Dynamics of Revolving D-branes

Hikaru Ohta

The Graduate University for Advanced Studies, SOKENDAI,

Tsukuba, Ibaraki 305-0801, Japan

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Abstract

In this thesis, the dynamics of revolving D-branes are discussed. In particular, the effective potential for them is calculated. Two kinds of methods are employed in calculating it. One of them is the improved perturbation method. Using the method, we can remove secular terms systematically to all orders of perturbation in the general perturbative calculation. The other is the partial modular transformation, which is to perform the modular transformation in a part of the moduli parameter. Thanks to this method, we can approximate the potential with the sum of contributions from both the open and closed massless modes. We discuss the possibility of the bound state consisting of revolving D-branes according to these calculations.

Contents

Chapter 1

Introduction

As widely known, superstring theory is a promising candidate for quantum gravity. In the history of the theory, there are some revolutions. Even among them, the discovery of the D-brane is remarkably significant [13], [14]. D-brane is the key subject to understanding the non-perturbative nature of string theory along with the various dualities. Since it was found in 1995 [1], a variety of works that investigate or make use of the D-branes have been published [12,17,18,25]. The important feature of D-brane is that open strings can have the end on it. The low energy effective theory of these open strings gives the field theory on the Dp-brane world volume. Considering the various configurations of D-branes, many attempts that give the geometrical understanding of supersymmetric gauge theories or realize our four-dimensional universe on it have been done [2–7, 16, 21]. When we consider multiple D-branes, their multiplicity determines the gauge symmetry and their location or trajectory gives the vacuum expectation value of scalars on the worldvolume theory. The supersymmetry on the worldvolume also depends on their configuration. From this perspective, we can discuss the gauge symmetry or supersymmetry and their breaking uniformly, in a geometrical way. Furthermore, since D-brane is a dynamical object, it is possible to construct the models such that the configuration is allowed by the string theory rather than set up by hand.

In spite of the elaborate investigations over the last 30 years, we hardly know the dynamics of D-brane yet. For simple configurations, like a static parallel case, tilted case or moving with a constant velocity case, we know how to calculate the effective potential which arises between D-branes [13], [14]. It is given as the one-loop effective potential of open strings stretched between D-branes or the amplitude for the exchange of massless fields between them. For the D-branes with acceleration, however, it had not known how to calculate the effective potential. Many of the models so far are based on the static D-branes that preserve the supersymmetry. For the static D-branes, the system which violates the supersymmetry becomes unstable unless performing special manipulations like an orientifolding. Considering the moving D-branes, we can innovate a new mechanism to stabilize D-brane systems and afford the possibility that spreads the directions to describe the open questions of our four-dimensional universe, like the hierarchy problem, the cosmological constant problem and so on.

In this thesis, as a first step to fully understand the D-brane dynamics and describe the four-dimensional theories on which, we concentrate on a pair of D-branes that revolving around each other in a flat spacetime. For static parallel D-branes, which is a BPS state, there is no potential because of the supersymmetry. By rotating them, the supersymmetry is broken and an attractive force arises. Naively, we can guess that these revolving D-branes compose a bound state due to balancing the attractive force that arises from strings stretched between them and the centrifugal force. If the distance between both of D-branes, which gives the vacuum expectation value of a scalar field of the D-brane worldvolume theory, is sufficiently smaller than the Planck scale or the string scale, it follows that this model describes the origin of the electroweak scale from the string theory perspective. Therefore, it is required to calculate the effective potential for these revolving D-branes for investigating whether a bound state exists. As I mentioned above, however, the method to calculate the D-brane effective potential is not sufficiently known yet. The main obstacle to calculate it is the difficulty of quantization of open strings between D-branes for general configurations. For open strings attached to accelerating D-branes, boundary conditions to become complicated, as explained in the following chapters. To avoid this difficulty, we can employ two kinds of methods. One of them is performing the perturbative calculation in the rotating coordinate [26]. By employing the rotating coordinate, boundary conditions for open strings become negative in exchange for the simplicity of the equation of motions. Then we can perform the mode expansion of operators and calculate the effective potential perturbatively. For perturbative calculation in the Heisenberg picture, however, there are secular terms that are a monotonic increasing function of the time and violate the validity of perturbation in operators and the Dyson series. Therefore, we developed the improved perturbation method by which secular terms are systematically removed and we can get the appropriate result. Using this method, we calculate the effective potential for revolving D-branes and see that it is governed by the distance r and the angular frequency ω. The other method is the partial modular transformation [32]. In general, string amplitudes have the modular invariance and interpreted in both of the open and closed channels. The partial modular transformation is performing the modular transformation in a part of the modular integral. Using this method, we can approximate the amplitude which gives the effective potential for D-branes with the sum of the contributions from the open massless modes and closed massless modes with good accuracy. Each contribution can be calculated in the low energy descriptions, namely, the super Yang-Mills theory for open strings and the supergravity for the closed strings. Therefore, we can obtain the effective potential without quantizing the open strings between D-branes exactly. Furthermore, we can investigate the short distance behavior of the effective potential which is significant to consider the possibility of bound states. Besides, from the worldvolume theory point of view, the contribution of closed massless modes is interpreted as the threshold correction of all of the massive open modes. We can see that the leading term of this correction is the fourth term of the relative velocity v of D-branes which decides the SUSY breaking scale for general trajectories. This fact also might be the key to understand the origin of the electroweak scale and its stability against the physics at the high energy scale.

This thesis is organized as follows. In chapter 2, we review the basics of string theory and D-brane. In particular, we calculate the effective potential between two D-branes for some simple configurations in both of the open and closed channels. In chapter 3 we focus on the D-branes which revolve around each other in the bosonic string theory. To calculate the effective potential, we employ the rotating coordinate and utilize the improved perturbation method. Then we consider the revolving D-branes in superstring theory. We show that the partial modular transformation enables us to calculate the effective potential without quantizing the open strings between these D-branes exactly in chapter 4. The existence of a resonant state is also discussed. Finally, in chapter 5 we conclude with a summary and future directions. Appendices are devoted to the details of the calculations.

Chapter 2

Foundation of D-brane dynamics

First, we will review the foundation of string theory, in particular the D-brane and the effective potential between them.

2.1 D-brane action

In this section, we will review the low energy effective action of D-branes, which governs the motion of D-branes in the spacetime. The D-brane is the extended object and open strings can be attached to it. Therefore, the low energy effective action of D-brane is described by the fields corresponding to the massless excitations of open strings. The bosonic part of the action for a Dp-brane is obtained as

$$S_{\text{DBI+CS}} = -T_p \int d^{p+1}\zeta \left[e^{-\Phi} \sqrt{-\det\left(\hat{g}_{\alpha\beta} + \hat{B}_{\alpha\beta} + 2\pi\alpha' \hat{F}_{\alpha\beta}\right)} + \hat{C}_{p+1} \right], \qquad (2.1.1)$$

where the first term is so called DBI action and the second term is Chern-Simons term. In this action, Φ, $g_{\mu\nu}$ and $C_{\mu_0\cdots\mu_p}$ are the dilaton, the gravitons and the R-R fields. $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the filed strength of the $U(1)$ field A_μ. In addition, the hat denotes the induced operators like $\hat{g}_{\alpha\beta} = g_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu$, where $\mu, \nu = 0, \cdots, 9$ and $\alpha, \beta = 0, \cdots, p$. The tension of D-branes T_p is determined as

$$\begin{aligned} T_p^2 &= \frac{\pi}{\kappa^2}(4\pi^2\alpha')^{3-p} \\ &= 4\pi^2(4\pi^2\alpha')^{-(p+1)} g^{-1} \end{aligned} \qquad (2.1.2)$$

so that the potential for the static parallel Dp-branes disappear in consequence of the supersymmetry. We should mention that A_I for $I = p+1, \cdots 9$ are related to the collective coordinate of D-branes X_I as $X_I = 2\pi\alpha' A_i$ by T-duality. Therefore, by solving the equations of motion of these fields, we can know how the D-brane moves in the spacetime.

The D-brane is considered as also the solitonic object of the supergravity theory which is obtained as low energy effective theory of the string theory. As we will see later, when we consider the potential for D-branes, exchanges of the massless fields between D-branes are important. The couplings between the massless fields of supergravity and the D-brane can be read off from this action with the background corresponding to the trajectory which is under the consideration. In order to read off the couplings and calculate the amplitude for exchanges of messless fields, it is convenience to employ the another flame. The above expression of the action is often called "the string frame". In this frame, the dilaton is exponentiated in the action of the supergravity. On the other hand, by rescaling the metric as

$$g_{\mu\nu}^{(str)} = e^{\frac{1}{4}\Phi} g_{\mu\nu}^{(E)}, \tag{2.1.3}$$

we obtain "the Einstein frame", in which the kinetic term of the dilaton in the action of the supergravity becomes canonical. In this frame, the low energy effective action of D-branes becomes

$$S_{\text{DBI+CS}} = -T_p \int d^{p+1}\zeta \left[e^{-\frac{p-3}{4}\Phi} \sqrt{-\det\left(\hat{g}_{\alpha\beta}^{(E)} + \hat{B}_{\alpha\beta}^{(E)} + 2\pi\alpha' \hat{F}_{\alpha\beta}^{(E)}\right)} + \hat{C}_{p+1} \right]. \tag{2.1.4}$$

We have seen the action of a single Dp-brane so far, however, we can consider multiple D-branes as well. For the multiple D-branes, corresponding to various open strings that have the ends on the different branes, the fields in the action becomes the matrices. Due to this adjustment, the worldvolume action is modified as

$$S_{\text{DBI}} = -T_p \int d^{p+1}\zeta \text{Tr} \left[e^{-\Phi} \sqrt{-\det\left(\hat{g}_{\alpha\beta} + \hat{B}_{\alpha\beta} + 2\pi\alpha' \hat{F}_{\alpha\beta}\right)} + \mathcal{O}\left([X_I, X_J]^2\right) \right], \tag{2.1.5}$$

where $I, J = p+1, \cdots 9$.

The action not only governs the D-brane motion in a target space, but also describes the physics on the D-brane worldvolume. Especially, the collective coordinates of D-branes are regarded as the scalar fields from the worldvolume point of view. From this aspect, the one-loop effective potential of the worldvolume theory gives the effective potential for D-branes. In order to calculate this, we often focus on the leading part of the α' expansion of the D-brane action. Actually, this effective theory becomes the

p-dimensional super Yang-Mills theory

$$S_{SYM} = \frac{1}{g_{SYM}^2}\int d^{p+1}x \; \mathrm{Tr}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}D_\mu\Phi_I D^\mu\Phi^I + \frac{1}{4}([\Phi_I,\Phi_J])^2 + \frac{i}{2}\bar{\Psi}\Gamma^\mu D_\mu\Psi + \frac{1}{2}\bar{\Psi}\Gamma^I[\Phi_I,\Psi]\right], \tag{2.1.6}$$

where Γ^μ, Γ^I are the Dirac matrices in ten dimensions. The maximally supersymmetric Yang-Mills theory in p+1 dimensions consists of gauge fields A_μ ($\mu = 0, 1, \cdots, p$), scalar fields Φ_I ($I = p+1, \cdots, 9$) which correspond to the collective coordinates of D-branes and fermions which are obtained from a Majorana-Weyl fermion Ψ in ten dimensions via the dimensional reduction. For the stack of N D-branes, the low energy action becomes the $U(N)$ SYM. As the distance between these branes becomes large, strings between them become heavier. Then, these massive modes are integrated out and the gauge symmetry of the low energy theory becomes small. For example, when a stack of N branes split into K and L branes, the gauge symmetry breaks as $U(N) \to U(K)\times U(L)$. Therefore, the scalars corresponding to the branes locations play the Higgs role. To fix the diagonal components of scalars by considering a certain configuration of D-branes means to give the expectation value of the Higgs field.

In order to calculate the potential between two D-branes, we should consider the $U(2)$ case. The way to calculate the potential for D-branes will be shown in the next section.

2.2 D-brane potential

The effective potential between D-branes is given as the corresponding string amplitude. It is calculated in the both of the open and closed channel due to the open-closed duality of the worldsheet theory. The one written in the form of one side can be translated in the other form by the modular transformation and the change of the integral variable (modular parameter). We can see that the open or closed string point of view is efficient for small or large distances respectively.

In this section, we will see how to calculate the potential in the open and closed channel respectively in some simple examples.

2.2.1 open channel

In the open string point of view, the potential between D-brane is given by the one-loop partition function of open strings stretched between them. It is written as

$$Z = \int_0^\infty \frac{dt}{2t} \mathrm{Tr}\left[e^{-2\pi s H}\right]. \tag{2.2.1}$$

For the simple configuration the boundary condition for open strings is sufficiently simple and we can quantize them. Then the Hamiltonian in the above expression is written down in the mode operators of open strings. Therefore, we can calculate the partition function exactly. First, we consider the static parallel Dp-Dp' system in the ten dimensional flat spacetime as the simplest example. The strings in the flat spacetime is described by the action

$$S_{str} = -\frac{1}{4\pi}\int_\Sigma d^2\sigma \left(\frac{1}{\alpha'}\partial_\alpha X_\mu \partial^\alpha X^\mu - i\bar{\psi}_\mu \rho_\alpha \partial^\alpha \psi^\mu\right), \tag{2.2.2}$$

or

$$S_{str} = -\frac{1}{4\pi}\int_\Sigma d^2\sigma \left(\frac{1}{\alpha'}\partial_\alpha X_\mu \partial^\alpha X^\mu - \psi_{\mu,-}\left(\partial_\sigma + \partial_t\right)\psi^\mu_- - \psi_{\mu,+}\left(\partial_\sigma - \partial_t\right)\psi^\mu_+\right) \tag{2.2.3}$$

in terms of the Weyl representation, where ρ_α is the two-dimensional Dirac matrices and we employ the conformal gauge for the metric. The worldsheet coordinate is spanned by $(\sigma_0 = t, \sigma_1 = \sigma)$. In the following, we employ the Euclidean signature by the wick rotation $\tau = it$.

The equations of motion are

$$(\partial_\sigma^2 + \partial_\tau^2)X^\mu(\sigma,\tau) = 0, \qquad (\partial_\sigma \mp i\partial_\tau)\psi^\mu_\pm(\sigma,\tau) = 0. \tag{2.2.4}$$

The boundary conditions for the open strings are obtained by the requirement that the surface terms vanish. For worldsheet bosons, they are given as

$$\begin{aligned} \text{Neumann:} \qquad & \partial_\sigma X_\mu(\tau,\sigma)|_{\sigma=0,\pi} = 0, \\ \text{Dirichlet:} \qquad & X_\mu(\tau,\sigma=0,\pi) = c_\mu, \end{aligned} \tag{2.2.5}$$

where c_μ are some constants which correspond to the D-brane location. In order to satisfy the equations of motion and the boundary conditions corresponding to static parallel D-branes, the mode expansions of worldsheet bosons are given as

$$\begin{aligned}
\text{N-N} : X^\mu(\tau,\sigma) &= x^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'}\sum_{n\neq 0}^{\infty} \frac{\alpha_n^\mu}{n} e^{-n\tau}\cos(n\sigma), \\
\text{D-D} : X^\mu(\tau,\sigma) &= \frac{y^\mu}{\pi}\sigma + \sqrt{2\alpha'}\sum_{n\neq 0}^{\infty} \frac{\alpha_n^\mu}{n} e^{-n\tau}\sin(n\sigma), \\
\text{N-D} : X^\mu(\tau,\sigma) &= i\sqrt{2\alpha'}\sum_{r\in Z+1/2}^{\infty} \frac{\alpha_r^\mu}{r} e^{-r\tau}\cos(r\sigma),
\end{aligned} \tag{2.2.6}$$

where N-N or D-D indicate that both sides of open strings stretched between D-branes satisfy the same Neumann or Dirichlet type boundary conditions and N-D means that the conditions for each end of open strings are different types. In the D-D expressions, y_i is the relative position of one of D-branes from the other. The mode operators satisfy the commutation relations $[\alpha_a^\mu, \alpha_b^\nu] = a\eta_{\mu\nu}\delta_{a,-b}$. Since integers p and p' can take only the same parity, for parallel D-branes, the number of N-D type direction $\#ND$ is always even. Due to the boundary conditions, left and right moving modes are no longer independent of each other.

In the following, we assume that $p \leq p'$. Substituting the mode expansions of X_μ into the Hamiltonian in the partition function, its contribution for the integrand of (2.2.1) is obtained as

$$\begin{aligned}
Z_B(t) =& V_{p+1}\int \frac{d^{p+1}k}{(2\pi)^{p+1}} e^{-2\pi\alpha' t k^2} e^{-\frac{R^2}{2\pi\alpha'}t} \\
&\times \mathrm{Tr}\left[e^{-2\pi t(8-\nu)\sum_{n=1}(\alpha_{-n}\alpha_n + n/2)} \cdot e^{-2\pi t\nu\sum_{r=1/2}(\alpha_{-r}\alpha_r + r/2)}\right] \\
=& \frac{V_{p+1}}{(8\pi^2\alpha')^{\frac{p+1}{2}}} t^{-\frac{p+1}{2}} e^{-\frac{R^2}{2\pi\alpha'}t} \left(e^{\frac{\pi t}{12}}\prod_{n=1}^{\infty}\frac{1}{1-e^{-2\pi n t}}\right)^{8-\nu}\left(e^{-\frac{\pi t}{24}}\prod_{r=1/2}^{\infty}\frac{1}{1-e^{-2\pi r t}}\right)^{\nu},
\end{aligned} \tag{2.2.7}$$

where we use the ν for the number of N-D directions instead of $\#NN$ for the simplicity of the expression. Here, we have already taken into account the contribution from the ghost mods arising from the gauge fixing of the diffeomorphism on the worldsheet. It cancels the unphysical contributions from the one timelike direction and one spacelike direction.

On the other hand, the Neumann type conditions for fermionic fields are given as

$$\psi_+^\mu(0,\tau) = \psi_-^\mu(0,\tau), \qquad \psi_+^\mu(\pi,\tau) = \begin{cases} \psi_-^\mu(\pi,\tau) & \text{Ramond} \\ -\psi_-^\mu(\pi,\tau) & \text{Neveu-Schwarz} \end{cases}. \tag{2.2.8}$$

The Dirichlet type conditions for fermions are determined as

$$\psi_+^\mu(0,\tau) = -\psi_-^\mu(0,\tau), \qquad \psi_+^\mu(\pi,\tau) = \begin{cases} -\psi_-^\mu(\pi,\tau) & \text{Ramond} \\ \psi_-^\mu(\pi,\tau) & \text{Neveu-Schwarz} \end{cases} \tag{2.2.9}$$

such that the worldsheet supersymmetry is preserved with correspondence to same type conditions for bosons. According to these boundary conditions, fermionic fields on the worldsheet are expanded as

$$\text{N-N}: \quad \psi_+^\mu = \sum_r \psi_r^\mu e^{-r(\tau+i\sigma)}, \qquad \psi_-^\mu = \sum_r \psi_r^\mu e^{-r(\tau-i\sigma)}, \tag{2.2.10}$$

$$\text{D-D}: \quad \psi_+^\mu = \sum_r \psi_r^\mu e^{-(r+1/2)(\tau+i\sigma)}, \qquad \psi_-^\mu = -\sum_r \psi_r^\mu e^{-(r+1/2)(\tau-i\sigma)}, \tag{2.2.11}$$

$$\text{N-D}: \quad \psi_+^\mu = \sum_r \psi_{r+1/2}^\mu e^{-(r+1/2)(\tau+i\sigma)}, \quad \psi_-^\mu = \sum_r \psi_{r+1/2}^\mu e^{-(r+1/2)(\tau-i\sigma)}, \tag{2.2.12}$$

where $r \in \mathbb{Z}$ for the R-sector and $r \in \mathbb{Z}+1/2$ for the NS-sector respectively. The anti-commutation relations that mode operators should satisfy are $\{\psi_{\pm,r}^\mu, \psi_{\pm,s}^\nu\} = \eta^{\mu\nu}\delta_{r,-s}$. Taking into account the contributions from the superghost, after the GSO projection, the fermionic contributions are obtained as

$$\begin{aligned}
Z_R(t) =& -\,\mathrm{Tr}\left[\frac{1+(-1)^F}{2}\, e^{-2\pi t(8-\nu)\sum_{n=1}(n\,\psi_{-n}\psi_n - n/2)} \cdot e^{-2\pi t\nu\sum_{r=1/2}(r\,\psi_{-r}\psi_r - r/2)}\right] \\
=& -\frac{1}{2}\left(\sqrt{2}\,e^{-\frac{\pi t}{12}}\prod_{n=1}^{\infty}\left(1+e^{-2\pi nt}\right)\right)^{8-\nu}\left(e^{\frac{\pi t}{24}}\prod_{r=1/2}^{\infty}\left(1+e^{-2\pi rt}\right)\right)^{\nu} \\
&+\frac{1}{2}\delta_{\nu,8}\left(e^{\frac{\pi t}{24}}\prod_{r=1/2}^{\infty}\left(1-e^{-2\pi rt}\right)\right)^{8}, \\
Z_{NS}(t) =& \mathrm{Tr}\left[\frac{1+(-1)^F}{2}\, e^{-2\pi t(8-\nu)\sum_{r=1/2}(r\,\psi_{-r}\psi_r - r/2)} \cdot e^{-2\pi t\nu\sum_{n=1}(n\,\psi_{-n}\psi_n - n/2)}\right] \\
=& \frac{1}{2}\left(e^{\frac{\pi t}{24}}\prod_{r=1/2}^{\infty}\left(1+e^{-2\pi rt}\right)\right)^{8-\nu}\left(\sqrt{2}\,e^{-\frac{\pi t}{12}}\prod_{n=1}^{\infty}\left(1+e^{-2\pi nt}\right)\right)^{\nu} \\
&-\frac{1}{2}\delta_{\nu,0}\left(e^{\frac{\pi t}{24}}\prod_{r=1/2}^{\infty}\left(1-e^{-2\pi rt}\right)\right)^{8}.
\end{aligned} \tag{2.2.13}$$

For $\nu \neq 0, 8$, the second term in the first line which includes factor $(-1)^F$ vanishes due to the sum over zero modes in the N-N or D-D type contributions for the R-sector and the N-D type contributions for the NS-sector. The factor $\sqrt{2}$ also comes from the zero mode sum. Whereas, for $\nu = 0, 8$, since there is no sum over the zero modes in one of the sectors, $(-1)^F$ term also contributes to the amplitude.

Combinning these contributions, we obtain the effective potential for the static paralel D-branes as

$$\begin{aligned} Z =& \frac{V_{p+1}}{(8\pi^2\alpha')^{\frac{p+1}{2}}} \int_0^\infty \frac{dt}{t} t^{-\frac{p+1}{2}} e^{-\frac{R^2}{2\pi\alpha'}t} \\ &\times \left[\left(\frac{\vartheta_{00}(0,it)}{\eta(it)^3} \right)^{\frac{8-\nu}{2}} \left(\frac{\vartheta_{10}(0,it)}{\vartheta_{01}(0,it)} \right)^{\frac{\nu}{2}} - \left(\frac{\vartheta_{10}(0,it)}{\eta(it)^3} \right)^{\frac{8-\nu}{2}} \left(\frac{\vartheta_{00}(0,it)}{\vartheta_{01}(0,it)} \right)^{\frac{\nu}{2}} \right. \\ &\qquad \left. -\delta_{\nu,0} \left(\frac{\vartheta_{01}(0,it)}{\eta(it)^3} \right)^4 + \delta_{\nu,8} \left(\frac{\vartheta_{01}(0,it)}{\vartheta_{01}(0,it)} \right)^4 \right]. \end{aligned} \tag{2.2.14}$$

Here, η- and ϑ- functions are defined as

$$\eta(a) = e^{\frac{\pi i a}{12}} \prod_{m=1}^{\infty} \left(1 - e^{2\pi i m a}\right), \tag{2.2.15}$$

$$\begin{aligned} \vartheta_{00}(a,b) =& \prod_{m=1}^{\infty} \left(1 - e^{2\pi i m b}\right) \left(1 + e^{2\pi i a} e^{2\pi i (m-1/2) b}\right) \left(1 + e^{-2\pi i a} e^{2\pi i (m-1/2) b}\right), \\ \vartheta_{01}(a,b) =& \prod_{m=1}^{\infty} \left(1 - e^{2\pi i m b}\right) \left(1 - e^{2\pi i a} e^{2\pi i (m-1/2) b}\right) \left(1 - e^{-2\pi i a} e^{2\pi i (m-1/2) b}\right), \\ \vartheta_{10}(a,b) =& 2 e^{\pi i b/4} \cos \pi a \prod_{m=1}^{\infty} \left(1 - e^{2\pi i m b}\right) \left(1 + e^{2\pi i (a + m b)}\right) \left(1 + e^{2\pi i (-a + m b)}\right), \\ \vartheta_{11}(a,b) =& -2 e^{\pi i b/4} \sin \pi a \prod_{m=1}^{\infty} \left(1 - e^{2\pi i m b}\right) \left(1 - e^{2\pi i (a + m b)}\right) \left(1 - e^{2\pi i (a + m b)}\right). \end{aligned} \tag{2.2.16}$$

For the ϑ-functions, Jacobi's abstruse equation

$$\vartheta_{00}(0,it)^4 - \vartheta_{01}(0,it)^4 - \vartheta_{10}(0,it)^4 = 0 \tag{2.2.17}$$

holds. From this expression, it is found that the potential vanishes when the number of ND type directions is a multiple of 4, $\#ND = 0, 4, 8$. It is understood as the cancelation of contributions from both of spacetime bosons and fermions, in consequence of the remained supersymmetry on the branes. In Type II string theory, there are 32 supercharges Q_α and $\tilde{Q}_\alpha$ for $\alpha = 1, \cdots 16$. When there is a Dp-brane, left and right moving modes are related by the boundary conditions. Therefore, only $Q_\alpha + (\beta^{(p)}\tilde{Q})_\alpha$, where $\beta^{(p)} := \prod_{i=p+1}^{9} \beta^i$ is a product of parity transformations β^i, are conserved and the supersymmetry becomes one-half on it. In the case that one more Dp'-brane exists, similar condition $Q_\alpha + (\beta^{(p')}\tilde{Q})_\alpha$ is imposed to supercharges. In consequence, the remained supersymmetry of whole theory are restricted to those that satisfy $Q_\alpha + (\beta^{(p)}\tilde{Q})_\alpha = Q_\alpha + (\beta^{(p')}\tilde{Q})_\alpha$, namely, that are invariant under $\beta^{ND} := \left(\beta^{(p)}\right)^{-1}\beta^{(p')}$ which corresponds to parity transformations for N-D directions. The transformation β_{ND} has the eigenvalue 1 only when $\#ND$ is a multiple of 4. This is why the potential vanishes when $\#ND = 0, 4, 8$. Furthermore, it is found that 8 supercharges are left, from the fact that β_{ND} is a traceless matrix on a 16-dimensional space and $\beta_{ND}^2 = 1$. As an exception, when $p = p'$, since $\beta_{ND} = 1$, there are 16 unbroken supersymmetry similar to single brane cases. Even for such supersymmetric combinations, when relative angles or velocity exist, the supersymmetry is broken and the potential arises as we will see below.

As a more general situation, we can consider relatively tilted D-branes. For example, let as consider two D3-branes located at $X_9 = 0$ and $X_9 = R$. We assume that one of the branes is expanded to X_a directions for $a = 0, \cdots, 3$ and another is tilted to X_a-X_{a+3} planes with angles θ_a for $a = 1, 2, 3$. Both branes are perpendicular to X_i for $i = 7, 8, 9$. In this system, boundary conditions for $X_3, \cdots, X_6$ on the tilted brane become

$$
\begin{aligned}
\partial_\sigma \left(\cos\theta_a X^a(\sigma,\tau) + \sin\theta_a X^{a+3}(\sigma,\tau)\right)|_{\sigma=\pi} =&0,\\
\sin\theta_a X^a(\pi,\tau) - \cos\theta_a X^{a+3}(\pi,\tau) =&0,\\
\cos\theta_a \psi_+^a(\pi,\tau) + \sin\theta_a \psi_+^{a+3}(\pi,\tau) \mp \left(\cos\theta_a\psi_-^a(\pi,\tau) + \sin\theta_a\psi_-^{a+3}(\pi,\tau)\right) =&0,\\
\sin\theta_a \psi_+^a(\pi,\tau) - \cos\theta_a \psi_+^{a+3}(\pi,\tau) \pm \left(\sin\theta_a\psi_-^a(\pi,\tau) - \cos\theta_a\psi_-^{a+3}(\pi,\tau)\right) =&0,
\end{aligned}
\tag{2.2.18}
$$

where upper and lower signatures are used for the R and NS sector respectively in fermionic conditions. To satisfy those conditions, mode expansions of worldsheet fields are given as

$$
\begin{aligned}
X^a(\sigma,\tau) =& i\sqrt{2\alpha'}\sum_{n=0}^{\infty}\left(\frac{\alpha^a_{n^a_+}}{n^a_+}e^{-n^a_+\tau}\cos n^a_+\sigma + \frac{\alpha^a_{n^a_-}}{n^a_-}e^{-n^a_-\tau}\cos n^a_-\sigma\right),\\
X^{a+3}(\sigma,\tau) =& i\sqrt{2\alpha'}\sum_{n=0}^{\infty}\left(\frac{\alpha^a_{n^a_+}}{n^a_+}e^{-n^a_+\tau}\sin n^a_+\sigma - \frac{\alpha^a_{n^a_-}}{n^a_-}e^{-n^a_-\tau}\sin n^a_-\sigma\right),
\end{aligned}
\tag{2.2.19}
$$

$$
\begin{aligned}
\psi^a_\pm(\sigma,\tau) =& \sum_r \left(\psi^a_{r^a_+} e^{-r_+(\tau\pm i\sigma)} + \psi^a_{r^a_-} e^{-r_-(\tau\pm i\sigma)}\right),\\
\psi^{a+3}_\pm(\sigma,\tau) =& -i\sum_r \left(\psi^a_{r^a_+} e^{-r_+(\tau\pm i\sigma)} - \psi^a_{r^a_-} e^{-r_-(\tau\pm i\sigma)}\right),
\end{aligned}
\tag{2.2.20}
$$

where $n^a_\pm(r^a_\pm) := n(r) \pm \theta^a/\pi$.

Actually, expressions for parallel branes are realized by setting angles to 0. In use of these expansions, we can obtain the one-loop potential for the branes as

$$V(R) \;=\; -\int_0^\infty \frac{dt}{t}(8\pi^2\alpha' t)^{-\frac{1}{2}} e^{-\frac{R^2}{2\pi\alpha'}t} \frac{i\prod_{a=1}^4 \vartheta_{11}(\frac{i}{\pi}\theta'_a t, it)}{\eta(it)^3\prod_{a=1}^3 \vartheta_{11}(\frac{i}{\pi}\theta_a t, it)}, \tag{2.2.21}$$

where

$$\begin{aligned} \theta'_1 &:= \frac{1}{2}(\theta_1+\theta_2+\theta_3), & \theta'_2 &:= \frac{1}{2}(\theta_1+\theta_2-\theta_3), \\ \theta'_3 &:= \frac{1}{2}(\theta_1-\theta_2+\theta_3), & \theta'_4 &:= \frac{1}{2}(\theta_1-\theta_2-\theta_3). \end{aligned} \tag{2.2.22}$$

For general angles, this potential does not vanish. As mentioned before, this fact is the consequence of the violation of supersymmetry on the branes.

The potential for branes with a constant relative velocity can be calculated by performing the Wick rotation to the tilted system. For two parallel Dp-branes, the one-loop amplitude of open strings is obtained as

$$\mathcal{A} = -i\frac{V_p}{(8\pi^2\alpha')^{p/2}}\int_0^\infty \frac{dt}{t}\, t^{-p/2}\, e^{-\frac{R^2}{2\pi\alpha'}t}\, \frac{\vartheta_{11}(ut/2\pi, it)^4}{\eta(it)^9\vartheta_{11}(ut/\pi, it)} \tag{2.2.23}$$

or, by the modular transformation which is discussed later,

$$\mathcal{A} = \frac{V_p}{(8\pi^2\alpha')^{p/2}}\int_0^\infty \frac{dt}{t}\, t^{(6-p)/2}\, e^{-\frac{R^2}{2\pi\alpha'}t}\, \frac{\vartheta_{11}(iu/2\pi, i/t)^4}{\eta(i/t)^9\vartheta_{11}(iu/\pi, i/t)}, \tag{2.2.24}$$

where $v = \tanh u$. The potential $V(R,v)$ defined such that its integral over the worldline gives the amplitude,

$$\mathcal{A} = -i\int_\infty^\infty d\tau\, V(R(\tau), v) \qquad \left(R(\tau)^2 := R^2 + v^2\tau^2\right), \tag{2.2.25}$$

is given as

$$\begin{aligned} V(R,v) =& i\frac{2vV_p}{(8\pi^2\alpha' t)^{(p+1)/2}}\int_0^\infty \frac{dt}{t}\, t^{(5-p)/2}\, e^{-\frac{R^2}{2\pi\alpha'}t}\, \frac{\vartheta_{11}(iu/2\pi, i/t)^4}{\eta(i/t)^9\vartheta_{11}(iu/\pi, i/t)} \\ =& -4^{1-p}\pi^{(5-3p)/2}\alpha'^{3-p}V_p\frac{v^4}{r^{7-p}}\Gamma\left(\frac{7-p}{2}\right) + \mathcal{O}(v^6). \end{aligned} \tag{2.2.26}$$

It can be found that the lowest order term of the velocity v is the quartic term. This behavior can be estimated by the dimensional analysis. We should mention that, however, above expression is valid only for $R \gtrsim \alpha'^{1/2}v^{1/2}$. This shall not apply for a smaller value of R.

Gauge theory calculations

For the low energy, the potential is given by the worldvolume theory described by the lightest modes. Especially, it is convenience to consider the α' expansion. As we have shown in 2.1, it becomes the p-dimensional Super Yang-Mills theory. Since we want to describe two Dp-branes revolving around each other, we choose the gauge group to be SU(2). The U(1) part describes the center of mass degrees of freedom and it is irrelevant in the present analysis.

The action including a gauge-fixing term and the associated ghost action is given by

$$\begin{aligned} S_{SYM} &= \frac{1}{g^2}\int d^{p+1}x \ \mathrm{Tr}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}D_\mu\Phi_I D^\mu\Phi^I + \frac{1}{4}([\Phi_I,\Phi_J])^2 \right. \\ &\qquad \left. + \frac{i}{2}\bar{\Psi}\Gamma^\mu D_\mu\Psi + \frac{1}{2}\bar{\Psi}\Gamma^I[\Phi_I,\Psi]\right] \\ &\quad -\frac{1}{2g^2}\int d^{p+1}x \ \mathrm{Tr}\left[(\partial^\mu A_\mu - i[B^I,\Phi_I])^2\right] \\ &\quad +\frac{1}{g^2}\int d^{p+1}x \ \mathrm{Tr}\left[\bar{c}(\partial^\mu D_\mu c - [B^I,[\Phi_I,c]])\right], \end{aligned} \tag{2.2.27}$$

where B_I are background fields for the scalars Φ_I, and c is the ghost field. We have chosen the background field gauge

$$\partial^\mu A_\mu - i[B_I,\Phi^I] = 0. \tag{2.2.28}$$

(See e.g. [29].) This is a natural gauge choice from the point of view of $\mathcal{N}=1$ SYM theory in ten dimensions.

We expand the action of eq.(2.2.27) around the background B_I by setting

$$A_\mu = a_\mu, \qquad \Phi_I = B_I + \phi_I, \qquad \Psi = \psi. \tag{2.2.29}$$

The relevant part S_2 of the action for obtaining the one-loop determinant consists of terms quadratic in the fluctuations a_μ, ϕ_I and ψ. It is given by

$$\begin{aligned} S_2 &= \frac{1}{g^2}\int d^{p+1}x \ \mathrm{Tr}\left[-\frac{1}{2}(\partial_\mu a_\nu)^2 + \frac{1}{2}[B_I,a_\mu]^2 - \frac{1}{2}(\partial_\mu\phi_I)^2 + \frac{1}{2}[B_I,\phi_J]^2 \right. \\ &\qquad + [B_I,B_J][\phi^I,\phi^J] + 2i\partial_\mu B_I[a^\mu,\phi^I] \\ &\qquad \left. + \frac{i}{2}\bar{\psi}\Gamma^\mu\partial_\mu\psi + \frac{1}{2}\bar{\psi}\Gamma^I[B_I,\psi] + \bar{c}\partial^\mu\partial_\mu c - \bar{c}[B^I,[B_I,c]]\right]. \end{aligned} \tag{2.2.30}$$

In the following, we are interested in a background configuration B_I corresponding to a motion of the Dp-branes. The background configuration B_I takes the form of

$$B_I = b_I(t)\sigma_3, \tag{2.2.31}$$

where $b_I(t)$ are functions of time t, describing the trajectories of the Dp-branes, and σ_i are Pauli matrices. It turns out that most of the terms in S_2 including B_I are the mass terms for the fluctuations. In addition, there is a mixing term of the gauge field a^μ and the scalar field ϕ^I, $2i\partial_\mu B_I[a^\mu, \phi^I]$ which gives a non-trivial effect of the background to the one-loop determinant.

The effective potential between the Dp-branes are induced by an open string stretched between them. Such an open string corresponds to the off-diagonal components of the fluctuations, which are proportional to $\sigma_{1,2}$. To compute the one-loop determinant relevant for the effective potential, we perform the Wick rotation

$$t = -i\tau, \qquad a_0 = ia_\tau, \qquad \Gamma^0 = -i\Gamma_\tau \tag{2.2.32}$$

to regularize the path integral, and set

$$a_m = \tilde{a}_m\sigma_+ + \tilde{a}_m^\dagger\sigma_-, \qquad \phi_I = \varphi_I\sigma_+ + \varphi_I^\dagger\sigma_-, \qquad \psi = \chi\sigma_+ + \tilde{\chi}\sigma_-, \tag{2.2.33}$$

where $\sigma_\pm := \frac{1}{2}(\sigma_1 \pm i\sigma_2)$ and $m = 1, \cdots, p, \tau$. Note that χ and $\tilde{\chi}$ are related to each other by the Majorana-Weyl condition of ψ. Inserting them into S_2, we obtain

$$\begin{aligned} S_2 \;=\; \frac{1}{g^2}\int d^{p+1}x \ \mathrm{Tr}\big[&|\partial_m a_n|^2 + 4(b_I)^2|a_m|^2 + |\partial_m\varphi_I|^2 + 4(b_I)^2|\varphi_J|^2 \\ &-4i\partial_m b_I(a_m\varphi_I^\dagger - a_m^\dagger\varphi_I) + i\bar{\chi}\Gamma_m\partial_m\chi - 2\bar{\chi}\Gamma_I b_I\chi \\ &+\bar{c}_+\partial^2 c_+ + \bar{c}_-\partial^2 c_- - 4(b_I)^2(\bar{c}_+c_+ + \bar{c}_-c_-)\big], \end{aligned} \tag{2.2.34}$$

where we denoted a_m instead of $\tilde{a}_m$ for notational simplicity.

2.2.2 closed channel

The effective potential for D-branes can be calculated also from the closed string point of view. It is given as the amplitude for the exchange of closed strings between D-branes. In this subsection, to calculate the potential, we construct the boundary state which corresponding to the D-brane first. Then, we will see the way to calculate the potential in use of the boundary states. The low energy description which given as the exchange of massless fields in the supergravity will be shown after that.

Boundary state

The state corresponding a D-brane is able to be constructed as a coherent state in the Hilbert space of closed strings. Here, we will see how to construct the such a state, which is called *the boundary state*, and that a boundary state couples to the massless fields in the supergravity appropriately. The boundary state can be used to compute the effective potential for a pair of D-brane as described later.

As explained in 2.1.5, the D-brane is an object on which open strings can have the end. Therefore, boundary conditions for open strings attached a D-brane is the Neumann type for the horizontal direction and the Dirichlet type for the perpendicular direction as

$$\partial_\sigma X^a(\sigma=0)=0, \qquad X^i(\sigma=0)=y^i, \tag{2.2.35}$$

where $a(i)$ is assigned for directions parallel (perpendicular) to the D-brane. Based on this boundary conditions, we can construct the boundary state, in use of the open-closed duality.

The closed strings satisfy (anti-)periodic boundary conditions

$$\begin{aligned} X^\mu(\sigma+2\pi,\tau) =&\ X^\mu(\sigma,\tau), \\ \psi^\mu(\sigma+2\pi,\tau) = \pm\,\psi^\mu(\sigma,\tau), &\qquad \tilde{\psi}^\mu(\sigma+2\pi,\tau)=\pm\tilde{\psi}^\mu(\sigma,\tau) \end{aligned} \tag{2.2.36}$$

and the mode expansions are

$$\begin{aligned} X^\mu(\sigma,\tau) &= x^\mu+\alpha' p^\mu t + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\left(\frac{\alpha_n^\mu}{n}e^{-n(\tau-i\sigma)}+\frac{\tilde{\alpha}_n^\mu}{n}e^{-n(\tau+i\sigma)}\right) \\ \psi^\mu(\sigma,\tau) &= e^{-i\pi/2}\sum_r \psi_r^\mu e^{-r(\tau-i\sigma)}, \qquad \tilde{\psi}^\mu(\sigma,\tau)=e^{i\pi/2}\sum_r\tilde{\psi}_r^\mu e^{-r(\tau+i\sigma)}. \end{aligned} \tag{2.2.37}$$

In order to transfer to the closed string point of view, we rotate the worldsheet coordinates (σ,τ) by $\pi/2$. Then the bosonic part of the boundary state $|B_X\rangle$ should satisfies

$$\partial_\tau X^a(\tau=0)|B_X\rangle=0, \qquad X^i(\tau=0)|B_X\rangle = y^i|B_X\rangle. \tag{2.2.38}$$

In terms of the mode operators,

$$\left(\alpha_n^a+\tilde{\alpha}_n^a\right)|B_X\rangle = 0, \qquad \left(\alpha_n^i-\tilde{\alpha}_n^i\right)|B_X\rangle = 0 \tag{2.2.39}$$

$$p^a|B_X\rangle = 0, \qquad q^i|B_X\rangle = y^i|B_X\rangle. \tag{2.2.40}$$

We can construct such a state as a coherent state of strings

$$|B_X\rangle \propto \delta^{(d-p-1)}\left(q^i-y^i\right)\prod_{n=1}^{\infty}\exp\left[-\frac{1}{n}\alpha_{-n}\cdot S\cdot\tilde{\alpha}_{-n}\right]|0\rangle, \tag{2.2.41}$$

where $S_{\mu\nu} := (\eta_{ab}, -\delta_{ij})$. The normalization factor is determined as $T_p/2$ by comparing the potentials for D-branes calculated in both of the open channel and closed channel.

On the other hand, fermionic part of the boundary state can be constructed as follows. The boundary conditions of the fermionic field on the open strings worldsheet for parallel directions are

$$\psi^\alpha(0,\tau) = \eta_1\, \tilde{\psi}^\alpha(0,\tau), \qquad \psi^\alpha(\pi,\tau) = \eta_2\, \tilde{\psi}^\alpha(\pi,\tau), \tag{2.2.42}$$

where $\eta_1 = \eta_2 = \pm 1$ for the R sector and $\eta_1 = -\eta_2 = \pm 1$ for the NS sector. The conditions for vertical directions are obtained by considering the invariance of physical states under the T-dual transformation. Any physical states should have the superconformal invariance

$$G_r|phys\rangle = \tilde{G}_r|phys\rangle = 0, \tag{2.2.43}$$

where superconformal generators are $G_r = \sum_{n=0}^{\infty} \alpha_{-n}\psi_{r+n}$. Since the bosonic modes transform as $\alpha_n^i \to \alpha_n^i$, $\tilde{\alpha}_n^i \to -\tilde{\alpha}_n^i$ under the T-dual transformation, fermionic modes must change as $\psi_r^i \to \psi_r^i$, $\tilde{\psi}_r^i \to -\tilde{\psi}_r^i$. Hence, the conditions for vertical directions are

$$\psi^i(0,\tau) = -\eta_1\, \tilde{\psi}^i(0,\tau), \qquad \psi^i(\pi,\tau) = -\eta_2\, \tilde{\psi}^i(\pi,\tau). \tag{2.2.44}$$

When we consider the boundary state at the one-loop level, in addition to the above conditions, fermionic modes must satisfy

$$\psi^\mu(\sigma,0) = -\eta_3\, \psi^\mu(0,T), \qquad \tilde{\psi}^\mu(\sigma,0) = -\eta_4\, \tilde{\psi}^\mu(\sigma,T), \tag{2.2.45}$$

which reflects the fact that the worldsheet is periodic for τ direction. By rotating the worldsheet coordinate, ψ, which have the conformal weight $\frac{1}{2}$, becomes $\psi^{\mu,closed} = e^{i\pi/4}\psi^\mu$, $\tilde{\psi}^{\mu,closed} = e^{-i\pi/4}\tilde{\psi}^\mu$. We should mention that only R-R sector and NS-NS sector are allowed for the boundary state in consequence of these conditions. Hereafter, we write the ψ^{closed} just as ψ. In consequence, we obtain the constraints for the fermoinic part of the boundary state as

$$\left(\psi^\mu(0,\sigma) - i\eta S^\mu_\nu \tilde{\psi}^\nu(0,\sigma)\right)|B_\psi,\eta\rangle = 0 \tag{2.2.46}$$

or

$$\left(\psi^\mu_r - i\eta S^\mu_\nu \tilde{\psi}^\nu_{-r}\right)|B_\psi,\eta\rangle = 0. \tag{2.2.47}$$

For NS-NS sector, we can construct the boundary state as well as the bosonic case

$$|B_\psi,\eta\rangle_{NS} = i\prod_{r=1/2}^{\infty} \exp\left[i\eta\psi_{-r}\cdot S\cdot\tilde{\psi}_{-r}\right]|0\rangle. \tag{2.2.48}$$

Whereas, for R-R sector, we need to take care of the zero mode. The state we want can be constructed as

$$|B_\psi, \eta\rangle_R = -\prod_{n=1}^{\infty} \exp\left[i\eta\psi_{-n}\cdot S\cdot\tilde{\psi}_{-n}\right]|B_\psi,\eta\rangle_R^{(0)}, \tag{2.2.49}$$

$$|B_\psi, \eta\rangle_R^{(0)} = \left(C\Gamma^0\cdots\Gamma^p\frac{1+i\eta\Gamma^{11}}{1+i\eta}\right)_{A\tilde{B}}|A\rangle|\tilde{B}\rangle. \tag{2.2.50}$$

Here, $|A\rangle|\tilde{B}\rangle$ is the spin state which satisfies

$$\psi_m^\mu|A\rangle|\tilde{B}\rangle = \tilde{\psi}_m^\mu|A\rangle|\tilde{B}\rangle = 0 \qquad (m>0), \tag{2.2.51}$$

$$\psi_0^\mu|A\rangle|\tilde{B}\rangle = \frac{1}{\sqrt{2}}(\Gamma^\mu)^A_C(1)^B_D|C\rangle|D\rangle, \qquad \tilde{\psi}_0^\mu|A\rangle|\tilde{B}\rangle = \frac{1}{\sqrt{2}}(\Gamma^{11})^A_C(\Gamma^\mu)^B_D|C\rangle|D\rangle. \tag{2.2.52}$$

The indices $A, B, \cdots$ runs $1, \cdots, 32$, and C,Γ^μ are the charge conjugate and the ten-dimensional Dirac matrices.

It is also necessary to take into account the bc-ghost and $\beta\gamma$-superghost. The ghost part of the boundary state is determined by the requirement to satisfy the BRST invariance $(Q_{BRST}+\tilde{Q}_{BRST})|B\rangle = 0$. Here, Q_{BRST} is the BRST charge operator, its explicit form is omitted here. In consequence, the ghost part is obtained as

$$|B_{bc}\rangle = \exp\left[\sum_{n=1}^{\infty}(c_{-n}\tilde{b}_{-n} - b_{-n}\tilde{c}_{-n})\right]\frac{c_0+\tilde{c}_0}{2}\,|q=1, q=1\rangle, \tag{2.2.53}$$

$$|B_{\beta\gamma},\eta\rangle = \begin{cases} \exp\left[i\eta\displaystyle\sum_{r=1/2}^{\infty}(\gamma_{-r}\tilde{\beta}_{-r} - \beta_{-r}\tilde{\gamma}_{-r})\right]|P=-1, P=-1\rangle \\ \exp\left[i\eta\left\{\displaystyle\sum_{n=1}^{\infty}(\gamma_{-n}\tilde{\beta}_{-n} - \beta_{-n}\tilde{\gamma}_{-n}) + \gamma_0\tilde{\beta}_0\right\}\right]|P=-1/2, P=-3/2\rangle. \end{cases} \tag{2.2.54}$$

In the last line, the upper expression is for the NS sector and the lower is for the R sector.

As the final step, we perform the GSO projection on the fermionic part of the boundary state $|B_F,\eta\rangle_{NS,R} \equiv |B_\psi,\eta\rangle_{NS,R}\times|B_{\beta\gamma},\eta\rangle_{NS,R}$. For the NS sector, the projected state is defined as

$$|B_F\rangle_{NS} \equiv \frac{1+(-1)^{F+G}}{2}\frac{1+(-1)^{\tilde{F}+\tilde{G}}}{2}|B_F,+\rangle_{NS}, \tag{2.2.55}$$

where $F = \sum_{r=1/2}^{\infty} \psi_{-r}\psi_r - 1$, $G = -\sum_{r=1/2}^{\infty}(\gamma_{-r}\beta_r + \beta_{-r}\gamma_r)$ are the fermion number and the superghost number respectively. According to the above expressions of the fermionic parts, it is found that this projected state for the NS sector is expressed as

$$|B_F\rangle_{NS} = \frac{1}{2}\Big(|B_F,+\rangle_{NS} - |B_F,-\rangle_{NS}\Big). \tag{2.2.56}$$

Similarly, the projected state for the R sector corresponding to the Dp-brane is

$$|B_F\rangle_R \equiv \frac{1+(-1)^{F+G+p+1}}{2}\frac{1+(-1)^{\tilde{F}+\tilde{G}}}{2}|B_F,+\rangle_R. \tag{2.2.57}$$

Here, p is a even number for type IIA and odd number for type IIB. Actuary, this state is expressed as

$$|B_F\rangle_R = \frac{1}{2}\Big(|B_F,+\rangle_R + |B_F,-\rangle_R\Big) \tag{2.2.58}$$

for both of type IIA and type IIB.

From the above expressions for each sector, we have constructed the boundary state

$$|B\rangle_{R,NS} = |B_B\rangle \times |B_F\rangle_{R,NS}. \tag{2.2.59}$$

Exchange of massless fields

At this point, we can calculate the effective potential for the D-branes by using the boundary state constructed in the above. It is given as the tree amplitude for the exchange of infinitely many closed string modes

$$\mathcal{A} = \langle B|D|B\rangle. \tag{2.2.60}$$

The propagator of closed strings D is

$$D = \left[\frac{2}{\alpha}'(L_0 + \tilde{L}_0 - 2a)\right]^{-1}, \tag{2.2.61}$$

where L_0 is the Virasoro zero modes and the Casimir energy a is 1/2 for the NS sector and 0 for the R sector. Here, similar to the open channel discussion, we consider the static parallel Dp-Dp' system. Then the tree amplitude (2.2.60) is obtained as

$$\begin{aligned}\mathcal{A} \;=\; & \frac{V_{\#NN}}{2}(8\pi^2\alpha')^{-\frac{\#NN}{2}}\int_0^\infty ds\, s^{-\frac{\#DD}{2}} e^{-\frac{y^2}{2\pi\alpha'}s^{-1}} \\ & \times\left[\frac{\vartheta_{00}(0,is)^{\frac{\rho}{2}}\vartheta_{01}(0,is)^{\frac{\rho}{2}}\Big(\vartheta_{00}(0,is)^{4-\rho}-\vartheta_{01}(0,is)^{4-\rho}\Big)}{\eta(is)^{12-\frac{3\rho}{2}}\vartheta_{10}(0,is)^{\frac{\rho}{2}}} - \frac{\vartheta_{10}(0,is)^4}{\eta(is)^{12}}\delta_{\rho,0}+\delta_{\rho,8}\right].\end{aligned} \tag{2.2.62}$$

Here, we use ρ instead of $\#ND$ for the simplicity of the expression. As we will see in the next subsection, this result is equivalent to the potential calculated in the open strings point of view.

In the case that there are relative angles or constant velocity, the potential can be calculated by imposing the appropriate conditions for the boundary states. The results are equivalent to open channel.

Supergravity calculation

In the case that the distance between D-branes is sufficiently large, as we will see in the next subsection, the potential is also obtained by the field theory calculation. It is given as the exchange of massless fields between the D-branes in the 10 dimensional supergravity. Here, we calculate the classical potential between Dp-branes induced by exchange of massless supergravity fields. The relevant fields are the graviton, dilaton and R-R (p+1)-field. The bosonic part of the action of Type II supergravity is given by

$$S_{\text{SUGRA}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x\sqrt{-g}\left[R + \frac{1}{2}(d\Phi)^2 + \frac{1}{2}(dC^{(p+1)})^2 + \cdots\right], \tag{2.2.63}$$

where the fields are normalized such that the kinetic terms become canonical. In other words, the Einstein frame is employed. Then the propagators are given by

$$\text{dilaton:} \qquad \Delta(x) := 2\kappa_{10}^2 \int \frac{d^{10}k}{(2\pi)^{10}} \frac{e^{ik\cdot x}}{k^2}, \tag{2.2.64}$$

$$\text{graviton:} \qquad \Delta_{\mu\nu;\rho\sigma}(x) := \left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \frac{1}{4}\eta_{\mu\nu}\eta_{\rho\sigma}\right)\Delta(x), \tag{2.2.65}$$

$$\text{R-R field:} \qquad \Delta_{\mu_0\cdots\mu_p;\nu_0\cdots\nu_p}(x) := \sum_{\sigma\in\mathcal{S}_{p+1}} \text{sgn}(\sigma)\, \eta_{\mu_0\nu_{\sigma(0)}} \cdots \eta_{\mu_p\nu_{\sigma(p)}} \Delta(x), \tag{2.2.66}$$

where the target space indices run over $0, 1 \cdots, 9$.

We then specify how these supergravity fields are coupled to D-branes. First, consider a general configuration of a pair of Dp-branes. Their trajectories $X^\mu(\zeta)$ and $\tilde{X}^\mu(\tilde{\zeta})$ are arbitrary, where ζ^α are the worldvolume coordinates with $\alpha = 0, 1, \cdots, p$. The interaction vertices of the Dp-brane with the supergravity fields can be read off from the DBI action with CS term (2.1.5). The vertices can be read off from the variations of this action. The relevant terms are

$$\text{dilaton:} \qquad \frac{p-3}{4}T_p \int d^{p+1}\zeta \, \sqrt{-\det \hat{\eta}_{\alpha\beta}}\, \delta\phi, \tag{2.2.67}$$

$$\text{graviton:} \qquad -\frac{1}{2}T_p \int d^{p+1}\zeta \, \sqrt{-\det \hat{\eta}_{\alpha\beta}}\, \hat{\eta}^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X^\nu \, \delta g_{\mu\nu}, \tag{2.2.68}$$

$$\text{R-R field:} \qquad \frac{T_p}{(p+1)!}\int d^{p+1}\zeta \, \epsilon^{\alpha_0\cdots\alpha_p}\partial_{\alpha_0}X^{\mu_0}\cdots\partial_{\alpha_p}X^{\mu_p}\,\delta C_{\mu_0\cdots\mu_p}, \tag{2.2.69}$$

where

$$\hat{\eta}_{\alpha\beta} := \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}, \qquad \hat{\eta}^{\alpha\beta}\hat{\eta}_{\beta\gamma} = \delta^\alpha_\gamma. \tag{2.2.70}$$

The dilaton vacuum expectation value is absorbed in the string coupling constant.

Using the above propagators and interaction vertices, we can obtain the contributions from the exchanges of each field. The dilaton exchange gives a contribution to the potential as

$$-\left(\frac{p-3}{4}\right)^2 T_p^2 \int d^{p+1}\zeta \int d^{p+1}\tilde{\zeta}\, \sqrt{-\det \hat{\eta}_{\alpha\beta}(X)}\sqrt{-\det \hat{\eta}_{\gamma\delta}(\tilde{X})}\Delta(X-\tilde{X}). \tag{2.2.71}$$

The graviton exchange gives a contribution to the potential as

$$\begin{aligned}&-\frac{1}{4}T_p^2 \int d^{p+1}\zeta \int d^{p+1}\tilde{\zeta}\sqrt{-\det \hat{\eta}_{\alpha\beta}(X)}\sqrt{-\det \hat{\eta}_{\gamma\delta}(\tilde{X})}\\ &\qquad\times \hat{\eta}^{\alpha\beta}(X)\partial_\alpha X^\mu \partial_\beta X^\nu \hat{\eta}^{\gamma\delta}(\tilde{X})\partial_\gamma \tilde{X}^\rho \partial_\delta X^\sigma \Delta_{\mu\nu,\rho\sigma}(X-\tilde{X}).\end{aligned} \tag{2.2.72}$$

The integrand can be simplified as follows:

$$\begin{aligned}& \hat{\eta}^{\alpha\beta}(X)\partial_\alpha X^\mu \partial_\beta X^\nu \hat{\eta}^{\gamma\delta}(\tilde{X})\partial_\gamma \tilde{X}^\rho \partial_\delta \tilde{X}^\sigma \Delta_{\mu\nu,\rho\sigma}(X-\tilde{X})\\
= \; & \hat{\eta}^{\alpha\beta}(X)\hat{\eta}^{\gamma\delta}(\tilde{X})\partial_\alpha X^\mu \partial_\beta X^\nu \partial_\gamma \tilde{X}^\rho \partial_\delta \tilde{X}^\sigma \left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \frac{1}{4}\eta_{\mu\nu}\eta_{\rho\sigma}\right)\Delta(X-\tilde{X})\\
= \; & \hat{\eta}^{\alpha\beta}(X)\hat{\eta}^{\gamma\delta}(\tilde{X})\left(2(\partial_\alpha X\cdot\partial_\gamma\tilde{X})(\partial_\beta X\cdot\partial_\delta\tilde{X}) - \frac{1}{4}(\partial_\alpha X\cdot\partial_\beta X)(\partial_\gamma\tilde{X}\cdot\partial_\delta\tilde{X})\right)\Delta(X-\tilde{X})\\
= \; & \left(2\hat{\eta}^{\alpha\beta}(X)(\partial_\beta X\cdot\partial_\delta\tilde{X})\hat{\eta}^{\delta\gamma}(\tilde{X})(\partial_\gamma\tilde{X}\cdot\partial_\alpha X) - \frac{(p+1)^2}{4}\right)\Delta(X-\tilde{X}).\end{aligned} \tag{2.2.73}$$

Therefore, we obtain

$$\begin{aligned}&\frac{(p+1)^2}{16}T_p^2 \int d^{p+1}\zeta \int d^{p+1}\tilde{\zeta}\, \sqrt{-\det \hat{\eta}_{\alpha\beta}(X)}\sqrt{-\det \hat{\eta}_{\gamma\delta}(\tilde{X})}\,\Delta(X-\tilde{X})\\
&-\frac{1}{2}T_p^2 \int d^{p+1}\zeta \int d^{p+1}\tilde{\zeta}\, \sqrt{-\det \hat{\eta}_{\alpha\beta}(X)}\sqrt{-\det \hat{\eta}_{\gamma\delta}(\tilde{X})}\\
&\qquad\times \hat{\eta}^{\alpha\beta}(X)(\partial_\beta X\cdot\partial_\delta\tilde{X})\hat{\eta}^{\delta\gamma}(\tilde{X})(\partial_\gamma\tilde{X}\cdot\partial_\alpha X)\Delta(X-\tilde{X}).\end{aligned} \tag{2.2.74}$$

On the other hand, the R-R field exchange gives a contribution to the potential as

$$
\begin{aligned}
-\frac{T_p^2}{((p+1)!)^2}\int d^{p+1}\zeta \int d^{p+1}\tilde{\zeta}\, \epsilon^{\alpha_0\cdots\alpha_p}\partial_{\alpha_0}X^{\mu_0}\cdots\partial_{\alpha_p}X^{\mu_p}\epsilon^{\beta_0\cdots\beta_p}\partial_{\beta_0}\tilde{X}^{\nu_0}\cdots\partial_{\beta_p}\tilde{X}^{\nu_p} \\
\times\eta_{\mu_0\cdots\mu_p;\nu_0\cdots\nu_p}\Delta(X-\tilde{X}),
\end{aligned}
\tag{2.2.75}
$$

where

$$
\eta_{\mu_0\cdots\mu_p;\nu_0\cdots\nu_p} \;:=\; \sum_{\sigma\in\mathcal{S}_{p+1}} \mathrm{sgn}(\sigma)\, \eta_{\mu_0\nu_{\sigma(0)}}\cdots\eta_{\mu_p\nu_{\sigma(p)}}. \tag{2.2.76}
$$

The integrand can be simplified as follows:

$$
\begin{aligned}
& \epsilon^{\alpha_0\cdots\alpha_p}\partial_{\alpha_0}X^{\mu_0}\cdots\partial_{\alpha_p}X^{\mu_p}\epsilon^{\beta_0\cdots\beta_p}\partial_{\beta_0}\tilde{X}^{\nu_0}\cdots\partial_{\beta_p}\tilde{X}^{\nu_p}\eta_{\mu_0\cdots\mu_p,\nu_0\cdots\nu_p} \\
= & \sum_{\sigma\in\mathcal{S}_{p+1}} \mathrm{sgn}\,(\sigma)\epsilon^{\alpha_0\cdots\alpha_p}\partial_{\alpha_0}X^{\mu_0}\cdots\partial_{\alpha_p}X^{\mu_p}\epsilon^{\beta_0\cdots\beta_p}\partial_{\beta_0}\tilde{X}^{\nu_0}\cdots\partial_{\beta_p}\tilde{X}^{\nu_p}\eta_{\mu_0\nu_{\sigma(0)}}\cdots\eta_{\mu_p\nu_{\sigma(p)}} \\
= & \sum_{\sigma\in\mathcal{S}_{p+1}} \mathrm{sgn}\,(\sigma)\epsilon^{\alpha_0\cdots\alpha_p}\partial_{\alpha_0}X^{\mu_0}\cdots\partial_{\alpha_p}X^{\mu_p} \\
& \qquad \times\mathrm{sgn}(\sigma)\,\epsilon^{\beta_{\sigma(0)}\cdots\beta_{\sigma(p)}}\partial_{\beta_{\sigma(0)}}\tilde{X}^{\nu_{\sigma(0)}}\cdots\partial_{\beta_{\sigma(p)}}\tilde{X}^{\nu_{\sigma(p)}}\eta_{\mu_0\nu_{\sigma(0)}}\cdots\eta_{\mu_p\nu_{\sigma(p)}} \\
= & \sum_{\sigma\in\mathcal{S}_{p+1}} \epsilon^{\alpha_0\cdots\alpha_p}\epsilon^{\beta_0\cdots\beta_p}\partial_{\alpha_0}X^{\mu_0}\cdots\partial_{\alpha_p}X^{\mu_p}\partial_{\beta_0}\tilde{X}^{\nu_0}\cdots\partial_{\beta_p}\tilde{X}^{\nu_p}\eta_{\mu_0\nu_0}\cdots\eta_{\mu_p\nu_p} \\
= & (p+1)!\,\epsilon^{\alpha_0\cdots\alpha_p}\epsilon^{\beta_0\cdots\beta_p}(\partial_{\alpha_0}X\cdot\partial_{\beta_0}\tilde{X})\cdots(\partial_{\alpha_p}X\cdot\partial_{\beta_p}\tilde{X}) \\
= & ((p+1)!)^2\,\det(\partial_\alpha X\cdot\partial_\beta\tilde{X}).
\end{aligned}
\tag{2.2.77}
$$

Therefore, we obtain

$$
-T_p^2\int d^{p+1}\zeta\int d^{p+1}\tilde{\zeta}\;\det(\partial_\alpha X\cdot\partial_\beta\tilde{X})\,\Delta(X-\tilde{X}). \tag{2.2.78}
$$

Summing them, the supergravity potential is totally given by

$$
-2\kappa_{10}^2\int d^{p+1}\zeta\int d^{p+1}\tilde{\zeta}\,\Delta(X-\tilde{X})\left(F_\Phi(X,\tilde{X})+F_g(X,\tilde{X})+F_C(X,\tilde{X})\right), \tag{2.2.79}
$$

where

$$
\begin{aligned}
F_\Phi(\zeta,\tilde{\zeta}) &= \left(\frac{p-3}{4}\right)^2 T_p^2\sqrt{-\det\hat{\eta}_{\alpha\beta}(X)}\sqrt{-\det\hat{\eta}_{\gamma\delta}(\tilde{X})}, & (2.2.80)\\
F_g(\zeta,\tilde{\zeta}) &= T_p^2\sqrt{-\det\hat{\eta}_{\alpha\beta}(X)}\sqrt{-\det\hat{\eta}_{\gamma\delta}(\tilde{X})}\left(-\frac{(p+1)^2}{16}\right. \\
&\qquad \left.+\frac{1}{2}\hat{\eta}^{\alpha\beta}(X)(\partial_\beta X\cdot\partial_\delta\tilde{X})\hat{\eta}^{\delta\gamma}(\tilde{X})(\partial_\gamma\tilde{X}\cdot\partial_\alpha X)\right), & (2.2.81)\\
F_C(\zeta,\tilde{\zeta}) &= T_p^2\det(\partial_\alpha X\cdot\partial_\beta\tilde{X}). & (2.2.82)
\end{aligned}
$$

Here, $F_\Phi(X,\tilde{X})$, $F_g(X,\tilde{X})$ and $F_C(X,\tilde{X})$ are contributions from the dilaton, graviton and RR-fields, respectively.

Ex: D1-branes at angle

As a simple check of our formula, let us consider a simple example of D1-branes at angle. Their trajectories are given by

$$\begin{aligned} X^\alpha &= \zeta^\alpha, \quad (\alpha = 0,1) \\ \tilde{X}^0 &= \tilde{\zeta}^0, \quad \tilde{X}^1 = \cos\phi\,\tilde{\zeta}^1, \quad \tilde{X}^2 = \sin\phi\,\tilde{\zeta}^1, \quad \tilde{X}^9 = r, \end{aligned} \tag{2.2.83}$$

and zero otherwise. It is easy to find that

$$\hat{\eta}_{\alpha\beta}(X) = \hat{\eta}_{\alpha\beta}(\tilde{X}) = \eta_{\alpha\beta}, \qquad \partial_\alpha X \cdot \partial_\beta \tilde{X} = \begin{bmatrix} -1 & 0 \\ 0 & \cos\phi \end{bmatrix}. \tag{2.2.84}$$

Then, we obtain

$$\begin{aligned} & \hat{\eta}^{\alpha\beta}(X)(\partial_\beta X \cdot \partial_\delta \tilde{X})\hat{\eta}^{\delta\gamma}(\tilde{X})(\partial_\gamma \tilde{X} \cdot \partial_\alpha X) \\ = & \ \mathrm{Tr}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & \cos\phi \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & \cos\phi \end{bmatrix} \\ = & \ 1 + \cos^2\phi, \end{aligned} \tag{2.2.85}$$

and

$$\det(\partial_\alpha X \cdot \partial_\beta \tilde{X}) = -\cos\phi. \tag{2.2.86}$$

Therefore,

$$\begin{aligned} & F_\Phi(X,\tilde{X}) + F_g(X,\tilde{X}) + F_C(X,\tilde{X}) \\ = & \ -\left(\frac{1-3}{4}\right)^2 T_p^2 - \left(-\frac{(1+1)^2}{16} + \frac{1+\cos^2\phi}{2}\right) T_p^2 + \cos\phi\,\rho_p^2 \\ = & \ -2T_1^2 \sin^2\frac{\phi}{2}. \end{aligned} \tag{2.2.87}$$

This reproduces the known result [14].

2.2.3 Modular invariance of the potential

The prominent feature of the potential for D-branes is the modular invariance. It plays the important role to deal with the revolving D-branes by using the partial modular transformation as we will see later.

The transformation of the modular parameter $t \to s = 1/t$ corresponds to move from the open channel to the closed channel, vice versa. The integrands in the above potential include some ϑ-functions and η-function. These functions transform as

$$\vartheta_{ij}(0, i/t) = t^{1/2}\vartheta_{ij}(0, it), \tag{2.2.88}$$

$$\eta(i/t) = t^{1/2}\eta(it), \tag{2.2.89}$$

for $(i,j) \neq (1,1)$, under the modular transformation $t \to 1/t$. In the use of this transformation properties, we can check that the above expressions in the open channel (2.2.14) and closed channel (2.2.62) switch to each other by the modular transformation. In both of the expressions, the oscillator part of the integrand can be expanded as

$$\sum_{n=0}^{\infty} c_n e^{-2\pi n t}, \tag{2.2.90}$$

where each term are the contributions from the each mode with mass level n. The small t region is the UV region where many massive modes contribute to the potential. In addition, the UV region in the one channel is corresponding to the IR region in the other channel.

Besides, it should be mentioned that, when we focusing on the large y^2 case, the closed string point of view is efficient. It is because of the factor $e^{-\frac{y^2}{2\pi\alpha'}s^{-1}}$ in the integrand. This factor suppress the contributions from the massive modes and the potential can be approximated by the contributions from the massless mode. On the other hand, in the open channel, the dependence of the distance is included as $e^{-\frac{y^2}{2\pi\alpha'}t}$, and the massive modes also contribute. This point is discussed in the 4.1 more precisely.

Chapter 3

Perturbative calculation of potential for revolving D-branes

3.1 Open strings stretched between revolving D-branes

3.1.1 Open strings in the rotational coordinate system

As a simple system of rotating D-branes in bosonic string, we consider two D0-branes in the flat space-time revolving around each other like a binary star. We assume that the orbits of the D0-branes lie on the x-y plane. Their positions (x_1, y_1) and (x_2, y_2) are changing with time and given by

$$\left\{\begin{array}{rcl} x_1(t) &=& r\cos\omega t, \\ y_1(t) &=& r\sin\omega t, \end{array}\right. \qquad \left\{\begin{array}{rcl} x_2(t) &=& -r\cos\omega t, \\ y_2(t) &=& -r\sin\omega t. \end{array}\right. \tag{3.1.1}$$

We then consider an open string stretched between these D0-branes. As explained in the introduction, we choose the rotating coordinate system for the target space-time in which the D0-branes are static. In the original coordinate system, the target space metric is simply given by

$$ds^2 = -dt^2 + dx^2 + dy^2 + (dx^i)^2 \tag{3.1.2}$$

where $i = 3, 4, \cdots, 25$. We then introduce the rotating coordinate system defined by

$$\begin{aligned} \tilde{t} &:= t, && (3.1.3)\\ \tilde{x} &:= x\cos\omega t + y\sin\omega t, && (3.1.4)\\ \tilde{y} &:= -x\sin\omega t + y\cos\omega t, && (3.1.5)\\ \tilde{x}^i &:= x^i. && (3.1.6) \end{aligned}$$

In this coordinate system, the orbits of the D0-branes (3.1.1) become static

$$\begin{cases} \tilde{x}_1(t) = r, \\ \tilde{y}_1(t) = 0, \end{cases} \qquad \begin{cases} \tilde{x}_2(t) = -r, \\ \tilde{y}_2(t) = 0 \end{cases} \tag{3.1.7}$$

but the metric takes the following non-diagonal form:

$$ds^2 = -d\tilde{t}^2 + d\tilde{x}^2 + d\tilde{y}^2 + 2\omega d\tilde{t}(\tilde{x}d\tilde{y} - \tilde{y}d\tilde{x}) + \omega^2(\tilde{x}^2 + \tilde{y}^2)d\tilde{t}^2 + (d\tilde{x}^i)^2. \tag{3.1.8}$$

Accordingly, the worldsheet action becomes

$$\begin{aligned} S = &-\frac{1}{4\pi\alpha'}\int d^2\sigma \Big[-\partial_\alpha\tilde{T}\partial^\alpha\tilde{T} + \partial_\alpha\tilde{X}\partial^\alpha\tilde{X} + \partial_\alpha\tilde{Y}\partial^\alpha\tilde{Y} + \partial_\alpha\tilde{X}^i\partial^\alpha\tilde{X}_i \\ &+2\omega\partial_\alpha\tilde{T}(\tilde{X}\partial^\alpha\tilde{Y} - \tilde{Y}\partial^\alpha\tilde{X}) + \omega^2(\tilde{X}^2 + \tilde{Y}^2)\partial_\alpha\tilde{T}\partial^\alpha\tilde{T}\Big]. \end{aligned} \tag{3.1.9}$$

By rescaling the fields as $\tilde{X}^\mu \to r\tilde{X}^\mu$, the action is given by

$$\begin{aligned} S = &-\frac{r^2}{4\pi\alpha'}\int d^2\sigma \Big[-\partial_\alpha\tilde{T}\partial^\alpha\tilde{T} + \partial_\alpha\tilde{X}\partial^\alpha\tilde{X} + \partial_\alpha\tilde{Y}\partial^\alpha\tilde{Y} + \partial_\alpha\tilde{X}^i\partial^\alpha\tilde{X}_i \\ &+2v\partial_\alpha\tilde{T}(\tilde{X}\partial^\alpha\tilde{Y} - \tilde{Y}\partial^\alpha\tilde{X}) + v^2(\tilde{X}^2 + \tilde{Y}^2)\partial_\alpha\tilde{T}\partial^\alpha\tilde{T}\Big], \end{aligned} \tag{3.1.10}$$

where $v := r\omega$ is the velocity of the D0-branes.[1] The boundary conditions for the rescaled fields, $\tilde{X}$ and $\tilde{Y}$, are simple in the new coordinate system,

$$\tilde{X}(\tau,\sigma) = \begin{cases} +1, & (\sigma = 0) \\ -1, & (\sigma = \pi) \end{cases} \qquad \tilde{Y}(\tau,\sigma) = \begin{cases} 0, & (\sigma = 0) \\ 0. & (\sigma = \pi) \end{cases} \tag{3.1.11}$$

However, the boundary condition for $\tilde{T}$ is still nontrivial. Indeed, the variation of the action with respect to $\tilde{T}$ gives the following boundary term:

$$\delta S\Big|_{bdy} = -\frac{r^2}{2\pi\alpha'}\delta\tilde{T}\left[-\partial_\sigma\tilde{T} + v(\tilde{X}\partial_\sigma\tilde{Y} - \tilde{Y}\partial_\sigma\tilde{X}) + v^2(\tilde{X}^2 + \tilde{Y}^2)\partial_\sigma\tilde{T}\right]\Big|_{bdy} \tag{3.1.12}$$

[1] Note that the relative velocity is $2v$ and the distance is $y = 2r$.

By using (3.1.11), $\tilde{T}$ must satisfy

$$\begin{aligned} (1-v^2)\partial_\sigma \tilde{T} - v\partial_\sigma \tilde{Y} &= 0, \qquad (\sigma = 0) \\ (1-v^2)\partial_\sigma \tilde{T} + v\partial_\sigma \tilde{Y} &= 0. \qquad (\sigma = \pi) \end{aligned} \tag{3.1.13}$$

These conditions are linear in the world sheet fields since we have substituted the boundary values of (3.1.11) for the nonlinear terms that would have appeared in (3.1.13). To simplify these conditions, we define a new field T as

$$T := \tilde{T} - \frac{v}{1-v^2} x(\sigma)\tilde{Y} \tag{3.1.14}$$

where

$$x(\sigma) := 1 - \frac{2\sigma}{\pi}. \tag{3.1.15}$$

Using the boundary condition (3.1.11) for $\tilde{Y}$, it can be shown that the field T satisfies the ordinary Neumann boundary condition $\partial_\sigma T|_{\sigma=0,\pi} = 0$. We also introduce a new field X

$$\tilde{X} = x(\sigma) + X. \tag{3.1.16}$$

Then $X|_{\sigma=0,\pi} = Y|_{\sigma=0,\pi} = 0$ are satisfied. For notational simplicity, we use tilde-less notations for $Y = \tilde{Y}$ and $X_i = \tilde{X}_i$ in the following discussions.

To summarize, the new world sheet fields X, Y, T and X_i (for $i = 3, \cdots 25$) satisfy the boundary conditions

$$X|_{\sigma=0,\pi} = 0, \quad Y|_{\sigma=0,\pi} = 0, \quad \partial_\sigma T|_{\sigma=0,\pi} = 0, \quad X_i|_{\sigma=0,\pi} = 0 \tag{3.1.17}$$

and the world sheet action is given by

$$\begin{aligned} S = & -\frac{r^2}{4\pi\alpha'} \int d^2\sigma \Bigg[- \dot{X}^2 - \dot{Y}^2 - (\dot{X}^i)^2 + \left(X' - \frac{2}{\pi} \right)^2 + (Y')^2 + (X^{i\prime})^2 \\ & + \Big[1 - v^2 \Big(\big(X + x(\sigma)\big)^2 + Y^2 \Big) \Big] \\ & \qquad \times \left[\left(\dot{T} + \frac{v x(\sigma)}{1-v^2} \dot{Y} \right)^2 - \left(T' + \frac{v}{1-v^2} \big(x(\sigma) Y' - \frac{2}{\pi} Y \big) \right)^2 \right] \\ & - 2v \left(\dot{T} + \frac{v x(\sigma)}{1-v^2} \dot{Y} \right) \Big(\big(X + x(\sigma)\big) \dot{Y} - Y \dot{X} \Big) \\ & + 2v \left(T' + \frac{v}{1-v^2} \big(x(\sigma) Y' - \frac{2}{\pi} Y \big) \right) \Big(\big(X + x(\sigma)\big) Y' - Y \big(X' - \frac{2}{\pi} \big) \Big) \Bigg]. \end{aligned} \tag{3.1.18}$$

In the rotating coordinate system, the action (3.1.18) becomes nonlinear in compensation for the simple boundary conditions. We analyze this theory perturbatively in the nonrelativistic limit $v \ll 1$.

3.1.2 Perturbative Hamiltonian with respect to v

The worldsheet Hamiltonian H can be obtained in the standard manner. It is decomposed into two parts:

$$H = H_{\rm rot}(X, Y, T) + H_{\rm free}(X^i), \tag{3.1.19}$$

where $H_{\rm rot}$ governs the subsystem consisting of X, Y and T while $H_{\rm free}$ is a free Hamiltonian for X^i. We now focus on the non-trivial part $H_{\rm rot}$. Let Π_X, Π_Y and Π_T denote the canonical momenta of X, Y and T, respectively. $H_{\rm rot}$ is given in perturbative series with respect to the velocity v as

$$H_{\rm rot} = H_0 + vV_1 + v^2 V_2 + \mathcal{O}(v^4), \tag{3.1.20}$$

where

$$\begin{aligned} H_0 &= \int_0^\pi d\sigma \left[\frac{\pi\alpha'}{r^2}(-\Pi_T^2 + \Pi_X^2 + \Pi_Y^2) + \frac{r^2}{4\pi\alpha'}\left\{ -(\partial_\sigma T)^2 + (\partial_\sigma X)^2 + (\partial_\sigma Y)^2 \right\} \right] \\ &\quad + \frac{r^2}{\pi^2\alpha'}, \end{aligned} \tag{3.1.21}$$

$$V_1 = \int_0^\pi d\sigma \left[\frac{2\pi\alpha'}{r^2}\Pi_T(X\Pi_Y - \Pi_X Y) + \frac{r^2}{2\pi\alpha'}\partial_\sigma T(X\partial_\sigma Y - \partial_\sigma X Y) + \frac{2r^2}{\pi^2\alpha'}\partial_\sigma T Y \right], \tag{3.1.22}$$

$$\begin{aligned} V_2 = \int_0^\pi d\sigma \Bigg[& \frac{\pi\alpha'}{r^2}\Big\{ -(X\Pi_Y - Y\Pi_X)^2 - x(\sigma)^2\Big(\Pi_T^2 + \Pi_Y^2\Big) \\ & - 2x(\sigma)\Big(\Pi_T^2 X + (X\Pi_Y - Y\Pi_X)\Pi_Y\Big)\Big\} \\ & + \frac{r^2}{4\pi\alpha'}\Bigg\{ (\partial_\sigma T)^2(X^2 + Y^2) + x(\sigma)^2\Big((\partial_\sigma T)^2 + (\partial_\sigma Y)^2\Big) + \frac{4}{\pi}x(\sigma)Y\partial_\sigma Y - \frac{12}{\pi^2}Y^2 \\ & + 2x(\sigma)\left((\partial_\sigma T)^2 X + X(\partial_\sigma Y)^2 - \partial_\sigma X Y \partial_\sigma Y\right) + \frac{4}{\pi}\left(\partial_\sigma X Y^2 - XY\partial_\sigma Y\right) \Bigg\} \Bigg]. \end{aligned} \tag{3.1.23}$$

In the next section, we quantize the Hamiltonian up to the second orders of v and calculate the one-loop partition function of the open string to obtain the potential between revolving D0-branes.

3.2 One-loop partition function

We are interested in the one-loop partition function of the rotating open string. It is given by

$$\begin{aligned} Z &= \int_0^\infty \frac{ds}{2s}\,\mathrm{Tr}\big[e^{-2\pi s(H_{\mathrm{rot}}-\frac{1}{8})}\big]\big(\eta(is)\big)^{-21} \\ &= \int_0^\infty \frac{ds}{2s}\,\mathrm{Tr}\big[e^{-2\pi s(H_{\mathrm{rot}}-1)}\big]\prod_{m=1}^{\infty}(1-e^{-2\pi m s})^{-21}. \end{aligned} \tag{3.2.1}$$

The contributions from the X_i fields in H_{free} $(i=3,\cdots,25)$ and the (b,c)-ghosts have been included in this expression. The one-loop determinant of a scalar field is written as

$$\det(\Delta+m^2)^{-1/2} = \exp\left[\int \frac{ds}{2s}\mathrm{Tr}e^{-(\Delta+m^2)s}\right] \equiv e^{-\mathcal{V}\mathcal{T}}, \tag{3.2.2}$$

where $\mathcal{T}$ is the time duration. Thus the open string one-loop partition function (3.2.1) gives the minus of an effective potential $\mathcal{V}(r,\omega)$, integrated over time[2], for the two D0-branes: $Z=-\mathcal{V}\mathcal{T}$. By the open-closed string duality, it is written as an exchange of a single closed string and approximately described by low energy closed string modes when the distance $2r$ is much larger than the string length. In section 3.3, we will instead investigate the short distance behavior of the effective potential $\mathcal{V}(r,\omega)$ at which open string low energy modes dominate.

In this section, we will calculate the partition function (3.2.1) perturbatively with respect to v. In the following, we perform the calculation in the Euclidean space by the Wick rotation of the world sheet variable from $T\to -iT$. Thus the velocity v is also analytically continued to iv.

3.2.1 Improved perturbation

In order to calculate the partition function, we need to know the energy spectrum of the Hamiltonian. We will perform perturbative calculations with respect to v, but even perturbatively it is not so straightforward since H_0, V_1, V_2 are not commutable with each other and we may need to diagonalize the Hamiltonian. Instead of explicitly diagonalizing it, we use a method of the improved perturbation [15]. This method is briefly reviewed in Appendix A. The basic idea is that we can systematically construct

[2] The stationary system of revolution is invariant under the time translation and the effective potential is given by removing the zero mode integral of T. When we compare the calculation with the constant velocity system in Sec.3.4 the integration needs a care.

a new Hamiltonian $H_0(v) = \sum_{n=0}^{\infty} v^n H_n$ such that it has the same eigenvalues as those of the original Hamiltonian $H = H_{\rm rot}$, but each term H_n commutes with H_0. Once H_n is explicitly given, it is straightforward to take a trace to obtain the partition function.

In the following, we calculate up to the second order of the v-expansion:

$$H_0(v) := H_0 + vH_1 + v^2 H_2 + \mathcal{O}(v^4). \tag{3.2.3}$$

The first term H_0 is given by the original free Hamiltonian in (3.1.22). As mentioned above, $H_0(v)$ shares the same eigenvalues with $H_{\rm rot}$ to all orders in v, so we can replace $H_{\rm rot}$ with $H_0(v)$ within the perturbation theory. Also note that H_1 and H_2 can be constructed such that they commute with H_0. In our case, their explicit forms are given by

$$H_1 \;:=\; V_{1,0}, \qquad H_2 := V_{2,0} - \sum_{m\neq 0} \frac{1}{m} V_{1,-m} V_{1,m}. \tag{3.2.4}$$

Here the operator V_i $(i = 1, 2)$ is decomposed into $V_{i,m}$ as

$$V_i = \sum_m V_{i,m}, \qquad [H_0, V_{i,m}] = m V_{i,m} \tag{3.2.5}$$

where m's are eigenvalues of the free Hamiltonian H_0. In particular, $V_{i,0}$ is the operator contained in V_i that are commutable with H_0.

3.2.2 Perturbative calculations of the trace

Now we calculate the trace in the partition function (3.2.1) with the Hamiltonian $H_{\rm rot}$ in (3.1.20). By using the method of improved perturbation, the difficulty of calculation ${\rm Tr} e^{-2\pi s H_{\rm rot}}$ due to the non-commutativity of H_0 and V_i is resolved by rewriting the trace in terms of a much easier improved Hamiltonian, $H_0(v)$. Due to the commutativity of H_n in $H_0(v)$, it can be expanded as

$$\begin{aligned}
{\rm Tr}\big[e^{-2\pi s H_{\rm rot}}\big] \;&=\; {\rm Tr}\big[e^{-2\pi s H_0(v)}\big] \\
&= {\rm Tr}\big[e^{-2\pi s H_0}\big] - 2\pi v s {\rm Tr}\big[e^{-2\pi s H_0} H_1\big] \\
&\quad -2\pi v^2 \left(s {\rm Tr}\big[e^{-2\pi s H_0} H_2\big] - \pi s^2 {\rm Tr}\big[e^{-2\pi s H_0} H_1^2\big]\right)
\end{aligned} \tag{3.2.6}$$

up to $\mathcal{O}(v^2)$. The traces are taken over the eigenstates $|\vec{\boldsymbol{n}}; k\rangle$ of H_0 where k is the momentum and $\vec{\boldsymbol{n}}$ represent the eigenmodes of harmonic oscillators. Since H_0 is a free Hamiltonian, and X, Y and T obey simple boundary conditions (3.1.17), the eigenvalues and eigenstates of H_0 are explicitly given. In the following, we recall them for fixing the notations.

The free Hamiltonian H_0 is given in terms of the mode operators as

$$H_0 \;=\; \alpha' p^2 + \frac{r^2}{\pi^2\alpha'} + \tilde{H}_0 \tag{3.2.7}$$

where

$$\tilde{H}_0 = \sum_{\mu=T,X,Y}\sum_{n=1}^{\infty} N_{\mu,n}, \qquad N_{\mu,n} := \;:\alpha^{\mu}_{-n}\alpha^{\mu}_{n}:\,. \tag{3.2.8}$$

For our conventions of the mode expansions, see Appendix B. Note that $N_{\mu,-n} = N_{\mu,n}$ hold. We denote the eigenstates of H_0 as $|\vec{\boldsymbol{n}};k\rangle$ which is given by

$$|\vec{\boldsymbol{n}};k\rangle = \prod_{m_X,m_Y,m_T=1}^{\infty} (\alpha^X_{-m_X})^{n^X_{m_X}}(\alpha^Y_{-m_Y})^{n^Y_{m_Y}}(\alpha^T_{-m_T})^{n^T_{m_T}}|0;k\rangle \tag{3.2.9}$$

and satisfies $N_{\mu,m_\mu}|\vec{\boldsymbol{n}};k\rangle = n^{\mu}_{m_\mu} m_\mu|\vec{\boldsymbol{n}};k\rangle$ and $p|\vec{\boldsymbol{n}};k\rangle = k|\vec{\boldsymbol{n}};k\rangle$.

With these notations, we can evaluate traces of the form:

$$\mathrm{Tr}\big[e^{-2\pi u H_0}O\big] \;=\; \mathcal{T} e^{-\frac{2r^2}{\pi\alpha'}u}\int\frac{dk}{2\pi}e^{-2\pi\alpha' u k^2}\,\mathrm{tr}[e^{-2\pi u\tilde{H}_0}\mathcal{O}] \tag{3.2.10}$$

where

$$\mathrm{tr}[e^{-2\pi u\tilde{H}_0}\mathcal{O}] = \sum_{\vec{\boldsymbol{n}}} e^{-2\pi u N(\vec{\mathbf{n}})}\langle\vec{\boldsymbol{n}};k|O|\vec{\boldsymbol{n}};k\rangle \tag{3.2.11}$$

and $N(\vec{\mathbf{n}}) := \sum_{\mu=T,X,Y}\sum_{m=1}^{\infty} m\, n^{\mu}_{m}$. In order to simplify the calculation, we denote $[O]_D$ as the diagonal element of O satisfying

$$\langle\vec{\boldsymbol{n}};k|O|\vec{\boldsymbol{n}};k\rangle \;=\; \langle\vec{\boldsymbol{n}};k|[O]_D|\vec{\boldsymbol{n}};k\rangle, \tag{3.2.12}$$

which are sufficient in the calculations (3.2.11). The remaining task to calculate the partition function is to calculate the diagonal elements of H_1, H_1^2 and H_2, and then to integrate over the Schwinger parameter s. In Section 3.2.3, we obtain $[H_1]_D$, $[H_1^2]_D$ and $[H_2]_D$, which enable us to calculate $\mathrm{Tr}[e^{-2\pi s H_0}H_{1,2}]$ and $\mathrm{Tr}[e^{-2\pi s H_0}H_1^2]$. Then in Section 3.2.4, we determine the one-loop partition function up to $\mathcal{O}(v^2)$.

3.2.3 Diagonal elements: $[H_1]_D$, $[H_1^2]_D$ and $[H_2]_D$

As defined in (3.2.4), the operator H_1 is the part of the operator V_1 (3.1.22) that commutes with H_0; namely $H_1|\text{state}\rangle \neq |\text{state}\rangle$, but has the same energy eigenvalues of H_0. Using the mode expansions given in Appendix B, it is straightforward to obtain

$$H_1 = -\frac{2i\alpha'}{r}p\sum_{k\neq 0}\frac{1}{k}\alpha^X_k\alpha^Y_{-k} + \frac{2i}{\pi}\sum_{k\neq 0}\frac{1}{k}\alpha^T_{-k}\alpha^Y_k\,. \tag{3.2.13}$$

For any $\vec{\boldsymbol{n}}$, the diagonal matrix element $\langle \vec{\boldsymbol{n}}; k|H_1|\vec{\boldsymbol{n}}; k\rangle$ vanishes and

$$[H_1]_D = 0. \tag{3.2.14}$$

Thus, we find that the trace $\mathrm{Tr}\big[e^{-2\pi\tau H_0}H_1\big]$ in (3.2.6) vanishes. Actually, this should be the case since the energy spectrum of the rotating open string is independent of the direction of the rotation, and the linear terms in v must vanish.

To determine $[H_1^2]_D$, we take the square of (3.2.13) and collect terms which commute with H_0. We find

$$\begin{aligned}[H_1^2]_D &= \frac{4(\alpha')^2}{r^2}p^2\left(2N_{XY}(2)+N_X(1)+N_Y(1)\right)\\ &\quad +\frac{4}{\pi^2}\left(2N_{TY}(2)+N_T(1)+N_Y(1)\right),\end{aligned} \tag{3.2.15}$$

where we defined

$$N_\mu(x) := \sum_{n=1}^{\infty}\frac{1}{n^x}N_{\mu,n}, \qquad N_{\mu\nu}(x) := \sum_{n=1}^{\infty}\frac{1}{n^x}N_{\mu,n}N_{\nu,n} \tag{3.2.16}$$

for $\mu,\nu = T, X, Y$. Note that the unperturbed states $|\vec{\boldsymbol{n}}; k\rangle$ are the eigenstates of these operators. By using the general formulae in Appendix D, we can sum over all the excited states of the open string and obtain the trace $\mathrm{Tr}\big[e^{-2\pi sH_0}H_1^2\big]$.

We now show the result of $[H_2]_D$. It is much more complicated since V_2 defined by (3.1.23) contains quartic terms in the world sheet variables, and we need to appropriately regularize the infinite sum appearing in the intermediate states. We leave the calculations in Appendix C, and show the final result here. $[H_2]_D$ is a sum of $[V_2]_D$ in (C.0.10) and $-\sum_m [V_{1,-m}V_{1,m}]_D/m$ in (C.0.11). Each of them contains many terms including various divergences. However, many terms in (C.0.10) and (C.0.11) are miraculously cancelled with each other, and the final result turns out to be quite simple;

$$\begin{aligned}[H_2]_D &= [V_2]_D - \sum_{m\neq 0}\frac{1}{m}[V_{1,-m}V_{1,m}]_D\\ &= \frac{\alpha'}{r^2}\left[2N_{XY}(2)+N_X(1)+N_Y(1)\right] - \frac{2}{\pi^2}(N_T(2)-N_Y(2))\\ &\quad -\frac{2}{\pi^2}(N_T(2)+N_Y(2)) - \frac{1}{3}\alpha' p^2 - \frac{2}{\pi^2}\zeta(1)\\ &\quad +\frac{\alpha'}{r^2}\left(N_X(0)+N_Y(0)+N_T(0)+\frac{1}{4}\zeta(0)\right).\end{aligned} \tag{3.2.17}$$

Since every term is written in terms of the operators $N_\mu(x)$ and $N_{\mu\nu}(x)$, we can use the general formulae in Appendix D to calculate the trace $\mathrm{Tr}\big[e^{-2\pi sH_0}H_2\big]$.

Several comments are in order. First, the last line is proportional to the non-zero modes of H_0 including the zero-point energy $\zeta(0)/4 = -3/24$. Thus it can be interpreted as the wave-function renormalization of T, X and Y fields with the following renormalization factor, $Z^{-1} = 1+\alpha'\omega^2$. In interacting theories, a one-particle state $a^\dagger|0\rangle$ is no longer an eigenstate of the Hamiltonian and we need to construct a state to include a Z-factor so that a state $Z^{1/2}\left(a^\dagger|0\rangle + |\text{multi}\rangle\right)$ is a properly normalized eigenstate of the interacting Hamiltonian. Here $|\text{multi}\rangle$ is a sum of multi-particle states and Z is interpreted as the probability of the eigenstate to be in the single-particle state $a^\dagger|0\rangle$. In the perturbation theory, Z-factor can be read from the renormalization of the free Hamiltonian, and in the present situation, it is given by

$$Z^{-1} = 1 + \alpha'\omega^2 > 1. \tag{3.2.18}$$

The same wave-function renormalization factor appears in a simpler calculation of the one-loop partition function of D0-branes at a constant relative motion discussed in Sec.3.4. Since the wave function renormalization gives $\mathcal{O}(v^2)$ correction to the coupling parameter v, the last line of $[H_2]_D$ does not contribute to the calculation of the effective potential up to $\mathcal{O}(v^2)$. Thus we should replace Eq. (3.2.17) with

$$\begin{aligned} [H_2]_D &= \frac{\alpha'}{r^2}\left[2N_{XY}(2) + N_X(1) + N_Y(1)\right] - \frac{4}{\pi^2}N_T(2) \\ &\quad -\frac{1}{3}\alpha' p^2 - \frac{2}{\pi^2}\zeta(1) \end{aligned} \tag{3.2.19}$$

in the following calculations.

Second, the final result turned out to be very simple after the miraculous cancellations between (C.0.10) and (C.0.11). Especially, the operator-valued terms with divergent coefficients, for example (C.0.8), cancel completely. This cancellation might be intimately related to the the renormalization property of the non-linear sigma model (3.1.10). At the one-loop level, the beta function of the target space metric is proportional to the Ricci tensor of the metric. Since the background metric of (3.1.10) is flat and Ricci tensor vanishes, the background metric of the action should not be renormalized even though divergence appears in the intermediate steps of the renormalization procedure.

Finally, the only remaining divergence appears in the zero-point energy which is independent of α'/r^2 in H_2. This zero-point energy must be also renormalized to obtain a sensible mass spectrum of the rotating open string, and we need to find the correct renormalization scheme to fix the finite part, say ϵ_0, of H_2;

$$-\frac{2}{\pi^2}\zeta(1) \longrightarrow \epsilon_0. \tag{3.2.20}$$

One possible way to fix ϵ_0 will be to check the BRST algebra of the worldsheet theory (3.1.10), as it determines the intercept for strings in the Minkowski space-time. Another possible way will be to examine the behavior of the one-loop partition function in the closed string channel, or in other words, at large distances. There must be the contribution from the massless graviton exchanged between the D0-branes which must give us the Newton potential. Since the large distance behavior of the partition function would depend on ϵ_0, the requirement for reproducing the Newton potential may choose the correct value for ϵ_0. In the following, we leave ϵ_0 to be an unknown parameter.

3.2.4 One-loop open string partition function

By using Eq. (3.2.6), Eqs. (3.2.14), (3.2.15), and (3.2.19), and various formulae derived in Appendix D, we can calculate the one-loop open string partition function (3.2.1). Many terms are miraculously cancelled and we have a very simple form

$$\begin{aligned} Z &= \mathcal{T}\int_0^\infty \frac{ds}{2s}(8\pi^2\alpha' s)^{-\frac{1}{2}} e^{-\frac{2r^2}{\pi\alpha'}s}\eta(is)^{-24}\left(1-\frac{1}{3}v^2\right)^{-\frac{1}{2}} \\ &\quad \times\left[1-2\pi v^2\left(-\frac{4}{\pi^2}s\sum_{n=1}^\infty \frac{n^{-1}q^n}{1-q^n}+\epsilon_0 s-\frac{4}{\pi}s^2\sum_{n=1}^\infty\frac{2q^n}{(1-q^n)^2}\right)\right]+\mathcal{O}(v^4), \end{aligned} \tag{3.2.21}$$

where $q := e^{-2\pi s}$ and $\eta(is) = q^{1/24}\prod_{m=1}^\infty(1-q^m)$. The derivation of this expression is summarized in Appendix D. In the $v \to 0$ limit, Eq. (3.2.21) is reduced to the partition function for D0-branes at rest. Then let us compare Eq. (3.2.21) with the partition function for D-branes moving with a constant relative velocity $2v$, which is given in Eq. (3.4.23). Some of the v^2-corrections in Eq. (3.2.21) that come from the non-zero modes, namely the third term of the v^2-corrections in the square bracket in Eq. (3.2.21), are exactly the same as the v^2-corrections in the constant-velocity case in Eq. (3.4.23). This term came from the $\mathcal{O}(v)$-mixing of T and Y in (3.2.13), which exists in both systems. Eq. (3.2.21), however, contains more v^2-corrections. That is, the first and the second terms in the curly bracket in Eq. (3.2.21) are peculiar only for the revolving case, and do not exist in Eq. (3.4.23). Thus we can say that the partition function for the revolving case contains not only the corrections due to the velocity v but also corrections due to the acceleration ω.

3.3 Effective potential at short distance

The one-loop partition function (3.2.21) of the open string gives the effective potential $\mathcal{V}(r,\omega) = -Z/\mathcal{T}$ between the two D0-branes induced by the exchange of a single closed string. In the present calculation, we are interested in the short distance behavior of the potential, namely $r \ll \sqrt{\alpha'}$. We expand the potential as a sum

$$\mathcal{V}(r,\omega) \;=\; \sum_{n=-1}^{\infty} \mathcal{V}_n(r,\omega), \tag{3.3.1}$$

where each term $\mathcal{V}_n(r,\omega)$ corresponds to the contribution of the open string states with the mass level $n+1$ to the partition function. n corresponds to the power of q in the q-expansion of the integrand of (3.2.21). Therefore, e.g., $\mathcal{V}_{-1}(r,\omega)$ is the contribution from the tachyon, and $\mathcal{V}_0(r,\omega)$ comes from the states which are massless when $v = r = 0$, and $\mathcal{V}_{n\geq 1}(r,\omega)$ are those from massive open string states. In the following, we ignore the tachyon contribution $\mathcal{V}_{-1}(r,\omega)$.

3.3.1 Massive contributions: $\mathcal{V}_{n\geq 1}(r,\omega)$

First let us consider contributions of the massive open string states. It is rather straightforward to evaluate the massive contributions $\mathcal{V}_n(r,\omega)$ with $n \geq 1$. For example, by expanding the partition function (3.2.21) with respect to q and take the linear terms in q, we obtain $\mathcal{V}_1(r,\omega)$ as

$$\begin{aligned} \mathcal{V}_1(r,\omega) \;&=\; -\int_0^\infty \frac{ds}{2s} f(s,r) e^{-2\pi s}\left[324 + \left(\frac{204}{\pi} - 648\epsilon_0\right) v^2 s + 432 v^2 s^2\right] \\ &\quad +\mathcal{O}(v^4), \end{aligned} \tag{3.3.2}$$

where

$$f(s,r) := \left(1 - \frac{1}{3}v^2\right)^{-\frac{1}{2}} \left(8\pi^2\alpha' s\right)^{-\frac{1}{2}} e^{-\frac{2r^2}{\pi\alpha'}s}. \tag{3.3.3}$$

The term $e^{-\frac{2r^2}{\pi\alpha'}s}$ is nothing but the effect of stretched open strings with distance $2r$, and the additional factor $\left(1 - \frac{1}{3}v^2\right)^{-\frac{1}{2}}$ comes from the v^2 correction to the momentum integration.

To determine $\mathcal{V}_1(r,\omega)$, we use the following formulae

$$\begin{aligned}
& -\int_0^\infty \frac{ds}{2s} f(s,r) e^{-2\pi s} s^k \\
= & -\left(1-\frac{1}{3}v^2\right)^{-\frac{1}{2}} (16\pi\alpha')^{-\frac{1}{2}} (2\pi)^{-k}\Gamma\left(k-\frac{1}{2}\right)\left(1+\frac{r^2}{\pi^2\alpha'}\right)^{-k+\frac{1}{2}} .
\end{aligned} \tag{3.3.4}$$

The integral with $k \leq \frac{1}{2}$ is defined by the analytic continuation for k. Using this formulae, we obtain

$$\begin{aligned}
\mathcal{V}_1(r,\omega) = & \frac{162}{\sqrt{\alpha'}}\left[1+\frac{1}{2}\left(1+\frac{-13+9\pi^2+27\pi\epsilon_0}{27}\alpha'\omega^2\right)\left(\frac{r^2}{\pi^2\alpha'}\right)\right. \\
& \left. -\frac{1}{8}\left(1+\frac{-44-18\pi^2+57\pi\epsilon_0}{27}\alpha'\omega^2\right)\left(\frac{r^2}{\pi^2\alpha'}\right)^2\right] \\
& +\mathcal{O}(\omega^4, r^6).
\end{aligned} \tag{3.3.5}$$

The potential $\mathcal{V}_1(r,\omega)$ is a generalization of the potential at rest. Since the energy eigenvalue of the first excited massive states are split by the interactions, $\mathcal{V}_1(r,\omega)$ is a sum of various contributions from the hypersplitted states. Note also that since our calculation is performed in the Wick rotated metric, the potential in the Lorentzian metric is obtained by replacing ω^2 with $-\omega^2$. The potential correctly reproduces the static limit at $\omega = 0$. The relevant part of the potential is written as a positive power series of $(r/l_{\rm str})^2$ and $(\omega/m_{\rm str})^2$;

$$\begin{aligned}
\mathcal{V}_2 \sim & \; m_{\rm str}\left(c_1 + c_2\left(\frac{w}{m_{str}}\right)^2+\cdots\right)\left(\frac{r}{l_{str}}\right)^2 \\
+ & \; m_{\rm str}\left(c_3 + c_4\left(\frac{w}{m_{str}}\right)^2+\cdots\right)\left(\frac{r}{l_{str}}\right)^4 + \cdots
\end{aligned} \tag{3.3.6}$$

where we defined $(2\pi\alpha')^{1/2} = l_{str} = 1/m_{\rm str}$ and $c_1 > 0$. As expected, the leading order potential is proportional to $m_{\rm str}^3 r^2$ and strongly attractive [16]. The second and higher terms in each bracket give angular-frequency corrections. The leading ω^2 correction to the effective potential is given by $m_{\rm str}\omega^2 r^2$. In the superstring case, the potential must vanish in the $\omega \to 0$ limit where D-branes are at rest. Thus this ω-dependent term gives the leading order massive state contribution to the potential.

Other contributions $\mathcal{V}_n(r,\omega)$ with $n \geq 2$ can be obtained similarly.

3.3.2 Massless contributions; $\mathcal{V}_0(r,\omega)$

Next, let us consider contributions from massless open string states $\mathcal{V}_0(r,\omega)$ given by the q^0 terms in the expansion of the partition function (3.2.21);

$$\mathcal{V}_0(r,\omega) \;=\; -\int_0^\infty \frac{ds}{2s} f(s,r)\left[24+\left(\frac{8}{\pi}-48\pi\epsilon_0\right)v^2 s+16v^2s^2\right]+\mathcal{O}(v^4). \qquad (3.3.7)$$

It is a sum of contributions from massless vector bosons $\alpha^\mu_{-1}|k\rangle$. This type of contributions to the effective potential, in particular in the case of D3-branes, would correspond to the Coleman-Weinberg type effective potential since its mass is given by the distance (i.e. moduli) between the branes. In the current setup, since the mass also depends on the angular frequency, the corresponding Coleman-Weinberg potential must be evaluated in presence of time-dependent scalar expectation value. Note that the result of the integral in (3.3.7) is singular at $r=0$ and the determination of $\mathcal{V}_0(r,\omega)$ needs some care.

The effective potential $\mathcal{V}_0(r,\omega)$ can be obtained if one knows the first four eigenvalues of $H_0(v)$, which we denote them by $E_i(k,r,\omega)$ $(i=0,\cdots,3)$;

$$\mathcal{V}_0(r,\omega) \;=-\int_0^\infty \frac{ds}{2s}\int\frac{dk}{2\pi}\left[\sum_{i=1}^{3} e^{-2\pi s(E_i(k,r,\omega)-1)}+21e^{-2\pi s E_0(k,r,\omega)}\right]. \qquad (3.3.8)$$

Since $H_0(v)$ commutes with H_0, these eigenvalues can be obtained by diagonalizing the upper-left four-by-four submatrix for $H_0(v)$. They are given in Appendix E and summarized as

$$\begin{aligned} H_0(v) \;&=\; \left(1-\frac{1}{3}v^2\right)\alpha' k^2+\frac{r^2}{\pi^2\alpha'}+1+v^2\epsilon_0 \\ &\quad+\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -\frac{4}{\pi^2}v^2 & -\frac{2\alpha'}{\pi r}kv^2 & \frac{2i}{\pi}v \\ 0 & -\frac{2\alpha'}{\pi r}kv^2 & \frac{\alpha'}{r^2}v^2 & \frac{2i\alpha'}{r}kv \\ 0 & -\frac{2i}{\pi}v & -\frac{2i\alpha'}{r}kv & \frac{\alpha'}{r^2}v^2 \end{bmatrix}+\mathcal{O}(v^3), \end{aligned} \qquad (3.3.9)$$

corresponding to the states $|\vec{\boldsymbol{n}};k\rangle$ with $N(\vec{\boldsymbol{n}})\le 1$.

The eigenvalues $E_i(k,r,\omega)$ $(i=0,\cdots,3)$ are given by

$$E_i(k,r,\omega) \;=\; \left(1-\frac{1}{3}v^2\right)\alpha' k^2+\frac{r^2}{\pi^2\alpha'}+1+v^2\epsilon_0+\mathcal{E}_i(k,r,\omega), \qquad (3.3.10)$$

where

$$\mathcal{E}_0(k,r,\omega) = -1+\mathcal{O}(v^4), \tag{3.3.11}$$

$$\mathcal{E}_1(k,r,\omega) = \frac{1}{\pi^2}v^2 h(k,r)^{-1}+\mathcal{O}(v^4), \tag{3.3.12}$$

$$\mathcal{E}_2(k,r,\omega) = 2v\frac{\sqrt{\alpha'}}{r}h(k,r)^{\frac{1}{2}}+\frac{1}{\pi^2}v^2\left[\frac{\pi^2\alpha'}{r^2}-2-\frac{1}{2}h(k,r)^{-1}\right]+\mathcal{O}(v^4), \tag{3.3.13}$$

$$\mathcal{E}_3(k,r,\omega) = -2v\frac{\sqrt{\alpha'}}{r}h(k,r)^{\frac{1}{2}}+\frac{1}{\pi^2}v^2\left[\frac{\pi^2\alpha'}{r^2}-2-\frac{1}{2}h(k,r)^{-1}\right]+\mathcal{O}(v^4), \tag{3.3.14}$$

and

$$h(k,r) := \alpha' k^2+\frac{r^2}{\pi^2\alpha'}. \tag{3.3.15}$$

The integral (3.3.8) is nothing but the Schwinger parameter representation of the partition function of $(1+0)$-dimensional particles whose energy is given by $2\pi E_i$.

Now we perform the integrations (3.3.8). For the eigenvalue $E_0(k,r,\omega)$, the integral can be performed easily. We obtain

$$\mathcal{V}_{00} = -\int_0^\infty \frac{ds}{2s}\int \frac{dk}{2\pi}\, e^{-2\pi u E_0(k,r,\omega)} = \sqrt{\frac{1+\epsilon_0\pi^2\alpha'\omega^2}{4\alpha'\left(1-\frac{1}{3}\omega^2 r^2\right)}\frac{r^2}{\pi^2\alpha'}}+\mathcal{O}(v^4). \tag{3.3.16}$$

The remaining integrals have more complicated forms. For the purpose of a rough estimate, we make the following approximation

$$h(k,r)^{\frac{1}{2}} \to \sqrt{\alpha'}|k|, \qquad v^2 h(k,r)^{-1}\to 0. \tag{3.3.17}$$

In this approximation, we obtain

$$\mathcal{V}_{01} = -\int_0^\infty \frac{ds}{2s}\int \frac{dk}{2\pi}\, e^{-2\pi s(E_1(k,r,\omega)-1)} \to \sqrt{\frac{1+\epsilon_0\pi^2\alpha'\omega^2}{4\alpha'\left(1-\frac{1}{3}\omega^2 r^2\right)}\frac{r^2}{\pi^2\alpha'}}, \tag{3.3.18}$$

$$
\begin{aligned}
\mathcal{V}_{02} + \mathcal{V}_{03} &= -\int_0^\infty \frac{ds}{2s} \int \frac{dk}{2\pi}\, e^{-2\pi s(E_2(k,r,\omega)-1)} - \int_0^\infty \frac{ds}{2s} \int \frac{dk}{2\pi}\, e^{-2\pi s(E_3(k,r,\omega)-1)} \\
&\to \sqrt{\frac{1+\left(\epsilon_0\pi^2 - 2\right)\alpha'\omega^2}{\alpha'\left(1-\dfrac{1}{3}\omega^2 r^2\right)} \frac{r^2}{\pi^2\alpha'}}. \qquad (3.3.19)
\end{aligned}
$$

The details of the integrations are given in Appendix F.

The effective potential induced by the massless modes becomes a sum of (3.3.16), (3.3.18) and (3.3.19),

$$\mathcal{V}_0 = \sum_{i=0}^{3} \mathcal{V}_{0i} \qquad (3.3.20)$$

and in the Lorentzian metric, ω^2 is replaced with $-\omega^2$. In the limit, $r \ll l_{str}$ and $\omega \ll m_{str}$, each term is written in the form of

$$m_{\rm str}\sqrt{\left(\frac{r}{l_{\rm str}}\right)^2 + C\left(\frac{r}{l_{\rm str}}\right)^2\left(\frac{\omega^2}{m_{\rm str}}\right)^2} \qquad (3.3.21)$$

At $\omega = 0$, the potential becomes r/l_s^2, which is proportional to the length of the stretched string.

The typical form of the effective potential induced by the massless state (3.3.21) is nothing but the Coleman-Weinberg potential for quantum particles in (1+0)-dimensions, which can be seen as follows. For comparison and future generalizations to Dp-branes, we will consider general cases in $(1+p)$-dimensions. In $(1+p)$-dimensional field theory, one-loop integral of a scalar field with mass m is given by

$$
\begin{aligned}
-\mathrm{Tr}\log(\Delta + m^2)^{-1/2} &= \int_\epsilon^\infty \frac{ds}{2s} \int \frac{d^{p+1}k}{(2\pi)^{p+1}} e^{-(k^2+m^2)s} \\
&\sim \int_\epsilon^\infty \frac{ds}{2s} s^{-\frac{p+1}{2}} e^{-m^2 s} \sim \begin{cases} (m^2)^{\frac{p+1}{2}} & p = \text{even} \\ (m^2)^{\frac{p+1}{2}}\log m^2 & p = \text{odd}. \end{cases} \qquad (3.3.22)
\end{aligned}
$$

If mass is given by vacuum expectation value of some scalar field ϕ, it gives the well-known Coleman-Weinberg potential. In our case, mass is generated by the distance r between D0-particles and the angular velocity ω. Thus the effective potential induced by open string massless states is given by the typical form (3.3.21), namely $p = 0$ case in (3.3.22).

3.4 D0-branes at constant velocities

In this section, we study a system of D0-branes at a constant relative velocity for comparison with the revolving case. One might think that the effective potential we obtained in the last section could be derived in a simpler manner by considering a D-brane system with a constant relative velocity. Namely, one may consider a system of two D0-branes whose trajectories are given by

$$\begin{cases} x_1(t) &= r, \\ y_1(t) &= vt, \end{cases} \qquad \begin{cases} x_2(t) &= -r, \\ y_2(t) &= -vt. \end{cases} \tag{3.4.1}$$

In the following, we refer this D0-brane system as the linear system, and the revolving D0-branes discussed so far as the revolving system. At the moment $t = 0$, the kinematic configuration of the linear system is the same as that of the revolving system, as far as the distance and the velocity are concerned. Therefore, one might expect that the effective potential for the revolving system would be obtained from the linear system at $t = 0$, at least at the order of perturbation we performed in the previous sections, and the corrections to the effective potential might coincide between these two systems. This is not the case since the effect of one D0-brane propagates with a finite speed and the effective potential depends on details of the trajectories of the other D0-brane at $t < 0$. Thus the interaction will depend not only on the velocity but on the acceleration at the moment of $t = 0$. In this section, we will show similarities and differences between these two systems.

Although the theory can be solved exactly in terms of a twisted boson [14], it is instructive to solve the worldsheet theory for the linear system in a non-trivial coordinate system similar to the rotational coordinate. We will observe that the resulting action has a similar form to the action (3.1.10) for the revolving system, which helps us to compare the two systems.

Consider the worldsheet theory of an open string with a boundary condition given by (3.4.1). We introduce the following coordinate system (in the Euclidean signature):

$$x' = x, \tag{3.4.2}$$

$$y' = y \cos \omega x - t \sin \omega x, \tag{3.4.3}$$

$$t' = y \sin \omega x + t \cos \omega x. \tag{3.4.4}$$

In this new coordinate system, the trajectories (3.4.1) of the D0-branes become simply

$$\begin{cases} x_1'(t) &= r, \\ y_1'(t) &= 0, \end{cases} \qquad \begin{cases} x_2'(t) &= -r, \\ y_2'(t) &= 0, \end{cases} \tag{3.4.5}$$

provided that ω satisfies

$$v = \tan r\omega. \tag{3.4.6}$$

Note that this relation between v and ω is the same as the one (3.1.1) for the revolving system up to $\mathcal{O}(v^2)$.

The worldsheet action in this coordinate system is

$$\begin{aligned} S = & -\frac{r^2}{4\pi\alpha'}\int d^2\sigma \Big[\partial_\alpha\tilde{T}\partial^\alpha\tilde{T} + \partial_\alpha\tilde{X}\partial^\alpha\tilde{X} + \partial_\alpha\tilde{Y}\partial^\alpha\tilde{Y} + \partial_\alpha\tilde{X}^i\partial^\alpha\tilde{X}_i \\ & +2v\partial_\alpha\tilde{X}(\tilde{T}\partial^\alpha\tilde{Y} - \tilde{Y}\partial^\alpha\tilde{T}) + v^2(\tilde{T}^2+\tilde{Y}^2)\partial_\alpha\tilde{X}\partial^\alpha\tilde{X} + \mathcal{O}(v^3)\Big]. \end{aligned} \tag{3.4.7}$$

Note that we have rescaled the fields by r. This is quite similar to the worldsheet action (3.1.10) for the revolving system.

The boundary conditions for $\tilde{X}$ and $\tilde{Y}$ are determined by (3.4.5). Define X by

$$\tilde{X} = x(\sigma) + X. \tag{3.4.8}$$

Then, X and $\tilde{Y}$ obey

$$X\Big|_{\sigma=0,\pi} = \tilde{Y}\Big|_{\sigma=0,\pi} = 0. \tag{3.4.9}$$

One can show that $\tilde{T}$ obeys the Neumann boundary condition;

$$\partial_\sigma\tilde{T}\Big|_{\sigma=0,\pi} = 0. \tag{3.4.10}$$

Note that, unlike the revolving system, we do not need any field redefinition like (3.1.14) to simplify the boundary condition for $\tilde{T}$. We define T by

$$\tilde{T} = \frac{t}{r} + T, \tag{3.4.11}$$

where t is the coordinate zero mode of $\tilde{T}$. In the following, we use Y instead of $\tilde{Y}$ for notational simplicity.

The Hamiltonian is given as

$$H_{\text{linear}} = H_0^{(l)} + vV_1^{(l)} + v^2V_2^{(l)} + \mathcal{O}(v^4), \tag{3.4.12}$$

where

$$
\begin{aligned}
H_0^{(l)} &= \int_0^\pi d\sigma \left[\frac{\pi\alpha'}{r^2}(\Pi_T^2 + \Pi_X^2 + \Pi_Y^2) + \frac{r^2}{4\pi\alpha'}\left\{(\partial_\sigma T)^2 + (\partial_\sigma X)^2 + (\partial_\sigma Y)^2\right\}\right] \\
&\quad + \frac{r^2}{\pi^2\alpha'} + \frac{v^2}{\pi^2\alpha'}t^2, \qquad (3.4.13) \\
V_1^{(l)} &= \int_0^\pi d\sigma \left[-\frac{2\pi\alpha'}{r^2}\Pi_X(\tilde{T}\Pi_Y - \Pi_T Y) + \frac{r^2}{2\pi\alpha'}\partial_\sigma X(\tilde{T}\partial_\sigma Y - \partial_\sigma T Y) + \frac{2r^2}{\pi^2\alpha'}\partial_\sigma T Y\right], \\
&\qquad (3.4.14) \\
V_2^{(l)} &= \int_0^\pi d\sigma \left[\frac{\pi\alpha'}{r^2}(\tilde{T}\Pi_Y - Y\Pi_T)^2 + \frac{r^2}{4\pi\alpha'}\left\{ (\partial_\sigma X)^2(\tilde{T}^2 + Y^2) + \frac{4}{\pi^2}(T^2 + Y^2) \right.\right. \\
&\qquad \left.\left. + \frac{8}{\pi^2 r}tT - \frac{4}{\pi}\partial_\sigma X(\tilde{T}^2 + Y^2)\right\}\right]. \qquad (3.4.15)
\end{aligned}
$$

Note that we have moved a term proportional to v^2t^2 from the interaction part $V_2^{(l)}$ to the free part $H_0^{(l)}$ since the sum $r^2 + v^2t^2$ is the distance squared between the D0-branes.[3] This is necessary to improve the behavior of the perturbative result for small v. From this Hamiltonian, using the improved perturbation theory in Appendix A, we can construct

$$
H_0^{(l)}(v) \ := \ H_0^{(l)} + vH_1^{(l)} + v^2 H_2^{(l)} + \mathcal{O}(v^4) \qquad (3.4.16)
$$

which is a counterpart of $H_0(v)$ (3.2.3) in the linear system. Note that the terms in the second line in (3.4.15) do not contribute to $H_2^{(l)}$, and we can drop them in the following discussions.

Recall that V_1 and V_2 in the Euclideanized theory for the revolving system are

$$
\begin{aligned}
V_1 &= \int_0^\pi d\sigma \left[-\frac{2\pi\alpha'}{r^2}\Pi_T(X\Pi_Y - \Pi_X Y) + \frac{r^2}{2\pi\alpha'}\partial_\sigma T(X\partial_\sigma Y - \partial_\sigma X Y) + \frac{2r^2}{\pi^2\alpha'}\partial_\sigma T Y\right], \\
&\qquad (3.4.17) \\
V_2 &= \int_0^\pi d\sigma \left[\frac{\pi\alpha'}{r^2}\left\{(X\Pi_Y - Y\Pi_X)^2 + x(\sigma)^2\Big(\Pi_Y^2 - \Pi_T^2\Big) \right.\right. \\
&\qquad \left.\left. + \frac{r^2}{4\pi\alpha'}\left\{ (\partial_\sigma T)^2(X^2 + Y^2) + \frac{8}{\pi^2}Y^2 + x(\sigma)^2\Big((\partial_\sigma T)^2 - (\partial_\sigma Y)^2\Big)\right\}\right]\right., \\
&\qquad (3.4.18)
\end{aligned}
$$

[3]Since the system of D0-branes at a constant relative motion does not have invariance under time translation, the zero-mode of T is no longer decoupled. It is a big difference from the revolving system noted in footnote 2, and we need to compare two systems at $t = 0$.

where only terms which contributes to H_1 and H_2 are shown above. The interaction terms $V_1^{(l)}$ and $V_2^{(l)}$ have a quite similar structure to their counterparts in the revolving system. However, there are some differences.

Apart from the potential term $v^2t^2/\pi^2\alpha'$ in the linear system in (3.4.13), there are two differences between the revolving and the linear systems. First, T and X are exchanged in many terms in the interactions V_1 and V_2. Since T and X obey different, Neumann and Dirichlet, boundary conditions, these terms give different contributions to $[H_2]_D$; namely the first line of (3.2.17) in the revolving system. Second, V_2 in the revolving case has the following terms

$$\int_0^\pi d\sigma \left[\frac{\pi\alpha'}{r^2}x(\sigma)^2(\Pi_Y^2 - \Pi_T^2) + \frac{r^2}{4\pi\alpha'}x(\sigma)^2\big((\partial_\sigma T)^2 - (\partial_\sigma Y)^2\big)\right], \tag{3.4.19}$$

which do not exist in the linear case $V_2^{(l)}$. The diagonal part of (3.4.19) which contributes to H_2 is given by the second line of (3.2.17) in $[H_2]_D$ in the revolving system. Therefore, the results for the linear system are actually different from those for the revolving system. There are other terms in V_2 which are absent in $V_2^{(l)}$, but they do not contribute to the results up to $\mathcal{O}(v^2)$.

After straightforward calculations in Appendix G, we obtain

$$H_0^{(l)} = \alpha' p^2 + \frac{v^2}{\pi^2\alpha'}t^2 + \frac{r^2}{\pi^2\alpha'} + \sum_{n=1}^{\infty}(N_{T,n} + N_{X,n} + N_{Y,n}) - \frac{1}{8}, \tag{3.4.20}$$

$$H_1^{(l)} = \frac{2i}{\pi}\sum_{n\neq 0}\frac{1}{n}\alpha_n^T\alpha_{-n}^Y. \tag{3.4.21}$$

For $[H_2^{(l)}]_D$ (see Appendix G), most of the terms are cancelled and the following terms for the wave function renormalization

$$[H_2^{(l)}] = \frac{\alpha'}{r^2}\left(\sum_n N_{T,n} + N_{X,n} + N_{Y,n} - \frac{3}{24}\right) \tag{3.4.22}$$

remains. It is the same as (3.2.18). Once the wave function is renormalized, it does not affect the final results of the partition function up to $\mathcal{O}(v^2)$. We have seen that the perturbative expansion of the linear system is similar to the revolving systems, but the above differences lead to the apparently different results.

Let us calculate the partition function of the linear system using $H_0^{(l)}(v)$. As we mentioned above, $H_0^{(l)}$ includes the term proportional to t^2. Therefore, the eigenvalues of the zero modes, t and p, no longer take continuous values. Instead, they form a harmonic oscillator with the angular frequency $2v/\pi$. The effect of $H_1^{(l)}$ in Eq. (3.4.21) to the partition function can be easily determined by diagonalizing it. We find that this gives shifts $\pm 2v/\pi$ to the eigenvalues of $H_0^{(l)}$. Since $H_2^{(l)}$ does not contribute to the partition function up to $\mathcal{O}(v^2)$, we see that $H_0^{(l)}(v)$ reproduces the correct partition function of the twisted boson [14]

$$\begin{aligned} Z &= \int_0^\infty \frac{ds}{2s} q^{-1} \frac{q^{\frac{v}{\pi}}}{1-q^{\frac{2v}{\pi}}} e^{-\frac{2r^2}{\pi\alpha'}s} \prod_{n=1}^\infty \left(1-q^{n+\frac{2v}{\pi}}\right)^{-1} \left(1-q^{n-\frac{2v}{\pi}}\right)^{-1} \left(1-q^n\right)^{-22} \\ &= \int_0^\infty \frac{ds}{2s} \frac{-2}{\sinh(2vs)} e^{-\frac{2r^2}{\pi\alpha'}s} \eta(is)^{-24} \left[1 - 2\pi v^2 \left(-\frac{4}{\pi} s^2 \sum_{n=1}^\infty \frac{2q^n}{(1-q^n)^2}\right)\right] + \mathcal{O}(v^4) \end{aligned} \tag{3.4.23}$$

up to $\mathcal{O}(v^2)$. This can be regarded as a consistency check of the calculation and the validity of the formalism we employed in this thesis. The v^2 corrections in the linear system give the third term in the v^2 corrections in the revolving system in Eq. (3.2.21). Note that the partition function itself is a function of v^2, but the energy eigenvalues are shifted by $\pm 2v/\pi$. This comes from the $\mathcal{O}(v)$-mixing between the world sheet variables, T and Y. Also note that after Wick rotation to the Lorentzian metric $v \to iv$, the integrand has zeros at an integer value of $2vs/\pi$, which generates an imaginary part in the partition function. It reflects open string pair creation [22]. Such imaginary part does not exist for the revolving system since the distance between D-branes are constant and non-adiabatic particle creation does not occur.

Chapter 4

Field theory calculation of effective potential for revolving D-branes

4.1 String Threshold Corrections in D-brane Models

We are interested in interaction potential between D-branes, which are relatively moving in a target space-time. At weak string coupling, we can obtain the potential by calculating the one-loop partition function of an open string stretched between the D-branes. For simple cases, we can quantize the stretched open strings and determine a closed form of the one-loop effective potential. But in many other cases where D-branes are accelerating, it is not possible to write the effective potential in a closed form, since open strings have complicated boundary conditions. For example, when two D-branes are revolving like a binary star, open string spectrum can be solved only perturbatively with respect to the relative velocity [26]. Thus, in order to calculate the potential between these D-branes, it is necessary to develop an alternative method. In this section, we propose an efficient method to obtain the interaction potential between generally moving D-branes, including threshold corrections of massive open string modes. The method was indicated in a seminal paper [25].

Schematically, the effective potential $V(R)$ is given as

$$V(R) \;=\; -\int_0^\infty \frac{dt}{t}\, e^{-\frac{R^2}{2\pi\alpha'}t} Z(t), \qquad (4.1.1)$$

where R is the distance between the D-branes. $Z(t)$ is the partition function of the stretched open string with the modulus (Schwinger parameter) t, where the factor $e^{-\frac{R^2}{2\pi\alpha'}t}$ due to the string tension is extracted. In many known examples the R-dependence only appears through $e^{-\frac{R^2}{2\pi\alpha'}t}$ and $Z(t)$ is R-independent, but generally it is not the case. The method for analyzing the effective potential at all ranges of R is based on a simple idea of separating the integration region into the UV region of $t \in [0,1]$ and the IR region of $t \in [1,\infty)$. The IR region for the open strings is dominantly given by the massless modes of open strings. If the modular transformation for $Z(t)$ can be explicitly performed, the UV region is mapped to the IR region of the dual closed strings and thus determines the large R behavior of the potential. But as we will see in the next section 4.1.1, it may also give sizable contributions to the small R behavior of the potential. They are the threshold corrections of infinitely many open string massive modes.

The UV region $[0,1]$ is dominantly described by the massless closed string modes, i.e., supergravity. The property holds even when the modular transformation is not explicitly given. Thus the open string one-loop amplitude of the UV region is approximated by using the supergravity calculations with an appropriate cutoff corresponding to $t \in [0,1]$.

4.1.1 Why are the string threshold corrections important?

In this section, we explain the method of *partial modular transformation.* It can provide a good approximation of the effective potential without directly performing the one-loop open string amplitude. Let us start from a toy example in the bosonic string theory. The model contains an open string tachyon and the potential is not well-defined for small R, but still it is a good example to see its efficiency and usefulness of the method. The effective potential of a pair of static parallel Dp-branes in the bosonic string theory is given by

$$V(R) \;=\; -\int_0^\infty \frac{dt}{t}\, e^{-\frac{R^2}{2\pi\alpha'}t}(8\pi^2\alpha' t)^{-\frac{1}{2}(p+1)}\eta(it)^{-24}. \qquad (4.1.2)$$

The integral contains contribution from the tachyon which makes the integral divergent at small R. We simply ignore it here. In the following sections, we will consider tachyon-free models whose effective potential is well-defined for all ranges of R.

First, let us consider the potential at large R. As usual, the asymptotic behavior of $V(R)$ at large R can be easily determined by using the modular transformation. Due to the exponential factor $e^{-\frac{R^2}{2\pi\alpha'}t}$, small t region dominantly contributes to the behavior at large R. After a modular transformation, we get

$$V(R) \;=\; -(8\pi^2\alpha')^{-\frac{1}{2}(p+1)}\int_0^\infty ds\, e^{-\frac{R^2}{2\pi\alpha'}s^{-1}} s^{\frac{1}{2}(p-25)}\eta(is)^{-24}. \qquad (4.1.3)$$

The large s region gives the dominant contribution at large R. Thus we expand the Dedekind eta function $\eta(is)$ as

$$\eta(is)^{-24} \;=\; \sum_{n=-1}^{\infty} d_n e^{-2\pi n s}, \qquad (4.1.4)$$

where $d_{-1} = 1$, $d_0 = 24$, $d_1 = 324$, $d_2 = 3200$ etc., and we retain only terms with small n. Again we ignore the closed tachyon contribution ($n = -1$) here. The $n = 0$ term gives

$$V(R) \;\sim\; -(4\pi)^{-\frac{1}{2}(p+1)}(2\pi\alpha')^{11-p}\Gamma(\tfrac{23-p}{2})R^{p-23}, \qquad (4.1.5)$$

which is a good approximation for large R, up to the tachyonic contribution. It corresponds to the exchange of massless closed string states, i.e., the dilaton and the graviton.

The behavior of $V(R)$ at small R, however, is more non-trivial. Similarly we can expand the $\eta(it)$ in eq.(4.1.2) by using the formula of eq.(4.1.4). Then, discarding the open string tachyon (d_{-1}), we may think that only the massless open string modes contribute to the behavior of $V(R)$ at small R. But actually it is not the case because all values of t, including large t, can contribute to the integral[1]. For example, the contribution from n-th excited states with the coefficient d_n in (4.1.4) gives the following contribution to the effective potential $V(R)$;

$$\begin{aligned}&\int_{1/\Lambda^2}^\infty \frac{dt}{t} e^{-\frac{R^2}{2\pi\alpha'}t} t^{-\frac{1}{2}(p+1)} e^{-2\pi n t} \\ &= \begin{cases} 2\Lambda - \sqrt{4\pi x} + \mathcal{O}(1/\Lambda), & (p=0) \\ \dfrac{\Lambda^4}{2} - \Lambda^2 x + \dfrac{3-2\gamma}{4}x^2 - \dfrac{1}{2}x^2\log(x/\Lambda^2) + \mathcal{O}(1/\Lambda^2), & (p=3) \end{cases}\end{aligned} \qquad (4.1.6)$$

where we have introduced the UV cutoff Λ (in unit of the string scale) and

$$x \;:=\; \frac{R^2}{2\pi\alpha'} + 2n\pi. \qquad (4.1.7)$$

The first and the second terms in the formula for $p = 3$ are nothing but the quartic and quadratic divergences in $d = 4$ quantum field theories. The third and the fourth terms

[1] It is known that a singular behavior of physical quantities can be extracted solely from the lightest open string states which become massless in the singular limit [25].

are the Coleman-Weinberg effective potential with a mass squared, $M^2 = x$. Since x increases with increasing n, massive open string modes give huge contributions to the low energy effective potential. Therefore, we cannot simply discard the contributions from massive open string states in determining the behavior of $V(R)$ for small R, even though they are heavy. We also need to take an appropriate treatment of the UV cutoff Λ appearing in the above formulas, which causes ambiguities of finite renormalizations of low energy observables.

In addition, the above calculations can provide behaviors of the potential $V(R)$ only for small R or large R regions. But, we are interested in the behavior of potential $V(R)$ in the whole ranges of R. In the next section, we propose an efficient method to evaluate $V(R)$ interpolating the small R and large R regions.

4.1.2 Partial modular transformation

We will now provide an efficient method to obtain a good approximation of $V(R)$ for all ranges of R. Interestingly, this method also resolves the issue of the UV divergences mentioned in the previous section.

Our method is based on the following rewriting of the potential of eq.(4.1.2):

$$\begin{aligned} V(R) &= -(8\pi^2\alpha')^{-\frac{1}{2}(p+1)}\left[\int_1^\infty \frac{dt}{t}\, e^{-\frac{R^2}{2\pi\alpha'}t} t^{-\frac{1}{2}(p+1)}\eta(it)^{-24} \right. \\ &\qquad \left. + \int_1^\infty ds\, e^{-\frac{R^2}{2\pi\alpha'}s^{-1}} s^{\frac{1}{2}(p-25)}\eta(is)^{-24}\right]. \end{aligned} \tag{4.1.8}$$

Here, we divided the integration region $[0,\infty)$ for t into $[0,1]$ and $[1,\infty)$, and perform the modular transformation for the first half region. An advantage of this rewriting is that, since $t, s \geq 1$ are satisfied, the Dedekind eta functions in the right-hand side can be replaced with a few terms in eq.(4.1.4) corresponding to light open (closed) string states, even for small R. For example,

$$\int_1^\infty \frac{dt}{t}\, e^{-\frac{R^2}{2\pi\alpha'}t} t^{-\frac{1}{2}(p+1)}\eta(it)^{-24} \;\to\; (\text{tachyon}) + 24\int_1^\infty \frac{dt}{t}\, e^{-\frac{R^2}{2\pi\alpha'}t} t^{-\frac{1}{2}(p+1)} \tag{4.1.9}$$

is a good approximation for all ranges of R.

Accuracy of the approximation can be estimated as follows. Using the expansion of eq.(4.1.4), the left-hand side of eq.(4.1.9) can be estimated as

$$\sum_{n=0}^\infty d_n \int_1^\infty \frac{dt}{t}\, e^{-\frac{R^2}{2\pi\alpha'}t} t^{-\frac{1}{2}(p+1)} e^{-2\pi n t} \;<\; \sum_{n=0}^\infty d_n e^{-2\pi n}\int_1^\infty \frac{dt}{t}\, e^{-\frac{R^2}{2\pi\alpha'}t} t^{-\frac{1}{2}(p+1)}. \tag{4.1.10}$$

Since $e^{-2\pi} = 0.001867$ is a very small number, the contributions from massive states are much smaller than those from the massless states. One might be worried that the exponential growth of d_n would invalidate this argument. However, it is known that d_n grows as $e^{4\pi\sqrt{n}}$, which is not large enough to overcome the suppression factor $e^{-2\pi n}$. The total contribution (without tachyon) to $V(R)$ turns out to be smaller than the massless state contribution times an infinite sum

$$\sum_{n=0}^{\infty} d_n e^{-2\pi n} \;=\; \eta(i)^{-24} - e^{2\pi} \;=\; 1.026\, d_0. \tag{4.1.11}$$

Therefore, the error due to discarding all massive open string states is less than 3%. Note that the smallness of the error is assured because we have introduced the cutoff at $t = 1$.

The second half in eq.(4.1.8) can be similarly approximated as

$$\sum_{n=0}^{\infty} d_n \int_1^{\infty} ds\, e^{-\frac{R^2}{2\pi\alpha'}s^{-1}} s^{\frac{1}{2}(p-25)} e^{-2\pi n s} \;<\; \sum_{n=0}^{\infty} d_n e^{-2\pi n} \int_1^{\infty} ds\, e^{-\frac{R^2}{2\pi\alpha'}s^{-1}} s^{\frac{1}{2}(p-25)}. \tag{4.1.12}$$

Therefore, retaining the contributions from massless closed string states gives a good approximation with the same accuracy as above. We emphasize that the accuracy of the approximation does not depend on R, so this approximation is valid for all range of R. If one needs a more precise approximation, one can retain the first excited states for both open and closed string channels. Then,

$$\sum_{n=1}^{\infty} d_n e^{-2\pi n} \;=\; \eta(i)^{-24} - e^{2\pi} - 24 \;=\; 1.019\, d_1 e^{-2\pi}, \tag{4.1.13}$$

shows that the expected error is about $0.019\, d_1 e^{-2\pi}/d_0 = 0.05\%$.

Several comments are in order. First the method is to sum the contributions from the open massless modes and the closed massless modes. If we did not introduce the Schwinger parameter cutoff, it would be a double counting. But as is clear from the procedure, it is not. Next, the expression is finite, as long as the square of the mass of the "tachyonic" state is positive. This implies that the issue of the UV divergences and ambiguities of finite renormalizations mentioned above are resolved by summing all open string massive contributions. Finally, in eq.(4.1.8), we separated the region of the moduli integration at $t = s = 1$, which is the fixed point of the modular transformation. If we separate the modulus at a different value, $t = 2$ and $s = 1/2$ for example, the suppression factor for the open string channel becomes $e^{-4\pi} = 3.487 \times 10^{-6}$ and the approximation becomes better. However, the suppression factor for the closed string channel becomes $e^{-\pi} = 0.04321$, giving a worse approximation. Hence the choice $t = s = 1$ seems to be optimal.

4.1.3 Another example: D3-branes at angle

As another example in the superstring case, we consider a pair of D3-branes at angle in Type IIB string theory. We follow the notations of the section 13.4 in [13, 14]. For $\theta_4 = 0$, the one-loop effective potential is given by

$$V(R) \;=\; -\int_0^\infty \frac{dt}{t}(8\pi^2\alpha' t)^{-\frac{1}{2}} e^{-\frac{R^2}{2\pi\alpha'}t} \frac{i\prod_{a=1}^4 \vartheta_{11}(\frac{i}{\pi}\theta'_a t, it)}{\eta(it)^3\prod_{a=1}^3 \vartheta_{11}(\frac{i}{\pi}\theta_a t, it)}, \tag{4.1.14}$$

where

$$\begin{aligned} \theta'_1 \;&:=\; \frac{1}{2}(\theta_1+\theta_2+\theta_3), \qquad \theta'_2 \;:=\; \frac{1}{2}(\theta_1+\theta_2-\theta_3), \\ \theta'_3 \;&:=\; \frac{1}{2}(\theta_1-\theta_2+\theta_3), \qquad \theta'_4 \;:=\; \frac{1}{2}(\theta_1-\theta_2-\theta_3). \end{aligned} \tag{4.1.15}$$

We assume that the angles θ_a are small so that the mass spectrum of the stretched open string is not largely deviated from that for the BPS configuration with $\theta_a = 0$. This integral is convergent for large t if

$$\sum_{a=1}^4 |\theta'_a| \;\leq\; \sum_{a=1}^3 |\theta_a| \tag{4.1.16}$$

is satisfied. This corresponds to the condition for the absence of open string tachyons. A solution of this condition is

$$\theta_1 \;=\; \theta_2 \;=\; \theta_3 \;=\; \theta \tag{4.1.17}$$

for any θ. The integral is always convergent for small t since there is no closed string tachyons. Therefore, the effective potential $V(R)$ with θ_a satisfying eq.(4.1.17) is well-defined for all ranges of R.

Similarly to the bosonic example in the previous section, the stringy result of eq.(4.1.14) can be approximated by a sum of open light and closed massless contri-

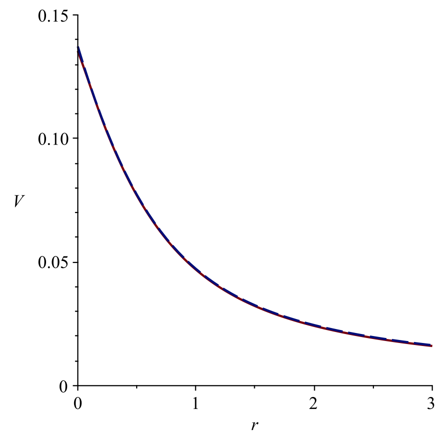

Figure 4.1: The effective potential $V(R)$ for D3-branes at angle with $\phi = \pi/12$ and $\alpha' = 1$. The exact effective potential in eq.(4.1.14) is drawn with red solid line. The blue broken line shows the potential using the approximate formula in eq.(4.1.18), which agrees very well with the exact one.

butions,

$$\tilde{V}(R) = \frac{1}{\sqrt{8\pi^2\alpha'}}\left[\int_1^\infty dt\, t^{-\frac{3}{2}} e^{-\frac{R^2}{2\pi\alpha'}t} \frac{2\sinh(\frac{3}{2}\phi t)\sinh^3(\frac{1}{2}\phi t)}{\sinh^3(\phi t)} + \int_1^\infty ds\, s^{-\frac{3}{2}} e^{-\frac{R^2}{2\pi\alpha'}s^{-1}} \frac{2\sin(\frac{3}{2}\phi)\sin^3(\frac{1}{2}\phi)}{\sin^3\phi}\right]. \quad (4.1.18)$$

This formula provides a good approximation of eq.(4.1.14). Indeed, the plot of $V(R)$ and $\tilde{V}(R)$ is shown in Figure 4.1. The error for the approximation is quite small for all range of R, which is difficult to see by naked eyes.

4.1.4 General Recipe

Let us summarize the method to give an efficient approximation to the effective potential $V(R)$ at all ranges of R. In the example in the previous section, given a modulus integral for the effective potential of interest in eq.(4.1.14), we divided the integration region into two, and performed the modular transformation for one of the integrals. Then, we retained only the contributions from light states to the integrals, open (nearly) massless states and closed massless states. The resulting expression gives a good approximation to the full effective potential for all ranges of R. Now we generalize the method to more complicated situations. To determine the approximate expression for the effective potential in the D-brane system, we did not actually need to know the full spectrum of the stretched open string. Only the information of the effective theories of the open massless states and the closed massless states are necessary. Namely the approximate effective potential is given as a sum of the SYM and the supergravity contributions;

$$\tilde{V}(R) \;=\; \tilde{V}_o(R) + \tilde{V}_c(R), \tag{4.1.19}$$

where $\tilde{V}_o(R)$ and $\tilde{V}_c(R)$ are schematically given as

$$\begin{aligned} \tilde{V}_o(R) \;&=\; -\int_1^\infty \frac{dt}{t} \int \frac{d^D k}{(2\pi)^D} \sum_{\text{light open}} e^{-2\pi t E_o(k) - \frac{R^2}{2\pi\alpha'} t}, \\ \tilde{V}_c(R) \;&=\; -\int_1^\infty ds \sum_{\substack{\text{massless} \\ \text{closed}}} \int \frac{d^{D'} k}{(2\pi)^{D'}} \langle B|l\rangle\langle l|B'\rangle e^{-2\pi s E_l(k) - \frac{R^2}{2\pi\alpha'} s^{-1}}. \end{aligned} \tag{4.1.20}$$

Here $|B\rangle$ and $|B'\rangle$ are the boundary states for the D-branes and $|l\rangle$ are the closed string massless states propagating between D-branes. Besides, the integration of the momentum and moduli parameter, the exponential factor $e^{-2\pi s E_l(k) - \frac{R^2}{2\pi\alpha'} s^{-1}}$, which arise as explained in 2.2.2, are explicitly expressed.

$\tilde{V}_o(R)$ is the Schwinger parametrization of the one-loop determinant for light open string states with the UV cutoff at the string scale. Thus, it can be obtained from the worldvolume theory of the D-brane system under consideration. Suppose that a D-brane configuration of interest is described by a classical field configuration in the worldvolume theory. Then, the one-loop calculation around the classical configuration gives the desired one-loop determinant. If we rewrite this in terms of the Schwinger parameter and put a suitable cutoff, we obtain $\tilde{V}_o(R)$ without performing any stringy calculations.

On the other hand, $\tilde{V}_c(R)$ is obtained from the massless closed string exchange between the D-branes. For general configurations of D-branes, see [27, 28]. This can be understood by noticing that the Schwinger parametrization of the massless propagator in D' dimensions is proportional to

$$\int \frac{d^{D'} k}{(2\pi)^{D'}} \frac{e^{ikx}}{k^2} \;=\; (4\pi)^{-\frac{D'}{2}} \int_0^\infty ds\, s^{-\frac{D'}{2}} e^{-\frac{1}{4} x^2 s^{-1}}. \tag{4.1.21}$$

The interaction vertex $\langle B|c\rangle$ of a D-brane to a closed string state is given by the corresponding Dirac-Born-Infeld (DBI) action with Chern-Simons (CS) term, provided that the trajectory of the D-brane is specified. Then, we obtain $\tilde{V}_c(R)$ by determining the appropriate tree amplitudes in supergravity, written in the Schwinger parametrization, and putting a suitable UV cutoff at the string scale.

Now, we have the recipe for a well-approximated expression to the full one-loop effective potential of a D-brane system, which includes threshold contributions from infinitely many massive open string modes:

1. Find a classical configuration in the worldvolume theory of a D-brane system under consideration, which corresponds to the D-brane configurations we are interested in. Then perform one-loop calculations around the classical configuration, and express the resulting one-loop determinant in terms of the Schwinger parameter t. UV cutoff in the t-integration is introduced.

2. Calculate the classical potential, mediated by the massless closed string states in supergravity, between the given configurations of D-branes. The coupling vertices are derived from the corresponding DBI action with CS term. Express the result in terms of the Schwinger parameter s and introduce the UV cutoff.

3. Normalize t and s such that the R-dependence appears in either of the form
$$e^{-\frac{R^2}{2\pi\alpha'}t}, \qquad e^{-\frac{R^2}{2\pi\alpha'}s^{-1}}, \tag{4.1.22}$$
and put the "cutoff" at $t, s = 1$. This corresponds to introducing the UV cutoff at the string mass scale $m_s = (2\pi\alpha')^{-1/2}$.

4. The sum of the above two expressions gives a good approximation $\tilde{V}(R)$ to the full effective potential $V(R)$ for all ranges of R.

If we interpret this recipe from the open string channel, what we have done amounts to summing all the stringy threshold corrections to the effective potential with a very good accuracy. This can be done by converting the threshold corrections into a contribution from the closed string massless states. The open-closed duality plays a key role in this calculation.

Note that the above calculations in the recipe can be performed even for *off-shell* configurations of D-brane systems, since the calculations are done in a gauge theory and a supergravity. Assuming that the open-closed duality holds off-shell, namely in the superstring field theory, we can expect that the above recipe enables us to study the effective potential for an off-shell configuration of D-branes.

We should also emphasize again that our recipe does not suffer from a double-counting problem, since we have introduced a cutoff in the Schwinger parameter integration. One can convince oneself of the validity of our recipe by examining the large R behavior of $\tilde{V}_o(R)$. Due to the factor $e^{-\frac{R^2}{2\pi\alpha'}t}$ and the cutoff at $t = 1$, $\tilde{V}_o(R)$ exponentially damps at large R, $V(R) \sim e^{-R^2/2\pi\alpha'}$. Therefore, the Newton potential appears only from $\tilde{V}_c(R)$. It is also important to notice that, although $\tilde{V}_c(R) \sim -1/R^{7-p}$ at large R, $\tilde{V}_c(R)$ is finite in the limit $R \to 0$, due to the cutoff at $s = 1$.

In the following, we apply our method to the revolving D-branes. In [26], we investigated this system based on a worldsheet theory of the stretched open string. However, there were several difficulties. One of them is the fact that the revolving configuration is off-shell at tree level, so that worldsheet calculations could have some troubles, for example an ambiguity for the renormalization procedure. We will see that our method in this thesis gives a quite reasonable finite result for the effective potential of the revolving D-branes, improving our previous investigation in [26].

4.2 Gauge theory calculations in revolving Dp-branes

In the following sections, we apply the recipe in the previous section to calculate the effective potential $\tilde{V}(R)$ for a system of Dp-branes revolving around each other. The distance $R = 2r$ is chosen to have mass dimension 1 and so is the radius of the revolution r. The radius with dimension -1 is given by $2\pi\alpha' r$. In this section, we set $2\pi\alpha' = 1$. To determine $\tilde{V}_o(2r)$, we perform a one-loop calculation around a suitable background field configuration in maximally supersymmetric Yang-Mills theory in $p+1$ dimensions[2]. For $\tilde{V}_c(2r)$, in the next section, we calculate the amplitude for the one-particle exchange between the revolving Dp-branes in Type II supergravity.

[2]The effective theory of Dp-branes is given by the DBI action with CS term and contains higher derivative corrections to the SYM theories. These higher dimensional vertices are suppressed by a factor $1/m_s$ and the corrections to the effective potential between D-branes can be neglected in the region $r < m_s$. Such suppression property is different from the threshold corrections of massive open string modes running in internal lines of the Feynman diagrams, which may give sizable contributions to the potential as discussed in section 4.1.1.

4.2.1 One-loop amplitude of SYM in revolving Dp-branes

We now apply the general formulae for the one-loop amplitude for the general background (2.2.34) to a specific background corresponding to the revolving Dp-branes ($p \leq 7$) in the 8-9 plane. The extended directions of Dp-branes are taken to be the same and thus always in parallel. The corresponding background configuration is given by

$$b_8 = r\cos\omega\tau, \qquad b_9 = r\sin\omega\tau, \tag{4.2.1}$$

and $b_I = 0$ otherwise, where ω is the angular frequency of the revolution and r is the radius of the circle on which the D-branes are revolving. Note that ω above has been analytically continued according to the Wick rotation of eq.(2.2.32). To recover the results in the Lorentzian signature, we will replace ω with $-i\omega$.

At first sight, since the quadratic action S_2 for the above stationary configuration b_I is τ-dependent, one may think that the one-loop determinant also depends on τ. Indeed, it is the case when D-branes are moving with a constant relative velocity [12,22]. But in the present situation, since the motion is stationary, the τ-dependence of the effective potential can be eliminated. By introducing new fields $\varphi_\pm$ defined by

$$\varphi_\pm := \frac{1}{\sqrt{2}}e^{\mp i\omega\tau}(\varphi_8 \pm \varphi_9), \tag{4.2.2}$$

the τ-dependence of the bosonic part of S_2 can be eliminated. Similarly, the τ-dependence of the fermionic part of S_2 can be eliminated by introducing

$$\theta := \exp\left[\frac{1}{2}\omega\tau\Gamma^{89}\right]\chi. \tag{4.2.3}$$

In terms of these new fields, the quadratic action S_2 becomes

$$S_2 = \frac{1}{g^2}\int d^{p+1}x\ [L_B + L_F + L_{\text{free}}] \tag{4.2.4}$$

where

$$\begin{aligned} L_B &= |(\partial_m + i\omega_m)\varphi_+|^2 + 4r^2|\varphi_+|^2 + |(\partial_m - i\omega_m)\varphi_-|^2 + 4r^2|\varphi_-|^2 \\ &\quad +|\partial_m a|^2 + 4r^2|a|^2 - 2\sqrt{2}r\omega\left(\varphi_- a^\dagger + \varphi_-^\dagger a - \varphi_+ a^\dagger - \varphi_+^\dagger a\right), \end{aligned} \tag{4.2.5}$$

$$L_F = i\bar{\theta}\Gamma_m\left(\partial_m - \frac{1}{2}\omega_m\Gamma^{89}\right)\theta - 2r\bar{\theta}\Gamma^8\theta, \tag{4.2.6}$$

$$L_{\text{free}} = |\partial_m a_i|^2 + 4r^2|a_i|^2 + \bar{c}_+\partial^2 c_+ - 4r^2\bar{c}_+c_+ + \bar{c}_-\partial^2 c_- - 4r^2\bar{c}_-c_-. \tag{4.2.7}$$

Here, we defined $\omega_m := \omega\delta_{m\tau}$ and $i = 1, 2, \cdots, p$. For notational simplicity, we used a instead of a_τ.

Now we can compute the one-loop determinant. Since the τ-dependence is no longer present, we can employ the momentum representation. Then, the bosonic Lagrangian L_B can be written as

$$\begin{aligned}
&(k^2+4r^2)\left|a(k) - \frac{2\sqrt{2}r\omega}{k^2+4r^2}(\varphi_-(k)-\varphi_+(k))\right|^2 \\
&+(k^2+\omega^2+4r^2+2\omega k_\tau)|\varphi_+(k)|^2 + (k^2+\omega^2+4r^2-2\omega k_\tau)|\varphi_+(k)|^2 \\
&-\frac{8(r\omega)^2}{k^2+4r^2}|\varphi_-(k)-\varphi_+(k)|^2,
\end{aligned} \tag{4.2.8}$$

where $k^2 = (k_m)^2$. The path integral for a can be easily performed, resulting in the determinant $\det(-\partial^2+4r^2)^{-1}$. To perform the path integral for $\varphi_\pm$, we need to diagonalize the matrix

$$\left(k^2+\omega^2+4r^2-\frac{8(r\omega)^2}{k^2+4r^2}\right)I_{2\times 2} + \begin{pmatrix} 2\omega k_\tau & \dfrac{8(r\omega)^2}{k^2+4r^2} \\ \dfrac{8(r\omega)^2}{k^2+4r^2} & -2\omega k_\tau \end{pmatrix}, \tag{4.2.9}$$

where $I_{2\times 2}$ is the diagonal matrix. Its eigenvalues are given by

$$E_{B\pm}(k) := k^2+\omega^2+4r^2-\frac{8(r\omega)^2}{k^2+4r^2} \pm \sqrt{4\omega^2k_\tau^2 + \left(\frac{8(r\omega)^2}{k^2+4r^2}\right)^2}. \tag{4.2.10}$$

Hence, the path integral for the bosonic field $\varphi_\pm$ gives $\det(E_{B+}(-i\partial))^{-1}\det(E_{B-}(-i\partial))^{-1}$.

Next, consider the fermionic part L_F. In the momentum representation, it can be written as

$$-\begin{pmatrix} \bar\theta_+ & \bar\theta_- \end{pmatrix}\begin{pmatrix} \Gamma_m(k_m+\frac{1}{2}\omega_m) & 2r\Gamma^8 \\ 2r\Gamma^8 & \Gamma_m(k_m-\frac{1}{2}\omega_m) \end{pmatrix}\begin{pmatrix} \theta_+ \\ \theta_- \end{pmatrix}, \tag{4.2.11}$$

where $\theta_\pm$ satisfy $i\Gamma^{89}\theta_\pm = \pm\theta_\pm$. The result of the path integral is given by the determinant of the following matrix

$$\begin{aligned}
&\begin{bmatrix} \left(k_m+\frac{1}{2}\omega_m\right)^2+4r^2 & 2r\omega\Gamma^{\tau 8} \\ 2r\omega\Gamma^{8\tau} & \left(k_m-\frac{1}{2}\omega_m\right)^2+4r^2 \end{bmatrix} \\
= &\begin{bmatrix} 1 & 0 \\ 0 & \Gamma^{8\tau} \end{bmatrix}\begin{bmatrix} \left(k_m+\frac{1}{2}\omega_m\right)^2+4r^2 & 2r\omega \\ 2r\omega & \left(k_m-\frac{1}{2}\omega_m\right)^2+4r^2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \Gamma^{\tau 8} \end{bmatrix},
\end{aligned} \tag{4.2.12}$$

which is the square of the matrix in eq.(4.2.11). The eigenvalues of this matrix are

$$E_{F\pm}(k) = k^2 + \frac{1}{4}\omega^2 + 4r^2 \pm \omega\sqrt{k_\tau^2 + 4r^2} \tag{4.2.13}$$

with multiplicity four for each of them. Therefore, the resulting determinant is given by $\det(E_{F+}(-i\partial))^4 \det(E_{F-}(-i\partial))^4$.

The remaining part L_{free} simply gives $\det(-\partial^2 + 4r^2)^{-5}$.

In summary, we obtain the one-loop determinant whose logarithm is given by

$$\begin{aligned}
&\log\Big[\det(-\partial^2 + r^2)^{-6} \det(E_{B+}(-i\partial))^{-1} \det(E_{B-}(-i\partial))^{-1} \\
&\qquad \times \det(E_{F+}(-i\partial))^4 \det(E_{F-}(-i\partial))^4\Big] \\
&= \int_{\Lambda^{-2}}^{\infty} \frac{dt}{t} \int \frac{d^{p+1}k}{(2\pi)^{p+1}} \Big[e^{-tE_{B+}(k)} + e^{-tE_{B-}(k)} \\
&\qquad -4\left(e^{-tE_{F+}(k)} + e^{-tE_{F-}(k)}\right) + 6\, e^{-t(k^2+4r^2)}\Big],
\end{aligned} \tag{4.2.14}$$

where Λ is a UV momentum cutoff with mass dimension 1. In section 4.2.2, it is fixed at $\Lambda = m_s$ following the recipe in section 4.1.4.

4.2.2 Effective potential $\tilde{V}_o(2r)$ from SYM

The contributions from the open light modes to the effective potential $\tilde{V}_o(2r)$ are given as a sum of the bosonic and fermionic ones,

$$\tilde{V}_o(2r) = \tilde{V}_{o,B}(2r) + \tilde{V}_{o,F}(2r) \tag{4.2.15}$$

where they are given by

$$\tilde{V}_{o,B}(2r) = -\int_{\Lambda^{-2}}^{\infty} \frac{dt}{t} \int \frac{d^{p+1}k}{(2\pi)^{p+1}} \left[e^{-tE_{B+}(k)} + e^{-tE_{B-}(k)} + 6e^{-t\left(k^2+4r^2\right)}\right], \tag{4.2.16}$$

$$\tilde{V}_{o,F}(2r) = 4\int_{\Lambda^{-2}}^{\infty} \frac{dt}{t} \int \frac{d^{p+1}k}{(2\pi)^{p+1}} \left[e^{-tE_{F+}(k)} + e^{-tE_{F-}(k)}\right]. \tag{4.2.17}$$

The ghost contribution is included in the bosonic part, $\tilde{V}_{o,B}(2r)$.

Let us now determine the cutoff parameter Λ following the recipe in section 4.1.4. The factor due to the string tension in the above expression is $\exp(-r^2 t/(\pi\alpha')^2)$ where

α' is recovered. Since $R = 2r$, the recipe tells us to choose the cutoff at $\tilde{t}_{\text{cutoff}} = 1$ when we rescale the variable t so that $\exp(-r^2 t/(\pi\alpha')^2) = \exp(-(2r)^2\tilde{t}/2\pi\alpha')$. Thus we choose $t = 2\pi\alpha'\tilde{t}$ and the momentum cutoff Λ can be fixed by the relation, $t_{\text{cutoff}} = \Lambda^{-2} = 2\pi\alpha'\tilde{t}_{\text{cutoff}} = 2\pi\alpha'$. Therefore, $\Lambda = m_s$.

Though Λ should be fixed as above, it is interesting to see the asymptotic behavior of $\tilde{V}_o(2r)$ at large r in the limit $\Lambda \to \infty$. By rescaling the integration variables, $\tilde{V}_o(2r)$ is rewritten as

$$\begin{aligned}\tilde{V}_o(2r) \quad = \quad & -r^{p+1}\int_{r^2\Lambda^{-2}}^{\infty}\frac{dt}{t}\int\frac{d^{p+1}k}{(2\pi)^{p+1}}e^{-t(k^2+4)}\Bigg[6 - 8e^{-\frac{\alpha^2}{4}t}\cosh\left(t\alpha\sqrt{k_\tau^2+4}\right) \\ & +2e^{-t\left(\alpha^2-\frac{8\alpha^2}{k^2+4}\right)}\cosh\left(t\sqrt{4\alpha^2k_\tau^2+\left(\frac{8\alpha^2}{k^2+4}\right)^2}\right)\Bigg], \qquad (4.2.18)\end{aligned}$$

where $\alpha := \omega/r$. This indicates that the $1/r$ expansion of this expression corresponds to the α expansion. We find that there are no terms with an odd power of α, as it should be, since the potential is independent of the direction of rotation with angular frequency ω. The $\mathcal{O}(\alpha^0)$ terms cancel trivially due to supersymmetry. The next $\mathcal{O}(\alpha^2)$ terms also cancel between $\tilde{V}_{o,B}(2r)$ and $\tilde{V}_{o,F}(2r)$;

$$-r^{p+1}\int_{r^2\Lambda^{-2}}^{\infty}\frac{dt}{t}\int\frac{d^{p+1}k}{(2\pi)^{p+1}}e^{-t(k^2+4)}\cdot 16\alpha^2\left(\frac{t}{k^2+4}-t^2\right)\xrightarrow{\Lambda\to 0} 0. \qquad (4.2.19)$$

Then, the leading non-vanishing terms are $\mathcal{O}(r^{p+1}\alpha^4)$, or equivalently $\mathcal{O}(v^4/r^{7-p})$, where $v := r\omega$. This behavior, which can be interpreted as the effective potential for Dp-branes at large r, is the same as the one expected from the supergravity calculation, which will be shown in the next section.

The effective potential $\tilde{V}_o(2r)$ in the Lorentzian signature is obtained by the replacement $\omega \to -i\omega$ after evaluating the integral in the Euclidean signature. Details are discussed in section 4.4. We briefly comment on some properties of the effective potential. For $r > l_s$, the effective potential represents an attractive force, which qualitatively agrees with the supergravity result. For small $r < l_s$, on the other hand, the effective potential behaves nontrivially as a function of r and ω. Many cancellations occur between bosons and fermions and we will show that, for $p = 3$, a minimum of the potential appears at a fixed value of ω.

4.3 Supergravity calculations in revolving Dp-branes

In this section, we calculate the classical potential $\tilde{V}_c(2r)$ by the one-particle exchanges of massless closed string modes.

We apply the result of eq.(2.2.79) to the revolving Dp-branes. The embedding functions X^μ and $\tilde{X}^\mu$ for the revolving Dp-branes are given by

$$\begin{aligned} X^\alpha &= \zeta^\alpha, \qquad X^8 = r\cos\omega\zeta^0, \qquad X^9 = r\sin\omega\zeta^0, \\ \tilde{X}^\alpha &= \tilde{\zeta}^\alpha, \qquad \tilde{X}^8 = -r\cos\omega\tilde{\zeta}^0, \qquad \tilde{X}^9 = -r\sin\omega\tilde{\zeta}^0. \end{aligned} \tag{4.3.1}$$

Inserting these functions into eq.(2.2.79) and performing some of the integrations, we obtain

$$\begin{aligned} \tilde{V}_c(2r) &= -\kappa_{10}^2 T_p^2 (4\pi)^{-\frac{10-p}{2}} \frac{v^4}{1+v^2} \int_{\tilde{\Lambda}^{-2}}^{\infty} ds\ s^{-\frac{10-p}{2}} \\ &\quad \times \int d\zeta \exp\left[-\frac{1}{4s}\left(\zeta^2 + 2r^2(1+\cos\omega\zeta)\right)\right](1+\cos\omega\zeta)^2, \end{aligned} \tag{4.3.2}$$

where $v = r\omega$. For details of the calculations, see Appendix H. Note that we have performed the Wick rotation of ζ^0 and $\tilde{\zeta}^0$ so that the integral is well-defined. ω is analytically continued as well.

Following the recipe in section 4.1.4, the cutoff $\tilde{\Lambda}$ is fixed as follows. The suppression factor due to the string tension in the above integrand is given by $\exp(-r^2/s)$. The cutoff is chosen at $\tilde{s} = 1$ when this factor is expressed as $\exp(-(2r)^2/(2\pi\alpha'\tilde{s}))$. Thus we take $s = \pi\alpha'\tilde{s}/2$ and $s_{\text{cutoff}} = \tilde{\Lambda}^{-2} = \pi\alpha'/2$. Hence $\tilde{\Lambda}$ needs to be fixed at $\tilde{\Lambda} = \sqrt{4/(2\pi\alpha')} = 2m_s$.

Several comments are in order. First, let us investigate the large r behavior of the potential with v fixed as a small value. The integral eq.(4.3.2) becomes

$$\tilde{V}_c(2r) = -(4\pi^2\alpha')^{3-p}(4\pi)^{-\frac{7-p}{2}}\Gamma\left(\tfrac{7-p}{2}\right)\frac{v^4}{r^{7-p}} + \mathcal{O}(v^6), \tag{4.3.3}$$

It reproduces the effective potential for two Dp-branes moving with the relative velocity $2v$ and the impact parameter $2r$, which can be calculated in string worldsheet theory (see eq.(13.5.7) in [14]). This provides a consistency check for our result in eq.(4.3.2).

We note that the potential from the supergravity calculation in eq.(4.3.3) is proportional to v^4/r^{7-p}. This behavior in case of $p = 0$ is well-known in the calculation of D0-brane scattering in the BFSS matrix theory [30, 31]. As mentioned at the end of section 4.2.2, the same potential can be reproduced from the SYM calculation, if we take the UV cutoff to infinity $\Lambda \to \infty$. In our calculation, Λ needs to be fixed at m_s in order to avoid the double counting, and the behavior of the Newton potential at large r is generated only by the supergravity calculation.

There is no chance of a bound state at large distances $r > l_s$. The potential is proportional to $-\omega^4 r^{p-3}$ and a very weak attractive potential. Indeed, if angular momentum of the revolving D-brane is conserved, ω is proportional to $1/r^2$. Then the potential is proportional to $-r^{p-11}$. Though it is attractive, the attractive force is too weak to balance with the repulsive centrifugal potential which is proportional to $1/r^2$.

Finally, note that the potential in eq.(4.3.2) is proportional to $v^4 = \omega^4 r^4$ and the $v^2 = \omega^2 r^2$ terms are cancelled. It is contrary to a naive expectation that there are large radiative corrections to the $\omega^2 r^2$ term in the effective potential: the supersymmetry breaking scale is given by ω. We come back to this property in the next section.

4.4 Effective potential at all ranges of r

We now investigate the behavior of the effective potential at all ranges of r by adding the contributions from SYM and supergravity; $\tilde{V}(2r) = \tilde{V}_o(2r) + \tilde{V}_c(2r)$. Here we assume that the angular frequency is small compared to the string scale, $\omega \ll m_s$ and the pair of Dp-branes are revolving slowly. We mainly focus on the $p = 3$ case. D0-branes are also interesting from the BFSS matrix theory point of view, since a threshold bound state is expected to arise [17, 18]. We leave its detailed analysis for future investigations.

In the following, we recover α' and the "distance" r is defined to have mass dimension 1. The gauge theory results turn out to be intact by regarding r as a quantity with mass dimension 1. For the supergravity result, we need to replace r with $2\pi\alpha' r$ in order to combine it with the gauge theory result for obtaining the effective potential in the worldvolume effective field theory.

The contributions from open light modes to the potential in Euclidean signature is given by a sum of these two contributions,

$$\begin{aligned}
\tilde{V}_{o,B}(2r) &= -\int_{\Lambda^{-2}}^{\infty} \frac{dt}{t} \int \frac{d^{p+1}k}{(2\pi)^{p+1}} e^{-(k^2+4r^2)t} \\
&\quad \times \left[6 + 2e^{-\omega^2 t + \frac{8(r\omega)^2}{k^2+4r^2}t} \cosh\left(t\sqrt{4\omega^2 k_\tau^2 + \left(\frac{8(r\omega)^2}{k^2+4r^2}\right)^2} \right) \right], \\
\tilde{V}_{o,F}(2r) &= 8\int_{\Lambda^{-2}}^{\infty} \frac{dt}{t} \int \frac{d^{p+1}k}{(2\pi)^{p+1}} e^{-(k^2+4r^2)t} e^{-\frac{1}{4}\omega^2 t} \cosh\left(t\sqrt{\omega^2 k_\tau^2 + 4(r\omega)^2} \right),
\end{aligned} \tag{4.4.1}$$

where the UV cutoff is fixed as $\Lambda = \sqrt{1/2\pi\alpha'} = m_s$. They are complicated integrals and the behaviors at small r and ω are nontrivial. We first look at some general behaviors. First, as discussed at the end of section 4.2.2, the potential is exponentially damped $\tilde{V}_o \sim e^{-4r^2/\Lambda^2}$ at large $r > \Lambda = m_s$. In the small r region, it behaves nontrivially, though the potential vanishes at $r = 0$. This can be seen by setting $r = 0$ in eq.(4.4.1).

$$\begin{aligned} \tilde{V}_{o,B}(2r) &= -\int_{\Lambda^{-2}}^{\infty} \frac{dt}{t} \int \frac{d^{p+1}k}{(2\pi)^{p+1}} e^{-k^2 t} \left[6 + 2e^{-\omega^2 t} \cosh\left(2\omega k_\tau t\right)\right], \\ \tilde{V}_{o,F}(2r) &= 8 \int_{\Lambda^{-2}}^{\infty} \frac{dt}{t} \int \frac{d^{p+1}k}{(2\pi)^{p+1}} e^{-k^2 t} e^{-\frac{1}{4}\omega^2 t} \cosh\left(\omega k_\tau t\right). \end{aligned} \tag{4.4.2}$$

Then, the ω dependence in each integral is removed by a shift of k_τ variable: $k_\tau \to k_\tau \pm \omega$ for the bosonic contribution and $k_\tau \to k_\tau \pm \omega/2$ for the fermionic contribution. We see that the bosonic and fermionic contributions are cancelled at $r = 0$ and $\tilde{V}_o(0) = 0$. Thus, the supersymmetry makes the potential non-singular at $r = 0$. Similarly the potential $\tilde{V}_o(2r)$ vanishes at $\omega = 0$.

The contributions from the supergravity $\tilde{V}_c(2r)$ in eq.(4.3.2) gives the Newton potential at large r and the threshold corrections to $\tilde{V}_o(2r)$ at small r. We discuss more details later, but here note that the potential is proportional to $v^4 = \omega^4 r^4$, and there are no terms proportional to v^2. As discussed in the introduction, since the supersymmetry breaking scale is given by ω, we may naively expect large threshold corrections proportional to $\omega^2 r^2$ from open string massive modes. In the present calculations, however, they are cancelled between infinitely many modes, and no terms like $\omega^2 r^2$ are generated for the moduli field r in the worldvolume effective field theory. It might be a stringy effect with infinitely many particles, and could not occur in ordinary quantum field theories. It is amusing if a similar mechanism would be applied to the hierarchy problem of the Higgs potential in the Standard Model.

4.4.1 Shape of the effective potential

In this section, in order to get an overview of the effective potential $\tilde{V}(2r)$, we expand the formulae in eq.(4.4.1) with respect to ω and perform the integrations. First, we look at $\tilde{V}_o(2r)$. From the integral representation of eq.(4.4.1), the expansion turns out to be an expansion with respect to ω/r. Thus the validity of the following expansion is restricted in the region $\omega < r$. This region is important for phenomenological applications [19, 20]. Details of the calculations are given in appendix I. After analytic continuation $\omega \to -i\omega$, we obtain the effective potential for $p = 3$ in the Lorentzian signature up to $\mathcal{O}(\omega^4)$;

$$\begin{aligned}\tilde{V}_{o,B}(2r) &= -\frac{\Lambda^4}{4\pi^2}\left[\left(1-\frac{4r^2}{\Lambda^2}\right)e^{-4r^2/\Lambda^2}+\frac{16r^4}{\Lambda^4}E_1(4r^2/\Lambda^2)\right]\\ &\quad -\omega^2\left[\frac{r^2}{\pi^2}e^{-4r^2/\Lambda^2}-\left(\frac{r^2}{\pi^2}+\frac{4r^4}{\pi^2\Lambda^2}\right)E_1(4r^2/\Lambda^2)\right]\\ &\quad -\omega^4\left[\left(\frac{1}{24\pi^2}+\frac{2r^2}{3\pi^2\Lambda^2}+\frac{10r^4}{3\pi^2\Lambda^4}\right)e^{-4r^2/\Lambda^2}-\left(\frac{6r^4}{\pi^2\Lambda^4}+\frac{40r^6}{3\pi^2\Lambda^6}\right)E_1(4r^2/\Lambda^2)\right]\\ &\quad +\mathcal{O}(\omega^6),\end{aligned}\tag{4.4.3}$$

$$\begin{aligned}\tilde{V}_{o,F}(2r) &= \frac{\Lambda^4}{4\pi^2}\left[\left(1-\frac{4r^2}{\Lambda^2}\right)e^{-4r^2/\Lambda^2}+\frac{16r^4}{\Lambda^4}E_1(4r^2/\Lambda^2)\right]\\ &\quad -\omega^2\left[\frac{r^2}{\pi^2}E_1(4r^2/\Lambda^2)\right]-\omega^4\left[\left(\frac{1}{48\pi^2}-\frac{r^2}{12\pi^2\Lambda^2}\right)e^{-4r^2/\Lambda^2}\right]+\mathcal{O}(\omega^6),\end{aligned}\tag{4.4.4}$$

where $E_n(x)$ are the exponential integral functions defined in eq.(I.0.9), whose small x behavior for $n = 1$ is given by

$$E_1(x) = -\gamma - \log x + x - \frac{x^2}{4} + \mathcal{O}(x^3). \tag{4.4.5}$$

Both of the bosonic and fermionic contributions have quartic and quadratic divergences but they are completely cancelled as expected. The sum gives the SYM contribution to the effective potential;

$$\begin{aligned}\tilde{V}_o(2r) &= -\frac{\omega^2r^2}{\pi^2}\left[e^{-4r^2/m_s^2}-\left(\frac{4r^2}{m_s^2}\right)E_1(4r^2/m_s^2)\right]\\ &\quad -\omega^4\left[\left(\frac{1}{16\pi^2}+\frac{7r^2}{12\pi^2m_s^2}+\frac{10r^4}{3\pi^2m_s^4}\right)e^{-4r^2/m_s^2}\right.\\ &\qquad\left.-\left(\frac{6r^4}{\pi^2m_s^4}+\frac{40r^6}{3\pi^2m_s^6}\right)E_1(4r^2/m_s^2)\right]+\mathcal{O}(\omega^6).\end{aligned}\tag{4.4.6}$$

Here we have replaced Λ by m_s. This formula is valid as far as the condition $\omega < r$ is satisfied. At large r it is exponentially damped and the potential is negative so that the corresponding force is attractive. From a general discussion, we saw that the potential vanishes $\tilde{V}_o(0) = 0$ at the origin. At small r (but $r > \omega$), the potential in eq.(4.4.6) behaves like the inverted harmonic potential, $-\omega^2r^2/\pi^2$, and the corresponding force is repulsive for a fixed ω. The next order term proportional to ω^4 seems to give a constant value at $r = 0$ and contradict with the general discussion $\tilde{V}_o(0) = 0$. However, it is simply because $r = 0$ at fixed ω is out of the validity region of the ω expansion in eq.(4.4.6).

In the region $r < \omega$, we can perform a different approximation of the integral for $\tilde{V}_o(2r)$ to estimate the shape of the potential. We set $r = \beta\omega$ and expand $\tilde{V}_o(2r)$ in terms of β. As a result, we obtain

$$\tilde{V}_o = \frac{\beta^2\omega^4}{\pi^2}\left(-\frac{m_s^2}{\omega^2}\left(1 - E_2(\omega^2/m_s^2)\right) + \int_{\omega^2/m_s^2}^{\infty}\frac{dt}{t}e^{-t/4}F(\tfrac{1}{2},\tfrac{3}{2};\tfrac{t}{4})\right) + \mathcal{O}(\beta^4). \tag{4.4.7}$$

Details of the calculations are given in the appendix J. The leading order behavior with respect to ω/m_s is the same as the above ω-expansion

$$\tilde{V}_o(2r) \sim -\frac{\omega^2 r^2}{\pi^2}. \tag{4.4.8}$$

Thus, as far as the leading behavior is concerned, eq.(4.4.6) seems to give a good approximation at small r.

The contributions from the supergravity calculations in eq.(4.3.2) can be also obtained by the ω expansion. In this case, the expansion is with respect to $v = \omega r$, and the validity holds as far as $\omega r < 1$ (here, the mass dimension of r is taken to be -1). Recall that, in this section, r is defined to have mass dimension 1. Thus we need to multiply r in eq.(4.3.2) by $1/2\pi\alpha' = m_s^2$. After expanding eq.(4.3.2) with respect to ω, the integrals can be easily performed and we obtain

$$\tilde{V}_c(2r) = -\frac{\omega^4}{16\pi^2}\left[1 - \left(1 + 4r^2/m_s^2\right)e^{-4r^2/m_s^2}\right] + \mathcal{O}(\omega^6) \tag{4.4.9}$$

At large r, it is approximated by

$$\tilde{V}_c(2r) \sim \frac{-\omega^4}{16\pi^2} = -\frac{v^4}{16\pi^2 r^4} \tag{4.4.10}$$

with $v = \omega r$, which reproduces the Newton-like potential for D3-branes in D=10. At small r, it becomes

$$\tilde{V}_c(2r) \sim -\frac{\omega^4 r^4}{2\pi^2 m_s^4}. \tag{4.4.11}$$

Note that a naively expected term $v^2 = \omega^2 r^2$ is absent and the potential starts from v^4. It has been known in the large r behavior of the D-brane potential, but it has also

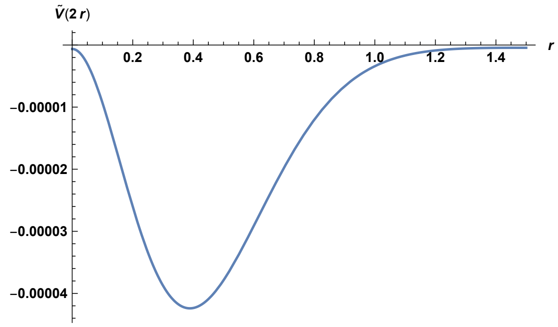

Figure 4.2: The shape of the effective potential $\tilde{V}(2r)$ (the sum of eqs.(4.4.6) and (4.4.9)) with $\omega = 0.1$ and $\Lambda = 1$.

an important implication in the small r behavior of the effective potential in the field theory of D-branes.

Now we sum the contributions from SYM and supergravity. The shape of the potential $\tilde{V}(2r)$ is drawn in figure 4.2 with ω fixed at 0.1. At large r, Newton potential is reproduced and the corresponding force is attractive. At small r, there is a minimum of the potential and the force is repulsive. In the next section, we briefly argue a possibility of a bound state. In the intermediate region of r, both of the SYM and supergravity contribute to the potential.

4.4.2 Can the revolving D3-branes form a bound state?

Finally we briefly argue whether there exists a bound state of revolving D3-branes with the potential $\tilde{V}(2r)$ studied above. Assume that the angular momentum is conserved and there are no quantum radiation. We then need to take into account the effect of the centrifugal potential for the D3-branes. Also it is necessary to study the behavior of the potential with fixing the angular momentum L of the D3-branes per unit volume, instead of the angular frequency ω.

The potential we need to study is given by

$$U(2r) := \frac{L^2}{4T_3 r^2} + \tilde{V}(2r) \tag{4.4.12}$$

with ω replaced with $L/T_3 r^2$. The relative distance and reduced mass for a unit volume is given by $2r$ and $T_3/2 = m_s^4/4\pi g_s$. Since our calculations are based on the one-loop string calculations, the string coupling constant should be smaller than 1. In such a situation, the potential $U(2r)$ behaves like in Figure 4.3, and there is no minimum, because the centrifugal potential is more dominant than the induced potential by the one-loop calculations. It excludes a possibility of forming a bound state for revolving two D3-branes as long as the string coupling is weak.

The situation is changed if we consider a stack of N D3-branes revolving around each other. Suppose that each of the revolving D3-branes are replaced with N D3-branes. Then, $\tilde{V}(2r)$ is multiplied by N^2, since there are N^2 open strings stretched between the two sets of D3-branes. On the other hand, the centrifugal potential is multiplied by N. Therefore, the potential $U(2r)$ is modified as

$$U_N(2r) := \frac{NL^2}{4T_3 r^2} + N^2\tilde{V}(2r). \tag{4.4.13}$$

For a sufficiently large N, the behavior of $U_N(2r)$ changes to the figure drawn in Figure 4.4, which is qualitatively different from $U(2r)$. The potential at small r in eq.(4.4.8) shows that the potential $U_N(r)$ falls off as r^{-2} for small r after replacing ω by $1/r^2$. It is still questionable if a stable bound state exists, but it is amusing that the potential shows different behavior at small r.

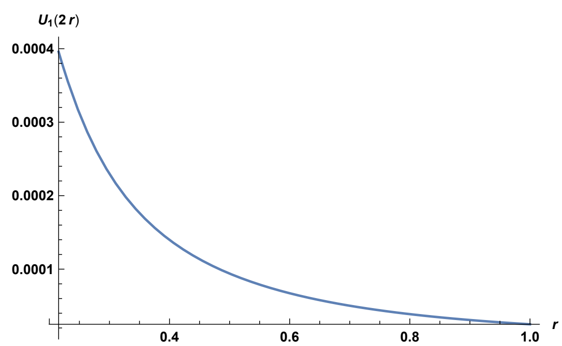

Figure 4.3: The shape of the potential $U(2r)$ with $T_3 = 1$, $L = 0.01$ and $\Lambda = 1$.

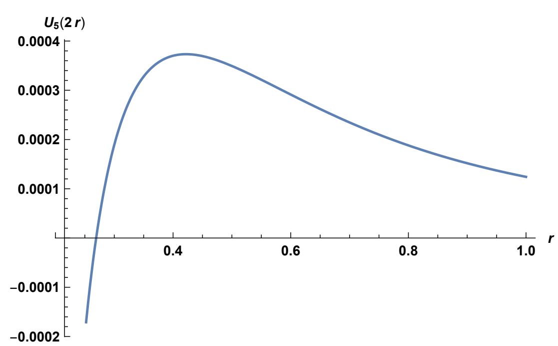

Figure 4.4: The shape of the potential $U_N(2r)$ with $T_3 = 1$, $L = 0.01$ and $\Lambda = 1, N = 5$.

Chapter 5

Conclusions

In this thesis, we have investigated the dynamics of revolving D-branes. The potential arising from the quantized open strings which have endpoints on the branes governs their motion. The obstacle to calculating the D-brane potential mainly comes from the difficulty to quantize open strings under the boundary condition corresponding to various D-brane motions. In order to overcome the obstacle and carry out the calculation, we can (i) quantize open strings by employing the appropriate coordinate in which the boundary conditions of the open strings become simpler or (ii) perform the calculation by utilizing the low energy field theory without quantizing strings exactly. We devised the two methods which founded on each of the directions.

First, we introduced the improved perturbation method as argued in Chapter 3. In the use of this method, we can eliminate the secular terms in the Dyson series for any order of perturbation. In the bosonic string theory, we have calculated the one-loop effective potential for the revolving D-branes. By employing the rotational coordinate, after some field redefinitions, the boundary conditions become simple. Then the world-sheet action becomes nonlinear in exchange for the simple boundary conditions. Using our method, we can analyze this system perturbatively in the absence of the secular terms and calculate the partition function, which gives the D-brane effective potential, without solving the equations of motion. In the bosonic case, it is found that the potential is the order of the string scale since there is no cancelation due to the supersymmetry.

We also calculated the D-brane potential by summing the contributions from both SUGRA and SYM theory in the use of the partial modular transformation. By performing the partial modular transformation, it is found that string amplitudes can be approximated by the sum of the one loop effective potential of the massless open strings and the tree amplitude of massless closed strings with the appropriate cut off. These contributions can be obtained by field theory calculation in both of the SYM for the open strings and the SUGRA for the closed strings. Therefore, we do not need to quantize the strings exactly to evaluate the D-brane potential. In fact, we have calculated the effective potential for revolving D-branes by using the method.

In consequence, we have obtained the one-loop effective potential for the revolving D-branes. Our calculation aims to investigate the possibility of the bound state consisting of revolving D-branes and argue the origin of the electroweak scale of the four-dimensional theory on the D-brane. Our calculation gives the one-loop correction to the D-brane potential for a certain classical motion, namely a revolution. The revolving motion is not the classical solution of the equation of motion of D-branes. Despite this, to verify the existence of the bound state, it is efficient to evaluate the quantum correction. We can expect that the potential becomes to have a stable point by balancing the quantum correction and the centrifugal potential which arises from the classical motion. The result in which centrifugal potential and our one-loop correction are combined is shown at the end of Chapter 5 with taking into account the conservation of the angular momentum. According to the result, although we can see that attractive force works at the small distance region for N stacks of D-branes, the existence of the bound state is still not revealed. As a future issue, we need to come back to this problem by taking into account the quantization of the collective coordinates of the D-branes.

Our motivation is to construct a model that breaks the supersymmetry at high energy and has the Higgs potential that realizes the stable EW scale. To this goal, the interesting fact is found in our investigation. The fact is that there is no quadratic term of the velocity in the contribution of the SUGRA side, which is the threshold correction of the massive states in the open string point of view, to the D-brane potential or the Higgs potential for any trajectory of D-branes. In the system that preserve the supersymmetry when branes are static, as our set up, the velocity determines the SUSY breaking scale. Therefore, this fact means that there is no correction from the infinitely many massive modes to the Higgs mass, which is usually expected to be large as the SUSY breaking scale becomes high energy and suggests the possibility that the Hierarchy is consistent with the SUSY breaking at high energy.

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Appendix A

Improved perturbation theory

In this appendix, we briefly review the improved perturbation theory developed in [15]. Consider a Hamiltonian of the form

$$H = H_0 + \lambda V_1 + \lambda^2 V_2. \tag{A.0.1}$$

In the interaction picture, we regard H_0 as the free part and the remaining part as the perturbation. Instead, we decompose H as

$$H = H_0(\lambda) + \lambda V(\lambda), \tag{A.0.2}$$

where

$$\begin{aligned} H_0(\lambda) &:= H_0 + \lambda H_1 + \lambda^2 H_2 + \lambda^3 H_3 + \cdots && \text{(A.0.3)} \\ V(\lambda) &:= V_1 - H_1 + \lambda(V_2 - H_2) - \lambda^2 H_3 - \cdots && \text{(A.0.4)} \end{aligned}$$

The operators H_1, H_2, H_3 etc. are to be determined later. One can construct a perturbation theory based on this decomposition up to any desired orders of perturbation with respect to λ. The time evolution of operators are given by a unitary operator $U(\lambda, t)$ which satisfy

$$\frac{d}{dt}U(\lambda,t) = -i\lambda V(\lambda,t)U(\lambda,t), \qquad V(\lambda,t) := e^{iH_0(\lambda)t}V(\lambda)e^{-iH_0(\lambda)t}. \tag{A.0.5}$$

The solution satisfying $U(\lambda, 0) = I$ is

$$\begin{aligned} U(\lambda,t) &= I + (-i\lambda)\int_0^t dt_1\, V(\lambda,t_1) \\ &\quad +(-i\lambda)^2\int_0^t dt_1\int_0^{t_1} dt_2\, V(\lambda,t_1)V(\lambda,t_2) \\ &\quad +(-i\lambda)^3\int_0^t dt_1\int_0^{t_1} dt_2\int_0^t dt_3\, V(\lambda,t_1)V(\lambda,t_2)V(\lambda,t_3) \\ &\quad +\mathcal{O}(\lambda^4). \end{aligned} \tag{A.0.6}$$

As proved in [15], this operator does not have any secular terms provided that H_n are appropriately chosen. To show the explicit expressions for them, let us introduce operators $V_{i,a}$ $(i = 1, 2)$ which satisfy

$$V_i = \sum_a V_{i,a}, \qquad [H_0, V_{i,a}] = \omega_a V_{i,a}. \tag{A.0.7}$$

Explicitly, they are given as

$$V_{i,a} = \sum_{E_m - E_n = \omega_a} |m\rangle\langle m|V_i|n\rangle\langle n|, \tag{A.0.8}$$

where $|n\rangle$ are the eigenstates of H_0. In terms of these operators, H_1 and H_2 are given as

$$H_1 = V_{1,0}, \tag{A.0.9}$$

$$H_2 = V_{2,0} - \sum_{a,b\neq 0} \frac{1}{\omega_a} \delta_{\omega_a+\omega_b} V_{1,b} V_{1,a}. \tag{A.0.10}$$

Note that these operators commute with H_0. For systematic derivations of higher H_n $(n \geq 3)$, see [15].

As a result, it turns out that the time-dependence of all operators in this system is given in terms of $e^{iH_0(\lambda)t}$ to all orders of perturbation theory. Therefore, the eigenvalues of $H_0(\lambda)$ should be the same as those of the full Hamiltonian H to all orders in λ.

Appendix B

Mode expansions

The worldsheet theory (3.1.10) becomes a free theory when $v = 0$. In this case, the quantization of the worldsheet fields are straightforward. In the Euclidean theory, their mode expansions at $\tau = 0$ are as follows:

$$T = \frac{t}{r} + i\frac{\sqrt{2\alpha'}}{r}\sum_{n\neq 0}\frac{\alpha_n^T}{n}\cos n\sigma, \tag{B.0.1}$$

$$X = \frac{\sqrt{2\alpha'}}{r}\sum_{n\neq 0}\frac{\alpha_n^X}{n}\sin n\sigma, \tag{B.0.2}$$

$$Y = \frac{\sqrt{2\alpha'}}{r}\sum_{n\neq 0}\frac{\alpha_n^Y}{n}\sin n\sigma, \tag{B.0.3}$$

$$\Pi_T = \frac{r}{\pi}p + \frac{r}{\pi\sqrt{2\alpha'}}\sum_{n\neq 0}\alpha_n^T\cos n\sigma, \tag{B.0.4}$$

$$\Pi_X = -\frac{ir}{\pi\sqrt{2\alpha'}}\sum_{n\neq 0}\alpha_n^X\sin n\sigma, \tag{B.0.5}$$

$$\Pi_Y = -\frac{ir}{\pi\sqrt{2\alpha'}}\sum_{n\neq 0}\alpha_n^Y\sin n\sigma. \tag{B.0.6}$$

Note that the radius r appears above because of the rescaling $\tilde{X}^\mu \to r\tilde{X}^\mu$ mentioned below (3.1.9). The commutation relations of the mode operators are

$$[\alpha_n^T, \alpha_m^T] = [\alpha_n^X, \alpha_m^X] = [\alpha_n^Y, \alpha_m^Y] = n\delta_{n+m}. \tag{B.0.7}$$

Appendix C

Calculation of $[H_2]_D$

In the appendix, we calculate the diagonal part of H_2, which is given as the following sum;

$$[H_2]_D = [V_2]_D - \sum_{m\neq 0} \frac{1}{m}[V_{1,-m}V_{1,m}]_D. \tag{C.0.1}$$

The calculations are complicated since V_2 defined by (3.1.23) and $(V_1)^2$ contain quartic terms in the world sheet variables, and we need to appropriately regularize the infinite sum appearing in the intermediate states. Let us first look at the following term in V_2:

$$\int_0^\sigma d\sigma\, \frac{\pi\alpha'}{r^2} X^2 \Pi_Y^2. \tag{C.0.2}$$

The diagonal part of this operator is obtained by substituting X^2 and Π_Y^2 with their diagonal parts given by

$$\begin{aligned} \left[X(\sigma)^2\right]_D &= \frac{2\alpha'}{r^2}\sum_{n\neq 0}\frac{1}{n}D_{X,n}\sin^2 n\sigma, & \text{(C.0.3)} \\ \left[\Pi_Y(\sigma)^2\right]_D &= \frac{r^2}{2\pi^2\alpha'}\sum_{n\neq 0} nD_{Y,n}\sin^2 n\sigma, & \text{(C.0.4)} \end{aligned}$$

where $D_{\mu,n}$ are defined by

$$D_{\mu,n} := \frac{1}{n}N_{\mu,n} + \theta(n), \qquad (\mu = T, X, Y) \tag{C.0.5}$$

and $\theta(n)$ is the step function;

$$\theta(n) := \begin{cases} 1, & (n > 0) \\ 0. & (n \leq 0) \end{cases} \tag{C.0.6}$$

A suitable regularization of the summations is necessary. As a result of the substitution, we obtain (see (C.0.26))

$$\begin{aligned}
&\left[\int_0^\pi d\sigma\, \frac{\pi\alpha'}{r^2} X^2 \Pi_Y^2\right]_D = \int_0^\pi d\sigma\, \frac{\pi\alpha'}{r^2} [X^2]_D [\Pi_Y^2]_D \\
= \;& \frac{\alpha'}{4r^2} \sum_{n,m\neq 0} \frac{m}{n} D_{X,n} D_{Y,m} + \frac{\alpha'}{4r^2} \sum_{n\neq 0} D_{X,n} D_{Y,n} - \frac{\alpha'}{8r^2} \sum_{n\neq 0} D_{X,n}.
\end{aligned} \tag{C.0.7}$$

By using the formulae (C.0.33), we see that the expression contains the following term

$$\frac{\alpha'}{4r^2} \left(2N_X(2) + \zeta(1)\right) \left(2N_Y(0) + \zeta(-1)\right), \tag{C.0.8}$$

where

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{C.0.9}$$

is the Riemann zeta function. Most of the infinite sums can be regularized by using $\zeta(-1) = -1/12$ or $\zeta(0) = -1/2$, but $\zeta(1)$ can not be regularized by the zeta function regularization. It will be, however, observed that this kind of divergences cancels among various terms within H_2.

Other terms in $[V_2]_D$ can be obtained in a similar manner. Using the diagonal parts of the products given in Appendix C.0.1, we find

$$\begin{aligned}
&[V_2]_D = \frac{\alpha'}{2r^2}\zeta(1)\left(2N_T(0) + N_X(0) + N_Y(0)\right) + \frac{\alpha'}{r^2}\zeta(1)\zeta(-1) \\
&+\frac{\alpha'}{r^2}\Bigg[N_{XY}(2) + \frac{1}{2}(N_{TX}(2) + N_{TY}(2)) \\
&+N_X(2)(N_Y(0) + N_T(0)) + N_Y(2)(N_X(0) + N_T(0)) \\
&+\zeta(-1)(N_X(2) + N_Y(2)) + \frac{1}{2}N_T(1) + \frac{3}{4}(N_X(1) + N_Y(1)) + \frac{1}{2}\zeta(0)\Bigg] \\
&-\frac{2}{\pi^2}(N_T(2) - N_Y(2)) - \frac{1}{3}\alpha' p^2.
\end{aligned} \tag{C.0.10}$$

For details of the calculation, see Appendix C.0.2.

Next we evaluate the second term in H_2 (3.2.4). Since the calculation is similarly performed, it is given in the Appendix C.0.3. The final result is

$$
\begin{aligned}
& -\sum_{m\neq 0}\frac{1}{m}[V_{1,-m}V_{1,m}]_D \\
= \; & -\frac{\alpha'}{2r^2}\zeta(1)\left(2N_T(0)+N_X(0)+N_Y(0)\right)-\frac{\alpha'}{r^2}\zeta(1)\zeta(-1)-\frac{2}{\pi^2}\zeta(1) \\
& +\frac{\alpha'}{r^2}\Big[N_{XY}(2)-\frac{1}{2}(N_{TX}(2)+N_{TY}(2)) \\
& -N_X(2)(N_Y(0)+N_T(0))-N_Y(2)(N_X(0)+N_T(0)) \\
& -\zeta(-1)(N_X(2)+N_Y(2))-\frac{1}{2}N_T(1)+\frac{1}{4}(N_X(1)+N_Y(1)) \\
& +N_X(0)+N_Y(0)+N_T(0)+\frac{1}{2}\zeta(0)^2\Big]-\frac{2}{\pi^2}(N_T(2)+N_Y(2)). \qquad \text{(C.0.11)}
\end{aligned}
$$

C.0.1 Diagonal parts

To derive the explicit form of $[H_2]_D$, we need the diagonal parts of products of fields. They are given as follows. For products including T and Π_T,

$$[\Pi_T(\sigma)\Pi_T(\sigma')]_D = \frac{r^2}{\pi^2}p^2+\frac{r^2}{2\pi^2\alpha'}\sum_{n\neq 0} nD_{T,n}\cos n\sigma\cos n\sigma', \qquad \text{(C.0.12)}$$

$$[\partial_\sigma T(\sigma)\partial_\sigma T(\sigma')]_D = \frac{2\alpha'}{r^2}\sum_{n\neq 0} nD_{T,n}\sin n\sigma\sin n\sigma', \qquad \text{(C.0.13)}$$

$$[\Pi_T(\sigma)\partial_\sigma T(\sigma')]_D = \frac{i}{\pi}\sum_{n\neq 0} nD_{T,n}\cos n\sigma\sin n\sigma', \qquad \text{(C.0.14)}$$

$$[\partial_\sigma T(\sigma)\Pi_T(\sigma')]_D = -\frac{i}{\pi}\sum_{n\neq 0} nD_{T,n}\sin n\sigma\cos n\sigma'. \qquad \text{(C.0.15)}$$

For products including X and Π_X,

$$[X(\sigma)X(\sigma')]_D = \frac{2\alpha'}{r^2}\sum_{n\neq 0}\frac{1}{n}D_{X,n}\sin n\sigma\sin n\sigma', \qquad \text{(C.0.16)}$$

$$[\Pi_X(\sigma)\Pi_X(\sigma')]_D = \frac{r^2}{2\pi^2\alpha'}\sum_{n\neq 0} nD_{X,n}\sin n\sigma\sin n\sigma', \qquad \text{(C.0.17)}$$

$$[\partial_\sigma X(\sigma)\partial_\sigma X(\sigma')]_D = \frac{2\alpha'}{r^2}\sum_{n\neq 0} nD_{X,n}\cos n\sigma\cos n\sigma', \qquad \text{(C.0.18)}$$

$$\qquad \text{(C.0.19)}$$

$$[X(\sigma)\Pi_X(\sigma')]_D = \frac{i}{\pi}\sum_{n\neq 0} D_{X,n}\sin n\sigma \sin n\sigma', \tag{C.0.20}$$

$$[\Pi_X(\sigma)X(\sigma')]_D = -\frac{i}{\pi}\sum_{n\neq 0} D_{X,n}\sin n\sigma \sin n\sigma', \tag{C.0.21}$$

$$[X(\sigma)\partial_\sigma X(\sigma')]_D = \frac{2\alpha'}{r^2}\sum_{n\neq 0} D_{X,n}\sin n\sigma \cos n\sigma', \tag{C.0.22}$$

$$[\partial_\sigma X(\sigma)X(\sigma')]_D = \frac{2\alpha'}{r^2}\sum_{n\neq 0} D_{X,n}\cos n\sigma \sin n\sigma', \tag{C.0.23}$$

$$[\Pi_X(\sigma)\partial_\sigma X(\sigma')]_D = -\frac{i}{\pi}\sum_{n\neq 0} nD_{X,n}\sin n\sigma \cos n\sigma', \tag{C.0.24}$$

$$[\partial_\sigma X(\sigma)\Pi_X(\sigma')]_D = \frac{i}{\pi}\sum_{n\neq 0} nD_{X,n}\cos n\sigma \sin n\sigma'. \tag{C.0.25}$$

Those including Y and Π_Y have the same form as above.

C.0.2 $[V_2]_D$

The diagonal part of (C.0.2) can be calculated as follows:

$$\begin{aligned}
&\int_0^\pi d\sigma\, \frac{\pi\alpha'}{r^2}[X^2]_D[\Pi_Y^2]_D \\
= &\ \frac{\alpha'}{r^2}\sum_{n\neq 0}\frac{1}{n}D_{X,n}\sum_{m\neq 0} mD_{Y,m}\cdot\frac{1}{\pi}\int_0^\pi d\sigma\, \sin^2 n\sigma \sin^2 m\sigma \\
= &\ \frac{\alpha'}{4r^2}\sum_{n,m\neq 0}\frac{m}{n}D_{X,n}D_{Y,m} + \frac{\alpha'}{8r^2}\sum_{n\neq 0}\frac{1}{n}D_{X,n}\left(nD_{Y,n} + (-n)D_{Y,-n}\right) \\
= &\ \frac{\alpha'}{4r^2}\sum_{n,m\neq 0}\frac{m}{n}D_{X,n}D_{Y,m} + \frac{\alpha'}{4r^2}\sum_{n\neq 0}D_{X,n}D_{Y,n} - \frac{\alpha'}{8r^2}\sum_{n\neq 0}D_{X,n}.
\end{aligned} \tag{C.0.26}$$

We have used the following identities

$$D_{X,-n} = -D_{X,n} + 1. \qquad (n \neq 0) \tag{C.0.27}$$

We introduce the following notations:

$$
\begin{aligned}
D^1_{\mu\nu} &:= \sum_{n,m\neq 0} \frac{m}{n} D_{\mu,n} D_{\nu,m}, && \text{(C.0.28)}\\
D^2_{\mu\nu} &:= \sum_{n\neq 0} D_{\mu,n} D_{\nu,n}, && \text{(C.0.29)}\\
D_\mu(x) &:= \sum_{n\neq 0} \frac{1}{n^x} D_{\mu,n}. && \text{(C.0.30)}
\end{aligned}
$$

In terms of these operators, (C.0.26) can be written as

$$
\int_0^\pi d\sigma\, \frac{\pi\alpha'}{r^2}[X^2]_D[\Pi_Y^2]_D = \frac{\alpha'}{4r^2}\left(D^1_{XY} + D^2_{XY} - \frac{1}{2}D_X(0)\right). \tag{C.0.31}
$$

The other terms contributing to $[V_2]_D$ are as follows:

$$
\begin{aligned}
\int_0^\pi d\sigma\, \frac{\pi\alpha'}{r^2}[Y^2]_D[\Pi_X^2]_D &= \frac{\alpha'}{4r^2}\left(D^1_{YX} + D^2_{YX} - \frac{1}{2}D_Y(0)\right),\\
-\int_0^\pi d\sigma\, \frac{2\pi\alpha'}{r^2}[X\Pi_X]_D[\Pi_Y Y]_D &= 0,\\
-\int_0^\pi d\sigma\, \frac{2\pi\alpha'}{r^2}[\Pi_X X]_D[Y\Pi_Y]_D &= 0,,\\
-\int_0^\pi d\sigma\, \frac{\pi\alpha'}{r^2}x(\sigma)^2[\Pi_T^2]_D &= -\frac{\alpha'}{3}p^2 - \frac{1}{12}D_T(-1) - \frac{1}{2\pi^2}D_T(1),
\end{aligned}
$$

$$
\begin{aligned}
\int_0^\pi d\sigma\, \frac{\pi\alpha'}{r^2}x(\sigma)^2[\Pi_Y^2]_D &= \frac{1}{12}D_Y(-1) - \frac{1}{2\pi^2}D_Y(1),\\
\int_0^\pi d\sigma\, \frac{r^2}{4\pi\alpha'}[(\partial_\sigma T)^2]_D[X^2]_D &= \frac{\alpha'}{4r^2}\left(D^1_{XT} + D^2_{XT} - \frac{1}{2}D_T(0)\right),\\
\int_0^\pi d\sigma\, \frac{r^2}{4\pi\alpha'}[(\partial_\sigma T)^2]_D[Y^2]_D &= \frac{\alpha'}{4r^2}\left(D^1_{YT} + D^2_{YT} - \frac{1}{2}D_T(0)\right),\\
\int_0^\pi d\sigma\, \frac{r^2}{4\pi\alpha'}x(\sigma)^2[(\partial_\sigma T)^2]_D &= \frac{1}{12}D_T(-1) - \frac{1}{2\pi^2}D_T(1),\\
-\int_0^\pi d\sigma\, \frac{r^2}{4\pi\alpha'}x(\sigma)^2[(\partial_\sigma Y)^2]_D &= -\frac{1}{12}D_Y(-1) - \frac{1}{2\pi^2}D_Y(1),\\
-\int_0^\pi d\sigma\, \frac{r^2}{4\pi\alpha'}\frac{4}{\pi}x(\sigma)[Y\partial_\sigma Y]_D &= -\frac{1}{\pi^2}D_Y(1),\\
\int_0^\pi d\sigma\, \frac{r^2}{4\pi\alpha'}\frac{12}{\pi^2}[Y^2]_D &= \frac{3}{\pi^2}D_Y(1).
\end{aligned}
$$

Summing all of them, $[V_2]_D$ is given as

$$
\begin{aligned}
[V_2]_D &= \frac{\alpha'}{4r^2}\left(D^1_{XY}+D^1_{YX}+D^1_{XT}+D^1_{YT}+2D^2_{XY}+D^2_{XT}+D^2_{YT}\right)\\
&\quad -\frac{1}{\pi^2}\left(D_T(1)-D_Y(1)\right)-\frac{1}{3}\alpha' p^2+\frac{\alpha'}{4r^2}. \qquad \text{(C.0.32)}
\end{aligned}
$$

To extract the divergent terms, we use the formulae:

$$
D^1_{XY} = \zeta(1)\left(2N_Y(0)+\zeta(-1)\right)+4N_X(2)N_Y(0)-\frac{1}{6}N_X(2), \qquad \text{(C.0.33)}
$$

$$
D^2_{XY} = 2N_{XY}(2)+N_X(1)+N_Y(1)+\zeta(0) \qquad \text{(C.0.34)}
$$

$$
D_T(1)-D_Y(1) = 2N_T(2)-2N_Y(2), \qquad \text{(C.0.35)}
$$

and so on. Here we use the zeta-function regularized values: $\zeta(-1) = -1/12$ and $\zeta(0) = -1/2$. By using these results, we obtain (C.0.10).

C.0.3 $[V_1V_1]_D$

The diagonal part of the second term can be calculated as follows. For example, let us consider the following term:

$$
-\sum_{m\neq 0}\frac{1}{m}\int_0^\pi d\sigma\frac{-2\pi\alpha'}{r^2}\left[\Pi_T X\Pi_Y(\sigma)\right]_{-m}\int_0^\pi d\sigma'\frac{-2\pi\alpha'}{r^2}\left[\Pi_T X\Pi_Y(\sigma')\right]_m \qquad \text{(C.0.36)}
$$

which contributes to H_2. The diagonal part of this operator can be written as follows:

$$
\begin{aligned}
&-\frac{2\alpha'}{r^2}\sum_{m,n,k,l\neq 0}\frac{nl}{mk}D_{T,n}D_{X,k}D_{Y,l}\delta_{n+k+l,m}\left[\frac{1}{\pi}\int_0^\pi d\sigma\ \cos n\sigma\sin k\sigma\sin l\sigma\right]^2\\
&-\frac{4(\alpha')^2}{r^2}p^2\sum_{m,k,l\neq 0}\frac{l}{mk}D_{X,k}D_{Y,l}\delta_{k+l,m}\left[\frac{1}{\pi}\int_0^\pi d\sigma\ \sin k\sigma\sin l\sigma\right]^2. \qquad \text{(C.0.37)}
\end{aligned}
$$

This expression can be obtained from (C.0.36) by substituting $\Pi_T(\sigma)\Pi_T(\sigma')$, $X(\sigma)X(\sigma')$ and $\Pi_Y(\sigma)\Pi_Y(\sigma')$ with their diagonal parts given in Appendix C.0.1, and then inserting $\delta_{n+k+l,m}$ and $\delta_{k+l,m}$ at appropriate places in the sums. Performing the integrals, we obtain

$$
\begin{aligned}
&\frac{\alpha'}{8r^2}\sum_{m,n,k,l\neq 0}\frac{nl}{mk}D_{T,n}D_{X,k}D_{Y,l}(-\delta_{2l,m}\delta_{l,k+n}-\delta_{2k,m}\delta_{k,l+n}-\delta_{2n,m}\delta_{n,k+l})\\
&-\frac{(\alpha')^2}{2r^2}p^2\sum_{k\neq 0}\frac{1}{k}D_{X,k}D_{Y,k}. \qquad \text{(C.0.38)}
\end{aligned}
$$

Again, this contains terms with divergent coefficients. For example,

$$\begin{aligned} & -\frac{\alpha'}{8r^2}\sum_{m,n,k,l\neq 0}\frac{nl}{mk}D_{T,n}D_{X,k}D_{Y,l}\delta_{2l,m}\delta_{l,k+n} \\ = \;& -\frac{\alpha'}{16r^2}\zeta(1)\left(2N_Y(0)+\zeta(-1)\right)+(\text{finite}). \end{aligned} \tag{C.0.39}$$

The other terms can be calculated similarly. In the following, we use an abbreviated notation, for example,

$$\begin{aligned} & (\Pi_T X\Pi_Y, \Pi_T X\Pi_Y) \\ := \;& -\sum_{m\neq 0}\frac{1}{m}\int_0^\pi d\sigma\,\frac{-2\pi\alpha'}{r^2}\left[\Pi_T X\Pi_Y(\sigma)\right]_{-m}\int_0^\pi d\sigma'\,\frac{-2\pi\alpha'}{r^2}\left[\Pi_T X\Pi_Y(\sigma')\right]_m. \end{aligned}$$

We also use the notations

$$\begin{aligned} D^{(\pm,\pm,\pm)}_{\mu\nu\rho} &:= \sum_{m,n,k,l\neq 0}\frac{nl}{mk}D_{\mu,n}D_{\nu,k}D_{\rho,l}\Delta_{(\pm,\pm,\pm)}, \\ \tilde{D}^{(\pm,\pm,\pm)}_{\mu\nu\rho} &:= \sum_{m,n,k,l\neq 0}\frac{n}{m}D_{\mu,n}D_{\nu,k}D_{\rho,l}\Delta_{(\pm,\pm,\pm)}, \\ D^3_{\mu\nu} &:= \sum_{k\neq 0}\frac{1}{k}D_{\mu,k}D_{\nu,k}, \end{aligned}$$

where

$$\Delta_{(\pm,\pm,\pm)} \equiv \left(\pm\delta_{2l,m}\delta_{l,k+n}\pm\delta_{2k,m}\delta_{k,l+n}\pm\delta_{2n,m}\delta_{n,k+l}\right).$$

Then the results are given as follows:

$$\begin{aligned} (\Pi_T X\Pi_Y, \Pi_T X\Pi_Y) &= -\frac{(\alpha')^2}{2r^2}p^2 D^3_{XY}+\frac{\alpha'}{8r^2}D^{(-,-,-)}_{TXY}, \\ (\Pi_T Y\Pi_X, \Pi_T Y\Pi_X) &= -\frac{(\alpha')^2}{2r^2}p^2 D^3_{YX}+\frac{\alpha'}{8r^2}D^{(-,-,-)}_{TYX}, \\ (\Pi_T X\Pi_Y, -\Pi_T Y\Pi_X) &= \frac{(\alpha')^2}{2r^2}p^2 D^3_{XY}+\frac{\alpha'}{8r^2}\tilde{D}^{(+,+,+)}_{TXY}, \\ (-\Pi_T Y\Pi_X, \Pi_T X\Pi_Y) &= \frac{(\alpha')^2}{2r^2}p^2 D^3_{YX}+\frac{\alpha'}{8r^2}\tilde{D}^{(+,+,+)}_{TYX}, \\ (\partial_\sigma TX\partial_\sigma Y, \partial_\sigma TX\partial_\sigma Y) &= \frac{\alpha'}{8r^2}D^{(-,-,-)}_{TXY}, \end{aligned}$$

$$\begin{aligned} (\partial_\sigma TY\partial_\sigma X, \partial_\sigma TY\partial_\sigma X) &= \frac{\alpha'}{8r^2}D^{(-,-,-)}_{TYX}, \\ (\partial_\sigma TX\partial_\sigma Y, -\partial_\sigma TY\partial_\sigma X) &= \frac{\alpha'}{8r^2}\tilde{D}^{(-,-,+)}_{TXY}, \\ (-\partial_\sigma TY\partial_\sigma X, \partial_\sigma TX\partial_\sigma Y) &= \frac{\alpha'}{8r^2}\tilde{D}^{(-,-,+)}_{TYX}, \end{aligned}$$

$$
\begin{aligned}
(\Pi_T X \Pi_Y, \partial_\sigma T X \partial_\sigma Y) &= \frac{\alpha'}{8r^2} D^{(-,+,-)}_{TXY}, \\
(\Pi_T Y \Pi_X, \partial_\sigma T Y \partial_\sigma X) &= \frac{\alpha'}{8r^2} D^{(-,+,-)}_{TYX}, \\
(\Pi_T X \Pi_Y, -\partial_\sigma T Y \partial_\sigma X) &= \frac{\alpha'}{8r^2} \tilde{D}^{(-,+,+)}_{TXY}, \\
(-\Pi_T Y \Pi_X, \partial_\sigma T X \partial_\sigma Y) &= \frac{\alpha'}{8r^2} \tilde{D}^{(-,+,+)}_{TYX}, \\
(\partial_\sigma T X \partial_\sigma Y, \Pi_T X \Pi_Y) &= \frac{\alpha'}{8r^2} D^{(-,+,-)}_{TXY}, \\
(\partial_\sigma T Y \partial_\sigma X, \Pi_T Y \Pi_X) &= \frac{\alpha'}{8r^2} D^{(-,+,-)}_{TYX}, \\
(-\partial_\sigma T Y \partial_\sigma X, \Pi_T X \Pi_Y) &= \frac{\alpha'}{8r^2} \tilde{D}^{(-,+,+)}_{TXY}, \\
(-\partial_\sigma T X \partial_\sigma Y, \Pi_T Y \Pi_X) &= \frac{\alpha'}{8r^2} \tilde{D}^{(-,+,+)}_{TYX}, \\
(\partial_\sigma T Y, \partial_\sigma T Y) &= -\frac{2}{\pi^2} D^3_{TY}.
\end{aligned}
$$

It turned out that most of the terms cancel miraculously. As a result, we find

$$
\begin{aligned}
-\sum_{m\neq 0} \frac{1}{m} [V_{1,-m} V_{1,m}]_D &= -\frac{\alpha'}{4r^2} \sum_{\substack{n,k\neq 0 \\ n+k\neq 0}} \frac{n}{k} \Big[D_{T,n} D_{X,k} D_{Y,n+k} + D_{T,n+k} D_{X,k} D_{Y,n} \\
&\qquad + D_{T,n} D_{Y,k} D_{X,n+k} + D_{T,n+k} D_{Y,k} D_{X,n} \Big] \\
&\quad + \frac{\alpha'}{2r^2} \sum_{\substack{k,l\neq 0 \\ k+l\neq 0}} D_{T,k+l} D_{X,k} D_{Y,l} - \frac{2}{\pi^2} \sum_{n\neq 0} \frac{1}{n} D_{T,n} D_{Y,n}.
\end{aligned}
\tag{C.0.40}
$$

To extract divergent terms, we use the following formulae:

$$
\sum_{n\neq 0} \frac{1}{n} D_{T,n} D_{Y,n} = N_T(2) + N_Y(2) + \zeta(1), \tag{C.0.41}
$$

$$\begin{aligned}
\sum_{\substack{n,k\neq 0\\ n+k\neq 0}} \frac{n}{k} D_{T,n} D_{X,k} D_{Y,n+k} &= \zeta(1)\left(2N_T(0)+\zeta(-1)\right) \\
&\quad +2N_X(2)N_Y(0)+2N_X(2)N_T(0)-N_{TX}(2) \\
&\quad -\frac{1}{6}N_X(2)-\frac{1}{2}N_X(1)-\frac{1}{2}N_X(0)-N_Y(0) \\
&\quad +\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{1}{k}\left(N_{Y,n}-N_{T,n}\right) \\
&\quad +\sum_{k=1}^{\infty}\sum_{n\neq 0,-k}\frac{1}{k(n+k)}N_{T,n}N_{Y,n+k}, \qquad \text{(C.0.42)} \\
\sum_{\substack{k,l\neq 0\\ k+l\neq 0}} D_{T,k+l} D_{X,k} D_{Y,l} &= N_{XY}(2)-N_{TX}(2)-N_{TY}(2)+N_X(0)+N_Y(0) \\
&\quad -N_T(1)+N_T(0)+\frac{1}{2}\zeta(0)^2 \qquad \text{(C.0.43)}
\end{aligned}$$

and so on. Using these results and the following identity,

$$\sum_{k=1}^{\infty}\sum_{n\neq 0,-k}\frac{1}{k(n+k)}N_{T,n}N_{Y,n+k}+\sum_{k=1}^{\infty}\sum_{n\neq 0,-k}\frac{1}{k(n+k)}N_{Y,n}N_{T,n+k} = N_{TY}(2),$$

we obtain the result of (C.0.11).

Appendix D

Useful formulae for traces and derivation of the partition function

In this appendix, we give general formulae for the traces such as

$$\mathrm{tr}[e^{-2\pi s\tilde{H}_0}N_\mu(x)], \qquad \mathrm{tr}[e^{-2\pi s\tilde{H}_0}N_{\mu\nu}(x)] \tag{D.0.1}$$

where $\tilde{H}_0$, $N_\mu(x)$, $N_{\mu\nu}(x)$ are defined in in Eq.(3.2.8) and Eq.(3.2.16). Using them, we derive the expression (3.2.21) for the partition function.

The calculation of these traces is reduced to considering a single harmonic oscillator

$$[\alpha_n, \alpha_{-n}] \ = \ n, \tag{D.0.2}$$

for which we determine

$$\mathrm{tr}_n \left[e^{-2\pi s N_n} N_n\right], \qquad N_n \ := \ \alpha_{-n}\alpha_n. \tag{D.0.3}$$

The trace tr_n is taken over the Fock space of α_n. It is easy to obtain

$$\mathrm{tr}_n \left[e^{-2\pi s N_n} N_n\right] \ = \ \frac{nq^n}{1-q^n}\cdot\frac{1}{1-q^n}, \tag{D.0.4}$$

where $q =^{-2\pi s}$. Using this formulae, we obtain

$$\mathrm{tr}\left[e^{-2\pi s\tilde{H}_0}N_\mu(x)\right] \ = \ \sum_{n=1}^{\infty}\frac{n^{1-x}q^n}{1-q^n}\cdot\prod_{m=1}^{\infty}(1-q^m)^{-3}, \tag{D.0.5}$$

$$\mathrm{tr}\left[e^{-2\pi s\tilde{H}_0}N_{\mu\nu}(x)\right] \ = \ \sum_{n=1}^{\infty}\frac{n^{2-x}q^{2n}}{(1-q^n)^2}\cdot\prod_{m=1}^{\infty}(1-q^m)^{-3}, \tag{D.0.6}$$

where we assumed $\mu \neq \nu$.

These formulae are sufficient to determine the partition function (3.2.1). For this purpose, it is helpful to notice the following formulae

$$\mathrm{tr}\left[e^{-2\pi s\tilde{H}_0}\left(2N_{\mu\nu}(2)+N_\mu(1)+N_\nu(1)\right)\right] = \sum_{n=1}^{\infty}\frac{2q^n}{(1-q^n)^2}\cdot\prod_{m=1}^{\infty}(1-q^m)^{-3}. \quad \text{(D.0.7)}$$

Then we obtain

$$\begin{aligned} s\,\mathrm{tr}\left[e^{-2\pi s\left(\tilde{H}_0-\frac{1}{8}\right)}H_2\right] &= s\left[\frac{\alpha'}{r^2}\sum_{n=1}^{\infty}\frac{2q^n}{(1-q^n)^2}-\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{n^{-1}q^n}{1-q^n}\right. \\ &\qquad \left.-\frac{1}{3}\alpha' k^2+\epsilon_0\right]\eta(is)^{-3}, \end{aligned} \quad \text{(D.0.8)}$$

$$-\pi s^2\mathrm{tr}\left[e^{-2\pi s\tilde{H}_0}H_1^2\right] = -\pi s^2\left(\frac{4(\alpha')^2}{r^2}k^2+\frac{4}{\pi^2}\right)\sum_{n=1}^{\infty}\frac{2q^n}{(1-q^n)^2}\eta(is)^{-3}. \quad \text{(D.0.9)}$$

Interestingly, after performing the k-integration, one more cancellation occurs between these two traces; i.e., the momentum integration of (D.0.9) becomes

$$\int\frac{dk}{2\pi}e^{-2\pi\alpha' sk^2}\mathrm{tr}\left[e^{-2\pi s\tilde{H}_0}H_1^2\right] = (8\pi^2\alpha' s)^{-\frac{1}{2}}\left(-\frac{\alpha'}{r^2}s-\frac{4}{\pi}s^2\right)\sum_{n=1}^{\infty}\frac{2q^n}{(1-q^n)^2}\eta(is)^{-3} \quad \text{(D.0.10)}$$

whose first term in the parenthesis is cancelled by the momentum integration of the first term in the square bracket of (D.0.9),

$$\int\frac{dk}{2\pi}e^{-2\pi\alpha' sk^2}\mathrm{tr}\left[e^{-2\pi s\tilde{H}_0}H_2\right] = (8\pi^2\alpha' s)^{-\frac{1}{2}}\cdot\frac{\alpha'}{r^2}s\sum_{n=1}^{\infty}\frac{2q^n}{(1-q^n)^2}\eta(is)^{-3}+\cdots. \quad \text{(D.0.11)}$$

Finally, the partition function becomes

$$\begin{aligned} &\int_0^{\infty}\frac{ds}{2s}(8\pi^2\alpha' s)^{-\frac{1}{2}}e^{-\frac{2r^2}{\pi\alpha'}s}\eta(is)^{-24}\left(1-\frac{1}{3}v^2\right)^{-\frac{1}{2}} \\ &\times\left[1-2\pi v^2\left(-\frac{4}{\pi^2}s\sum_{n=1}^{\infty}\frac{n^{-1}q^n}{1-q^n}+\epsilon_0 s-\frac{4}{\pi}s^2\sum_{n=1}^{\infty}\frac{2q^n}{(1-q^n)^2}\right)\right]+\mathcal{O}(v^4). \end{aligned} \quad \text{(D.0.12)}$$

Appendix E

Matrix elements for low lying states

We determine the matrix elements $\langle \vec{\boldsymbol{n}}; k|O|\vec{\boldsymbol{n}}'; k\rangle$ for $O = H_1^2, H_2$, in order to calculate the eigenvalues $E_i(k, r, \omega)$ of $H_0(v)$ we need in subsection 3.3.2. The states $|\vec{\boldsymbol{n}}\rangle$ we consider in the following satisfy $N(\vec{\boldsymbol{n}}) \leq 1$.

First for the ground state with $N(\vec{\boldsymbol{n}}) = 0$, all of n_n^α are zero. Therefore, we obtain

$$\langle \vec{\boldsymbol{n}}; k|H_1^2|\vec{\boldsymbol{n}}; k\rangle = 0, \tag{E.0.1}$$

$$\langle \vec{\boldsymbol{n}}; k|H_2|\vec{\boldsymbol{n}}; k\rangle = -\frac{1}{3}\alpha' k^2 + \epsilon_0. \tag{E.0.2}$$

After returning to the Lorentzian metric, $-v^2 H_2/\alpha'$ gives the v^2 corrections to the energy of the tachyonic state.

Then we consider the first excited states with $N(\vec{\boldsymbol{n}}) = 1$, which correspond to the massless open string states excited by world sheet variables T, X, Y; so there are three states specified by the integers (n_1^T, n_1^X, n_1^Y). We denote these states by

$$\vec{\boldsymbol{n}}_1 = (1,0,0), \quad \vec{\boldsymbol{n}}_2 = (0,1,0), \quad \vec{\boldsymbol{n}}_3 = (0,0,1). \tag{E.0.3}$$

The diagonal matrix elements can be obtained easily. The results are

$$\langle \vec{\mathbf{n}}_1; k|H_2|\vec{\mathbf{n}}_1; k\rangle = -\frac{1}{3}\alpha' k^2 - \frac{4}{\pi^2} + \epsilon_0, \tag{E.0.4}$$

$$\langle \vec{\mathbf{n}}_2; k|H_2|\vec{\mathbf{n}}_2; k\rangle = \frac{\alpha'}{r^2} - \frac{1}{3}\alpha' k^2 + \epsilon_0, \tag{E.0.5}$$

$$\langle \vec{\mathbf{n}}_3; k|H_2|\vec{\mathbf{n}}_3; k\rangle = \frac{\alpha'}{r^2} - \frac{1}{3}\alpha' k^2 + \epsilon_0. \tag{E.0.6}$$

We also need the off-diagonal matrix elements. To obtain them, we must use H_1 and H_2, not just their diagonal parts $[H_1^2]_D$ and $[H_2]_D$. Using the expression (3.2.13), we find that the only non-zero off-diagonal elements of H_1 are

$$\langle \vec{\boldsymbol{n}}_1; k|H_1|\vec{\boldsymbol{n}}_3; k\rangle = \frac{2i}{\pi}, \qquad \langle \vec{\boldsymbol{n}}_2; k|H_1|\vec{\boldsymbol{n}}_3; k\rangle = \frac{2i\alpha'}{r}k, \tag{E.0.7}$$

and their complex conjugates.

It turns out that most of terms in H_2 do not contribute to the off-diagonal elements for the states with $N(\vec{\boldsymbol{n}}) = 1$. This can be seen by considering the following matrix element

$$\langle 0|\alpha_m \alpha_n|1\rangle, \qquad |1\rangle \ := \ |n_1 = 1\rangle. \tag{E.0.8}$$

The matrix elements for $(m, n) = (1, 0), (0, 1)$ are non-vanishing only if $\langle 0|\alpha_0|0\rangle$ is non-zero, and the other matrix elements vanish. This implies that terms with a zero mode in H_2 give non-zero off-diagonal matrix elements.

Among the worldsheet fields, Π_T is the only field which has a zero mode. It turns out that

$$-\int_0^\pi d\sigma \, \frac{2\pi\alpha'}{r^2} x(\sigma)\Pi_T^2 X. \tag{E.0.9}$$

is the only term in V_2 which gives the off-diagonal matrix element

$$\langle \vec{\boldsymbol{n}}_1; k| \left[-\int_0^\pi d\sigma \, \frac{2\pi\alpha'}{r^2} x(\sigma)\Pi_T^2 X \right] |\vec{\boldsymbol{n}}_2; k\rangle \ = \ -\frac{2\alpha'}{\pi r}k. \tag{E.0.10}$$

and its complex conjugate.

The terms in $-\sum_{m\neq 0} V_{1,-m}V_{1,m}/m$ which could possibly give off-diagonal matrix elements are

$$-\sum_{m\neq 0} \frac{1}{m} \int_0^\pi d\sigma \frac{-2\pi\alpha'}{r^2} \left[\Pi_T \left(X\Pi_Y - Y\Pi_X\right)\right]_{-m} \int_0^\pi d\sigma' \frac{2r^2}{\pi^2\alpha'} \left[\partial_\sigma T Y\right]_m \tag{E.0.11}$$

and its Hermitian conjugate. In fact, we find that the off-diagonal matrix elements of these terms vanish.

In summary, the matrix elements of $H_0(v)$ for the states with $N(\vec{\boldsymbol{n}}) \leq 1$ are

$$\left(1 - \frac{1}{3}v^2\right)\alpha' k^2 + \frac{r^2}{\pi^2\alpha'} + v^2\epsilon_0 + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 - \frac{4}{\pi^2}v^2 & -\frac{2\alpha'}{\pi r}kv^2 & \frac{2i}{\pi}v \\ 0 & -\frac{2\alpha'}{\pi r}kv^2 & 1 + \frac{\alpha'}{r^2}v^2 & \frac{2i\alpha'}{r}kv \\ 0 & -\frac{2i}{\pi}v & -\frac{2i\alpha'}{r}kv & 1 + \frac{\alpha'}{r^2}v^2 \end{bmatrix}. \tag{E.0.12}$$

Back to the Lorentzian metric and multiplying $-v^2/\alpha'$, these expressions give v^2-dependent corrections to the mass matrix of the open string massless states, whose 0-th order energy is given by $k^2 + r^2/\pi^2\alpha'^2$. Since H_1 and H_2 mix these 3 states, we need to diagonalize the matrix to obtain the energy eigenvalues. It is given in subsection 3.3.2.

Appendix F

Integration for $\mathcal{V}_{02}$ and $\mathcal{V}_{03}$

Recall that $E_2(k,r,\omega)-1$ is given as

$$\begin{aligned} E_2(k,r,\omega)-1 &= \left(1-\frac{1}{3}\omega^2 r^2\right)\alpha' k^2 + (1+\epsilon_0\pi^2\alpha'\omega^2)\frac{r^2}{\pi^2\alpha'} \\ &\quad +2\sqrt{\alpha'}\omega h(k,r)^{\frac{1}{2}} + \alpha'\omega^2 - 2\alpha'\omega^2\frac{r^2}{\pi^2\alpha'} - \frac{\omega^2 r^2}{2\pi^2}h(k,r)^{-1}. \end{aligned} \tag{F.0.1}$$

The integral

$$\mathcal{V}_{02} = -\int_0^\infty \frac{ds}{2s}\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\, e^{-2\pi s(E_2(k,r,\omega)-1)} \tag{F.0.2}$$

is very complicated. To see the behavior of $\mathcal{V}_{02}$, we replace

$$\begin{aligned} E_2(k,r,\omega)-1 &\to \left(1-\frac{1}{3}\omega^2 r^2\right)\alpha' k^2 + (1+\epsilon_0\pi^2\alpha'\omega^2)\frac{r^2}{\pi^2\alpha'} \\ &\quad +2\alpha'\omega|k| + \alpha'\omega^2 - 2\alpha'\omega^2\frac{r^2}{\pi^2\alpha'}. \end{aligned} \tag{F.0.3}$$

Then, the integral becomes

$$\begin{aligned} \mathcal{V}_{02} &\to -2\int_0^\infty \frac{ds}{2s}\, e^{-2\pi s\left[\alpha'\omega^2+(1+(\epsilon_0\pi^2-2)\alpha'\omega^2)\frac{r^2}{\pi^2\alpha'}\right]} \\ &\quad \times\int_0^\infty \frac{dk}{2\pi} e^{-2\pi s\alpha'\left[\left(1-\frac{1}{3}\omega^2 r^2\right)k^2+2\omega k\right]}. \end{aligned} \tag{F.0.4}$$

The expression for $\mathcal{V}_{03}$ is obtained by flipping the sign of ω. It can be written as

$$\begin{aligned} \mathcal{V}_{03} &\to -2\int_0^\infty \frac{ds}{2s}\, e^{-2\pi s\left[\alpha'\omega^2+(1+(\epsilon_0\pi^2-2)\alpha'\omega^2)\frac{r^2}{\pi^2\alpha'}\right]} \\ &\quad \times\int_{-\infty}^0 \frac{dk}{2\pi} e^{-2\pi s\alpha'\left[\left(1-\frac{1}{3}\omega^2 r^2\right)k^2+2\omega k\right]}. \end{aligned} \tag{F.0.5}$$

The sum of them can be easily integrated and results in (3.3.19).

Appendix G

$[H_2^{(l)}]_D$ for the linear system

$[H_2^{(l)}]_D$ for the linear system can be calculated in a similar way as the revolving system. The difference is that $H_0^{(l)}$ involves the term proportional to t^2 and the zero mode forms a harmonic oscillator. In terms of the creation/annihilation operators of the harmonic oscillator, t and p are written as

$$t = \sqrt{\frac{\pi\alpha'}{2v}}(\alpha_0 + \alpha_0^\dagger), \qquad p = -i\sqrt{\frac{v}{2\pi\alpha'}}(\alpha_0 - \alpha_0^\dagger). \tag{G.0.1}$$

Then the free Hamiltonian for the zero modes in $H_0^{(l)}$ is given by $\frac{2v}{\pi}(N_0 + \frac{1}{2})$, where $[\alpha_0, \alpha_0^\dagger] = 1$ and $N_0 := \alpha_0^\dagger \alpha_0$.

For products including $\tilde{T}$ and Π_T, the diagonal parts are given as

$$\begin{aligned}
[\Pi_T(\sigma)\Pi_T(\sigma')]_D &= \frac{vr^2}{\pi^3\alpha'}\left(N_0 + \frac{1}{2}\right) + \frac{r^2}{2\pi^2\alpha'}\sum_{n\neq 0} nD_{T,n}\cos n\sigma \cos n\sigma', \\
\left[\tilde{T}(\sigma)\tilde{T}(\sigma')\right]_D &= \frac{\pi\alpha'}{vr^2}\left(N_0 + \frac{1}{2}\right) + \frac{2\alpha'}{r^2}\sum_{n\neq 0}\frac{1}{n}D_{T,n}\cos n\sigma \cos n\sigma', \\
[\partial_\sigma T(\sigma)\partial_\sigma T(\sigma')]_D &= \frac{2\alpha'}{r^2}\sum_{n\neq 0} nD_{T,n}\sin n\sigma \sin n\sigma', \\
\left[\Pi_T(\sigma)\tilde{T}(\sigma')\right]_D &= -\frac{i}{2\pi} - \frac{i}{\pi}\sum_{n\neq 0} D_{T,n}\cos n\sigma \cos n\sigma', \\
\left[\tilde{T}(\sigma)\Pi_T(\sigma')\right]_D &= \frac{i}{2\pi} + \frac{i}{\pi}\sum_{n\neq 0} D_{T,n}\cos n\sigma \cos n\sigma'.
\end{aligned}$$

For X, Π_X and Y, Π_Y, mode expansions are given in the same form as those in the revolving system.

Let us first look at differences between the revolving and the linear systems. Since T and X obey different, Neumann and Dirichlet, boundary conditions, these terms give different contributions to the partition function. For example, for the revolving system, $[H_2]$ has a contribution

$$
\begin{aligned}
\int_0^\pi d\sigma\, \frac{\pi\alpha'}{r^2}[X^2][\Pi_Y^2] &= \frac{\alpha'}{4r^2}\left[\zeta(1)\Big(2N_Y(0)-\frac{1}{12}\Big)+4N_X(2)N_Y(0)-\frac{1}{6}N_X(2)\right] \\
&\quad +\frac{\alpha'}{4r^2}\left[2N_{XY}(2)+N_X(1)+N_Y(1)-\frac{1}{2}-\frac{1}{2}\zeta(0)\right],
\end{aligned}
$$

while for the linear system, the corresponding contribution to $[H_2^{(l)}]$ is

$$
\begin{aligned}
\int_0^\pi d\sigma\, \frac{\pi\alpha'}{r^2}[T^2][\Pi_Y^2] &= \frac{\alpha'}{4r^2}\left[\zeta(1)\Big(2N_Y(0)-\frac{1}{12}\Big)+4N_X(2)N_Y(0)-\frac{1}{6}N_X(2)\right] \\
&\quad -\frac{\alpha'}{4r^2}\left[2N_{XY}(2)+N_X(1)+N_Y(1)-\frac{1}{2}-\frac{1}{2}\zeta(0)\right],
\end{aligned}
$$

in which the second line has the opposite sign. Note that the divergent terms are common in two cases. This turns out to be the case for all divergent terms.

In a way similar to the revolving system, each terms contributing to $[V_2^{(l)}]_D$ are calculated as follows:

$$
\begin{aligned}
\int_0^\pi d\sigma\, \frac{\pi\alpha'}{r^2}[\tilde{T}^2]_D[\Pi_Y^2]_D &= \frac{\alpha'}{4r^2}\left(D_{TY}^1-D_{TY}^2+\frac{1}{2}D_T(0)\right) \\
&\quad +\frac{\pi\alpha'}{4vr^2}\left(N_0+\frac{1}{2}\right)D_Y(-1), \\
\int_0^\pi d\sigma\, \frac{\pi\alpha'}{r^2}[Y^2]_D[\Pi_T^2]_D &= \frac{\alpha'}{4r^2}\left(D_{YT}^1-D_{YT}^2+\frac{1}{2}D_Y(0)\right) \\
&\quad +\frac{v\alpha'}{\pi r^2}\left(N_0+\frac{1}{2}\right)D_Y(1), \\
-\int_0^\pi d\sigma\, \frac{\pi\alpha'}{r^2}[\tilde{T}\Pi_T]_D[\Pi_Y Y]_D &= 0, \\
-\int_0^\pi d\sigma\, \frac{\pi\alpha'}{r^2}[\Pi_Y Y]_D[\tilde{T}\Pi_T]_D &= 0,
\end{aligned}
$$

$$\begin{aligned}
\int_0^\pi d\sigma\, \frac{r^2}{4\pi\alpha'}[(\partial_\sigma X)^2]_D[\tilde{T}^2]_D &= \frac{\alpha'}{4r^2}\left(D_{XT}^1 + D_{XT}^2 - \frac{1}{2}D_T(0)\right) \\
&\quad + \frac{\pi\alpha'}{4vr^2}\left(N_0 + \frac{1}{2}\right) D_X(-1), \\
\int_0^\pi d\sigma\, \frac{r^2}{4\pi\alpha'}[(\partial_\sigma X)^2]_D[Y^2]_D &= \frac{\alpha'}{4r^2}\left(D_{XY}^1 - D_{XY}^2 + \frac{1}{2}D_Y(0)\right), \\
\int_0^\pi d\sigma\, \frac{r^2}{4\pi\alpha'}\frac{4}{\pi^2}[T^2]_D &= \frac{1}{\pi^2}D_T(1), \\
\int_0^\pi d\sigma\, \frac{r^2}{4\pi\alpha'}\frac{4}{\pi^2}[Y^2]_D &= \frac{1}{\pi^2}D_Y(1).
\end{aligned}$$

Summing all of them, and using the same formulae as the revolving system, we get

$$\begin{aligned}
[V_2]_D &= \zeta(1)\left[\frac{2}{\pi^2} + \frac{\alpha'}{2r^2}\left(N_T(0) + 2N_X(0) + N_Y(0) - \frac{1}{6}\right)\right] \\
&\quad + \frac{\alpha'}{r^2}\Big[-N_{TY}(2) + \frac{1}{2}N_{TX}(2) - \frac{1}{2}N_{XY}(2) \\
&\qquad + N_T(0)N_Y(2) + N_T(2)N_Y(0) + N_T(2)N_X(0) + N_X(0)N_Y(2) \\
&\qquad - \frac{1}{12}N_T(2) - \frac{1}{12}N_Y(2) - \frac{1}{4}N_T(1) - \frac{3}{4}N_Y(1) + \frac{1}{8}\Big] \\
&\quad + \frac{\pi\alpha'}{2vr^2}\left(N_0 + \frac{1}{2}\right)\left(N_X(0) + N_Y(0) - \frac{1}{12}\right) + \frac{2}{\pi^2}\big(N_T(2) + N_Y(2)\big) \\
&\quad + \mathcal{O}(v).
\end{aligned} \tag{G.0.2}$$

We need to take some care when calculating the contribution to $[V_1^{(l)}V_1^{(l)}]_D$ that includes zero modes. For example, there is a following term in $(\Pi_X\tilde{T}\Pi_Y, \Pi_X\tilde{T}\Pi_Y)$:

$$\begin{aligned}
&-\frac{\pi\alpha'}{2vr^2}\sum_{m=-\infty}^{\infty}\sum_{n,k\neq 0}\left(\frac{\alpha_0\alpha_0^\dagger}{2v/\pi + m} + \frac{\alpha_0^\dagger\alpha_0}{-2v/\pi + m}\right) \\
&\qquad\qquad \times nkD_{X,n}D_{Y,k}\delta_{m,n+k}\left[\frac{1}{\pi}\int d\sigma \sin n\sigma \sin k\sigma\right]^2 \\
&= -\frac{\pi^2\alpha'}{16v^2r^2}\sum_{n\neq 0} n^2\big(D_{X,n}D_{Y,n} - D_{X,n}\big) \\
&\quad -\frac{\pi\alpha'}{8vr^2}\sum_{n\neq 0}\frac{k^2}{2v/\pi + 2n}\Big[D_{X,}D_{Y,n} + N_0\big(D_{X,n} + D_{Y,n} - 1\big)\Big].
\end{aligned}$$

The terms includes no zero modes can be calculated in the same way as the revolving system. Thus each contributions are obtained as follows:

$$\begin{aligned}
\left(\Pi_X\tilde{T}\Pi_Y,\Pi_X\tilde{T}\Pi_Y\right) &= -\frac{\alpha'}{8r^2}D^{(+,+,+)}_{XTY}-\frac{\pi^2\alpha'}{16v^2r^2}\sum_{n\neq 0}n^2\big(D_{X,n}D_{Y,n}-D_{X,n}\big)\\
&\quad -\frac{\pi\alpha'}{8vr^2}\sum_{n\neq 0}n^2\frac{D_{X,n}D_{Y,n}+N_0\big(D_{X,n}+D_{Y,n}-1\big)}{2v/\pi+2n},\\
(-\Pi_X\Pi_TY,-\Pi_X\Pi_TY) &= -\frac{\alpha'}{8r^2}D^{(+,+,+)}_{XYT}-\frac{\alpha'}{4r^2}\sum_{n\neq 0}\big(D_{X,n}D_{Y,n}-D_{X,n}\big)\\
&\quad -\frac{v\alpha'}{2\pi r^2}\sum_{n\neq 0}\frac{D_{X,n}D_{Y,n}+N_0\big(D_{X,n}+D_{Y,n}-1\big)}{2v/\pi+2n},
\end{aligned}$$

$$\begin{aligned}
\left(\partial_\sigma X\tilde{T}\partial_\sigma Y,\partial_\sigma X\tilde{T}\partial_\sigma Y\right) &= -\frac{\alpha'}{8r^2}D^{(+,+,+)}_{XTY}-\frac{\pi^2\alpha'}{16v^2r^2}\sum_{n\neq 0}n^2\big(D_{X,n}D_{Y,n}-D_{X,n}\big)\\
&\quad -\frac{\pi\alpha'}{8vr^2}\sum_{n\neq 0}n^2\frac{D_{X,n}D_{Y,n}+N_0\big(D_{X,n}+D_{Y,n}-1\big)}{2v/\pi+2n},\\
(-\partial_\sigma X\partial_\sigma TY,-\partial_\sigma X\partial_\sigma TY) &= -\frac{\alpha'}{8r^2}D^{(+,+,+)}_{XYT},\\
(\partial_\sigma TY,\partial_\sigma TY) &= -\frac{2}{\pi^2}D^3_{TY},
\end{aligned}$$

$$\begin{aligned}
\left(\Pi_X\tilde{T}\Pi_Y,-\Pi_X\Pi_TY\right) &= \left(-\Pi_X\Pi_TY,\Pi_X\tilde{T}\Pi_Y\right)\\
&= \frac{\alpha'}{8r^2}\tilde{D}^{(+,+,+)}_{XTY}-\frac{\pi\alpha'}{8vr^2}(2N_0+1)\sum_{n\neq 0}n\big(D_{X,n}D_{Y,n}-D_{X,n}\big)\\
&\quad +\frac{\alpha'}{4r^2}\sum_{n\neq 0}n\frac{D_{X,n}D_{Y,n}+N_0\big(D_{X,n}+D_{Y,n}-1\big)}{2v/\pi+2n},\\
\left(\Pi_X\tilde{T}\Pi_Y,\partial_\sigma X\tilde{T}\partial_\sigma Y\right) &= \left(\partial_\sigma X\tilde{T}\partial_\sigma Y,\Pi_X\tilde{T}\Pi_Y\right)\\
&= -\frac{\alpha'}{8r^2}D^{(+,-,+)}_{XTY}+\frac{\pi^2\alpha'}{16v^2r^2}\sum_{n\neq 0}n^2\big(D_{X,n}D_{Y,n}-D_{X,n}\big)\\
&\quad -\frac{\pi\alpha'}{8vr^2}\sum_{n\neq 0}n^2\frac{D_{X,n}D_{Y,n}+N_0\big(D_{X,n}+D_{Y,n}-1\big)}{2v/\pi+2n},\\
\left(\Pi_X\tilde{T}\Pi_Y,-\partial_\sigma X\partial_\sigma TY\right) &= \left(-\partial_\sigma X\partial_\sigma TY,\Pi_X\tilde{T}\Pi_Y\right)\\
&= -\frac{\alpha'}{8r^2}\tilde{D}^{(+,-,-)}_{XTY},
\end{aligned}$$

$$
\begin{aligned}
\Big(-\Pi_X\Pi_T Y, \partial_\sigma X\tilde{T}\partial_\sigma Y\Big) &= \Big(\partial_\sigma X\tilde{T}\partial_\sigma Y, -\Pi_X\Pi_T Y\Big) \\
&= \frac{\alpha'}{8r^2}\tilde{D}^{(+,-,+)}_{XTY} + \frac{\pi\alpha'}{8vr^2}(2N_0+1)\sum_{n\neq 0} n\big(D_{X,n}D_{Y,n} - D_{X,n}\big) \\
&\quad + \frac{\alpha'}{4r^2}\sum_{n\neq 0} n\frac{D_{X,n}D_{Y,n} + N_0\big(D_{X,n}+D_{Y,n}-1\big)}{2v/\pi + 2n}, \\
(-\Pi_X\Pi_T Y, -\partial_\sigma X\partial_\sigma T Y) &= (-\partial_\sigma X\partial_\sigma T Y, -\Pi_X\Pi_T Y) \\
&= \frac{\alpha'}{8r^2}D^{(-,+,-)}_{XYT}, \\
\Big(\partial_\sigma X\tilde{T}\partial_\sigma Y, -\partial_\sigma X\partial_\sigma T Y\Big) &= \Big(-\partial_\sigma X\partial_\sigma T Y, \partial_\sigma X\tilde{T}\partial_\sigma Y\Big) \\
&= -\frac{\alpha'}{8r^2}\tilde{D}^{(+,+,-)}_{XTY}.
\end{aligned}
$$

Again, most of the terms cancel. The remaining contributions are

$$
\begin{aligned}
& -\sum_{m\neq 0}\frac{1}{m}\left[V^{(l)}_{1,-m}V^{(l)}_{1,m}\right]_D \\
= \quad & -\zeta(1)\left[\frac{2}{\pi^2} + \frac{\alpha'}{2r^2}\Big\{N_T(0) + 2N_X(0) + N_Y(0) - \frac{1}{6}\Big\}\right] \\
& -\frac{\alpha'}{r^2}\Bigg[-N_{TY}(2) + \frac{1}{2}N_{TX}(2) - \frac{1}{2}N_{XY}(2) \\
& \qquad + N_T(0)N_Y(2) + N_T(2)N_Y(0) + N_T(2)N_X(0) + N_X(0)N_Y(2) \\
& \qquad - \frac{1}{12}N_T(2) - \frac{1}{12}N_Y(2) - \frac{1}{4}N_T(1) - \frac{3}{4}N_Y(1) \\
& \qquad - N_T(0) - N_X(0) - N_Y(0) + \frac{1}{4}\Bigg] \\
& -\frac{\pi\alpha'}{2vr^2}\left(N_0 + \frac{1}{2}\right)\left(N_X(0) + N_Y(0) - \frac{1}{12}\right) + \frac{2}{\pi^2}\big(N_T(2) + N_Y(2)\big) \\
& + \mathcal{O}(v).
\end{aligned} \tag{G.0.3}
$$

As a result, summing up (G.0.2) and (G.0.3), we find

$$
[H_2^{(l)}]_D = \frac{\alpha'}{r^2}\left[N_T(0) + N_X(0) + N_Y(0) - \frac{1}{8}\right], \tag{G.0.4}
$$

up to higher orders of v.

Appendix H

Supergravity potential between revolving branes

The trajectories are given as

$$
\begin{aligned}
X^\alpha &= \zeta^\alpha, & X^8 &= r\cos\omega\zeta^0, & X^9 &= r\sin\omega\zeta^0, \\
\tilde{X}^\alpha &= \tilde{\zeta}^\alpha, & \tilde{X}^8 &= -r\cos\omega\tilde{\zeta}^0, & \tilde{X}^9 &= -r\sin\omega\tilde{\zeta}^0,
\end{aligned}
\tag{H.0.1}
$$

where $\alpha = 0, 1, \cdots, p$, and $X^\mu, \tilde{X}^\mu = 0$ otherwise. We obtain

$$
\hat{\eta}_{\alpha\beta}(X) \;=\; \begin{bmatrix} -1+v^2 & 0 \\ 0 & \mathbf{1}_p \end{bmatrix} \;=\; \hat{\eta}_{\alpha\beta}(\tilde{X}). \tag{H.0.2}
$$

Therefore,

$$
\hat{\eta}^{\alpha\beta}(X) \;=\; \begin{bmatrix} (-1+v^2)^{-1} & 0 \\ 0 & \mathbf{1}_p \end{bmatrix} \;=\; \hat{\eta}^{\alpha\beta}(\tilde{X}), \tag{H.0.3}
$$

and

$$
\sqrt{-\det\hat{\eta}_{\alpha\beta}(X)} \;=\; \sqrt{1-v^2} \;=\; \sqrt{-\det\hat{\eta}_{\alpha\beta}(\tilde{X})}. \tag{H.0.4}
$$

We also obtain

$$
\partial_\alpha X\cdot\partial_\beta\tilde{X} \;=\; \begin{bmatrix} -1-v^2\cos\omega(\zeta^0-\tilde{\zeta}^0) & 0 \\ 0 & \mathbf{1}_p \end{bmatrix}. \tag{H.0.5}
$$

Then

$$
\hat{\eta}^{\alpha\beta}(X)(\partial_\beta X\cdot\partial_\delta\tilde{X})\hat{\eta}^{\delta\gamma}(\tilde{X})(\partial_\gamma\tilde{X}\cdot\partial_\alpha X) \;=\; \frac{(1+v^2\cos\omega(\zeta^0-\tilde{\zeta}^0))^2}{(1-v^2)^2}+p, \tag{H.0.6}
$$

$$
\det(\partial_\alpha X\cdot\partial_\beta\tilde{X}) \;=\; -(1+v^2\cos\omega(\zeta^0-\tilde{\zeta}^0)). \tag{H.0.7}
$$

Now, we find

$$F_\Phi(X,\tilde{X}) = T_p^2\left(\frac{p-3}{4}\right)^2(1-v^2), \tag{H.0.8}$$

$$F_g(X,\tilde{X}) = T_p^2\left\{-\frac{(p+1)^2}{16}(1-v^2)+\frac{1}{2}\left[\frac{(1+v^2\cos\omega(\zeta^2-\tilde{\zeta}^0))^2}{1-v^2}+p(1-v^2)\right]\right\} \tag{H.0.9}$$

$$F_C(X,\tilde{X}) = -\rho_p^2(1+v^2\cos\omega(\zeta^2-\tilde{\zeta}^0)). \tag{H.0.10}$$

They give

$$F_\Phi(X,\tilde{X})+F_g(X,\tilde{X})+F_C(X,\tilde{X}) = T_p^2\frac{v^4}{2(1-v^2)}\Big(1+\cos\omega(\zeta^0-\tilde{\zeta}^0)\Big)^2. \tag{H.0.11}$$

Then, the effective potential becomes

$$\begin{aligned}
&-2\kappa^2\int d^{p+1}\zeta\int d^{p+1}\tilde{\zeta}\,\Delta(X-\tilde{X})\Big(F_\Phi(X,\tilde{X})+F_g(X,\tilde{X})+F_C(X,\tilde{X})\Big)\\
=\;& -\kappa^2T_p^2\frac{v^4}{1-v^2}\int d^{p+1}\zeta\int d^{p+1}\tilde{\zeta}\,\Delta(X-\tilde{X})\Big(1+\cos\omega(\zeta^0-\tilde{\zeta}^0)\Big)^2.
\end{aligned} \tag{H.0.12}$$

The integral can be rewritten as follows.

$$\begin{aligned}
&\int d^{p+1}\zeta\int d^{p+1}\tilde{\zeta}\,\Delta(X-\tilde{X})\Big(1+\cos\omega(\zeta^0-\tilde{\zeta}^0)\Big)^2\\
=\;& V_p\int d\zeta^0\int d\tilde{\zeta}^0\int\frac{d^{10-p}k}{(2\pi)^{10-p}}\frac{1}{k^2}\Big(1+\cos\omega(\zeta^0-\tilde{\zeta}^0)\Big)^2\\
&\qquad\times\exp\Big(ik_\tau(\zeta^0-\tilde{\zeta}^0)+ik_9r(\cos\omega\zeta^0+\cos\omega\tilde{\zeta}^0)+ik_9r(\sin\omega\zeta^0+\sin\omega\tilde{\zeta}^0)\Big)\\
=\;& V_p\int d\zeta^0\int d\tilde{\zeta}^0(4\pi)^{-\frac{10-p}{2}}\int_0^\infty ds\,s^{-\frac{10-p}{2}}\Big(1+\cos\omega(\zeta^0-\tilde{\zeta}^0)\Big)^2\\
&\qquad\times\exp\left(-\frac{1}{4s}\Big[-(\zeta^0-\tilde{\zeta}^0)^2+r^2(2+2\cos\omega(\zeta^0-\tilde{\zeta}^0)\Big]\right)\\
=\;& V_{p+1}(4\pi)^{-\frac{10-p}{2}}\int d\zeta\int_0^\infty ds\,s^{-\frac{10-p}{2}}e^{-\frac{1}{4s}\left[-\zeta^2+2r^2(1+\cos\omega\zeta)\right]}\left(1+\cos\omega\zeta\right)^2,
\end{aligned} \tag{H.0.13}$$

where $\zeta := \zeta^0-\tilde{\zeta}^0$. To make this integral well-defined, we perform the Wick rotation $\zeta\to -i\zeta$ and the analytic continuation $\omega\to i\omega$. The result is given in (4.3.2) in section 4.3.

Appendix I

ω expansion of SYM potential $\tilde{V}_o(r)$

In this appendix, we evaluate the SYM potential by expanding it with respect to ω/r. Thus its validity is restricted to $\omega < r$. The contributions to the effective potential from bosons and the ghost are

$$\begin{aligned}
\tilde{V}_{o,B} &= -\int_{\Lambda^{-2}}^{\infty}\frac{dt}{t}\int\frac{d^{p+1}k}{(2\pi)^{p+1}}e^{-t(k^2+4r^2)} \\
&\quad\times\left[6+2e^{-t\omega^2+t\frac{8(r\omega)^2}{k^2+4r^2}}\cosh\left(t\sqrt{4\omega^2k_\tau^2+\left(\frac{8(r\omega)^2}{k^2+4r^2}\right)^2}\right)\right] \\
&= \int_{\Lambda^{-2}}^{\infty}\frac{dt}{t}\int\frac{d^{p+1}k}{(2\pi)^{p+1}}e^{-t(k^2+4r^2)}\left[-8+\omega^2\left(2t-4k_\tau^2t^2-\frac{16r^2t}{k^2+4r^2}\right)\right. \\
&\quad\left.+\omega^4\left(-t^2+4k_\tau^2t^3-\frac{4}{3}k_\tau^4t^4-\frac{32k_\tau^2r^2t^3-16r^2t^2}{k^2+4r^2}-\frac{128r^4t^2}{(k^2+4r^2)^2}\right)\right] \\
&\quad+\mathcal{O}(\omega^6)
\end{aligned} \tag{I.0.1}$$

Those from fermions are

$$\begin{aligned}
\tilde{V}_{o,F} &= 4\int_{\Lambda^{-2}}^{\infty}\frac{dt}{t}\int\frac{d^{p+1}k}{(2\pi)^{p+1}}e^{-t(k^2+4r^2)}e^{-t\cdot\frac{\omega^2}{4}}\cdot 2\cosh\left(t\sqrt{\omega^2k_\tau^2+4(r\omega)^2}\right) \\
&= \int_{\Lambda^{-2}}^{\infty}\frac{dt}{t}\int\frac{d^{p+1}k}{(2\pi)^{p+1}}e^{-t(k^2+4r^2)}\left[8+\omega^2\left(-2t+4k_\tau^2t^2+16r^2t^2\right)\right. \\
&\quad\left.+\omega^4\left(\frac{1}{4}t^2-k_\tau^2t^3-4r^2t^3+\frac{1}{3}k_\tau^4t^4+\frac{8}{3}k_\tau^2r^2t^4+\frac{16}{3}r^4t^4\right)\right] \\
&\quad+\mathcal{O}(\omega^6).
\end{aligned} \tag{I.0.2}$$

In the following, we drop the $\mathcal{O}(\omega^0)$ terms since they trivially cancel between bosons and fermions. For the other terms, the t-integration can be done easily.

The bosonic contribution becomes

$$\begin{aligned}
&\int \frac{d^{p+1}k}{(2\pi)^{p+1}} e^{-(k^2+4r^2)/\Lambda^2} \left[\omega^2 \left(\frac{2\Lambda^2 - 4k_\tau^2}{\Lambda^2(k^2+4r^2)} - \frac{16r^2+4k_\tau^2}{(k^2+4r^2)}\right)\right. \\
&+\omega^4 \left(-\frac{3\Lambda^4 - 12\Lambda^2 k_\tau^2 + 4k_\tau^4}{3\Lambda^6(k^2+4r^2)} + \frac{16\Lambda^2 r^2 - 32k_\tau^2 r^2 - \Lambda^4 + 8k_\tau^2\Lambda^2 - 4k_\tau^4}{\Lambda^4(k^2+4r^2)^2}\right. \\
&\left.\left.-\frac{128r^4 - 16\Lambda^2 r^2 + 64k_\tau^2 r^2 - 8k_\tau^2\Lambda^2 + 8k_\tau^4}{\Lambda^2(k^2+4r^2)^3} - \frac{128r^4 + 64k_\tau^2 r^2 + 8k_\tau^4}{(k^2+4r^2)^4}\right)\right] \\
&+\mathcal{O}(\omega^6).
\end{aligned} \tag{I.0.3}$$

The fermionic contribution becomes

$$\begin{aligned}
&\int \frac{d^{p+1}k}{(2\pi)^{p+1}} e^{-(k^2+4r^2)/\Lambda^2} \left[\omega^2 \left(\frac{16r^2 - 2\Lambda^2 + 4k_\tau^2}{\Lambda^2(k^2+4r^2)} + \frac{16r^2+4k_\tau^2}{(k^2+4r^2)^2}\right)\right. \\
&+\omega^4 \left(\frac{64r^4 - 48\Lambda^2 r^2 + 32k_\tau^2 r^2 + 3\Lambda^4 - 12k_\tau^2\Lambda^2 + 4k_\tau^4}{12\Lambda^6(k^2+4r^2)}\right. \\
&+\frac{64r^4 - 32\Lambda^2 r^2 + 32k_\tau^2 r^2 + \Lambda^4 - 8k_\tau^2\Lambda^2 + 4k_\tau^4}{4\Lambda^4(k^2+4r^2)^2} \\
&\left.\left.+\frac{32r^4 - 8\Lambda^2 r^2 + 16k_\tau^2 r^2 - 2k_\tau^2\Lambda^2 + 2k_\tau^4}{\Lambda^2(k^2+4r^2)^3} + \frac{32r^4 + 16k_\tau^2 r^2 + 2k_\tau^4}{(k^2+4r^2)^4}\right)\right] \\
&+\mathcal{O}(\omega^6).
\end{aligned} \tag{I.0.4}$$

In the following, we focus on $p = 3$. By the rotational symmetry, k_τ^2 in the integrand can be replaced with $\frac{1}{4}k^2$. To deal with k_τ^4, we employ the polar coordinates for the momentum. Then

$$\begin{aligned}
\int \frac{d^4k}{(2\pi)^4} f(k^2) k_\tau^4 &= \frac{1}{(2\pi)^4} \int_0^\infty d\kappa\, \kappa^3 f(\kappa^2)\kappa^4 \cdot 4\pi \int_0^\pi d\theta \sin^2\theta \cos^4\theta \\
&= \frac{1}{(2\pi)^4} \int_0^\infty d\kappa\, \kappa^3 f(\kappa^2)\kappa^4 \cdot \frac{\pi^2}{4} \\
&= \int \frac{d^4k}{(2\pi)^4} f(k^2) \cdot \frac{1}{8}k^4.
\end{aligned} \tag{I.0.5}$$

Using this rewriting, the bosonic contribution becomes

$$\int \frac{d^4k}{(2\pi)^4} e^{-(k^2+4r^2)/\Lambda^2} \left[\omega^2 \left(-\frac{1}{\Lambda^2} + \frac{4r^2+\Lambda^2}{\Lambda^2(k^2+4r^2)} - \frac{12r^2}{(k^2+4r^2)^2} \right) \right.$$
$$+\omega^4 \left(-\frac{k^2+4r^2}{6\Lambda^6} + \frac{8r^2+3\Lambda^2}{6\Lambda^6} - \frac{8r^4+24\Lambda^2 r^2}{3\Lambda^6(k^2+4r^2)} \right.$$
$$\left.\left. +\frac{24r^2}{\Lambda^4(k^2+4r^2)^2} - \frac{80r^4}{\Lambda^2(k^2+4r^2)^3} - \frac{80r^4}{(k^2+4r^2)^4} \right) \right] + \mathcal{O}(\omega^6). \quad \text{(I.0.6)}$$

The fermionic contribution becomes

$$\int \frac{d^4k}{(2\pi)^4} e^{-(k^2+4r^2)/\Lambda^2} \left[\omega^2 \left(\frac{1}{\Lambda^2} + \frac{12r^2-\Lambda^2}{\Lambda^2(k^2+4r^2)} + \frac{12r^2}{(k^2+4r^2)^2} \right) \right.$$
$$+\omega^4 \left(\frac{k^2+4r^2}{24\Lambda^6} + \frac{8r^2-3\Lambda^2}{24\Lambda^6} + \frac{10r^4-6\Lambda^2 r^2}{3\Lambda^6(k^2+4r^2)} \right.$$
$$\left.\left. +\frac{10r^4-4\Lambda^2r^2}{\Lambda^4(k^2+4r^2)^2} + \frac{20r^4-4\Lambda^2r^2}{\Lambda^2(k^2+4r^2)^3} + \frac{20r^4}{(k^2+4r^2)^4} \right) \right] + \mathcal{O}(\omega^6). \quad \text{(I.0.7)}$$

The k-integration can be done as follows.

$$\begin{aligned}
& \int \frac{d^4k}{(2\pi)^4} e^{-(k^2+4r^2)/\Lambda^2} \frac{1}{(k^2+4r^2)^n} \\
= \; & \frac{2\pi^2}{(2\pi)^4} \int_0^\infty d\kappa\, \kappa^3 e^{-(\kappa^2+4r^2)/\Lambda^2} \frac{1}{(\kappa^2+4r^2)^n} \\
= \; & \frac{1}{16\pi^2} \int_0^\infty du\, e^{-(u+4r^2)/\Lambda^2} \frac{u}{(u+4r^2)^n} \\
= \; & \frac{1}{16\pi^2} \int_{4r^2}^\infty du\, e^{-u/\Lambda^2} \frac{u-4r^2}{u^n} \\
= \; & \frac{1}{16\pi^2} (4r^2)^{2-n} \int_1^\infty du\, e^{-4r^2u/\Lambda^2} \frac{u-1}{u^n} \\
= \; & \frac{1}{16\pi^2} (4r^2)^{2-n} \left(E_{n-1}(4r^2/\Lambda^2) - E_n(4r^2/\Lambda^2) \right),
\end{aligned} \quad \text{(I.0.8)}$$

where $E_n(x)$ are defined as

$$E_n(x) \; := \; \int_1^\infty du\, \frac{e^{-xu}}{u^n}. \quad \text{(I.0.9)}$$

For $n \leq 0$, they are elementary functions:

$$E_0(x) = \frac{1}{x} e^{-x}, \qquad E_{-1}(x) = \frac{x+1}{x^2} e^{-x}, \qquad E_{-2}(x) = \frac{x^2+2x+2}{x^3} e^{-x}, \quad \text{(I.0.10)}$$

etc. Note that $E_n(x)$ with $n > 1$ satisfy

$$\begin{aligned} E_n(x) &= -\frac{e^{-xu}}{(n-1)u^{n-1}}\Big|_1^\infty - \frac{x}{n-1}\int_1^\infty du\,\frac{e^{-xu}}{u^{n-1}} \\ &= \frac{e^{-x}}{n-1} - \frac{x}{n-1}E_{n-1}(x). \end{aligned} \tag{I.0.11}$$

Using these recursion relations, the effective potential can be written in terms of $E_1(x)$ and elementary functions. The bosonic contribution becomes

$$\begin{aligned} &\omega^2\left[\frac{r^2}{\pi^2}e^{-4r^2/\Lambda^2} - \left(\frac{r^2}{\pi^2} + \frac{4r^4}{\pi^2\Lambda^2}\right)E_1(4r^2/\Lambda^2)\right] \\ &+\omega^4\left[\left(-\frac{1}{24\pi^2} - \frac{2r^2}{3\pi^2\Lambda^2} - \frac{10r^4}{3\pi^2\Lambda^4}\right)e^{-4r^2/\Lambda^2} + \left(\frac{6r^4}{\pi^2\Lambda^4} + \frac{40r^6}{3\pi^2\Lambda^6}\right)E_1(4r^2/\Lambda^2)\right] \\ &+\mathcal{O}(\omega^6). \end{aligned} \tag{I.0.12}$$

The fermionic contribution becomes

$$\omega^2\left[\frac{r^2}{\pi^2}E_1(4r^2/\Lambda^2)\right] + \omega^4\left[\left(-\frac{1}{48\pi^2} + \frac{r^2}{12\pi^2\Lambda^2}\right)e^{-4r^2/\Lambda^2}\right] + \mathcal{O}(\omega^6). \tag{I.0.13}$$

The sum of these two contributions is

$$\begin{aligned} &\omega^2\left[\frac{r^2}{\pi^2}e^{-4r^2/\Lambda^2} - \frac{4r^2}{\pi^2\Lambda^2}E_1(4r^2/\Lambda^2)\right] \\ &+\omega^4\left[\left(-\frac{1}{16\pi^2} - \frac{7r^2}{12\pi^2\Lambda^2} - \frac{10r^4}{3\pi^2\Lambda^4}\right)e^{-4r^2/\Lambda^2} + \left(\frac{6r^4}{\pi^2\Lambda^4} + \frac{40r^6}{3\pi^2\Lambda^6}\right)E_1(4r^2/\Lambda^2)\right] \\ &+\mathcal{O}(\omega^6). \end{aligned} \tag{I.0.14}$$

Performing the analytic continuation of ω, this becomes

$$\begin{aligned} &-\omega^2\left[\frac{r^2}{\pi^2}e^{-4r^2/\Lambda^2} - \frac{4r^2}{\pi^2\Lambda^2}E_1(4r^2/\Lambda^2)\right] \\ &-\omega^4\left[\left(\frac{1}{16\pi^2} + \frac{7r^2}{12\pi^2\Lambda^2} + \frac{10r^4}{3\pi^2\Lambda^4}\right)e^{-4r^2/\Lambda^2} - \left(\frac{6r^4}{\pi^2\Lambda^4} + \frac{40r^6}{3\pi^2\Lambda^6}\right)E_1(4r^2/\Lambda^2)\right] \\ &+\mathcal{O}(\omega^6). \end{aligned} \tag{I.0.15}$$

The ω-independent terms which we have dropped at the beginning are, as noted, trivially cancelled between bosons and fermions,

$$\frac{\Lambda^4}{16\pi^2}E_3(4r^2/\Lambda^2)\cdot(-8+8) = 0. \tag{I.0.16}$$

Appendix J

r expansion of SYM potential $\tilde{V}_o(r)$

In the region $r < \omega$, the expansion of the effective potential in Appendix I is no longer valid and we need another method to approximate it. In this appendix, we set $r = \beta\omega$ and approximate the effective potential in terms of β-expansion. Thus this evaluation of the effective action is valid in the region of $r < \omega$. The bosonic and fermionic contributions to the effective action for $p = 3$ are rewritten as

$$\begin{aligned}\tilde{V}_{o,B}(\omega,\beta) &= -\omega^4 \int_{\omega^2/\Lambda^2}^{\infty} \frac{dt}{t} \int \frac{d^4k}{(2\pi)^4} e^{-Bt} \left[6 + 2e^{-t} e^{t\frac{8\beta^2}{B}} \cosh\left(2t\sqrt{k_\tau^2 + \left(\frac{4\beta^2}{B}\right)^2} \right) \right] \\ \tilde{V}_{o,F}(\omega,\beta) &= 8\omega^4 \int_{\omega^2/\Lambda^2}^{\infty} \frac{dt}{t} \int \frac{d^4k}{(2\pi)^4} e^{-Bt} e^{-t/4} \cdot \cosh\left(t\sqrt{k_\tau^2 + 4\beta^2} \right) \end{aligned} \qquad \text{(J.0.1)}$$

where $B = k^2 + 4\beta^2$. Since the integral

$$-\omega^4 \int_{\omega^2/\Lambda^2}^{\infty} \frac{dt}{t} \int \frac{d^4k}{(2\pi)^4} e^{-Bt} \left[6 + 2e^{-t}\cosh(2tk_\tau) - 8e^{-t/4}\cosh(tk_\tau) \right] \qquad \text{(J.0.2)}$$

which is obtained by setting $\beta = 0$ in the integrands except for the factor e^{-Bt} in the each of contributions vanishes, we can subtract it from the total potential. Therefore the total potential can be written as

$$\tilde{V}_o(\omega,\beta) = \tilde{V}'_{o,B}(\omega,\beta) + \tilde{V}'_{o,F}(\omega,\beta), \qquad \text{(J.0.3)}$$

where

$$
\begin{aligned}
\tilde{V}'_{o,B}(\omega,\beta) &= -\omega^4 \int_{\omega^2/\Lambda^2}^{\infty} \frac{dt}{t} \int \frac{d^4k}{(2\pi)^4} e^{-Bt} 2e^{-t} \\
&\quad \times \left[e^{t\frac{8\beta^2}{B}} \cosh\left(2t\sqrt{k_\tau^2 + \left(\frac{4\beta^2}{B}\right)^2} \right) - \cosh(2tk_\tau) \right], \\
\tilde{V}'_{o,F}(\omega,\beta) &= 8\omega^4 \int_{\omega^2/\Lambda^2}^{\infty} \frac{dt}{t} \int \frac{d^4k}{(2\pi)^4} e^{-Bt} e^{-t/4} \\
&\quad \times \left[\cosh\left(t\sqrt{k_\tau^2 + 4\beta^2} \right) - \cosh(tk_\tau) \right].
\end{aligned} \tag{J.0.4}
$$

We now expand the square brackets in each of the above equations with respect to t and pick up the terms proportional to β^2. We find

$$
\begin{aligned}
e^{t\frac{8\beta^2}{B}} \sum_{n=0}^{\infty} \frac{4^n}{(2n)!} t^{2n} \left(k_\tau^2 + \frac{16\beta^4}{B^2} \right)^n \Bigg|_{\beta^2} &= 8\frac{\beta^2}{k^2} \sum_{n=0}^{\infty} \frac{4^n}{(2n)!} t^{2n+1} k_\tau^{2n}, \\
\sum_{n=0}^{\infty} \frac{1}{(2n)!} t^{2n} \left(k_\tau^2 + 4\beta^2 \right)^n \Bigg|_{\beta^2} &= 4\beta^2 \sum_{n=0}^{\infty} \frac{n+1}{(2n+2)!} t^{2n+2} k_\tau^{2n}.
\end{aligned} \tag{J.0.5}
$$

Then, by rescaling the integration variable k as

$$
\begin{aligned}
\int \frac{d^4k}{(2\pi)^4} \frac{e^{-tk^2} k_\tau^{2n}}{k^2} &= t^{-(n+1)} \int \frac{d^4k}{(2\pi)^4} \frac{e^{-k^2} k_\tau^{2n}}{k^2}, \\
\int \frac{d^4k}{(2\pi)^4} e^{-tk^2} k_\tau^{2n} &= t^{-(n+2)} \int \frac{d^4k}{(2\pi)^4} e^{-k^2} k_\tau^{2n},
\end{aligned} \tag{J.0.6}
$$

the leading order terms in (J.0.4) become

$$
\begin{aligned}
\tilde{V}'_{o,B}(\omega,\beta) &= -16\beta^2\omega^4 \sum_{n=0}^{\infty} \frac{4^n}{(2n)!} \int_{\omega^2/\Lambda^2}^{\infty} \frac{dt}{t} e^{-t} t^n \int \frac{d^4k}{(2\pi)^4} \frac{e^{-k^2} k_\tau^{2n}}{k^2} + \mathcal{O}(\beta^4), \\
\tilde{V}'_{o,F}(\omega,\beta) &= 16\beta^2\omega^4 \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \int_{\omega^2/\Lambda^2}^{\infty} \frac{dt}{t} e^{-t/4} t^n \int \frac{d^4k}{(2\pi)^4} e^{-k^2} k_\tau^{2n} + \mathcal{O}(\beta^4).
\end{aligned} \tag{J.0.7}
$$

We can perform the momentum integrations as

$$
\begin{aligned}
\int \frac{d^4k}{(2\pi)^4} \frac{e^{-k^2} k_\tau^{2n}}{k^2} &= \frac{1}{16\pi^2} \cdot \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(n+1)}, \\
\int \frac{d^4k}{(2\pi)^4} e^{-k^2} k_\tau^{2n} &= \frac{1}{16\pi^2} \cdot \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}}.
\end{aligned} \tag{J.0.8}
$$

We find that the summation can be performed as follows: for $\tilde{V}'_{o,B}(\omega,\beta)$,

$$\begin{aligned}
& \sum_{n=0}^{\infty}\frac{4^n}{(2n)!}\int_{\omega^2/\Lambda^2}^{\infty}\frac{dt}{t}e^{-t}t^n\int\frac{d^4k}{(2\pi)^4}\frac{e^{-k^2}k_\tau^{2n}}{k^2} \\
= & \frac{1}{16\pi^2}\int_{\omega^2/\Lambda^2}^{\infty}\frac{dt}{t}e^{-t}\sum_{n=0}^{\infty}\frac{4^n}{(2n)!}\frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}}t^n \\
= & \frac{1}{16\pi^2}\int_{\omega^2/\Lambda^2}^{\infty}\frac{dt}{t}e^{-t}\sum_{n=0}^{\infty}\frac{4^n}{(2n)!}\frac{(2n)!}{4^n(n+1)!}t^n \\
= & \frac{1}{16\pi^2}\int_{\omega^2/\Lambda^2}^{\infty}\frac{dt}{t}e^{-t}\frac{e^t-1}{t} \\
= & \frac{1}{16\pi^2}\frac{\Lambda^2}{\omega^2}\left(1-E_2(\omega^2/\Lambda^2)\right), \qquad \text{(J.0.9)}
\end{aligned}$$

and for $\tilde{V}'_{o,F}(\omega,\beta)$,

$$\begin{aligned}
& \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\int_{\omega^2/\Lambda^2}^{\infty}\frac{dt}{t}e^{-t/4}t^n\int\frac{d^4k}{(2\pi)^4}e^{-k^2}k_\tau^{2n} \\
= & \frac{1}{16\pi^2}\int_{\omega^2/\Lambda^2}^{\infty}\frac{dt}{t}e^{-t/4}\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}}t^n \\
= & \frac{1}{16\pi^2}\int_{\omega^2/\Lambda^2}^{\infty}\frac{dt}{t}e^{-t/4}\sum_{n=0}^{\infty}\frac{2\sqrt{\pi}}{4^{n+1}n!\,\Gamma(n+\frac{3}{2})}\frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}}t^n \\
= & \frac{1}{16\pi^2}\int_{\omega^2/\Lambda^2}^{\infty}\frac{dt}{t}e^{-t/4}\cdot\frac{1}{2}\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+\frac{3}{2})}\frac{1}{n!}\left(\frac{t}{4}\right)^n \\
= & \frac{1}{16\pi^2}\int_{\omega^2/\Lambda^2}^{\infty}\frac{dt}{t}e^{-t/4}F(\tfrac{1}{2},\tfrac{3}{2};\tfrac{t}{4}). \qquad \text{(J.0.10)}
\end{aligned}$$

Therefore, the total potential becomes

$$\tilde{V}_o = \frac{\beta^2\omega^4}{\pi^2}\left(-\frac{\Lambda^2}{\omega^2}\left(1-E_2(\omega^2/\Lambda^2)\right)+\int_{\omega^2/\Lambda^2}^{\infty}\frac{dt}{t}e^{-t/4}F(\tfrac{1}{2},\tfrac{3}{2};\tfrac{t}{4})\right)+\mathcal{O}(\beta^4). \qquad \text{(J.0.11)}$$

If we also assume $\omega \ll \Lambda$, then

$$-\frac{\Lambda^2}{\omega^2}\left(1-E_2(\omega^2/\Lambda^2)\right) = \log\frac{\omega^2}{\Lambda^2}-1+\gamma+\mathcal{O}(\omega^2/\Lambda^2), \qquad \text{(J.0.12)}$$

and

$$\begin{aligned}\int_{\omega^2/\Lambda^2}^{\infty}\frac{dt}{t}e^{-t/4}F(\tfrac{1}{2},\tfrac{3}{2};\tfrac{t}{4}) &= E_1(\omega^2/4\Lambda^2)+\int_0^{\infty}\frac{dt}{t}e^{-t/4}\left(F(\tfrac{1}{2},\tfrac{3}{2};\tfrac{t}{4})-1\right)\\ &\quad+\int_0^{\omega^2/\Lambda^2}\frac{dt}{t}e^{-t/4}\left(F(\tfrac{1}{2},\tfrac{3}{2};\tfrac{t}{4})-1\right)\\ &= -\gamma-\log\frac{\omega^2}{4\Lambda^2}+2-\log 4+\mathcal{O}(\omega^2/\Lambda^2)\end{aligned} \quad (J.0.13)$$

imply

$$\tilde{V}_o=\frac{\omega^4\beta^2}{\pi^2}=\frac{\omega^2 r^2}{\pi^2} \quad (J.0.14)$$

in the $r<\omega<\Lambda$ region. The potential in eq.(4.4.8) is obtained by analytical continuation back to the Lorentzian signature.

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