m^2 + w_1(2k \cdot P_{N-1} + P_{N-1}^2)]^2} \\
&= (-1)^N \left\{ \prod_{i=1}^{N-3} \frac{1}{(k + P_i)^2 + m^2} \right\} \frac{\Gamma(2)}{\Gamma(1)^2} \frac{\Gamma(3)}{\Gamma(1)\Gamma(2)} \int_0^1 dw_1 \int_0^1 dw_2 \\
&\quad \times \frac{(1 - w_2)}{[k^2 + m^2 + w_1(1 - w_2)(2k \cdot P_{N-1} + P_{N-1}^2) + w_2(2k \cdot P_{N-2} + P_{N-2}^2)]^3} \\
&\quad \vdots \\
&= (-1)^N \frac{\Gamma(N)}{\Gamma(1)^N} \left\{ \prod_{i=1}^{N-1} \int_0^1 dw_i (1 - w_i)^{i-1} \right\} \left[k^2 + m^2 + \sum_{i=1}^{N-1} w_i \left\{ \prod_{j=i}^{N-1} (1 - w_j) \right\} (2k \cdot P_{N-i} + P_{N-i}^2) \right]^{-N}, \tag{I.3}
\end{aligned}$$

where we use momentum conservation $P_N = \sum_j^N (p_j + q_j) = 0$.

Using $l = k + \sum_{i=1}^{N-1} w_i \left\{ \prod_{j=i}^{N-1} (1 - w_j) \right\} P_{N-i}$,

$$l^2 = k^2 + \sum_{i=1}^{N-1} w_i \left\{ \prod_{j=i}^{N-1} (1 - w_j) \right\} 2k \cdot P_{N-i} + \left(\sum_{i=1}^{N-1} w_i \left\{ \prod_{j=i}^{N-1} (1 - w_j) \right\} 2k \cdot P_{N-i,\mu} \right)^2. \tag{I.4}$$

Thus,

$$\begin{aligned}
& \int \frac{d^d k}{(2\pi)^d} \prod_{i=1}^N \left\{ \frac{-1}{(k + \sum_{j=1}^i (p_j + q_j))^2 + m^2} \right\} \\
&= (-1)^N \Gamma(N) \left\{ \prod_{i=1}^{N-1} \int_0^1 dw_i (1 - w_i)^{i-1} \right\} \int \frac{d^d l}{(2\pi)^d} \left(\frac{1}{l^2 + m^2 + (c - c') P^2} \right)^N, \tag{I.5}
\end{aligned}$$

where $P_{i,\mu} = c_{i,\mu}|P|$ and

$$c(\{w_i\}) = \sum_{i=1}^{N-1} w_i \left\{ \prod_{j=i}^{N-1} (1-w_j) \right\} c_i^2, \quad c'(\{w_i\}) = \left(\sum_{i=1}^{N-1} w_i \left\{ \prod_{j=i}^{N-1} (1-w_j) \right\} c_{i,\mu} \right)^2. \quad (\text{I.6})$$

Using a identity

$$\int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2 + \Delta)^N} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(N - \frac{d}{2})}{\Gamma(N)} \left(\frac{1}{\Delta} \right)^{N - \frac{d}{2}}, \quad (\text{I.7})$$

we eventually obtain

$$\begin{aligned} & \int \frac{d^d k}{(2\pi)^d} \prod_{i=1}^N \left\{ \frac{-1}{(k + \sum_{j=1}^i (p_j + q_j))^2 + m^2} \right\} \\ &= (-1)^N \mu^{d-2N} \frac{\Gamma(N - \frac{d}{2})}{(4\pi)^{d/2}} \left\{ \prod_{i=1}^{N-1} \int_0^1 dw_i (1-w_i)^{i-1} \right\} \left(\frac{\mu^2}{m^2 + (c-c')P^2} \right)^{N - \frac{d}{2}}. \\ & \xrightarrow{d=4-\epsilon} (-1)^N \mu^{4-\epsilon-2N} \frac{\Gamma(N - 2 + \frac{\epsilon}{2})}{(4\pi)^{2-\epsilon/2}} \left\{ \prod_{i=1}^{N-1} \int_0^1 dw_i (1-w_i)^{i-1} \right\} \left(\frac{\mu^2}{m^2 + (c-c')P^2} \right)^{N-2+\frac{\epsilon}{2}}. \\ & \approx (-1)^N \mu^{-\epsilon} \frac{\Gamma(N-2)}{(4\pi)^2} \left\{ \prod_{i=1}^{N-1} \int_0^1 dw_i (1-w_i)^{i-1} \right\} \left(\frac{1}{m^2 + (c-c')P^2} \right)^{N-2+\frac{\epsilon}{2}} + \mathcal{O}(\epsilon). \end{aligned} \quad (\text{I.8})$$

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