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# The Gravitational Higgs Phenomena and the Vacuum Instability

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# Abstract

The stability problem of the electroweak vacuum has been theoretically discussed and widely investigated for quite a long time. However, the recent measurements of the Higgs boson mass $m_h$ and the top quark mass $m_t$ strongly suggest that the current Higgs vacuum is not stable and would eventually decay into a true Planck-scale vacuum. From the standard analysis of the instanton method with the best fit values of parameters of the Standard Model, the lifetime of the electroweak vacuum exceeds that of the Universe. Therefore, it was thought that the metastability of the Higgs vacuum does not have any serious impact on the observed Universe. However, the situation drastically changes when the quantum influence of the gravity can not be ignored. In particular, whether cosmological inflation of the early Universe or evaporations of the black holes are compatible with the stability problem of the Higgs vacuum has recently attracted significant interest in the high-energy community.

In this thesis, I summarize my past research in which I have done so far mainly on the Higgs vacuum stability in gravitational background or cosmological situations. The central issue for this research theme is how gravitationally induced Higgs fluctuation affects the stability of the electroweak vacuum. By using some techniques of quantum field theory in curved spacetime which is consistent with the semiclassical approach of quantum gravity, we derive gravitationally induced vacuum fluctuation and standard effective potential in curved spacetime. However, we clearly show that the induced vacuum fluctuation modifies the standard effective potential [1,2] and the spacetime itself as the gravitational backreaction [3]. Based on the formulation, we investigate the electroweak vacuum stability in various background spacetimes or cosmological situations as during inflation corresponding to the de-Sitter spacetime [1,2], after inflation like the preheating or reheating stage [4] and around evaporating black holes [5].

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# Contents

# Chapter 1

# Introduction

Recent discovery of the Higgs boson at the large hadron collider (LHC) has been recognized as a major breakthrough in particle physics [6, 7] and established the Standard Model (SM). However, there are still many unanswered questions or deepest puzzles. Even in the context of SM, we have been plagued with many problems: What is the origin of the electroweak symmetry breaking ? Why the observed Higgs mass is unnaturally small against the Plank scale ? Whether does the Higgs field participates in cosmic inflation and why does the SM Higgs vacuum seem to be metastable ? These problems are important clues to discover new physics beyond the SM and understand cosmological history of our Universe.

The stability problem of the Higgs vacuum has been theoretically discussed for quite a long time. However, the recent measurements of the observed Higgs boson mass $m_h$ [8–11] and the top quark mass $m_t$ [12] strongly suggest that the SM Higgs potential without no additional new physics develops an instability below the Planck scale. Thus, our living electroweak vacuum is not stable and would eventually collapse. For the present best fit values of the SM parameters, the decay ratio of the Higgs vacuum using the analysis of the instanton methods in quantum field theory (QFT) is exceedingly small compared with the age of observed Universe [13–15]. Fortunately, the metastability of the electroweak vacuum does not seem to require additional new physics beyond the SM.

However, the situation drastically changes when quantum gravitational effects can not be ignored. The strong gravitational background generates the large vacuum fluctuation of the Higgs field which triggers a collapse of the electroweak false vacuum [16–21]. Most subtle situation is the inflationary Universe where the large fluctuation of the Higgs field can be generated during inflation. In the case where the Higgs field can be effectively regarded as massless scalar fields like inflaton itself, the Higgs vacuum fluctuation enlarges up to the Hubble scale. Thus, if the large inflationary Higgs fluctuation overcomes the barrier of the effective Higgs potential, it triggers off a catastrophic vacuum collapse of the Universe. Furthermore, even at the end of the inflation, the large vacuum fluctuation of the Higgs field can be generated via parametric or tachyonic resonance during the preheating stage, and poses a threat to the Higgs vacuum stability. The thermal fluctuation at the reheating stage can also trigger a false vacuum decay, but the effects can be somewhat relaxed by the thermal corrections to the Higgs potential. The evaporating black holes which emit thermal Hawking radiation also raise a serious problem about the Higgs vacuum stability.

These issues are the subject of this thesis. Several aspects about the Higgs vacuum stability in gravitational background or cosmological situations have already been discussed in literature, but here we focus on and clearly demonstrate how the vacuum fluctuation of the Higgs field affects the electroweak vacuum stability. The gravitational effects of the Higgs fluctuation on the electroweak vacuum are twofold. On one side, the Higgs fluctuation on gravitational background can stabilize or destabilize the effective potential as the backreaction. On the other side, the local and inhomogeneous Higgs field originated from the fluctuation can generate true vacuum bubbles or domains and triggers off a false vacuum decay. Whether the electroweak false vacuum survives or not in various gravitational backgrounds and cosmological situations can be determined by these effects although it has some essential difficulties to analyze the Higgs fluctuation in gravitational background.

In standard QFT, the vacuum field fluctuation is formally described by the two-point correlation function, which has troublesome ultraviolet (UV) divergences. Thus, a regularization or renormalization must be required. In ordinary Minkowski spacetime, these UV divergences can be eliminated by standard renormalization methods. However, in curved spacetime where gravity curves the spacetime, it is much trouble to perform the renormalization and estimate the vacuum fluctuation due to ambiguity of vacuum states and gravitational particle creations. In this thesis we adopt some techniques of QFT in curved spacetime [22–30] corresponding to the semiclassical approach of quantum gravity (QG). Here we derive the effective potential in curved spacetime with the gravitational backreaction and clearly demonstrate how the Higgs vacuum fluctuation affects the stability of the electroweak vacuum. Based on this semiclassical analysis, we investigate the electroweak vacuum stability in various background spacetimes or cosmological situations as during inflation corresponding to the de-Sitter spacetime, after inflation like the preheating and reheating stage and around evaporating black holes.

## 1.1 Summary of the Research

This research project focus on the Higgs vacuum stability in various gravitational backgrounds or cosmological situations. The central issues for this project are how the Higgs vacuum fluctuation affects the stability of the electroweak vacuum and to analyze the gravitational vacuum fluctuation of the Higgs field in curved spacetime. By using some techniques of QFT in curved spacetime corresponding to the semiclassical approach of quantum gravity (QG), we derive the standard effective potential in curved spacetime. However, we clearly show that gravitational vacuum fluctuation modifies the standard effective potential [1, 2] and the spacetime itself as the gravitational backreaction [3]. Based on the above considerations and discussions, we investigate the electroweak vacuum stability during inflation [1, 2], after inflation in particular the preheating stage [4] and around evaporating black holes [5].

## 1.2 Organization of the Thesis

This thesis is constructed as follows:

In Chapter 2 we review a formulation of the SM and consider some famous puzzles beyond the SM. We will discuss the naturalness problems of the cosmological constant or the Higgs boson mass, and formulation of the Coleman-Weinberg effective potential. These originate from quantum zero-point vacuum energy and we review the Casimir effect as the famous example of its existence. Next we will discuss the renormalization group (RG) running of the Higgs self-coupling which provides rich information about the high-energy scale physics. Especially, for the best fit values of the SM parameters, the running Higgs self-coupling becomes negative at the high-energy scale and raise a vacuum stability problem. Here we will provide a comprehensive review of the electroweak vacuum stability.

In Chapter 3 we will review some techniques of QFT in curved spacetime and semiclassical gravity. These are entitled semiclassical approach to the quantum gravity (QG) where the matter fields only are quantized but the gravity field does not. Those provide quantitative depictions of some quantum gravitational phenomena. Here we consider the renormalization of the quantum fluctuation in curved spacetime and derive the standard effective potential in curved spacetime. However, we clearly show that the gravitationally induced vacuum fluctuation modifies the standard effective potential and the spacetime itself as the backreaction.

In Chapter 4 we will investigate the gravitationally induced vacuum fluctuation of scalar field in de-Sitter and Schwarzschild spacetime by using adiabatic regularization or point-splitting regularization. The induced vacuum fluctuation of scalar field in de-Sitter spacetime enlarges in proportion to the Hubble scale. On the other hand, the gravitationally induced vacuum fluctuation in Schwarzschild spacetime glows up to the Hawking temperature. However, in the Schwarzschild spacetime there are three well-defined vacua like the Boulware vacuum, Unruh vacuum and Hartle-Hawking vacuum, and therefore careful consideration must be required.

In Chapter 5 we will briefly discuss false vacuum decay in curved background. The quantum effects of the gravitationally induced vacuum fluctuation are twofold. First, the induced vacuum fluctuation of scalar field stabilizes or destabilizes the effective potential as the backreaction effect. Second, the inhomogeneous fluctuation of the scalar field can generate true vacuum bubbles or domains and triggers off a collapse of the false vacuum. Whether the false vacuum state in curved background become stable or not can be determined by these two effects. In this chapter we discuss quantitatively the gravitational phase transition.

In Chapter 6 we will thoroughly investigate electroweak vacuum stability in gravitational background. First, we will derive the effective Higgs potential modified by the gravitationally induced vacuum fluctuation and explain how one in flat spacetime change in curved spacetime. Next, we will introduce the stochastic formalism using the induced vacuum fluctuation to discuss a false vacuum decay in curved spacetime. Based on the above formulation, we investigate the electroweak vacuum stability during inflation corresponding to de-Sitter spacetime, after inflation like the preheating or reheating stage and around evaporating black holes.

Chapter 7 is devoted to the conclusion of this thesis.

# Chapter 2

# The Standard Model and Beyond

The ATLAS and CMS experiments at the large hadron collider (LHC) in CERN [6,7] discovered the famous Higgs boson that is the last missing particle of the Standard Model (SM). This discovery has been recognized as a greatest accompaniments of particle physics in the past few decades and completely established the SM. In this chapter, we review a formulation of the SM and its problems suggesting a new physics. Especially, we will discuss the stability problem of the electroweak vacuum in more detail.

## 2.1 The Standard Model

The SM is formally based on the quantum field theory (QFT) and incorporates three fundamental forces, the electromagnetic, the weak and the strong forces involving different types of gauge bosons. The SM with the Higgs boson is a complete theory at the low energy scale and has been done remarkable experimental confirmations. In this section we will review the formulation of SM and discuss the Brout-Englert-Higgs (BEH) mechanism in quite some detail.

### 2.1.1 The Standard Model Lagrangian

The SM is formulated as the gauge theory under the gauge group $G_{\mathrm{SM}} = SU(3)_C \times SU(2)_L \times U(1)_Y$. The fundamental forces of the SM can be propagated by gauge bosons: one $B$ boson in $U(1)_Y$, three $W$ bosons in $SU(2)_L$ and eight gluons in $SU(3)_C$. However, the electroweak symmetry $SU(2)_L \times U(1)_Y$ is spontaneously broken to $U(1)_{EM}$ by the vacuum expectation value (VEV) of the Higgs field [31]. The Higgs field is only a scalar field in the SM and assumed $(\mathbf{1}, \mathbf{2}, +1/2)$ under $SU(3)_C \times SU(2)_L \times U(1)_Y$. The Lagrangian of the SM is given as follows:

$$\mathcal{L}_{\mathrm{SM}} = \mathcal{L}_{\mathrm{gauge}} + \mathcal{L}_{\mathrm{matter}} + \mathcal{L}_{\mathrm{Higgs}}, \tag{2.1}$$

where:

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4}B^{\mu\nu}B_{\mu\nu} - \frac{1}{4}\sum_{a=1}^{3} W^a_{\mu\nu}W^{a\mu\nu} - \frac{1}{4}\sum_{a=1}^{8} G^a_{\mu\nu}G^{a\mu\nu}, \tag{2.2}$$

$$\begin{aligned}\mathcal{L}_{\text{matter}} =& \overline{Q_i}i\gamma^\mu\left(\partial_\mu - \frac{g'}{6}iB_\mu - \frac{g}{2}i\sigma^a W^a_\mu - \frac{g_s}{2}i\lambda^a G^a_\mu\right)Q_i \\ &+ \overline{U_i}i\gamma^\mu\left(\partial_\mu - \frac{2g'}{3}iB_\mu - \frac{g_s}{2}i\lambda^a G^a_\mu\right)U_i + \overline{D_i}i\gamma^\mu\left(\partial_\mu + \frac{g'}{3}iB_\mu - \frac{g_s}{2}i\lambda^a G^a_\mu\right)D_i \\ &+ \overline{L_i}i\gamma^\mu\left(\partial_\mu + \frac{g'}{2}iB_\mu - \frac{g}{2}i\sigma^a W^a_\mu\right)L_i + \overline{E_i}i\gamma^\mu\left(\partial_\mu + ig'B_\mu\right)E_i,\end{aligned} \tag{2.3}$$

where $\lambda^a$ is the Gell-Mann matrix $\lambda^a$ and $\sigma^a$ is the Pauli matrix. The gauge fields for gauge symmetries $U(1)_Y$, $SU(2)_L$ and $SU(3)_C$ are denoted as $B_\mu$, $W_\mu$ and $G_\mu$ with couplings $g'$, $g$ and $g_s$. On the other hand $Q$, $U$ or $D$ express quarks, and $L$ or $E$ are leptons with the generation matrix indices $i$ and $j$. The two $SU(2)$ doublets can be constructed as $AB = A^T i\sigma^2 B$. These field strengths $X^a_{\mu\nu}$ are defined by

$$X^a_{\mu\nu} = \partial_\mu X^a_\nu - \partial_\nu X^a_\mu - ig_X f^{abc}X^b_\mu X^c_\nu. \tag{2.4}$$

### 2.1.2 The Brout-Englert-Higgs Mechanism

The BEH mechanism is one of the most elegant idea in the theoretical particle physics [32–37]. The electroweak symmetry $SU(2)_L \times U(1)_Y$ is spontaneously broken to $U(1)_{EM}$ by the Higgs VEV and the Standard Model particles acquire their masses. The masses of the SM particles like $W/Z$ bosons, charged leptons, quarks and the Higgs boson itself are completely determined by the strength of the Higgs interactions and its VEV. The Higgs Lagrangian of the SM is given as follows,

$$\mathcal{L}_{\text{Higgs}} = \mathcal{L}_{\text{kinetic}} + \mathcal{L}_{\text{potential}} + \mathcal{L}_{\text{Yukawa}}, \tag{2.5}$$

where $\mathcal{L}_{\text{kinetic}}$, $\mathcal{L}_{\text{potential}}$ and $\mathcal{L}_{\text{Yukawa}}$ are kinetic, potential and Yukawa part of the Higgs Lagrangian. The Higgs kinetic part is written as

$$\mathcal{L}_{\text{kinetic}} = (D_\mu H)^\dagger D^\mu H = \left(\partial_\mu H - ig_2 W_\mu H - \frac{1}{2}ig_Y B_\mu H\right)^\dagger\left(\partial^\mu H - ig_2 W^\mu H - \frac{1}{2}ig_Y B^\mu H\right). \tag{2.6}$$

The Higgs potential is assumed as a wine-bottle or mexican-hat type form

$$\mathcal{L}_{\text{potential}} = \mu^2 H^\dagger H + \frac{\lambda}{2}\left(H^\dagger H\right)^2 = \frac{\lambda}{2}\left(H^\dagger H - \frac{-\mu^2}{\lambda}\right)^2 + \frac{\mu^4}{2\lambda}, \tag{2.7}$$

where the Higgs mass is negative to be $\mu^2 < 0$. The origin of the Higgs potential is unstable, and therefore, the Higgs field classically rolls down to the stable vacuum state and gets the the vacuum expectation value (VEV),

$$\langle 0|H|0\rangle = v_{\text{EW}} = \sqrt{\frac{-\mu^2}{\lambda}}. \tag{2.8}$$

The Higgs VEV spontaneously breaks the electroweak symmetry $SU(2)_L \times U(1)_Y$ to the electromagnetic symmetry $U(1)_{EM}$. The Higgs field can be decomposed as a four-component scalar field,

$$H = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} = \begin{pmatrix} 0 \\ v_{\rm EW} \end{pmatrix} + \begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}}\left(h + iG^0\right) \end{pmatrix}, \tag{2.9}$$

where $h$ is the SM Higgs boson discovered at the LHC and $G$ is the Nambu-Goldstone bosons. The spontaneous electroweak symmetry braking via the Higgs VEV makes the three $W$ bosons massive observed as the $W_\pm$ bosons and the $Z$ boson as follows:

$$Z_\mu = \frac{1}{\sqrt{g^2 + g'^2}}\left(gW_\mu^3 - g'B_\mu\right) = \cos\theta_W W_\mu^3 - \sin\theta_W B_\mu, \tag{2.10}$$

$$A_\mu = \frac{1}{\sqrt{g^2 + g'^2}}\left(g'W_\mu^3 + gB_\mu\right) = \sin\theta_W W_\mu^3 + \cos\theta_W B_\mu, \tag{2.11}$$

$$W_\mu^\pm = \frac{1}{\sqrt{2}}\left(W_\mu^1 \mp iW_\mu^2\right), \tag{2.12}$$

where $\theta_W$ is the Weinberg angle. The Higgs VEV generates the masses of the SM particles like $W/Z$ bosons, charged leptons, quarks and the Higgs boson itself,

$$m_h^2 = 2\lambda v_{\rm EW}^2, \quad M_Z^2 = \frac{g'^2 + g^2}{2} v_{\rm EW}^2, \quad M_W^2 = \frac{g^2}{2} v_{\rm EW}^2. \tag{2.13}$$

The Higgs VEV completely determines the electroweak scale and $v_{\rm EW}$ has been given by the measurement of the mass of the $W^\pm$ bosons.

$$G_F = \frac{g^2\sqrt{2}}{8m_W^2} = \frac{\sqrt{2}}{2v_{\rm EW}^2}, \tag{2.14}$$

where the Fermi constant $G_F$ was experimentally known with a very good accuracy and therefore, the value of Higgs VEV could be estimated as $v_{\rm EW} \sim 246$ GeV. However, the self-coupling $\lambda$ can be only predicted by the measurement of the Higgs boson mass. The mass of the fermions $m_f = y_f v_{\rm EW}/2$ are generated after spontaneous symmetry breaking. The couplings $y_f$ are called Yukawa couplings and determines the observed values of fermion masses.

## 2.2 Beyond the Standard Model

The SM is almost complete and satisfying theory at the low energy scale. Even if we extrapolate the SM up to the Planck scale adopted to be $M_{\rm Pl} \sim 10^{19}$ GeV, it remains theoretically consistent and no new physics must be required. However, there are many long standing questions and puzzles in particle physics and cosmology. Phenomenologically, there are various experimental evidences which the SM can not explain correctly: dark matter, dark energy, horizon or flatness problem, baryon asymmetry and non-zero neutrino mass. Theoretically, there are more troublesome problems, such as the naturalness or fine-tuning problem of the Higgs boson or the cosmological constant, the consistent formulation of quantum gravity (QG), the origin of the electroweak symmetry breaking and the stability problem of the electroweak vacuum. These notorious problems are important clues to understand new physics beyond the SM and the cosmological history of our Universe.

Let us list these problems beyond the SM as follows:

- Dark matter: Non-baryonic dark matter in the Universe
- Dark energy: Accelerated expansion of the Universe
- Baryon asymmetry in the Universe
- Horizon problem: Homogeneity of the Universe although there are many causally disconnected patches
- Flatness problem: Energy density of the Universe so close to the critical density
- Energy balance of the Universe: Why are the contributions of the dark energy, dark matter and baryonic matter comparable in size today ?
- Quantum Gravity: What is the consistent theory of QG ?
- Non-zero neutrino masses from neutrino flavor oscillations
- Strong CP violation: Why is $\theta_{\rm QCD}$ extremely small ?
- What is the origin of the Higgs sector of the SM ?
- Naturalness problem: why are the Higgs boson mass or the cosmological constant so small compared with the Planck scale ?
- Vacuum stability: Is the Higgs (electroweak) vacuum stable or not ?

Despite remarkable experimental confirmations, the SM can not explain many problems. Therefore we are convinced that there would exist new physics beyond the SM. Below let us explain phenological or theoretical problems in quite some detail.

The gravity around our solar system can be correctly described by the Einstein's general relativity with the SM, but it fail at the larger scales. As is well known, the matter of the SM is not enough to explain the rotation of the galaxies. The unknown matter is called dark matter and we do not know what it is at all. Moreover, various cosmological observations like the distant type-Ia supernovae show that the expansion of the Universe is accelerating. This fact can be explained by postulating that the current Universe is dominated by a mysterious vacuum energy called the dark energy. The cosmological observations of the cosmic microwave background (CMB) by the WMAP (Wilkinson Microwave Anisotropy Probe) satellite and the recent Planck satellite have clarified that almost content of the present Universe is formed by the mysterious dark matter and dark energy.

Furthermore there exist old cosmological problems like horizon or flatness problem. These puzzles can be explained by the inflation which is the hypothetical exponential expansion of the

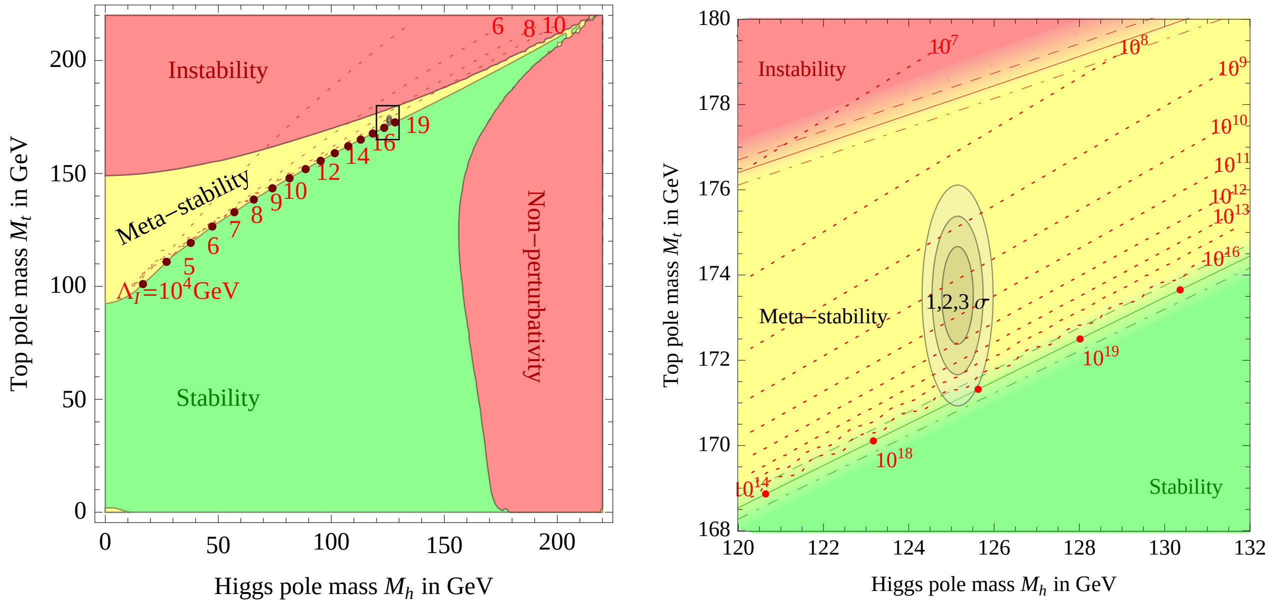


Figure 2.1: The SM phase diagram in terms of the Higgs boson mass $M_h$ and the top quark mass $M_t$ shows the instability, metastability, and stability region of the electroweak vacuum. The grey areas expresses 1, 2 and $3\sigma$ uncertainties for $M_h$ and $M_t$. The uncertainties of $M_h$ and $M_t$ still allow for the absolute stability region and there is a need for more precise measurements of these masses to decide the state of the electroweak vacuum stability. This figure is cited from Ref. [14].

early Universe. Most of the framework of the inflation requires some new scalar particles beyond the SM. On the other hand, there exits another unsolved problem, the baryon asymmetry of the observed Universe. The contents of the SM can not explain a huge amount of baryons against anti-baryons. The desired mechanism is named baryogenesis which requires new particles or new interactions beyond the SM. Most of these cosmological problems like dark matter, dark energy, inflation and even baryogenesis are closely related to the gravity and therefore a modification of the gravity can solve some of these problems. However it would be highly improbable that the modified gravity solves all these problems.

These facts require some new physics beyond the SM. However, there is another motivation to extend the SM from the theoretical point of view. One of the most notorious issues is about quantum gravity (QG). The SM does not include the gravity and the Einstein's general relativity is not satisfactory from the viewpoint of the QFT. In principle, we should quantize the gravity to discuss quantum phenomena beyond the Planck scale. However, QG has essentially non-renormalizable and non-unitary properties, and therefore we do not have any consistent theory for QG although various attractive theories have been investigated.

The most troublesome issues within the SM is so-called naturalness problem. The Higgs field is not protected from quantum radiative corrections and closely related with the high-energy scale physics such as GUT, QG and string theory. Certainly the quantum radiative corrections to the Higgs mass $\mu_H$ have quadratic divergences and require unnatural fine-tuning,

$$\Delta\mu_H^2 \simeq \frac{\alpha}{(4\pi)^2}\left(\Lambda_{UV}^2 + \cdots\right), \tag{2.15}$$

where $\alpha$ is the coupling of the SM particles and $\Lambda_{\rm UV}$ is the ultraviolet cut-off scale. If we assume that the SM is valid up to the Planck scale $M_{\rm Pl} \sim 10^{19}$ GeV, we encounter incredible fine-tuning cancellation between the bare Higgs mass $\mu_H^2$ and the quantum corrections $\Delta\mu_H^2$. This is called the fine-tuning problem or naturalness problem. The leading candidate to solve the issue is e.g. supersymmetry, extra-dimensions and compositeness. However, these proposals has been suffered from the observed Higgs boson mass $m_h^{\rm obs} = 125$ GeV and the current experimental constraints on the new physics. The fine-tuning problem of the the Higgs mass and the cosmological constant has remained one of the most serious puzzles of the particle physics.

On the other hand, the stability of the Higgs potential is one of the most important problem beyond the SM. For the present best fit values of the Higgs mass $m_h$ [8–11] and the top quark mass $m_t$ [12] the SM Higgs potential develops an instability below the Planck scale. This instability suggests that our living electroweak vacuum is a false vacuum and would eventually collapse. The life-time of the Higgs vacuum via the standard analysis of QFT is exceedingly small compared with the age of the Universe under these conditions [13–15]. However there are several experimental and theoretical uncertainties, and furthermore it has recently pointed out that the above situation drastically changes when the gravitational effects can not be ignored [16–21]. The stability of the electroweak vacuum has a grate impact on the entire history of the Universe and provide some hints of the new physics beyond the SM. These issues are the central subjects of this thesis.

## 2.3 The Quantum Zero-point Vacuum Energy

The SM which is the self-consistent framework for elementary particle physics is formally based on the QFT. Formally, the QFT [38] is constructed as an enormously large collection of the quantum harmonic oscillators. Thus, the vacuum energy receives divergent zero-point energy with various quantum fields,

$$E_{\rm zero} = \frac{\pm 1}{2}\sum_{\rm spin}\sum_{k}\omega_k \longrightarrow \infty, \tag{2.16}$$

where $\omega_k = \sqrt{k^2 + m^2}$ and $m$ is the masses of quantum fields. Theoretically, the quantum zero-point corrections can be renormalized by bare parameters and one fixes the physical values so that they agree with the observations. Therefore, the QFT makes no prediction for the physical values of the vacuum energy [39] although the fine-tuning between the bare parameters and the quantum corrections would be still serious and leads to the naturalness problems of the cosmological constant and the Higgs boson mass.

On the other hand difference of the divergent zero-point energy $\Delta E_{\rm zero}$ has already been recognized to provide the observable effects. Famously, the Casimir effect [40] can be described by using electromagnetic zero-point energy between two parallel conducting plates and has been experimentally detected [41]. The difference of divergent energy with the boundary conditions becomes finite Casimir energy [42] and these phenomena have indeed provided an important hint of this problem.

Now, let us consider a massless scalar field between two parallel plates to impose the Dirichlet boundary condition,

$$\phi\left(z=0\right)=\phi\left(z=a\right)=0. \tag{2.17}$$

where the boundary condition discretizes the modes $k=n\pi/a$ and the zero-point energy on this condition can be given by

$$E_{\text{zero}}=\frac{1}{2}\sum_{k}\omega_{k}=\frac{1}{2}\sum_{n=0}^{\infty}\int\frac{d^{2}k}{\left(2\pi\right)^{2}}\sqrt{k^{2}+\left(\frac{n\pi}{a}\right)^{2}}, \tag{2.18}$$

Next, let us adopt so-called zeta function regularization method and use following mathematical formula,

$$\int_{0}^{\infty}\frac{dt}{t}t^{-\alpha}e^{-zt}=\Gamma\left(-\alpha\right)z^{\alpha},\quad\int d^{d}ke^{-tk^{2}}=\left(\frac{\pi}{t}\right)^{d/2}, \tag{2.19}$$

where $d$ is the complex dimension of the spacetime and the left expression is called the proper time integral. Using these formula, we can get the following expression,

$$\begin{aligned}E_{\text{zero}}&=\frac{1}{2}\sum_{n=0}^{\infty}\int\frac{d^{d}k}{\left(2\pi\right)^{d}}\int_{0}^{\infty}\frac{dt}{t\cdot\Gamma\left(-1/2\right)}t^{-1/2}e^{-t\left(k^{2}+\left(n\pi/a\right)^{2}\right)}\\&=-\frac{1}{4\sqrt{\pi}}\frac{1}{\left(4\pi\right)^{d/2}}\sum_{n=0}^{\infty}\int_{0}^{\infty}\frac{dt}{t}t^{-1/2-d/2}e^{-tn^{2}\pi^{2}/a^{2}}\end{aligned} \tag{2.20}$$

Proceeding the calculation we obtain

$$\begin{aligned}E_{\text{zero}}&=-\frac{1}{4\sqrt{\pi}}\frac{1}{\left(4\pi\right)^{d/2}}\left(\frac{\pi}{a}\right)^{1+d}\Gamma\left(-\frac{d+1}{2}\right)\sum_{n=0}^{\infty}n^{d+1}\\&=-\frac{1}{4\sqrt{\pi}}\frac{1}{\left(4\pi\right)^{d/2}}\left(\frac{\pi}{a}\right)^{1+d}\Gamma\left(-\frac{d+1}{2}\right)\zeta\left(-d-1\right)\end{aligned} \tag{2.21}$$

Now we take analytic continuation to remove the divergences and get the Riemann zeta function,

$$\Gamma\left(\frac{z}{2}\right)\zeta\left(z\right)\pi^{-z/2}=\Gamma\left(\frac{1-z}{2}\right)\zeta\left(1-z\right)\pi^{-(1-z)/2} \tag{2.22}$$

The quantum zero-point energy on the Dirichlet boundary condition can be written as

$$\begin{aligned}E_{\text{Casimir}}&=\lim_{d\to2}\left\{-\frac{1}{2^{d+2}\pi^{d/2+1}}\frac{1}{a^{d+1}}\Gamma\left(1+\frac{d}{2}\right)\zeta\left(2+d\right)\right\}\\&=-\frac{\pi^{2}}{1440\cdot a^{3}}\end{aligned} \tag{2.23}$$

where the zero-point divergences are systematically removed by the analytic continuation of the zeta function $\zeta(z)$. The finite and negative zero-point energy is so-called Casimir energy and observable as the attractive force between two parallel plates at small distances,

$$F_{\text{Casimir}} = -\frac{\partial E_{\text{Casimir}}}{\partial a} = -\frac{\pi^2}{480 \cdot a^4} \tag{2.24}$$

where $F_{\text{Casimir}}$ is a famous Casimir force per unit area. In the electromagnetic fields between two parallel conductive plates, the Casimir force can be written as $F_{\text{Casimir}} = -\pi^2/(240 \cdot a^4)$ [40] where we count the two polarization states of the photon. The Casimir effect has been confirmed in many experiments [43–45] and strongly depend on the size, geometry and topology of the given boundaries. On the other hand, theoretically, whether the observations of the Casimir force prove the reality of the zero-point energy or not has been still under debate [46] because the Casimir force can alternatively be computed without invoking the zero-point electromagnetic energy as the standard perturbative methods of QED [46] like the Lamb shift and the van der Waals interactions [47–49]. The question of the existence of the zero-point energy is outside the scope of this thesis, but the existence is crucial for constructing the Coleman-Weinberg effective potential. In next section we consider the fine-tuning problems of the cosmological constant or the Higgs mass and the formulation of the Coleman-Weinberg effective potential.

### 2.3.1 The Naturalness of the Cosmological Constant

The quantum vacuum energy density is given by the following momentum integral

$$\rho_{\text{zero}} = \frac{E_{\text{zero}}}{\text{Volume}} = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\sqrt{k^2+m^2}. \tag{2.25}$$

Taking the momentum cut-off $\Lambda_{\text{UV}}$, the divergences of the quantum zero-point energy density are regularized as

$$\rho_{\text{zero}} = \frac{1}{2}\int^{\Lambda_{\text{UV}}} \frac{d^3k}{(2\pi)^3}\sqrt{k^2+m^2} \tag{2.26}$$

$$= \frac{\Lambda_{\text{UV}}^4}{16\pi^2} + \frac{m^2\Lambda_{\text{UV}}^2}{16\pi^2} + \frac{m^4}{64\pi^2}\log\left(\frac{m^2}{\Lambda_{\text{UV}}^2}\right) + \cdots, \tag{2.27}$$

which has quartic or quadratic divergences. If the cut-off $\Lambda_{\text{UV}}$ physically corresponds to the Planck scale $M_{\text{Pl}}$, these divergence terms imply that the bare cosmological constant vacuum term $\rho_\Lambda = \Lambda/8\pi G_N$ defined by the cosmological constant $\Lambda$ and the Newton's constant $G_N$ must be fine-tuned to the current vacuum energy density. Thus, there is an huge discrepancy between the theoretical prediction and the observed dark energy obtained from current cosmological data [50–54]. This is known as the (old) cosmological constant problem, which is recognized as one of the most profound puzzles in theoretical physics [55–64].

Let us adopt the dimensional regularization and the quantum energy density is regularized,

$$\rho_{\text{zero}} = \frac{m^4}{64\pi^2}\left[\ln\left(\frac{m^4}{\mu^2}\right) - \frac{1}{\epsilon} - \log 4\pi + \gamma - \frac{3}{2}\right], \tag{2.28}$$

where $\mu$ is the subtraction scale of dimensional regularization, $\epsilon$ is the regularization parameter and $\gamma$ is the Euler's constant. The divergence terms of $\epsilon$ are absorbed by $\rho_\Lambda$. Next, let us split the bare term $\rho_\Lambda$ to be $\rho_\Lambda = \rho_\Lambda(\mu) + \delta\rho_\Lambda$ and the counterterm $\delta\rho_\Lambda$ of the cosmological constant in $\overline{\text{MS}}$ scheme can be written as

$$\delta\rho_\Lambda = \frac{m^4}{4(4\pi)^2}\left(\frac{1}{\epsilon} + \log 4\pi - \gamma\right), \tag{2.29}$$

where the counterterm $\delta\rho_\Lambda$ depends on the regularization or renormalization scheme. By absorbing the divergence terms into the counterterm $\delta\rho_\Lambda$, we obtain a finite renormalized expression,

$$\rho_{\text{vacuum}} = \rho_\Lambda + \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\sqrt{k^2+m^2} \tag{2.30}$$

$$= \rho_\Lambda(\mu) + \delta\rho_\Lambda + \frac{m^4}{64\pi^2}\left[\ln\left(\frac{m^4}{\mu^2}\right) - \frac{1}{\epsilon} - \log 4\pi + \gamma - \frac{3}{2}\right] \tag{2.31}$$

$$= \rho_\Lambda(\mu) + \frac{m^4}{64\pi^2}\left(\ln\frac{m^4}{\mu^2} - \frac{3}{2}\right). \tag{2.32}$$

The divergences of the zero-point energy density can be formally removed by the normal ordering of the operators or absorbed by the renormalization of the cosmological constant. However, we encounter incredible fine-tuning of the cosmological constant. In the framework of the SM, we can write down the vacuum energy density as follows:

$$\rho_{\text{vacuum}} = \rho_\Lambda + \rho_{\text{EW}} + \rho_{\text{QCD}} + \sum_i \frac{n_i m_i^4}{64\pi^2}\left(\ln\frac{m_i^4}{\mu^2} - \frac{3}{2}\right) + \mathcal{O}\left(\Lambda_{\text{UV}}^4\right) + \cdots, \tag{2.33}$$

where $\rho_{\text{EW}} \sim 10^8$ GeV$^4$ and $\rho_{\text{QCD}} \sim 10^{-2}$ GeV$^4$ express classical vacuum energies originating from electroweak symmetry breaking or chiral symmetry breaking, respectively. The logarithm terms express quantum corrections of the SM particles where $n_i$ and $m_i$ are the number of degrees of freedom and the mass of the SM particles $i$, respectively. The current physical value of the energy density observed as the dark energy, $\rho_{\text{dark}} \simeq 2.5 \times 10^{-47}$ GeV$^4$ is unacceptably smaller than the theoretical predictions of those energy densities. Despite enormous efforts, there are no satisfactory scenario to solve this fine-tuning issue and derive the dark energy $\rho_{\text{dark}} \simeq 2.5 \times 10^{-47}$ GeV$^4$ which is far from the electroweak, GUT and Planck scale.

### 2.3.2 The Coleman-Weinberg Effective Potential

Let us consider the Coleman-Weinberg (CW) effective potential [65] which is constructed by the zero-point energy. Let us consider the scalar potential,

$$V(\phi) = \rho_\Lambda + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4. \tag{2.34}$$

where $\lambda$ is self-interaction coupling of the scalar field $\phi$. In this set-up the quantum zero-point energy is written as

$$V_{\text{zero}}(\phi) = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\sqrt{k^2 + M^2(\phi)}, \tag{2.35}$$

where $M(\phi) = V''(\phi)^{1/2} = \sqrt{m^2 + 3\lambda\phi^2}$ and the CW effective potential is formally constructed as follows:

$$\begin{aligned} V_{\text{eff}}(\phi) &= V(\phi) + V_{\text{zero}}(\phi) \\ &= \rho_\Lambda(\mu) + \frac{m^2(\mu)}{2}\phi^2 + \frac{\lambda(\mu)}{4}\phi^4 + \delta\rho_\Lambda + \frac{\delta m^2}{2}\phi^2 + \frac{\delta\lambda}{4}\phi^4 \\ &\quad + \frac{M^4(\phi)}{64\pi^2}\left[\ln\left(\frac{M^2(\phi)}{\mu^2}\right) - \frac{1}{\epsilon} - \log 4\pi + \gamma - \frac{3}{2}\right] \\ &= \rho_\Lambda(\mu) + \frac{m^2(\mu)}{2}\phi^2 + \frac{\lambda(\mu)}{4}\phi^4 + \frac{M^4(\phi)}{64\pi^2}\left[\ln\left(\frac{M^2(\phi)}{\mu^2}\right) - \frac{3}{2}\right], \end{aligned} \tag{2.36}$$

where $\delta\rho_\Lambda$, $\delta m$, $\delta\lambda$ are the counterterms of the couplings. The variances of the zero-point energy via the scalar field vales $\phi$ modify the classical potential $V(\phi)$ and the couplings, and sometimes the ground state to be minima of net energy density changes against what we would expect from the classical potential.

The famous example of the CW mechanism is a model of the massless scalar QED which is described by

$$\mathcal{L}_{\text{MSQ}} = -\frac{1}{4}(F_{\mu\nu})^2 + \frac{1}{2}(D_\mu\phi)^2 - \frac{1}{4!}\lambda\phi^4, \tag{2.37}$$

where the scalar field $\phi$ is complex, the electromagnetic field tensor is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and the covariant derivative containing the electric charge $e$ is $D_\mu = \partial_\mu - eA_\mu$. The one-loop effective potential of the massless scalar is given by

$$V_{\text{eff}}(\phi) = \frac{\lambda}{4!}\phi^4 + \left(\frac{5\lambda^2}{1152\pi^2} + \frac{3e^4}{64\pi^2}\right)\phi^4\left(\ln\frac{\phi^2}{\mu^2} - \frac{25}{6}\right), \tag{2.38}$$

where $\mu$ corresponds to the renormalization scale and we can simply absorb the factor of $-25/6$ into the definition of $\mu$. By setting the scale to the VEV as $\mu = v_\phi$ and neglecting the term $\lambda^2$, we obtain a well-known effective potential as follows

$$V_{\text{eff}}(\phi) = \frac{3e^4}{64\pi^2}\phi^4\left(\ln\frac{\phi^2}{v_\phi^2} - \frac{1}{2}\right). \tag{2.39}$$

Although the classical potential $V(\phi)$ has no mass term, the spontaneous symmetry breaking can occur due to the quantum radiative correction. Therefore the quantum radiative correction has physical and phenomenological meanings.

### 2.3.3 The Naturalness of the Standard Model Higgs Boson

Besides the problem of the cosmological constant there is a same problem in the Higgs sector of the SM. The electroweak hierarchy or naturalness problem [66–68] has been discussed for past decades and motivated some TeV scale new physics e.g. supersymmetry, extra-dimensions and compositeness. Let us briefly review the Higgs naturalness problem.

The Higgs mass of the SM receives quadratic quantum corrections from loop diagrams. The contribution at one-loop diagram can be estimated by

$$\mu^2_{\text{physical}} \simeq \mu^2_H + \frac{\alpha}{(4\pi)^2}\left(\Lambda^2_{\text{UV}} + \cdots\right), \tag{2.40}$$

where $\alpha$ is $\mathcal{O}(1)$ coupling and $\Lambda_{\text{UV}}$ is the UV cut-off scale. The Higgs mass term has any symmetry and therefore we encounter incredible fine-tuning cancellation between the bare Higgs mass $\mu^2_H$ and the quantum corrections from the cut-off scale $\Lambda_{\text{UV}}$. One of possibility to solve the unnaturalness is that the cut-off scale $\Lambda_{\text{UV}}$ is near $\mathcal{O}(10^2)$ GeV, and the standard QFT breaks down above the scale.

One of the most elegant solution is the supersymmetry (SUSY) being a fundamental symmetry between fermions and bosons. The SUSY provides two scalar boson for a respective fermion and cancels the quadratic divergences via the super-partners. However, it must be broken softly by some breaking scale $M_{\text{SUSY}}$ because we have no super-partners at the low-energy scale. Thus, the SUSY contributions to the Higgs mass can be estimated as

$$\mu^2_{\text{physical}} \simeq \mu^2_H + \frac{\alpha}{(4\pi)^2} M^2_{\text{SUSY}} \log\frac{\Lambda^2_{\text{UV}}}{M^2_{\text{EW}}} + \cdots, \tag{2.41}$$

where we expect that the SUSY breaking scale $M_{\text{SUSY}}$ must be near the electroweak scale $M_{\text{EW}}$ to relax the hierarchy. But, the above expectation has been suffered from the observed Higgs boson mass and the current bound for SUSY particles at the LHC. Furthermore, the case of the cosmological constant already breaks the above expectation. The naturalness of the the Higgs mass or the cosmological constant has remained one of the most serious problem and there are no entirely satisfactory solutions to these problems.

## 2.4 Phenomenology of the Higgs Vacuum Stability

In this section we turn to the more realistic problem of the Higgs sector. The renormalization group (RG) running of the Higgs self-coupling provides rich information about the high-energy physics and gave allowed regions of the Higgs mass before its discovery. The observed Higgs boson with $m_h \approx 125\,\text{GeV}$ has no Landau pole problem or triviality and therefore it seems to require a drastic modifications of the SM up to the Planck scale. However, the vacuum stability of the Higgs potential has remained a real problem. For the present best fit values of the Higgs mass $m_h$ and the top quark mass $m_t$, our electroweak vacuum has been living in the edge between the stability and the instability. Thus, the metastability of the Higgs vacuum has recently attracted significant interest in the high-energy community. In this section we review a phenomenology of the running Higgs self-coupling and provide a comprehensive review of the the vacuum stability of the Higgs.

### 2.4.1 The RG running of the Higgs self-coupling

The renormalization group (RG) is the standard tool in QFT and the RG running of the coupling provides a detailed depiction about the high-energy physics. The one-loop RG equations in the

SM are given by

$$\frac{d\lambda}{dt} = \frac{1}{(4\pi)^2}\left(\lambda\left(-9g^2 - 3g'^2 + 12y_t^2\right) + 24\lambda^2 + \frac{3}{4}g^4 + \frac{3}{8}\left(g^2 + g'^2\right)^2 - 6y_t^4\right), \tag{2.42}$$

$$\frac{dy_t}{dt} = \frac{1}{(4\pi)^2}\left(\frac{9}{2}y_t^3 + y_t\left(-\frac{9}{4}g^2 - \frac{17}{12}g'^2 - 8g_s^2\right)\right), \tag{2.43}$$

$$\frac{dg_1}{dt} = \frac{1}{(4\pi)^2}\left(-\frac{19}{6}g^3\right), \quad \frac{dg'}{dt} = \frac{1}{(4\pi)^2}\left(\frac{41}{6}g'^3\right), \quad \frac{dg_s}{dt} = \frac{1}{(4\pi)^2}\left(-7g_s^3\right), \tag{2.44}$$

where $t \equiv \ln(\mu/\mu_0)$ is the running parameter. For simplicity we consider the one-loop RG equation of the Higgs self-coupling $\lambda$ and neglect other coupling contributions as

$$\frac{d\lambda}{dt} \simeq \frac{3\lambda^2}{2\pi^2}, \tag{2.45}$$

which lead to the simple solution

$$\lambda(\mu) = \frac{\lambda(\mu_0)}{1 - \frac{3\lambda(\mu_0)}{2\pi^2}\ln(\mu/\mu_0)}. \tag{2.46}$$

where the denominator goes to zero in the limit $\mu \to \Lambda_{\rm LP}$,

$$\Lambda_{\rm LP} \simeq \mu_0 \exp\left(\frac{2\pi^2}{3\lambda(\mu_0)}\right) \simeq v \exp\left(\frac{4\pi^2 v^2}{3m_h^2}\right). \tag{2.47}$$

The Landau pole is the coupling constant (interaction strength) becomes infinite and therefore we cannot rely on the perturbative theory to work. Since $\Lambda_{\rm LP}$ approximately given by Eq. (2.47) gives the maximum scale where our perturbation theory is valid, we need to reside allowed bound of the Higgs Landau pole or triviality. If we require that our theory is perturbative all the way up to the Planck scale $M_{\rm Pl} \sim 10^{19}\,\text{GeV}$, the maximum allowed Higgs mass becomes 180 GeV, which is consistent with the observed Higgs mass $m_h \approx 125\,\text{GeV}$.

Let us go back to the full renormalization group (RG) equation of Eq. (2.44) and consider the stability bound of the Higgs mass where we worry about whether the RG running of the Higgs self coupling $\lambda$ can lead to the vacuum instability or not. In the small coupling regime where the Higgs self-coupling $\lambda$ is around zero, we can simplify the one-loop RG equation of $\lambda$ as follows:

$$\frac{d\lambda}{d\log\mu} = \frac{1}{16\pi^2}\left(-6y_t^4 + \frac{3}{4}g^4 + \frac{3}{8}\left(g^2 + g'^2\right)^2 + \mathcal{O}(\lambda)\right) \tag{2.48}$$

$$\Longleftrightarrow \lambda(\mu) \simeq \lambda(v) + \frac{1}{16\pi^2}\left(-12y_t^4 + \frac{3}{8}g^4 + \frac{3}{16}\left(g^2 + g'^2\right)^2 + \mathcal{O}(\lambda)\right)\log\frac{\mu^2}{v^2}, \tag{2.49}$$

where $y_t = \sqrt{2}m_t/v$. The negative contributions of the running Higgs self-coupling $\lambda(\mu)$ only come from the top quark mass $m_t$. The negative Higgs self-coupling as $\lambda(\mu) < 0$ imply that our electroweak vacuum is not stable and finally decay to the more stable true vacuum. The instability scale $\Lambda_I$ where the Higgs self-coupling becomes zero as $\lambda(\Lambda_I) = 0$ can be given by

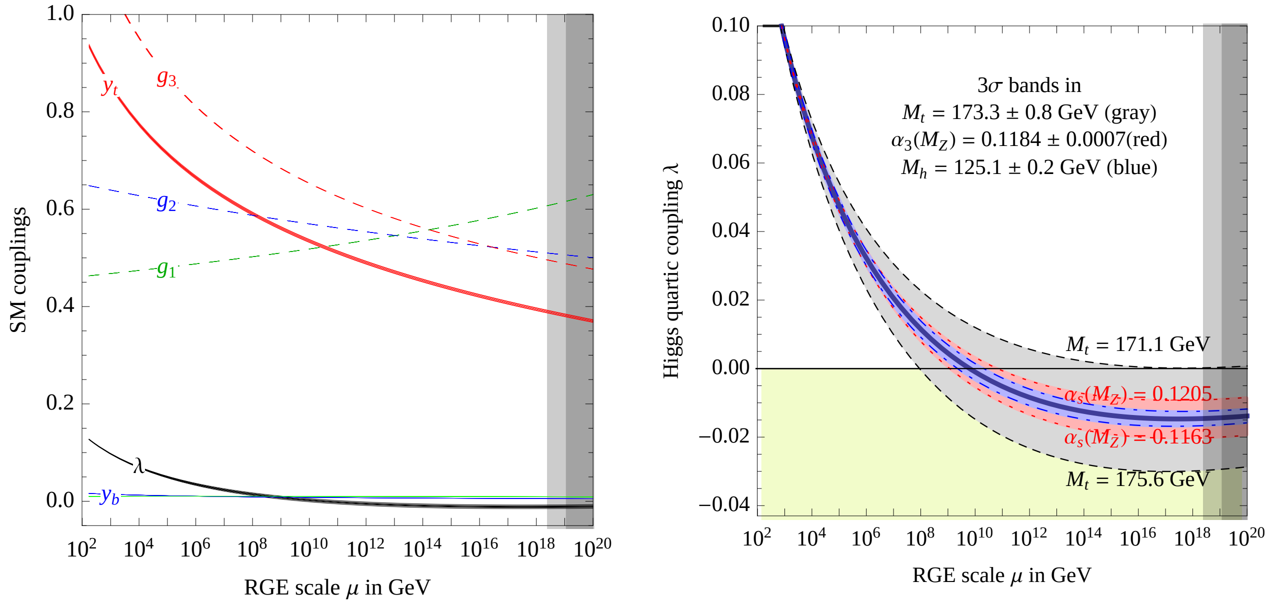


Figure 2.2: RG running of the Standard Model couplings from the electroweak scale to the Planck scale. The $3\sigma$ uncertainties of $M_h$, $M_t$ and $\alpha_3(m_Z)$ are shown by the colored bounds. The above plot is cited from by Ref. [14].

Eq. (2.49). If the Higgs mass is $m_h \approx 60\,\text{GeV}$, our electroweak vacuum is unstable unless the new physics appear around TeV scale and stabilize the Higgs potential. On the other hand, if the Higgs mass is greater than about $m_h \approx 130\,\text{GeV}$ there is no vacuum stability up to the Planck scale. The range of the Higgs mass 130 GeV $\lesssim m_h \lesssim$ 180 GeV is called as the nightmare scenario where we have no Landau pole and vacuum stability problem in the SM Higgs. From this point of view, the observed Higgs boson with $m_h \approx 125\,\text{GeV}$ does not seem to require a drastic modifications of the Higgs sector in the Standard Model. Certainly, there is no Landau pole problem or triviality in observed Higgs boson. However, the vacuum stability of the Higgs potential has been found to remain a real problem.

### 2.4.2 The Metastability of the Electroweak Vacuum

The RG running of the Higgs self-coupling affects the stability of the electroweak vacuum. The exact stability condition of the Higgs mass $m_h$ can be obtained by the analysis of the three-loop RG equations and given by [14, 15]

$$m_h > 129.6 \text{ GeV} + 2.0\,(m_t - 173.34 \text{ GeV}) - \frac{\alpha_3(m_Z) - 0.1184}{0.0014}\,\text{GeV} \pm 0.3 \text{ GeV}, \tag{2.50}$$

where the error bars originate from both experimental and theory uncertainties. By combining in various theoretical uncertainties and experimental errors, we can get the bound

$$m_h > 129.6 \text{ GeV} \pm 1.5 \text{ GeV}. \tag{2.51}$$

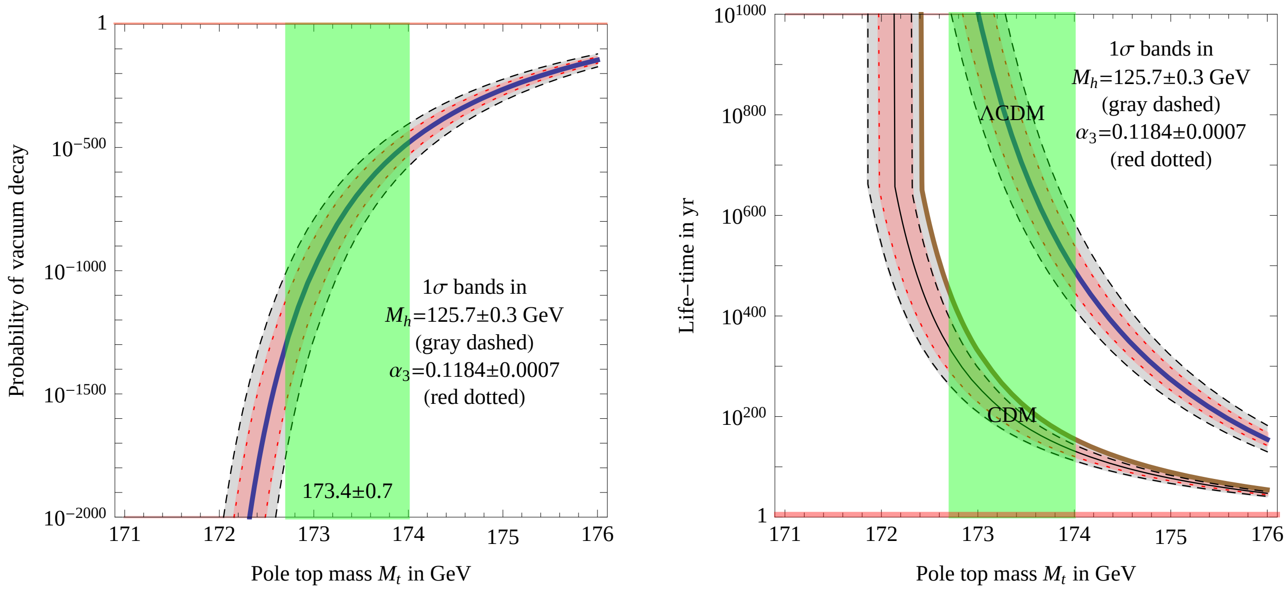


Figure 2.3: Decay probability and lifetime of the electroweak vacuum as the function of $M_h$, $M_t$ and $\alpha_3\,(m_Z)$. The above plot is cited from Ref. [14].

This result show that the vacuum stability of the SM is excluded at $2.8\sigma$. However, there are large uncertainty originating from $m_t$ in Eq. (2.50) and therefore, the measurement of the top quark mass $m_t$ has a great impact on whether the electroweak vacuum is stable or not.

Since the Higgs mass $m_h$ has been measured and the experimental error is already small at the LHC, the stability condition of the Higgs potential would be better to write the condition of $m_t$. The vacuum stability condition of the top quark mass $m_t$ can be given by

$$m_t > 171.53 \text{ GeV} \pm 0.42 \text{ GeV}, \tag{2.52}$$

where we combined in theoretical and experimental uncertainties on $m_h$ and $\alpha_3$. The instability scale $\Lambda_{\mathrm{I}}$ can be defined by the maximum value of the effective Higgs potential as $\Lambda_{\mathrm{I}} \equiv (\max V_{\mathrm{eff}}(\phi))^{1/4}$. The instability scale $\Lambda_{\mathrm{I}}$ is numerically given by

$$\log_{10}\frac{\Lambda_{\mathrm{I}}}{\text{GeV}} = 9.5 + 0.7\,(m_h - 125.15 \text{ GeV}) - 1.0\,(m_t - 173.34 \text{ GeV}) + 0.6\frac{\alpha_3\,(m_Z) - 0.1184}{0.0014}. \tag{2.53}$$

The current values of the Higgs boson mass $m_h = 125.09 \pm 0.21$ (stat) $\pm\, 0.11$ (syst) GeV [8–11] and the top quark mass $m_t = 172.44 \pm 0.13$ (stat)$\pm 0.47$ (syst) GeV [12] suggest that the effective Higgs potential develops an instability at the high scale $\Lambda_I = 10^{10} \sim 10^{11}$ GeV. Therefore, if there are no new physics to stabilize the Higgs potential, the current electroweak vacuum is not stable and finally cause a catastrophic decay through quantum tunneling. The decay rate of the electroweak vacuum per unit time and unit volume is estimated by instanton methods [69–71],

$$dp/(dV \cdot dt) = \frac{S_{\mathrm{E}}^2}{4\pi^2} \left| \frac{\det' \left[-\partial^2 + V''(\phi)\right]}{\det \left[-\partial^2 + V''(\phi)\right]} \right|^{-1/2} \exp(-S_{\mathrm{E}}), \tag{2.54}$$

where $S_{\mathrm{E}}$ is the action of the Euclidean tunneling bounce solution and the prefactor can be replaced with width of the barrier $\Lambda_{\mathrm{I}}$. Simplifying the effective Higgs potential to be $V_{\mathrm{eff}}(\phi) = -|\lambda_{\mathrm{eff}}(\phi)|\phi^4/4$ the bounce action becomes $S_{\mathrm{E}} \simeq -8\pi^2/(3|\lambda_{\mathrm{eff}}(\Lambda_{\mathrm{I}})|)$. Therefore we can get the approximate decay rate to be $dp/(dV \cdot dt) \approx \Lambda_{\mathrm{I}}^4 \exp[-2600/(|\lambda_{\mathrm{eff}}(\Lambda_{\mathrm{I}})|/0.01)]$. The volume-factor can be estimated by the age of the Universe $\tau_{\mathrm{U}}^4 \approx (e^{140}/M_{\mathrm{Pl}})^4$. Fortunately, the decay probability is extremely small $p \ll 1$ with the age of our Universe (see Fig. 2.3) and therefore, the electroweak vacuum lives in the metastability region [13, 14, 72, 73].

### 2.4.3 The Cosmological Stability of the Electroweak Vacuum

However, the recent investigations [1, 2, 4, 16, 17, 74–83] show that the electroweak vacuum metastability is incompatible with large-field inflation models. It is well-known that the vacuum field fluctuation $\langle\delta\phi^2\rangle$ rapidly glows up to Hubble scale. Therefore, if the large inflationary fluctuation $\langle\delta\phi^2\rangle$ of the Higgs field overcomes the barrier of the Higgs effective potential $V_{\mathrm{eff}}(\phi)$, it can triggers off a vacuum transition to the negative Planck-scale true vacuum. Furthermore, even after the inflation, the large Higgs fluctuation can be generated via parametric resonance or tachyonic resonance, and that is potentially serious [4, 84–88]. Besides that, the decay of the false Higgs vacuum can be enhanced in Schwarzschild background [5, 18–21, 89–92], and therefore, the existence of the evaporating black holes might not favor the metastability of the Higgs vacuum. Therefore, whether cosmological inflation of the early Universe or evaporations of the black holes are compatible with the stability problem of the Higgs vacuum has recently attracted significant interest, and the gravitationally induced vacuum fluctuation of the Higgs is crucial factors on this problem.

# Chapter 3

# Quantum Field Theory in Curved Spacetime

The modern gravitational physics is mainly based on the classical Einstein's theory. However, sometimes in the early Universe and the black hole physics, we face the necessity to properly handle the quantum phenomena involving the gravity. In principle, the quantum gravitational phenomena should be discussed in the framework of quantum gravity (QG) theory where gravity fields are also quantized together with matter fields. But owing essentially to the non-renormalizable and non-unitary properties, we have not yet got any consistent theory although there exist many attractive approaches for QG . Furthermore, some gravitational quantum phenomena being much smaller than the Planck scale can be successfully described by the semiclassical approach where matter fields only are quantized but gravity does not. The semiclassical approach of QG can be formally classified into several steps as follows:

- QFT in curved spacetime: the background spacetime is classic but the matter fields are quantized where we ignore the backreaction of the matter fields on the spacetime. The formulation and technique of QFT in curved spacetime are well-known and this approach is considered as the first approximation to QG.

- Semiclassical gravity: the spacetime is still teated as the classical background, but include the gravitational backreaction. The generalized Einsteins equations in the semiclassical gravity are written as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N \langle T_{\mu\nu} \rangle \,, \tag{3.1}$$

  where the vacuum expectation values $\langle T_{\mu\nu} \rangle$ of the energy momentum tensor $T_{\mu\nu}$. The semiclassical equations provide the quantum backreaction of the matter fields on the background spacetime. The theory of the semiclassical gravity which includes the gravitational high-order derivative terms as $R^2$, $R_{\mu\nu}R^{\mu\nu}$ or $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ is completely renormalizable.

In this section, we consider the semiclassical approach and review some techniques or methods of the QFT in curved spacetime and the semiclassical gravity.

## 3.1 The Einstein-Hilbert Action

First of all, let us briefly introduce the gravitational action in the semiclassical gravity. The Einstein-Hilbert action is the well-know and simplest action to describe the gravity. However, more complicated action involving higher order derivatives is possible and required to the renormalization of the QFT in curved spacetime. The total gravitational action can be given by

$$S_{\text{total}} \equiv -\frac{1}{16\pi G_N} \int d^4x\sqrt{-g}\,(R+2\Lambda) + S_{\text{HG}} + S_{\text{matter}}, \tag{3.2}$$

where $G_N$ is the Newton's gravitational constant and $\Lambda$ is the bare cosmological constant.

The first action is the standard Einstein-Hilbert action with the cosmological constant $\Lambda$, $S_{\text{HG}}$ is high-order derivative gravitational action and $S_{\text{matter}}$ is the matter action. The high-order gravitational action $S_{\text{HG}}$, being required to construct a renormalizable theory in curved spacetime, can be written by

$$S_{\text{HG}} \equiv -\int d^4x\sqrt{-g}\left(a_1R^2 + a_2C^2 + a_3E + a_4\Box R\right), \tag{3.3}$$

where $a_1, a_2, a_3, a_4$ express the high-order derivative gravitational couplings, $C^2 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + (1/3)\,R^2$ is the square of the Weyl tensor and $E = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ is known as the Gauss-Bonnet term. The gravitational actions via the principle of least action yields general Einstein's equations,

$$\frac{1}{8\pi G_N}G_{\mu\nu} + \rho_\Lambda g_{\mu\nu} + a_1H^{(1)}_{\mu\nu} + a_2H^{(2)}_{\mu\nu} + a_3H^{(3)}_{\mu\nu} = T_{\mu\nu}, \tag{3.4}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor and $H^{(1)}_{\mu\nu}$, $H^{(2)}_{\mu\nu}$ or $H^{(3)}_{\mu\nu}$ are tensors including the high-order derivative terms $R^2$, $R_{\mu\nu}R^{\mu\nu}$ or $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$. $T_{\mu\nu}$ is the energy momentum tensor formally defined by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}. \tag{3.5}$$

## 3.2 The Cosmological Dynamics of the Universe

The cosmological dynamics of the Universe is described in the assumption called cosmological principle that the Universe is spatially isotropic and and homogeneous and does not have any special point and direction. That is consistent with the cosmological observations and the observed Universe is almost isotropic and homogeneous at the large scale.

The geometry of the spatially isotropic and homogeneous spacetime where $g_{\mu\nu}$ is the metric can be written by Friedmann-Lemaitre-Robertson-Walker (FLRW) metric:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right], \tag{3.6}$$

where $a(t) = a = 1 + z$ is the scale factor which relates with cosmic time $t$ and redshift $z$. The coordinate $(r, \theta, \varphi)$ expresses comving coordinate system and $K$ is the curvature parameter. The positive, zero, and negative values of $K$ correspond to closed, flat, and hyperbolic Universe.

The Einstein's equations describing the cosmological dynamics of the Universe are formally given by

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}, \tag{3.7}$$

where $R_{\mu\nu}$ is the Riemann tensor and $R$ is the Ricci scalar. For the homogeneous and isotropic Universe we take the perfect fluid approximation of the energy momentum tensor,

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}, \tag{3.8}$$

where $\rho$ is the energy density, $p$ is the pressure and $u_\mu$ is the four velocity of fluid. By introducing a parameter $w$ depending on the kind of matter, we get a relation $p = w\rho$. For the matter, we have $w = 0$ and for the radiation we have $w = 1$.

The Einstein's equations reduce to two dynamical equations, determining the time evolution of the Universe as follows:

$$\left(\frac{\dot{a}}{a}\right)^2 \equiv H^2(t) = \frac{8\pi G_N}{3}\rho - \frac{K}{a^2} + \frac{\Lambda}{3}, \tag{3.9}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}(\rho + 3p) + \frac{\Lambda}{3}, \tag{3.10}$$

which are called the Friedmann equations representing the time evolution of scale factor or the entropy conservation of the Universe, and $H(t)$ is the Hubble parameter. Using Eq. (3.9) with $K = 0$ and $\Lambda = 0$ the total energy density $\rho_{c0}$ in the present Universe can be given by

$$\rho_{c0} = \frac{3H_0}{8\pi G_N}, \tag{3.11}$$

which is called critical energy density and $H_0$ is the Hubble constant. The inverse quantity $H_0^{-1}$ is so-called the Hubble time which is roughly the age of the Universe.

By using the Friedmann equations of Eq. (3.9) and Eq. (3.10), we get the time evolution of the energy density $\rho(t)$ as

$$\dot{\rho} = -3\left(\frac{\dot{a}}{a}\right)\rho(1 + \omega), \tag{3.12}$$

which leads to

$$\rho(t) \propto a^{-3(1+\omega)}, \tag{3.13}$$

where we assume that $w$ is constant. The energy density $\rho(t)$ of the matter and radiation becomes $\rho_m(t) \propto a^{-3}$ and $\rho_r(t) \propto a^{-4}$ respectively. This fact is intuitively clear. The matter density depends on the volume, but the radiation being a relativistic matter has an additional energy dependence on scale. By introducing present values of the matter, radiation, curvature and cosmological constant $\rho_{m0}$, $\rho_{r0}$, $\rho_{\Lambda 0}$ and $\rho_{K0}$, we can write the evolution of energy density $\rho(t)$ as follows

$$\rho_m = \rho_{m0} a^{-3}, \ \rho_r = \rho_{r0} a^{-4}, \ \rho_\Lambda = \rho_{\Lambda 0}, \ \rho_K = \rho_{K0} a^{-2}. \tag{3.14}$$

Let us introduce so-called density parameter as

$$\Omega_m = \frac{\rho_m}{\rho_{c0}}, \ \Omega_r = \frac{\rho_r}{\rho_{c0}}, \ \Omega_\Lambda = \frac{\Lambda}{3H_0}, \ \Omega_K = \frac{K}{H_0}, \tag{3.15}$$

and rewrite the Friedmann equations of Eq. (3.9) via these density parameter as

$$\left(\frac{H(t)}{H_0}\right)^2 = \frac{\Omega_{m0}}{a^3} + \frac{\Omega_{r0}}{a^4} + \Omega_{\Lambda 0} - \frac{\Omega_{K0}}{a^2}. \tag{3.16}$$

In the present Universe with the Hubble constant $H_0$ we can simplify Eq. (3.16) as follows

$$1 = \Omega_{m0} + \Omega_{r0} + \Omega_{\Lambda 0} - \Omega_{K0}, \tag{3.17}$$

where the matter component $\Omega_{m0}$ is strongly dominate compared with the radiation component $\Omega_{r0}$. From the latest Planck satellite data the observed Universe is spatially flat and the curvature component $\Omega_{K0}$ is nearly zero. The total density of the matter obtained from various sources such as the CMB, gravitational lensing and structure formation of galaxy clusters is about 30% including the baryonic matter and the dark matter, and the missing total energy being about 70% comes from the unknown energy component which is called the dark energy.

The cosmological constant $\Lambda$ allowed by the general coordinate invariance in general relativity provides another contribution on the cosmological dynamics of the Universe. For the cosmological constant component which has a negative pressure we have $w = -1$ and the energy density becomes constant to be $\rho_\Lambda(t) = \text{const}$. The cosmological constant $\Lambda$ can be also considered as the vacuum energy density term $\rho_{\text{vacuum}}$, given by the vacuum expectation vales of the energy momentum tensor $\langle T_{\mu\nu} \rangle = g_{\mu\nu} \rho_{\text{vacuum}}$. The vacuum energy comes from the minima of various scalar potential and the quantum corrections. From the viewpoint of QFT, we can not distinguish the cosmological constant $\Lambda$ and $\rho_{\text{vacuum}}$ whose relations are written as

$$\rho_\Lambda = \frac{\Lambda}{8\pi G_N} = \rho_{\text{vacuum}}. \tag{3.18}$$

However, as discussed in the previous section quantum corrections originate from the zero-point energy and therefore has notoriously and badly divergences

$$\rho_{\text{vacuum}} \propto \frac{\pm 1}{2} \int^\infty \frac{d^3k}{(2\pi)^3} \sqrt{k^2 + m^2}, \tag{3.19}$$

where the quantum corrections become proportional to $\Lambda_{\text{UV}}^4$ if we introduce cut off the integration at some scale $\Lambda_{\text{UV}}$. But the cut-off scale $\Lambda_{\text{UV}}^4$ which could be physically inspired by some new physics beyond the standard model still is unacceptably and extremely larger than the scale of the dark energy.

The cosmological constant term $\rho_\Lambda$ is still unknown, but it can determine the cosmological dynamics of the Universe which is summarized by

$$\ddot{a} = -\frac{4\pi G_N}{3}\left(\rho\left(1+3\omega\right)-2\rho_\Lambda\right)a\left(t\right). \tag{3.20}$$

From this equation we know that the matter with $w = 0$ and the radiation $w = 1$ decelerate the expansion of the Universe whereas the cosmological constant term $\rho_\Lambda$ exponentially accelerates its expansion. The value of $\rho_\Lambda$ can have both signs because the quantum zero-point energy has opposite signs for bosons and fermions. For the positive case the Universe exponentially expands and approaches de Sitter Universe but for the negative case it becomes Anti-de Sitter Universe and finally collapses. The current cosmological data seem consistent with the positive cosmological constant and suggest the Universe with the fine-tuning cosmological constant.

## 3.3 Review of the Primordial Inflation

The current Universe is found to be expanding from the supernova observation [93]. The cosmological observations show us the time evolution of the Universe and the existence of the extremely high density plasma. In this high temperature and density era, the Universe was opaque filled with very active particles and radiation. Due to the cosmic expansion, the high temperature and density plasma is finally cooled until neutral atoms are formed, and the Universe becomes transparent. The radiation from this period became observable and is now observed as the cosmic microwave background (CMB). The CMB is precisely measured by COBE [94], WMAP [95], and Planck [96] satellite mission.

However, the observed CMB is very homogeneous and isotropic, and therefore, the pure interpretation of this observation leads to the horizon problem. Another issue is about the critical density of the Universe. In general, small perturbations from the critical density increase in time and become a large effect. Thus, it is suggested that the past critical density has to be fine-tuned in order to be compatible with the observed values. This is known as the flatness problem. Finally, many theories predict the formation of exotic particles like magnetic monopoles in the early Universe, but such exotic particles have not been observed. These cosmological problems are elegantly solved by assuming that the early Universe experienced a period of exponential expansion known as inflation.

Inflation which provides elegant explanations for the horizon, flatness and monopoles problems, and also generate seeds of primordial density perturbations finally initiating the formation of galaxies and large-scale structure is the most reasonable scenarios for the early Universe. In this section, we review a scenario of the standard inflation.

### 3.3.1 Slow-roll Inflation

From the Klein-Gordon and Friedmann equation we can obtain

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \tag{3.21}$$

$$3M_{\rm pl}^2 H^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{3.22}$$

$$\frac{\ddot{a}}{a} = \frac{1}{3M_{\rm pl}^2}\left[V(\phi) - \dot{\phi}^2\right]. \tag{3.23}$$

In order to generate the accelerated expansion we consider $V(\phi) \gg \dot{\phi}^2$ and impose the following conditions to continue sufficiently long

$$\left|\ddot{\phi}\right| \ll \left|H\dot{\phi}\right|,\ |V'(\phi)|. \tag{3.24}$$

For the above conditions we can simplify the Klein-Gordon equation

$$3H\dot{\phi} \simeq -V'(\phi). \tag{3.25}$$

Inserting this equation into $V(\phi) \gg \dot{\phi}^2$ we can obtain the following slow-roll parameters

$$\epsilon \equiv \frac{M_{\rm pl}^2}{2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 \ll 1, \quad |\eta| \equiv \left|M_{\rm pl}^2\left(\frac{V''(\phi)}{V(\phi)}\right)\right| \ll 1. \tag{3.26}$$

These two slow-roll parameters $\epsilon$ and $\eta$ provides a useful tool to investigate the inflaton potential. For $\epsilon, |\eta| \simeq 1$ the slow-roll conditions brake down and the cosmic inflation finally ends.

Let us consider the primordial density perturbations generated during inflation and their relations with slow-roll parameters. Due to the quasi de Sitter expansion of the spacetime, quantum vacuum fluctuations inside the horizon are stretched toward the super-horizon scale and becomes finally classic. The primordial curvature and tensor perturbations are generated thorough this process. The scalar and tensor power spectrum are given by

$$P_s \simeq \left(\frac{H_*}{2\pi M_{pl}}\right)^2 \frac{1}{2\epsilon}, \quad P_t \simeq \frac{2}{\pi^2}\frac{H_*^2}{M_{pl}^2}. \tag{3.27}$$

From these scalar and tensor power spectrum the tensor-to-scalar ratio $r$ closed related with the scale of the inflation is defined as

$$r \equiv \frac{P_t}{P_s} \simeq 16\epsilon, \tag{3.28}$$

and scalar and tensor spectral index are defined as

$$\frac{d\ln P_s(k)}{dk} \equiv n_s - 1 \simeq 2\eta - 6\epsilon, \quad \frac{d\ln P_t(k)}{dk} \equiv n_t \simeq -2\epsilon. \tag{3.29}$$

The measurements of $n_s$ and $r$ determines the inflaton potential. The CMB observations already restricts the scalar spectral index and the tensor-to-scalar ratio [96]; $n_s = 0.9603 \pm 0.0073$ and

$r < 0.11$. The observed $n_s$ suggests the nearly scale invariant spectrum and provides the strong evidence of the inflation model. Note that not all models of inflation predict the spectrum of perturbations originating from the inflaton field. The simple model is so called curvaton scenario [97–99] where the primordial perturbations are given by another (curvaton) field.

The e-folding number at the observable scale $k$ is defined as,

$$N_k \equiv \ln \frac{a_{\rm end}}{a_k}. \tag{3.30}$$

The precise value of $N_k$ which corresponds to the observable perturbations is determined by the inflation scale, the reheating process and the post-inflationary dynamics. The e-folding number $N_k$ on scale $k$ corresponding to observable perturbations has been shown as [100–102],

$$N_k = 62 - \ln \frac{k}{a_0 H_0} - \ln \frac{10^{16}{\rm GeV}}{V_k^{1/4}} + \ln \frac{V_k^{1/4}}{V_{\rm end}^{1/4}} - \frac{1}{3} \ln \frac{V_{\rm end}^{1/4}}{\rho_{\rm reh}^{1/4}}, \tag{3.31}$$

where $a_0 H_0$ indicate current horizon scale, $V_k$ is the inflaton potential on scale $k$, $V_{\rm end}$ is the inflaton potential at the end of inflation of the order $10^{16}$ GeV, $\rho_{\rm reh}$ is the energy density of the reheating which should satisfy the following relation,

$$\rho_{\rm reh} = \frac{\pi^2}{30} g_* \left(T_{\rm reh}\right) T_{\rm reh}^4. \tag{3.32}$$

Since $\rho_{\rm reh} \leq V_{\rm end} \simeq 10^{16}$ GeV, the reheating temperature is constrained as $T_{\rm reh} \lesssim 10^{16}$ GeV. The relationship between $N_k$ and the reheating temperature $T_{\rm reh}$ can be expressed as

$$N_k = 56.2 - \ln \frac{k}{a_0 H_0} + \frac{1}{3} \ln \frac{T_{\rm reh}}{10^8 \ {\rm GeV}}. \tag{3.33}$$

In the standard inflation models, it is necessary to achieve at least 60 e-folds for the horizon and flatness problems. Solving the horizon and flatness problems requires the total e-folding number to be,

$$N_{\rm tot} \equiv \ln \frac{a_{\rm end}}{a_{\rm start}} = \int_t^{t_{\rm end}} H dt = \int_\phi^{\phi_{\rm end}} \frac{H}{\dot\phi} d\phi \simeq \int_{\phi_{\rm end}}^{\phi} \frac{d\phi}{M_{\rm pl}} \frac{1}{\sqrt{2\epsilon}} \gtrsim 60. \tag{3.34}$$

which described the total duration of the inflation.

## 3.3.2 Reheating after Inflation

Let us discuss the reheating process after inflation. After the end of inflation, the inflaton field oscillates near the minimum of its potential and produces a huge amount of elementary particles, which interact with each other and eventually form a thermal plasma. The reheating process is generally classified into several stages. In the first stage, the classical, coherently-oscillating inflaton field may give rise to the production of massive bosons due to parametric resonance. In most cases, this first stage occurs extremely rapidly. This nonthermal period is called preheating [103], and is different from the subsequent stages of reheating and thermalization. Parametric resonance in the preheating stage may sometimes produce topological defects or lead to nonthermal phase transitions [104]. During the reheating stage, most of the inflaton energy is transferred to the thermal energy of elementary particles. The reheating process finishes approximately when $H \approx \Gamma_{\rm tot}$. Thus, the reheating temperature can be expressed as

$$T_{\rm reh} = \left(\frac{90}{\pi^3 g_*}\right)^{1/4} \sqrt{M_{\rm pl}\Gamma_{\rm tot}}, \tag{3.35}$$

where $g_*$ is the number of relativistic degrees of freedom.

## 3.4 Particle Creation in Curved Spacetime

Generally, we can determine a unique vacuum state in flat spacetime. But in curved spacetime the vacuum state is not fixed uniquely. The concept of particles in curved spacetime becomes ambiguous and the physical interpretation becomes much more difficult. To clarify these matters, we consider the Klein-Gordon equation for the scalar field $\phi$ which is written as follows

$$\Box\phi\left(t,x\right) + \left(m^2 + \xi R\right)\phi\left(t,x\right) = 0, \tag{3.36}$$

where $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu = 1/\sqrt{-g}\partial_\mu\left(\sqrt{-g}\partial^\mu\right)$ express the generally covariant d'Alembertian operator and $\xi$ is the non-minimal curvature coupling constant. There are two popular choices for $\xi$ which are minimal coupling ($\xi = 0$) and conformal coupling ($\xi = 1/6$). The conformal coupling case in the massless limit leads to the conformally invariant scalar field theory.

Let us treat the scalar field $\phi$ as the field operator acting on the vacuum states and the field operator $\phi$ can be written by a sum of annihilation and creation operators

$$\phi\left(t,x\right) = \sum_k \left\{a_k\phi_k\left(t,x\right) + a_k^\dagger\phi_k^*\left(t,x\right)\right\}, \tag{3.37}$$

where the creation and annihilation operators of $\phi_k$ satisfy the commutation relations

$$\left[a_k, a_{k'}\right] = \left[a_k^\dagger, a_{k'}^\dagger\right] = 0, \quad \left[a_k, a_{k'}^\dagger\right] = \delta_{kk'}, \tag{3.38}$$

where the vacuum state $|0\rangle$ is defined as $a_k|0\rangle = 0$. For convenience, we introduce the scalar product which builds a complete set of orthonormal mode functions $\phi_k$ in the curved spacetime

$$\left(\phi_k, \phi_{k'}\right) = i\int_\Sigma d\Sigma^\mu\sqrt{-g_\Sigma}\left[\phi_k^*\left(\partial_\mu\phi_k\right) - \left(\partial_\mu\phi_k^*\right)\phi_k,\right] \tag{3.39}$$

where $d\Sigma^\mu = n^\mu d\Sigma$ is defined by the timelike unit vector $n^\mu$ and the volume element $d\Sigma$. These orthonormal mode conditions can be simplified as

$$\left(\phi_k, \phi_{k'}\right) = \delta_{kk'}. \tag{3.40}$$

In the flat spacetime, we can take a unique choice of $\{\phi_k\}$ being a complete set of positive norm solutions and defines a unique Minkowski vacuum state due to the Lorentz symmetry. However, the situation drastically changes in curved spacetime. There is no unique choice of $\{\phi_k\}$ and no unique vacuum state. Thus, the notion of the particles becomes ambiguous and the physical interpretation becomes more trouble.

Let us consider a specific spacetime which is asymptotically flat in the past and in the future, but the intermediate region is non-flat. We assume that $\{\phi_k\}$ is positive frequency solutions in the past in-region and $\{\varphi_k\}$ is positive frequency solutions in the future out-region. In the asymptotical spacetime, the orthonormal mode conditions can be given by

$$(\phi_k, \phi_{k'}) = (\varphi_k, \varphi_{k'}) = \delta_{kk'}, \tag{3.41}$$
$$(\phi_k^*, \phi_{k'}^*) = (\varphi_k^*, \varphi_{k'}^*) = -\delta_{kk'}, \tag{3.42}$$
$$(\phi_k, \phi_{k'}^*) = (\varphi_k, \varphi_{k'}^*) = 0. \tag{3.43}$$

Let us expand the past in-modes in terms of the future out-modes

$$\phi_k(t, x) = \sum_j \left\{\alpha_{kj}\varphi_j(t, x) + \beta_{kj}\varphi_j^*(t, x)\right\}. \tag{3.44}$$

By inserting this expansion into the orthonormal mode conditions, we can obtain the following relation. Let us expand the past in-modes in terms of the future out-modes

$$\sum_j \left\{\alpha_{kj}\alpha_{k'j}^* - \beta_{kj}\beta_{k'j}^*\right\} = \delta_{kk'}, \tag{3.45}$$
$$\sum_j \left\{\alpha_{kj}\alpha_{k'j} - \beta_{kj}\beta_{k'j}\right\} = 0. \tag{3.46}$$

The scalar field operator $\phi$ can be expanded by the mode function of $\{\phi_k\}$ and $\{\varphi_k\}$ as follows

$$\phi(t, x) = \sum_k \left\{a_k\phi_k(t, x) + a_k^\dagger\phi_k^*(t, x)\right\} \tag{3.47}$$
$$= \sum_k \left\{b_k\varphi_k(t, x) + b_k^\dagger\varphi_k^*(t, x)\right\}, \tag{3.48}$$

where $a_k$ and $a_k^\dagger$ are annihilation and creation operators for the in-region whereas $b_k$ and $b_k^\dagger$ express annihilation and creation operators for the out-region. Note that the in-vacuum state $|0\rangle_{\text{in}}$ corresponding to the no particle initial-state is defined by $a_k|0\rangle_{\text{in}} = 0$ and the out-vacuum state $|0\rangle_{\text{out}}$ is defined by $b_k|0\rangle_{\text{out}} = 0$. The annihilation and creation operators of the in-state are related with the out-state operator by the Bogolubov transformation

$$a_k = \sum_j \left\{\alpha_{kj}^* b_j - \beta_{kj}^* b_j^\dagger\right\}, \tag{3.49}$$
$$b_j = \sum_k \left\{\alpha_{kj} a_k + \beta_{kj} a_k^\dagger\right\}, \tag{3.50}$$

where $\alpha_{kj}$ and $\beta_{kj}$ are called the Bogolubov coefficients. Let us consider a specific situation where there are no particles before the gravitational field is turned on. Adopting the Heisenberg picture in this system, the in-vacuum $|0\rangle_{\text{in}}$ is invariant for all time. However, the number density operator $n_j$ depends on the time and becomes $n_j = b_j^\dagger b_j$ in the out-region. Therefore, the created number of particles in the gravitational background can be given by

$$\langle n_j \rangle =_{\rm in} \langle 0| b_j^\dagger b_j |0\rangle_{\rm in} = \sum_k |\beta_{kj}|^2, \tag{3.51}$$

where non-zero Bogolubov coefficients $\beta_{kj}$ lead to the gravitational particle creation.

## 3.5 Effective Potential in Curved Spacetime

In this section, we review the effective potential in curved spacetime using the adiabatic expansion method and discuss how quantum vacuum fluctuation construct the effective potential in the curved spacetime. The effective potential in curved spacetime has some different points from the Minkowski spacetime and those facts originate from the renormalization of the curved spacetime and the gravitational particle creation.

### 3.5.1 Quantum Vacuum Fluctuation

For convenience, we introduce the action of the scalar field in curved spacetime,

$$S[\phi] = -\int d^4x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + V(\phi)\right), \tag{3.52}$$

where we assume the simple form for the scalar potential $V(\phi)$ as

$$V(\phi) = \frac{1}{2}\left(m^2 + \xi R\right)\phi^2 + \frac{\lambda}{4}\phi^4. \tag{3.53}$$

The Klein-Gordon equation for the scalar field $\phi$ can be given by

$$\Box\phi(t,x) + V'(\phi(t,x)) = 0, \tag{3.54}$$

For convenience, we decompose the scalar field operator $\phi$ into the classic field and the quantum field as follows

$$\phi(t,x) = \phi(t,x) + \delta\phi(t,x), \tag{3.55}$$

where $\langle 0|\delta\phi(t,x)|0\rangle = 0$. By dividing the bare parameters into the renormalized parameters and the counterterms as $m^2 = m^2(\mu) + \delta m^2$, $\xi = \xi(\mu) + \delta\xi$ and $\lambda = \lambda(\mu) + \delta\lambda$, we can obtain the Klein-Gordon equations in the one-loop approximation as

$$\begin{aligned}\Box\phi + \left(m^2(\mu) + \delta m^2\right)\phi + \left(\xi(\mu) + \delta\xi\right)R\phi \\ + 3\left(\lambda(\mu) + \delta\lambda\right)\left\langle\delta\phi^2\right\rangle\phi + \left(\lambda(\mu) + \delta\lambda\right)\phi^3 = 0,\end{aligned} \tag{3.56}$$

$$\left(\Box + m^2(\mu) + \xi(\mu)R + 3\lambda(\mu)\phi^2\right)\delta\phi = 0. \tag{3.57}$$

The quantum scalar field $\delta\phi$ is decomposed into each $k$ modes as,

$$\delta\phi\left(t,x\right)=\int d^3k\left(a_k\delta\phi_k\left(t,x\right)+a_k^{\dagger}\delta\phi_k^*\left(t,x\right)\right), \tag{3.58}$$

where the creation and annihilation operators of $\delta\phi_k$ satisfy the commutation relations

$$\left[a_k,a_{k'}\right]=\left[a_k^{\dagger},a_{k'}^{\dagger}\right]=0,\quad\left[a_k,a_{k'}^{\dagger}\right]=\delta\left(k-k'\right), \tag{3.59}$$

where the in-vacuum state $|0\rangle$ is defined as $a_k|0\rangle=0$ and depends on the boundary conditions of the mode functions $\delta\phi_k$. The different boundary conditions of $\delta\phi_k$ are related with different initial state of the vacuum. The scalar product in curved spacetime to build a complete set of orthonormal mode functions can be given by

$$\left(\delta\phi_k,\delta\phi_{k'}\right)=i\int_{\Sigma}d\Sigma^{\mu}\sqrt{-g_{\Sigma}}\left[\delta\phi_k^*\left(\partial_\mu\delta\phi_k\right)-\left(\partial_\mu\delta\phi_k^*\right)\delta\phi_k\right]. \tag{3.60}$$

The above orthonormal mode conditions can be simplified as

$$\left(\delta\phi_k,\delta\phi_{k'}\right)=\delta\left(k-k'\right). \tag{3.61}$$

The two-point correlation function $\left\langle\delta\phi^2\right\rangle$ of the scalar field being consistent with the quantum vacuum fluctuation can be given by

$$\langle 0|\,\delta\phi^2\,|0\rangle=\int d^3k|\delta\phi_k\left(t,x\right)|^2, \tag{3.62}$$

which has quadratic and logarithmic divergences and requires a regularization, e.g. cut-off regularization or dimensional regularization, and we must cancel them using the counterterms of the various couplings.

For simplicity, we consider the Friedmann-Lemaitre-Robertson-Walker (FLRW) spacetime

$$ds^2=-dt^2+a^2\left(t\right)\left\{dr^2+r^2\left(d\theta^2+\sin^2\theta d\varphi^2\right)\right\}, \tag{3.63}$$

where $a\left(t\right)$ is the scale factor and we introduce the conformal time $\eta$ defined by $d\eta=dt/a$. The vacuum fluctuation $\left\langle\delta\phi^2\right\rangle$ of the scalar field can be given by

$$\langle 0|\,\delta\phi^2\,|0\rangle=\frac{1}{2\pi^2C\left(\eta\right)}\int_0^\infty dkk^2|\delta\chi_k|^2, \tag{3.64}$$

where $C\left(\eta\right)=a^2\left(\eta\right)$ and $\delta\phi_k\left(\eta,x\right)$ is given by

$$\delta\phi_k\left(\eta,x\right)=\frac{e^{ik\cdot x}}{\left(2\pi\right)^{3/2}\sqrt{C\left(\eta\right)}}\delta\chi_k\left(\eta\right), \tag{3.65}$$

The Klein-Gordon equation for the rescaled field $\delta\chi_k\left(\eta\right)$ can be given by

$$\delta\chi_k''+\Omega_k^2\left(\eta\right)\delta\chi_k=0, \tag{3.66}$$

where

$$\Omega_k^2(\eta) = k^2 + C(\eta)\left(m^2 + 3\lambda\phi^2 + (\xi - 1/6)R\right). \tag{3.67}$$

The orthonormal condition of Eq. (3.61) for the rescaled field $\delta\chi_k(\eta)$ is given by

$$\delta\chi_k\delta\chi_k'^* - \delta\chi_k'\delta\chi_k^* = i. \tag{3.68}$$

The Klein-Gordon equation of Eq. (3.66) is consistent with the differential equation of the harmonic oscillator with the time-dependent mass. Therefore, we can rewrite the mode function $\delta\chi_k(\eta)$ by using the two complex function $\alpha_k(\eta)$ and $\beta_k(\eta)$,

$$\delta\chi_k(\eta) = \frac{1}{\sqrt{2\Omega_k(\eta)}}\left\{\alpha_k(\eta)\delta\varphi_k(\eta) + \beta_k(\eta)\delta\varphi_k^*(\eta)\right\}, \tag{3.69}$$

where $\delta\varphi_k(\eta)$ are defined as

$$\delta\varphi_k(\eta) = \exp\left\{-i\int^\eta \Omega_k(\eta_1)\,d\eta_1\right\}. \tag{3.70}$$

The orthonormal condition condition of Eq. (3.68) can be given by

$$|\alpha_k(\eta)|^2 - |\beta_k(\eta)|^2 = 1. \tag{3.71}$$

From Eq. (3.66) and Eq. (3.69), we can obtain the relations for $\alpha_k(\eta)$ and $\beta_k(\eta)$ as the following

$$\alpha_k' = \frac{1}{2}\frac{\Omega_k'}{\Omega_k}\beta_k\delta\varphi_k^{*2}(\eta), \quad \beta_k' = \frac{1}{2}\frac{\Omega_k'}{\Omega_k}\alpha_k\delta\varphi_k^2(\eta). \tag{3.72}$$

The initial conditions for $\alpha_k(\eta_0)$ and $\beta_k(\eta_0)$ are consistent with the choice of the in-vacuum state. From Eq. (3.64) and Eq. (3.69) the vacuum fluctuation $\langle\delta\phi^2\rangle$ of the scalar field can be given by

$$\langle\delta\phi^2\rangle = \frac{1}{4\pi^2 C(\eta)}\int_0^\infty dk k^2\Omega_k^{-1}\left\{1 + 2|\beta_k|^2 + \alpha_k\beta_k^*\delta\varphi_k^2 + \alpha_k^*\beta_k\delta\varphi_k^{*2}\right\}. \tag{3.73}$$

For convenience, let us introduce $n_k$ and $z_k$ as the following

$$n_k = |\beta_k|^2, \quad z_k = \alpha_k\beta_k^*\delta\varphi_k^2. \tag{3.74}$$

The quantity $n_k = |\beta_k(\eta)|^2$ can be interpreted as the number density created in curved spacetime. Therefore, the number density $N(\eta)$ of created particles and the corresponding energy density $\rho(\eta)$ can be given by

$$N(\eta) = \frac{1}{2\pi^2 a^3(\eta)}\int_0^\infty dk k^2|\beta_k|^2, \quad \rho(\eta) = \frac{1}{2\pi^2 a^4(\eta)}\int_0^\infty dk k^2\Omega_k|\beta_k|^2.$$

From Eq. (3.74), $n_k$ and $z_k$ satisfy the following differential equations as

$$n_k' = \frac{\Omega_k'}{\Omega_k}\mathrm{Re}z_k, \quad z_k' = \frac{\Omega_k'}{\Omega_k}\left(n_k + \frac{1}{2}\right) - 2i\Omega_k z_k. \tag{3.75}$$

In order to solve Eq. (3.75), we must take adequately the initial conditions of $n_k$ and $z_k$. For simplicity, we choose the following condition

$$n_k(\eta_0) = z_k(\eta_0) = 0, \tag{3.76}$$

which corresponds to the conformal vacuum at the time $\eta_0$ owing to $\alpha_k(\eta_0) = 1$, $\beta_k(\eta_0) = 0$. To obtain the vacuum fluctuation $\langle \delta\phi^2 \rangle$ of the scalar field, we must solve adequately Eq. (3.75) with the above condition and insert $n_k$ and $z_k$ into Eq. (3.73). By using $n_k$ and $z_k$, we can gain the following expression of the vacuum fluctuation,

$$\begin{aligned} \langle \delta\phi^2 \rangle &= \frac{1}{4\pi^2 C(\eta)} \int_0^\infty dk k^2 \Omega_k^{-1} \{1 + 2n_k + 2\mathrm{Re} z_k\} \\ &= \langle \delta\phi^2 \rangle^{(0)} + \frac{1}{4\pi^2 C(\eta)} \int_0^\infty dk k^2 \Omega_k^{-1} \{2n_k + 2\mathrm{Re} z_k\}, \end{aligned} \tag{3.77}$$

where $\langle \delta\phi^2 \rangle^{(0)}$ has UV (quadratic and logarithmic) divergences to be

$$\begin{aligned} \langle \delta\phi^2 \rangle^{(0)} &= \frac{1}{4\pi^2 C(\eta)} \int_0^\infty dk k^2 \Omega_k^{-1} \\ &= \frac{1}{4\pi^2 C(\eta)} \int_0^\infty \frac{dk k^2}{\sqrt{k^2 + C(\eta)\left(m^2 + 3\lambda\phi^2 + (\xi - 1/6) R\right)}} \longrightarrow \infty, \end{aligned} \tag{3.78}$$

whose expressions are close similar to the standard Minkowski spacetime. Therefore, we can regularize the divergence of $\langle \delta\phi^2 \rangle^{(0)}$ by using the cut-off regularization or the dimensional regularization and offset them via the counterterms of the couplings. By using the dimensional regularization, we can obtain the following regularized expression of $\langle \delta\phi^2 \rangle^{(0)}$ as

$$\langle \delta\phi^2 \rangle^{(0)} = \frac{M^2(\phi)}{16\pi^2} \left[ \ln\left( \frac{M^2(\phi)}{\mu^2} \right) - \frac{1}{\epsilon} - \log 4\pi - \gamma - 1 \right], \tag{3.79}$$

where:

$$M^2(\phi) = m^2(\mu) + 3\lambda(\mu)\phi^2 + (\xi(\mu) - 1/6) R, \tag{3.80}$$

where $\mu$ is the renormalization parameter and $\gamma$ is the Euler-Mascheroni constant. The counterterms $\delta m^2$, $\delta\xi$ and $\delta\lambda$ must cancel these divergences as follows

$$\delta m^2 = \frac{3\lambda(\mu) m^2(\mu)}{16\pi^2} \left( \frac{1}{\epsilon} + \log 4\pi + \gamma \right) + \cdots, \tag{3.81}$$

$$\delta\xi = \frac{3\lambda(\mu)}{16\pi^2} \left( \xi(\mu) - \frac{1}{6} \right) \left( \frac{1}{\epsilon} + \log 4\pi + \gamma \right) + \cdots, \tag{3.82}$$

$$\delta\lambda = \frac{9\lambda(\mu)}{16\pi^2} \left( \frac{1}{\epsilon} + \log 4\pi + \gamma \right) + \cdots. \tag{3.83}$$

By using the regularized expression of Eq. (3.79) and these counterterms, we can construct the one-loop effective evolution equation of the scalar field $\phi$ as follows:

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V_{\text{eff}}(\phi)}{\partial\phi} = 0, \tag{3.84}$$

where $V_{\rm eff}(\phi)$ is the standard one-loop effective potential in curved spacetime as follows

$$V_{\rm eff}(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{2}\xi R\phi^2 + \frac{\lambda}{4}\phi^4 + \frac{M^4(\phi)}{64\pi^2}\left[\ln\left(\frac{M^2(\phi)}{\mu^2}\right) - \frac{3}{2}\right], \tag{3.85}$$

where:

$$M^2(\phi) = m^2(\mu) + 3\lambda(\mu)\phi^2 + (\xi(\mu) - 1/6)R. \tag{3.86}$$

### 3.5.2 Gravitationally Induced Vacuum Fluctuation

The renormalized expression of Eq. (3.79) contracts the standard effective potential in curved spacetime. However, the effective potential of Eq. (3.105) is formally modified by gravitational backreaction. The backreaction effect originates from gravitationally induced vacuum fluctuation corresponding to the gravitational particle creation in curved spacetime.

In this subsection, we consider the gravitationally induced vacuum fluctuation in FLRW spacetime. In order to obtain the vacuum fluctuation of $\langle\delta\phi^2\rangle$ in curved spacetime we must solve numerically and analytically Eq. (3.75) with the adequate vacuum condition. For convenience, let us adopt the adiabatic (WKB) expansion method which is valid in large mass, large momentum mode or slowly-varying background as

$$\left|\Omega'_k/\Omega_k^2\right| \ll 1. \tag{3.87}$$

By using the adiabatic (WKB) expansion method, $n_k$ and $z_k$ are approximately given by

$$n_k = n_k^{(2)} + n_k^{(4)} + \cdots, \tag{3.88}$$

$$\mathrm{Re}z_k = \mathrm{Re}z_k^{(2)} + \mathrm{Re}z_k^{(4)} + \cdots, \tag{3.89}$$

where superscripts $(i)$ express the adiabatic order. The second-order adiabatic expressions can be written as

$$n_k^{(2)} = \frac{1}{16}\frac{\Omega'^2_k}{\Omega_k^4}, \quad \mathrm{Re}z_k^{(2)} = \frac{1}{8}\frac{\Omega''_k}{\Omega_k^3} - \frac{1}{4}\frac{\Omega'^2_k}{\Omega_k^4}. \tag{3.90}$$

The forth-order adiabatic expressions can be written as

$$n_k^{(4)} = -\frac{\Omega'_k\Omega'''_k}{32\Omega_k^6} + \frac{\Omega''^2_k}{64\Omega_k^6} + \frac{5\Omega'^2_k\Omega''_k}{32\Omega_k^7} - \frac{45\Omega'^4_k}{256\Omega_k^8}, \tag{3.91}$$

$$\mathrm{Re}z_k^{(4)} = -\frac{\Omega''''_k}{32\Omega_k^5} + \frac{11\Omega'_k\Omega'''_k}{32\Omega_k^6} - \frac{115\Omega'^2_k\Omega''_k}{64\Omega_k^7} + \frac{7\Omega''^2_k}{32\Omega_k^6} + \frac{45\Omega'^4_k}{32\Omega_k^8}. \tag{3.92}$$

By using the adiabatic (WKB) expansion method, we can get the following expression of the vacuum fluctuation $\langle\delta\phi^2\rangle$,

$$\langle\delta\phi^2\rangle = \frac{1}{4\pi^2 C(\eta)}\int_0^\infty dk k^2\Omega_k^{-1} + \langle\delta\phi^2\rangle^{(2)} + \langle\delta\phi^2\rangle^{(4)} + \cdots, \tag{3.93}$$

where $\left\langle \delta\phi^2 \right\rangle^{(2n)}$ express the gravitationally induced vacuum fluctuation

$$\left\langle \delta\phi^2 \right\rangle^{(2n)} = \frac{1}{4\pi^2 C(\eta)} \int_0^\infty dk k^2 \Omega_k^{-1} \left\{ 2n_k^{(2n)} + 2\mathrm{Re} z_k^{(2n)} \right\}. \tag{3.94}$$

From Eq. (3.90) and (3.94), the second-order induced vacuum fluctuation can be given by

$$\left\langle \delta\phi^2 \right\rangle^{(2)} = \frac{1}{16\pi^2 C(\eta)} \int_0^\infty dk k^2 \Omega_k^{-1} \left\{ \frac{\Omega''_k}{\Omega_k^3} - \frac{3}{2} \frac{\Omega'_k}{\Omega_k^4} \right\}. \tag{3.95}$$

The gravitationally induced vacuum fluctuation $\left\langle \delta\phi^2 \right\rangle^{(2)}$ of Eq. (3.95) can be written as

$$\left\langle \delta\phi^2 \right\rangle^{(2)} = \frac{1}{16\pi^2 C(\eta)} \int_0^\infty dk k^2 \Omega_k^{-1} \left\{ \frac{\left( \bar{M}\bar{M}'' + \bar{M}'^2 \right)}{\Omega_k^4} - \frac{5}{2} \frac{\bar{M}^2 \bar{M}''}{\Omega_k^6} \right\}, \tag{3.96}$$

where:

$$\bar{M}^2 = C(\eta) M^2(\phi). \tag{3.97}$$

The high-order adiabatic expressions of $\left\langle \delta\phi^2 \right\rangle^{(2n)}$ could be finite and therefore, there is no need to regularize the integral of the high-order adiabatic expressions. The corresponding integrals can converge to the finite values as follows:

$$F(\alpha) \equiv \int_0^\infty dk k^2 \left( k^2 + \bar{M}^2 \right)^{-\alpha} = \frac{\bar{M}^{3-2\alpha}}{2} \frac{\Gamma(3/2)\Gamma(\alpha - 3/2)}{\Gamma(\alpha)}, \tag{3.98}$$

where the above expression is valid for $\alpha > 3/2$. By using the formula of Eq. (3.98), the gravitationally induced vacuum fluctuation $\left\langle \delta\phi^2 \right\rangle^{(2)}$ of Eq. (3.95) can be written as follows

$$\begin{aligned} \left\langle \delta\phi^2 \right\rangle^{(2)} &= \frac{1}{16\pi^2 C(\eta)} \left\{ \left( \bar{M}\bar{M}'' + \bar{M}'^2 \right) F\left(\frac{5}{2}\right) - \frac{5}{2} \bar{M}^2 \bar{M}'' F\left(\frac{7}{2}\right) \right\} \\ &= \frac{1}{48\pi^2 C(\eta)} \frac{\bar{M}''}{\bar{M}}. \end{aligned} \tag{3.99}$$

Let us construct the second-order expression of the proper time $t$ as follows

$$\begin{aligned} \left\langle \delta\phi^2 \right\rangle^{(2)} &= \frac{1}{48\pi^2} \left\{ \frac{a''}{a^3} + 2\frac{a'}{a^2}\frac{M'}{M} + \frac{1}{a^2}\frac{M''}{M} \right\} \\ &= \frac{1}{48\pi^2} \left\{ \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + 3\frac{\dot{a}}{a}\frac{\dot{M}}{M} + \frac{\ddot{M}}{M} \right\} \\ &= \frac{1}{48\pi^2} \left\{ \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{3}{2}\frac{\dot{a}}{a} \frac{(\xi - 1/6)\dot{R} + 6\lambda\phi\dot{\phi}}{m^2 + (\xi - 1/6)R + 3\lambda\phi^2} \right. \\ &\left. - \frac{1}{4} \frac{\left( (\xi - 1/6)\dot{R} + 6\lambda\phi\dot{\phi} \right)^2}{\left( m^2 + (\xi - 1/6)R + 3\lambda\phi^2 \right)^2} + \frac{1}{2} \frac{(\xi - 1/6)\ddot{R} + 6\lambda\left( \phi\ddot{\phi} + \dot{\phi}^2 \right)}{m^2 + (\xi - 1/6)R + 3\lambda\phi^2} \right\}. \end{aligned} \tag{3.100}$$

For simplicity, we consider the conformal coupling case $\xi = 1/6$. In this case, the second-order expression of the induced vacuum fluctuation can be written as

$$\begin{aligned}\left\langle \delta\phi^2 \right\rangle^{(2)} =& \frac{1}{48\pi^2}\left\{ \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{3}{2}\frac{\dot{a}}{a}\frac{6\lambda\phi\dot{\phi}}{m^2+3\lambda\phi^2} - \frac{1}{4}\left(\frac{6\lambda\phi\dot{\phi}}{m^2+3\lambda\phi^2}\right)^2 + \frac{1}{2}\frac{6\lambda\phi\ddot{\phi}+6\lambda\dot{\phi}^2}{m^2+3\lambda\phi^2}\right\} \\ =& \frac{1}{48\pi^2}\left\{ \frac{R}{6} + \frac{3H}{2}\frac{6\lambda\phi\dot{\phi}}{m^2+3\lambda\phi^2} - \frac{1}{4}\left(\frac{6\lambda\phi\dot{\phi}}{m^2+3\lambda\phi^2}\right)^2 + \frac{1}{2}\frac{6\lambda\phi\ddot{\phi}+6\lambda\dot{\phi}^2}{m^2+3\lambda\phi^2}\right\}. \end{aligned} \tag{3.101}$$

If we can neglect the time-derivative terms to be $\dot{\phi} \simeq \ddot{\phi} \simeq 0$, the gravitationally induced vacuum fluctuation of Eq. (3.101) can be simplified as

$$\left\langle \delta\phi^2 \right\rangle^{(2)} = \frac{R}{288\pi^2}. \tag{3.102}$$

In massive conformal coupling case ($\xi = 1/6$ and $m \gtrsim H$), the gravitationally induced vacuum fluctuation in FLRW spacetime can be given by

$$\left\langle \delta\phi^2 \right\rangle^{(2n)} = \frac{R}{288\pi^2} + \mathcal{O}\left(R^2\right) + \cdots. \tag{3.103}$$

From here we discuss how the gravitationally induced vacuum fluctuation modify the standard effective potential in curved spacetime. For convenience, let us rewrite the vacuum fluctuation of Eq. (3.77) in curved spacetime as follows:

$$\begin{aligned}\left\langle \delta\phi^2 \right\rangle =& \frac{1}{4\pi^2 C(\eta)}\int_0^\infty dk k^2 \Omega_k^{-1}\left\{1 + 2n_k + 2\mathrm{Re}z_k\right\} \\ =& \frac{1}{4\pi^2 C(\eta)}\int_0^\infty dk k^2 \Omega_k^{-1} + \frac{1}{4\pi^2 C(\eta)}\int_0^\infty dk k^2 \Omega_k^{-1}\left\{2n_k + 2\mathrm{Re}z_k\right\}, \end{aligned} \tag{3.104}$$

where the first expression describes quantum vacuum fluctuation and constructs the effective potential of Eq. (3.105) as previously discussed, whereas the second describes gravitationally induced vacuum fluctuation which expresses the particle creation. Therefore, we must reconstruct the effective potential by including the gravitational vacuum fluctuation and the modified expression of the effective potential can be given by [1, 2]

$$\begin{aligned} V_{\text{eff}}(\phi) =& \frac{1}{2}m^2(\mu)\phi^2 + \frac{1}{2}\xi(\mu)R\phi^2 + \frac{\lambda(\mu)}{4}\phi^4 \\ &+ \frac{3\lambda(\mu)}{2}\left\langle\delta\phi^2\right\rangle_{\text{gravity}}\phi^2 + \frac{M^4(\phi)}{64\pi^2}\left[\ln\left(\frac{M^2(\phi)}{\mu^2}\right) - \frac{3}{2}\right], \end{aligned} \tag{3.105}$$

where $\left\langle\delta\phi^2\right\rangle_{\text{gravity}}$ is the gravitationally induced vacuum fluctuation written as

$$\left\langle\delta\phi^2\right\rangle_{\text{gravity}} = \frac{1}{4\pi^2 C(\eta)}\int_0^\infty dk k^2 \Omega_k^{-1}\left\{2n_k + 2\mathrm{Re}z_k\right\}. \tag{3.106}$$

The expression of Eq. (3.105) does include the effective mass from the gravitational vacuum fluctuation and was first discussed in Ref. [105]. Next let us discuss some issues of the renormalization scale $\mu$. Generally speaking, we take the renormalization scale $\mu$ so as to suppress the high order log-corrections about $\log\left(M^2(\phi)/\mu^2\right)$. In the Minkowski spacetime to be $R = 0$, we usually take the renormalization scale to be $\mu \approx \phi$. The renormalization scale $\mu$ corresponds to the phenomenological energy scale described as the effective mass of the scalar field. Although log-corrections in Eq. (3.105) does not include the gravitational vacuum fluctuation terms, the high-order expressions would have these terms and therefore the renormalization scale can be taken as $\mu^2 \approx \phi^2 + \langle \delta\phi^2 \rangle_{\text{gravity}} + R$. Through the above consideration, we can obtain the exact effective potential in curved spacetime including the gravitational backreaction.

## 3.6 Semiclassical Gravity and Gravitational Backreaction

The semiclassical gravity is a consistent framework to describe quantum gravitational phenomena where only matter fields are quantized, while the metric is still treated as a classic. The semiclassical gravity provides a satisfactory description below the Planck scale [22], and there are many successful examples. Despite the non-renormalizable properties of QG, this approach is completely renormalizable and crucial for understanding the backreaction effect of quantum fluctuations onto the spacetime and the evaporation of the back holes [106]. The semiclassical gravity is formally described by the general Einstein's equations

$$\frac{1}{8\pi G_N} G_{\mu\nu} + \rho_\Lambda g_{\mu\nu} + a_1 H^{(1)}_{\mu\nu} + a_2 H^{(2)}_{\mu\nu} + a_3 H^{(3)}_{\mu\nu} = \langle T_{\mu\nu} \rangle \,, \tag{3.107}$$

In this section we consider the semiclassical gravity and the gravitational backreaction.

### 3.6.1 Renormalized Vacuum Energy Density and Gravitational Backreaction in Curved Spacetime

Let us consider the renormalization of the energy momentum tensor in curved spacetime. The standard energy momentum tensor $T_{\mu\nu}$ can be given as follows [107]

$$\begin{aligned} T_{\mu\nu} = &(1-2\xi)\,\partial_\mu\phi\partial_\nu\phi + \left(2\xi - \frac{1}{2}\right) g_{\mu\nu}\partial^\mu\phi\partial_\mu\phi - 2\xi\phi\nabla\partial_\nu\phi \\ &+ 2\xi g_{\mu\nu}\phi\Box\phi - \xi G_{\mu\nu}\phi^2 + \frac{1}{2}m^2 g_{\mu\nu}\phi^2, \end{aligned} \tag{3.108}$$

whose diagonal and non-vanishing components are $T_{00}$ and $T_{11} = T_{22} = T_{33}$. We introduce the trace of the energy momentum tensor $T^\alpha_\alpha$ which satisfy the relation $T_{ii} = 1/3\left(T_{00} - C(\eta)\,T^\alpha_\alpha\right)$. The vacuum expectation values $\langle T_{\mu\nu} \rangle$ of the energy momentum tensor for the rescaled mode function $\delta\chi_k(\eta)$ are given as follows [107]

$$\begin{aligned}\langle T_{00}\rangle = \frac{1}{4\pi^2 C(\eta)}\int dk k^2\Bigg[&|\delta\chi'_k|^2 + \omega_k^2|\delta\chi_k|^2 \\ &+\left(\xi-\frac{1}{6}\right)\left(3D\left(\delta\chi_k\delta\chi_k^{*'}+\delta\chi_k^*\delta\chi'_k\right)-\frac{3}{2}D^2\delta\chi_k^2\right)\Bigg],\end{aligned} \tag{3.109}$$

$$\begin{aligned}\langle T_\alpha^\alpha\rangle = \frac{1}{2\pi^2 C^2(\eta)}\int dk k^2\Bigg[&Cm^2|\delta\chi_k|^2 + 6\left(\xi-\frac{1}{6}\right)\left(|\delta\chi_k|^2-\frac{1}{2}D\left(\delta\chi_k\delta\chi_k^{*'}+\delta\chi_k^*\delta\chi'_k\right)\right. \\ &\left.-\omega_k^2|\delta\chi_k|^2-\frac{1}{2}D'|\delta\chi_k|^2-\left(\xi-\frac{1}{6}\right)\left(3D'+\frac{3}{2}D^2\right)|\delta\chi_k|^2\right)\Bigg],\end{aligned} \tag{3.110}$$

where $D(\eta) = C'(\eta)/C(\eta)$ and the corresponding vacuum energy density and vacuum pressure can be given by $\rho_{\text{vacuum}} = \langle T_{00}\rangle/C(\eta)$ and $p_{\text{vacuum}} = \langle T_{ii}\rangle/C(\eta)$. For comparison let us rewrite the vacuum expectation value $\langle\delta\phi^2\rangle$ of the scalar field which is given by

$$\langle\delta\phi^2\rangle = \frac{1}{2\pi^2 C(\eta)}\int_0^\infty k^2|\delta\chi_k|^2 dk, \tag{3.111}$$

From here let us adopt the adiabatic (WKB) approximation for the mode function $\delta\chi_k(\eta)$. The WKB approximation of the mode function $\delta\chi_k(\eta)$ can be given by

$$\delta\chi_k(\eta) = \frac{1}{\sqrt{2W_k(\eta)C(\eta)}}\exp\left(-i\int W_k(\eta)\,d\eta\right), \tag{3.112}$$

where $W_k(\eta)$ is given by

$$W_k^2 = \Omega_k^2 - \frac{1}{2}\frac{W_k''}{W_k} + \frac{3}{4}\frac{(W_k')^2}{W_k^2}. \tag{3.113}$$

We can obtain the WKB solution by solving iteratively Eq. (3.113) and the zero-order WKB solution $W_k^0(\eta)$ is given by

$$\left(W_k^0\right)^2 = \Omega_k^2. \tag{3.114}$$

The first-order WKB solution $W_k^1(\eta)$ is given by

$$\left(W_k^1\right)^2 = \Omega_k^2 - \frac{1}{2}\frac{(W_k^0)''}{W_k^0} + \frac{3}{4}\frac{\left(W_k^{0\prime}\right)^2}{(W_k^0)^2}. \tag{3.115}$$

The high-order WKB solution $W_k(\eta)$ can be given by

$$\begin{aligned} W_k \simeq\ & \omega_k + \frac{3(\xi-1/6)}{4\omega_k}\left(2D'+D^2\right) - \frac{m^2C}{8\omega_k^3}\left(D'+D^2\right) + \frac{5m^4C^2D^2}{32\omega^5} \\ & + \frac{m^2C}{32\omega_k^5}\left(D''' + 4D'D + 3D'^2 + 6D'D^2 + D^4\right) \\ & - \frac{m^4C^2}{128\omega_k^7}\left(28D''D + 19D'^2 + 122D'^2 + 47D^4\right) \\ & + \frac{221m^6C^3}{256\omega_k^9}\left(D'D^2+D^4\right) - \frac{1105m^8C^4D^4}{2048\omega_k^{11}} - \frac{(\xi-1/6)}{8\omega_k^3}\left(3D''' + 3D''D + 3D'^2\right) \\ & + (\xi-1/6)\frac{m^2C}{32\omega_k^5}\left(30D''D + 18D'^2 + 57D'D^2 + 9D^4\right) \\ & - (\xi-1/6)\frac{75m^4C^2}{128\omega_k^7}\left(2D'D^2+D^4\right) - \frac{(\xi-1/6)^2}{32\omega_k^3}\left(36D'^2 + 36D'D^2 + 9D^4\right) + \cdots . \end{aligned} \tag{3.116}$$

For convenience, let us reconsider the vacuum expectation value $\langle \delta\phi^2 \rangle$ of the scalar field in the conformally coupled case by using the WKB solution of Eq. (3.116). For conformally couple case $(\xi = 1/6)$ the WKB solution is given by

$$\begin{aligned} W_k &= \omega_k - \frac{m^2C}{8\omega_k^3}\left(D'+D^2\right) + \frac{5m^4C^2D^2}{32\omega^5} + \cdots , \\ &= \omega_k - \frac{1}{8}\frac{m^2C''}{\omega_k^3} + \frac{5}{32}\frac{m^4(C')^2}{\omega_k^5} + \cdots . \end{aligned} \tag{3.117}$$

where $\omega_k = \sqrt{k^2 + C(\eta)m^2}$. Thus, we can obtain the following expression

$$\frac{1}{W_k} \simeq \frac{1}{\omega_k} + \frac{1}{8}\frac{m^2C''}{\omega_k^5} - \frac{5}{32}\frac{m^4(C')^2}{\omega_k^7} + \cdots . \tag{3.118}$$

From Eq. (3.118) the vacuum expectation value $\langle \delta\phi^2 \rangle$ of the scalar field can be given by

$$\langle \delta\phi^2 \rangle = \frac{1}{4\pi^2 C(\eta)}\left[\int_0^\infty \frac{k^2}{\omega_k}dk + \frac{m^2C''}{8}\int_0^\infty \frac{k^2}{\omega_k^5}dk + \frac{5m^4(C')^2}{32}\int_0^\infty \frac{k^2}{\omega_k^7}dk + \cdots\right], \tag{3.119}$$

where the first term diverges but the second or third term become finite,

$$\begin{aligned} \langle \delta\phi^2 \rangle &= \frac{1}{4\pi^2 C(\eta)}\int_0^\infty \frac{dk k^2}{\sqrt{k^2 + C(\eta)m^2}} + \left[\frac{m^2C''}{8}\int_0^\infty \frac{k^2}{\omega_k^5}dk + \frac{5m^4(C')^2}{32}\int_0^\infty \frac{k^2}{\omega_k^7}dk + \cdots\right] \\ &= \frac{1}{4\pi^2 C(\eta)}\int_0^\infty \frac{dk k^2}{\sqrt{k^2 + C(\eta)m^2}} - \frac{1}{96\pi^2 C(\eta)}\left[\frac{1}{2}\left(\frac{C'}{C}\right)^2 - \frac{C''}{C}\right] + \cdots \\ &= \frac{m^2}{16\pi^2}\left[\ln\left(\frac{m^2}{\mu^2}\right) - \frac{1}{\epsilon} - \ln 4\pi + \gamma - 1\right] + \frac{R}{288\pi^2} + \cdots , \end{aligned} \tag{3.120}$$

where we adopt the dimensional regularization and the second term is consistent with Eq. (3.102).

Let us return the vacuum expectation values $\langle T_{\mu\nu}\rangle$ of the energy momentum tensor and the expression of the WKB solution can be given by [107]

$$
\begin{aligned}
\langle T_{00}\rangle =& \frac{1}{8\pi^2 C(\eta)}\int dk k^2\Bigg[2\omega_k + \frac{C^2m^4D^2}{16\omega_k^5} - \frac{C^2m^4}{64\omega_k^7}\left(2D''D - D'^2 + 4D'D^2 + D^4\right) \\
&+ \frac{7C^3m^6}{64\omega_k^9}\left(D'D^2 + D^4\right) - \frac{105C^4m^8D^4}{1024\omega_k^{11}} \\
&+ \left(\xi - \frac{1}{6}\right)\left(-\frac{3D^2}{2\omega_k} - \frac{3Cm^2D^2}{2\omega_k^3} + \frac{Cm^2}{8\omega_k^5}\left(6D''D - 3D'^2 + 6D'D^2\right)\right. \\
&\left. - \frac{C^2m^4}{64\omega_k^7}\left(120D'D^2 + 105D^4\right) + \frac{105C^3m^6D^4}{64\omega_k^9}\right) \\
&+ \left(\xi - \frac{1}{6}\right)^2\left(-\frac{1}{16\omega_k^3}\left(72D''D - 36D'^2 - 27D^4\right) + \frac{Cm^2}{8\omega_k^5}\left(54D'D^2 + 27D^4\right)\right)\Bigg],
\end{aligned}
\tag{3.121}
$$

$$
\begin{aligned}
\langle T^\alpha_\alpha\rangle =& \frac{1}{8\pi^2 C^2(\eta)}\int dk k^2\Bigg[\frac{Cm^2}{\omega_k} + \frac{C^2m^4}{8\omega_k^5}\left(D' + D^2\right) - \frac{5C^3m^6D^2}{32\omega_k^7} \\
&- \frac{C^2m^4}{32\omega_k^7}\left(D''' + 4D''D + 3D'^2 + 6D'D^2 + D^4\right) \\
&+ \frac{C^3m^6}{128\omega_k^9}\left(28D''D + 21D'^2 + 126D'D^2 + 49D^4\right) - \frac{231C^4m^8}{256\omega_k^{11}}\left(D'D^2 + D^4\right) \\
&+ \frac{1155C^5m^{10}D^4}{2048\omega_k^{13}} + \left(\xi - \frac{1}{6}\right)\left(-\frac{3D'}{\omega_k} - \frac{Cm^2}{\omega_k^3}\left(3D' + \frac{3}{4}D^2\right)\right. \\
&+ \frac{9C^2m^4D^2}{4\omega_k^5} + \frac{Cm^2}{4\omega_k^5}\left(3D''' + 6D''D + \frac{9}{2}D'^2 + 3D'D^2\right) \\
&- \frac{C^2m^4}{32\omega_k^7}\left(120D''D + 90D'^2 + 390D'D^2 + 105D^4\right) \\
&\left.+ \frac{C^3m^6}{128\omega_k^9}\left(1680D'D^2 + 1365D^4\right) - \frac{945C^4m^8D^4}{128\omega_k^{11}}\right) + \left(\xi - \frac{1}{6}\right)^2\left(-\frac{1}{4\omega_k^3}\left(18D''' - 27D'D^2\right)\right. \\
&\left.+ \frac{Cm^2}{32\omega_k^5}\left(432D''D + 324D'^2 + 648D'D^2 + 27D^4\right) - \frac{C^2m^4}{16\omega_k^7}\left(270D'D^2 + 135D^4\right)\right)\Bigg],
\end{aligned}
\tag{3.122}
$$

where we consider the forth-order WKB approximation. As previously discussed in the vacuum expectation value $\langle\delta\phi^2\rangle$ of the scalar field, the high-order WKB terms are finite and the divergence of the energy momentum tensor come from the lowest-order terms,

$$
\begin{aligned}
\langle T_{00}\rangle_{\mathrm{low-order}} =& \frac{1}{8\pi^2 C(\eta)}\int dk k^2\Bigg[2\omega_k - \frac{3}{2}D^2\left(\xi - \frac{1}{6}\right)\left(\frac{1}{\omega_k} + \frac{Cm^2}{\omega_k^3}\right) \\
&- \frac{1}{16\omega_k^3}\left(\xi - \frac{1}{6}\right)^2\left(72D''D - 36D'^2 - 27D^4\right)\Bigg],
\end{aligned}
\tag{3.123}
$$

$$\begin{aligned}\langle T^{\alpha}_{\alpha}\rangle_{\rm low-order} = & \frac{1}{8\pi^2 C^2(\eta)}\int dk k^2\left[\frac{Cm^2}{\omega_k} - \left(\xi - \frac{1}{6}\right)\left(\frac{3D'}{\omega_k} + \frac{Cm^2}{\omega_k^3}\left(3D' + \frac{3}{4}D^2\right)\right)\right. \\ & \left. - \frac{1}{4\omega_k^3}\left(\xi - \frac{1}{6}\right)^2\left(18D''' - 27D'D^2\right)\right].\end{aligned} \tag{3.124}$$

The divergent momentum integrals can be simplified as follows

$$I(0,n) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\omega_k^n} = \int \frac{d^3k}{(2\pi)^3}\frac{1}{(k^2 + a^2m^2)^{n/2}}.$$

We regulate these integrals of the spatial dimensions $3 - 2\epsilon$ to be

$$I(\epsilon,n) = \int \frac{d^{3-2\epsilon}k}{(2\pi)^{3-2\epsilon}}\frac{(a\mu)^{2\epsilon}}{\omega_k^n} = \frac{(am)^{3-n}}{8\pi^{3/2}}\frac{\Gamma\left(\epsilon - \frac{3-n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\left(\frac{4\pi\mu^2}{m^2}\right)^{\epsilon}.$$

By using the dimensional regularization the divergent terms of $\langle T_{00}\rangle$ can be given by

$$\begin{aligned}\langle T_{00}\rangle_{\rm low-order} = & -\frac{m^4 C}{64\pi^2}\left[\frac{1}{\epsilon} + \frac{3}{2} - \gamma + \ln 4\pi + \ln\frac{\mu^2}{m^2}\right] \\ & - \frac{3m^2D^2}{32\pi^2}\left(\xi - \frac{1}{6}\right)\left[\frac{1}{\epsilon} + \frac{1}{2} - \gamma + \ln 4\pi + \ln\frac{\mu^2}{m^2}\right] \\ & - \frac{1}{256\pi^2 C}\left(\xi - \frac{1}{6}\right)^2\left(72D''D - 36D'^2 - 27D^4\right)\left[\frac{1}{\epsilon} - \gamma + \ln 4\pi + \ln\frac{\mu^2}{m^2}\right].\end{aligned} \tag{3.125}$$

The high-order WKB terms of $\langle T_{00}\rangle$ are finite and describe the gravitationally induced vacuum fluctuation on curved spacetime

$$\begin{aligned}\langle T_{00}\rangle_{\rm high-order} = & \frac{m^2D^2}{384\pi^2} - \frac{1}{2880\pi^2 C}\left(\frac{3}{2}D''D - \frac{3}{4}D'^2 - \frac{3}{8}D^4\right) \\ & + \frac{1}{256\pi^2 C}\left(\xi - \frac{1}{6}\right)\left(8D''D - 4D'^2 - 3D^4\right) + \frac{1}{64\pi^2 C}\left(\xi - \frac{1}{6}\right)^2\left(18D'D^2 + 9D^4\right).\end{aligned} \tag{3.126}$$

Thus, the vacuum expectation values $\langle T_{\mu\nu}\rangle$ of the energy momentum tensor in curved spacetime are given by the following divergent low-order contributions and the finite high-order contributions,

$$\langle T_{\mu\nu}\rangle = \langle T_{\mu\nu}\rangle_{\rm low-order} + \langle T_{\mu\nu}\rangle_{\rm high-order}. \tag{3.127}$$

The unphysical divergences of $\langle T_{\mu\nu}\rangle$ can be removed by the cancellation of the bare coupling constants of the general Einstein's equations

$$\frac{1}{8\pi G_N}G_{\mu\nu} + \rho_\Lambda g_{\mu\nu} + a_1 H^{(1)}_{\mu\nu} + a_2 H^{(2)}_{\mu\nu} + a_3 H^{(3)}_{\mu\nu} = \langle T_{\mu\nu}\rangle. \tag{3.128}$$

Now, we divide the low-order WKB terms of $\langle T_{\mu\nu}\rangle$ into the divergent terms and the finite terms as follows

$$\langle T_{\mu\nu}\rangle = \langle T_{\mu\nu}\rangle_{\rm div} + \langle T_{\mu\nu}\rangle_{\rm finite} + \langle T_{\mu\nu}\rangle_{\rm high-order}. \tag{3.129}$$

The bare gravitational coupling constants $G_N$, $\rho_\Lambda$, $a_1$ can be divided into the finite parts and the counter parts to be

$$G_N = G_N(\mu) + \delta G_N, \tag{3.130}$$

$$\rho_\Lambda = \rho_\Lambda(\mu) + \delta\rho_\Lambda, \tag{3.131}$$

$$a_1 = a_1(\mu) + \delta a_1. \tag{3.132}$$

The divergences of $\langle T_{\mu\nu}\rangle$ are absorbed by the counter terms of the gravitational couplings $\delta G_N$, $\delta\rho_\Lambda$, $\delta a_1$ as follows

$$\begin{aligned}\langle T_{00}\rangle_{\rm div} &= \frac{1}{8\pi\delta G_N}G_{00} + \delta\rho_\Lambda g_{00} + \delta a_1 H^{(1)}_{00} + \delta a_2 H^{(2)}_{00} + \delta a_3 H_{00} \\ &= \frac{1}{8\pi\delta G_N}\left(-\frac{3}{4}D^2\right) + \delta\rho_\Lambda(C) + \delta a_1\left(\frac{-72D''D + 36D'^2 + 27D^4}{8C}\right) + \cdots .\end{aligned} \tag{3.133}$$

Thus, the gravitational counter terms must satisfy the following conditions $\delta G_N$, $\delta\rho_\Lambda$, $\delta a_1$,

$$\delta\rho_\Lambda = \frac{m^4}{64\pi^2}\left[\frac{1}{\epsilon} - \gamma + \ln 4\pi\right], \tag{3.134}$$

$$\frac{1}{8\pi\delta G_N} = -\frac{m^2}{8\pi^2}\left(\xi - \frac{1}{6}\right)\left[\frac{1}{\epsilon} - \gamma + \ln 4\pi\right], \tag{3.135}$$

$$\delta a_1 = -\frac{1}{32\pi^2}\left(\xi - \frac{1}{6}\right)^2\left[\frac{1}{\epsilon} - \gamma + \ln 4\pi\right]. \tag{3.136}$$

We can get the effective Einstein's equations as follows

$$\frac{1}{8\pi G_N(\mu)}G_{\mu\nu} + \rho_\Lambda(\mu)\, g_{\mu\nu} + a_1(\mu)\, H^{(1)}_{\mu\nu} + a_2(\mu)\, H^{(2)}_{\mu\nu} + a_3(\mu)\, H^{(3)}_{\mu\nu} = \langle T_{\mu\nu}\rangle_{\rm ren}, \tag{3.137}$$

where we write the renormalized energy momentum tensor of $\langle T_{\mu\nu}\rangle_{\rm ren}$ as follows

$$\langle T_{\mu\nu}\rangle_{\rm ren} = \langle T_{\mu\nu}\rangle_{\rm finite} + \langle T_{\mu\nu}\rangle_{\rm gravity}. \tag{3.138}$$

The finite contributions of the energy momentum tensor $\langle T_{00}\rangle_{\rm finite}$ can be given by

$$\begin{aligned}\langle T_{00}\rangle_{\rm finite} = &\frac{m^4 C}{64\pi^2}\left(\ln\frac{m^2}{\mu^2} - \frac{3}{2}\right) + \frac{3m^2D^2}{32\pi^2}\left(\xi - \frac{1}{6}\right)\left(\ln\frac{m^2}{\mu^2} - \frac{1}{2}\right) \\ &+ \frac{1}{256\pi^2 C}\left(\xi - \frac{1}{6}\right)^2\left(72D''D - 36D'^2 - 27D^4\right)\left(\ln\frac{m^2}{\mu^2}\right).\end{aligned} \tag{3.139}$$

The renormalized vacuum energy density can be given as follows:

$$\begin{aligned}
\rho_{\text{vacuum}} &= \langle T_{00}\rangle_{\text{ren}}/C \\
&= \frac{m^4}{64\pi^2}\left(\ln\frac{m^2}{\mu^2}-\frac{3}{2}\right)+\frac{3m^2D^2}{32\pi^2C}\left(\xi-\frac{1}{6}\right)\left(\ln\frac{m^2}{\mu^2}-\frac{1}{2}\right) \\
&+\frac{1}{256\pi^2C^2}\left(\xi-\frac{1}{6}\right)^2\left(72D''D-36D'^2-27D^4\right)\left(\ln\frac{m^2}{\mu^2}\right) \\
&+\frac{m^2D^2}{384\pi^2C}-\frac{1}{2880\pi^2C^2}\left(\frac{3}{2}D''D-\frac{3}{4}D'^2-\frac{3}{8}D^4\right) \\
&+\frac{1}{256\pi^2C^2}\left(\xi-\frac{1}{6}\right)\left(8D''D-4D'^2-3D^4\right) \\
&+\frac{1}{64\pi^2C^2}\left(\xi-\frac{1}{6}\right)^2\left(18D'D^2+9D^4\right),
\end{aligned} \tag{3.140}$$

In matter dominated Universe to be $a(\eta)=\eta^2/9$, the renormalized vacuum energy density can be written as

$$\begin{aligned}
\rho_{\text{vacuum}} = &\frac{m^4}{64\pi^2}\left(\ln\frac{m^2}{\mu^2}-\frac{3}{2}\right)+\frac{3m^2}{8\pi^2}\left(\xi-\frac{1}{6}\right)H^2\left(\ln\frac{m^2}{\mu^2}-\frac{1}{2}\right)-\frac{81}{64\pi^2}\left(\xi-\frac{1}{6}\right)^2H^4\left(\ln\frac{m^2}{\mu^2}\right) \\
&+\frac{m^2H^2}{96\pi^2}+\frac{H^4}{768\pi^2}-\frac{9}{64\pi^2}\left(\xi-\frac{1}{6}\right)H^4+\frac{9}{8\pi^2}\left(\xi-\frac{1}{6}\right)^2H^4.
\end{aligned} \tag{3.141}$$

On the other hand, in de Sitter Universe as $a(\eta)=-1/H\eta$, the renormalized vacuum energy density can be obtained by

$$\begin{aligned}
\rho_{\text{vacuum}} = &\frac{m^4}{64\pi^2}\left(\ln\frac{m^2}{\mu^2}-\frac{3}{2}\right)+\frac{3m^2}{8\pi^2}\left(\xi-\frac{1}{6}\right)H^2\left(\ln\frac{m^2}{\mu^2}-\frac{1}{2}\right) \\
&+\frac{m^2H^2}{96\pi^2}-\frac{H^4}{960\pi^2}+\frac{9}{2\pi^2}\left(\xi-\frac{1}{6}\right)^2H^4.
\end{aligned} \tag{3.142}$$

# Chapter 4

# Gravitationally Induced Vacuum Fluctuation

The vacuum fluctuation of the scalar field in curved spacetime can be divided into quantum vacuum fluctuation and gravitationally induced vacuum fluctuation. The former has UV divergences and construct the standard effective potential. The latter corresponds to the particle creation effect on curved spacetime. The gravitationally induced vacuum fluctuation of the scalar field $\phi$ in curved spacetime can be formally written as follows:

$$\left\langle \delta\phi^2 \right\rangle_{\text{gravity}} = \frac{1}{4\pi^2 C(\eta)} \int_0^\infty dk k^2 \Omega_k^{-1} \{2n_k + 2\text{Re}z_k\}, \tag{4.1}$$

where $n_k$ and $z_k$ are defined by the Bogolubov coefficients as previously discussed. In this section we discuss the gravitationally induced vacuum fluctuation on various spacetimes.

## 4.1 Gravitationally Induced Vacuum Fluctuation in FLRW Spacetime

The Friedmann-Lemaitre-Robertson-Walker (FLRW) metric describes the spacetime geometry of a homogeneous and isotropic Universe, and is given by

$$g_{\mu\nu} = \text{diag}\left(-1, \frac{a^2(t)}{1-Kr^2}, a^2(t)\, r^2, a^2(t)\, r^2 \sin^2\theta\right), \tag{4.2}$$

where $a = a(t)$ is the scale factor and $K$ is the spatial curvature parameter. For the spatially flat spacetime, we can simply take $K = 0$ and the Ricci scalar is given by

$$R = 6\left[\left(\frac{\dot{a}}{a}\right)^2 + \left(\frac{\ddot{a}}{a}\right)\right] = 6\left(\frac{a''}{a^3}\right), \tag{4.3}$$

where $\eta$ is the conformal time and defined by $d\eta = dt/a$. In radiation dominated Universe, the scale factor becomes $a(t) = t^{1/2}$ and the Ricci scalar is expressed to be $R = 0$. On the other hand, the scale factor becomes $a(t) = t^{2/3}$ in matter dominated Universe and the Ricci scalar is expressed to be $R = 3H^2$. Finally, in de Sitter Universe, the scale factor becomes $a(t) = e^{Ht}$ and the Ricci scalar is expressed to be $R = 12H^2$.

For simplicity we consider the massive conformal coupling case ($\xi = 1/6$ and $m \gtrsim H$) and the gravitationally induced vacuum fluctuation can be summarize as follows:

$$\left\langle \delta\phi^2 \right\rangle_{\mathrm{gravity}} = \frac{R}{288\pi^2} + \mathcal{O}\left(R^2\right) + \mathcal{O}\left(R^3\right) + \cdots \tag{4.4}$$

$$\simeq \begin{cases} 0, & \text{(the radiation Universe)} \\ H^2/96\pi^2, & \text{(the matter Universe)} \\ H^2/24\pi^2. & \text{(the de Sitter Universe)} \end{cases} \tag{4.5}$$

Note that the massless scalar field cannot satisfy the adiabatic (WKB) condition of Eq. (3.87),

$$\frac{\Omega'_k}{\Omega_k^2} \simeq \frac{2H}{m} \ll 1, \tag{4.6}$$

where we assume $m = \mathrm{const}$ for simplicity. Thus, the adiabatic (WKB) approximation can not be adopted in the massless case. In the non-adiabatic case, the gravitationally induced vacuum fluctuation can generally enlarged to be

$$\left\langle \delta\phi^2 \right\rangle_{\mathrm{gravity}} \gtrsim \mathcal{O}\left(H^2\right). \tag{4.7}$$

The above situation occurs during inflation for the massless scalar field or during preheating stage of the parametric resonance.

### 4.1.1 The scalar field background

In the general cosmological situation, the scalar fields dynamically change and do not stagnate for all times. The dynamical variation of some scalar fields provides the varying effective mass and leads to the real particle productions or the induced vacuum fluctuation. Even in the slowly varying scalar field background, the induced vacuum fluctuations are non-negligible. Now, we consider the induced vacuum fluctuation in the slowly varying scalar field background following the literature [108].

For convenience, we rewrite Eq. (4.1) in order to estimate the induced vacuum field fluctuation on the varying field background

$$\left\langle \delta\phi^2 \right\rangle_{\mathrm{gravity}} = \frac{1}{4\pi^2 C(\eta)} \int_0^\infty dk k^2 \Omega_k^{-1} \left\{2n_k + 2\mathrm{Re} z_k\right\} \tag{4.8}$$

where $n_k$ and $z_k$ are determined by the differential equations,

$$n'_k = \frac{\Omega'_k}{\Omega_k}\mathrm{Re} z_k, \quad z'_k = \frac{\Omega'_k}{\Omega_k}\left(n_k + \frac{1}{2}\right) - 2i\Omega_k z_k. \tag{4.9}$$

Let us assume the vacuum satisfying the initial conditions: $n_k(\eta_0) = z_k(\eta_0) = 0$. In this situation, we can get the following equations,

$$n_k(\eta) = \int_{\eta_0}^{\eta} d\eta_1 \int_{\eta_0}^{\eta_1} d\eta_2 \frac{\Omega'_k(\eta_1)}{\Omega_k(\eta_1)} \frac{\Omega'_k(\eta_2)}{\Omega_k(\eta_2)} \times \cos\{2\int_{\eta_2}^{\eta_1} d\eta_3 \Omega_k(\eta_3)\} \left(\frac{1}{2} + n_k(\eta_2)\right) \tag{4.10}$$

$$\mathrm{Re} z_k(\eta) = \int_{\eta_0}^{\eta} d\eta_1 \frac{\Omega'_k(\eta_1)}{\Omega_k(\eta_1)} \cos\{2\int_{\eta_1}^{\eta} d\eta_2 \Omega_k(\eta_2)\} \left(\frac{1}{2} + \int_{\eta_0}^{\eta_1} d\eta_3 \frac{\Omega'_k(\eta_3)}{\Omega_k(\eta_3)} \mathrm{Re} z_k(\eta_3)\right) \tag{4.11}$$

For simplicity, we assume the following condition as

$$\left|\int_{\eta_0}^{\eta} d\eta_1 \frac{\Omega'_k(\eta_1)}{\Omega_k(\eta_1)}\right| \ll 1 \tag{4.12}$$

which corresponds to the small time-difference of $\bar{M}^2(\eta)$ as follows

$$\left|\bar{M}^2(\eta) - \bar{M}^2(\eta_0)\right| \ll 2\bar{M}^2(\eta) \tag{4.13}$$

In this assumption, we can approximate these equations of Eq. (4.10) and Eq. (4.11),

$$n_k(\eta) \simeq 0, \quad \mathrm{Re} z_k(\eta) \simeq \frac{1}{2}\int_{\eta_0}^{\eta} d\eta_1 \frac{\Omega'_k(\eta_1)}{\Omega_k(\eta_1)} \cos\{2\int_{\eta_1}^{\eta} d\eta_2 \Omega_k(\eta_2)\}. \tag{4.14}$$

Furthermore, we can approximate Eq. (4.14) as the following

$$\begin{aligned} \mathrm{Re} z_k(\eta) &\simeq \frac{1}{2}\int_{\eta_0}^{\eta} d\eta_1 \frac{\bar{M}(\eta_1)\bar{M}'(\eta_1)}{\Omega_k^2(\eta_1)} \cos\{2\int_{\eta_1}^{\eta} d\eta_2 \Omega_k(\eta_2)\} \\ &\simeq \frac{1}{2\Omega_k^2(\eta)} \int_{\eta_0}^{\eta} d\eta_1 \bar{M}(\eta_1)\bar{M}'(\eta_1) \cos\{2\Omega_k(\eta)(\eta-\eta_1)\}. \end{aligned} \tag{4.15}$$

From Eq. (4.8), we get the induced vacuum fluctuation of the scalar field,

$$\begin{aligned} \langle \delta\phi^2 \rangle_{\text{gravity}} &= \frac{1}{2\pi^2 C(\eta)} \int_0^\infty dk k^2 \Omega_k^{-1} \{n_k + \mathrm{Re} z_k\} \\ &\simeq \frac{1}{2\pi^2 C(\eta)} \int_0^\infty dk k^2 \Omega_k^{-3} \int_{\eta_0}^{\eta} d\eta_1 \bar{M}(\eta_1)\bar{M}'(\eta_1) \cos\{2\Omega_k(\eta)(\eta-\eta_1)\}. \end{aligned} \tag{4.16}$$

By performing the partial integration, we have the following expression,

$$\begin{aligned} \langle \delta\phi^2 \rangle_{\text{gravity}} &\simeq \frac{\bar{M}^2(\eta)}{8\pi^2 C(\eta)} \left(\bar{M}^2(\eta_0) - \bar{M}^2(\eta)\right) \int_0^\infty dk k^2 \Omega_k^{-3} \\ &+ \frac{1}{4\pi^2 C(\eta)} \int_0^\infty dk \Omega_k^{-1} \int_{\eta_0}^{\eta} d\eta_1 \bar{M}(\eta_1)\bar{M}'(\eta_1) \cos\{2\Omega_k(\eta)(\eta-\eta_1)\} \\ &+ \frac{\bar{M}^2(\eta)}{4\pi^2 C(\eta)} \int_0^\infty dk \Omega_k^{-2} \int_{\eta_0}^{\eta} d\eta_1 \left(\bar{M}^2(\eta_1) - \bar{M}^2(\eta_0)\right) \sin\{2\Omega_k(\eta)(\eta-\eta_1)\}, \end{aligned} \tag{4.17}$$

which equivalent to the result using the perturbation technique [105]. By performing the integration, we can get the following expression

$$\begin{aligned}\left\langle\delta\phi^2\right\rangle_{\text{gravity}} \simeq & \frac{1}{8\pi^2 a^2(\eta)}\left(\bar{M}^2(\eta_0)-\bar{M}^2(\eta)\right) \\ & -\frac{1}{8\pi^2 a^2(\eta)}\int_{\eta_0}^{\eta} d\eta_1 \bar{M}(\eta_1)\bar{M}'(\eta_1) N_0\left(2\bar{M}(\eta-\eta_1)\right) \\ & +\frac{\bar{M}^2(\eta)}{8\pi^2 a^2(\eta)}\int_{\eta_0}^{\eta} d\eta_1 (\eta-\eta_1)\left(\bar{M}^2(\eta_1)-\bar{M}^2(\eta_0)\right) F\left(2\bar{M}(\eta-\eta_1)\right)\end{aligned}$$

where $N_0(x)$ is the Bessel function, $F(x)$ are combination of Bessel and Struve functions defined in Ref. [105] and $\bar{M}(\eta)$ is described by the varying background field as $\bar{M}(\eta)\simeq 3\lambda a(\eta)\phi^2(\eta)$. When the expansion of the Universe is slow and the background scalar field $\phi(\eta)$ evolve quickly on the cosmological timescale, the induced vacuum fluctuation enlarges in proportion to $\bar{M}(\eta)$. The above expression corresponds to the first-order adiabatic approximation of Eq. (3.93) where the odd-order adiabatic number density is exactly zero as $n_k^{(2n+1)}=0$. As previously discussed, the second-order approximation of Eq. (3.99) is given by

$$\left\langle\delta\phi^2\right\rangle_{\text{gravity}} = \frac{1}{48\pi^2 a^2(\eta)}\frac{\bar{M}''(\eta)}{\bar{M}(\eta)} \tag{4.18}$$

where $\bar{M}^2(\eta)=a^2(\eta)\left(m^2+3\lambda\phi^2+(\xi-1/6)R\right)$. Thus, we obtain the following expression of the second-order adiabatic induced fluctuation,

$$\begin{aligned}\left\langle\delta\phi^2\right\rangle_{\text{gravity}} =& \frac{1}{48\pi^2}\left\{\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}+\frac{3}{2}\frac{\dot{a}}{a}\frac{(\xi-1/6)\dot{R}+6\lambda\phi\dot{\phi}}{m^2+(\xi-1/6)R+3\lambda\phi^2}\right. \\ & \left.-\frac{1}{4}\frac{\left((\xi-1/6)\dot{R}+6\lambda\phi\dot{\phi}\right)^2}{\left(m^2+(\xi-1/6)R+3\lambda\phi^2\right)^2}+\frac{1}{2}\frac{(\xi-1/6)\ddot{R}+6\lambda\left(\phi\ddot{\phi}+\dot{\phi}^2\right)}{m^2+(\xi-1/6)R+3\lambda\phi^2}\right\}.\end{aligned} \tag{4.19}$$

For the large background scalar field $\phi(t)$ where we can safely neglect the mass terms or the non-minimal curvature terms, the second-order expression of the induced vacuum fluctuation can be given by

$$\left\langle\delta\phi^2\right\rangle_{\text{gravity}} \simeq \frac{1}{48\pi^2}\left\{\frac{1}{6}R+\frac{3H\dot{\phi}}{\phi}+\frac{\ddot{\phi}}{\phi}\right\}. \tag{4.20}$$

If the curvature effects of the spacetime are negligible and the background scalar field evolves quickly as $\phi(t)\sim e^{-M(\phi)t}$ or $\phi(t)\sim\sin(M(\phi)t)$, the induced vacuum fluctuation on the varying scalar field background can be approximated from Eq. (4.19) and Eq. (4.20) as

$$\left\langle\delta\phi^2\right\rangle_{\text{gravity}} \simeq \frac{M_{\text{eff}}^2(\phi)}{48\pi^2}. \tag{4.21}$$

If the scalar field has the large effective mass $M_{\text{eff}}(\phi)$ and the background scalar field develops rapidly on the cosmological timescale, the induced vacuum fluctuation of the scalar field glow in proportional to $M_{\text{eff}}(\phi)$.

### 4.1.2 The multiple scalar field background

On the other hand, if there are other coherent or classical scalar fields $S$ which couples the scalar field with the coupling $\lambda_{\phi S}$, the effective mass of the scalar field is generated as $m_{\phi S}^2 = \lambda_{\phi S} S^2$. The above situation is cosmological realistic during or after inflation. In this case, the scalar field acquires the effective mass as $\bar{M}^2(\eta) = a^2(\eta)\left(m^2 + 3\lambda\phi^2 + \lambda_{\phi S}S^2 + (\xi - 1/6)R\right)$ and the second-order adiabatic vacuum fluctuation can be written as

$$\begin{aligned}\left\langle \delta\phi^2 \right\rangle_{\text{gravity}} = \frac{1}{48\pi^2}\Bigg\{ &\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{3}{2}\frac{\dot{a}}{a}\frac{(\xi - 1/6)\dot{R} + 6\lambda\phi\dot{\phi} + 2\lambda_{\phi S}S\dot{S}}{m^2 + (\xi - 1/6)R + 3\lambda\phi^2 + \lambda_{\phi S}S^2} \\ &- \frac{1}{4}\frac{\left((\xi - 1/6)\dot{R} + 6\lambda\phi\dot{\phi} + 2\lambda_{\phi S}S\dot{S}\right)^2}{\left(m^2 + (\xi - 1/6)R + 3\lambda\phi^2 + \lambda_{\phi S}S^2\right)^2} \\ &+ \frac{1}{2}\frac{(\xi - 1/6)\ddot{R} + 6\lambda\left(\phi\ddot{\phi} + \dot{\phi}^2\right) + 2\lambda_{\phi S}\left(S\ddot{S} + \dot{S}^2\right)}{m^2 + (\xi - 1/6)R + 3\lambda\phi^2 + \lambda_{\phi S}S^2}\Bigg\}. \end{aligned} \tag{4.22}$$

For the large background scalar field $S(t)$, the second-order adiabatic vacuum fluctuation can be given by

$$\left\langle \delta\phi^2 \right\rangle_{\text{gravity}} \simeq \frac{1}{48\pi^2}\left\{ \frac{1}{6}R + \frac{3H\dot{S}}{S} + \frac{\ddot{S}}{S} \right\}. \tag{4.23}$$

The evolution of the background scalar field $S(t)$ is determined by the effective potential $V_{\text{eff}}(S)$. Thus, the induced vacuum fluctuation on the varying background scalar field can be given by

$$\left\langle \delta\phi^2 \right\rangle_{\text{gravity}} \simeq \frac{M_{\text{eff}}^2(S)}{48\pi^2}, \tag{4.24}$$

where $M_{\text{eff}}(S)$ is the effective mass of $S$ and defined by $V_{\text{eff}}''(S) = M_{\text{eff}}^2(S)$. The gravitationally induced vacuum fluctuation of the scalar field can become as large as the scalar curvature and the masses of some scalar fields in FLRW Spacetime.

## 4.2 Gravitationally Induced Vacuum Fluctuation in De-Sitter Spacetime

In the non-anabatic case, e.g small mass or rapid varying background, we must usually solve Eq. (3.75) with the suitable in-vacuum. However, it is hard task to calculate the induced vacuum fluctuation in this method. When we assume unspecified initial conditions or vacuum, we use the following expression of $z_k$ [108],

$$\begin{aligned} z_k(\eta) = &\int_{\eta_0}^{\eta} d\eta_1 \frac{\Omega'_k(\eta_1)}{\Omega_k(\eta_1)}\left(n_k(\eta_1) + \frac{1}{2}\right) \times \exp\left\{-2i\int_{\eta_1}^{\eta} d\eta_2 \Omega_k(\eta_2)\right\} \\ &+ z_k(\eta_0)\exp\left\{-2i\int_{\eta_0}^{\eta} d\eta_2\Omega_k(\eta_2)\right\} \end{aligned} \tag{4.25}$$

where we must solve Eq. (3.75) and inset into Eq. (4.1), and therefore, there is usually no other way except numerical calculations to obtain the induced vacuum fluctuation. However, if we obtain the exact mode function $\delta\chi(\eta)$ from the Klein-Gordon equation given by Eq. (3.66), we can obtain the induced vacuum fluctuation $\langle\delta\phi^2\rangle_{\rm gravity}$ using the adiabatic regularization or the point-splitting regularization method.

### 4.2.1 Adiabatic Regularization Method

In this section, we review the adiabatic regularization [22, 109–115] which is the extremely powerful method to get the gravitationally induced vacuum fluctuation even in the non-adiabatic regime. The adiabatic regularization is not the mathematical method to regularize divergent integrals as dimensional regularization or cut-off regularization. As previously discussed, the divergences of $\langle\delta\phi^2\rangle$ originate from the lowest-order adiabatic (WKB) mode, and therefore, we can remove these divergences by subtracting the lowest-order adiabatic (WKB) expression of Eq. (3.78) from $\langle\delta\phi^2\rangle$. Thus, we get the gravitationally induced vacuum fluctuation $\langle\delta\phi^2\rangle_{\rm gravity}$ as the following

$$\begin{aligned}\langle\delta\phi^2\rangle_{\rm gravity} = \langle\delta\phi^2\rangle - \langle\delta\phi^2\rangle^{(0)} &= \frac{1}{4\pi^2 C(\eta)}\int_0^\infty dk k^2\Omega_k^{-1}\left\{2n_k + 2{\rm Re}z_k\right\} \\ &= \frac{1}{4\pi^2 C(\eta)}\left[\int_0^\infty dk 2k^2|\delta\chi_k|^2 - \int_0^\infty dk k^2\Omega_k^{-1}\right]\end{aligned} \tag{4.26}$$

where we must get the exact mode function $\delta\chi_k(\eta)$ with appropriate in-vacuum. This method is well-known and equivalent to the point-splitting regularization method to regularize divergences via the point separation in the two-point function.

Let us consider the gravitationally induced vacuum fluctuation for the massless minimally coupling scalar field ($m = 0$ and $\xi = 0$) in de Sitter spacetime (for the detail discussions see e.g. Ref. [114, 115]). In this case, the rescaled mode function $\delta\chi_k(\eta)$ can be exactly given by

$$\delta\chi_k(\eta) = \frac{1}{\sqrt{2k}}\left\{\alpha_k\delta\varphi_k(\eta) + \beta_k\delta\varphi_k^*(\eta)\right\}, \tag{4.27}$$

where

$$\delta\varphi_k(\eta) = e^{-ik\eta}\left(1 + \frac{1}{ik\eta}\right). \tag{4.28}$$

In the massless minimally coupled case, the vacuum expectation value $\langle\delta\phi^2\rangle$ of the scalar field has not only UV divergences but also infrared (IR) divergences. For simplicity let us assume that the Universe changes from the radiation-dominated Universe to the de Sitter Universe in order to avoid the IR divergences

$$a(\eta) = \begin{cases} 2 - \frac{\eta}{\eta_0}, & (\eta < \eta_0) \\ \frac{\eta}{\eta_0}, & (\eta > \eta_0) \end{cases} \tag{4.29}$$

where $\eta_0 = -1/H$ and we choose the following mode function as the in-vacuum state

$$\delta\chi_k = e^{-ik\eta}/\sqrt{2k}. \tag{4.30}$$

during the radiation-dominated Universe ($\eta < \eta_0$). By requiring the matching conditions $\delta\chi_k(\eta)$ and $\delta\chi'_k(\eta)$ at the time $\eta = \eta_0$, we can obtain the corresponding coefficients of the mode function to be

$$\alpha_k = 1 + \frac{H}{ik} - \frac{H^2}{2k^2}, \quad \beta_k = -\frac{H^2}{2k^2} e^{\frac{2ik}{H}} = \alpha_k + \frac{2ik}{3H} + \mathcal{O}\left(\frac{k^2}{H^2}\right). \tag{4.31}$$

By using $\alpha_k$ and $\beta_k$ of Eq. (4.31) we can obtain the suitable mode function of $\delta\chi_k(\eta)$. For small $k$ modes in de Sitter Universe ($\eta > \eta_0$), we can approximate the mode function as the following

$$|\delta\chi_k|^2 = \frac{1}{2k}\left[\left(\frac{2}{3H\eta} + 2 + \frac{H^2\eta^2}{6}\right)^2 + \mathcal{O}\left(\frac{k^2}{H^2}\right) + \cdots\right]. \tag{4.32}$$

which does not have IR divergences because $k^2|\delta\chi_k|^2 \approx \mathcal{O}(k)$. For large $k$ modes, we can get the following expression

$$|\delta\chi_k|^2 = \frac{1}{2k}\left[1 + \frac{1}{k^2\eta^2} - \frac{H^2}{k^2}\cos\left(2k\left(1/H + \eta\right)\right) + \mathcal{O}\left(\frac{H^3}{k^3}\right) + \cdots\right]. \tag{4.33}$$

Now we we must require the cut-off of $k$ mode satisfying the adiabatic (WKB) condition $\Omega_k^2 > 0$ to be $k > \sqrt{2}/|\eta| = \sqrt{2}aH$. From Eq. (4.26) we can get the gravitationally induced vacuum fluctuation as follows

$$\begin{aligned}
\left\langle\delta\phi^2\right\rangle_{\text{gravity}} &= \lim_{\Lambda\to\infty}\frac{1}{4\pi^2 C(\eta)}\left[\int_0^\Lambda 2k^2|\delta\chi_k|^2 dk - \int_{\sqrt{2}/|\eta|}^\Lambda dk k^2\Omega_k^{-1}\right] \\
&= \lim_{\Lambda\to\infty}\frac{1}{4\pi^2 C(\eta)}\left[\int_0^\Lambda 2k^2|\delta\chi_k|^2 dk - \int_{\sqrt{2}/|\eta|}^\Lambda \frac{k^2}{\sqrt{k^2 - 2/\eta^2}} dk\right] \\
&= \lim_{\Lambda\to\infty}\frac{1}{4\pi^2 C(\eta)}\left[\int_0^\Lambda 2k^2|\delta\chi_k|^2 dk - \int_{\sqrt{2}/|\eta|}^\Lambda \left(k + \frac{1}{k\eta^2} + \cdots\right) dk\right].
\end{aligned} \tag{4.34}$$

For large $k$ modes, we can use Eq. (4.33) and subtract the UV divergences as

$$\lim_{\Lambda\to\infty}\frac{1}{4\pi^2 C(\eta)}\left[\int_{\sqrt{2}/|\eta|}^\Lambda \left(k + \frac{1}{k\eta^2}\right) dk - \int_{\sqrt{2}/|\eta|}^\Lambda \left(k + \frac{1}{k\eta^2}\right) dk\right] = 0. \tag{4.35}$$

Thus, we get the following expression of the gravitationally induced vacuum fluctuation,

$$\begin{aligned}
\left\langle\delta\phi^2\right\rangle_{\text{gravity}} =& \frac{1}{2\pi^2 C(\eta)}\int_0^{\sqrt{2}/|\eta|} k^2|\delta\chi_k|^2 dk + \frac{\eta^2 H^2}{4\pi^2}\int_{\sqrt{2}/|\eta|}^\infty \left(-\frac{H^2}{k^2}\cos\left(2k\left(1/H + \eta\right)\right)\right. \\
&\left. + \mathcal{O}\left(\frac{H^3}{k^3}\right) + \cdots\right) k dk.
\end{aligned} \tag{4.36}$$

At the late cosmic-time ($\eta \simeq 0$ is consistent with $N = Ht \gg 1$), we have the following approximation

$$\begin{aligned} \left\langle \delta\phi^2 \right\rangle_{\text{gravity}} &\simeq \frac{\eta^2 H^2}{2\pi^2} \int_0^{\sqrt{2}/|\eta|} k^2 |\delta\chi_k|^2 dk, \\ &\simeq \frac{1}{9\pi^2} \int_0^H k dk + \frac{H^2}{4\pi^2} \int_H^{\sqrt{2}/|\eta|} \frac{1}{k} dk, \end{aligned} \tag{4.37}$$

where we approximate the mode function $\delta\chi_k(\eta)$ from Eq. (4.32) and Eq. (4.33),

$$|\delta\chi_k|^2 \simeq \begin{cases} \frac{1}{2k}\left(\frac{2}{3H\eta} + 2 + \frac{H^2\eta^2}{6}\right)^2 & (0 \leq k \leq H) \\ \frac{1}{2k}\left(1 + \frac{1}{k^2\eta^2}\right) & \left(H \leq k \leq \sqrt{2}/|\eta|\right) \end{cases} \tag{4.38}$$

Therefore, we can finally obtain the well-know de-Sitter vacuum fluctuation as follows

$$\begin{aligned} \left\langle \delta\phi^2 \right\rangle_{\text{gravity}} &\simeq \frac{H^2}{18\pi^2} + \frac{H^2}{4\pi^2}\left(\frac{1}{2}\log 2 + Ht\right) \\ &\simeq \frac{H^3}{4\pi^2} t, \end{aligned} \tag{4.39}$$

which grows as cosmic-time proceeds.

Next, we consider the massless minimally coupled scalar field ($\xi = 0$ and $m \ll H$) in de Sitter spacetime. This is cosmologically important situation in order to understand the origin of the primordial perturbations or the self-backreaction of the inflaton field in the inflationary Universe (see, e.g. Ref. [116,117]). In this case, the rescaled mode function $\delta\chi_k(\eta)$ can be given by

$$\delta\chi_k(\eta) = \sqrt{\frac{\pi}{4}} \eta^{1/2} \left\{ \alpha_k H_\nu^{(2)}(k\eta) + \beta_k H_\nu^{(1)}(k\eta) \right\}, \tag{4.40}$$

where:

$$\nu \equiv \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \simeq \frac{3}{2} - \frac{m^2}{3H^2}. \tag{4.41}$$

where $H_\nu^{(1,2)}(k\eta)$ are the Hankel functions.

As previously discussed, let us assume the transition from the radiation-dominated Universe to the de Sitter Universe and require the matching conditions at $\eta = \eta_0$ to determine the Bogoliubov coefficients

$$\alpha_k = \frac{1}{2i}\sqrt{\frac{\pi k\eta_0}{2}} \left( \left(-i + \frac{H}{2k}\right) H_\nu^{(1)}(k\eta_0) - H_\nu^{(1)'}(k\eta_0) \right) e^{ik/H}, \tag{4.42}$$

$$\beta_k = -\frac{1}{2i}\sqrt{\frac{\pi k\eta_0}{2}} \left( \left(-i + \frac{H}{2k}\right) H_\nu^{(2)}(k\eta_0) - H_\nu^{(2)'}(k\eta_0) \right) e^{ik/H}. \tag{4.43}$$

Thus, the gravitationally induced vacuum fluctuation form Eq. (4.26) can be written as follows

$$\begin{aligned} \left\langle \delta\phi^2 \right\rangle_{\text{gravity}} &= \lim_{\Lambda\to\infty} \frac{1}{4\pi^2 C\left(\eta\right)} \left[ \int_0^\Lambda 2k^2 |\delta\chi_k|^2 dk - \int_{\sqrt{2}/|\eta|}^\Lambda dk k^2 \Omega_k^{-1} \right] \\ &= \frac{\eta^2 H^2}{2\pi^2} \int_0^H k^2 |\delta\chi_k|^2 dk + \frac{\eta^2 H^2}{2\pi^2} \int_H^{\sqrt{2}/|\eta|} k^2 |\delta\chi_k|^2 dk. \end{aligned} \tag{4.44}$$

The divergence parts of $\left\langle \delta\phi^2 \right\rangle_{\text{gravity}}$ exactly cancel as previously discussed

$$\lim_{\Lambda\to\infty} \frac{1}{4\pi^2 C\left(\eta\right)} \left[ \int_{\sqrt{2}/|\eta|}^\Lambda 2k^2 |\delta\chi_k|^2 dk - \int_{\sqrt{2}/|\eta|}^\Lambda dk k^2 \Omega_k^{-1} \right] \tag{4.45}$$

where we take the adiabatic (WKB) mode cut-off as $k > \sqrt{2 - m^2/H^2}/\left|\eta\right| \simeq \sqrt{2}/\left|\eta\right|$. It is difficult task in the massive case than the massless case to obtain exactly the gravitationally induced vacuum fluctuation from Eq. (4.42) and Eq. (4.43). However, by using the asymptotic behavior of the Hankel functions, we can easily get the gravitationally induced vacuum fluctuation via the adiabatic regularization method (for the detail discussions, see Ref. [114, 115]).

By using the following formula of the Hankel functions we can get

$$H_\nu^{(1,2)\prime}\left(k\eta_0\right) = H_{\nu-1}^{(1,2)}\left(k\eta_0\right) - \frac{\nu}{k\eta_0} H_\nu^{(1,2)}\left(k\eta_0\right), \tag{4.46}$$

and the Bessel function of the first kind is defined by $J_\nu = (H_\nu^{(1)} + H_\nu^{(2)})/2$. Thus we obtain the following expression

$$\left|\alpha_k - \beta_k\right| = \sqrt{\frac{\pi k}{2H}} \left| J_{\nu-1}\left(k\eta_0\right) + \left(i - \frac{H}{2k} + \frac{\nu H}{k}\right) J_\nu\left(k\eta_0\right) \right| \tag{4.47}$$

For small $k$ modes, the Bessel function and the Hankel function asymptotically behave as follows

$$J_\nu\left(k\eta_0\right) \simeq \frac{1}{\Gamma\left(\nu+1\right)} \left(\frac{k\eta_0}{2}\right)^\nu \tag{4.48}$$

$$H_\nu^{(2)}\left(k\eta_0\right) \simeq -H_\nu^{(1)}\left(k\eta_0\right) \simeq \frac{i}{\pi}\Gamma\left(\nu\right) \left(\frac{k\eta_0}{2}\right)^{-\nu} \tag{4.49}$$

Thus, we can obtain the following expression of the mode function $\delta\chi_k\left(\eta\right)$

$$\left|\delta\chi_k\left(\eta\right)\right|^2 \simeq \frac{2}{9k} \left(H\left|\eta\right|\right)^{1-2\nu} \qquad \left(0 \le k \le H\right) \tag{4.50}$$

For large $k$ modes, we can approximate the Bogoliubov coefficients as $\alpha_k \simeq 1$ and $\beta_k \simeq 0$ and get the following mode function

$$\delta\chi_k\left(\eta\right) \simeq \sqrt{\frac{\pi}{4}} \eta^{1/2} H_\nu^{(2)}\left(k\eta\right) \tag{4.51}$$

Thus, we obtain the following expression

$$|\delta\chi_k|^2 \simeq \frac{|\eta|}{16}\left(\frac{k\,|\eta|}{2}\right)^{-2\nu} \qquad \left(H \le k \le \sqrt{2}/\,|\eta|\right). \tag{4.52}$$

From Eq. (4.50) and Eq. (4.52), the induced vacuum fluctuation can be given by

$$\begin{aligned}\left\langle\delta\phi^2\right\rangle_{\rm gravity} &\simeq \frac{(H\,|\eta|)^{3-2\nu}}{9\pi^2}\int_0^H k dk + \frac{H^2\,|\eta|^{3-2\nu}}{4\pi^2\cdot 2^{3-2\nu}}\int_H^{\sqrt{2}/|\eta|} k^{2-2\nu}dk \qquad (4.53)\\ &\simeq \frac{H^2}{18\pi^2}e^{-\frac{2m^2t}{3H^2}} + \frac{3H^2}{8\pi^2m^2}\left(1-e^{-\frac{2m^2t}{3H^2}}\right). \qquad (4.54)\end{aligned}$$

In de Sitter spacetime, the gravitationally induced vacuum fluctuation $\left\langle\delta\phi^2\right\rangle_{\rm gravity}$ via the adiabatic regularization can be summarized as follows [1]

$$\left\langle\delta\phi^2\right\rangle_{\rm gravity} \simeq \begin{cases} \frac{H^3t}{4\pi^2}, & (m=0,\ \xi=0)\\ \frac{H^2}{18\pi^2}e^{-\frac{2m^2t}{3H^2}} + \frac{3H^2}{8\pi^2m^2}(1-e^{-\frac{2m^2t}{3H^2}}), & (m\ll H,\ \xi\ll 1/6)\\ \frac{H^2}{24\pi^2}. & (m\gtrsim H,\ \xi\gtrsim 1/6)\end{cases} \tag{4.55}$$

The induced vacuum fluctuation described by Eq. (4.39) and Eq. (4.55) are equivalent to the gravitational particle creation in curved spacetime, and therefore, generated fluctuation remains on the cosmological timescale. However, if the created particles can decay into other particles, the created vacuum fluctuation would disappear on the particle decay timescale.

### 4.2.2 Point-splitting Regularization Method

The point-splitting regularization is the method of regularizing divergences as the point separation in the two-point function, and has been studied in detail in Ref. [118, 119]. In this section, we review the gravitationally induced vacuum fluctuation using the point-splitting regularization, and compare the results in the previous section. The regularized vacuum expectation value can be expressed as [119]

$$\begin{aligned}\left\langle\delta\phi^2\right\rangle_{\rm reg} =& -16\pi^2\epsilon^2 + \frac{R}{576\pi^2} + \frac{1}{16\pi^2}\left[m^2 + \left(\xi-\frac{1}{6}\right)R\right]\\ &\left[\ln\left(\frac{\epsilon^2\mu^2}{12}\right) + \ln\left(\frac{R}{\mu^2}\right) + 2\gamma - 1 + \psi\left(\frac{3}{2}+\nu\right) + \psi\left(\frac{3}{2}-\nu\right)\right].\end{aligned} \tag{4.56}$$

where we take Bunch-Davies vacuum state, $\epsilon$ is the regularization parameter which corresponds with the point separation, $\mu$ is the renormalization scale, $\gamma$ is Euler's constant and $\psi(z) = \Gamma'(z)\,/\,\Gamma(z)$ is the digamma function. By proceeding the renormalization method, we obtain the following expression

$$\begin{aligned}\left\langle\delta\phi^2\right\rangle_{\rm gravity} =& \frac{1}{16\pi^2}\Bigg\{-m^2\ln\left(\frac{12m^2}{\mu^2}\right)\\ &+\left[m^2 + \left(\xi-\frac{1}{6}\right)R\right]\left[\ln\left(\frac{R}{\mu^2}\right) + \psi\left(\frac{3}{2}+\nu\right) + \psi\left(\frac{3}{2}-\nu\right)\right]\Bigg\}.\end{aligned} \tag{4.57}$$

where the additive constant $\psi\left(\frac{3}{2}\pm\nu\right)$ is chosen so that $\left\langle\delta\phi^2\right\rangle_{\text{gravity}} = 0$ at the radiation-dominated Universe $R=0$. In the massive scalar field, we can set simply the renormalization scale to be $\mu^2 = 12m^2$ and get

$$\left\langle\delta\phi^2\right\rangle_{\text{gravity}} = \frac{1}{16\pi^2}\left[m^2+\left(\xi-\frac{1}{6}\right)R\right]\left[\ln\left(\frac{R}{12m^2}\right)+\psi\left(\frac{3}{2}+\nu\right)+\psi\left(\frac{3}{2}-\nu\right)\right] \tag{4.58}$$

In the minimal coupling case $\xi=0$ and we take massless limit $m\to 0$

$$\left\langle\delta\phi^2\right\rangle_{\text{gravity}} \to \frac{R^2}{384\pi^2 m^2} = \frac{3H^4}{8\pi^2 m^2}. \tag{4.59}$$

which is consistent with Eq. (4.54).

Next let us discuss the extremely massive case $m \gg H$ and the digamma function $\psi(z)$ for $z\gg 1$ can be approximated to be [118]

$$\text{Re}\,\psi\left(\frac{3}{2}+iz\right) = \log z + \frac{11}{24z^2} - \frac{127}{960z^4}+\cdots. \tag{4.60}$$

We obtain the following formula,

$$\ln\left(\frac{H^2}{m^2}\right)+\psi\left(\frac{3}{2}+\nu\right)+\psi\left(\frac{3}{2}-\nu\right) \approx \ln\left(\frac{H^2}{m^2}\right)+\psi\left(\frac{3}{2}+i\frac{m}{H}\right)+\psi\left(\frac{3}{2}-i\frac{m}{H}\right) \tag{4.61}$$

$$\approx \frac{11}{12}\frac{H^2}{m^2}-\frac{127}{480}\frac{H^4}{m^4}+\cdots. \tag{4.62}$$

Thus, the gravitationally induced vacuum fluctuation of the extremely massive scalar field for $m\gg H$ is given as follows:

$$\left\langle\delta\phi^2\right\rangle_{\text{gravity}} = \frac{1}{16\pi^2}\left[m^2+\left(\xi-\frac{1}{6}\right)R\right]\left(\frac{11}{12}\frac{H^2}{m^2}-\frac{127}{480}\frac{H^4}{m^4}+\cdots\right) \simeq \mathcal{O}\left(H^2\right), \tag{4.63}$$

which is consistent with Eq. (4.5). Thus, the point-splitting regularization is equivalent to the previous adiabatic regularization.

## 4.3 Gravitationally Induced Vacuum Fluctuation in Schwarzschild Spacetime

The metric of the Schwarzschild spacetime is written by

$$ds^2 = -\left(1-\frac{2M_{\text{BH}}}{r}\right)dt^2 + \frac{dr^2}{1-2M_{\text{BH}}/r} + r^2\left(d\theta^2+\sin^2\theta d\varphi^2\right), \tag{4.64}$$

which covers the exterior region of the spacetime $r > 2M_{\text{BH}}$ where $M_{\text{BH}}$ is the black-hole mass. In the metric of the Schwarzschild spacetime, there is a singularity at the horizon $r = 2M_{\text{BH}}$, which

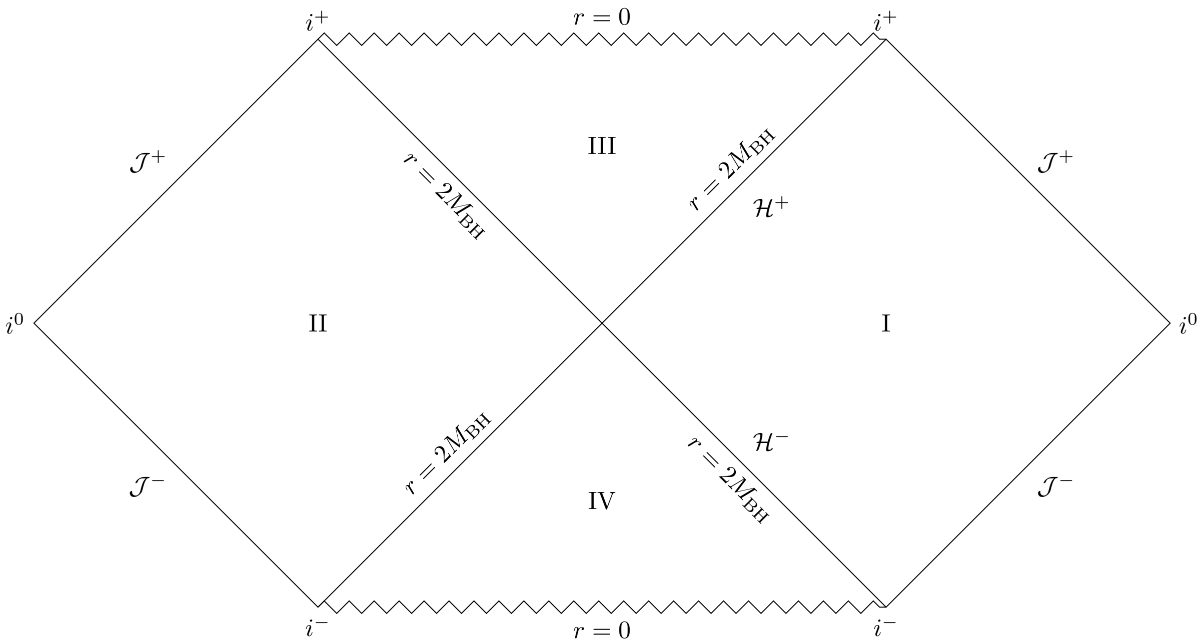


Figure 4.1: The Penrose-Carter diagram of the maximally extended Schwarzschild manifold. Regions I or II are asymptotically flat, Region III is the black hole, and Region IV is the white hole. $\mathcal{H}^+$ corresponds to the future black hole horizon, $\mathcal{H}^-$ is the past black hole horizon, $\mathcal{J}^+$ corresponds to the future null infinity and $\mathcal{J}^-$ is the past null infinity [5].

can be removed by transforming to Kruskal coordinates. By taking the Kruskal coordinates, we obtain the following metric

$$ds^2 = \frac{32M_{\rm BH}^3}{r}e^{-r/2M_{\rm BH}}dUdV + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{4.65}$$

where these coordinates $U$ and $V$ are formally given by

$$U = -M_{\rm BH}e^{-u/4M_{\rm BH}}, \quad u = t - r - 2M_{\rm BH}\ln\left(\frac{r}{2M_{\rm BH}} - 1\right), \tag{4.66}$$

$$V = M_{\rm BH}e^{v/4M_{\rm BH}}, \quad v = t + r + 2M_{\rm BH}\ln\left(\frac{r}{2M_{\rm BH}} - 1\right). \tag{4.67}$$

The Schwarzschild coordinates cover only a part of the spacetime, whereas the Kruskal coordinates cover the extended spacetime and becomes regular at the horizon $r = 2M_{\rm BH}$. These features of the Schwarzschild geometry are summarized in Penrose-Carter diagrams as Fig. 4.1. Generally, there are no unique vacua in curved spacetime, and we must take appropriate vacuum states. In Schwarzschild spacetime, there are three well defined vacua, namely: Boulware vacuum (vacuum polarization around a static star) [120, 121], Unruh vacuum (black hole evaporation) [122] and Hartle-Hawking vacuum (black hole in thermal equilibrium) [123] which correspond to the vacuum definitions on the respective coordinate systems.

Let us consider the massless scalar field $\phi(t,x)$ and then the corresponding Klein-Gordon equation can be given as

$$\left[-\partial_\mu g^{\mu\nu}\sqrt{-g}\partial_\nu\right]\phi(t,x)=0, \tag{4.68}$$

where we drop the curvature term $\xi R\phi^2$ because the Ricci scalar becomes $R=0$ in Schwarzschild spacetime. Note that the Kretschmann scalar $K$ constructed of two Riemann tensors is non-zero to be $K=R_{abcd}R^{abcd}=48M^2/r^6$ and therefore the curvature effects are not zero.

The scalar field $\phi(t,r,\theta,\varphi)$ in Schwarzschild spacetime can be decomposed into

$$\phi(t,r,\theta,\varphi)=\int_0^\infty d\omega\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(a_{\omega lm}u^{in}_{\omega lm}+a^\dagger_{\omega lm}u^{in*}_{\omega lm}+b_{\omega lm}u^{out}_{\omega lm}+b^\dagger_{\omega lm}u^{out*}_{\omega lm}\right), \tag{4.69}$$

where these mode functions $u^{in}_{\omega lm}$ and $u^{out}_{\omega lm}$ defines the vacuum state that $a_{\omega lm}|0\rangle=b_{\omega lm}|0\rangle=0$ which corresponds to the initial and final conditions. In Schwarzschild spacetime, these mode functions $u^{in}_{\omega lm}$ and $u^{out}_{\omega lm}$ for the massless scalar field are given by

$$u^{in}_{\omega lm}=(4\pi\omega)^{-1/2}R^{in}_l(r;\omega)\,Y_{lm}(\theta,\varphi)\,e^{-i\omega t}, \tag{4.70}$$

$$u^{out}_{\omega lm}=(4\pi\omega)^{-1/2}R^{out}_l(r;\omega)\,Y_{lm}(\theta,\varphi)\,e^{-i\omega t}. \tag{4.71}$$

These radial functions $R^{in}_l(r;\omega)$ and $R^{out}_l(r;\omega)$ have the well-known asymptotic forms,

$$R^{in}_l(r;\omega)\simeq\begin{cases}B_l(\omega)\,r^{-1}e^{-i\omega r_*} & (r\to 2M_{\mathrm{BH}})\\ r^{-1}e^{-i\omega r_*}+A^{in}_l(\omega)\,r^{-1}e^{i\omega r_*} & (r\to\infty)\end{cases},$$

$$R^{out}_l(r;\omega)\simeq\begin{cases}r^{-1}e^{i\omega r_*}+A^{out}_l(\omega)\,r^{-1}e^{-i\omega r_*} & (r\to 2M_{\mathrm{BH}})\\ B_l(\omega)\,r^{-1}e^{i\omega r_*} & (r\to\infty)\end{cases}$$

where $A^{in}_l(\omega)$, $A^{out}_l(\omega)$ and $B_l(\omega)$ are the reflection and transmission coefficients [24]. The Boulware vacuum $|0_{\mathrm{B}}\rangle$ is defined by taking ingoing and outgoing modes to be positive frequency with respect to the Killing vector $\partial_t$ of the Schwarzschild metric [120] and corresponds to the interpretation of the scattering theory. This state closely reproduces the Minkowski vacuum state at infinity due to the fact $\langle 0_{\mathrm{B}}|\,\delta\phi^2\,|0_{\mathrm{B}}\rangle\to 1/r^2$ in the limit $r\to\infty$. However, the Boulware vacuum is singular on the event horizons $r=2M_{\mathrm{BH}}$ and hence not valid near the black-hole horizon. Thus, the Boulware vacuum is considered to be the appropriate state which describes the gravitationally induced vacuum fluctuation around a static star, not black hole.

The two-point correlation function $\langle\delta\phi^2\rangle$ for the scalar field in the Boulware vacuum $|0_{\mathrm{B}}\rangle$ can be given by [124, 125]:

$$\langle 0_{\mathrm{B}}|\delta\phi^2(x)|0_{\mathrm{B}}\rangle=\frac{1}{16\pi^2}\int_0^\infty\frac{d\omega}{\omega}\times\left[\sum_{l=0}^{\infty}(2l+1)\left[\left|R^{in}_l(r;\omega)\right|^2+\left|R^{out}_l(r;\omega)\right|^2\right]\right], \tag{4.72}$$

where the sum of these radial functions $R_l^{in}(r;\omega)$ and $R_l^{out}(r;\omega)$ have the asymptotic forms,

$$\sum_{l=0}^{\infty}(2l+1)\left|R_l^{in}(r;\omega)\right|^2 \sim \begin{cases} \frac{\sum_{l=0}^{\infty}(2l+1)|B_l(\omega)|^2}{4M_{\rm BH}^2} & (r\to 2M_{\rm BH}) \\ 4\omega^2 & (r\to\infty) \end{cases},$$
$$\sum_{l=0}^{\infty}(2l+1)\left|R_l^{out}(r;\omega)\right|^2 \sim \begin{cases} \frac{4\omega^2}{1-2M_{\rm BH}/r} & (r\to 2M_{\rm BH}) \\ \frac{\sum_{l=0}^{\infty}(2l+1)|B_l(\omega)|^2}{r^2} & (r\to\infty) \end{cases}. \tag{4.73}$$

As previously discussed, the two-point correlation function $\left\langle\delta\phi^2\right\rangle$ of Eq. (4.72) has clearly UV divergences and requires the regularization. There exist several regularization methods to remove the divergences of the integral, but here we adopt the point-splitting regularization. Let us consider $\delta\phi^2(x)\to\delta\phi(x)\,\delta\phi(x')$ temporarily and afterwards take the coincident limit $x'\to x$. Using the point-splitting regularization we get the gravitationally induced vacuum fluctuation

$$\left\langle\delta\phi^2(x)\right\rangle_{\rm gravity} = \lim_{x'\to x}\left[\langle\delta\phi(x)\delta\phi(x')\rangle - \langle\delta\phi(x)\delta\phi(x')\rangle_{\rm div}\right], \tag{4.74}$$

where $\langle\delta\phi(x)\delta\phi(x')\rangle_{\rm div}$ express the divergence part and is namely the DeWitt-Schwinger counter-term, which is generally given by [126]

$$\langle\delta\phi(x)\delta\phi(x')\rangle_{\rm div} = \frac{1}{8\pi^2\sigma} + \frac{m^2+(\xi-1/6)R}{8\pi^2}\left[\gamma+\frac{1}{2}\ln\left(\frac{\mu^2\sigma}{2}\right)\right] - \frac{m^2}{16\pi^2} + \frac{1}{96\pi^2}R_{\alpha\beta}\frac{\sigma^{;\alpha}\sigma^{;\beta}}{\sigma}, \tag{4.75}$$

where $\sigma$ is the biscalar associated with the short geodesic and $\gamma$ express the Euler-Mascheroni constant. Note that the UV divergences do not contribute to the gravitational particle production and therefore the renormalized two-point correlation function exactly express the gravitationally induced vacuum fluctuation. In Schwarzschild metric for the massless scalar field, we can simplify the DeWitt-Schwinger counter-term of $\langle\delta\phi(x)\delta\phi(x')\rangle_{\rm div}$ and get

$$\langle\delta\phi(x)\delta\phi(x')\rangle_{\rm div} = \frac{1}{8\pi^2\sigma}. \tag{4.76}$$

For simplicity when we take the time separation to be $x=(t,r,\theta,\varphi)$ and $x'=(t+\epsilon,r,\theta,\varphi)$ the gravitationally induced vacuum fluctuation $\left\langle\delta\phi^2\right\rangle$ in the Boulware vacuum $|0_{\rm B}\rangle$ is given by

$$\left\langle\delta\phi^2(x)\right\rangle_{\rm gravity} = \lim_{\epsilon\to 0}\left[\frac{1}{16\pi^2}\int_0^\infty\frac{e^{-i\omega\epsilon}}{\omega}\left[\sum_{l=0}^{\infty}(2l+1)\left|R_l^{in}(r;\omega)\right|^2\left|R_l^{out}(r;\omega)\right|^2\right]d\omega - \frac{1}{8\pi^2\sigma(\epsilon)}\right]. \tag{4.77}$$

By taking the second-order geodesic expansion we get the following expression [126]

$$\sigma(\epsilon) = -\frac{1-2M_{\rm BH}/r}{2}\epsilon^2 - \frac{M_{\rm BH}^2(1-2M_{\rm BH}/r)}{24r^4}\epsilon^4 + O\left(\epsilon^5\right), \tag{4.78}$$

where $\epsilon^{-2}$ satisfy the following relation

$$\epsilon^{-2} = -\int_0^\infty\omega e^{i\omega\epsilon}d\omega. \tag{4.79}$$

Using Eq. (4.77), Eq. (4.78) and Eq. (4.79), we obtain the gravitationally induced vacuum fluctuation of the Boulware vacuum $|0_{\mathrm{B}}\rangle$ as follows [125]

$$\left\langle 0_{\mathrm{B}}|\delta\phi^2(x)|0_{\mathrm{B}}\right\rangle_{\mathrm{gravity}} = \frac{1}{16\pi^2}\int_0^\infty \frac{d\omega}{\omega}\left[\sum_{l=0}^{\infty}(2l+1)\left[\left|R_l^{in}(r;\omega)\right|^2+\left|R_l^{out}(r;\omega)\right|^2\right]-\frac{4\omega^2}{1-2M_{\mathrm{BH}}/r}\right] - \frac{M_{\mathrm{BH}}^2}{48\pi^2 r^4\left(1-2M_{\mathrm{BH}}/r\right)}.$$

For the Boulware vacuum $|0_{\mathrm{B}}\rangle$, the asymptotic expression of the gravitationally induced vacuum fluctuation $\left\langle\delta\phi^2\right\rangle_{\mathrm{gravity}}$ can be given by

$$\left\langle 0_{\mathrm{B}}|\delta\phi^2(x)|0_{\mathrm{B}}\right\rangle_{\mathrm{gravity}} \longrightarrow \infty \quad (r\to 2M_{\mathrm{BH}}),$$
$$\left\langle 0_{\mathrm{B}}|\delta\phi^2(x)|0_{\mathrm{B}}\right\rangle_{\mathrm{gravity}} \longrightarrow 1/r^2 \quad (r\to \infty),$$

which is singular on the event horizons $r = 2M_{\mathrm{BH}}$ and ill-defined near the black-hole horizon.

In vacuum expectation values $\langle T_{\mu\nu}\rangle$ of the energy momentum tensor has been given by Ref. [125,127–132] and shown similar properties to $\left\langle\delta\phi^2\right\rangle_{\mathrm{graivty}}$. The renormalized energy momentum tensor $\langle T_{\mu\nu}\rangle_{\mathrm{gravity}}$ in Boulware vacuum $|0_{\mathrm{B}}\rangle$ has the following asymptotic forms

$$\left\langle 0_{\mathrm{B}}|T_\mu^\nu|0_{\mathrm{B}}\right\rangle_{\mathrm{gravity}} \longrightarrow -\frac{1}{30\cdot 2^{12}\pi^2 M_{\mathrm{BH}}^4\left(1-2M_{\mathrm{BH}}/r\right)^2}\begin{pmatrix}-1 & 0 & 0 & 0\\ 0 & 1/3 & 0 & 0\\ 0 & 0 & 1/3 & 0\\ 0 & 0 & 0 & 1/3\end{pmatrix} \quad (r\to 2M_{\mathrm{BH}}),$$
$$\left\langle 0_{\mathrm{B}}|T_\mu^\nu|0_{\mathrm{B}}\right\rangle_{\mathrm{gravity}} \longrightarrow 1/r^6 \quad (r\to\infty),$$

which produces a negative energy divergence at the horizon $r = 2M_{\mathrm{BH}}$. Therefore, the usual interpretation of the above result is that the Boulware vacuum $|0_{\mathrm{B}}\rangle$ is considered to be the appropriate vacuum state around a static star and not a black hole.

Next, let us discuss the Unruh vacuum $|0_{\mathrm{U}}\rangle$ which considered to be appropriate vacua which describe evaporating black hole formed by gravitational collapse [125]. The Unruh vacuum $|0_{\mathrm{U}}\rangle$ is formally defined by taking ingoing modes to be positive frequency with respect to $\partial_t$, but outgoing modes to be positive frequency with respect to Kruskal coordinate $\partial_U$ [122]. The Unruh vacuum corresponds to the state where the black hole radiates at the Hawking temperature $T_{\mathrm{H}} = 1/8\pi M_{\mathrm{BH}}$ in empty space, and therefore, the gravitationally induced vacuum fluctuation $\left\langle\delta\phi^2\right\rangle_{\mathrm{gravity}}$ approaches thermal Hawking fluctuation near the black-hole horizon: $\langle 0_{\mathrm{U}}|\,\delta\phi^2\,|0_{\mathrm{U}}\rangle \to \mathcal{O}\left(T_{\mathrm{H}}^2\right)$ in the limit $r\to 2M_{\mathrm{BH}}$.

The two-point correlation function $\left\langle\delta\phi^2\right\rangle$ for the Unruh vacuum $|0_{\mathrm{U}}\rangle$ is given by [124,125],

$$\left\langle 0_{\mathrm{U}}|\delta\phi^2(x)|0_{\mathrm{U}}\right\rangle = \frac{1}{16\pi^2}\int_0^\infty \frac{d\omega}{\omega}\left[\sum_{l=0}^{\infty}(2l+1)\left[\left|R_l^{in}(r;\omega)\right|^2+\coth\left(\frac{\pi\omega}{\kappa}\right)\left|R_l^{out}(r;\omega)\right|^2\right]\right], \tag{4.80}$$

where $\kappa = (4M_{\rm BH})^{-1}$ is the surface gravity of the black hole and the factor of $\coth\left(\frac{\pi\omega}{\kappa}\right)$ originating from the thermal features of the outgoing modes. The gravitationally induced vacuum fluctuation for the Unruh vacuum $|0_{\rm U}\rangle$ can be written as

$$\left\langle 0_{\rm U}|\delta\phi^2(x)|0_{\rm U}\right\rangle_{\rm gravity} = \frac{1}{16\pi^2}\int_0^\infty \frac{d\omega}{\omega}\left[\sum_{l=0}^\infty (2l+1)\left[\left|R_l^{in}(r;\omega)\right|^2 + \coth\left(\frac{\pi\omega}{\kappa}\right)\left|R_l^{out}(r;\omega)\right|^2\right] - \frac{4\omega^2}{1-2M_{\rm BH}/r}\right] - \frac{M_{\rm BH}^2}{48\pi^2 r^4(1-2M_{\rm BH}/r)}. \tag{4.81}$$

For the Unruh vacuum, we obtain asymptotic expression of the gravitationally induced vacuum fluctuation to be

$$\left\langle 0_{\rm U}|\delta\phi^2(x)|0_{\rm U}\right\rangle_{\rm gravity} \longrightarrow \frac{1}{192\pi^2 M_{\rm BH}^2} - \frac{1}{32\pi^2 M_{\rm BH}^2}\int_0^\infty \frac{d\omega\omega\sum_{l=0}^\infty (2l+1)|B_l(\omega)|^2}{\omega\left(e^{2\pi\omega/\kappa}-1\right)} \quad (r\to 2M_{\rm BH}),$$
$$\left\langle 0_{\rm U}|\delta\phi^2(x)|0_{\rm U}\right\rangle_{\rm gravity} \longrightarrow 1/r^2 \quad (r\to\infty).$$

The Hartle-Hawking Vacuum $|0_{\rm HH}\rangle$ is formally defined by taking ingoing modes to be positive frequency with respect to $\partial_V$, and outgoing modes to be positive frequency with respect to the Kruskal coordinate $\partial_U$ [123]. In the Hartle-Hawking vacuum $|0_{\rm HH}\rangle$, we obtain the two-point correlation functions,

$$\left\langle 0_{\rm HH}|\delta\phi^2(x)|0_{\rm HH}\right\rangle = \frac{1}{16\pi^2}\int_0^\infty \frac{d\omega}{\omega}\left[\coth\left(\frac{\pi\omega}{\kappa}\right)\sum_{l=0}^\infty (2l+1)\left[\left|R_l^{in}(r;\omega)\right|^2 + \left|R_l^{out}(r;\omega)\right|^2\right]\right]. \tag{4.82}$$

For the Hartle-Hawking vacuum $|0_{\rm HH}\rangle$ we get the asymptotic expression of the gravitationally induced vacuum fluctuation $\left\langle\delta\phi^2\right\rangle_{\rm gravity}$,

$$\left\langle 0_{\rm HH}|\delta\phi^2(x)|0_{\rm HH}\right\rangle_{\rm gravity} \longrightarrow \frac{1}{192\pi^2 M_{\rm BH}^2} \quad (r\to 2M_{\rm BH}),$$
$$\left\langle 0_{\rm HH}|\delta\phi^2(x)|0_{\rm HH}\right\rangle_{\rm gravity} \longrightarrow T_{\rm H}^2/12 \quad (r\to\infty),$$

where the gravitationally induced vacuum fluctuation $\left\langle\delta\phi^2\right\rangle$ approaches the thermal fluctuation $\langle 0_{\rm HH}|\,\delta\phi^2\,|0_{\rm HH}\rangle \to T_{\rm H}^2/12$ at infinity $r\to\infty$. Therefore, the Hartle-Hawking vacuum corresponds to a black hole in thermal equilibrium at $T_{\rm H} = 1/8\pi M_{\rm BH}$.

The analytic approximations of $\left\langle\delta\phi^2\right\rangle_{\rm gravity}$ and $\langle T_{\mu\nu}\rangle_{\rm gravity}$ in Schwarzschild spacetime for the various vacua (Boulware, Unruh vacuum and Hartle-Hawking) and the various fields of spin 0, 1/2 and 1 have been investigated by Ref. [125,127–142]. The gravitationally induced vacuum contributions of the various fields are proportional to the inverse of the black-hole mass $M_{\rm BH}$ near the black-hole horizon and approximately become thermal with the Hawking temperature $T_{\rm H}$. This fact originates from that $\left\langle\delta\phi^2\right\rangle_{\rm gravity}$ and $\langle T_{\mu\nu}\rangle_{\rm gravity}$ are generated by the quantum gravitational effects around the black hole.

# Chapter 5

# Gravitationally Induced Vacuum Phase Transition

The gravitational effects of the induced vacuum fluctuation of the various fields on the vacuum stability are twofold. On one side, the gravitationally induced vacuum fluctuation can destabilize the effective potential as the backreaction. On the other side, the inhomogeneous vacuum field fluctuation can generate true vacuum bubbles or domains and triggers off a collapse of the false vacuum. The decay of the false vacuum in various gravitational backgrounds or cosmological situations has these two scenarios. In this chapter we discuss the gravitationally induced vacuum phase transition in more detail.

## 5.1 Gravitational Phase Transition

In this section we consider the gravitational second order phase transition where gravitationally induced vacuum fluctuation destabilizes the effective potential.

For simplicity we restrict our attention to the scalar field theory and then let us consider a simple effective potential $V(\phi)$ where the scalar field $\phi$ couples the extra scalar field $\varphi$ with the interaction coupling $g$ as follows:

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4 - \frac{g}{2}\varphi^2\phi^2. \tag{5.1}$$

where the self-interaction coupling $\lambda$ is positive. The gravitationally induced vacuum fluctuations are classic and modify the above scalar potential as the following

$$\begin{aligned} V(\phi) &= \frac{1}{2}\left(m^2 + \lambda\left\langle\delta\phi^2\right\rangle_{\text{gravity}} - g\left\langle\delta\varphi^2\right\rangle_{\text{gravity}}\right)\phi^2 + \frac{\lambda}{4}\phi^4 \\ &\approx -g\left\langle\delta\varphi^2\right\rangle_{\text{gravity}}\phi^2 + \frac{\lambda}{4}\phi^4, \end{aligned} \tag{5.2}$$

For large vacuum fluctuation of $\left\langle\delta\varphi^2\right\rangle_{\text{gravity}}$ the origin of the effective potential $V(\phi)$ does not stable and therefore the scalar field classically rolls down to the new vacuum state. Dynamical behavior of the scalar field in cosmological spacetime is determined by Klein-Gordon equations

$$\Box\phi(t) + V'(\phi(t)) = 0. \tag{5.3}$$

We rewrite the Klein-Gordon equation as the following

$$\ddot{\phi}(t) + 3H\dot{\phi}(t) + V'(\phi(t)) = 0. \tag{5.4}$$

If we can approximate the effective potential as $V'(\phi) = -g\left\langle\delta\varphi^2\right\rangle_{\text{gravity}}\phi$, the dynamics of the scalar field $\phi(t)$ can be described as follows

$$\phi(t) \propto \exp\left\{\frac{t}{2}\left(-3H + \sqrt{9H^2 + 4g\left\langle\delta\varphi^2\right\rangle_{\text{gravity}}}\right)\right\} \tag{5.5}$$

## 5.2 Gravitational False Vacuum Decay

In this section we consider the gravitational first order phase transition. To investigate the decay of the false vacuum in gravitational background there are different approaches to calculate the probability. The quantum tunneling method in gravitational background is formally calculated by the Coleman-de Luccia (CdL) formalism [143] which corresponds to true vacuum bubble nucleation in false vacuum. The decay rate of the vacuum in gravitational background is given by

$$\Gamma = A\exp(-B), \tag{5.6}$$

where $A$ is a prefactor and $B$ is given by the difference between the action of the bounce solution and the action of the false vacuum as follows:

$$B = S_E(\phi) - S_E(\phi_{fv}) \tag{5.7}$$

which is determined by the Euclidean action:

$$S_E[\phi, g_{\mu\nu}] = \int d^4x\sqrt{-g}\left[\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi + V(\phi) - \frac{M_{\text{Pl}}^2}{2}R\right]. \tag{5.8}$$

To discuss instanton mediated vacuum transitions in gravitational background we consider the Euclidean analogue of cosmological spacetime:

$$ds^2 = d\chi^2 + a^2(\chi)d\Omega_3^2, \tag{5.9}$$

where $\chi^2 = t^2 + r^2$, $a(\chi)$ is the Euclidean scale factor and $d\Omega_3^2$ is the metric of a 3-sphere. The equations of motion in this case are given by

$$\ddot{\phi} + \frac{3\dot{a}}{a}\dot{\phi} - V'(\phi) = 0 \tag{5.10}$$

$$\dot{a}^2 = 1 + \frac{a^2}{3M_{\text{P}}^2}\left(\frac{\dot{\phi}^2}{2} - V(\phi)\right). \tag{5.11}$$

By solving these equations the decay exponent $B$ can be taken as

$$B = \frac{24\pi^2 M_{\text{P}}^4}{V(\phi_{\text{fv}})} - 2\pi^2\int_0^\infty d\chi a^3(\chi)V(\phi(\chi)) \tag{5.12}$$

For simplicity let us consider de-Sitter spacetime. The trivial solution of the Euclidean equations of motion assumes that the scalar field stays on the top of the potential. That is known as the Hawking-Moss instanton [144],

$$\phi(\chi) = \phi_{\rm max}, \;\; a(\chi) = \frac{\sqrt{3}M_{\rm pl}}{V\left(\phi_{\rm max}\right)^{1/2}} \sin\left(\frac{V\left(\phi_{\rm max}\right)^{1/2}}{\sqrt{3}M_{\rm pl}}\chi\right), \tag{5.13}$$

For this solution the decay exponent $B$ is given by

$$B = 24\pi^2 M_{\rm Pl}^4 \left(\frac{1}{V(\phi_{\rm fv})} - \frac{1}{V(\phi_{\rm max})}\right). \tag{5.14}$$

In the limit $|V(\phi_{\rm max}) - V(\phi_{\rm fv})| \ll |V(\phi_{\rm fv})|$ the decay exponent $B$ can approximately become

$$B \simeq \frac{8\pi^2 V(\phi_{\rm max})^4}{3H^4}, \tag{5.15}$$

which represents the probability that thermal fluctuation pushes the scalar field $\phi$ on the Hubble volume from the false vacuum to the top of the potential at the Gibbons-Hawking temperature $T_{\rm GH} = H/2\pi$. On the other hand there are other bounce solutions known as Coleman-de Luccia (CdL) instanton. The CdL instanton can be interpreted as that thermal fluctuation pushes $\phi$ partially and the pushed $\phi$ goes out to true vacuum via quantum tunneling. The Hawking-Moss instanton and the CdL instanton has applicable ranges respectively. If the effective potential barrier is sufficiently small compared with the Hubble scale, the CdL instanton does not necessarily exist [145] and the gravitational vacuum transition is described by the Hawking-Moss instanton. Note that the Hawking-Moss transition should be interpreted as an entire Hubble-volume tunneling [145, 146], and therefore the transition occurs on only a Hubble patch, and not the entire Universe.

There is an another established method as a more intuitive formalism of dealing with the gravitational vacuum transition which is so-called stochastic formalism [147]. This formalism quantitatively captures the vacuum transitions induced by the thermal de-Sitter fluctuation. In de-Sitter spacetime the quantum fluctuation can be given by

$$\delta\phi \approx T_{\rm GH} = \frac{H}{2\pi}. \tag{5.16}$$

where $T_{\rm GH}$ is the Gibbons-Hawking temperature. The stochastic approach treats the thermal de-Sitter fluctuation as a random noise term of the stochastic equation and can be summarized as the Langevin equation and the Fokker-Planck equation

$$\frac{\partial P\left(\phi\right)}{\partial t} = \frac{\partial}{\partial\phi}\left[\frac{V'(\phi)}{3H}P\left(\phi\right) + \frac{H^3}{8\pi^2}\frac{\partial P\left(\phi\right)}{\partial\phi}\right]. \tag{5.17}$$

where $P\left(\phi\right)$ is the probability distribution function. At the late time static solution for the probability distribution function $P\left(\phi\right)$ completely corresponds to the transition probability of the Hawking-Moss instanton as

$$P\left(\phi\right) \simeq \exp\left(-\frac{8\pi^2 V(\phi)^4}{3H^4}\right) \tag{5.18}$$

However, the Fokker-Planck approach becomes more useful than the Hawking-Moss instanton because it takes the bounce solution from the top of the potential. For instance the Hawking-Moss instanton does not capture multiple transitions across the barrier. More generally if we can recognize the de-Sitter quantum fluctuation as the Gaussian noise, the probability density function $P(\phi)$ can be given by

$$P(\phi) = \frac{1}{\sqrt{2\pi \left\langle \delta\phi^2 \right\rangle_{\text{gravity}}}} \exp\left(-\frac{\phi^2}{2\left\langle \delta\phi^2 \right\rangle_{\text{gravity}}}\right), \tag{5.19}$$

For instance if we consider de-Sitter spacetime and the chaotic potential $V(\phi) = m^2\phi^2/2$, the gravitationally induced vacuum fluctuation and probability density function $P(\phi)$ are given by

$$\left\langle \delta\phi^2 \right\rangle_{\text{gravity}} = \frac{3H^4}{8\pi^2 m^2} \longrightarrow P(\phi) = \frac{m}{H^2}\sqrt{\frac{4\pi}{3}} \exp\left(-\frac{4\pi^2 m^2 \phi^2}{3H^4}\right) \tag{5.20}$$

which corresponds to the the Fokker-Planck equation with the chaotic potential $V(\phi) = m^2\phi^2/2$. If we get the gravitationally induced vacuum fluctuation $\left\langle \delta\phi^2 \right\rangle_{\text{gravity}}$ by using some regularization methods in curved spacetime, we can calculate the probability of the vacuum decay transition. From now we assume that the probability distribution function of the gravitationally induced vacuum fluctuation $\left\langle \delta\phi^2 \right\rangle_{\text{gravity}}$ is Gaussian [17]:

$$P(\phi) = \frac{1}{\sqrt{2\pi \left\langle \delta\phi^2 \right\rangle_{\text{gravity}}}} \exp\left(-\frac{\phi^2}{2\left\langle \delta\phi^2 \right\rangle_{\text{gravity}}}\right). \tag{5.21}$$

By using Eq. (5.21), the probability that the false vacuum survives can be given as

$$P(\phi < \phi_{\text{max}}) \equiv \int_{-\phi_{\text{max}}}^{\phi_{\text{max}}} P(\phi)\, d\phi, \tag{5.22}$$

$$= \operatorname{erf}\left(\frac{\phi_{\text{max}}}{\sqrt{2\left\langle \delta\phi^2 \right\rangle_{\text{gravity}}}}\right). \tag{5.23}$$

where $\phi_{\text{max}}$ is defined as the maximal value of the potential $V(\phi)$. On the other hand, the probability that the inhomogeneous scalar field falls into true vacuum can be expressed as

$$P(\phi > \phi_{\text{max}}) = 1 - \operatorname{erf}\left(\frac{\phi_{\text{max}}}{\sqrt{2\left\langle \delta\phi^2 \right\rangle_{\text{gravity}}}}\right) \simeq \frac{\sqrt{2\left\langle \delta\phi^2 \right\rangle_{\text{gravity}}}}{\pi\phi_{\text{max}}} \exp\left(-\frac{\phi_{\text{max}}^2}{2\left\langle \delta\phi^2 \right\rangle_{\text{gravity}}}\right). \tag{5.24}$$

Then, the constraint from the gravitational vacuum decay can be represented by

$$\mathcal{V} \cdot P(\phi > \phi_{\text{max}}) < 1, \tag{5.25}$$

where we take $\mathcal{V}$ to be the volume of the domains in which the gravitationally induced vacuum fluctuation $\langle\delta\phi^2\rangle_{\rm gravity}$ governs. By substituting Eq. (5.24) into Eq. (5.25), we can simplify the above constraint of gravitational vacuum decay,

$$\frac{\langle\delta\phi^2\rangle_{\rm gravity}}{\phi_{\rm max}^2} < \frac{1}{2}\left(\log\mathcal{V}\right)^{-1}, \tag{5.26}$$

The most uncertain thing in the stochastic formalism is how to determine the volume factor $\mathcal{V}$. In inflationary Universe, the volume factor of $\mathcal{V}$ can be given by

$$\mathcal{V} \simeq e^{3N_{\rm hor}} \tag{5.27}$$

where $N_{\rm hor}$ is the e-folding number which can be $N_{\rm hor} \simeq N_{\rm CMB} \simeq 60$. By substituting Eq. (5.27) into Eq. (5.26) we obtain following relation of the vacuum stability

$$\frac{\langle\delta\phi^2\rangle_{\rm gravity}}{\phi_{\rm max}^2} < \frac{1}{6N_{\rm hor}}. \tag{5.28}$$

However, the gravitationally induced vacuum decay around evaporating black holes can be more trouble. The induced vacuum fluctuation in Unruh vacuum $|0_{\rm U}\rangle$ approximately approach the Hawking temperature $T_{\rm H}$ near the horizon. But at the infinity the vacuum fluctuation attenuates rapidly and becomes zero. Therefore, the gravitationally induced vacuum fluctuation $\langle\delta\phi^2\rangle_{\rm gravity}$ around the evaporating black hole can be summarized as follows:

$$\langle\delta\phi^2\rangle_{\rm gravity} \simeq \begin{cases} \mathcal{O}\left(T_{\rm H}^2\right) & (r \to 2M_{\rm BH}) \\ 0 & (r \to \infty) \end{cases}. \tag{5.29}$$

Thus it is obvious that we can not take the entire volume of the Universe as $\mathcal{V}$ because the induced vacuum fluctuation approaches zero far from the black hole and the large vacuum fluctuation exists only near the black-hole horizon. Therefore, the volume factor $\mathcal{V}$ can be given by $\mathcal{V} = \mathcal{N}_{\rm EBH}\cdot\mathcal{O}\left(1\right)$ where $\mathcal{N}_{\rm EBH}$ is the number of the evaporating or evaporated black holes during the cosmological history of the Universe. By using Eq. (5.26), we obtain the number constraint of the evaporating black holes as follows:

$$\frac{\langle\delta\phi^2\rangle_{\rm gravity}}{\phi_{\rm max}^2} < \frac{1}{2}\left(\log\mathcal{N}_{\rm EBH}\right)^{-1}, \tag{5.30}$$

In next chapter we discuss the electroweak vacuum stability in gravitational background using the stability conditions of Eq. (5.28) and Eq. (5.30)

# Chapter 6

# Electroweak Vacuum Stability in Curved Spacetime

In this chapter we discuss electroweak vacuum stability in curved spacetime. First, we derive the SM effective Higgs potential in curved spacetime and than explain how the gravitationally induced vacuum fluctuation of the SM particles modify the effective Higgs potential. Next, we discuss false vacuum decay on the gravitational background using the stochastic formalism based on the Higgs vacuum fluctuation. We investigate the electroweak vacuum stability in various spacetimes or cosmological situations as during inflation corresponding to the de-Sitter spacetime, after inflation in particular the preheating stage and around evaporating black holes.

## 6.1 Effective Higgs Potential in Curved Spacetime

Let us consider the SM Higgs effective potential in curved spacetime. The one-loop effective Higgs potential $V_{\text{eff}}(\phi)$ where $\phi$ is the Higgs field can be given as follows [78]

$$\begin{aligned} V_{\text{eff}}(\phi) &= \rho_\Lambda(\mu) + \frac{1}{2}m_\phi^2(\mu)\phi^2 + \frac{1}{2}\xi_\phi(\mu)R\phi^2 + \frac{\lambda_\phi(\mu)}{4}\phi^4 \\ &+ \sum_{i=W,Z,t,G,H} \frac{n_i}{64\pi^2} M_i^4(\phi)\left[\log\frac{M_i^2(\phi)}{\mu^2} - C_i\right], \end{aligned} \tag{6.1}$$

where we take 't Hooft-Landau gauge and $\overline{\text{MS}}$ scheme, and $M_i^2(\phi)$ is given by

$$M_i^2(\phi) = \kappa_i\phi^2 + \kappa_i' + \theta_i R. \tag{6.2}$$

The coefficients $n_i$, $\kappa_i$, $\kappa_i'$ and $\theta_i$ are given by Table I of Ref. [78]. The mass terms of $W/Z$ bosons, top quark, Nambu-Goldstone bosons, and Higgs boson are given by

$$\begin{aligned} &m_W^2 = \frac{1}{4}g^2\phi^2,\ m_Z^2 = \frac{1}{4}\left[g^2 + g'^2\right]\phi^2,\ m_t^2 = \frac{1}{2}y_t^2\phi^2, \\ &m_G^2 = m_\phi^2 + \lambda_\phi\phi^2,\ m_H^2 = m_\phi^2 + 3\lambda_\phi\phi^2, \end{aligned}$$

where $g$, $g'$, $y_t$ are the $SU(2)_L$, $U(1)_Y$, top Yukawa couplings and $\lambda_\phi$ is the Higgs self-coupling. The $\beta$-function for the non-minimal coupling $\xi(\mu)$ in the Standard Model without gravity loops is given by

$$\beta_\xi = \frac{1}{(4\pi)^2}(\xi - 1/6)\left(6\lambda + 3y_t^2 - \frac{3}{4}g'^2 - \frac{9}{4}g^2\right). \tag{6.3}$$

The RG running of the non-minimal coupling $\xi(\mu)$ can be obtained by integrating $\beta_\xi$

$$\xi(\mu) = \frac{1}{6} + \left(\xi_{\rm EW} - \frac{1}{6}\right)F(\mu), \tag{6.4}$$

where $F(\mu)$ is the function depending on the renormalization scale $\mu$. If we have the nearly minimal coupling $\xi_{\rm EW} \lesssim O(10^{-2})$ at the electroweak scale, the running non-minimal coupling $\xi(\mu)$ becomes negative at some scale [78]. On the other hand, we can take the initial condition of the running non-minimal coupling $\xi(\mu)$ at the Planck scale [17]. If the curvature $R$ is larger than the instability scale to be $R > \Lambda_I$ and then $\xi(R) < 0$, the effective Higgs potential in curved spacetime is unstable: $V'_{\rm eff}(\phi) \lesssim 0$ and the homogeneous Higgs field $\phi$ on the entire Universe rolls down to the negative-energy Planck-scale true vacuum.

However, we must include the backreaction from the gravitationally induced vacuum fluctuation of the Higgs field in the effective Higgs potential. The modified effective Higgs potential including the gravitational Higgs vacuum fluctuation can be given by [1,2]:

$$\begin{aligned} V_{\rm eff}(\phi) = \rho_\Lambda(\mu) + \frac{1}{2}m_\phi^2(\mu)\phi^2 + \frac{1}{2}\xi_\phi(\mu)R\phi^2 + \frac{\lambda_\phi(\mu)}{4}\phi^4 + \frac{3\lambda(\mu)}{2}\left\langle\delta\phi^2\right\rangle_{\rm gravity}\phi^2 \\ + \sum_{i=W,Z,t,G,H}\frac{n_i}{64\pi^2}M_i^4(\phi)\left[\log\frac{M_i^2(\phi)}{\mu^2} - C_i\right], \end{aligned} \tag{6.5}$$

where the induced Higgs fluctuation stabilize or destabilize the effective Higgs potential in curved spacetime. However, the vacuum fluctuation of the $W/Z$ bosons and the top quark can raise the effective Higgs potential, and therefore, the effective Higgs potential $V_{\rm eff}(\phi)$ including the vacuum fluctuation of the various SM fields can be written as follows:

$$\begin{aligned} V_{\rm eff}(\phi) = \rho_\Lambda(\mu) + \frac{1}{2}m_\phi^2(\mu)\phi^2 + \frac{1}{2}\xi_\phi(\mu)R\phi^2 + \frac{\lambda_\phi(\mu)}{4}\phi^4 + \frac{3\lambda(\mu)}{2}\left\langle\delta\phi^2\right\rangle_{\rm gravity}\phi^2 \\ + \frac{g^2(\mu)}{8}\left\langle\delta W^2\right\rangle_{\rm gravity}\phi^2 + \frac{[g^2(\mu)+g'^2(\mu)]}{8}\left\langle\delta Z^2\right\rangle_{\rm gravity}\phi^2 + \frac{y_t^2(\mu)}{4}\left\langle\delta t^2\right\rangle_{\rm gravity}\phi^2 + \\ + \sum_{i=W,Z,t,G,H}\frac{n_i}{64\pi^2}M_i^4(\phi)\left[\log\frac{M_i^2(\phi)}{\mu^2} - C_i\right], \end{aligned} \tag{6.6}$$

where the renormalization scale $\mu$ can be taken as $\mu^2 \approx \phi^2 + \left\langle\delta\phi^2\right\rangle_{\rm gravity} + R$ to suppress the high order log-corrections as previously discussed. Through the above consideration, we obtain the exact effective Higgs potential including gravitational backreactions in curved spacetime.

If we assume that gravitational fluctuations of the Higgs, $W/Z$ bosons and top quark are approximately equivalent to thermal fluctuations with the Gibbons-Hawking temperature $T_{\rm GH}$,

$$\left\langle\delta\phi^2\right\rangle_{\rm gravity} \approx \left\langle\delta W^2\right\rangle_{\rm gravity} \approx \left\langle\delta Z^2\right\rangle_{\rm gravity} \approx \left\langle\delta t^2\right\rangle_{\rm gravity} \approx \mathcal{O}\left(T_{\rm GH}^2\right), \tag{6.7}$$

the effective Higgs potential in curved spacetime reproduces the thermal result. Thus, roughly speaking, the vacuum stability of the Higgs in the de-Sitter and Schwarzschild spacetime becomes consistent with the thermal case. However, the gravitationally induced vacuum fluctuation of the SM fields can not be always approximated as the thermal fluctuation with the Gibbons-Hawking temperature $T_{\rm GH}$ as previously discussed.

## 6.2 Electroweak Vacuum Stability during Inflation

In this section, we investigate the electroweak vacuum stability during inflation. The stability of the false Higgs vacuum is determined by the behavior of the homogeneous Higgs field and the inhomogeneous Higgs field from gravitationally induced vacuum fluctuation.

The Higgs field phenomenologically acquires various effective masses during inflation, e.g. the inflaton-Higgs coupling $\lambda_{\phi S}$ provides an extra contribution to the Higgs mass $m_{\rm eff}^2 = \lambda_{\phi S} S^2$ where $S$ is the inflaton field. Let us restrict our attention to the simple case that the Higgs field only couples to the gravity via the non-minimal Higgs-gravity coupling $\xi(\mu)$ and we disregard other inflationary effective mass terms. For convenience, we consider the effective mass $m_{\rm eff}^2 = \xi(\mu) R = 12\xi(\mu) H_{\rm inf}^2$ and use the results of Eq. (4.55), the gravitationally induced vacuum fluctuation is written as

$$\left\langle \delta\phi^2 \right\rangle_{\rm ren} \simeq \begin{cases} \frac{H_{\rm inf}^2}{32\pi^2 \xi(\mu)} & (\xi(\mu) \ll \mathcal{O}(10^{-1})) \\ \frac{H_{\rm inf}^2}{24\pi^2} & (\xi(\mu) \gtrsim \mathcal{O}(10^{-1})) \end{cases} \tag{6.8}$$

where $H_{\rm inf}$ is the Hubble parameter during inflation. From here we infer the sign of $V_{\rm eff}(\phi)$ by using the relations $\xi(\mu) R < |\lambda(\mu) \left\langle \delta\phi^2 \right\rangle_{\rm gravity}|$ where we assume $\mu \simeq \left(12H_{\rm inf}^2 + \left\langle \delta\phi^2 \right\rangle_{\rm gravity}\right)^{1/2} > \Lambda_I$ [1]. If we consider $\xi(\mu) R = \xi(\mu) 12 H_{\rm inf}^2$, $\lambda(\mu) \left\langle \delta\phi^2 \right\rangle_{\rm gravity} \simeq \lambda(\mu) H_{\rm inf}^2 / 32\pi^2 \xi(\mu)$ and $\lambda(\mu) \simeq -0.01$, we obtain the constraint on the non-minimal coupling to be $\xi(\mu) \lesssim \mathcal{O}(10^{-3})$ where $H_{\rm inf} > \sqrt{32\pi^2 \xi(\mu)}\, \Lambda_I$. In this case, the global Higgs field $\phi$ goes out to the negative Planck-energy vacuum state, and therefore, the excursion of the Higgs field $\phi$ to the Planck-scale true vacuum can terminate inflation and cause an immediate collapse of the Universe.

On the other hand, the gravitational Higgs fluctuation $\left\langle \delta\phi^2 \right\rangle_{\rm graivty}$ can directly cause vacuum transitions of the Universe [1, 2, 4, 16, 17, 74–83]. If local Higgs fields overcome the barrier of the potential, the local Higgs fields classically roll down into the Planck-energy true vacuum and catastrophic Anti-de Sitter (AdS) domains are formed. Note that not all AdS domains formed during inflation threaten the existence of the Universe [17, 82], which highly depends on the evolution of the AdS domains at the end of inflation (see Ref. [17] for the details). The AdS domains can either shrink or expand eating other regions of the electroweak vacuum. Although the high-scale inflation can generate more expanding AdS domains than shrinking domains during inflation, such domains never overcome the inflationary expansion of the Universe, i.e.,

[1] In the case $\left(12H_{\rm inf}^2 + \left\langle \delta\phi^2 \right\rangle_{\rm gravity}\right)^{1/2} < \Lambda_I$, the quartic term $\lambda(\mu)\phi^4/4$ becomes positive unless $\phi > \Lambda_I$. Therefore, the homogeneous Higgs field $\phi$ cannot classically go out to the Planck-scale vacuum state. However, it is possible to produce AdS domains or bubbles via the gravitational Higgs fluctuation shown in (5.28).

one AdS domain cannot terminate the inflation of the Universe [2]. However, after inflation, some AdS domains expand and consume the entire Universe. Thus, the existence of AdS domains is problematic and so we focus on the conditions not to be generated during or after inflation. From Eq. (5.28) the stability condition of the Higgs vacuum during inflation is given by

$$\frac{\left\langle \delta\phi^2 \right\rangle_{\text{gravity}}}{\phi_{\text{max}}^2} < \frac{1}{6N_{\text{hor}}}. \tag{6.9}$$

where the gravitationally induced Higgs fluctuation is given by Eq. (6.8) and $\phi_{\text{max}}$ is the maximum value of the effective Higgs potential. The effective Higgs potential with the relatively large effective mass $m_{\text{eff}}$ can be approximately given by

$$V_{\text{eff}}(\phi) \simeq \frac{1}{2} m_{\text{eff}}^2 \phi^2 \left( 1 - \frac{1}{2} \left( \frac{\phi}{\phi_{\text{max}}} \right)^2 \right), \tag{6.10}$$

where $\phi_{\text{max}}$ is estimated to be

$$\phi_{\text{max}} = \sqrt{-m_{\text{eff}}^2/\lambda}. \tag{6.11}$$

In the numerical analysis, then we can approximate the maximal field value to be $\phi_{\text{max}} \simeq 10 \cdot m_{\text{eff}}$. By using Eq. (6.9) and Eq. (6.11) we can obtain the constraint of the non-minimal coupling $\xi(\mu) \lesssim \mathcal{O}\left(10^{-2}\right)$ where $H_{\text{inf}} > \sqrt{32\pi^2 \xi(\mu)}\, \Lambda_I$. Thus, the Anti-de Sitter (AdS) domain or bubble formations from the high-scale inflation can be avoided if the relatively large non-minimal Higgs-gravity coupling is introduced. Here, we summarize the conclusions obtained in this section as follows [1]:

- For $\xi(\mu) \lesssim \mathcal{O}\left(10^{-3}\right)$ and $H_{\text{inf}} > \sqrt{32\pi^2 \xi(\mu)}\, \Lambda_I$, the Higgs effective potential is destabilized by the gravitational Higgs fluctuation during inflation where $\lambda(\mu) \left\langle \delta\phi^2 \right\rangle_{\text{gravity}}$ overcome the stabilization mass term $\xi(\mu)R$. In this case, the effective potential becomes negative as $V'_{\text{eff}}(\phi) \lesssim 0$, the excursion of the Higgs field to the Planck-energy vacuum terminates the inflation and cause a collapse of the Universe.

- For $\mathcal{O}\left(10^{-3}\right) \lesssim \xi(\mu) \lesssim \mathcal{O}\left(10^{-2}\right)$ and $H_{\text{inf}} > \sqrt{32\pi^2 \xi(\mu)}\, \Lambda_I$, The curvature mass $\xi(\mu)R = \xi(\mu) 12 H_{\text{inf}}^2$ can stabilize the Higgs effective potential during inflation. Thus, the global dynamics of the Higgs field can be suppressed. However, the gravitational Higgs fluctuation generates some Anti-de Sitter (AdS) domains or bubbles, which finally cause the vacuum transition of the Universe.

- For $\xi(\mu) \gtrsim \mathcal{O}\left(10^{-2}\right)$ or $H_{\text{inf}} < \sqrt{32\pi^2 \xi(\mu)}\, \Lambda_I$, the Higgs effective potential stabilizes and the Anti-de Sitter (AdS) domains or bubbles are not formed during inflation.

[2] The expansion of AdS domains or bubbles never takes over the expansion of the inflationary dS space [17], and therefore, it is impossible that one AdS domain terminates the inflation of the Universe. However, if the non-inflating domains or the AdS domains dominates all the space of the Universe [148], the inflating space would crack, and the inflation of all the space of the Universe finally comes to an end.

The relatively large non-minimal Higgs-gravity coupling as $\xi(\mu) \gtrsim \mathcal{O}(10^{-2})$ can stabilize the effective Higgs potential and suppress the formations of the AdS domains or bubbles during inflation. Other than that, we can simply avoid this situation by assuming the inflaton-Higgs coupling $\lambda_{\phi S}$ [75] or other scalar field interaction couplings. So far we only consider gravitational backreaction of the Higgs fluctuation during inflation. However, as previously discussed in Eq. (6.42) gravitationally induced vacuum fluctuations of $W/Z$ bosons and the top quark enlarge up to the Hubble scale $H_{\text{inf}}$ and contribute to the Higgs potential as the positive masses

$$\begin{aligned} m_{\text{eff}}^2 \simeq\ & \xi_\phi(\mu) 12 H_{\text{inf}}^2 + \lambda(\mu) \left\langle \delta\phi^2 \right\rangle_{\text{gravity}} + \frac{1}{2} g^2(\mu) \left\langle \delta W^2 \right\rangle \\ & + \frac{1}{2}\left[g^2(\mu) + g'^2(\mu)\right] \left\langle \delta Z^2 \right\rangle_{\text{gravity}} + y_t^2(\mu) \left\langle \delta t^2 \right\rangle_{\text{gravity}}, \end{aligned} \tag{6.12}$$

If the gravitationally induced vacuum fluctuations of $W/Z$ bosons and the top quark are larger than the Higgs one, the effective Higgs potential can be stabilized

$$\left\langle \delta\phi^2 \right\rangle_{\text{gravity}} \lesssim \left\langle \delta W^2 \right\rangle_{\text{gravity}}, \ \left\langle \delta Z^2 \right\rangle_{\text{gravity}}, \ \left\langle \delta t^2 \right\rangle_{\text{gravity}} \Longrightarrow \text{Stabilized} \tag{6.13}$$

Since these gravitational fluctuations of $W/Z$ bosons and the top quark depend on their mass as well as Higgs field, they can be suppressed by their non-minimal coupling. Thus we obtain the constraints of their non-minimal coupling as follows:

$$\xi(\mu) \gtrsim \xi_W(\mu),\ \xi_Z(\mu),\ \xi_t(\mu) \Longrightarrow \text{Stabilized} \tag{6.14}$$

## 6.3 Electroweak Vacuum Stability after Inflation

In this section we discuss the electroweak vacuum stability after inflation. For simplicity let us consider the non-minimal Higgs-gravity coupling case. After inflation, the effective mass $\xi(\mu)R$ via the non-minimal coupling drops rapidly and sometimes become negative. Thus, the effect of the stabilization via $\xi(\mu)R$ disappears and the Higgs effective potential becomes rather unstable due to $\lambda(\mu) \left\langle \delta\phi^2 \right\rangle_{\text{gravity}}$ and $\xi(\mu)R$. Furthermore, the non-minimal Higgs-gravity coupling can generate the large Higgs fluctuation via so-called tachyonic resonance. Thus, the Higgs potential is destabilized, or some Anti-de Sitter (AdS) domains or bubbles are formed during subsequent preheating stage [4, 84–88].

At the end of the inflation, the inflaton field $S$ begins coherently oscillating near the minimum of the inflaton potential $V_{\text{inf}}(S)$ and produces extremely a huge amount of massive bosons via the parametric or tachyonic resonance. This temporal non-thermal stage is called preheating [103], and is essentially different from the subsequent stages of the reheating and the thermalization. For simplicity, we approximate the inflaton potential as the quadratic form

$$V_{\text{inf}}(S) = \frac{1}{2} m_S^2 S^2. \tag{6.15}$$

In this case, the inflaton field $S$ classically oscillates as

$$S(t) = \Phi \sin(m_S t), \quad \Phi = \sqrt{\frac{8}{3}} \frac{M_{\text{pl}}}{m_S t}, \tag{6.16}$$

where the reduced Planck mass is $M_{\rm pl} = 2.4 \times 10^{18}$ GeV. If the inflaton field $S$ dominates the energy density and the pressure of the Universe, i.e., during inflation or preheating stage, the scalar curvature $R(t)$ can be written by

$$R(t) = \frac{1}{M_{\rm pl}^2}\left[4V_{\rm inf}(S) - \dot{S}^2\right], \tag{6.17}$$

$$\simeq \frac{m_S^2 \Phi^2}{M_{\rm pl}^2}\left(3\sin^2(m_S t) - 1\right). \tag{6.18}$$

When the inflaton field $S$ oscillates as Eq. (6.16), the effective mass $\xi(\mu)R$ drastically changes between positive and negative values. Therefore, the Higgs field fluctuations grows extremely rapidly via the tachyonic resonance, which is called geometric preheating [149, 150].

The general equation for $k$ modes of the Higgs field during preheating is given as follows:

$$\frac{d^2\left(a^{3/2}\delta\phi_k\right)}{dt^2} + \left(\frac{k^2}{a^2} + V'_{\rm eff}(\phi) + \frac{1}{M_{\rm pl}^2}\left(\frac{3}{8} - \xi\right)\dot{S} - \frac{1}{M_{\rm pl}^2}\left(\frac{3}{4} - 4\xi\right)V(S)\right)\left(a^{3/2}\delta\phi_k\right) = 0. \tag{6.19}$$

Eq. (6.19) can be reduced to the following Mathieu equation

$$\frac{d^2\left(a^{3/2}\delta\phi_k\right)}{dz^2} + (A_k - 2q\cos 2z)\left(a^{3/2}\delta\phi_k\right) = 0, \tag{6.20}$$

where we take $z = m_S t$ and $A_k$ and $q$ are given as

$$A_k = \frac{k^2}{a^2 m_S^2} + \frac{V'_{\rm eff}(\phi)}{m_S^2} + \frac{\Phi^2}{2M_{\rm pl}^2}\xi, \tag{6.21}$$

$$q = \frac{3\Phi^2}{4M_{\rm pl}^2}\left(\xi - \frac{1}{4}\right). \tag{6.22}$$

The solutions of the Mathieu equation via non-minimal coupling in Eq. (6.20) show tachyonic (broad) resonance when $q \gtrsim 1$ or narrow resonance when $q < 1$. In tachyonic resonance regime where $q \gtrsim 1$, i.e. $\Phi^2\xi \gtrsim M_{\rm pl}^2$, the tachyonic resonance extremely amplifies the Higgs fluctuation immediately right after inflation. During the preheating stage, $A_k$ and $q$ are $z$-dependent function, making Eq. (6.20) difficult to derive analytical estimation. If we take $m_S \simeq 7 \times 10^{-6} M_{\rm pl}^2$ assuming chaotic inflation with a quadratic potential, we can numerically obtain the condition of the tachyonic resonance as $\xi(\mu) \gtrsim \mathcal{O}(10)$ (see Fig. 6.3 for the details). In narrow resonance regime, where $q < 1$, i.e. $\Phi^2\xi < M_{\rm pl}^2$, the tachyonic resonance cannot occur, and therefore, the Higgs field fluctuation decreases due to the expansion of the Universe. Here, we briefly summarize the results of the Higgs vacuum fluctuation after inflation as follows:

$$\begin{cases} \left\langle\delta\phi^2\right\rangle_{\rm gravity} \gg \mathcal{O}\left(H^2(t)\right), & \left(\Phi^2\xi \gtrsim M_{\rm pl}^2\right) \\ \left\langle\delta\phi^2\right\rangle_{\rm gravity} \simeq \mathcal{O}\left(H^2(t)\right). & \left(\Phi^2\xi < M_{\rm pl}^2\right) \end{cases} \tag{6.23}$$

In the same way as the inflation stage, when the tachyonic resonance happens, it is clear that the effective Higgs potential becomes negative $V'_{\rm eff}(\phi) \lesssim 0$ due to the inequality $\xi(\mu)R(t) <$

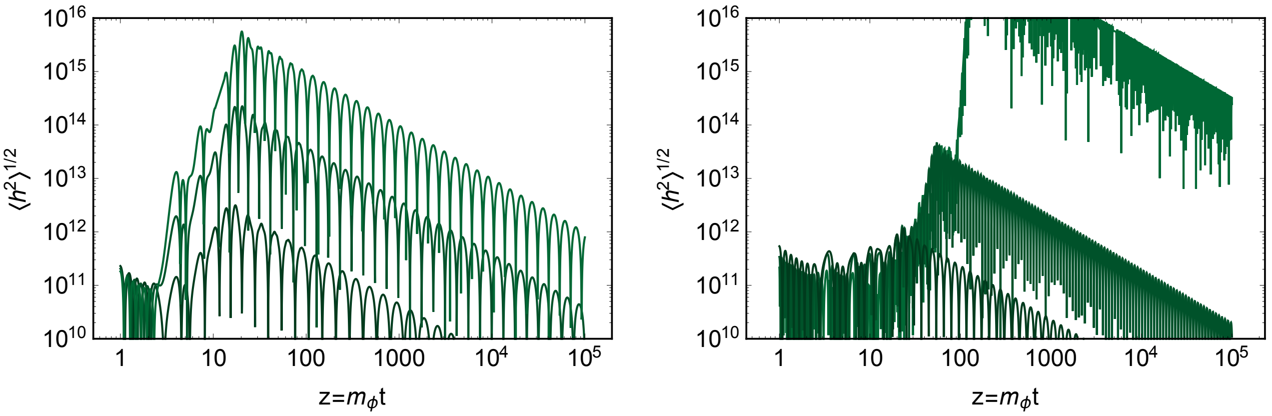


Figure 6.1: Left: Higgs fluctuation in non-minimal gravity-Higgs coupling case. In the lower, middle and upper curves we have used the nonminimal couplings $\xi = 10^{1.4}$, $\xi = 10^{1.6}$ and $\xi = 10^{1.8}$ respectively. Tachyonic resonance occurs strongly for $\xi \gtrsim 10^{1.6}$. Right: Higgs fluctuation in inflaton-Higgs coupling case. In the lower, middle and upper curves we have used the inflaton-Higgs coupling $\lambda_{\phi S} = 10^{-4.4}, 10^{-4}$ and $10^{-3.6}$. Broad resonance occurs strongly for $\lambda_{\phi S} \gtrsim 10^{-4}$. This figure is cited from Ref. [4].

$|\lambda(\mu)\left\langle\delta\phi^2\right\rangle_{\text{gravity}}|$, the Higgs field dynamically goes out to the Planck-energy vacuum state. On the other hand, it cannot happen the same situation in the narrow resonance where $\xi(\mu)R(t) < |\lambda(\mu)\left\langle\delta\phi^2\right\rangle_{\text{gravity}}|$ because the Higgs fluctuation are suppressed $\left\langle\delta\phi^2\right\rangle_{\text{gravity}} \simeq O\left(H^2(t)\right)$ due to the expansion of the Universe, the scalar curvature decreases as $|R(t)| \simeq 3H^2(t)$.

However, the scalar curvature $R(t)$ shown in (6.18) oscillates during each cycle $t \simeq 1/m_S$. The stabilization of $\xi(\mu)R(t)$ generally does not work after inflation, because $\xi(\mu)R(t)$ changes sign during each oscillation cycle. If the oscillation time-scale $t \simeq 1/m_S$ is relatively long, the curvature term $\xi(\mu)R(t)$ can accelerate dynamical motion of the Higgs field $\phi(t)$ immediately at the end of the inflation. Briefly we discuss the development of the coherent Higgs field $\phi(t)$ after inflation. In one oscillation time-scale $t \simeq 1/m_S$, we simply approximate the effective mass as $m_{\text{eff}}^2 \simeq \xi(\mu)R(t) \approx -\xi(\mu)3H_{\text{end}}^2$. By using Eq. (5.5), the dynamical behavior of the Higgs field $\phi(t)$ immediately after infation can be written as

$$\phi(t) \simeq \phi_{\text{end}} \cdot e^{\left(3\xi(\mu)H_{\text{end}}^2\right)t/3H_{\text{end}}}, \tag{6.24}$$

$$\simeq \phi_{\text{end}} \cdot e^{\left(3\xi(\mu)H_{\text{end}}^2/3H_{\text{end}}m_S\right)}, \tag{6.25}$$

$$\simeq \phi_{\text{end}} \cdot e^{\left(\xi(\mu)H_{\text{end}}/m_S\right)}, \tag{6.26}$$

where the local Higgs field $\phi_{\text{end}}$ after inflation is generally not zero, and corresponds to the Higgs field fluctuations $\left\langle\delta\phi^2\right\rangle_{\text{gravity}} \simeq O\left(H_{\text{end}}^2\right)$. Thus, if we assume $\phi(t) > \phi_{\max}$ and $H_{\text{end}}/m_S \gtrsim (\log 10\sqrt{3\xi(\mu)})/\xi(\mu)$ [3], the almost local Higgs fields $\phi(t)$ produced after inflation go out to the negative Planck-vacuum state and cause a collapse of the Universe. Furthermore, the inflation produces an enormous amount of causally disconnected horizon-size domains and our observable

[3] By using $\phi_{\max} \simeq 10m_{\text{eff}}$, we can approximate $\phi_{\max} \simeq 10\sqrt{\xi(\mu)\,|R(t)|} \simeq 10H_{\text{end}}\sqrt{3\xi(\mu)}$

Universe contains $e^{3N_{\rm hor}}$ of them. Thus, we can consider one domain which has the large Higgs field fluctuations $6N_{\rm hor}\left\langle\delta\phi^2\right\rangle_{\rm gravity}$ by using Eq. (5.28). The classical dynamics of the Higgs field on such domains can be given by

$$\phi\left(t\right) \simeq \sqrt{6N_{\rm hor}\left\langle\delta\phi^2\right\rangle_{\rm ren}}\cdot e^{\left(3\xi(\mu)H_{\rm end}^2\right)t/3H_{\rm end}}, \tag{6.27}$$

$$\simeq 10H_{\rm end}\cdot e^{\left(\xi(\mu)H_{\rm end}/m_S\right)}, \tag{6.28}$$

where we take $N_{\rm hor}=60$. Therefore, if we have $\phi\left(t\right)>\phi_{\rm max}$ i.e., $H_{\rm end}/m_S\gtrsim(\log\sqrt{3\xi\left(\mu\right)})/\xi\left(\mu\right)$, the coherent Higgs field $\phi\left(t\right)$ on such domain goes out to the negative Planck-vacuum state and forms the AdS domains, which finally cause the vacuum transition of the Universe. That conclusion depends strongly on the non-minimal coupling $\xi(\mu)$, the oscillation time-scale $t\simeq 1/m_S$ and the Hubble scale $H_{\rm end}$ at the end of the inflation. Thus, the large non-minimal Higgs-gravity coupling $\xi(\mu)$ can destabilize the behavior of the coherent Higgs field after the end of the inflation. However, if the curvature oscillation is very fast, the curvature mass-term $\xi(\mu)R\left(t\right)$ cannot generate the exponential growth of the coherent Higgs field $\phi\left(t\right)$ after inflation. Here, we summarize the conclusions obtained by the above discussion as follows [1]:

- For $H_{\rm end}>\Lambda_I$ and $H_{\rm end}/m_S\gtrsim(\log 10\sqrt{3\xi\left(\mu\right)})/\xi\left(\mu\right)$, the almost local Higgs fields $\phi\left(t\right)$ generated at the end of the inflation exponentially grow and finally go out to the Planck-energy vacuum state, which leads to the collapse of the Universe.

- For $H_{\rm end}>\Lambda_I$ and $H_{\rm end}/m_S\gtrsim(\log\sqrt{3\xi\left(\mu\right)})/\xi\left(\mu\right)$, the coherent Higgs field $\phi\left(t\right)$ on one horizon-size domain exponentially grows at the end of the inflation and forms AdS domains or bubbles, which finally cause the vacuum transition of the Universe.

- In tachyonic resonance regime $\Phi^2\xi\gtrsim M_{\rm pl}^2$, the Higgs fluctuation increases extremely . Thus, the Higgs potential is destabilized as $V'_{\rm eff}\left(\phi\right)\lesssim 0$, and the excursion of the homogeneous Higgs field $\phi\left(t\right)$ to the negative vacuum state occurs during preheating stage and cause the collapse of the Universe.

- In narrow resonance regime $\Phi^2\xi<M_{\rm pl}^2$, the Higgs fluctuations decrease, and therefore, it is improbable to destabilize the effective potential during preheating stage.

The relative large non-minimal Higgs-gravity coupling $\xi(\mu)\gtrsim\mathcal{O}\left(10^{-2}\right)$ can stabilize the effective Higgs potential and suppress the formations of the AdS domains or bubbles during inflation. However, after inflation, the effective mass-term $\xi(\mu)R$ via the non-minimal coupling drops rapidly, sometimes become negative and lead to the exponential growth of the Higgs field $\phi\left(t\right)$ at the end of inflation, or the large Higgs fluctuation via the tachyonic resonance during preheating stage. Therefore, the non-minimally coupling $\xi(\mu)$ can not seem to prevent the catastrophic scenario during or after inflation. After all, if the inflation of our Universe receives large Hubble scale $H>\Lambda_I$, meaning the relatively large tensor-to-scalar ratio $r_T$ from the cosmic microwave background radiation, the safety of our electroweak vacuum is inevitably threatened during inflation or after inflation. However, we can avoid this situation by assuming the inflaton-Higgs couplings $\lambda_{\phi S}$ [75], the inflationary stabilizations [151, 152], or the high-order corrections from GUT or Planck-scale new physics [153–156]. Furthermore, we comment that the gravitationally induced vacuum fluctuation of the SM except the Higgs can stabilize the electroweak vacuum.

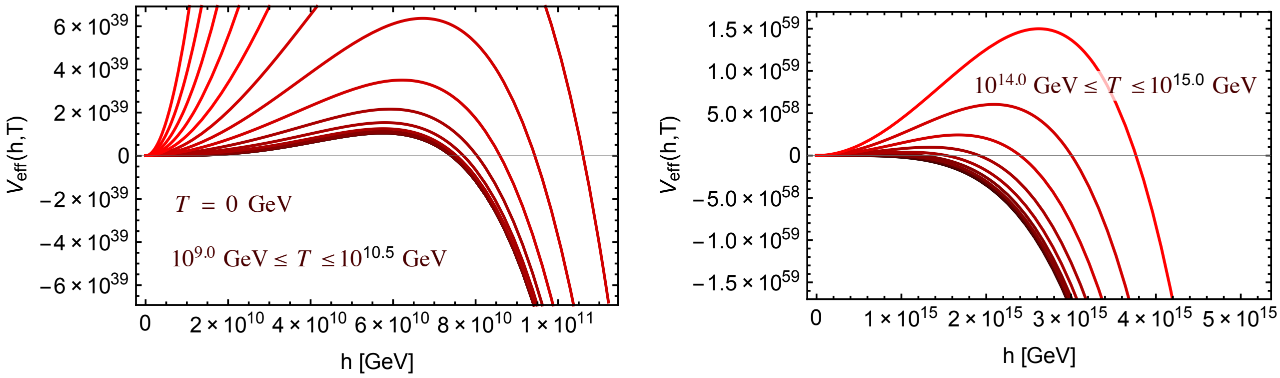


Figure 6.2: Left: SM Higgs effective potential at finite temperature for $T = 0$ GeV and $10^{9.0}$ GeV $\leq T \leq 10^{10.5}$ GeV on the present best-fit values of $M_h$ and $M_t$. Right: SM Higgs effective potential at finite temperature for $10^{14.0}$ GeV $\leq T \leq 10^{15.0}$ GeV. The Higgs field value corresponding to the maximum of the Higgs potential is $\phi_{\rm max} = 2.62\ T$ for $T = 10^{15.0}$ GeV. This figure is cited from Ref. [4].

In the rest of this section we discuss the electroweak vacuum stability during reheating stage. After inflation, the inflaton field $S$ oscillates and produces a huge amount of elementary particles. These particles produced during preheating stage interact with each other and eventually form a thermal plasma. Thermal effects during reheating stage raise the effective Higgs potential via the extra effective mass $m_{\rm eff}^2 = O\left(T^2\right)$.

The one-loop thermal corrections to the Higgs effective potential is given as follows [157,158]:

$$\Delta V_{\rm eff}\left(\phi, T\right) = \sum_{i=W,Z,t} \frac{n_i T^4}{2\pi^2} \int_0^\infty dk k^2 \ln\left(1 \mp e^{-\sqrt{k^2 + M_i^2(\phi)/T^2}}\right) \tag{6.29}$$

$$= \sum_{i=W,Z} n_i J_B\left(M_i\left(\phi\right), T\right) + \sum_{i=t} n_i J_F\left(M_i\left(\phi\right), T\right). \tag{6.30}$$

where we concentrate on the contributions from $W$ and $Z$ bosons and the top quark, and $J_B$ ($J_F$) is the thermal bosonic (fermionic) function. In Fig. 6.3 we plot the SM Higgs effective potential at finite temperature for a range of temperatures. From this figure, we see that although the high-temperature effects raise the effective potential, it cannot be stabilized up to high energy scales unless new physics emerges below the Planck scale. Thus, if the Higgs field get over $\phi_{\rm max}$ during inflation or preheating stage, the generated Higgs field cannot go back to the electroweak vacuum by this high temperature effects.

In high-temperature limit ($T \gg M_i\left(\phi\right)$), the thermal bosonic (fermionic) function $J_B$ ($J_F$) can be approximately written as

$$\begin{aligned} J_B\left(M_i\left(\phi\right), T\right) &\simeq -\frac{\pi^2 T^4}{90} + \frac{M_i^2\left(\phi\right) T^2}{24} - \frac{M_i^3\left(\phi\right) T}{12\pi} - \frac{M_i^4\left(\phi\right)}{64\pi^2} \log\frac{M_i^2\left(\phi\right)}{a_B T^2}, \\ J_F\left(M_i\left(\phi\right), T\right) &\simeq \frac{7}{8}\frac{\pi^2 T^4}{90} - \frac{M_i^2\left(\phi\right) T^2}{48} - \frac{M_i^4\left(\phi\right)}{64\pi^2} \log\frac{M_i^2\left(\phi\right)}{a_F T^2}, \end{aligned} \tag{6.31}$$

where we omit the terms which are independent of $\phi$ and $\log a_B \simeq 5.408$ or $\log a_F \simeq 2.635$. As is well known the one-loop corrections to the Higgs potential in the high-temperature limit $(T \gg M_i(\phi))$ can be approximately written as

$$\Delta V_{\text{eff}}(\phi, T) \simeq \frac{c(T)T^2}{2}\phi^2 + \frac{d(T)T}{3}\phi^3 + \frac{\lambda(T)}{4}\phi^4, \tag{6.32}$$

where

$$c(T) = \frac{3g^2 + g'^2 + 4y_t^2}{16}, \quad d(T) = \frac{6g^3 + 3(g^2 + g'^2)^{3/2}}{32\pi}, \tag{6.33}$$
$$\lambda(T) = \frac{3}{64\pi^2}\left(-\frac{g^4}{2}\log\frac{m_W^2(h)}{a_B T^2} - \frac{\left(g^2 + g'^2\right)^2}{4}\log\frac{m_Z^2(\phi)}{a_B T^2} + 4y_t^4\log\frac{m_t^2(\phi)}{a_F T^2}\right).$$

The thermal fluctuation of the Higgs field can be given as [147, 159–161]

$$\langle\delta\phi^2\rangle_T = \frac{1}{2\pi^2}\int_0^\infty \frac{k^2 dk}{\sqrt{k^2 + m_{\text{eff}}^2}\left[\exp\left(\frac{\sqrt{k^2+m_{\text{eff}}^2}}{T}\right) - 1\right]} \tag{6.34}$$
$$\simeq \frac{T^2}{12} - \frac{m_{\text{eff}}T}{4\pi}, \tag{6.35}$$

where the thermal Higgs mass is $m_{\text{eff}} = c^{1/2}(T)T$ and numerically we obtain $c(T) \simeq 0.2$ using the RGE of the SM. The stability condition of the Higgs vacuum during reheating is given by

$$\frac{\langle\delta\phi^2\rangle_T}{\phi_{\max}^2(T)} < \frac{1}{6N_{\text{hor}}}. \tag{6.36}$$

where $e^{3N_{\text{hor}}}$ corresponds to the physical volume of our universe at the end of the inflation. For simplicity let us rewrite the stability condition as

$$\frac{6N_{\text{hor}}\langle\delta\phi^2\rangle_T}{\phi_{\max}^2(T)} < 1. \tag{6.37}$$

The maximum of the Higgs potential is moved out to larger values when thermal corrections are taken into account, and numerically we have found that $\phi_{\max}$ can be estimated as $\phi_{\max}(T) = 2 \sim 6\ T$. In Fig. 6.3, we show $\phi_{\max}(T)/T$ by using the effective Higgs potential at the high temperature and plot $6N_{\text{hor}}\langle h^2\rangle_T / h_{\max}^2(T)$. If we set $N_{\text{hor}} = 60$, the stability condition of Eq. (6.37) gives us the following upper bound on the temperature $T$ as follows:

$$T < 2.4 \times 10^{10}\text{GeV}. \tag{6.38}$$

It has been thought that the thermal Higgs fluctuation does not destabilize the standard Higgs vacuum because the probability for the thermal decay using the instanton methods is sufficiently small [162–165]. This discrepancy is due to different analysis methods. In general, the analysis of stochastic method and instanton method is approximately consistent in results, but strictly

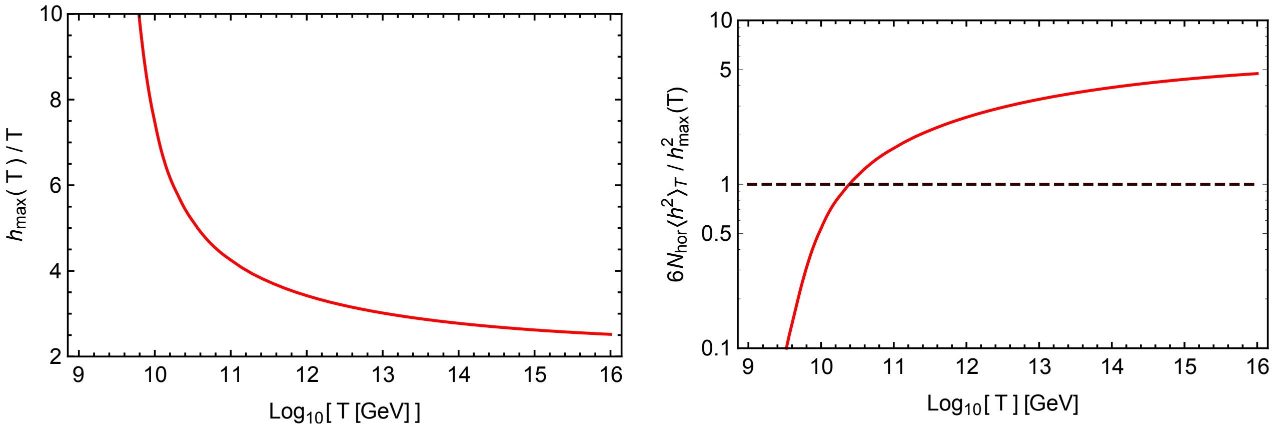


Figure 6.3: Left: We plot of $h_{\text{max}}(T)/T$ by using the Standard Model effective Higgs potential at finite temperature. Right: We plot $6N_{\text{hor}}\langle h^2\rangle_T/h^2_{\text{max}}(T)$ and obtain the constraint of the temperature $T < 2.4\times 10^{10}$GeV. This figure is cited from Ref. [4].

speaking deviated results are obtained. Although, in this thesis, we don't conclude whether the thermal Higgs fluctuation destabilize or not, it is necessary to investigate thoroughly the thermal vacuum metastability during the reheating era. In the rest of this section, we assume that the thermal Higgs fluctuation destabilizes the standard Higgs vacuum and show how the scale of the Hubble parameter $H$ is restricted in this case.

The reheating process finishes approximately when $H \approx \Gamma_{\text{tot}}$ and the reheating temperature can be expressed as

$$T_{\text{reh}} \approx \left(\frac{90}{\pi^3 g_*}\right)^{1/4}\sqrt{M_{\text{pl}}\Gamma_{\text{tot}}}, \tag{6.39}$$

where $g_*$ is the number of relativistic degrees of freedom. It is known that the reheating temperature $T_{\text{reh}}$ is not the maximal temperature, unless the reheating process is instantaneous. After inflation although still sub-dominant, the decay products from the oscillating inflaton field can become thermalized and produce a so-called dilute plasma. Thus the maximal temperature $T_{\text{max}}$ can be estimated by [166–168]

$$T_{\text{max}} \approx \left(\frac{3}{8}\right)^{2/5}\left(\frac{40}{\pi^2}\right)^{1/8}\frac{g_*^{1/8}(T_{\text{reh}})}{g_*^{1/4}(T_{\text{max}})}M_{\text{pl}}^{1/4}H_{\text{end}}^{1/4}T_{\text{reh}}^{1/2}, \tag{6.40}$$

where the reduced Planck mass $M_{\text{pl}} = 2.4\times 10^{18}$ GeV and $g_*(T)$ is the number of relativistic degrees of freedom at the temperature $T$. By using constraints (6.38) and assuming $T_{\text{reh}} < T_{\text{max}}$, we can obtain the upper bound on the Hubble scale $H$ as a function of $T_{\text{reh}}$.

## 6.4 Electroweak Vacuum Stability around Evaporating Black Hole

The black holes emit thermal radiation at the Hawking temperature $T_{\rm H} = 1/8\pi G_N M_{\rm BH}$ [106] due to the quantum effects on strong gravitational field. The quantum gravitational effects around the black hole determines the fate of the evaporating black hole which is still unknown and closely related with the information loss puzzle [169], and furthermore, leads to the spontaneous symmetry restoration [170] or the false vacuum decay around the black hole [171–173], which bring cosmological singular possibility. Especially recent discussion about the electroweak vacuum stability around the evaporating black hole has been growing. There are no general mechanisms to prevent the formation of such small black holes which finally evaporate during the history of the Universe. Furthermore, the primordial black holes (PBH) are formulated by primordial density fluctuations [174–176] which has strong impacts of the Higgs stability [92, 177]. However, there is some controversy about whether one evaporating black hole can be a trigger of the false vacuum decay on the Higgs vacuum. It is because the backreaction of the thermal Hawking radiation can not be ignored in the Higgs stability problem [90–92, 178].

In the literature [18–20], the false vacuum decay in the Schwarzschild black hole has been investigated by the Coleman-De Luccia (CDL) formalism [143] with the zero-temperature effective Higgs potential. However, it is reasonable intuitively to assume the high-temperature effective Higgs potential [157, 158, 163, 165, 179] instead of the zero-temperature potential in the environment of the thermal Hawking flux. Generally, the thermal corrections stabilize the effective Higgs potential, and furthermore, the false vacuum decay from strong gravitational field can only happen around the black hole horizon. That is approximately we can obtain the vacuum decay ratio of the Minkowski spacetime only far from the black hole, which is extremely small. From this viewpoint, the probability of vacuum decay around the black hole can be expected to be lower than what was considered in the literature [18–20]. In this section, we discuss the electroweak vacuum stability around evaporating block hole by using stochastic approach to the Schwarzschild background which was also discussed in literature [180].

As previously discussed the gravitationally induced vacuum fluctuation in the Unruh vacuum $|0_{\rm U}\rangle$ which is appropriate vacuum state around evaporating black hole, approximately approach the thermal Hawking temperature $T_{\rm H}$ near the horizon. But at the infinity the gravitational induced fluctuation attenuates rapidly and approaches zero. We summarize the gravitationally induced vacuum fluctuation $\left\langle \delta\phi^2 \right\rangle_{\rm gravity}$ of the Higgs field around the evaporating black hole as follows:

$$\left\langle \delta\phi^2 \right\rangle_{\rm gravity} \simeq \begin{cases} \mathcal{O}\left(T_{\rm H}^2\right) & (r \to 2M_{\rm BH}) \\ 0 & (r \to \infty) \end{cases} . \tag{6.41}$$

As previously discussed in Chapter 4 the gravitationally induced vacuum fluctuation of the Higgs, $W$ and $Z$ bosons and the top quark approximately approach the Hawking thermal fluctuations near the black hole horizon

$$\left\langle \delta\phi^2 \right\rangle_{\rm gravity} \approx \left\langle \delta W^2 \right\rangle_{\rm graivty} \approx \left\langle \delta Z^2 \right\rangle_{\rm gravity} \approx \left\langle \delta t^2 \right\rangle_{\rm gravity} \approx \mathcal{O}\left(T_{\rm H}^2\right).$$

The gravitationally induced Higgs fluctuation works to push down the Higgs potential due to the negative running Higgs self-coupling, whereas the induced fluctuation of the gauge bosons and fermions raise the effective Higgs potential as follows:

$$V_{\text{eff}}(\phi) = \rho_\Lambda(\mu) + \frac{1}{2}m_\phi^2(\mu)\phi^2 + \frac{1}{2}\xi_\phi(\mu)R\phi^2 + \frac{\lambda_\phi(\mu)}{4}\phi^4 + \frac{3\lambda(\mu)}{2}\left\langle\delta\phi^2\right\rangle_{\text{gravity}}\phi^2$$
$$+ \frac{g^2(\mu)}{8}\left\langle\delta W^2\right\rangle_{\text{gravity}}\phi^2 + \frac{\left[g^2(\mu)+g'^2(\mu)\right]}{8}\left\langle\delta Z^2\right\rangle_{\text{gravity}}\phi^2 + \frac{y_t^2(\mu)}{4}\left\langle\delta t^2\right\rangle_{\text{gravity}}\phi^2 +$$
$$+ \sum_{i=W,Z,t,G,H}\frac{n_i}{64\pi^2}M_i^4(\phi)\left[\log\frac{M_i^2(\phi)}{\mu^2} - C_i\right], \tag{6.42}$$

It is clear that the Higgs potential around evaporating black hole reproduce the thermal one, and therefore the Higgs potential can be stabilized even around evaporating block hole. Thus, the maximal field value $\phi_{\text{max}}$ can be estimated as follows

$$\phi_{\text{max}}^2 \simeq \mathcal{O}\left(10^2\right)\cdot\left(\frac{g^2\left\langle\delta W^2\right\rangle_{\text{gravity}}}{4} + \frac{\left[g^2+g'^2\right]\left\langle\delta Z^2\right\rangle_{\text{gravity}}}{4} + \frac{y_t^2\left\langle\delta t^2\right\rangle_{\text{gravity}}}{2} + \frac{\lambda_{\text{eff}}\left\langle\delta\phi^2\right\rangle_{\text{gravity}}}{2}\right)$$

By using Eq. (5.25), we derive the constraint of the number of the evaporating (primordial) black holes not to cause the Higgs vacuum collapse as follows:

$$\mathcal{V}\cdot P\left(\phi > \phi_{\text{max}}\right) \simeq \frac{\mathcal{N}_{\text{PBH}}\sqrt{2\left\langle\delta\phi^2\right\rangle_{\text{gravity}}}}{\pi\phi_{\text{max}}}\exp\left(-\frac{\phi_{\text{max}}^2}{2\left\langle\delta\phi^2\right\rangle_{\text{gravity}}}\right)$$
$$\approx \mathcal{N}_{\text{PBH}}\cdot e^{-\mathcal{O}(100)} \lesssim 1, \tag{6.43}$$

Thus, we obtain the constraint on the number of the primordial black holes as $\mathcal{N}_{\text{PBH}} \lesssim \mathcal{O}\left(10^{43}\right)$ which is extremely huge in order to threaten the Higgs metastable vacuum. Thus, one evaporating black hole can not cause serious problems in vacuum stability of the standard model Higgs case.

The total number of the evaporating black hole (or the PBHs) strongly depends on the cosmological models at the early Universe, and therefore, let us consider the upper bound on the yield of the PBHs $Y_{\text{PBH}}/ \equiv n_{\text{PBH}}/s$ as follows.

$$Y_{\text{PBH}} = \frac{n_{\text{PBH}}}{s} = \frac{\mathcal{N}_{\text{PBH}}}{s_0/H_0^3} \lesssim \mathcal{O}\left(10^{-43}\right), \tag{6.44}$$

where $s_0$ denotes the entropy density at present ($\approx \left(3\times10^{-4}\text{ eV}\right)^3$), and $H_0$ the current Hubble constant ($\approx 10^{-33}$ eV). $Y_{\text{PBH}}$ is constant from the formation time to the evaporation time. Let us transform this bound into an upper bound on $\beta$, which is defined by taking values at the formation of the PBH to be

$$\beta \equiv \left.\frac{\rho_{\text{PBH}}}{\rho_{\text{tot}}}\right|_{\text{formation}}, \tag{6.45}$$

where $\rho_{\text{PBH}}$ and $\rho_{\text{tot}}$ are the energy density of the PBHs and the total energy density of the Universe including the PBHs at the formation, respectively. It is remarkable that $\beta$ means the number of the PBHs per the horizon volume at the formation ($\beta \sim n_{\rm PBH}/H^3$). Then we have a relation,

$$\beta \sim 10^{30} \frac{n_{\rm PBH}}{s} \left( \frac{m_{\rm PBH}}{10^{15}{\rm g}} \right)^{3/2}. \tag{6.46}$$

Combining this relation with (6.44), we obtain

$$\begin{aligned} \beta &\lesssim \mathcal{O}\left(10^{-12}\right) \left( \frac{m_{\rm PBH}}{10^{15}{\rm g}} \right)^{3/2}, \\ &\lesssim \mathcal{O}\left(10^{-21}\right) \left( \frac{m_{\rm PBH}}{10^{9}{\rm g}} \right)^{3/2}. \end{aligned} \tag{6.47}$$

This bound can be stronger than the known one for $m_{\rm PBH} \lesssim 10^9$g [181].

But, at the final stage of the evaporation of the black hole, the black-hole mass $M_{BH}$ becomes extremely small and the Hawking temperature $T_{\rm H}$ approaches to the Planck scale: $M_{\rm Pl} \sim 10^{19}$ GeV. Thus, the UV corrections of the beyond Standard Model (BSM) and the QG can not be ignored at the last stage of the evaporation and undoubtedly contribute to the Higgs vacuum stability. In the rest of this section, we discuss how the Planck scale physics affect the electroweak vacuum stability around evaporating black hole. If the Hawking temperature approaches to the Planck scale: $T_{\rm H} \to \mathcal{O}\left(M_{\rm Pl}\right)$, the Planck scale corrections determines the stability of the vacuum. Therefore, if the Planck scale physics destabilize the Higgs potential, the electroweak vacuum decay can occur by even a single evaporating black hole.

Now, let us consider the effective Higgs potential with the Planck scale corrections. For convenience, we add two higher dimension operators $\phi^6$ and $\phi^8$ to the effective Higgs potential as follows:

$$V_{\rm eff}\left(\phi\right) = \frac{\lambda_{\rm eff}(\phi)}{4}\phi^4 + \frac{\delta\lambda_{\rm bsm}}{4}\phi^4 + \frac{\lambda_6}{6}\frac{\phi^6}{M_{\rm Pl}^2} + \frac{\lambda_8}{8}\frac{\phi^8}{M_{\rm Pl}^4} + \cdots, \tag{6.48}$$

where $\delta\lambda_{\rm bsm}$ express the running corrections, $\lambda_6$ and $\lambda_8$ dimensionless coupling constants. The higher-dimension corrections of $\lambda_6$ and $\lambda_8$ are usually negligible although these corrections can affect the false vacuum decay via the quantum tunneling.

However, at the final stage of the evaporation of the black hole where $T_{\rm H} \to \mathcal{O}\left(M_{\rm Pl}\right)$, these higher-dimension contributions of $\lambda_6$ and $\lambda_8$ can not be neglected and have a strong impact on the Higgs vacuum stability around the evaporating black hole. As previously discussed, the Higgs potential is modified by the thermal Hawking fluctuation around the black hole:

$$\begin{aligned} V_{\rm eff}\left(\phi\right) = &\frac{1}{2}\left( \lambda_{\rm eff}T_{\rm H}^2 + \kappa^2 T_{\rm H}^2 + \frac{\lambda_6 T_{\rm H}^4}{M_{\rm Pl}^2} + \frac{\lambda_8 T_{\rm H}^6}{M_{\rm Pl}^4} + \cdots \right)\phi^2 \\ &+\frac{1}{4}\left( \lambda_{\rm eff} + \delta\lambda_{\rm bsm} + \frac{\lambda_6 T_{\rm H}^2}{M_{\rm Pl}^2} + \frac{\lambda_8 T_{\rm H}^4}{M_{\rm Pl}^4} + \cdots \right)\phi^4 + \cdots, \end{aligned} \tag{6.49}$$

where the Planck scale quantum corrections govern the effective Higgs potential at $T_{\rm H} \to \mathcal{O}\left(M_{\rm Pl}\right)$. If the Higgs potential is destabilized by these corrections via $\left\langle \delta\phi^2 \right\rangle_{\rm gravity} \simeq \mathcal{O}\left(T_{\rm H}^2\right)$ and the Higgs potential becomes negative to be $\partial V_{\text{eff}}(\phi)/\partial\phi < 0$, the local Higgs fields around the evaporating black hole classically roll down into true vacuum. The Higgs Anti-de Sitter (AdS) domains or bubbles whose sizes are about the black-hole horizon are formed. Not all Higgs AdS domains threaten the existence of the Universe, which highly depends on their evolutions. However, some Higgs AdS domains generally expand eating other regions of the electroweak vacuum and finally consume the entire Universe. Thus, even a single evaporating black hole can be catastrophic for the stability of the Higgs vacuum via the extremely high Hawking temperature $T_{\text{H}} \to \mathcal{O}(M_{\text{Pl}})$ although this possibility strongly depends on the BSM or the Planck scale physics and the detail of the evaporation of the black hole.

# Chapter 7

# Conclusions and Discussion

In this thesis, we have considered the Higgs vacuum stability in various gravitational background or cosmological situations as during inflation corresponding to the de-Sitter spacetime, after inflation in particular the preheating stage and around evaporating black hole. Several discussions or considerations about the Higgs vacuum stability in the gravitational background have already been investigated in the literature, but here we focus on and clarify how gravitationally induced vacuum fluctuation affects the electroweak vacuum stability. This thesis based on my works [1–5] provided a comprehensive description of the gravitationally induced Higgs vacuum decay and reached new conclusion.

The curved spacetime generates the gravitationally induced vacuum fluctuation of the Higgs field which triggers a collapse of the false electroweak vacuum. However, in the past research the influence of the gravitational Higgs fluctuation was not clearly or directly discussed. We have demonstrated that the gravitational effects of the Higgs fluctuation on the vacuum stability are twofold. The gravitationally induced Higgs fluctuation can destabilize the effective Higgs potential as backreaction effects. On the other side, the Higgs fluctuation which overcomes the effective potential can create true vacuum bubbles or domains and triggers off a collapse of the Higgs vacuum. Whether the Higgs vacuum in various gravitational backgrounds or cosmological situations becomes stable or not can be determined by these twofold effects.

Furthermore we have developed a novel method of the gravitationally induced false vacuum decay. The false vacuum decay in curved spacetime is usually studied by the Coleman-de Luccia (CdL) formalism which corresponds to true vacuum bubble nucleation on false vacuum via the gravity field, but the Euclidean solution has ambiguity and still some discussions about gravitationally induced decay. Comparing with this instanton method, the stochastic approach represented by the Langevin equation and the Fokker-Planck equation is a more intuitive formalism dealing with the gravitationally induced false vacuum decay. We developed improved this formalism by using the probability density function $P(\phi)$ and the two-point correlation function $\left\langle \delta\phi^2 \right\rangle$ which express the quantum vacuum fluctuation in the standard QFT [1,2].

The quantum vacuum fluctuation formally described by $\left\langle \delta\phi^2 \right\rangle$ has some UV divergences and therefore some regularizations or renormalizations must be required. In curved spacetime it is not so simple to treat these renormalization issues and consider the gravitationally induced vacuum fluctuation which corresponds to the gravitational particle creations. In this thesis we have adopted some techniques of the QFT in curved spacetime like the adiabatic (WKB) approximation, the adiabatic or point-splitting regularization methods and investigated the gravitationally induced vacuum fluctuation of the Higgs field in de-Sitter spacetime or Schwarzschild spacetime. We have derived the standard effective potential in curved spacetime by using these techniques. But we have noticed that the gravitationally induced vacuum fluctuation modifies the standard effective potential [1,2] and the standard effective potential [1,2] and the background spacetime itself as the gravitational backreaction [3]. Based on the above formulation, we have investigated the Higgs vacuum stability in various background spacetimes or cosmological situations.

The de-Sitter expansion of the Universe during inflation can enlarge the large vacuum fluctuation of the Higgs field up to the Hubble scale $H_{\text{inf}}$ when the Higgs field can be effectively regarded as the massless scalar field. If the inflationary fluctuation of the Higgs field overcomes the barrier of the Higgs effective potential, it triggers off a vacuum collapse of the early Universe. The relative large non-minimal curvature coupling $\xi(\mu)$ avoids the serious scenarios. We have shown the inflationary Higgs vacuum stability with the non-minimal curvature coupling $\xi(\mu)$ as follows [1]:

- For $\xi(\mu) \lesssim O(10^{-3})$ and $H_{\text{inf}} \gg \Lambda_I$, the Higgs effective potential during inflation is destabilized and the potential barrier disappears. The dynamical excursion of the global Higgs field to the negative Planck-vacuum state terminates the inflation of the Universe.

- For $O(10^{-3}) \lesssim \xi(\mu) \lesssim O(10^{-2})$ and $H_{\text{inf}} \gg \Lambda_I$, The curvature mass $\xi(\mu)R = 12\xi(\mu)H_{\text{inf}}^2$ stabilizes the Higgs potential during inflation. But the gravitationally induced Higgs fluctuation generates some Anti-de Sitter (AdS) domains or bubbles, and therefore the vacuum transition of the Universe would finally occur after inflation.

- For $\xi(\mu) \gtrsim O(10^{-2})$ or $H_{\text{inf}} < \Lambda_I$, the Higgs effective potential stabilizes and any Anti-de Sitter (AdS) domains or bubbles would not be formed during inflation.

The relative large non-minimal coupling $\xi(\mu) \gtrsim O(10^{-2})$ can stabilize the Higgs effective potential and suppress the formations of the AdS domains during inflation. Furthermore the gravitationally induced vacuum fluctuation of other SM fields could also stabilize the Higgs vacuum as the de-Sitter thermalization. On the other hand, after inflation $\xi(\mu)R$ drops rapidly, sometimes become negative and lead to the large vacuum fluctuations of the Higgs field via the tachyonic resonance during preheating stage [4]. But this situation depends on the inflation models and the couplings. The thermal Higgs fluctuation at the reheating stage also triggers a false vacuum decay, but the effects can be somewhat relaxed by the thermal corrections to the Higgs potential. The evaporating black hole which emits thermal Hawking radiation also raise a same serious problem about the Higgs vacuum stability. Especially the primordial black holes (PBH) are formulated by primordial density fluctuations which has strong impacts of the Higgs stability and give new cosmological constrains [5].

# Appendix A

# Renormalization Group Equation in Standard Model

In this Appendix, we summarize the renormalization group equation in the SM. The $\beta$ functions for a generic coupling parameter X are defined through the relation

$$\frac{dX\left(t\right)}{dt}=\sum_{i}\beta_{X}^{(i)}. \tag{A.1}$$

The $\beta$ functions and anomalous dimension $\gamma$ at one-loop order are given as follows

$$\begin{aligned}
\beta_{\lambda}^{(1)} &= \frac{1}{\left(4\pi\right)^2}\left[\lambda\left(-9g^2-3g'^2+12y_t^2\right)+24\lambda^2+\frac{3}{4}g^4+\frac{3}{8}\left(g^2+g'^2\right)^2-6y_t^4\right], \\
\beta_{y_t}^{(1)} &= \frac{1}{\left(4\pi\right)^2}\left[\frac{9}{2}y_t^3+y_t\left(-\frac{9}{4}g^2-\frac{17}{12}g'^2-8g_s^2\right)\right], \\
\beta_{g}^{(1)} &= \frac{1}{\left(4\pi\right)^2}\left[-\frac{19}{6}g^3\right], \quad \beta_{g'}^{(1)} = \frac{1}{\left(4\pi\right)^2}\left[\frac{41}{6}g'^3\right], \quad \beta_{g_s}^{(1)} = \frac{1}{\left(4\pi\right)^2}\left[-7g_s^3\right], \\
\gamma^{(1)} &= \frac{1}{\left(4\pi\right)^2}\left[3y_t^2-\frac{9g^2}{4}-\frac{3g'^2}{4}\right]
\end{aligned} \tag{A.2}$$

The $\beta$ functions and anomalous dimension $\gamma$ at two-loop order are given as follows

$$\begin{aligned}
\beta_\lambda^{(2)} &= \frac{1}{(4\pi)^4}\Bigg[-312\lambda^3 - 144\lambda^2 y_t^2 + 36\lambda^2\left(3g^2 + g'^2\right) - 3\lambda y_t^4 + \lambda y_t^2\left(\frac{45}{2}g^2 + \frac{85}{6}g'^2 + 80g_s^2\right) \\
&- \frac{73}{8}\lambda g^4 + \frac{39}{4}\lambda g^2 g'^2 + \frac{629}{24}\lambda g'^4 + 30y_t^6 - 32y_t^4 g_s^2 - \frac{9}{4}y_t^2 g^4 - \frac{8}{3}y_t^4 g'^2 \\
&+ \frac{21}{2}y_t^2 g^2 g'^2 - \frac{19}{4}y_t^2 g'^4 + \frac{305}{16}g^6 - \frac{289}{48}g^4 g'^2 - \frac{559}{48}g^2 g'^4 - \frac{379}{48}g'^6\Bigg], \\
\beta_{y_t}^{(2)} &= \frac{1}{(4\pi)^4}\Bigg[y_t\Bigg(-12y_t^4 + y_t^2\left(\frac{225}{16}g^2 + \frac{131}{16}g'^2 + 36g_s^2 - 12\lambda\right) + \frac{1187}{216}g'^4 \\
&- \frac{3}{4}g^2 g'^2 + \frac{19}{9}g'^2 g_s^2 - \frac{23}{4}g^4 + 9g^2 g_s^2 - 108g_s^4 + 6\lambda^2\Bigg)\Bigg], \\
\beta_g^{(2)} &= \frac{1}{(4\pi)^4}\left[g^3\left(\frac{35}{6}g^2 + \frac{3}{2}g'^2 + 12g_s^2 - \frac{3}{2}y_t^2\right)\right], \\
\beta_{g'}^{(2)} &= \frac{1}{(4\pi)^4}\left[g'^3\left(\frac{9}{2}g^2 + \frac{199}{18}g'^2 + \frac{44}{3}g_s^2 - \frac{17}{6}y_t^2\right)\right], \\
\beta_{g_s}^{(2)} &= \frac{1}{(4\pi)^4}\left[{g_s}^3\left(\frac{9}{2}g^2 + \frac{11}{6}g'^2 - 26g_s^2 - 2y_t^2\right)\right] \\
\gamma^{(2)} &= \frac{1}{(4\pi)^4}\left[6\lambda^2 - \frac{27}{4}y_t^4 + \frac{5}{2}\left(\frac{9}{4}g^2 + \frac{17}{12}g'^2 + 8g_s^2\right)y_t^2 - \frac{271}{32}g^4 + \frac{9}{16}g^2 g'^2 + \frac{431}{96}g'^4\right].
\end{aligned} \tag{A.3}$$

# Appendix B

# Geometrical tensors in FLRW metric

In the FLRW metric, the Ricci tensor and the Ricci scalar are given as follows [107]

$$
\begin{aligned}
R_{00} &= \frac{3}{2}D', \quad R_{11} = -\frac{1}{2}\left(D' + D^2\right), \quad R = \frac{3}{C}\left(D' + \frac{1}{2}D^2\right), \\
G_{00} &= -\frac{3}{4}D^2, \quad G_{ii} = D' + \frac{1}{4}D^2, \\
H^{(1)}_{00} &= \frac{9}{C}\left(\frac{1}{2}D'^2 - D''D + \frac{3}{8}D^4\right), \\
H^{(1)}_{ii} &= \frac{3}{C}\left(2D''' - D''D + \frac{1}{2}D'^2 - 3D'D^2 + \frac{3}{8}D^4\right).
\end{aligned}
\tag{B.1}
$$

# Appendix C

# Adiabatic (WKB) Approximation Method

In this Appendix we review the adiabatic (WKB) approximation method following the literature [108]. To obtain the gravitationally induced vacuum fluctuation we must solve the following equations with the initial conditions:

$$n_k' = \frac{\Omega_k'}{\Omega_k}\mathrm{Re}z_k, \quad z_k' = \frac{\Omega_k'}{\Omega_k}\left(n_k + \frac{1}{2}\right) - 2i\Omega_k z_k. \tag{C.1}$$

For simplicity we assume $z_k = u_k + iv_k$, i.e $u_k = \mathrm{Re}z_k$ and $v_k = \mathrm{Im}z_k$. By using these relations we rewrite Eq. (C.1) as the following

$$n_k' = \frac{\Omega_k'}{\Omega_k}u_k, \tag{C.2}$$

$$u_k' = \frac{\Omega_k'}{\Omega_k}\left(n_k + \frac{1}{2}\right) + 2\Omega_k v_k, \tag{C.3}$$

$$v_k' = -2\Omega_k u_k. \tag{C.4}$$

Here, we introduce a single formal adiabatic parameter $T$ and a rescaling time variable $\tau \equiv \eta/T$. The adiabatic (WKB) condition is restated by

$$\frac{d}{d\eta}\Omega\left(\eta/T\right) = \frac{1}{T}\frac{d}{d\tau}\Omega\left(\tau\right), \tag{C.5}$$

where $T \to \infty$. By using this procedure we rewrite Eq. (C.2), Eq. (C.3) and Eq. (C.4) as follows:

$$\frac{1}{T}n_k' = \frac{1}{T}\frac{\Omega_k'}{\Omega_k}u_k, \tag{C.6}$$

$$\frac{1}{T}u_k' = \frac{1}{T}\frac{\Omega_k'}{\Omega_k}\left(n_k + \frac{1}{2}\right) + 2\Omega_k v_k, \tag{C.7}$$

$$\frac{1}{T}v_k' = -2\Omega_k u_k. \tag{C.8}$$

Next we expand $n_k$, $u_k$ and $v_k$ in inverse powers of $T$ as

$$n_k = n_k^{(0)} + \frac{1}{T} n_k^{(1)} + \frac{1}{T^2} n_k^{(2)} + \cdots , \tag{C.9}$$

$$u_k = u_k^{(0)} + \frac{1}{T} u_k^{(1)} + \frac{1}{T^2} u_k^{(2)} + \cdots , \tag{C.10}$$

$$v_k = v_k^{(0)} + \frac{1}{T} v_k^{(1)} + \frac{1}{T^2} v_k^{(2)} + \cdots , \tag{C.11}$$

where superscripts $(i)$ express the adiabatic order and the zeroth order expressions are given by

$$n_k^{(0)} = \text{const}, \quad u_k^{(0)} = 0, \quad v_k^{(0)} = 0, \tag{C.12}$$

where we solved Eq. (C.6), Eq. (C.7) and Eq. (C.8) with an iterative procedure. The above integration constant is determined by the initial conditions for $n_k(\eta_0)$, and $z_k(\eta_0)$ which corresponds to the choice of the initial vacuum state. For $n_k(\eta_0) = z_k(\eta_0) = 0$, the zeroth-order adiabatic number density $n_k^{(0)}$ is zero. For the first adiabatic order, we can obtain the following expression

$$n_k^{(1)} = 0, \quad u_k^{(1)} = 0, \quad v_k^{(1)} = -\frac{1}{2}\frac{\Omega'_k}{\Omega_k^2}\left(n_k^{(0)} + \frac{1}{2}\right), \tag{C.13}$$

where the odd-order adiabatic number density is zero. Next we write the second order adiabatic expressions as follows

$$n_k^{(2)} = \frac{1}{16}\frac{\Omega'^2_k}{\Omega_k^4}, \quad u_k^{(2)} = \frac{1}{8}\frac{\Omega''_k}{\Omega_k^3} - \frac{1}{4}\frac{\Omega'^2_k}{\Omega_k^4}, \quad v_k^{(2)} = 0, \tag{C.14}$$

In the same way, the third order adiabatic expressions are given by

$$n_k^{(3)} = 0, \quad u_k^{(3)} = 0, \tag{C.15}$$

$$v_k^{(3)} = \frac{1}{16\Omega_k^4}\left(\Omega'''_k - 7\frac{\Omega'_k\Omega''_k}{\Omega_k} + \frac{15}{2}\frac{\Omega'^3_k}{\Omega_k^2}\right), \tag{C.16}$$

Finally, the forth order adiabatic expressions are given by

$$n_k^{(4)} = -\frac{\Omega'_k\Omega'''_k}{32\Omega_k^6} + \frac{\Omega''^2_k}{64\Omega_k^6} + \frac{5\Omega'^2_k\Omega''_k}{32\Omega_k^7} - \frac{45\Omega'^4_k}{256\Omega_k^8}, \tag{C.17}$$

$$u_k^{(4)} = -\frac{\Omega''''_k}{32\Omega_k^5} + \frac{11\Omega'_k\Omega'''_k}{32\Omega_k^6} - \frac{115\Omega'^2_k\Omega''_k}{64\Omega_k^7} + \frac{7\Omega''^2_k}{32\Omega_k^6} + \frac{45\Omega'^4_k}{32\Omega_k^8}, \tag{C.18}$$

$$v_k^{(4)} = 0 \tag{C.19}$$

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