Triangulation of the Amplituhedron from Sign Flips

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Abstract

One of the most fundamental and important physical observables in any QFT is the scattering amplitude, which describes the scattering processes of elementary particles. Recently there is much theoretical progress in understanding and computing scattering amplitudes and these lead the discovery of new mathematical structures “Positive geometry”.

Many of the recent developments have been driven from the $\mathcal{N} = 4$ Super-Yang-Mills theory (SYM) in the planar limit. The first and exciting example of the positive geometry is the amplituhedron, which is obtained from the planar $\mathcal{N} = 4$ SYM. The amplituhedron is a purely geometric object which defines the scattering amplitude in planar $\mathcal{N} = 4$ SYM. It is conjectured that the scattering amplitude (loop integrand) of planar $\mathcal{N} = 4$ SYM at any loop order is given by a “canonical form” on the amplituhedron which has logarithmic singularities on all of its boundaries.

To date there is one completely general and in principle straightforward way to obtain the canonical form: by triangulating the amplituhedron into elementary cells for which the canonical form is easy to compute, and subsequently summing the individual pieces. However, it is difficult to triangulate general amplituhedron because of its non-trivial structure. From this, obtaining a systematic way to triangulate the general amplituhedron is an open problem.

In this thesis, we investigate a systematic way to triangulate the general amplituhedron. Once we triangulate the amplituhedron, we can obtain the canonical form. Recently the topological definition of the amplituhedron “sign flip definition” is proposed. We find that by using this topological definition, we can triangulate 2-loop MHV amplituhedron and obtain the canonical form. We also find that this definition makes it possible to triangulate 1-loop NMHV amplituhedron and reveals new representations of the 1-loop NMHV integrand that we have never known from the planar $\mathcal{N} = 4$ SYM

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Introduction

The traditional formulation of quantum field theory (QFT) is constructed from the two principles: Locality and Unitarity [1]. The standard calculations of the scattering amplitudes from lagrangians or path integrals make these two principles manifest. However, because of this, a large amount of unphysical redundancies (field redefinitions, gauge redundancies) are introduced.

In 1985, Park and Taylor discovered a surprisingly simple result for the tree scattering amplitude of six gluons by using the spinor-helicity variables [2]. In the original Feynman diagram calculation, there are 220 Feynman diagrams and 100 pages of the calculation are needed. They also generalized this for n gluons tree amplitude and this result showed the same simplicity [3]. This is totally hidden in the original Feynman diagram calculation. This means that the original formalism of QFT is completely hiding some properties of the physical observables we have never known. To reveal these hidden properties, new alternative methods for calculations both tree-level and loop amplitudes are proposed. In 2005 Britto, Cachazo, Feng, and Witten found new on-shell recursion relations (BCFW) for tree-level amplitudes of gluons [4], [5]. These recursion relations are derived from the contour integrations on the complex momentum and the factorization property which comes from the unitarity.

Recently many developments have been driven in the planar limit of the $\mathcal{N} = 4$ Super Yang-Mills (SYM). It has been known that this theory has super conformal symmetry. In 2008, the hidden symmetry "Dual super conformal symmetry" is discovered [6], which is not manifest in the ordinal formulation and had not been found for long years. Combining these two symmetries, the infinite symmetry "Yangian symmetry" is constructed [7]. The discovery of these hidden symmetries motivated to find new alternative representations that make all symmetries manifest. In 2009, the new variable "Momentum twistor variable" is proposed by Hodges [8]. When scattering amplitudes are written in this variable, all symmetries become manifest. In 2010, the BCFW recursion relations are generalized into the integrand of loop amplitudes in planar $\mathcal{N} = 4$ SYM by using this momentum twistor space [9]. By using this, all loop integrands are systematically and efficiently constructed from lower-loop integrands recursively. In this BCFW representation, there are non-local spurious poles that have never appeared in the ordinal Feynman diagrams expansion. Instead of these spurious poles, this representation makes all symmetries of the amplitude manifest term-by-term, including the Yangian symmetry which is hidden in the original formulation.

The important progress is the discovery of the geometric structure of the scattering amplitudes. The first example of this geometric picture is found by Hodges [8]. He found that the tree NMHV amplitude in this momentum twistor variable can be interpreted as a volume of a polytope and the BCFW recursion relation for the tree NMHV amplitude is interpreted as a triangulation of this polytope. In 2012, the connection between on-shell scattering amplitudes in planar $\mathcal{N} = 4$ SYM and the fundamental object in algebraic geometry "Positive Grassmannian $G_+(k, n)$" is found [10]. In 2013, Arkani-Hamed and Trnka generalized these geometric pictures and found a completely new geometric object: Amplituhedron [11]. This is defined as a generalization of the positive Grassmannian, this means that in the definition of the amplituhedron is purely geometric. It is conjectured that the scattering amplitude (loop integrand) of planar $\mathcal{N} = 4$ SYM at any loop order is given by a "canonical form" on the amplituhedron which has logarithmic singularities on all of its boundaries. The BCFW recursion relation of all loop integrands is interpreted as one of the triangulation of the amplituhedron. The remarkable point is that Unitarity and Locality of the scattering amplitudes are derived from the "positivity", which is a property of the amplituhedron. This is the first example of the connection between the physical observable and new geometric structure "positive geometry" [12]. Recently many example of this relation are found: the wavefunction of the universe and the cosmological polytope [13], tree amplitudes in the bi-adjoint ϕ^3 scalar theory [14] and its generalization [15–18] and the conformal bootstrap and the cyclic polytope [19].

The amplituhedron has been explored from a variety of perspectives in the past few years [20–28]. The amplituhedron gives a geometric picture for tree amplitudes and loop integrands as a canonical form of the amplituhedron. Calculating amplitudes or loop integrands starting from the amplituhedron requires the construction of the canonical form associated to the geometry. To do this, we need to triangulate the amplituhedron into a more simple one that it is easy to obtain the form. However, already at tree level and much more at loop level, the geometry of the amplituhedron is highly non-trivial and because of this, it is difficult to triangulate general amplituhedron and it remains an open problem.

We focus on this problem and investigate the triangulation of the general amplituhedron by using the topological definition of the amplituhedron. Recently the topological definition of the amplituhedron is proposed [29]. In this definition, the amplituhedron is defined from the boundary inequalities and the sign flip characterization. This new definition gives us a completely new and clear understanding of the geometry of the loop amplituhedron. For example, in the MHV case, the higher loop amplituhedron is decomposed into the one loop MHV amplituhedron and conditions of the positivity among the different loop momentum variables. This extremely simple picture of the loop MHV amplituhedron makes it easy to consider the triangulation of the amplituhedron. The topological definition of the N^kMHV loop amplituhedron is more interesting. For example, N^kMHV one loop amplituhedron is constructed as an intersection of the two lower-dimensional amplituhedra. This is not obvious from the original definition. The remarkable point is that we can triangulate the lower-dimensional amplituhedron by using this topological definition [12, 29]. These will lead us to a triangulation of the 1-loop NMHV amplituhedron.

In this thesis, we investigate the triangulation of the loop amplituhedron. First, we consider the 2-loop MHV amplituhedron. We see that the 2-loop MHV amplituhedron can be triangulated by using the topological definition. From this, we obtain the canonical form of the n-point 2-loop MHV amplituhedron. The representation of the 2-loop MHV integrand from this canonical form looks completely different from the BCFW representation which obtained from the planar $\mathcal{N} = 4$ SYM. This is a new feature that starts from the 2-loop level, the 1-loop MHV case the canonical form obtained from the geometry is corresponding to the BCFW representation.

Next, we consider the 1-loop NMHV amplituhedron. We obtain an explicit representation of the n-point 1-loop NMHV amplituhedron as a product of two lower-dimensional amplituhedra by using the topological definition. This is a completely new representation that we have never known from the planar $\mathcal{N} = 4$ SYM or BCFW triangulation. From this, we triangulate this 1-loop NMHV amplituhedron explicitly and obtain the canonical form. We see that this canonical form is expressed as a product of two canonical forms of the lower-dimensional amplituhedra. We will also give another new representation of the 1-loop NMHV amplituhedron, "super-local representation". The super-local means both of external poles and internal poles are local. In this representation, the positivity of this form is manifest term-by-term. The positivity of the canonical form is related to the existence of a "dual amplituhedron". This positivity suggests the existence of a dual amplituhedron for the 1-loop NMHV amplituhedron.

This thesis is organized as follows. In section 1, we briefly review the basic notions of scattering amplitudes and the BCFW recursion relation. In section 2, we introduce the main object in this thesis: the amplituhedron. First, we review two definitions of the amplituhedron: a generalization of the convex polygon and the topological definition. Then we introduce the canonical form and see how to extract scattering amplitudes from this form. In section 3, we give an explicit triangulation of the 2-loop MHV amplituhedron by using the topological definition. And we introduce another geometric object: 2-loop log amplituhedron, whose canonical form is corresponding to the log of the 2-loop MHV integrand. In section 4, we give an explicit expression of the 1-loop NMHV amplituhedron as the product of the lower-dimensional amplituhedron and then we construct the canonical form of this representation by using the "sign flip triangulation". From this expression, we obtain a new representation "super-local representation" for the 1-loop NMHV integrand.

Chapter 1

Scattering amplitudes in planar $\mathcal{N} = 4$ SYM

In this thesis, we focus on scattering amplitudes in planar $\mathcal{N} = 4$ SYM. In section 1.1, we briefly review the basic notions for scattering amplitudes. In section 1.2, we review the tree amplitudes and loop integrands in planar $\mathcal{N} = 4$ SYM. And we apply the BCFW recursion relation for loop integrands. In section 1.3, we review the polytope picture of the tree amplitude in planar $\mathcal{N} = 4$ SYM introduced by Hodges.

1.1 Basic notions for scattering amplitudes

The scattering amplitude A is defined as an inner product of two asymptotic states, the initial state and the final state:

$$A = \langle \text{out}; t = +\infty | \text{in}; t = -\infty \rangle_S = \langle \text{out} | S | \text{in} \rangle_H, \tag{1.1.1}$$

where $\langle \dots \rangle_S$ means the Schrödinger picture and $\langle \dots \rangle_H$ means the Heisenberg picture. The operator S in the Heisenberg picture is called S-matrix. In the traditional formulation of the quantum field theory, we can calculate this scattering amplitude A from the Lagrangian by using Feynman diagrams expansion. Once we obtain the amplitude, we can calculate the differential cross-section $\frac{d\sigma}{d\Omega} \sim |A|^2$. Finally the cross-section σ can be obtained by integration of $\frac{d\sigma}{d\Omega}$ over angles and multiply appropriate symmetry factors. These quantities σ and $\frac{d\sigma}{d\Omega}$ are the observables of the particle physics experiments. However, it is clear that the on-shell scattering amplitude A is the building block of these observables. These on-shell amplitudes

A are the subject of this thesis.

1.1.1 Color decomposition

Here we introduce the color decomposition of the amplitude. Generally, scattering amplitudes in (s)YM have kinematic degrees of freedom and color degrees of freedom. Here we see that these two degrees of freedoms can be decomposed not only tree level but also loop level amplitudes in the large N limit. The color dependence of scattering amplitudes arises from contractions of the structure constants of $SU(N)$. First, we normalize the generators as $\mathrm{Tr}(T^aT^b) = \delta^{ab}$ and $[T^a, T^b] = i\tilde{f}^{abc}T^c$. It can be shown that

$$i\tilde{f}^{abc} = \mathrm{Tr}(T^aT^bT^c) - \mathrm{Tr}(T^bT^aT^c). \tag{1.1.2}$$

And there is $SU(N)$ Fierz identity:

$$(T^a)_i^{\ j}(T^a)_k^{\ l} = \delta_i^l\delta_k^j - \frac{1}{N}\delta_i^j\delta_k^l \tag{1.1.3}$$

From these relations, the color contributions can be written as products of traces of generator. Any n-point tree amplitude involving any particles that transform in the adjoint of the gauge group becomes

$$A_n^{\text{full,tree}}(\{p_i, h_i, a_i\}) = g^{n-2} \sum_{\text{perm}\sigma} A_n[1^{h_1}\sigma(2^{h_2}\cdots n^{h_n})]\mathrm{Tr}(T^{a_1}T^{\sigma(a_2}\cdots T^{a_n)}). \tag{1.1.4}$$

where each particle is labeled by its on-shell momentum p_i, helicity h_i and color index a_i. This is called "color-decomposition" and A_n is the partial amplitude. In this tree case, there are only single trace structures. For loop amplitudes, there are multi trace structures in addition to the simple single trace. However, there exists a limit of gauge theory in which only single trace structures appear at every loop level. This is called the "large N limit". The l-loop amplitude in $N \to \infty$ is written as

$$A_n^{\text{full},l\text{-loop}}(\{p_i, h_i, a_i\}) = g^{n-2}(g^2N)^l \sum_{\text{perm}\sigma} A_n^{l\text{-loop}}[1^{h_1}\sigma(2^{h_2}\cdots n^{h_n})]\mathrm{Tr}(T^{a_1}T^{\sigma(a_2}\cdots T^{a_n)}). \tag{1.1.5}$$

In this thesis, we usually focus on this limit and consider partial amplitudes.

1.1.2 Spinor helicity variable

When we consider scattering amplitudes, we need to impose on-shell conditions by hand. To make this condition manifest, the "spinor-helicity variable" is introduced. First, we provide a two-dimensional matrix representation of the four-momentum as

$$p^{\dot{a}a} = p^{\mu}(\bar{\sigma}_{\mu})^{\dot{a}a} = p_{\mu}(\bar{\sigma}^{\mu})^{\dot{a}a} \tag{1.1.6}$$

where $(\bar{\sigma}_{\mu})^{\dot{a}a} = (1, \sigma_1, \sigma_2, \sigma_3)^{\dot{a}a}$. These spinor indices are raised and lowered by the $\epsilon_{ab}, \epsilon_{\dot{a}\dot{b}}$ and

$$p_{\dot{a}a} = \epsilon_{ab}\epsilon_{\dot{a}\dot{b}}(\bar{\sigma}_{\mu})^{\dot{b}b}p^{\mu} = (\bar{\sigma}_{\mu})_{\dot{a}a}p^{\mu}. \tag{1.1.7}$$

If a 2×2 matrix has vanishing determinant, it can be written as a product of two 2-component vectors: $\lambda_a, \tilde{\lambda}_{\dot{a}}$ as

$$p^{\dot{a}a} = \tilde{\lambda}^{\dot{a}}\lambda^{a}, \quad p_{a\dot{a}} = \lambda_a\tilde{\lambda}_{\dot{a}}. \tag{1.1.8}$$

This $\lambda_a, \tilde{\lambda}_{\dot{a}}$ are called "spinor-helicity variables". Even if the four-momentum p^{μ} is constrained by the on-shell condition, these spinor-helicity variables are unconstrained variables. Here we introduce a shorthand notation "angle, square representation",

$$\lambda_{ia} \to |i\rangle_a, \quad \lambda_i^a \to \langle i|^a, \quad \tilde{\lambda}_i^{\dot{a}} \to |i]^{\dot{a}}, \quad \tilde{\lambda}_{i\dot{a}} \to [i|_{\dot{a}}. \tag{1.1.9}$$

In this thesis, we will use this notation.

The Lorentz invariants which made from these variables are expressed as

$$\epsilon_{ab}\lambda_i^a\lambda_j^b = \lambda_i^a\lambda_{ja} = \langle ij\rangle, \quad \epsilon_{\dot{a}\dot{b}}\tilde{\lambda}_i^{\dot{b}}\tilde{\lambda}_j^{\dot{a}} = \tilde{\lambda}_{i\dot{a}}\tilde{\lambda}_j^{\dot{a}} = [ij]. \tag{1.1.10}$$

Scattering amplitudes are constructed from the Lorentz invariants. The momentum conservation can be written as

$$\sum_{i=1}^{n} p_i = 0 \to \sum_{i=1}^{n}\lambda_i\tilde{\lambda}_i = 0 \to \sum_{i=1}^{n}\langle ai\rangle[ib] = 0 \tag{1.1.11}$$

here we take contractions with arbitrary λ_a and $\tilde{\lambda}_b$. We can see that relations (1.1.8) are invariant under any phase transformation

$$\lambda^a \to e^{i\phi}\lambda^a, \quad \tilde{\lambda}^{\dot{a}} \to e^{-i\phi}\tilde{\lambda}^{\dot{a}}. \tag{1.1.12}$$

We can identify this $U(1)$ transformation as the little group transformation of the massless particle. Then this $U(1)$ transformation is called "little group scaling". The standard convention is that the spinors λ and $\tilde{\lambda}$ carry helicities $-1/2$ and $+1/2$, respectively. For complex momenta, the angle and square spinors are independent, then the little group scaling is extended to any non-zero complex number t

$$\lambda^a \to t\lambda^a, \quad \tilde{\lambda}^{\dot{a}} \to t^{-1}\tilde{\lambda}^{\dot{a}}. \tag{1.1.13}$$

When we consider scattering amplitudes for spin-1 massless particles, we need to introduce polarization vectors $\epsilon_\pm(p_i)$ for each particle. These vectors satisfy some conditions

$$\epsilon_\pm(p)\cdot p = 0, \quad \epsilon_\pm(p)\cdot(\epsilon_\pm(p))^* = -1, \quad \epsilon_\pm(p)\cdot(\epsilon_\mp(p))^* = 0. \tag{1.1.14}$$

We can write these polarization vectors in spinor-helicity variables as

$$\epsilon_-^\mu(p;q) = -\frac{\langle p|\gamma^\mu|q]}{\sqrt{2}[qp]}, \quad \epsilon_+^\mu(p;q) = -\frac{\langle q|\gamma^\mu|p]}{\sqrt{2}\langle qp\rangle} \tag{1.1.15}$$

where γ is Gamma-matrix and q is an arbitrary reference spinor. The arbitrariness in the choice of reference spinor reflects gauge invariance $\epsilon_\pm^\mu(p) \to \epsilon_\pm^\mu(p) + Cp_\mu$ This does not change the on-shell amplitude A_n. When summing over all diagrams, the final result is independent of the choices of the reference spinor q.

Next, we see some examples of the gluon scattering amplitudes in this spinor-helicity variable. It can be shown that both tree gluon amplitudes with all-plus (or all minus) helicity and with only one particle has different helicity vanish [30]:

$$A_n^{\text{tree}}(1^\pm, 2^\pm, \ldots, n^\pm) = 0, \quad A_n^{\text{tree}}(1^\mp, 2^\pm, \ldots, n^\pm) = 0. \tag{1.1.16}$$

Notice that (1.1.16) holds for $n \geq 4$ case. The three-point amplitudes are exceptional: $A_n^{\text{tree}}(i^-, j^\mp, k^+) \neq 0$. From (1.1.16) we can see the first non-zero helicity amplitudes are those involving two negative helicity gluons. These amplitudes are called maximally helicity violating (MHV) amplitudes. The MHV n-point gluon amplitude in the spinor-helicity variables is given as

$$A_n^{\text{tree}}(1^+, \ldots, i^-, \ldots, j^-, \ldots, n^+) = \delta^4(\sum_{i=1}^n p_i)\frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}. \tag{1.1.17}$$

This is known as Parke-Taylor formula [3]. The amplitudes involving three negative-helicity gluons are called next-to-MHV (NMHV). In general we denote with N^{k}MHV amplitude which have $k+2$ negative-helicity gluons and $n-k-2$ positive-helicity gluons. When an amplitude has $n-2$ negative helicity gluons and 2 of positive helicity, it is called anti-MHV.

1.1.3 Recursion relation for tree amplitudes

Here we consider the analytically continuing the momentum of scattering amplitudes into the complex plane. The important point is that three-point on-shell amplitudes vanish when the momentum is real. From the momentum conservation,

$$p_1 \cdot p_2 = \frac{1}{2}(p_1 + p_2)^2 = 0, \quad p_1 \cdot p_3 = p_2 \cdot p_3 = 0. \tag{1.1.18}$$

In spinor helicity variables, these relations become

$$\langle 12\rangle[21] = \langle 13\rangle[31] = \langle 23\rangle[32] = 0. \tag{1.1.19}$$

Spinor helicity variables associated with real momenta are not independent, then all brackets vanish. This means that we can not write down a non-zero expression for a three-point amplitude. However, if the momenta are complex, spinor helicity variables $\lambda, \tilde{\lambda}$ are independent and $\langle ab\rangle[ba] = 0$ implies that one of the brackets is zero. If $[12] = 0$, from the momentum conservation, other brackets vanish $[23], [31] = 0$. Similarly, when $\langle 12\rangle = 0$, other angle brackets equal to zero. These two conditions mean that a three-point amplitude depends on only square brackets or angle brackets. Then, from the little group scaling and some dimensional analysis, the three-point MHV and anti-MHV gluon amplitudes are completely fixed up to an overall constant:

$$A_3^{\text{MHV}}(1^-, 2^-, 3^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}, \quad A_3^{\text{anti-MHV}}(1^+, 2^+, 3^-) = -\frac{[12]^4}{[12][23][31]} \tag{1.1.20}$$

We can use these three-point amplitudes as elementary blocks to construct a recursion relation for higher-point tree on-shell amplitudes.

The idea of the on-shell recursion relation is to complexify the momenta of the external particles and use the complex analysis. First we introduce complex vectors $r_i^\mu, \ i = 1, \cdots, n$ such that

1. $\sum_{i=1}^n r_i^\mu = 0$,
2. $r_i \cdot r_j = 0$ fot all $i, j = 1, 2, \ldots, n$,
3. $p_i \cdot r_i = 0$ for each i.

And we define n shifted momenta

$$\hat{p}_i^\mu \equiv p_i^\mu + z r_i^\mu, \quad z \in \mathbb{C} \tag{1.1.21}$$

This shifted momenta satisfy the momentum conservation $\sum_{i=1}^n \hat{p}_i = 0$ and the on-shell condition $\hat{p}_i^2 = 0$. When we consider a subset of this shifted momenta $\hat{P}_I^\mu \equiv \sum_{i\in I} \hat{p}_i^\mu$, this is linear in z:

$$\hat{P}_I^2 = P_I^2 + 2z P_I \cdot R_I, \tag{1.1.22}$$

where $P_I^\mu \equiv \sum_{i\in I} p_i^\mu$ and $R_I = \sum_{i\in I} r_i$. We write this as

$$\hat{P}_I^2 = -\frac{P_I^2}{z_I}(z - z_I), \quad z_I = -\frac{P_I^2}{2P_i \cdot R_i}. \tag{1.1.23}$$

Let $\hat{A}_n(z)$ be a n-point tree amplitude in terms of the shifted momenta $\hat{p}_i$. In this tree level case, there are only simple poles at z_I which come from the shifted propagators $1/\hat{P}_I^2$. By using the Cauchy's theorem, the unshifted amplitude is given as

$$A_n = -\sum_{z_I} \mathrm{Res} \frac{\hat{A}_n(z)}{z} + B_n, \tag{1.1.24}$$

where B_n is the residue of the pole at $z = \infty$. At a z_I pole, the propagator $1/\hat{P}_I^2$ goes on-shell, and the shifted amplitude factorize into two on-shell parts $\hat{A}_L, \hat{A}_R$:

$$\mathrm{Res}_{z=z_I} \frac{\hat{A}_n(z)}{z} = -\hat{A}_L(z_I) \frac{1}{P_I^2} \hat{A}_R(z_I). \tag{1.1.25}$$

Each $\hat{A}_L, \hat{A}_R$ have a fewer number of the external particles than the original amplitude.

The boundary term B_n from the pole at infinity has no similar expression in terms of lower-point amplitudes and there is not a systematic way to obtain this term. The systematic calculation of this boundary term has been investigated, for example [31]. Here we will not see the detail of this calculation since this boundary term does not appear in the $\mathcal{N} = 4$ SYM.

The BCFW shift is one of the shifts which discussed above. This shift involves two momenta shift

$$\begin{aligned} &\lambda_i \to \lambda_i, \quad \tilde{\lambda}_i \to \tilde{\lambda}_i + z\tilde{\lambda}_j \\ &\lambda_j \to \lambda_j - z\lambda_i, \quad \tilde{\lambda}_j \to \tilde{\lambda}_j, \end{aligned}$$

and no other spinors are shifted. This shift is called $[i,j\rangle$-BCFW shift. Let us consider the $[n,1\rangle$-BCFW shift. In this choice, only propagators involving $\hat{p}_1$ or $\hat{p}_n$ have a pole at some z. For example, we introduce $P_i = p_1 + \cdots + p_{i-1}$, the shifted propagator $1/\hat{P}_i^2$ can be written as

$$\frac{1}{\hat{P}_i^2} = \frac{1}{P_i^2 - z\langle n|P_i|1]} \tag{1.1.26}$$

and this is singular at $z = z_{P_i} = P_i^2/\langle n|P_i|1]$. On this pole z_{P_i},

$$\lim_{z\to z_{P_i}} \frac{\hat{A}_n^{\text{tree}}(z)}{z} = -\frac{1}{(z-z_{P_i})}\sum_{s=\pm} A_L(\hat{1},2,\cdots,i-1,-\hat{P}_i^s)|_{z_{P_i}})\frac{1}{P_i^2}A_R(\hat{P}_i^{-s},i,\cdots,\hat{n}). \tag{1.1.27}$$

Then from (1.1.24), the amplitude is given as

$$A_n^{\text{tree}} = \sum_{i=3}^{n-1}\sum_{s=\pm} A_L(\hat{1},2,\cdots,i-1,-\hat{P}_i^s)|_{z_{P_i}}\frac{1}{P_i^2}A_R(\hat{P}_i^{-s},i,\cdots,\hat{n}). \tag{1.1.28}$$

This is the BCFW recursion relation obtained from the $[n,1\rangle$-BCFW shift.

1.2 $\mathcal{N}=4$ SYM

In this thesis, we investigate scattering amplitudes in the $\mathcal{N}=4$ super Yang-Mills theory with $SU(N)$ gauge group. This theory describes sixteen states: two gluons, eight fermion states, and six scalars. All of these particles can interact with each other in many different combinations. This means that we can obtain many different amplitudes. In this section, we introduce a super-amplitude which gives a compact and simple description of the amplitudes of all the particles in the $\mathcal{N}=4$ SYM by using the $\mathcal{N}=4$ supermultiplet of massless states. Then we review the BCFW recursion relation for this super-amplitude and loop integrands in the planar limit.

1.2.1 $\mathcal{N}=4$ supermultiplets

Here we see how to obtain the massless representations of the $\mathcal{N}=4$ SUSY algebra and construct the massless supermultiplet. First, the $\mathcal{N}=4$ SUSY algebra is given as

$$\{q_a^A, \bar{q}_{B\dot{a}}\} = \delta_B^A p_{a\dot{a}} \tag{1.2.1}$$

where $A, B = 1, 2, 3, 4$ and $q, \bar{q}$ are supersymmetric generators. In the massless case, we can choose the Lorentz frame in which $p\mu = (p, 0, 0, p)$, then the relation (1.2.1) becomes

$$\{q^A_a, \bar{q}_{B\dot{a}}\} = \delta^A_B (1 + \sigma_3)_{a\dot{a}} p, \tag{1.2.2}$$

and this is reduced to the Clifford algebra

$$\{q^A_1, \bar{q}_{B\dot{1}}\} = 2\delta^A_B p, \quad \{q^A_1, \bar{q}_{B\dot{2}}\} = 0, \quad \{q^A_2, \bar{q}_{B\dot{1}}\} = 0, \quad \{q^A_2, \bar{q}_{B\dot{2}}\} = 0. \tag{1.2.3}$$

The massless states are labeled by their helicity and the eigenvalue of the Lorentz generator J_{12}. For chiral spinors, the Lorentz generator J_{12} is $\frac{1}{2}(\sigma_3)_a^b$ and the helicity is $\{q^A_1, q^A_2\} = \{1/2, -1/2\}$. For anti-chiral spinors, the Lorentz generator J_{12} is $\frac{1}{2}(\bar{\sigma}_3)_{\dot{a}}^{\dot{b}}$ and the helicity is $\{\bar{q}_{A\dot{1}}, \bar{q}_{A\dot{2}}\} = \{-1/2, 1/2\}$.

Next, we define a vacuum state $|h\rangle$ which has helicity h as

$$q^A_1|h\rangle = q^A_2|h\rangle = q_{A\dot{2}}|h\rangle = 0, \quad J_{12}|h\rangle = h|h\rangle. \tag{1.2.4}$$

The massless supermultiplet is obtained by applying the four creation operators $\bar{q}_{A\dot{1}}$ to $|h\rangle$:

State	Helicity	Multiplicity
$\|h\rangle$	h	1
$\bar{q}_{A\dot{1}}\|h\rangle$	$h - 1/2$	4
$\bar{q}_{A\dot{1}}\bar{q}_{B\dot{1}}\|h\rangle$	$h - 1$	6
$\epsilon^{ABCD}\bar{q}_{A\dot{1}}\bar{q}_{B\dot{1}}\bar{q}_{C\dot{1}}\|h\rangle$	$h - 3/2$	4
$\epsilon^{ABCD}\bar{q}_{A\dot{1}}\bar{q}_{B\dot{1}}\bar{q}_{C\dot{1}}\bar{q}_{D\dot{1}}\|h\rangle$	$h - 2$	1

By choosing $h = 1$, we can obtain the CPT self-conjugate supermultiplet which describing massless particles of helicities $\pm 1, \pm 1/2, 0$. We write these states as

$$h = \pm 1 : |G^{\pm}\rangle, \ h = \frac{1}{2} : |\Gamma_A\rangle, \ h = -\frac{1}{2} : |\bar{\Gamma}^A\rangle, \ h = 0 : |S_{AB}\rangle. \tag{1.2.5}$$

In the previous section, we used a special frame, then the Lorentz invariance is broken. Here we construct the supermultiplet in a manifestly Lorentz covariant way and introduce a super-state. First, we rewrite the SUSY algebra (1.2.1) by using the spinor-helicity variables:

$$\{q^A_a, \bar{q}_{B\dot{a}}\} = \delta^A_B |p\rangle_a [p|_{\dot{a}} \to \{q^A, \bar{q}_B\} = \delta^A_B. \tag{1.2.6}$$

where we define $q^A, \bar{q}_B$ as $q^A_a = |p\rangle_a q^A$ and $\bar{q}_{B\dot{a}} = [p|_{\dot{a}} \bar{q}_B$. This is the Lorentz covariant projection of the q_a^A, and $\bar{q}_{B\dot{a}}$. From this, we can see that $q^A, \bar{q}_A$ are interpreted as the covariant analogs of the annihilation operator q_1^A and the creation operator $\bar{q}_{A\dot{1}}$. There is another Lorentz covariant projection $q'^A = \langle p|^a q_a^A$. We can see that the projections $q'^A, \bar{q}'_A$ anticommute with each other and with the rest of the generators. From this, we can interpret these as the covariant analogs of the q_2^A and $\bar{q}_{A\dot{2}}$. It is known that this algebras are realized by using Grassmann variables η^A:

$$q^A = \eta^A, \quad \bar{q}_A = \frac{\partial}{\partial \eta^A}. \tag{1.2.7}$$

Next, we define a super-state

$$\begin{aligned}|\Phi\rangle = |G^+\rangle + \eta^A|\Gamma_A\rangle + \frac{1}{2}\eta^A\eta^B|S_{AB}\rangle + \frac{1}{3!}\eta^A\eta^B\eta^C\epsilon_{ABCD}|\bar{\Gamma}^D\rangle \\ + \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D\epsilon_{ABCD}|G^-\rangle.\end{aligned} \tag{1.2.8}$$

We can obtain the states of the multiplet by using the generators (1.2.7). For example, the state with $h = 1$ is obtained as $|\Phi\rangle|_{\eta=0} = |G^+\rangle$. The next state is obtained by applying the creation operator $\bar{q}_A$: $\bar{q}_A|\Phi\rangle|_{\eta=0} = |\Gamma_A\rangle$.

1.2.2 Super-amplitudes in $\mathcal{N} = 4$ SYM

Next, we introduce a superamplitude $\mathcal{A}_n(\Phi_1, \cdots, \Phi_n)$ which gives a compact representation of the scattering amplitudes of all the particles in the $\mathcal{N} = 4$ SYM. First we define a superfield Φ as

$$\begin{aligned}\Phi(p,\eta) = G^+(p) + \eta^A\Gamma_A(p) + \frac{1}{2}\eta^A\eta^B S_{AB}(p) + \frac{1}{3!}\eta^A\eta^B\eta^C\epsilon_{ABCD}\bar{\Gamma}^D(p) \\ + \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D\epsilon_{ABCD}G^-(p).\end{aligned} \tag{1.2.9}$$

The fields appearing in this superfield have to be thought as annihilation operators. These operators produce the exciting states when acting on the vacuum $\langle 0|$. Then a superamplitude $\mathcal{A}_n(\Phi_1, \cdots, \Phi_n)$ is defined as the S-matrix between the vacuum $|0\rangle$ and n outgoing states which are created by the superfields Φ_i. It depends on the on-shell momentum p_i and Grassmann variables η_{iA} for each particle $i = 1, \cdots, n$. First, we consider the supersymmetry generator

$$q_a^A = \sum_{i=1}^n q_{ia}^A = \sum_{i=1}^n |i\rangle_a \eta_i^A. \tag{1.2.10}$$

The supersymmetry invariance for the vacuum requires that this generator annihilate the superamplitude $q^A_a \mathcal{A}_n = 0$. Then we can deduce the superamplitude as

$$\mathcal{A}_n = \delta^4(\sum_{i=1}^{n} p_i)\delta^{0|8}(\sum_{i=1}^{n} |i\rangle_a \eta_i^A)\mathcal{P}_n(\lambda, \tilde{\lambda}, \eta), \tag{1.2.11}$$

where $\delta^{0|8}(\sum_{i=1}^{n} |i\rangle_a \eta_i^A) = \prod_{a=1,2}\prod_{A=1}^{4} |i\rangle_a \eta_i^A$ is a Grassmann delta function and $\mathcal{P}_n$ is a polynomial in Grassmann variables η_i^A. We can obtain any amplitude from this superamplitude by using Grassmann differential. For example,

$$A_n(|G_1^+\rangle \dots |G_i^-\rangle \dots |G_j^-\rangle \dots |G_n^+\rangle) = \left(\prod_{A=1}^{4} \frac{\partial}{\partial \eta_{iA}}\right)\left(\prod_{B=1}^{4} \frac{\partial}{\partial \eta_{jB}}\right)\mathcal{A}_n(|\Phi_1\rangle, \cdots, |\Phi\rangle_n)|_{\eta_{kC}=0}. \tag{1.2.12}$$

Since there is $SU(4)$ R-symmetry, the superamplitude is expanded to a sum of degree $4(K+2)$ polynomials in η_{iA} as

$$\mathcal{A}_n = \delta^4(\sum_{i=1}^{n} p_i)\delta^{0|8}(\sum_{i=1}^{n} |i\rangle_a \eta_i^A)\left[\mathcal{P}_n^{(0)} + \mathcal{P}_n^{(4)} + \mathcal{P}_n^{(8)} + \cdots + \mathcal{P}_n^{(4n-16)}\right], \tag{1.2.13}$$

We call the order K of the Grassmann polynomial as N^KMHV sector. Then the superamplitude can be expanded as

$$\mathcal{A}_n = \mathcal{A}_n^{\text{MHV}} + \mathcal{A}_n^{\text{NMHV}} + \mathcal{A}_n^{\text{N}^2\text{MHV}} + \cdots + \mathcal{A}_n^{\text{anti-MHV}} \tag{1.2.14}$$

where we call the order $n-4$ of the Grassmann polynomial as anti-MHV sector.

1.2.3 Momentum twistor

We have seen that the on-shell condition is manifestly resolved by using the spinor-helicity variables. There is another condition: the momentum conservation. In the previous section, we saw that the superamplitudes have two delta functions: $\delta^4(\sum_{i=1}^{n} p_i)$ and $\delta^{0|8}(\sum_{i=1}^{n} |i\rangle_a \eta_i^A)$. Here we introduce new dual variables y_i, θ_i such that

$$y_i^{\dot{a}a} - y_{i+1}^{\dot{a}a} = p_i^{\dot{a}a}, \quad |\theta_{iA}\rangle - |\theta_{i+1,A}\rangle = q_{iA}^{\dagger} = |i\rangle\eta_{iA}. \tag{1.2.15}$$

The momentum conservation $\sum_{i=1}^{n} p_i = 0$ and the super momentum conservation $\sum_{i=1}^{n} q_{iA}^{\dagger} = 0$ correspond to the periodicity condition that $y_{n+1} = y_1$, $|\theta_{n+1,A}\rangle = |\theta_{1,A}\rangle$. By using this dual variable, the momentum conservation is manifest, however, the on-shell condition $(y_i -$

$y_{i+1})^2 = 0$ is not manifest. We can expect that by combining these two variables: the spinor-helicity variable and the dual variable, we can construct a new variable that makes these two conditions manifest. Before constructing this variable, we see the relation of this dual variable and the hidden symmetry.

These dual variables make it possible to exhibit a hidden symmetry "dual conformal symmetry". First observation of this symmetry was given in the calculation of the four-gluon MHV amplitude [32], [33]. Additional evidence was obtained both at weak coupling [34], [35], [36] and at strong coupling [37], [38]. Then in it was proven that all scattering amplitudes in $\mathcal{N} = 4$ SYM are dual superconformal invariants [6].

Next, we introduce a new variable "Momentum twistor variable". By using this dual variables, we introduce another variable $[\mu_i|^a, \chi_i^A$ defined as

$$\langle i|_{\dot{a}} y_i^{\dot{a}a} = \langle i|_{\dot{a}} y_{i+1}^{\dot{a}a} \equiv [\mu|_i^a, \quad \chi_i^A = \langle i\theta_{iA}\rangle = \langle i\theta_{i+1,A}\rangle \tag{1.2.16}$$

We write the new four-component spinor which made from a pair of spinors $[\mu_i|^a$ and $|i\rangle^{\dot{a}}$ as $Z_i^I \equiv (|i\rangle^{\dot{a}}, [\mu_i|^a)$ with $I = (\dot{a}, a)$. These Z_i^I are called "momentum twistors". When the fermionic χ_i^A is included, this is called "momentum supertwistor" $\mathcal{Z}_i^{\mathcal{A}}$ where $\mathcal{A} = (\dot{a}, a, A)$. The important point is that these new variables transform linearly under the dual conformal transformation. By using this momentum twistor, we can make a dual conformal invariant by contracting four Z^I with the Levi-Civita tensor

$$\langle ijkl\rangle \equiv \epsilon_{IJKL} Z_i^I Z_j^J Z_k^K Z_l^L. \tag{1.2.17}$$

The relation (1.2.16) imply that $|\mu_i] \to t_i|\mu_i]$ under the little group scaling (1.1.13). From this, the momentum twistors transform as $Z_i^I \to t_i Z_i^I$. This means that the momentum twistors are defined projectively.

Since the dual variable y_i is defined as (1.2.15), a point Z_i^I is determined by the line of two points y_i^μ and y_{i+1}^μ. On the other hand, the dual variable is written as

$$y_i^{\dot{a}a} = \frac{|i\rangle^{\dot{a}}[\mu_{i-1}|^a - |i-1\rangle^{\dot{a}}[\mu_i|^a}{\langle i-1 i\rangle}. \tag{1.2.18}$$

This means that a point in this y space is determined by a line $(Z_{i-1}^I Z_i^I)$ in Z space. Then a point in the dual space maps to a line in the momentum twistor space, and vice versa. This is the reason why this variables Z_i^I is called momentum twistor variables. If we choose n points Z_i in this momentum twistor space, there are n lines defined by consecutive points

(Z_i, Z_{i+1}). From (1.2.18), each line (Z_i, Z_{i+1}) is mapped to the point in the dual space y_i and the relation (1.2.16) requires that the points y_i and y_{i+1} are null-separated. This means that the corresponding momenta $p_i = y_i - y_{i+1}$ are on-shell. Then this momentum twistor variables make both conditions: the on-shell condition and the momentum conservation manifest.

Next, we see the relation between physical poles and this momentum twistor variables. The physical poles are written as

$$\frac{1}{(p_i + p_{i+1} + p_{i+2} + \cdots + p_{j-2} + p_{j-1})^2} = \frac{1}{y_{ij}^2}, \tag{1.2.19}$$

where

$$y_{ij} \equiv y_i - y_j = p_i + p_{i+1} + \cdots + p_{j-1}. \tag{1.2.20}$$

In momentum twistor space, this dual variable is written as

$$y_{ij}^2 = \frac{\langle i-1, i, j-1, j\rangle}{\langle i-1, i\rangle\langle j-1, j\rangle}. \tag{1.2.21}$$

Then the physical poles $y_{ij}^2 = 0$ mean that $\langle i-1, i, j-1, j\rangle = 0$.

1.2.4 BCFW recursion relation for $\mathcal{N} = 4$ SYM

Here we consider the BCFW recursion relation for tree amplitudes in the $\mathcal{N} = 4$ SYM. In the super symmetric theory case, we need to generalize the $[i, j\rangle$ BCFW shift to the $[i, j\rangle$ BCFW super-shift

$$\tilde{\lambda}_i \to \tilde{\lambda}_i + z\tilde{\lambda}_j, \quad \lambda_j \to \lambda_j - z\lambda_i, \quad \eta_i \to \eta_i + z\eta_j. \tag{1.2.22}$$

When we involve the shifts of the Grassmann variables η, the supermomentum $q = \sum_i \lambda_i \eta_i$ is conserved. From this, this generalization is natural from the point of view of SUSY. Following the same step as non-SUSY BCFW, we can obtain the superamplitude as

$$\mathcal{A}_n^{\text{tree}} = \sum_{i=1}^{n-1} \int d^4\eta_{\hat{P}_i} \mathcal{A}_L(\hat{1}, 2, \cdots, i-1, -\hat{P}_i)|_{z_{P_i}} \frac{1}{P_i^2} \mathcal{A}_R(\hat{P}_i), i, \cdots, n-1\hat{n})|_{z_{P_i}}. \tag{1.2.23}$$

The difference between non-SUSY and SUSY BCFW is the integration over the η. As in the non-SUSY case, we need to sum over all possible states that can be exchanged on the internal line. In the $\mathcal{N} = 4$ SYM case, this includes all 16 states. In terms of component amplitudes, the particle on the internal line depends on the external states of the amplitude. First, if all external states are gluon, the internal line is also a gluon and we need to sum over the helicities as

$$\left[\left(\prod_{A=1}^{4}\frac{\partial}{\partial\eta_{\hat{P}A}}\right)\hat{\mathcal{A}}_L\right]\frac{1}{P^2}\hat{\mathcal{A}}_R+\hat{\mathcal{A}}_L\frac{1}{P^2}\left[\left(\prod_{A=1}^{4}\frac{\partial}{\partial\eta_{\hat{P}A}}\right)\hat{\mathcal{A}}_R\right]\Bigg|_{\eta_{\hat{P}A}=0}. \tag{1.2.24}$$

If a gluino is exchanged, we need to move one of the four Grassmann derivatives from $\hat{\mathcal{A}}_L$ to $\hat{\mathcal{A}}_R$ in both the first and second terms. In a scalar exchange case, two Grassmann derivatives act on $\hat{\mathcal{A}}_L$ and the another two derivatives act on $\hat{\mathcal{A}}_R$. All of this is summarised by

$$\left(\prod_{A=1}^{4}\frac{\partial}{\partial\eta_{\hat{P}A}}\right)\left[\hat{\mathcal{A}}_L\frac{1}{P^2}\hat{\mathcal{A}}_R\right]\Bigg|_{\eta_{\hat{P}A}=0}=\int d^4\eta_{\hat{P}}\hat{\mathcal{A}}_L\frac{1}{P^2}\hat{\mathcal{A}}_R. \tag{1.2.25}$$

An important point is that the boundary term is always vanishes in the supersymmetric recursion relation. By using this BCFW recursion relation, it is possible to obtain all tree amplitudes $\mathcal{A}_n^{\mathrm{N}^k\mathrm{MHV}}$ [39]. The MHV tree superamplitude is given as

$$\mathcal{A}_n^{\mathrm{MHV}}=\frac{\delta^4(\sum_i\lambda_i\tilde{\lambda}_i)\delta^{0|8}(\sum_i\lambda_i\eta_i)}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}. \tag{1.2.26}$$

We can prove this by using the BCFW. The NMHV tree superamplitude is

$$\mathcal{A}_n^{\mathrm{NMHV}}=\frac{\delta^4(\sum_i\lambda_i\tilde{\lambda}_i)\delta^{0|8}(\sum_i\lambda_i\eta_i)}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}\sum_{k=j+2}^{n-1}\sum_{j=2}^{n-3}R_{njk}, \tag{1.2.27}$$

where R_{nij} is called R-invariants which defined as

$$R_{njk}=\frac{\delta^{(4)}(\langle j-1,j,k-1,k\rangle\chi_{nA}+\text{cyclic})}{\langle n,j-1,j,k-1\rangle\langle j-1,j,k-1,k\rangle\langle j,k-1,k,n\rangle\langle k-1,k,n,j-1\rangle\langle k,n,j-1,j\rangle}. \tag{1.2.28}$$

The important point is that in this representation, each R-invariant has spurious poles. From the previous section, the physical poles are written as $\langle i-1,i,j-1,j\rangle$, then for example the pole $\langle n,j-1,j,k-1\rangle$ is not physical pole. These spurious poles are canceled in the sum of the R-invariants. This is the important feature of the BCFW. We can rewrite the amplitudes into compact expressions, but the cost is the appearance of spurious poles. This means that in the BCFW representation the locality is not manifest.

1.2.5 BCFW for loop integrands in planar $\mathcal{N}=4$ SYM

The analytic structure of loop level amplitudes is more complicated than for tree-level amplitudes. The tree-amplitude is a simple rational function and has only single poles. However, the loop-amplitude is expressed as generalized logarithms and special functions and has branch cuts in addition to poles. From this, we focus on the loop integrand: the rational function inside the loop momentum integration. This integrand is a rational function with poles from the propagators.

When we consider the BCFW shift for the loop integrand, there is a problem that does not appear in the tree case. In the loop level amplitude, the loop momenta l_i are just integrating variables that we can change these variables which gives the same integrated result. However, if we consider the BCFW shifts for the integrand, this reparametrization gives different functions in z. For example, we consider the 1-loop 4-point box integrand

$$I_4(1,2,3,4) = \frac{1}{l^2(l-p_1)^2(l-p_1-p_2)^2(l+p_4)^2}. \tag{1.2.29}$$

We consider the equivalent parametrization $I_4' = I_4(l \to l+p_1)$ and take the BCFW shift for p_1 and p_2,

$$\begin{aligned} I_4(\hat{1},\hat{2},3,4) &= \frac{1}{l^2(l-p_1-zq)^2(l-p_1-p_2)^2(l+p_4)^2} \\ I_4'(\hat{1},\hat{2},3,4) &= \frac{1}{(l+p_1+zq)^2 l^2 (l-p_2+zq)^2(l-p_2-p_3+zq)^2}. \end{aligned}$$

These two integrands are different, then the BCFW recursion relation for the integrands is ill-defined.

However, this problem is resolved in supersymmetric theories in the planar limit. This means that in the planar limit, the loop momenta in the integrand is defined unambiguously. Here we define the dual variable y_0 for the loop momentum as $l = y_1 - y_0$. This is the same as the relation $p_i = y_i - y_{i+1}$. When the ordering of the external particles is well-defined based on the color-ordering, this definition has no ambiguity. In the loop level, this requires that all diagrams be planar.

Then we consider the BCFW recursion relation for the planar integrand. Here we use the momentum supertwistor $\mathcal{Z}_i^A$. The dual variable y is expressed as a line in momentum twistor space. Then we take y_0 to be mapped to a line $(Z_A Z_B)$ in the momentum twistor space. The explicit representation of the BCFW recursion relation for all loop integrands in planar $\mathcal{N}=4$ SYM is given in [9] as

$$
\begin{aligned}
M_{n,k,l}(1,\cdots,n) = M_{n-1,k,l}&(1,\cdots,n-1)\\
&+\sum_{n_L,k_L,l_L;j}[j\ j+1\ n-1\ n\ 1]\times M^R_{n_R,k_R,l_R}(1,\cdots,j,I_j)\times M^L_{n_L,k_L,l_L}(I_j,j+1,\cdots,\hat{n}_j)\\
&+\int_{\mathrm{GL}(2)}[AB\ n-1\ n\ 1]\times M_{n+2,k+1,l-1}(1,\cdots,\hat{n}_{AB},\hat{A},B)
\end{aligned}
$$

where $n_L+n_R=n,\ k_L+k_R=k,\ l_L+l_R=l$ and

$$
\begin{aligned}
&[a,b,c,d,e]=\frac{\delta^{0|4}(\eta_a\langle bcde\rangle+\text{cyclic})}{\langle abcd\rangle\langle bcde\rangle\langle cdea\rangle\langle deab\rangle\langle eabc\rangle}\\
&\hat{n}_j=(n-1\ n)\cap(j\ j+1\ 1),\qquad I_j=(j\ j+1)\cap(n-1\ n\ 1),\\
&\hat{n}_{AB}=(n-1\ n)\cap(A\ B\ 1),\qquad \hat{A}=(A\ B)\cap(n-1\ n\ 1).
\end{aligned}
\tag{1.2.30}
$$

We can obtain all loop integrands form this BCFW recursion relation. For example, the 1-loop MHV integrand is given as

$$
A^{\text{1-loop}}_{\text{MHV}}(1,2,\cdots,n)=\sum_{i<j}\frac{\langle AB(1ii+1)\cap(1jj+1)\rangle}{\langle AB1i\rangle\langle AB1i+1\rangle\langle ABii+1\rangle\langle AB1j\rangle\langle AB1j+1\rangle\langle ABjj+1\rangle},
\tag{1.2.31}
$$

here we omit to write the MHV tree factor.

1.3 Polytope picture for NMHV amplitudes

In this section, we review the polytope picture for NMHV tree amplitudes which was introduced by Hodges [8].

1.3.1 Volume of a n-simplex

First, we start from the simplest example. We consider a triangle in a 2-dimensional space which made from the three vertices $(x_1,y_1),(x_2,y_2),(x_3,y_3)$. The area of this triangle can be written as

$$
V_2=\frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\\ 1 & 1 & 1\end{vmatrix}.
\tag{1.3.1}
$$

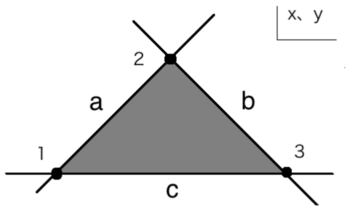


Figure 1.1: Area of a triangle

We define three 3-vectors and a reference vector as

$$W_{iI} = \begin{pmatrix} x_i \\ y_i \\ 1 \end{pmatrix}, \quad \mathcal{Z}_0^I = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{1.3.2}$$

By using these 3-vectors, the area can be written as

$$V_2 = \frac{1}{2} \frac{\langle 1, 2, 3 \rangle}{(\mathcal{Z}_0 \cdot W_1)(\mathcal{Z}_0 \cdot W_2)(\mathcal{Z}_0 \cdot W_3)}. \tag{1.3.3}$$

Next, we define a dual space $\mathcal{Z}_a^I$ as

$$\mathcal{Z}^I W_I = 0. \tag{1.3.4}$$

These dual points $\mathcal{Z}_a^I$ are related to lines in W space. Let us label the three edges of the triangle as a, b and c as Figure 1.1. Then each W_{iI} is characterized by two lines in the dual space. These dual points make a triangle and this is called "dual triangle". By using these dual points, we can calculate the area of the dual triangle. For example, W_{1I} is the intersection of lines a and c, then this W_{1I} satisfies $\mathcal{Z}_a^I W_{1I} = \mathcal{Z}_c^I W_{1I} = 0$. We can solve these constraints and W_{1I} is written as $W_{1I} = \epsilon_{IJK} \mathcal{Z}_c^J \mathcal{Z}_a^K$. Then the area of the dual triangle is given as

$$V_2 = \frac{1}{2} \frac{\langle a, b, c \rangle^2}{\langle 0, a, b \rangle \langle 0, b, c \rangle \langle 0, c, a \rangle} \equiv [a, b, c], \tag{1.3.5}$$

where $\langle a, b, c \rangle \equiv \epsilon_{IJK} \mathcal{Z}_a^I \mathcal{Z}_b^J \mathcal{Z}_c^K$. We can generalize this expression to the volume of "dual" n-simplex in $\mathbb{CP}^n$ as

$$[\mathcal{Z}_{i_1}, \cdots, \mathcal{Z}_{i_n+1}] = \frac{1}{n!} \frac{\langle i_1, i_2, \ldots, i_{n+1} \rangle^n}{\langle 0, i_1, i_2, \ldots, i_n \rangle \langle 0, i_2, \ldots, i_{n+1} \rangle \cdots \langle 0, i_{n+1}, i_1, \ldots, i_{n-1} \rangle}. \tag{1.3.6}$$

1.3.2 NMHV tree amplitudes

In the momentum twistor space, the BCFW representation of the NMHV tree amplitude is written as

$$\mathcal{A}_n^{\text{NMHV}} = \mathcal{A}_n^{\text{MHV}} \sum_{k=j+2}^{n-1} \sum_{j=2}^{n-3} [n, j-1, j, k-1, k], \tag{1.3.7}$$

where $[n, j-1, j, k-1, k] = R_{njk}$. Let us define the 5-component vector

$$\mathcal{Z}_i^{\mathcal{I}} = \begin{pmatrix} Z_i^I \\ \chi_i \cdot \psi \end{pmatrix}, \quad \mathcal{I} = 1, \cdots, 5, \tag{1.3.8}$$

where $\chi_i \cdot \psi = \chi_i^A \cdot \psi_A$ and ψ_A is an $SU(4)$ auxiliary Grassmann variable. This ψ common for all external particles. Then we define $\langle i, j, k, l, m\rangle$ as the contraction of five of these 5-vectors with a 5-indexed Levi-Civita tensor. By using this 5-bracket, the R-invariant can be written as

$$[i, j, k, l, m] = \frac{1}{4!} \int d^4\psi \frac{\langle i, j, k, l, m\rangle^4}{\langle 0, i, j, k, l\rangle\langle 0, j, k, l, m\rangle\langle 0, k, l, m, i\rangle\langle 0, l, m, i, j, k\rangle\langle 0, m, i, j, k\rangle}, \tag{1.3.9}$$

where we have introduced the auxiliary reference 5-vector

$$\mathcal{Z}_0^{\mathcal{I}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{1.3.10}$$

Since the integrand is invariant under $\mathcal{Z}_i^{\mathcal{I}} \to t_i \mathcal{Z}_i^{\mathcal{I}}$, the 5-vector is projectively and we can interpret this as the coordinates of points in projective space $\mathbb{CP}^4$.

From this, we can see that the integrand of the R-invariant (1.3.9) is the volume of a "dual" 4-simplex in $\mathbb{CP}^4$. For example, the five point amplitude is written as

$$\mathcal{A}_5^{\text{NMHV}} = \mathcal{A}_5^{\text{MHV}} \times [1, 2, 3, 4, 5]. \tag{1.3.11}$$

Then, up to the MHV factor $\mathcal{A}_5^{\text{MHV}}$, this is the volume of a 4-simplex in $\mathbb{CP}^4$. Next, the BCFW representation of the six point amplitude based on $[2, 3\rangle$ super-shift is

$$\frac{\mathcal{A}_6^{\text{NMHV}}}{\mathcal{A}_6^{\text{MHV}}} = [1, 2, 3, 4, 5] + [1, 2, 3, 5, 6] + [1, 3, 4, 5, 6]. \tag{1.3.12}$$

This is the sum of the volumes of the 4-simplices. This sum of the volumes corresponds to the volume of the six-vertices dual polytope in $\mathbb{CP}^4$. From this, we can interpret this the six-point amplitude as a volume of the six-vertices dual polytope. When we triangulate this polytope into the simplices, then the volume becomes a sum of the volumes of each simplex such as (1.3.12). This means that the BCFW representation is interpreted as a triangulation of the polytope. Of cause, there is another triangulation of this polytope such as $[1,2,3,4,6]+[2,3,4,5,6]+[1,2,4,5,6]$. This triangulation is the BCFW representation of the six-point amplitude based on $[3,2\rangle$ super-shift.

The polytope interpretation described here is valid for NMHV n-point tree superamplitudes. This was generalized into the 1-loop n-point MHV integrands in [40].

Chapter 2

The Amplituhedron

In the previous section, we see that the BCFW recursion relation gives all loop level integrands in planar $\mathcal{N} = 4$ SYM. And the NMHV amplitudes can be interpreted as a volume of a polytope. In 2013, Arkani-Hamed and Trnka generalized these geometric pictures for all loop integrands and found a completely new geometric object: Amplituhedron [11]. In this section, we will explain the definition of the amplituhedron and how to extract scattering amplitudes from this geometric object.

2.1 Definition of the amplituhedron

The original definition of the amplituhedron is given as a generalization of a convex polygon to the Grassmannian. This can be interpreted as a generalization of the vertex-centered description of the convex polygon. The second definition is a topological definition of the amplituhedron. This can be interpreted as a generalization of the face-centered description: definition of the polygon by the collection of inequalities associated with the boundaries. We briefly review these two definitions.

2.1.1 Generalization of the convex polygon

First we review the original definition of the amplituhedron. Let us start with a triangle in real two dimensional space $\mathbb{RP}^2$. Any point in this space is expressed as a linear combination of the vertices Z_i^I $(I = 1, 2, 3)$ of the triangle,

$$Y^I = c_1 Z_1^I + c_2 Z_2^I + c_3 Z_3^I. \tag{2.1.1}$$

The interior of the triangle is parametrized as all the triplet $(c_1, c_2, c_3)/\mathrm{GL}(1)$ with all ratios c_a/c_b are positive, so that all c_a are positive or negative. For simplicity, here we take all of the coefficients are positive: $c_a > 0$.

We can consider two generalizations of this construction. On the one hand, we can interpret this triangle as a 2-dimensional simplex and go to higher-dimensional simplices in higher-dimensional spaces. On the other hand, we can consider polygons in the two-dimensional plane. First, we generalize the triangle to an $(n-1)$ dimensional simplex in general projective space. The interior of this simplex is expressed as a collection $(c_1, c_2, \cdots, c_n)/\mathrm{GL}(1)$ with $c_a > 0$. We can further generalize this to the space of k-dimensional planes in n-dimensional space. This is the Grassmannian $G(k, n)$ which is a collection of n k-dimensional vectors,

$$C = \begin{pmatrix} c_1^1,\ c_2^1, \cdots, c_n^1 \\ c_1^2,\ c_2^2, \cdots, c_n^2 \\ \vdots \\ c_1^k,\ c_2^k, \cdots, c_n^k \end{pmatrix} /\mathrm{GL}(k) = (\vec{c}_1, \vec{c}_2, \cdots, \vec{c}_n)/\mathrm{GL}(k). \tag{2.1.2}$$

The positivity giving us the interior of a simplex can be generalized to the Grassmannian: all ordered minors of this C are positive,

$$\langle c_{a_1} \cdots c_{a_k} \rangle > 0 \text{ for } a_1 < \cdots < a_k. \tag{2.1.3}$$

The Grassmannian which satisfies this positivity is called "Positive Grassmannian $G_+(k, n)$".

Next, we consider a polygon in the two-dimensional plane with vertices $Z_1^I, \cdots, Z_n^I$. There is a well-defined notion of the interior that exists when the polygon is "convex". This convexity for vertices is that all ordered minors of the $3 \times n$ matrix constructed from vertices are positive:

$$\langle Z_{a_1} Z_{a_2} Z_{a_3} \rangle > 0 \text{ for } a_1 < a_2 < a_3. \tag{2.1.4}$$

The $k \times n$ matrix which satisfies this positivity is called "the positive matrix $M_+(k, n)$". Then the interior of the convex polygon is given by a set of points as

$$Y^I = c_1 Z_1^I + c_2 Z_2^I + \cdots + c_n Z_n^I = c_a Z_a^I \text{ with } c_a > 0. \tag{2.1.5}$$

This can be interpreted as an intersection of two positive space:

$$(c_1, \cdots, c_n) \subset G_+(1, n), \quad (Z_1, \cdots, Z_n) \subset M_+(3, n). \tag{2.1.6}$$

This can be generalized to higher m-dimensional space as

$$Y^I = c_a Z_a^I \quad \text{where} \quad (c_1, \cdots, c_n) \subset G_+(1,n), \quad (Z_1, \cdots, Z_n) \subset M_+(1+m,n). \tag{2.1.7}$$

We can further generalize this into the Grassmannian. We consider $(k+m)$ dimensional vectors Z_a^I for $I = 1, \cdots, k+m$ where we restrict $n \leq (k+m)$. And we introduce the space of k dimensional planes in this $(k+m)$ dimensional space $Y \subset G(k, k+m)$,

$$Y_\alpha^I, \;\; \alpha = 1, \cdots, k, \;\; I = 1, \cdots, k+m. \tag{2.1.8}$$

We consider a subspace of $G(k, k+m)$ which is determined as

$$Y_\alpha^I = C_{\alpha a} Z_a^I \tag{2.1.9}$$

where

$$C_{\alpha a} \subset G_+(k,n), \quad Z_a^I \subset M_+(k+m,n). \tag{2.1.10}$$

This space is called "the generalized tree amplituhedron $\mathcal{A}_{n,k}^{(m)}(Z)$. The case of $m = 4$, this is called the tree amplituhedron.

The tree amplituhedron is defined as a generalization of the interior of the convex polygon. We can see that the polygon is the $k = 1, m = 2$ case and $n = k + m$ tree amplituhedron is corresponding to the positive Grassmannian. This definition is purely geometric, however, we will see that this tree amplituhedron gives tree scattering amplitudes in $\mathcal{N} = 4$ SYM.

Here we can define the loop amplituhedron as a generalization of the tree amplituhedron as follows. First, we consider L 2-dimensional planes $\mathcal{L}_{(i)}$ in 4-dimensional space complement of Y. This $\mathcal{L}$ is the different linear combination of the external data $\mathcal{Z}$

$$\mathcal{L}_{(i)\alpha}^I = D_{a\alpha(i)} Z_a^I \tag{2.1.11}$$

where D is the positive Grassmannian $G_+(2,n)$. Then the full amplituhedron $\mathcal{A}_{n,k,l}$ is the space of all Y, $\mathcal{L}_{(i)}$ of the form

$$Y_\alpha^I = C_{\alpha a} Z_a^I, \qquad \mathcal{L}_{(i)\alpha}^I = D_{a\alpha(i)} Z_a^I \tag{2.1.12}$$

where all ordered minors of the matrix

$$\begin{pmatrix} D_{(i_1)} \\ \vdots \\ D_{(i_l)} \\ C \end{pmatrix} \tag{2.1.13}$$

are positive.

Next, we see the boundaries of the amplituhedron. First, we consider the simplest example: $k=1, m=2$ polygon case. In order to look at the boundaries, we consider $\langle Yij \rangle$. When this bracket is positive for all Y, the line $(Z_i Z_j)$ is a boundary. In this case, we can expand Y as $Y = c_1 Z_1 + \cdots + c_n Z_n$, then this bracket becomes as

$$\langle Yij \rangle = \sum_{a=1}^{n} \langle Z_a Z_i Z_j \rangle. \tag{2.1.14}$$

From the positivity of C and Z, when i, j are consecutive, all terms are positive and this bracket is manifestly positive:

$$\langle Yii+1 \rangle = \sum_{a=1}^{n} \langle Z_a Z_i Z_i + 1 \rangle > 0. \tag{2.1.15}$$

Then the boundaries are lines $(Z_i Z_{i+1})$ and this is consistent that the boundaries of the polygon are given as consecutive lines. We can extend this to higher k, m cases. For example, we consider the $k=1, m=4$ case. Similarly, as the case of the polygon, the bracket $\langle Yijkl \rangle$ becomes as

$$\langle Y Z_i Z_j Z_k Z_l \rangle = \sum_{a=1}^{n} \langle Z_a Z_i Z_j Z_k Z_l \rangle. \tag{2.1.16}$$

Only when (i, j, k, l) is $(i, i+1, j, j+1)$, this is manifestly positive:

$$\langle Y Z_i Z_{i+1} Z_j Z_{j+1} \rangle = \sum_{a=1}^{n} \langle Z_a Z_i Z_{i+1} Z_j Z_{j+1} \rangle > 0. \tag{2.1.17}$$

From this, the boundaries are planes $(i, i+1, j, j+1)$. For the case of general k, we can see that

$$\langle Y_1 \cdots Y_k Z_i Z_{i+1} Z_j Z_{j+1} \rangle = \sum_{a_1 < \cdots < a_k} \langle c_{a_1} \cdots c_{a_k} \rangle \langle Z_{a_1} \cdots Z_{a_k} Z_i Z_{i+1} Z_j Z_{j+1} \rangle > 0. \tag{2.1.18}$$

2.1.2 Sign flip definition

Another definition is a topological definition called "sign flip definition" [29]. The sign flip definition is a generalization of the "face-centered" description of the polytope: which is the definition of the polytope by the collection of inequalities associated with the facet of the polytope. However, in the case of the amplituhedron, in addition to the boundary inequalities, the sign flip characterization is needed. Let us consider the $m = 2$ tree amplituhedron. All the co-dimension one boundaries are expressed as $\langle Yii+1\rangle$. However, the space defined as $\langle Yii+1\rangle > o$ does not correspond to the $m = 2$ amplituhedron. For example, we consider the $k = 2, m = 2, n = 4$ case, the amplituhedron is corresponding to the positive Grassmannian $G_+(2,4)$. The inequalities of the boundaries $\langle Y12\rangle, \langle Y23\rangle, \langle Y34\rangle, \langle Y14\rangle$ are positive, however, from the Plucker relations we can see that

$$\langle Y13\rangle\langle Y24\rangle = \langle Y12\rangle\langle Y34\rangle + \langle Y23\rangle\langle Y14\rangle > 0. \tag{2.1.19}$$

We can not fix the signs of $\langle Y13\rangle, \langle Y24\rangle$ from the boundary inequalities. The signs of both $\langle Y13\rangle, \langle Y24\rangle$ are negative in the amplituhedron, then the boundary inequalities are insufficient to define the amplituhedron.

To see how to define the amplituhedron as a collection of the inequalities, we consider the simplest case: $m = 1$ amplituhedron. We project the $k+1$ dimensional vectors Z_a^I through the k plane Y and obtain a configuration of 1-dimensional vectors Z'_a. Since we are projecting through Y, this Y is mapped to the origin in this 1-dimensional space. We look at the number of times the path $(12), (23), \cdots, (n-1n)$ jumps over the origin Y. This number is equivalent to the number of the sign flips in the brackets $\{\langle Y1\rangle, \langle Y2\rangle, \cdots, \langle Yn\rangle\}$. In the case of $n = k+1$, the signs of all $\langle Ya\rangle$ are fixed. From this case, we can see the explicit number of the sign flips for the amplituhedron: the $m = 1, k$ amplituhedron has exactly k sign flips. We can extends this for general n:

Y is in the $m = 1, k$ amplituhedron if
the sequence $\{\langle Y1\rangle, \cdots, \langle Yn\rangle\}$ has precisely k sign flips.

Similarly we can obtain the sign flip definition for general m amplituhedron [29]. The sign flip definition of the $m = 2$ tree amplituhedron is

Y is in the $m = 2$ amplituhedron iff
$\langle Yii+1\rangle > 0$ and the sequence $\{\langle Y12\rangle, \cdots, \langle Y1n\rangle\}$ has precisely k sign flips.

We can define $m = 4$ amplituhedron similarly as

$$Y \text{ is in the } m=4 \text{ amplituhedron iff}$$
$$\langle Yii+1jj+1\rangle > 0 \text{ and the sequence } \{\langle Y1234\rangle, \cdots, \langle Y123n\rangle\} \text{ has precisely } k \text{ sign flips.}$$

Next, we consider the generalization of this sign flip definition to the loop amplituhedron. Here we write the loop momenta in the l-loop integrands as $(AB)_\gamma$ where $\gamma = 1, \cdots, l$. The boundaries are $\langle (YAB)_\gamma ii+1\rangle$, $\langle Yii+1jj+1\rangle$ and $\langle Y(AB)_{\gamma_1}(AB)_{\gamma_2}\rangle$. The sign flip definition of the loop amplituhedron is given as

$$\begin{aligned} \langle (YAB)_\gamma ii+1\rangle > 0,\ \langle Yii+1jj+1\rangle > 0,\ \langle Y(AB)_{\gamma_1}(AB)_{\gamma_2}\rangle > 0 \\ \{(\langle YAB)_\gamma 12\rangle, \cdots, \langle (YAB)_\gamma 1n\rangle\} \text{ has } k+2 \text{ sign flips} \\ \langle Y1234\rangle > 0, \cdots, \langle Y123n\rangle \text{ has } k \text{ sign flips} \end{aligned} \tag{2.1.20}$$

where γ is the number of loops.

2.2 Canonical form and scattering amplitudes

Here we see how to extract scattering amplitudes from the amplituhedron. Let us define a "canonical form", a differential form defined by the requirement that it has logarithmic singularities on all the boundaries of the amplituhedron. Let's consider the simplest example: triangle. We can write the interior of the triangle as $Y = Z_1 + c_2 Z_2 + c_3 Z_3$, the boundaries are reached as $c_i \to 0$. Then the canonical form is given as

$$\Omega = \frac{dc_1}{c_1}\frac{dc_2}{c_2}. \tag{2.2.1}$$

This can be written as

$$\Omega_2 = \frac{\langle YdYdY\rangle\langle 123\rangle^2}{\langle Y12\rangle\langle Y23\rangle\langle Y31\rangle}. \tag{2.2.2}$$

Similarly the canonical form for the $m=4, k=1, n=5$ amplituhedron is given as

$$\Omega_4 = \frac{\langle Yd^4Y\rangle\langle 12345\rangle^4}{\langle Y1234\rangle\langle Y2345\rangle\langle Y3451\rangle\langle Y4512\rangle\langle Y5123\rangle}. \tag{2.2.3}$$

Next, we consider the convex polygon P. The canonical form of this polygon can be obtained by triangulating this and summing all the canonical forms for each triangle as

$$\Omega_P = \sum_i \Omega_{1ii+1}. \tag{2.2.4}$$

The canonical form can be written as

$$\Omega^{(m)}_{n,k} = \prod_{\alpha=1}^{k} \langle Y_1 \cdots Y_k d^m Y_\alpha \rangle \Omega'^{(m)}_{n,k} \tag{2.2.5}$$

where $\Omega'^{(m)}_{n,k}$ is a rational function. The scattering amplitude $\mathcal{M}_{n,k}$ is extracted from the $m = 4$ canonical form $\Omega^{(4)}_{n,k}$. Here we consider the tree case. First we fix the $k+4$ dimensional external data Z^I as 4-dimensional kinematic part z^i and k-dimensional part $\phi^j_A \cdot \eta^A$ as

$$Z^I = (z^i, \phi^1_A \cdot \eta^A, \cdots, \phi^k_A \cdot \eta^A) \tag{2.2.6}$$

for $i, A = 1, \cdots, 4$ and ϕ and η are Grassmann parameters. To obtain the $\mathcal{M}_{n,k}$, we localize the form $\Omega'^{(4)}_{n,k}$ to Y_0 and integrate over the ϕ as

$$\frac{\mathcal{M}_{n,k}(z_a, \eta_a)}{\mathcal{M}_{n,0}(z_a, \eta_a)} = \int d^4\phi_1 \cdots \int d^4\phi_k \Omega'^{(4)}_{n,k}(Y_0, Z_a). \tag{2.2.7}$$

For example, we consider the canonical form (2.2.3),

$$\begin{aligned} &\int d^4\phi \frac{\langle 12345 \rangle^4}{\langle Y_0 1234 \rangle \langle Y_0 2345 \rangle \langle Y_0 3451 \rangle \langle Y_0 4512 \rangle \langle Y_0 5123 \rangle} \\ &= \frac{\delta^{0|4}(\langle 1234 \rangle \eta_5 + \text{cyclic permutations})}{\langle 1234 \rangle \langle 2345 \rangle \langle 3451 \rangle \langle 4512 \rangle \langle 5123 \rangle}. \end{aligned} \tag{2.2.8}$$

This is just the 5-point NMHV tree amplitude. We can obtain the loop integrand from the canonical form of the loop amplituhedron similarly as

$$\frac{\mathcal{M}_{n,k}(z_a, \eta_a, \mathcal{L}_{\gamma(i)})}{\mathcal{M}_{n,0}(z_a, \eta_a)} = \int d^4\phi_1 \cdots \int d^4\phi_k \prod_{i=1}^{L} \langle \mathcal{L}_{1(i)} \mathcal{L}_{2(i)} d^2 \mathcal{L}_{1(i)} \rangle \langle \mathcal{L}_{1(i)} \mathcal{L}_{2(i)} d^2 \mathcal{L}_{2(i)} \rangle \int \Omega'^{(4)}_{n,k}(Y_0, Z_a, \mathcal{L}_{\gamma(i)}).$$

2.3 Sign flip triangulation

In this section, we see that the sign flip pattern gives a natural triangulation of the amplituhedron. First we consider the tree $m = 1, k = 1$ case. From the definition of sign flips, $\{\langle Y1 \rangle, \cdots, \langle Yn \rangle\}$ has 1 sign flip. We denote the place where the sign flip takes place j; $\langle Yj \rangle < 0$ and $\langle Yj+1 \rangle > 0$. Now we can expand Y on some basis $\mathcal{Z}_A, \mathcal{Z}_B$ as $Y = \mathcal{Z}_A + x\mathcal{Z}_B$. In order to describe the $m = 1$ cell where the sign flip occurs at j, it is convenient to choose $\mathcal{Z}_A = \mathcal{Z}_j$, $\mathcal{Z}_B = \mathcal{Z}_{j+1}$. From the sign flip conditions, we must have $x_j > 0$ and conversely, every Y of this form with $x > 0$ will belong to this cell. Then the canonical form for this sign flip pattern is

$$\Omega_j = \frac{dx_j}{x_j} \tag{2.3.1}$$

and the full form of $m = 1$ $k = 1$ amplituhedron is

$$\Omega = \sum_{1\le j\le n-1} \frac{dx_j}{x_j}. \tag{2.3.2}$$

This is the triangulation of the $m = 1, k = 1$ tree amplituhedron from sign flips. We can similarly triangulate for general k. The region in the $m = 1, k$ amplituhedron where $\{\langle Y1\rangle \cdots \langle Yn\rangle\}$ flips in slots $(j_1, \cdots j_k)$ is covered by

$$Y = (\mathcal{Z}_{j_1} + x_1\mathcal{Z}_{j_1+1})(\mathcal{Z}_{j_2} + x_2\mathcal{Z}_{j_2+1})\cdots(\mathcal{Z}_{j_k} + x_k\mathcal{Z}_{j_k+1}) \ \text{ with } x_k > 0. \tag{2.3.3}$$

The k-form related to each cell is

$$\Omega^{\{j_1,\cdots j_k\}} = \prod_{\alpha=1}^{k} d\log x_\alpha = \prod_{\alpha=1}^{k} d\log\frac{\langle Y\mathcal{Z}_{i_\alpha+1}\rangle}{\langle Y\mathcal{Z}_{i_\alpha}\rangle} \tag{2.3.4}$$

and the full form is

$$\Omega = \sum_{2\le j_1\le j_2\le\cdots\le j_k\le n-1} \Omega^{\{j_1,\cdots j_k\}}. \tag{2.3.5}$$

The dimension of each cell is k and this is the triangulation of $\mathcal{A}(m = 1, k, n)$ into non-redundant cells. Similarly the $m = 2$ amplituhedron can be triangulated from the sign flips. For $m = 2$ case, the sequence $\{\langle Y12\rangle, \langle Y13\rangle, \cdots, \langle Y1n\rangle\}$ has k sign flip in the slots $(j_1, \cdots j_k)$. Then we can expand Y as

$$Y = (+\mathcal{Z}_1 + x_1\mathcal{Z}_{j_1} + y_1\mathcal{Z}j_1 + 1)(-\mathcal{Z}_1 + x_2\mathcal{Z}_{j_2} + y_2\mathcal{Z}j_2 + 1)\cdots((-1)^k\mathcal{Z}_1 + x_k\mathcal{Z}_{j_k} + y_k\mathcal{Z}j_k + 1) \tag{2.3.6}$$

with $x_k, y_k > 0$. The k-form for each cell is

$$\begin{aligned}\Omega^{\{j_1,\cdots j_k\}} &= \prod_{\alpha=1}^{k} d\log x_\alpha d\log y_\alpha = \prod_{\alpha=1}^{k} d\log\frac{\langle Y1i_\alpha\rangle}{\langle Yi_\alpha i_\alpha+1\rangle} d\log\frac{\langle Y1i_\alpha+1\rangle}{\langle Yi_\alpha i_\alpha+1\rangle} \\ &= [1, j_1, j_1+1; 1, j_2, j_2+1; \cdots; 1, j_k, j_k+1]\end{aligned} \tag{2.3.7}$$

where

$$[i_1, i_2, i_3; \cdots; k_1, k_2, k_3] = \frac{\langle Yd^2Y_1\rangle\langle Yd^2Y_2\rangle\langle Yd^2Y_3\rangle\langle (Y_1Y_2Y_3)\cap(i_1i_2i_3)\cap\cdots\cap(k_1k_2k_3)\rangle}{\langle Yi_1i_2\rangle\langle Yi_2i_3\rangle\langle Yi_3i_1\rangle\cdots\langle Yk_3k_1\rangle}. \tag{2.3.8}$$

Then the full form is

$$\Omega_{n\text{-pt}}^{m=2,k} = \sum_{2\leq j_1\leq j_2\leq\cdots\leq j_k\leq n-1} [1, j_1, j_1+1; 1, j_2, j_2+1; \cdots; 1, j_k, j_k+1]. \tag{2.3.9}$$

The dimension of each cell is $2k$ and this is the triangulation of $\mathcal{A}(m=2,k,n)$. The important case is $k=2$, it is isomorphic to the 1-loop MHV amplituhedron $\mathcal{A}(k=0,n,m=4,l=1)$. The full canonical form of this amplituhedron $\mathcal{A}(2,n,2)$ from sign flips is

$$\begin{aligned}\Omega_{n\text{-pt}}^{m=2,k=2} &= \sum_{2\leq i\leq j\leq n-1} [1,i,i+1;1,j,j+1] \\ &= \sum_{2\leq i\leq j\leq n-1} \frac{\langle Yd^2Y_1\rangle\langle Yd^2Y_2\rangle\langle Y_1Y_2(1ii+1)\cap(1jj+1)\rangle^2}{\langle Y1i\rangle\langle Y1i+1\rangle\langle Yii+1\rangle\langle Y1j\rangle\langle Y1j+1\rangle\langle Yjj+1\rangle}.\end{aligned} \tag{2.3.10}$$

Take $(Y_1,Y_2)\leftrightarrow(A,B)$, this form corresponds to the Kermit representation of 1-loop MHV integrand.

For $m=1$ and $m=2$ case, we have seen that the sign flip pattern gives us a triangulation of the amplituhedron. However, for the case of $m=4$, there isn't a relation between the triangulation and the sign flip pattern.

Chapter 3

The 2-loop MHV Amplituhedron

In this section, we see that it is possible to triangulate the 2-loop MHV amplituhedron $\mathcal{A}(k=0,n,l=2)$ using the sign flip definition. The sign flip definition of the 2-loop MHV amplituhedron is

$$
\begin{aligned}
&\langle ABii+1\rangle > 0, \quad \langle CDii+1\rangle > 0 \\
&\{\langle AB12\rangle, \langle AB13\rangle, \cdots, \langle AB1n\rangle\} \ \text{ has 2 sign flip} \\
&\{\langle CD12\rangle, \langle CD13\rangle, \cdots, \langle CD1n\rangle\} \ \text{ has 2 sign flip} \\
&\langle ABCD\rangle > 0,
\end{aligned}
\tag{3.0.1}
$$

(A,B) and (C,D) are the loop momentum for each amplituhedron. From this, we can see that the 2-loop MHV amplituhedron is constructed by two 1-loop MHV amplituhedron (AB), (CD) and a further constraint $\langle ABCD\rangle > 0$. The important fact is that even if we consider the general n-point, there is only one constraint $\langle ABCD\rangle > 0$. Because of this, to obtain the canonical form we need to solve only one constraint and it is very easy rather than the $Y = C \cdot Z$ description. We construct the triangulation of the 2-loop MHV amplituhedron from the sign flip definition and compare with the BCFW.

3.1 Triangulation of 2-loop MHV Amplituhedron

3.1.1 Four point case

First we consider the simplest case, 2-loop 4-point MHV amplituhedron. From the sign flip definition (2.1.20), it is constructed from the two 1-loop 4-point MHV. The sign flip definition of the 1-loop amplituhedron is

$$\begin{aligned}
&\langle ABii+1\rangle > 0, \\
&\{\langle AB12\rangle, \langle AB13\rangle, \langle AB14\rangle\} \text{ has 1 sign flip.}
\end{aligned} \tag{3.1.1}$$

From this, there is only 1 sign flip pattern

$$\{\langle AB12\rangle, \langle AB13\rangle, \langle AB14\rangle\} = \{+,-,+\}. \tag{3.1.2}$$

Then we can expand the loop momentum as

$$Z_A = Z_1 + x_1 Z_2 + w_1 Z_3, \quad Z_B = -Z_1 + y_1 Z_3 + z_1 Z_4. \tag{3.1.3}$$

From the sign flip condition, the region of these variables are $x_1, w_1, y_1, z_1 > 0$. In the view of the $Y_\alpha = C_{\alpha a}\mathcal{Z}_a$ description, the C-matrix of this sign flip pattern is

$$C = \begin{pmatrix} 1 & x_1 & w_1 & 0 \\ -1 & 0 & y_1 & z_1 \end{pmatrix}. \tag{3.1.4}$$

Boundary of this pattern is $x_1 \to 0, w_1 \to 0, y_1 \to 0, z_1 \to 0$, then the canonical form is

$$\Omega_4^{l=1} = \frac{dx_1}{x_1}\frac{dw_1}{w_1}\frac{dy_1}{y_1}\frac{dz_1}{z_1} = \frac{\langle ABd^2A\rangle\langle ABd^2B\rangle\langle 1234\rangle^2}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB14\rangle}. \tag{3.1.5}$$

This form corresponds to the form of the 4-point 1-loop MHV amplituhedron obtained from the $Y = C \cdot Z$ description [11,41]. Next we consider the 2-loop 4-point MHV amplituhedron. This is constructed from the two 1-loop amplituhedron and a constraint $\langle ABCD\rangle > 0$. We can parametrize these two 1-loop amplituhedron as

$$\begin{aligned}
Z_A &= Z_1 + x_1 Z_2 + w_1 Z_3, \quad Z_B = -Z_1 + y_1 Z_3 + z_1 Z_4 \\
Z_C &= Z_1 + x_2 Z_2 + w_2 Z_3, \quad Z_D = -Z_1 + y_2 Z_3 + z_2 Z_4 \\
&\text{with} \quad x_1, w_1, y_1, z_1, x_2, w_2, y_2, z_2 > 0.
\end{aligned} \tag{3.1.6}$$

In view of the $\mathcal{Y} = \mathcal{C} \cdot Z$ description, the C-matrix is

$$C = \begin{pmatrix} 1 & x_1 & w_1 & 0 \\ -1 & 0 & y_1 & z_1 \\ 1 & x_2 & w_2 & 0 \\ -1 & 0 & y_2 & z_2 \end{pmatrix}. \tag{3.1.7}$$

Under this parametrization, the constraint become

$$\langle ABCD \rangle = \langle 1234 \rangle \{(x_1 - x_2)(y_1 z_2 - y_2 z_1) + (z_1 - z_2)(w_1 x_2 - w_2 x_1)\} > 0. \tag{3.1.8}$$

From this condition, these parameters are bounded further. Without loss of generality, we can take $y_1 z_2 - y_2 z_1 > 0$. Then from $\langle ABCD \rangle > 0$,

$$x_1 > x_2 - \frac{(z_1 - z_2)(w_1 x_2 - w_2 x_1)}{y_1 z_2 - y_2 z_1} = x_2 - a. \tag{3.1.9}$$

Therefore there are 4 cases depending on the signs of $(z_1 - z_2),\ (w_1 x_2 - w_2 x_1)$. For example, the case of $(z_1 - z_2) > 0,\ (w_1 x_2 - w_2 x_1) > 0$, the regions of these variables are

$$\begin{aligned} x_2 + a > x_1 > 0, \quad w_1 > \frac{x_1}{x_2} w_2, \quad y_1 > \frac{z_1}{z_2} y_2, \quad z_1 > z_2 \\ x_2 > 0, \quad w_2 > 0, \quad y_2 > 0, \quad z_2 > 0. \end{aligned} \tag{3.1.10}$$

Compare with (3.1.6), the regions of these parameters are further bounded because of this constraint. Then there are 9 boundaries

$$\begin{aligned} (x_1 \to x_2 + a, \quad x_1 \to 0), \quad w_1 \to \tfrac{x_1}{x_2} w_2, \quad y_1 \to \tfrac{z_1}{z_2} y_2, \quad z_1 \to z_2 \\ x_2 \to 0, \quad w_2 \to 0, \quad y_2 \to 0, \quad z_2 \to 0. \end{aligned} \tag{3.1.11}$$

We can obtain the canonical form for this case. For example, the region of x_1 is $0 < x_1 < x_2 + a$, then the form for x_1 is

$$\frac{1}{x_1} - \frac{1}{x_1 - x_2 - a}. \tag{3.1.12}$$

Then the canonical form for this case is

$$\Omega = \frac{1}{x_2} \left(\frac{1}{x_1} - \frac{1}{x_1 - x_2 - a} \right) \frac{1}{w_1 - \frac{x_1}{x_2} w_2} \frac{1}{w_2} \frac{1}{y_1 - \frac{z_1}{z_2} y_2} \frac{1}{y_2} \frac{1}{z_1 - z_2} \frac{1}{z_2}. \tag{3.1.13}$$

There are 4 patterns depending on the signs of $(z_1 - z_2)$, $(w_1 x_2 - w_2 x_1)$. The forms related to these 4 patterns can be constructed similarly

$$
\begin{aligned}
\Omega_1 &= \frac{1}{x_1 - x_2 + a}\frac{1}{x_2}\left(\frac{1}{w_1} - \frac{1}{w_1 - \frac{x_1}{x_2}w_2}\right)\frac{1}{w_2}\frac{1}{y_1 - \frac{z_1}{z_2}y_2}\frac{1}{y_2}\frac{1}{z_1 - z_2}\frac{1}{z_2} \\
\Omega_2 &= \frac{1}{x_1 - x_2 + a}\frac{1}{x_2}\frac{1}{w_1 - \frac{x_1}{x_2}w_2}\frac{1}{w_2}\frac{1}{y_1 - \frac{z_1}{z_2}y_2}\frac{1}{y_2}\left(\frac{1}{z_1} - \frac{1}{z_1 - z_2}\right)\frac{1}{z_2} \\
\Omega_3 &= \frac{1}{x_1}\left(\frac{1}{x_2} - \frac{1}{x_2 - x_1 - a}\right)\frac{1}{w_1 - \frac{x_1}{x_2}w_2}\frac{1}{w_2}\frac{1}{y_1 - \frac{z_1}{z_2}y_2}\frac{1}{y_2}\frac{1}{z_1 - z_2}\frac{1}{z_2} \\
\Omega_4 &= \frac{1}{x_1}\left(\frac{1}{x_2} - \frac{1}{x_2 - x_1 - a}\right)\left(\frac{1}{w_1} - \frac{1}{w_1 - \frac{x_1}{x_2}w_2}\right)\frac{1}{w_2}\frac{1}{y_1 - \frac{z_1}{z_2}y_2}\frac{1}{y_2}\left(\frac{1}{z_1} - \frac{1}{z_1 - z_2}\right)\frac{1}{z_2}.
\end{aligned} \tag{3.1.14}
$$

The remaining four cases $y_1 z_2 - y_2 z_1 < 0$ are obtained that swap $1 \leftrightarrow 2$. The sum of these 8 form is

$$
\Omega_{4\mathrm{pt}}^{l=2} = \frac{dx_1 dx_2 dw_1 dw_2 dy_1 dy_2 dz_1 dz_2}{x_1 x_2 w_1 w_2 y_1 y_2 z_1 z_2}\frac{(x_1 y_1 z_2 + x_2 y_2 z_1 + x_2 w_1 z_1 + x_1 w_2 z_2)}{\{(x_1 - x_2)(y_1 z_2 - y_2 z_1) + (z_1 - z_2)(w_1 x_2 - w_2 x_1)\}}. \tag{3.1.15}
$$

To translate it into the momentum twistor, we need to solve (3.1.6) for $x_1, x_2, \cdots, z_2$

$$
\begin{aligned}
x_1 &= -\frac{\langle AB13\rangle}{\langle AB23\rangle}, \quad w_1 = \frac{\langle AB12\rangle}{\langle AB23\rangle}, y_1 = \frac{\langle AB14\rangle}{\langle AB34\rangle}, \quad z_1 = -\frac{\langle AB13\rangle}{\langle AB34\rangle} \\
x_2 &= -\frac{\langle CD13\rangle}{\langle CD23\rangle}, \quad w_2 = \frac{\langle CD12\rangle}{\langle CD23\rangle}, y_2 = \frac{\langle CD14\rangle}{\langle CD34\rangle}, \quad z_2 = -\frac{\langle CD13\rangle}{\langle CD34\rangle}.
\end{aligned} \tag{3.1.16}
$$

Then the full form in the momentum twistor space is

$$
\begin{aligned}
\Omega_{4\mathrm{pt}}^{l=2} &= \frac{\langle 1234\rangle^3 \langle ABd^2A\rangle\langle ABd^2B\rangle\langle CDd^2C\rangle\langle CDd^2D\rangle}{\langle AB12\rangle\langle AB14\rangle\langle AB23\rangle\langle AB34\rangle\langle ABCD\rangle\langle CD12\rangle\langle CD14\rangle\langle CD23\rangle\langle CD34\rangle} \\
&\times \Big\{\langle AB34\rangle\langle CD12\rangle + \langle AB23\rangle\langle CD14\rangle + \langle AB14\rangle\langle CD23\rangle + \langle AB12\rangle\langle CD34\rangle\Big\}.
\end{aligned} \tag{3.1.17}
$$

The dimension of this amplituhedron is 8, therefore in this 4-point case, it is just a non-redundant cell. Of cause it can be obtained from the $Y = C \cdot Z$ description directly [41] and our result is corresponding to this $Y = C \cdot Z$ result. Next we see that the higher point 2-loop MHV amplituhedron can be triangulated into the non-redundant dimension 8 cells.

3.1.2 Five point case

Next we consider the 5-point amplitude. The 2-loop 5-point MHV amplituhedron is constructed from the two 1-loop 5-point MHV amplituhedron and a further constraint. In the 1-loop $n=5, k=2$ amplitude, there are 3 patterns of sign flips as

$$\{\langle AB12\rangle, \langle AB13\rangle, \langle AB14\rangle, \langle AB15\rangle\} = \{+,-,+,+\} \text{ or } \{+,-,-,+\} \text{ or } \{+,+,-,+\}. \tag{3.1.18}$$

Then we can parametrize for each pattern as

$$\begin{cases} Z_A = Z_1 + x_1 Z_2 + w_1 Z_3 \\ Z_B = -Z_1 + y_1 Z_3 + z_1 Z_4 \end{cases} (2,3) \text{ pattern },\ \begin{cases} Z_A = Z_1 + x_1 Z_2 + w_1 Z_3 \\ Z_B = -Z_1 + y_1 Z_4 + z_1 Z_5 \end{cases} (2,4) \text{ pattern},$$
$$\begin{cases} Z_A = Z_1 + x_1 Z_3 + w_1 Z_4 \\ Z_B = -Z_1 + y_1 Z_4 + z_1 Z_5 \end{cases} (3,4) \text{ pattern}. \tag{3.1.19}$$

Then depending on which pattern (3.1.19) we choose, there are $3\times 3 = 9$ patterns in the 2-loop amplituhedron. We can expect that the full form of the 2-loop 5-point MHV amplituhedron is obtained by the sum of these forms related to each 9 pattern. Each form can be obtained similarly as the 4-pt case, and the explicit calculation is given in the appendix and here we will write only the results. The case of $(2,3)\times(2,3)$ is same as the 4-pt case. The case of $(3,4)\times(3,4)$,

$$\begin{aligned}
\Omega_{3434} &= \frac{dx_1 dx_2 dw_1 dw_2 dy_1 dy_2 dz_1 dz_2}{x_1 x_2 w_1 w_2 y_1 y_2 z_1 z_2} \frac{\langle 1345\rangle}{\langle ABCD\rangle}(x_1 y_1 z_2 + x_2 y_2 z_1 + x_2 w_1 z_1 + x_1 w_2 z_2) \\
&= \frac{\langle 1345\rangle^3 \langle ABd^2A\rangle\langle ABd^2B\rangle\langle CDd^2C\rangle\langle CDd^2D\rangle}{\langle AB13\rangle\langle AB15\rangle\langle AB34\rangle\langle AB45\rangle\langle ABCD\rangle\langle CD13\rangle\langle CD15\rangle\langle CD34\rangle\langle CD45\rangle} \\
&\times \left\{\langle AB45\rangle\langle CD13\rangle + \langle AB34\rangle\langle CD15\rangle + \langle AB15\rangle\langle CD34\rangle + \langle AB13\rangle\langle CD45\rangle\right\}
\end{aligned} \tag{3.1.20}$$

The case of $(2,4)\times(3,4)$,

$$\begin{aligned}
\Omega_{2434} &= \frac{\langle 123A_4\rangle\langle 134C_4\rangle\langle ABd^2A\rangle\langle ABd^2B\rangle\langle CDd^2C\rangle\langle CDd^2D\rangle}{\left\{\begin{matrix}\langle AB12\rangle\langle AB13\rangle\langle AB14\rangle\langle AB15\rangle\langle AB23\rangle\langle AB45\rangle \\ \times\langle ABCD\rangle\langle CD13\rangle\langle CD14\rangle^2\langle CD15\rangle\langle CD34\rangle\langle CD45\rangle\end{matrix}\right\}} \\
&\times \Big\{\langle 123A_4\rangle(\langle AB45\rangle\langle CD13\rangle\langle CD14\rangle + \langle AB15\rangle\langle CD34\rangle\langle CD14\rangle) \\
&\quad -\langle 345A_2\rangle\langle AB14\rangle\langle CD14\rangle\langle CD15\rangle + \langle 123C_4\rangle\langle CD14\rangle\langle AB45\rangle\langle AB13\rangle\Big\}.
\end{aligned} \tag{3.1.21}$$

The case of $(2,3)\times(3,4)$,

$$\Omega_{2334} = \frac{\langle 123A_3\rangle\langle 134C_4\rangle\langle ABd^2A\rangle\langle ABd^2B\rangle\langle CDd^2C\rangle\langle CDd^2D\rangle}{\left\{\begin{array}{c}\langle AB12\rangle\langle AB13\rangle^2\langle AB14\rangle\langle AB23\rangle\langle AB34\rangle\\ \times\langle ABCD\rangle\langle CD13\rangle\langle CD14\rangle^2\langle CD15\rangle\langle CD34\rangle\langle CD45\rangle\end{array}\right\}}$$
$$\times\ \Big\{\langle AB13\rangle\langle 123C_4\rangle\langle CD4A_3\rangle - \langle AB13\rangle\langle AB14\rangle\langle CD13\rangle\langle 234C_4\rangle$$
$$+\langle CD14\rangle\langle 145A_2\rangle\langle CD3A_3\rangle - \langle AB14\rangle\langle AB23\rangle\langle CD13\rangle\langle CD14\rangle\langle 1345\rangle\Big\}. \tag{3.1.22}$$

The case of $(2,4)\times(2,4)$,

$$\Omega_{2424} = \frac{\langle 123A_4\rangle\langle 123C_4\rangle\langle ABd^2A\rangle\langle ABd^2B\rangle\langle CDd^2C\rangle\langle CDd^2D\rangle}{\left\{\begin{array}{c}\langle AB12\rangle\langle AB13\rangle\langle AB14\rangle\langle AB15\rangle\langle AB23\rangle\langle AB45\rangle\langle ABCD\rangle\\ \times\langle CD12\rangle\langle CD13\rangle\langle CD14\rangle\langle CD15\rangle\langle CD23\rangle\langle CD45\rangle\end{array}\right\}}$$
$$\times\ \Big\{\langle 123A_4\rangle(\langle AB12\rangle\langle CD13\rangle\langle CD45\rangle + \langle AB15\rangle\langle CD14\rangle\langle CD23\rangle)$$
$$+\langle 123C_4\rangle(\langle AB13\rangle\langle AB45\rangle\langle CD12\rangle + \langle AB14\rangle\langle AB23\rangle\langle CD15\rangle)$$
$$+\langle 2345\rangle(\langle AB12\rangle\langle AB15\rangle\langle CD13\rangle\langle CD14\rangle + \langle AB13\rangle\langle AB14\rangle\langle CD12\rangle\langle CD15\rangle)\Big\}. \tag{3.1.23}$$

We use the symbols that

$$A_i \equiv (AB)\cap(1ii+1), \quad C_k \equiv (CD)\cap(1kk+1). \tag{3.1.24}$$

The remaining patterns are $(3,4)\times(2,3),(2,4)\times(2,3),(3,4)\times(2,4)$. These forms can be obtained from $\Omega_{2334},\Omega_{2324},\Omega_{2434}$ that swap $AB\leftrightarrow CD$. We obtain all 9 forms and we can calculate the sum of these forms

$$\Omega^{l=2,\text{MHV}}_{\text{5-pt}} = \Omega_{2323}+\Omega_{2424}+\Omega_{3434}+\Omega_{2324}+\Omega_{2334}+\Omega_{2434}+\Omega_{2423}+\Omega_{3423}+\Omega_{3424}. \tag{3.1.25}$$

Each form Ω_{ijkl} has spurious poles $\langle AB13\rangle,\langle AB14\rangle,\langle CD13\rangle,\langle CD14\rangle$, we can see that all of these are canceled and remain only the physical poles in the full form and this result is corresponding to the BCFW representation. From this result, we can see that the 2-loop 5-point MHV amplituhedron is triangulated into the 9 cells related to each sign flip pattern, and these cells are 8-dimensional cells $G_+(4,4)$.

In the case of the BCFW, each cell of the 2-loop 5-point MHV amplitude has also the spurious poles not only like $\langle AB13\rangle,\langle AB14\rangle,\langle CD13\rangle,\langle CD14\rangle$, but also more complicate poles from taking the forward limit. Therefore this triangulation has a different structure compared with the BCFW triangulation.

3.1.3 n-point case

Next we consider the general n-pt case. First we consider the 1-loop n-point MHV amplituhedron. There are $\frac{1}{2}(n-3)(n-2)$ sign flip patterns from the way to chose i, j that

$$i, j = 2, 3, \cdots, n-1, \quad i < j. \tag{3.1.26}$$

When sign flip occurs at i, j slots, we can parametrize the loop momentum as

$$Z_A = Z_1 + xZ_i + wZ_{i+1}, Z_B = -Z_1 + yZ_j + zZ_{j+1}, \tag{3.1.27}$$

and the canonical form of this pattern is

$$\Omega_{ij} = \frac{dx}{x}\frac{dw}{w}\frac{dy}{y}\frac{dz}{z}. \tag{3.1.28}$$

Then the full form is

$$\Omega = \sum_{\substack{i,j=2,3,\cdots,n-1 \\ i<j}} \Omega_{ij}. \tag{3.1.29}$$

Next we consider the 2-loop n-point MHV amplituhedron. There are $[\frac{1}{2}(n-3)(n-2)]^2$ sign flip patterns in the 2-loop n-point MHV amplituhedron depending on the way to chose i, j, k, l that

$$i, j, k, l = 2, 3, \cdots, n-1, \quad i < j, k < l. \tag{3.1.30}$$

We can expand as

$$\begin{cases} Z_A = Z_1 + x_1 Z_i + w_1 Z_{i+1}, \\ Z_B = -Z_1 + y_1 Z_j + z_1 Z_{j+1} \end{cases} \begin{cases} Z_C = Z_1 + x_2 Z_k + w_2 Z_{k+1}, \\ Z_D = -Z_1 + y_2 Z_l + z_2 Z_{l+1} \end{cases}. \tag{3.1.31}$$

From the constraint $\langle ABCD \rangle > 0$, these parameters are bounded. The region of these parameters are depending on the other parameters and $\langle ijkl \rangle$, however, the sign of this determinant changes depending on the relation between (i, j) and (k, l). More precisely, the sign is depending on the order of i, j, k, l, if $i < j < k < l$, then $\langle ijkl \rangle > 0$. Therefore we need to determine the order of i, j, k, l to calculate each form. This order of i, j, k, l can be divided into 13 groups as

$$
\begin{aligned}
&i<k<l<j\cdots(1), \quad && i<k<j<l\cdots(2), \quad i<j<k<l\cdots(3), \quad i=k<l<j\cdots(4),\\
&i=k<j=l\cdots(5), && i=k<j<l\cdots(6), \quad i<k<j=l\cdots(7), \quad i<j=k<l\cdots(8),\\
&k<i=l<j\cdots(9), && k<i<l<j\cdots(10), \quad k<i<j=l\cdots(11), \quad k<i<j<l\cdots(12),\\
&k<l<i<j\cdots(13).
\end{aligned}
\tag{3.1.32}
$$

We can compute the forms for each case in the same way as the 5-point case. The case of (1), C-matrix is

$$
C=\begin{pmatrix}
1 & \cdots & i & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
-1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & j & \cdots\\
1 & \cdots & \cdots & \cdots & k & \cdots & \cdots & \cdots & \cdots & \cdots\\
-1 & \cdots & \cdots & \cdots & \cdots & \cdots & l & \cdots & \cdots & \cdots
\end{pmatrix}
\tag{3.1.33}
$$

where

$$
(i,i+1\to x_1,w_1)\ \ (j,j+1\to y_1,z_1)\ \ (k,k+1\to x_2,w_2)\ \ (l,l+1\to y_2,z_2)\ \ ,\cdots=0. \tag{3.1.34}
$$

The canonical form of this case in the momentum twistor space is

$$
\Omega^1_{ijkl}=\frac{\omega^{1'}_{ijkl}\langle 1ii+1A_j\rangle\langle 1kk+1C_l\rangle\langle ABd^2A\rangle\langle ABd^2B\rangle\langle CDd^2C\rangle\langle CDd^2D\rangle}{\langle AB1i\rangle\langle AB1i+1\rangle\langle AB1j\rangle\langle AB1j+1\rangle\langle ABCD\rangle\langle CD1k\rangle\langle CD1k+1\rangle\langle CD1l\rangle\langle CD1l+1\rangle} \tag{3.1.35}
$$

where

$$
\omega^{1'}_{ijkl}=\frac{\langle ABii+1\rangle\langle A_jC_kC_l1\rangle+\langle A_iA_jC_kC_l\rangle}{\langle ABii+1\rangle\langle ABjj+1\rangle\langle CDkk+1\rangle\langle CDll+1\rangle}. \tag{3.1.36}
$$

Again we use the symbols (3.1.24). The canonical forms for another case can be obtained similarly. We give all the canonical forms and the explicit calculation of the case of (1) in the appendix.

Then the full form of the 2-loop n-pt MHV amplituhedron is

$$
\Omega^{n\text{-pt 2-loop}}_{\text{MHV}}=\sum_{\substack{i,j,k,l=2,3,\cdots,n-1\\ i<k<l<j}}\Omega^1_{ijkl}+\sum_{i<k<j<l}\Omega^2_{ijkl}+\sum_{i<j<k<l}\Omega^3_{ijkl}+\cdots+\sum_{k<l<i<j}\Omega^{13}_{ijkl}. \tag{3.1.37}
$$

Similarly for the 5-pt case, these cells have spurious poles. However, all of these poles are canceled and remain only physical poles. We compared this result and the BCFW representation numerically and we checked that these results are corresponding up to at least 22-pt.

From this results, we can see that the 2-loop n-pt MHV amplituhedron is triangulated into the $[\frac{1}{2}(n-3)(n-2)]^2$ 8-dimension cells and this triangulation is obtained directly from the geometry.

3.2 More 2-loop Objects

3.2.1 Log of the 2-loop MHV Amplitude

In this section we consider the log of the 2-loop MHV amplitude. The expansion of the amplitude is

$$\mathcal{A} = 1 + gA_1 + g^2 A_2 + g^3 A_3 + \cdots . \tag{3.2.1}$$

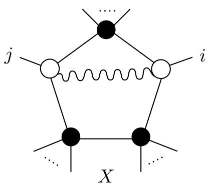
Then the expansion of the logarithm of the amplitude is

$$\mathcal{S} = \log \mathcal{A} = gS_1 + g^2 S_2 + g^3 S_3 + \cdots \tag{3.2.2}$$

where S_L is a sum of A_L and products of lower-loop amplitude,

$$S_1 = A_1, \quad S_2 = A_2 - \frac{1}{2}A_1^2, \quad S_3 = A_3 - A_2 A_1 + \frac{1}{3}A_1^3, \quad \cdots , \tag{3.2.3}$$

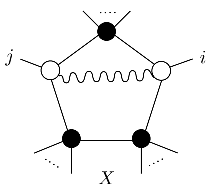
therefore the first non-trivial part is the 2-loop log amplitude. The 2-loop log amplitude can be expressed simply as a non-planar cyclic sum of the double pentagon diagram because of the simple relation between the square of the 1-loop pentagon diagram and the 2-loop double pentagon diagram [42]. The 1-loop pentagon diagram is



$$= \frac{\langle AB(i-1ii+1) \cap (j-1jj+1)\rangle \langle Xij\rangle}{\langle ABX\rangle \langle ABi-1i\rangle \langle ABii+1\rangle \langle ABj-1j\rangle \langle ABjj+1\rangle} \tag{3.2.4}$$

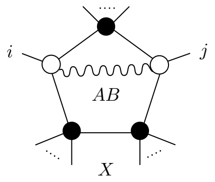
and the 1-loop MHV amplitude is

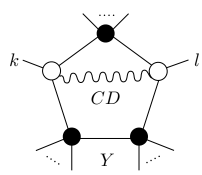
$$\mathcal{A}^{1-\text{loop}}_{\text{MHV}} = \sum_{i<j} \left\{ \right. $$

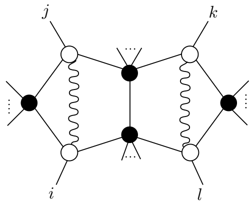


$$\left. \right\} . \tag{3.2.5}$$

The relation between this pentagon diagram and the double pentagon diagram is

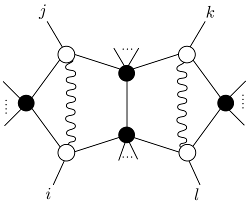
$$\sum_{i<j}$$


$$\times \sum_{k<l}$$


$$= \sum_{i<j,k<l}$$


$$, \tag{3.2.6}$$

where the double pentagon diagram is given as

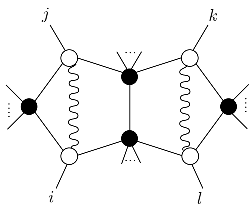


$$\begin{aligned} &= \frac{\langle AB(i{-}1ii{+}1)\cap(j{-}1jj{+}1)\rangle\langle ijkl\rangle}{\langle ABi{-}1i\rangle\langle ABii{+}1\rangle\langle ABj{-}1j\rangle\langle ABjj{+}1\rangle\langle ABCD\rangle} \\ &\quad\times\frac{\langle CD(k{-}1kk{+}1)\cap(l{-}1ll{+}1)\rangle}{\langle CDk{-}1k\rangle\langle CDkk{+}1\rangle\langle CDl{-}1l\rangle\langle CDll{+}1\rangle} \\ &= Q_{ijkl}. \end{aligned} \tag{3.2.7}$$

The left side is just $(\mathcal{A}_{\mathrm{MHV}}^{1-\mathrm{loop}})^2$ and the right side contains not only the planar diagrams $i<j<k<l$ but also the non-planar diagrams; for example, $i<k<j<l$. From (3.2.3), the log of the 2-loop amplitude is

$$-[\log\mathcal{A}]_{\mathrm{MHV}}^{2-\mathrm{loop}} = \frac{1}{2}(\mathcal{A}_{\mathrm{MHV}}^{1-\mathrm{loop}})^2 - \mathcal{A}_{\mathrm{MHV}}^{2-\mathrm{loop}}. \tag{3.2.8}$$

This means that the sum of all non-planar double pentagon diagrams times minus sign gives us the log of the 2-loop amplitude [42]

$$[\log\mathcal{A}]_{\mathrm{MHV}}^{2-\mathrm{loop}} = -\sum_{i<k<j<l<i}$$


$$= -\sum_{i<k<j<l<i} Q_{ijkl}. \tag{3.2.9}$$

For example, the log of the 4-pt amplitude is

$$\begin{aligned} &[\log\mathcal{A}]_{\mathrm{MHV}}^{2-\mathrm{loop,\ 4-pt}} = -Q_{1324} \\ = &\frac{\langle 1234\rangle^3(\langle AB13\rangle\langle CD24\rangle+\langle AB24\rangle\langle CD13\rangle)}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB14\rangle\langle ABCD\rangle\langle CD12\rangle\langle CD23\rangle\langle CD34\rangle\langle CD14\rangle}. \end{aligned} \tag{3.2.10}$$

Next we consider the log of the amplitude from the geometrical view. First we consider the region of the log of the amplitude. In the case of the 2-loop MHV, the definition of the amplituhedron is

$$\mathcal{L}_i = D_i \cdot Z, \quad i = 1, 2 \tag{3.2.11}$$

and

$$\langle D_{(1)} D_{(2)} \rangle > 0. \tag{3.2.12}$$

We consider another case: the square of the 1-loop MHV amplituhedron. This is defined similarly

$$\mathcal{L}_i = D_i \cdot Z, \quad i = 1, 2. \tag{3.2.13}$$

However, this has no positivity condition. From (3.2.8), the region of the minus log of the amplitude is $\langle D_{(1)} D_{(2)} \rangle < 0$. This pattern can be extended to all higher loop [41]. Then the question is that is it possible to obtain the log of the 2-loop MHV amplitude from the geometry

$$\mathcal{L}_i = D_i \cdot Z, \quad i = 1, 2, \quad \text{and} \quad \langle D_{(1)} D_{(2)} \rangle < 0 \tag{3.2.14}$$

and the canonical form of this geometry? In this section we construct the canonical form of this space (3.2.14) and see that it is corresponding to the log of the 2-loop MHV amplitude. To obtain the canonical form, we use the sign flip definition of this geometry

$$\begin{aligned}
&\langle ABii+1 \rangle > 0, \quad \langle CDii+1 \rangle > 0 \\
&\{\langle AB12 \rangle, \langle AB13 \rangle, \cdots, \langle AB1n \rangle\} \ \text{ has 2 sign flip} \\
&\{\langle CD12 \rangle, \langle CD13 \rangle, \cdots, \langle CD1n \rangle\} \ \text{ has 2 sign flip} \\
&\langle ABCD \rangle < 0
\end{aligned} \tag{3.2.15}$$

and call this geometry "2-loop MHV log amplituhedron". From this definition, we can see that the 2-loop n-point MHV log amplituhedron is constructed from the two 1-loop MHV amplituhedron and a negative constraint $\langle ABCD \rangle < 0$. Then there are $[\frac{1}{2}(n-3)(n-2)]^2$ sign flip patterns in the 2-loop n-point MHV log amplituhedron depending on the way to chose i, j, k, l that

$$i, j, k, l = 2, 3, \cdots, n-1, \quad i < j, k < l. \tag{3.2.16}$$

We can expand the loop momentum $(Z_A, Z_B), (Z_C, Z_D)$ as (3.1.31) and the order of i, j, k, l is divided into 13 groups as (3.1.32). Once we get the order of i, j, k, l, then we can calculate the canonical form similarly. For example, the canonical form for the case of (1) is

$$\Omega^1_{ijkl}[\log] = \frac{dx_1 dx_2 \cdots dz_1 dz_2}{x_1 x_2 w_1 w_2 y_1 y_2 z_1 z_2} \frac{-1}{(az_2 - bw_1 - cx_1 - dw_2 + ey_2)} \times \omega^1_{ijkl}[\log] \tag{3.2.17}$$

where

$$\begin{aligned}
\omega^1_{ijkl}[\log] &= x_2y_1\langle 1i+1kj\rangle + x_2z_1\langle 1i+1kj+1\rangle + y_1y_2\langle 1i+1lj\rangle + y_2z_1\langle 1i+1lj+1\rangle \\
&+ y_1z_2\langle 1i+1l+1j\rangle + z_1z_2\langle 1i+1l+1j+1\rangle + x_2y_1\langle 1ikj\rangle + x_2z_1\langle 1ikj+1\rangle \\
&+ w_2y_1\langle 1ik+1j\rangle + w_2z_1\langle 1ik+1j+1\rangle + y_1y_2\langle 1ilj\rangle + y_2z_1\langle 1il+1j\rangle \\
&+ y_1z_2\langle 1il+1j\rangle + z_1z_2\langle 1il+1j+1\rangle + w_1y_1\langle 1i+1k+1j\rangle \\
&+ w_1z_1\langle 1i+1k+1j+1\rangle \qquad (3.2.18)
\end{aligned}$$

and

$$\begin{aligned}
a &= x_1x_2\langle 1ikl+1\rangle + w_2x_1\langle 1ik+1l+1\rangle + w_1x_2\langle 1i+1kl+1\rangle + w_1w_2\langle 1i+1k+1l+1\rangle \\
&\quad +x_2y_1\langle 1kl+1j\rangle + x_2z_1\langle 1kl+1j+1\rangle + w_2y_1\langle 1k+1l+1j\rangle + w_2z_1\langle 1k+1l+1j+1\rangle \\
&\quad +x_1x_2y_1\langle ikl+1j\rangle + x_1x_2z_1\langle ikl+1j+1\rangle + w_2x_1y_1\langle ik+1l+1j\rangle \\
&\quad +w_2x_1z_1\langle ik+1l+1j+1\rangle + w_1x_2y_1\langle i+1kl+1j\rangle + w_1x_2z_1\langle i+1kl+1j+1\rangle \\
&\quad +w_1w_2y_1\langle i+1k+1l+1j\rangle + w_1w_2z_1\langle i+1k+1l+1j+1\rangle \\
b &= x_2y_1\langle 1i+1kj\rangle + x_2z_1\langle 1i+1kj+1\rangle + y_1y_2\langle 1i+1lj\rangle + y_2z_1\langle 1i+1lj+1\rangle \\
&\quad +y_1z_2\langle 1i+1l+1j\rangle + z_1z_2\langle 1i+1l+1j+1\rangle \\
c &= x_2y_1\langle 1ikj\rangle + x_2z_1\langle 1ikj+1\rangle + w_2y_1\langle 1ik+1j\rangle + w_2z_1\langle 1ik+1j+1\rangle + y_1y_2\langle 1ilj\rangle \\
&\quad +y_2z_1\langle 1il+1j\rangle + y_1z_2\langle 1il+1j\rangle + z_1z_2\langle 1il+1j+1\rangle \\
d &= w_1y_1\langle 1i+1k+1j\rangle + w_1z_1\langle 1i+1k+1j+1\rangle \\
e &= x_1x_2\langle 1ikl\rangle + w_2x_1\langle 1ik+1l\rangle + w_1x_2\langle 1i+1kl\rangle + w_1w_2\langle 1i+1k+1l\rangle + x_2y_1\langle 1klj\rangle \\
&\quad +x_2z_1\langle 1klj+1\rangle + w_2y_1\langle 1k+1lj\rangle + w_2z_1\langle 1k+1lj+1\rangle + x_1x_2y_1\langle iklj\rangle \\
&\quad +x_1x_2z_1\langle iklj+1\rangle + w_2x_1y_1\langle ik+1lj\rangle + w_2x_1z_1\langle ik+1lj+1\rangle + w_1x_2y_1\langle i+1klj\rangle \\
&\quad +w_1x_2z_1\langle i+1klj+1\rangle + w_1w_2y_1\langle i+1k+1lj\rangle + w_1w_2z_1\langle i+1k+1lj+1\rangle. \qquad (3.2.19)
\end{aligned}$$

We can calculate all forms for each pattern and the explicit form is written in the appendix. Then the full form of the 2-loop n-pt MHV log amplituhedron is

$$\Omega[\log[\mathcal{A}^{n\text{-pt 2-loop}}_{\text{MHV}}]] = \sum_{\substack{i,j,k,l=2,3,\cdots,n-1 \\ i<k<l<j}} \Omega^1_{ijkl}[\log] + \sum_{i<k<j<l} \Omega^2_{ijkl}[\log] + \cdots + \sum_{k<l<i<j} \Omega^{13}_{ijkl}[\log]. \tag{3.2.20}$$

Then we can compare with this result and the non-planar sum of the double pentagon diagrams (3.2.9) and we checked that these results are corresponding up to at least 22-pt.

In the case of the 2-loop MHV amplitude, we can obtain the log of the amplitude from the canonical form on the well-defined space as (3.2.14). However, the important point is that in the case of the 3-loop or higher loop, we can not define the log of the amplitude as a canonical form on the well-defined space. This means that the 2-loop MHV case is a special that we can define the log of the amplitude geometrically.

3.2.2 Square of the Amplituhedron and Positivity

Next we consider the decomposition of the square of the 1-loop MHV amplituhedron. From (3.2.13), the square of the 1-loop MHV amplituhedron is decomposed into the amplituhedron and the log amplituhedron

$$\begin{aligned}\big(\mathcal{L}_i = D_i \cdot Z\big) &= \big(\mathcal{L}_i = D_i \cdot Z,\ \ \langle D_{(1)} D_{(2)}\rangle > 0\big) \\ &+ \big(\mathcal{L}_i = D_i \cdot Z,\ \ \langle D_{(1)} D_{(2)}\rangle < 0\big)\end{aligned} \tag{3.2.21}$$

for $i = 1, 2$. We can see this decomposition directly from the canonical form. For example, the 4-point case, the canonical form of the amplitude and log of the amplitude is

$$\begin{aligned}\Omega[\mathcal{A}] &= \frac{dx_1 dx_2 dw_1 dw_2 dy_1 dy_2 dz_1 dz_2}{x_1 x_2 w_1 w_2 y_1 y_2 z_1 z_2} \frac{x_1 z_2 + x_2 z_1 + w_1 y_2 + w_2 y_1}{\{(x_1 - x_2)(z_2 - z_1) + (w_1 - w_2)(y_2 - y_1)\}} \\ \Omega[\log \mathcal{A}] &= \frac{dx_1 dx_2 dw_1 dw_2 dy_1 dy_2 dz_1 dz_2}{x_1 x_2 w_1 w_2 y_1 y_2 z_1 z_2} \frac{-(x_1 z_1 + x_2 z_2 + w_1 y_1 + w_2 y_2)}{\{(x_1 - x_2)(z_2 - z_1) + (w_1 - w_2)(y_2 - y_1)\}}.\end{aligned} \tag{3.2.22}$$

Then

$$\Omega[\mathcal{A}] + \Omega[\log \mathcal{A}] = \frac{dx_1 dx_2 dw_1 dw_2 dy_1 dy_2 dz_1 dz_2}{x_1 x_2 w_1 w_2 y_1 y_2 z_1 z_2}. \tag{3.2.23}$$

This is just the canonical form of the square of the 1-loop MHV amplituhedron (3.2.13). We can be confirmed that it holds for general n-point case from the explicit representation of the canonical form.

The interesting feature is that the numerator of the canonical form of the 2-loop MHV amplituhedron is the positive part of $\langle ABCD\rangle$ and the numerator of the log amplitude is the negative part. For example, the 4-pt case,

$$\langle ABCD\rangle = \langle 1234\rangle\{x_1 z_2 + x_2 z_1 + w_1 y_2 + w_2 y_1 - (x_1 z_1 + x_2 z_2 + w_1 y_1 + w_2 y_2)\}. \tag{3.2.24}$$

From the condition that (A, B) and (C, D) are the 1-loop MHV amplituhedron, we can see that

$$\langle 1234\rangle, x_1, x_2, w_1, w_2, \cdots, z_2 > 0. \tag{3.2.25}$$

Then $\langle ABCD\rangle$ is decomposed to

$$\langle ABCD\rangle = A^+ + A^- \tag{3.2.26}$$

where

$$A^+ = \langle 1234\rangle(x_1z_2+x_2z_1+w_1y_2+w_2y_1), \quad A^- = -\langle 1234\rangle(x_1z_1+x_2z_2+w_1y_1+w_2y_2) \tag{3.2.27}$$

and A^+ is positive, A^- is negative. From (3.2.22),

$$\Omega[\mathcal{A}] = \frac{dx_1dx_2dw_1dw_2dy_1dy_2dz_1dz_2}{x_1x_2w_1w_2y_1y_2z_1z_2}\frac{A^+}{\langle ABCD\rangle} \tag{3.2.28}$$

$$\Omega[\log\mathcal{A}] = \frac{dx_1dx_2dw_1dw_2dy_1dy_2dz_1dz_2}{x_1x_2w_1w_2y_1y_2z_1z_2}\frac{A^-}{\langle ABCD\rangle}, \tag{3.2.29}$$

we use the relation

$$\langle 1234\rangle\{(x_1-x_2)(z_2-z_1)+(w_1-w_2)(y_2-y_1)\} = \langle ABCD\rangle. \tag{3.2.30}$$

From the n-point forms of the amplitude and the log amplitude, we can see that this holds for general n-point case. $\langle ABCD\rangle$ is decomposed into the positive and negative parts even for the n-pt case. For example, the pattern (1) for (3.1.32),

$$\langle ABCD\rangle = az_2 - bw_1 - cx_1 - dw_2 + ey_2 \tag{3.2.31}$$

where a, b, c, d, e are defined as (3.2.19) and these are positive. Then the positive and negative parts is

$$A^+ = az_2 + ey_2, \quad A^- = -(bw_1 + cx_1 + dw_2). \tag{3.2.32}$$

The canonical form of the 2-loop amplitude and the log of the amplitude for this pattern (1) is

$$\Omega^1_{ijkl}[\mathcal{A}] = \frac{dx_1dx_2dw_1dw_2dy_1dy_2dz_1dz_2}{x_1x_2w_1w_2y_1y_2z_1z_2}\frac{A^+}{\langle ABCD\rangle} \tag{3.2.33}$$

$$\Omega^1_{ijkl}[\log\mathcal{A}] = \frac{dx_1dx_2dw_1dw_2dy_1dy_2dz_1dz_2}{x_1x_2w_1w_2y_1y_2z_1z_2}\frac{A^-}{\langle ABCD\rangle} \tag{3.2.34}$$

and we can see that this holds for all another patterns of (3.1.32). From this result and

$A^+ > 0$, the form of the n-pt 2-loop MHV amplituhedron is positive. Addition to this, in the form of the log amplitude, $A^- < 0$ and $\langle ABCD \rangle < 0$. Then the log of the amplitude is also positive. The positivity of the canonical form is related to the existence of a "dual amplituhedron" [20]. Then this is the another prove of the positivity of the canonical form directly.

Chapter 4

The 1-loop NMHV Amplituhedron

In the previous section, we have seen that the 2-loop MHV amplituhedron is triangulated by using the sign flip triangulation. In this section, we consider the 1-loop NMHV amplituhedron. Since there is no isomorphism between this 1-loop NMHV and the $m = 2$ amplituhedron, then we cannot use the same way with the MHV case. However, from the sign flip definition, we can see that the 1-loop N^kMHV amplituhedron is constructed from the $m = 2, k + 2$ amplituhedron and $m = 2, k$ amplituhedron which intersecting with the N^kMHV tree amplituhedron. This means that even higher k case, the amplituhedron is constructed from the two $m = 2$ amplituhedra and once we obtain this representation, we can triangulate by using the sign flip triangulation. In section 4.1, we see how to construct the 1-loop MNHV amplituhedron as a product of two $m = 2$ amplituhedra and construct explicitly. In section 4.2, we introduce the super-local representation of the 1-loop NMHV amplituhedron.

4.1 6-2 Representation of the 1-loop NMHV Amplituhedron

4.1.1 Amplituhedron as a Product of $m = 2$ Amplituhedra

The sign flip definition of the 1-loop NMHV amplituhedron is

$$\begin{aligned} \langle (YAB)ii+1\rangle > 0, \langle Yii+1jj+1\rangle > 0 \\ \{\langle (YAB)12\rangle, \cdots, \langle (YAB)1n\rangle\} \text{ has 3 sign flips} \\ \{\langle Y1234\rangle, \langle Y1235\rangle, \cdots, \langle Y123n\rangle\} \text{ has 1 sign flips.} \end{aligned} \tag{4.1.1}$$

From this definition, we can see that this 1-loop NMHV amplituhedron is written as a product of two $m = 2$ amplituhedra; $m = 2, k = 3$ amplituhedron (YAB) and the polygon which is the intersection of the plane (YAB) and the tree amplituhedron given by the convex hull of the external data for $k = 1$. This means that the form of this 1-loop NMHV amplituhedron is expressed as "6-form $\times$ 2-form", where the 6-form is the canonical form for (YAB) amplituhedron, and the 2-form is the one for the intersecting polygon. The important point is that in this representation, there is no difference between Y and AB variables. From this, we write the (YAB) plane as $(Y_1Y_2Y_3)$ plane. Usually, we write this form as "4-form $\times$ 4-form" from the BCFW: one 4-form is depended on Y (which is corresponding to the R-invariant), another 4-form for the loop momentum (AB). Then this 6-form$\times$2-form representation has a completely different structure than the BCFW. We call this representation as "6-2 representation".

Next, we see that what vertices make this "intersecting polygon". In the case of 1-loop NMHV amplituhedron, we need to consider the intersection of a 2-plane $(Y_1Y_2Y_3)$ and a 4-dimensional polytope with vertices Z_i. The boundaries of this pentagon are determined by the intersection of the plane $(Y_1Y_2Y_3)$ and the facets of the cyclic polytope: $\langle ii+1jj+1\rangle$. A vertex comes from the intersection of the plane and a triplet who share three indices of two boundaries. For example, the triplet which is determined as a intersection of two boundaries $(ii+1jj+1), (ii+1j+1j+2)$ is $(ii+1j+1)$. Explicitly, the boundary of this polytope $(ii+1jj+1)$ intersects with a 2-plane with a line

$$\begin{aligned}(Y_1Y_2Y_3)\cap(ii+1jj+1) &= (Z_iZ_{i+1})\langle Yjj+1\rangle + (Z_{i+1}Z_j)\langle Yj+1i\rangle \\ &+ (Z_jZ_{j+1})\langle Yii+1\rangle + (Z_{j+1}Z_i)\langle Yi+1j\rangle. \end{aligned} \tag{4.1.2}$$

where $\langle Yij\rangle = \langle Y_1Y_2Y_3ij\rangle$. The triplet $(ii+1j)$ intersects with a 2-plane with a point

$$(Y_1Y_2Y_3)\cap(ii+1j) = Z_i\langle Yi+1j\rangle + Z_{i+1}\langle Yji\rangle + Z_j\langle Yii+1\rangle. \tag{4.1.3}$$

This point is in the interior of this polytope if all of these coefficients are positive,

$$\langle Yii+1\rangle, \langle Yi+1j\rangle, \langle Yji\rangle > 0. \tag{4.1.4}$$

This means that the vertices of the intersecting polygon satisfy this condition. From this, the vertices of the intersecting polygon are written as triplets (a, b, c) which satisfy (4.1.4). The case of more general dimension is discussed in [19].

Once we obtain the shape of the intersecting polygon, we can write the canonical form of it.

For example, the 2-form of the triangle whose vertices are $\{\hat{i}, \hat{j}, \hat{k}\} = \{(i_1i_2i_3), (j_1j_2j_3), (k_1k_2k_3)\}$ is

$$\Omega^{m=2,k=1}_{\text{3-pt}(i_1i_2i_3)(j_1j_2j_3)(k_1k_2k_3)} = \frac{\langle yd^2y\rangle\langle\hat{i}\hat{j}\hat{k}\rangle^2}{\langle y\hat{i}\hat{j}\rangle\langle y\hat{j}\hat{k}\rangle\langle y\hat{k}\hat{i}\rangle} \tag{4.1.5}$$

where y is a point on the $(Y_1Y_2Y_3)$ plane inside this triangle. Next, we rewrite this 6-2 representation into the (YAB) space. In (YAB) space, the line $(\hat{i}\hat{j})$ on the plane $(Y_1Y_2Y_3)$ is just the intersection of two boundaries of the cyclic polytope $(i_1i_2i_3) \cap (j_1j_2j_3)$. Similarly $(\hat{i}\hat{j}\hat{k})$ is just the intersection of three boundaries $(i_1i_2i_3) \cap (j_1j_2j_3) \cap (k_1k_2k_3)$. From this, the explicit relations of brackets in the $(Y_1Y_2Y_3), y$ space and in the (YAB)space are

$$\begin{aligned}
\langle Y_1Y_2Y_3ij\rangle &\rightarrow \langle YABij\rangle, \\
\\
\langle y\hat{i}\hat{j}\rangle &\rightarrow \langle YAB(i_1i_2i_3) \cap (j_1j_2j_3)\rangle \\
&= \langle YABi_1i_2\rangle\langle Yi_3j_1j_2j_3\rangle + \langle YABi_2i_3\rangle\langle Yi_1j_1j_2j_3\rangle \\
&+ \langle YABi_3i_1\rangle\langle Yi_2j_1j_2j_3\rangle, \\
\\
\langle\hat{i}\hat{j}\hat{k}\rangle &\rightarrow \langle(YAB) \cap (i_1i_2i_3) \cap (j_1j_2j_3) \cap (k_1k_2k_3)\rangle \\
&= \begin{vmatrix} \langle YAi_1i_2i_3\rangle & \langle YA(j_1j_2j_3)\rangle & \langle YA(k_1k_2k_3)\rangle \\ \langle ABi_1i_2i_3\rangle & \langle AB(j_1j_2j_3)\rangle & \langle AB(k_1k_2k_3)\rangle \\ \langle BYi_1i_2i_3\rangle & \langle BY(j_1j_2j_3)\rangle & \langle BY(k_1k_2k_3)\rangle \end{vmatrix}.
\end{aligned} \tag{4.1.6}$$

And the measure changes as

$$\langle Yd^2Y_1\rangle\langle Yd^2Y_2\rangle\langle Yd^2Y_3\rangle\langle yd^2y\rangle \rightarrow \langle Yd^4Y\rangle\langle YABd^2A\rangle\langle YABd^2B\rangle. \tag{4.1.7}$$

We can generalize this to the 1-loop N^{k}MHV amplituhedron $\mathcal{A}^{\text{1-loop}}_{n,k}$. From the sign flip definition, we can see that $\mathcal{A}^{\text{1-loop}}_{n,k}$ is constructed from the $m = 2, k + 2$ amplituhedron and $m = 2, k$ amplituhedron which intersecting with the N^{k}MHV tree amplituhedron. Then the form of $\mathcal{A}^{\text{1-loop}}_{n,k}$ becomes $2(k + 2) \times 2k$ form and we call this as $2(k + 2)$-$2k$ representation.

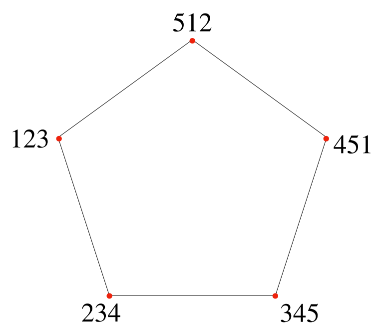


Figure 4.1: 5-pt intersecting pentagon

4.1.2 Five point case

In this section, we construct the 6-2 representation of the 1-loop NMHV amplituhedron explicitly. First we consider the simplest example; 5-pt case. In this simplest case, the $m = 2, k = 3$ amplituhedron is just the $G_+(3,5)$ positive Grassmannian $\mathcal{A}^{m=2,k=3}_{\text{5-pt}}(1,2,3,4,5)$. Next, we consider the shape of the intersecting polygon. The edges of this pentagon come from the boundaries of the cyclic polytope

$$(1234), (2345), (3451), (4512), (5123). \tag{4.1.8}$$

The triplets who share three indices of two boundaries are

$$(123), (234), (345), (451), (512), (124), (134), (135), (235), (245), (135). \tag{4.1.9}$$

A triplet (a, b, c) becomes a vertex of this polygon if the condition (4.1.4) is satisfied. From this, we can see that only $(123), (234), (345), (451), (512)$ are vertices of the polygon and the shape of this polygon is Figure 4.1. We denote these vertices as $\{(512), (123), (234), (345), (451)\} = \{\hat{1}, \hat{2}, \hat{3}, \hat{4}, \hat{5}\}$ and y as the point on this polygon. From this, we can see that the intersecting polygon is this pentagon and this is $m = 2, k = 1, n = 5$ amplituhedron where the vertices are $\{\hat{1}, \hat{2}, \hat{3}, \hat{4}, \hat{5}\}$. From this, the 6-2 representation of the 5-pt 1-loop NMHV amplitude is

$$\mathcal{A}^{l=1,k=1}_{\text{5-pt}}(1,2,3,4,5) = \mathcal{A}^{m=2,k=3}_{\text{5-pt}}(1,2,3,4,5) \times \mathcal{A}^{m=2,k=1}_{\text{5-pt}}(\hat{1}, \hat{2}, \hat{3}, \hat{4}, \hat{5}) \tag{4.1.10}$$

This is corresponding to the representation which is obtained from the "Momentum twistor diagram" [43]. From this 6-2 representation, we can see that the geometric factor of the measure of the 1-loop NMHV amplituhedron which is discussed in [43] is the intersecting

$m = 2, k = 1$ amplituhedron.

Next, we consider the canonical form of this 6-2 representation of the 5-pt case. In this case, $\mathcal{A}^{m=2,k=3}_{\text{5-pt}}$ is just the $G_+(3,5)$ positive Grassmannian and the 6-form is

$$\Omega_6^{\text{5pt}} = \frac{\langle 12345\rangle^2 \langle Y d^2 Y_1\rangle\langle Y d^2 Y_2\rangle\langle Y d^2 Y_3\rangle}{\langle Y12\rangle\langle Y23\rangle\langle Y34\rangle\langle Y45\rangle\langle Y51\rangle}. \tag{4.1.11}$$

To obtain the canonical form of the intersecting pentagon, we need to triangulate this. The form of this pentagon is written as

$$\Omega_2^{\text{5pt}} = \langle y d^2 y\rangle \times \left(\frac{\langle\hat{1}\hat{2}\hat{3}\rangle^2}{\langle y\hat{1}\hat{2}\rangle\langle y\hat{2}\hat{3}\rangle\langle y\hat{1}\hat{3}\rangle} + \frac{\langle\hat{1}\hat{3}\hat{4}\rangle^2}{\langle y\hat{1}\hat{3}\rangle\langle y\hat{3}\hat{4}\rangle\langle y\hat{4}\hat{5}\rangle} + \frac{\langle\hat{1}\hat{4}\hat{5}\rangle^2}{\langle y\hat{1}\hat{4}\rangle\langle y\hat{4}\hat{5}\rangle\langle y\hat{1}\hat{5}\rangle}\right). \tag{4.1.12}$$

Then the full form of the 5-pt 1-loop NMHV amplituhedron is

$$\Omega^{\text{5pt}} \quad = \quad \Omega_6^{\text{5pt}} \times \Omega_2^{\text{5pt}} \tag{4.1.13}$$

We can transform into (YAB) space by using (4.1.6) as

$$\begin{aligned}
\Omega^{\text{5pt}} \quad &= \quad \frac{\langle 12345\rangle^2\langle Y d^4 Y\rangle\langle YABd^2A\rangle\langle YABd^2B\rangle}{\langle YAB12\rangle\langle YAB23\rangle\langle YAB34\rangle\langle YAB45\rangle\langle YAB51\rangle} \\
&\times \quad \left\{\frac{\langle YAB12\rangle\langle YAB23\rangle\langle 12345\rangle^2}{\langle Y1235\rangle\langle Y1234\rangle\langle YAB13\rangle\langle YAB(125)\cap(234)\rangle}\right. \\
&+ \quad \frac{\langle YAB45\rangle\langle YAB15\rangle\langle 12345\rangle^2}{\langle Y3451\rangle\langle Y4512\rangle\langle YAB14\rangle\langle YAB(512)\cap(345)\rangle} \\
&+ \quad \left.\frac{\langle YAB34\rangle\langle YAB25\rangle^2\langle 12345\rangle^2}{\langle Y2345\rangle\langle Y3451\rangle\langle YAB13\rangle\langle YAB45\rangle\langle YAB(125)\cap(234)\rangle}\right\}.
\end{aligned} \tag{4.1.14}$$

This is corresponding to the 5-pt 1-loop NMHV amplituhedron.

Of cause we can triangulate the pentagon in another way. If we triangulate this pentagon by the lines of $(\hat{5}\hat{2})$, $(\hat{5}\hat{3})$, the form of this pentagon is

$$\Omega_2^{\text{5pt}} = \langle y d^2 y\rangle \times \left(\frac{\langle\hat{5}\hat{1}\hat{2}\rangle^2}{\langle y\hat{5}\hat{1}\rangle\langle y\hat{1}\hat{2}\rangle\langle y\hat{5}\hat{2}\rangle} + \frac{\langle\hat{5}\hat{2}\hat{3}\rangle^2}{\langle y\hat{5}\hat{2}\rangle\langle y\hat{2}\hat{3}\rangle\langle y\hat{5}\hat{3}\rangle} + \frac{\langle\hat{5}\hat{3}\hat{4}\rangle^2}{\langle y\hat{5}\hat{3}\rangle\langle y\hat{3}\hat{4}\rangle\langle y\hat{5}\hat{4}\rangle}\right) \tag{4.1.15}$$

We can rewrite this form into the (YAB) space as

$$\Omega^{5\text{pt}} = \langle YABd^2A\rangle\langle YABd^2B\rangle\langle Yd^4Y\rangle$$
$$\times\left\{\frac{\langle 12345\rangle^4}{\langle Y1245\rangle\langle Y1235\rangle\langle YAB23\rangle\langle YAB34\rangle\langle YAB45\rangle\langle YAB(145)\cap(123)\rangle}\right.$$
$$+\frac{\langle 12345\rangle^4}{\langle Y1345\rangle\langle Y2345\rangle\langle YAB12\rangle\langle YAB23\rangle\langle YAB15\rangle\langle YAB(145)\cap(234)\rangle}$$
$$\left.+\frac{\langle 12345\rangle^4\langle YAB14\rangle^2}{\langle Y1234\rangle\langle YAB12\rangle\langle YAB34\rangle\langle YAB45\rangle\langle YAB15\rangle\langle YAB(145)\cap(123)\rangle\langle YAB(145)\cap(234)\rangle}\right\}.$$

This is just the BCFW representation of the 5-pt 1-loop NMHV amplituhedron. From this, the BCFW triangulation for this 5-pt case is interpreted as one of the triangulation of the intersecting pentagon. However, this simple relation between the sign flip triangulation and the BCFW holds only in the 5-pt case.

4.1.3 Six point case

Next we consider 6-pt case. The triplets who share three indices of two boundaries are

$$\begin{aligned}&(1,2,3),(1,2,4),(1,2,5),(1,3,4),(2,3,4),(2,3,5),(2,3,6),(3,4,5),(3,4,6)\\&(4,5,6),(5,6,1),(6,1,2),(2,4,5),(3,5,6),(2,5,6),(4,6,1),(1,3,6),(1,4,5).\end{aligned} \tag{4.1.16}$$

First we consider the $(Y_1Y_2Y_3)$ amplituhedron. From the sign flip definition, this is decomposed into four cells as

	$\langle Y12\rangle$	$\langle Y13\rangle$	$\langle Y14\rangle$	$\langle Y15\rangle$	$\langle Y16\rangle$
$\mathcal{A}^6_{234}$	$+$	$-$	$+$	$-$	$-$
$\mathcal{A}^6_{235}$	$+$	$-$	$+$	$+$	$-$
$\mathcal{A}^6_{245}$	$+$	$-$	$-$	$+$	$-$
$\mathcal{A}^6_{345}$	$+$	$+$	$-$	$+$	$-$

Let's consider $\mathcal{A}_{234}$ cell. From the signs of brackets $\langle Yii+1\rangle, \langle Y1i\rangle$, we can see that $(1,2,3),\ (1,2,5),\ (2,3,4),\ (3,4,5),\ (1,4,5)$ can become the vertices of the polygon. However, some other vertices

$$(2,3,6),\ (3,4,6),\ (4,5,6),\ (6,1,2),\ (2,5,6),\ (4,6,1) \tag{4.1.17}$$

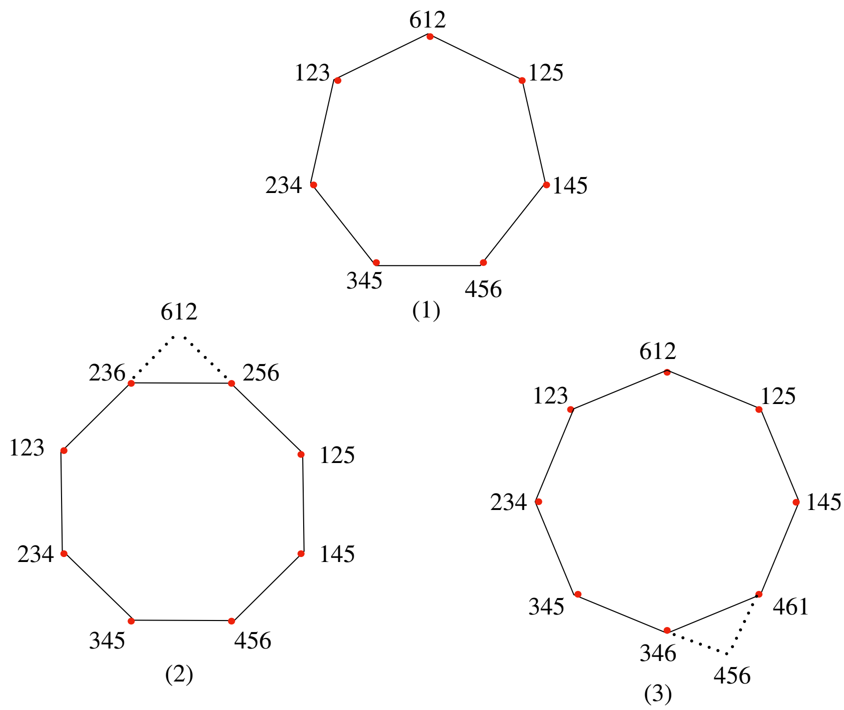


Figure 4.2: polygons for each cell

can become vertices depending on the signs of other brackets

$$\langle Y26\rangle,\ \langle Y36\rangle,\ \langle Y46\rangle. \tag{4.1.18}$$

The possible patterns of the signs of these brackets are

$\langle Y26\rangle$	$\langle Y36\rangle$	$\langle Y46\rangle$	vertices	pentagon
$+$	$+$	$-$	$(6,1,2),(4,5,6)$	(1)
$+$	$-$	$-$	$(6,1,2),(4,5,6)$	
$-$	$+$	$-$	$(2,3,6),(2,5,6),(4,5,6)$	(2)
$+$	$-$	$+$	$(3,4,6),(4,5,6),(6,1,2)$	(3)

For each sign patterns, there are different polygons as Figure 4.2. This means that we need to further triangulate these sign flip cells by other brackets to obtain the intersecting polygon. Similarly we can obtain the pentagons for other cells. The explicit results and calculations are written in appendix B.1 and the final result is

$$\begin{aligned}
\mathcal{A}^{l=1,k=1}_{\text{6-pt}} &= \mathcal{A}^{m=2,k=3}_{\text{6-pt}}(1,2,3,4,5,6)\times\mathcal{A}^{m=2,k=1}_{\text{6-pt}}(\hat{1},\hat{2},\hat{3},\hat{4},\hat{5},\hat{6}) \\
&+ \sum_{1\leq i\leq 6}\mathcal{A}^{m=2,k=3}_{\text{5-pt}}(i+2,i+3,\cdots,i-1,i)\times\mathcal{A}^{m=2,k=1}_{\text{3-pt}((ii+1i+2),(ii+2i+3),(ii+2i-1))}.
\end{aligned} \tag{4.1.19}$$

This is the 6-2 representation of the 6-pt 1-loop NMHV amplituhedron.

To obtain the canonical form, we need to triangulate the $m=2, k=3$ amplituhedron. We can triangulate all $m=2$ amplituhedron by using the "sign flip triangulation" [29], and we can construct the canonical form. The canonical form of the $m=2$ amplituhedron from the sign flip triangulation is given as (2.3.9). From this, we can obtain the canonical form of the 6-2 representation of the 6-pt case as

$$\begin{aligned}
\Omega^{6\times 2}_{\text{6-pt}} &= \Omega^{m=2,k=3}_{\text{6-pt}(61)}\times\sum_{2\leq i\leq 5}\Omega^{m=2,k=1}_{\text{3-pt}(1)(i)(i+1)} \\
&+ \sum_{1\leq i\leq 6}\Omega^{m=2,k=3}_{\text{5-pt}(ii+2)}\times\Omega^{m=2,k=1}_{\text{3pt}(ii+2i-1)(ii+2i+1)(ii+2i+3)}
\end{aligned} \tag{4.1.20}$$

here we use the notation that

$$\Omega^{m=2,k}_{n\text{-pt}(ab)} = \sum_{b+1\leq i_1<\cdots<i_k\leq a-1}[b,i_1,i_1+1;\cdots;b,i_k,i_k+1] \tag{4.1.21}$$

and $\Omega^{m=2,k=1}_{\text{3-pt}}$ is given as (4.1.5). We can transform into (YAB) space similarly by using (4.1.6). We write the explicit representation in this (YAB) space in appendix C. We have checked that the sum of all of these cells are corresponding to the BCFW representation of the 6-pt 1-loop NMHV amplituhedron.

4.1.4 n-point case

To go to the higher point case, we need to further triangulate sign flip cells of the $(Y_1Y_2Y_3)$ amplituhedron. Let consider the 234 cell in 7-pt. The 234 cell means that the cell which has 3 sign flips at $\langle Y12\rangle, \langle Y13\rangle, \langle Y14\rangle$. To obtain the vertices of the intersecting polygon, we need to triangulate by the signs of other brackets as $\{(24),(25),(26),(27),(35),(36),\cdots,(57)\}$ where (ij) means $\langle Yij\rangle$. The number of possible patterns is 10 and there are polygons for each cell. When we go to a higher point, the number of the cells and the intersecting polygons for each cell become very large, then it is difficult to obtain the 6-2 representation for the general

n-pt amplituhedron from this way. However, we have already seen that the 6-2 representation of the 6-pt case is simple. This simplicity holds for not only 6-pt but also higher point case. For example, from the straightforward calculation, the 7-pt and 8-pt results are

$$\begin{aligned}\mathcal{A}^{l=1,k=1}_{7\text{-pt}} &= \mathcal{A}^{m=2,k=3}_{7\text{-pt}}(1,2,3,4,5,6,7) \times \mathcal{A}^{m=2,k=1}_{7\text{-pt}}(\hat{1},\hat{2},\hat{3},\hat{4},\hat{5},\hat{6},\hat{7}) \\ &+ \sum_{1\le i\le 7}\sum_{2\le k\le 3} \mathcal{A}^{m=2,k=3}_{(8-k)\text{-pt}}(i+k,i+k+1,\cdots,i) \\ &\times \left(\mathcal{A}^{m=2,k=1}_{3\text{pt}(ii+ki-1)(ii+ki+k-1)(ii+ki+1)} + \mathcal{A}^{m=2,k=1}_{3\text{pt}(ii+ki-1)(ii+ki+1)(ii+ki+k+1)}\right)\end{aligned} \tag{4.1.22}$$

$$\begin{aligned}\mathcal{A}^{l=1,k=1}_{8\text{-pt}} &= \mathcal{A}^{m=2,k=3}_{8\text{-pt}}(1,2,\cdots,8) \times \mathcal{A}^{m=2,k=1}_{8\text{-pt}}(\hat{1},\hat{2},\cdots,\hat{8}) \\ &+ \sum_{1\le i\le 8}\sum_{2\le k\le 4} \left(\mathcal{A}^{m=2,k=3}_{(9-k)\text{-pt}}(i+k,i+k+1,\cdots,i) - \mathcal{A}^{m=2,k=3}_{(k+1)\text{-pt}}(i,i+1,\cdots,i+k)\right) \\ &\times \left(\mathcal{A}^{m=2,k=1}_{3\text{pt}(ii+ki-1)(ii+ki+k-1)(ii+ki+1)} + \mathcal{A}^{m=2,k=1}_{3\text{pt}(ii+ki-1)(ii+ki+1)(ii+ki+k+1)}\right)\end{aligned} \tag{4.1.23}$$

From these results, we can suppose that the 6-2 representation of the general n-pt amplituhedron is written as

$$\begin{aligned}\mathcal{A}^{l=1,k=1}_{n\text{-pt}} &= \mathcal{A}^{m=2,k=3}_{n\text{-pt}}(1,2,\cdots,n) \times \mathcal{A}^{m=2,k=1}_{n\text{-pt}}(\hat{1},\hat{2},\cdots,\hat{n}) \\ &+ \frac{1}{2}\sum_{1\le i\le n}\sum_{2\le k\le n-2} \left(\mathcal{A}^{m=2,k=3}_{(n-k+1)\text{-pt}}(i+k,i+k+1,\cdots,i) - \mathcal{A}^{m=2,k=3}_{(k+1)\text{-pt}}(i,i+1,\cdots,i+k)\right) \\ &\times \left(\mathcal{A}^{m=2,k=1}_{3\text{pt}(ii+ki-1)(ii+ki+k-1)(ii+ki+1)} + \mathcal{A}^{m=2,k=1}_{3\text{pt}(ii+ki-1)(ii+ki+1)(ii+ki+k+1)}\right).\end{aligned} \tag{4.1.24}$$

To check this formula is true, we need to obtain the canonical form of this 6-2 representation and compare it with another expression like BCFW. We can similarly construct the canonical form by using the sign flip triangulation and the result is

$$\begin{aligned}\Omega^{6\times 2}_{n\text{-pt}} &= \Omega^{m=2,k=3}_{n\text{-pt}(n1)} \times \sum_{2\le i\le n-1} \Omega^{m=2,k=1}_{3\text{-pt}(1)(i)(i+1)} \\ &+ \frac{1}{2}\sum_{1\le i\le n}\sum_{2\le k\le n-2} \left(\Omega^{m=2,k=3}_{(n-k+1)\text{-pt}(ii+k)} - \Omega^{m=2,k=3}_{(k+1)\text{-pt}(i+ki)}\right) \\ &\times \left(\Omega^{m=2,k=1}_{3\text{pt}(ii+ki-1)(ii+ki+k-1)(ii+ki+1)} + \Omega^{m=2,k=1}_{3\text{pt}(ii+ki-1)(ii+ki+1)(ii+ki+k+1)}\right).\end{aligned} \tag{4.1.25}$$

We have checked that this formula is consistent with the BCFW up to at least 22-pt numerically. This canonical form is expressed as a product of two canonical forms of the $m = 2$ amplituhedra. This is a completely different structure than the BCFW triangulation, which is written as a product of R-invariant and 1-loop MHV Kermit [9].

4.2 Super-Local Form and Positivity

In this section, we see the "super-local" representation of the 1-loop NMHV amplitude. There is the local representation of the loop amplitudes [42]. For example, the local representation of the 1-loop NMHV amplituhedron is

$$\begin{aligned}
\mathcal{A}^{\text{1-loop}}_{\text{NMHV}} &= \sum_{i<j<k<i} \frac{\langle YAB(i-1ii+1)\cap\Sigma_{ijk}\rangle}{\langle YABX\rangle\langle YABi-1i\rangle\langle YABii+1\rangle\langle YABjj+1\rangle\langle YABkk+1\rangle} \\
&\qquad \times[i,j,j+1,k,k+1] \\
&+ \sum_{i<j<i} \frac{\langle YAB(i-1ii+1)\cap(j-1jj+1)\rangle\langle Xij\rangle}{\langle YABX\rangle\langle YABi-1i\rangle\langle YABii+1\rangle\langle YABj-1j\rangle\langle YABjj+1\rangle} \\
&\qquad \times\mathcal{A}^{\text{tree}}_{\text{NMHV}}(j,j+1,\cdots,i-1,i)
\end{aligned} \tag{4.2.1}$$

where X is a reference bi-twistor and

$$\begin{aligned}
[i,j,k,l,m] &= \frac{\langle ijklm\rangle^4}{\langle Yijkl\rangle\langle Yjklm\rangle\langle Yklmi\rangle\langle Ylmij\rangle\langle Ymijk\rangle} \\
\Sigma_{ijk} &= \frac{1}{2}[(jj+1(ikk+1)\cap X)-(kk+1(ijj+1)\cap X)].
\end{aligned} \tag{4.2.2}$$

This expression involves the R-invariants which have spurious poles as a function of the external particle momenta. This means that the only poles involving the loop integration variables are local.

Here we obtain another representation: "Super-local representation". The super-local means both of external poles and internal poles are local. From the 6-2 representation, the 1-loop NMHV amplituhedron is constructed from $m=2,k=3$ and $m=2,k=1$ amplituhedra. We know the local triangulation for a $m=2,k=1$

$$\Omega^{m=2,k=1}_{n\text{pt}} = \sum_i \frac{\langle 12i\rangle\langle i-1ii+1\rangle}{\langle y12\rangle\langle yi-1i\rangle\langle yii+1\rangle}. \tag{4.2.3}$$

and for a $m=2,k=3$

$$\Omega^{m=2,k=3}_{n\text{pt}(n1)} = \sum_{j_1,j_2,j_3} \frac{\langle 12j_1j_2j_3\rangle\langle Y(j_1)\cap(j_2)\cap(j_3)\rangle}{\langle Y12\rangle\langle Yj_1-1j_1\rangle\langle Yj_1j_1+1\rangle\cdots\langle Yj_3j_3+1\rangle}, \tag{4.2.4}$$

where we use the notation that

$$\langle Y(j_1)\cap(j_2)\cap(j_3)\rangle \equiv \langle Y(j_1-1j_1j_1+1)\cap(j_2-1j_2j_2+1)\cap(j_3-1j_3j_3+1)\rangle \quad (4.2.5)$$

From this, we can rewrite a term $\Omega^{m=2,k=3}_{n\text{-pt}}\times\Omega^{m=2,k=1}_{n\text{-pt}}$ of (4.1.25) as local. Next, we consider $\Omega^{m=2,k=3}_{(n-k+1)\text{-pt}(ii+k)}$, $\Omega^{m=2,k=3}_{(k+1)\text{-pt}(i+ki)}$. We can also rewrite this term by using (4.2.4). The important point is that these terms have spurious pole $\langle Yii+k\rangle$ for $\Omega^{m=2,k=3}_{(n-k+1)\text{-pt}(ii+k)}$ and $\langle Yi+ki\rangle$ for $\Omega^{m=2,k=3}_{(k+1)\text{-pt}(i+ki)}$. The last remain part is

$$\Omega^{m=2,k=1}_{3\text{pt}(ii+ki-1)(ii+ki+k-1)(ii+ki+1)}+\Omega^{m=2,k=1}_{3\text{pt}(ii+ki-1)(ii+ki+1)(ii+ki+k+1)}. \quad (4.2.6)$$

This is the canonical form of the $m=2,k=1,n=4$ amplituhedron whose vertices are $\{(ii+ki-1),(ii+ki+k-1),(ii+ki+1),(ii+ki+k+1)\}$. The local representation of this form in the $(Y_1Y_2Y_3)$ space is

$$\begin{aligned}&\frac{\langle Yii+k\rangle\langle i,i+k,i-1,i+k+1,i+1\rangle\langle i,i+k,i+k+1,i+1,i+k-1\rangle}{\langle Yi-1ii+ki+k+1\rangle\langle Yii+1i+ki+k+1\rangle\langle Yii+1i+ki+k-1\rangle}\\ +\;&\frac{\langle Yii+k\rangle\langle i,i+k,i-1,i+k+1,i+k-1\rangle\langle i,i+k,i+1,i+k-1,i-1\rangle}{\langle Yi-1ii+ki+k+1\rangle\langle Yii+1i+ki+k-1\rangle\langle Yi-1ii+k-1i+k\rangle}.\end{aligned} \quad (4.2.7)$$

This has only physical poles and the important point is that this has a $\langle Yii+k\rangle$ on its denominator and because of this, the spurious pole is canceled. This means that all terms of (4.1.25) are local. The explicit super-local form of the 1-loop NMHV amplituhedron in

(YAB) space is

$$\begin{aligned}
\Omega^{6\times 2}_{n\text{-pt}} =& \sum_{\substack{2\le j_1<j_2\\<j_3\le n-1}} \frac{\langle 12j_1j_2j_3\rangle\langle YAB(j_1)\cap(j_2)\cap(j_3)\rangle}{\langle YAB12\rangle\langle YABj_1-1j_1\rangle\langle j_1j_1+1\rangle\cdots\langle YABj_3j_3+1\rangle}\\
&\times\sum_{1\le i\le n}\frac{\langle YAB(n12)\cap(123)\cap(i-1ii+1)\rangle\langle i-2i-1ii+1i+2\rangle}{\langle YAB12\rangle\langle Yn123\rangle\langle Yi-2i-1ii+1\rangle\langle Yi-1ii+1i+2\rangle}\\
&+\frac{1}{2}\sum_{\substack{2\le k\le n-2\\1\le i\le n}}\left(\sum_{\substack{i+k+1\le j_1<j_2\\<j_3\le i-1}}\frac{\langle i+kj_1j_2j_3i\rangle\langle YAB(j_1)\cap(j_2)\cap(j_3)\rangle}{\langle YABi+ki+k+1\rangle\langle YABi+k+1i+k+2\rangle\cdots\langle YABi-1i\rangle}\right.\\
&\qquad\left.+\sum_{\substack{i+1\le j_1<j_2\\<j_3\le i+k-1}}\frac{\langle ij_1j_2j_3i+k\rangle\langle YAB(j_1)\cap(j_2)\cap(j_3)\rangle}{\langle YABii+1\rangle\langle YABi+1i+2\rangle\cdots\langle YABi+k-1i+k\rangle}\right)\\
&\times\left(\frac{\langle i-1,i,i+1,i+k,i+k+1\rangle\langle i,i+1,i+k-1,i+k,i+k+1\rangle}{\langle Yi-1ii+ki+k+1\rangle\langle Yii+1i+ki+k+1\rangle\langle Yii+1i+ki+k-1\rangle}\right.\\
&\left.+\frac{\langle i-1,i,i+k-1,i+k,i+k+1\rangle\langle i-1,i,i+1,i+k-1,i+k\rangle}{\langle Yi-1ii+ki+k+1\rangle\langle Yii+1i+ki+k-1\rangle\langle Yi-1ii+k-1i+k\rangle}\right).
\end{aligned} \tag{4.2.8}$$

We write the explicit form of this super-local form of the 6-pt case in appendix C.

Next, we consider the positivity of this form. This super-local form has only physical pole $\langle YABii+1\rangle$, $\langle Yii+1jj+1\rangle$. From the definition of the amplituhedron, all of these physical poles are positive. We can prove that $\langle YAB(j_1)\cap(j_2)\cap(j_3)\rangle>0$ for $j_1<j_2<j_3$ from the positivity properties of the determinants of minors as

$$\langle abc(j_1)\cap(j_2)\cap(j_3)\rangle>0 \quad \text{for} \quad a<b<c,\ j_1<j_2<j_3. \tag{4.2.9}$$

The detail is discussed in [12]. From these properties and the positivity of the all the ordered minors $\langle ijklm\rangle>0$ for $i<j<k<l<m$, we can see that the super-local representation (4.2.8) is positive.

Chapter 5

Conclusion

We have investigated the triangulation of the 2-loop MHV amplituhedron and 1-loop NMHV amplituhedron. The crucial point is that the sign flip definition gives a new interpretation of the loop amplituhedron. From this definition, we can see that the higher loop MHV amplituhedron is decomposed into the one loop MHV amplituhedron and conditions of the positivity among condition, the N^{k}MHV loop amplituhedron is constructed as an intersection of the two lower-dimensional amplituhedra. By using this fact, we have obtained the triangulation of these amplituhedra.

First, we have obtained the canonical form of the n-point 2-loop MHV amplituhedron from this triangulation. We found that the representation of the 2-loop MHV integrand from this canonical form looks completely different from the BCFW representation. This is a new feature that starts from the 2-loop level. We have also obtained the n-point 2-loop MHV log integrand from the geometry that constructed from the two 1-loop MHV amplituhedron and the "negativity".

Next, we have obtained an explicit representation of the n-point 1-loop NMHV amplituhedron as a product of two lower-dimensional $m = 2$ amplituhedra. From this, we triangulated this 1-loop NMHV amplituhedron explicitly and obtained the canonical form. We also have obtained the new representation of the 1-loop NMHV amplituhedron: super-local representation, which means both external poles and internal poles are local. This super-local representation makes the positivity of this 1-loop NMHV amplituhedron manifest term-by-term. The positivity of the canonical form is related to the existence of a "dual amplituhedron". Then this will give clues to the existence of the dual amplituhedron for the 1-loop NMHV amplituhedron.

There are many open questions for future studies. The natural generalization is to go to the higher loop MHV amplituhedron. From the sign flip definition, the general L-loop MHV amplituhedron is decomposed into L 1-loop MHV amplituhedra and $\frac{1}{2}L(L-1)$ positivity conditions. We can apply the same method of the 2-loop case for this general L-loop, however, it is difficult to find the region of the parameters which satisfy all positivity conditions. Once we triangulate the higher loop MHV amplituhedron, we can obtain the canonical form and this form will give us a new structure of the integrand.

Generalization of the 6-2 representation to the higher k one-loop amplituhedron is also interesting. From the sign flip definition, the 1-loop N^kMHV amplituhedron is constructed from the $m = 2, k+2$ amplituhedron and $m = 2, k$ amplituhedron which intersecting with the N^kMHV tree amplituhedron. This means that even higher k case, the amplituhedron is constructed from the two $m = 2$ amplituhedra. Once we obtain this representation, we can obtain the canonical form of this 1-loop N^kMHV amplituhedron by using the sign flip triangulation of the $m = 2$ amplituhedron.

These generalizations lead us to consider the L-loop N^kMHV amplituhedron. From the sign flip definition, there are L $m = 2, k+2$ amplituhedron $(YA_1B_1), \cdots, (YA_LB_L)$ and $m = 2, k$ amplituhedron which intersecting with the N^kMHV tree amplituhedron. In addition to this, there is the further condition for the positivity $\langle YA_iB_iA_jB_j\rangle > 0$. We hope to revisit these problems in the future.

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Appendix A

Explicit Calculation of the 2-loop MHV Amplituhedron

A.1 5-point case

The case of $(2,3)\times(2,3)$,

$$\begin{cases} Z_A = Z_1 + x_1 Z_2 + w_1 Z_3 \\ Z_B = -Z_1 + y_1 Z_3 + z_1 Z_4 \end{cases} \quad \begin{cases} Z_C = Z_1 + x_2 Z_2 + w_2 Z_3 \\ Z_D = -Z_1 + y_2 Z_3 + z_2 Z_4 \end{cases}$$

Therefore it is same as 4-pt case. C-matrix is (3.1.7) and the form is (3.1.17).
Next the case of $(3,4)\times(3,4)$,

$$\begin{cases} Z_A = Z_1 + x_1 Z_3 + w_1 Z_4 \\ Z_B = -Z_1 + y_1 Z_4 + z_1 Z_5 \end{cases} \quad \begin{cases} Z_C = Z_1 + x_2 Z_3 + w_2 Z_4 \\ Z_D = -Z_1 + y_2 Z_4 + z_2 Z_5 \end{cases}$$

Then

$$\langle ABCD\rangle = \langle 1345\rangle\{(x_1-x_2)(y_1z_2-y_2z_1)+(z_1-z_2)(w_1x_2-w_2x_1)\} \tag{A.1.1}$$

It is almost same as the case of $(2,3)\times(2,3)$. The only difference is $\langle 1345\rangle$, thus there are 8 forms and the sum of these forms is

$$\begin{aligned}
\Omega_{3434} &= \frac{dx_1dx_2dw_1dw_2dy_1dy_2dz_1dz_2}{x_1x_2w_1w_2y_1y_2z_1z_2}\frac{\langle 1345\rangle}{\langle ABCD\rangle}(x_1y_1z_2+x_2y_2z_1+x_2w_1z_1+x_1w_2z_2)\\
&= \frac{\langle 1345\rangle^3\langle ABd^2A\rangle\langle ABd^2B\rangle\langle CDd^2C\rangle\langle CDd^2D\rangle}{\langle AB13\rangle\langle AB15\rangle\langle AB34\rangle\langle AB45\rangle\langle ABCD\rangle\langle CD13\rangle\langle CD15\rangle\langle CD34\rangle\langle CD45\rangle}\\
&\times\Big\{\langle AB45\rangle\langle CD13\rangle+\langle AB34\rangle\langle CD15\rangle+\langle AB15\rangle\langle CD34\rangle+\langle AB13\rangle\langle CD45\rangle\Big\}
\end{aligned} \tag{A.1.2}$$

The case of $(2,3)\times(2,4)$, two 1-loop amplituhedron are parametrized as

$$\begin{cases} Z_A = Z_1+x_1Z_2+w_1Z_3 \\ Z_B = -Z_1+y_1Z_3+z_1Z_4 \end{cases} \quad \begin{cases} Z_C = Z_1+x_2Z_2+w_2Z_3 \\ Z_D = -Z_1+y_2Z_4+z_2Z_5 \end{cases}$$

In view of the $Y=C\cdot Z$ description, the C-matrix is

$$C = \begin{pmatrix} 1 & x_1 & w_1 & 0 & 0 \\ -1 & 0 & y_1 & z_1 & 0 \\ 1 & x_2 & w_2 & 0 & 0 \\ -1 & 0 & 0 & y_2 & z_2 \end{pmatrix}. \tag{A.1.3}$$

The constraint is

$$\begin{aligned}
\langle ABCD\rangle &= (x_1-x_2)\{y_1y_2\langle 1234\rangle+y_1z_2\langle 1235\rangle+z_1z_2\langle 1245\rangle\}\\
&+ (x_1w_2-x_2w_1)\{(y_2-z_1)\langle 1234\rangle+z_2\langle 1235\rangle-z_1z_2\langle 2345\rangle\}\\
&+ (w_1-w_2)z_1z_2\langle 1345\rangle
\end{aligned} \tag{A.1.4}$$

From $\langle ABCD\rangle>0$,

$$\begin{aligned}
x_1 &> x_2-\frac{(x_1w_2-x_2w_1)\{(y_2-z_1)\langle 1234\rangle+z_2\langle 1235\rangle-z_1z_2\langle 2345\rangle\}+(w_1-w_2)z_1z_2\langle 1345\rangle}{y_1y_2\langle 1234\rangle+y_1z_2\langle 1235\rangle+z_1z_2\langle 1245\rangle}\\
&= x_2-a
\end{aligned} \tag{A.1.5}$$

The region of x_1 is depends on the sign of a. When $a<0$,

$$x_1>x_2-a,\ w_2<w_1-\frac{(x_1w_2-x_2w_1)\{(y_2-z_1)\langle 1234\rangle+z_2\langle 1235\rangle-z_1z_2\langle 2345\rangle\}}{z_1z_2\langle 1345\rangle}=w_1-b \tag{A.1.6}$$

Similarly, the region of w_1 is depends on the sign of b. When $b<0$, there are 2 cases that

$$\begin{cases} w_1 < w_2 - b, \quad \text{and} \quad x_2 < \frac{w_2}{w_1}x_1, \quad z_1 > \frac{y_2\langle 1234\rangle + z_2\langle 1235\rangle}{\langle 1234\rangle + z_2\langle 2345\rangle} = c \\ \qquad\qquad \text{or} \\ w_1 < w_2 - b, \quad \text{and} \quad x_2 > \frac{w_2}{w_1}x_1, \quad z_1 < c \end{cases} \tag{A.1.7}$$

There are 8 cases depending on the signs of a,b. The forms for these cases can be obtained similarly as 4-point case,

$$\begin{aligned}
\Omega_1 &= \frac{1}{x_1-x_2+a}\left(\frac{1}{x_2}-\frac{1}{x_2-\frac{w_2}{w_1}x_1}\right)\left(\frac{1}{w_1}-\frac{1}{w_1-w_2+b}\right)\frac{1}{w_2}\frac{1}{y_1}\frac{1}{y_2}\frac{1}{z_1-c}\frac{1}{z_2} \\
\Omega_2 &= \frac{1}{x_1-x_2+a}\frac{1}{x_2-\frac{w_2}{w_1}x_1}\left(\frac{1}{w_1}-\frac{1}{w_1-w_2+b}\right)\frac{1}{w_2}\frac{1}{y_1}\frac{1}{y_2}\left(\frac{1}{z_1}-\frac{1}{z_1-c}\right)\frac{1}{z_2} \\
\Omega_3 &= \frac{1}{x_1-x_2+a}\left(\frac{1}{x_2}-\frac{1}{x_2-\frac{w_2}{w_1}x_1}\right)\frac{1}{w_1}\frac{1}{w_2-w_1-b}\frac{1}{y_1}\frac{1}{y_2}\left(\frac{1}{z_1}-\frac{1}{z_1-c}\right)\frac{1}{z_2} \\
\Omega_4 &= \frac{1}{x_1-x_2+a}\frac{1}{x_2-\frac{w_2}{w_1}x_1}\frac{1}{w_1}\frac{1}{w_2-w_1-b}\frac{1}{y_1}\frac{1}{y_2}\frac{1}{z_1-c}\frac{1}{z_2} \\
\Omega_5 &= \left(\frac{1}{x_2}-\frac{1}{x_2-x_1-a}\right)\frac{1}{x_1-\frac{w_1}{w_2}x_2}\frac{1}{w_1-w_2+b}\frac{1}{w_2}\frac{1}{y_1}\frac{1}{y_2}\frac{1}{z_1-c}\frac{1}{z_2} \\
\Omega_6 &= \left(\frac{1}{x_2}-\frac{1}{x_2-x_1-a}\right)\left(\frac{1}{x_1}-\frac{1}{x_1-\frac{w_1}{w_2}x_2}\right)\frac{1}{w_1-w_2+b}\frac{1}{w_2}\frac{1}{y_1}\frac{1}{y_2}\left(\frac{1}{z_1}-\frac{1}{z_1-c}\right)\frac{1}{z_2} \\
\Omega_7 &= \left(\frac{1}{x_2}-\frac{1}{x_2-x_1-a}\right)\frac{1}{x_1-\frac{w_1}{w_2}x_2}\frac{1}{w_1}\left(\frac{1}{w_2}-\frac{1}{w_2-w_1-b}\right)\frac{1}{y_1}\frac{1}{y_2}\left(\frac{1}{z_1}-\frac{1}{z_1-c}\right)\frac{1}{z_2} \\
\Omega_8 &= \left(\frac{1}{x_2}-\frac{1}{x_2-x_1-a}\right)\left(\frac{1}{x_1}-\frac{1}{x_1-\frac{w_1}{w_2}x_2}\right)\frac{1}{w_1}\left(\frac{1}{w_2}-\frac{1}{w_2-w_1-b}\right)\frac{1}{y_1}\frac{1}{y_2}\frac{1}{z_1-c}\frac{1}{z_2}
\end{aligned} \tag{A.1.8}$$

For

$$\begin{aligned}
a &= \frac{(x_1w_2-x_2w_1)\{(y_2-z_1)\langle 1234\rangle + z_2\langle 1235\rangle - z_1z_2\langle 2345\rangle\} + (w_1-w_2)z_1z_2\langle 1345\rangle}{y_1y_2\langle 1234\rangle + y_1z_2\langle 1235\rangle + z_1z_2\langle 1245\rangle} \\
b &= \frac{(x_1w_2-x_2w_1)\{(y_2-z_1)\langle 1234\rangle + z_2\langle 1235\rangle - z_1z_2\langle 2345\rangle\}}{z_1z_2\langle 1345\rangle} \\
c &= \frac{y_2\langle 1234\rangle + z_2\langle 1235\rangle}{\langle 1234\rangle + z_2\langle 2345\rangle}
\end{aligned} \tag{A.1.9}$$

Then sum of these 8 forms is

$$\begin{aligned}
\Omega_{2324} &= \frac{dx_1dx_2dw_1dw_2dy_1dy_2dz_1dz_2}{x_1x_2w_1w_2y_1y_2z_1z_2}\frac{1}{\langle ABCD\rangle}\{\langle 1234\rangle(x_1w_2y_2+x_1y_1y_2+x_2w_1z_1) \\
&+ \langle 1235\rangle x_1z_2(w_2+y_1)+z_1z_2(\langle 1345\rangle w_1+\langle 1245\rangle x_1+\langle 2345\rangle x_2w_1)\}
\end{aligned} \tag{A.1.10}$$

Rewrite it into the momentum twistor,

$$\begin{aligned}
\Omega_{2324} &= \frac{\langle 123A_3\rangle\langle 123C_4\rangle\langle ABd^2A\rangle\langle ABd^2B\rangle\langle CDd^2C\rangle\langle CDd^2D\rangle}{\left\{\begin{array}{c}\langle AB12\rangle\langle AB13\rangle^2\langle AB14\rangle\langle AB23\rangle\langle AB34\rangle\\ \times\langle ABCD\rangle\langle CD12\rangle\langle CD13\rangle\langle CD14\rangle\langle CD15\rangle\langle CD23\rangle\langle CD45\rangle\end{array}\right\}}\\
&\times \Big\{\langle 123A_3\rangle(\langle AB12\rangle\langle CD13\rangle\langle CD45\rangle + \langle AB15\rangle\langle CD14\rangle\langle CD23\rangle)\\
&\quad \langle 123C_4\rangle(\langle AB13\rangle\langle AB34\rangle\langle CD12\rangle + \langle AB13\rangle\langle AB14\rangle\langle CD23\rangle)\\
&\quad -\langle 1235\rangle\langle AB13\rangle\langle AB14\rangle\langle CD14\rangle\langle CD23\rangle - \langle 2345\rangle\langle AB12\rangle\langle AB13\rangle\langle CD12\rangle\langle CD14\rangle\Big\}.
\end{aligned} \tag{A.1.11}$$

We use these symbols

$$\begin{aligned}
(AB)\cap(1ii+1) &= -Z_1\langle ii+1AB\rangle - Z_i\langle i+1AB1\rangle - Z_{i+1}\langle AB1i\rangle \equiv A_i\\
(CD)\cap(1ii+1) &= -Z_1\langle ii+1CD\rangle - Z_i\langle i+1CD1\rangle - Z_{i+1}\langle CD1i\rangle \equiv C_i
\end{aligned} \tag{A.1.12}$$

The case of $(2,4)\times(3,4)$,

$$\begin{cases} Z_A = Z_1 + x_1 Z_2 + w_1 Z_3\\ Z_B = -Z_1 + y_1 Z_4 + z_1 Z_5\end{cases} \qquad \begin{cases} Z_C = Z_1 + x_2 Z_3 + w_2 Z_4\\ Z_D = -Z_1 + y_2 Z_4 + z_2 Z_5\end{cases}$$

C-matrix is

$$C = \begin{pmatrix} 1 & x_1 & w_1 & 0 & 0\\ -1 & 0 & 0 & y_1 & z_1\\ 1 & 0 & x_2 & w_2 & 0\\ -1 & 0 & 0 & y_2 & z_2\end{pmatrix} \tag{A.1.13}$$

$$\begin{aligned}
\langle ABCD\rangle &= (z_2 - z_1)(\langle 1345\rangle w_1 w_2 + \langle 1235\rangle x_1 x_2 + \langle 1245\rangle x_1 w_2)\\
&+ (z_1 y_2 - z_2 y_1)\{\langle 1345\rangle(x_2 - w_1) - \langle 1245\rangle x_1 + \langle 2345\rangle x_1 x_2\}\\
&+ (y_2 - y_1)\langle 1234\rangle x_1 x_2
\end{aligned} \tag{A.1.14}$$

From$\langle ABCD\rangle > 0$,

$$\begin{aligned}
z_2 &> z_1 - \frac{(z_1 y_2 - z_2 y_1)\{\langle 1345\rangle(x_2 - w_1) - \langle 1245\rangle x_1 + \langle 2345\rangle x_1 x_2\} + (y_2 - y_1)\langle 1234\rangle x_1 x_2}{x_1 x_2\langle 1234\rangle}\\
&= z_1 - a
\end{aligned} \tag{A.1.15}$$

The region of z_2 is depends on the sign of a. When $a < 0$,

$$z_2 > z_1 - a, \text{ and } y_2 < y_1 - \frac{(z_1y_2 - z_2y_1)\{\langle 1345\rangle(x_2 - w_1) - \langle 1245\rangle x_1 + \langle 2345\rangle x_1x_2\}}{x_1x_2\langle 1234\rangle} = y_1 - b \tag{A.1.16}$$

Similarly, the region of y_2 is depends on the sign of b. When $b < 0$, there are 2 cases that

$$y_2 < y_1 - b, \quad \text{and} \quad \begin{cases} z_1 > \frac{y_1}{y_2}z_2, \quad x_2 < \frac{w_1\langle 1345\rangle + x_1\langle 1245\rangle}{x_1\langle 2345\rangle + \langle 1345\rangle} = c \\ \qquad \text{or} \\ z_1 < \frac{y_1}{y_2}z_2, \quad x_2 > c \end{cases} \tag{A.1.17}$$

There are 8 cases depending on the signs of a,b. Then the forms for these cases are

$$\begin{aligned}
\Omega_1 &= \frac{1}{x_1}\left(\frac{1}{x_2} - \frac{1}{x_2 - c}\right)\frac{1}{w_1}\frac{1}{w_2}\frac{1}{y_1}\left(\frac{1}{y_2} - \frac{1}{y_2 - y_1 + b}\right)\frac{1}{z_1 - \frac{y_1}{y_2}z_2}\frac{1}{z_2 - z_1 + a} \\
\Omega_2 &= \frac{1}{x_1}\frac{1}{x_2 - c}\frac{1}{w_1}\frac{1}{w_2}\frac{1}{y_1}\left(\frac{1}{y_2} - \frac{1}{y_2 - y_1 + b}\right)\left(\frac{1}{z_1} - \frac{1}{z_1 - \frac{y_1}{y_2}z_2}\right)\frac{1}{z_2 - z_1 + a} \\
\Omega_3 &= \frac{1}{x_1}\frac{1}{x_2 - c}\frac{1}{w_1}\frac{1}{w_2}\frac{1}{y_1 - y_2 - b}\frac{1}{y_2}\frac{1}{z_1 - \frac{y_1}{y_2}z_2}\frac{1}{z_2 - z_1 + a} \\
\Omega_4 &= \frac{1}{x_1}\left(\frac{1}{x_2} - \frac{1}{x_2 - c}\right)\frac{1}{w_1}\frac{1}{w_2}\frac{1}{y_1}\frac{1}{y_2 - y_1 + b}\left(\frac{1}{z_1} - \frac{1}{z_1 - \frac{y_1}{y_2}z_2}\right)\frac{1}{z_2 - z_1 + a} \\
\Omega_5 &= \frac{1}{x_1}\left(\frac{1}{x_2} - \frac{1}{x_2 - c}\right)\frac{1}{w_1}\frac{1}{w_2}\frac{1}{y_1}\frac{1}{y_2 - y_1 + b}\left(\frac{1}{z_1} - \frac{1}{z_1 - \frac{y_1}{y_2}z_2}\right)\left(\frac{1}{z_2} - \frac{1}{z_2 - z_1 + a}\right) \\
\Omega_6 &= \frac{1}{x_1}\frac{1}{x_2 - c}\frac{1}{w_1}\frac{1}{w_2}\frac{1}{y_1}\frac{1}{y_2 - y_1 + b}\left(\frac{1}{z_1} - \frac{1}{z_1 - \frac{y_1}{y_2}z_2}\right)\frac{1}{z_2 - z_1 + a} \\
\Omega_7 &= \frac{1}{x_1}\frac{1}{x_2 - c}\frac{1}{w_1}\frac{1}{w_2}\left(\frac{1}{y_1} - \frac{1}{y_1 - y_2 - b}\right)\frac{1}{y_2}\left(\frac{1}{z_1} - \frac{1}{z_1 - \frac{y_1}{y_2}z_2}\right)\left(\frac{1}{z_2} - \frac{1}{z_2 - z_1 + a}\right) \\
\Omega_8 &= \frac{1}{x_1}\left(\frac{1}{x_2} - \frac{1}{x_2 - c}\right)\frac{1}{w_1}\frac{1}{w_2}\left(\frac{1}{y_1} - \frac{1}{y_1 - y_2 - b}\right)\frac{1}{y_2}\left(\frac{1}{z_1} - \frac{1}{z_1 - \frac{y_1}{y_2}z_2}\right)\frac{1}{z_2 - z_1 + a}
\end{aligned} \tag{A.1.18}$$

For

$$\begin{aligned}
a &= \frac{(z_1y_2 - z_2y_1)\{\langle 1345\rangle(x_2 - w_1) - \langle 1245\rangle x_1 + \langle 2345\rangle x_1x_2\} + (y_2 - y_1)\langle 1234\rangle x_1x_2}{x_1x_2\langle 1234\rangle} \\
b &= \frac{(z_1y_2 - z_2y_1)\{\langle 1345\rangle(x_2 - w_1) - \langle 1245\rangle x_1 + \langle 2345\rangle x_1x_2\}}{x_1x_2\langle 1234\rangle} \\
c &= \frac{w_1\langle 1345\rangle + x_1\langle 1245\rangle}{x_1\langle 2345\rangle + \langle 1345\rangle}
\end{aligned} \tag{A.1.19}$$

Then sum of these 8 forms is

$$\begin{aligned}\Omega_{2434} &= \frac{dx_1dx_2dw_1dw_2dy_1dy_2dz_1dz_2}{x_1x_2w_1w_2y_1y_2z_1z_2}\frac{1}{\langle ABCD\rangle}\{\langle 1345\rangle(w_1w_2z_2+w_1y_1z_2+x_2y_2z_1) \\ &+ \langle 1235\rangle x_1x_2z_2+\langle 1245\rangle(x_1w_2z_2+x_1y_1z_2)+\langle 1234\rangle x_1x_2y_2+\langle 2345\rangle x_1x_2y_2z_1\}\end{aligned} \tag{A.1.20}$$

In the momentum twistor,

$$\begin{aligned}\Omega_{2434} &= \frac{\langle 123A_4\rangle\langle 134C_4\rangle\langle ABd^2A\rangle\langle ABd^2B\rangle\langle CDd^2C\rangle\langle CDd^2D\rangle}{\left\{\begin{array}{c}\langle AB12\rangle\langle AB13\rangle\langle AB14\rangle\langle AB15\rangle\langle AB23\rangle\langle AB45\rangle \\ \times\langle ABCD\rangle\langle CD13\rangle\langle CD14\rangle^2\langle CD15\rangle\langle CD34\rangle\langle CD45\rangle\end{array}\right\}} \\ &\times \Big\{\langle 123A_4\rangle(\langle AB45\rangle\langle CD13\rangle\langle CD14\rangle+\langle AB15\rangle\langle CD34\rangle\langle CD14\rangle) \\ &\quad -\langle 345A_2\rangle\langle AB14\rangle\langle CD14\rangle\langle CD15\rangle+\langle 123C_4\rangle\langle CD14\rangle\langle AB45\rangle\langle AB13\rangle\Big\}\end{aligned} \tag{A.1.21}$$

The case of $(2,3)\times(3,4)$,

$$\begin{cases} Z_A = Z_1+x_1Z_2+w_1Z_3 \\ Z_B = -Z_1+y_1Z_3+z_1Z_4\end{cases} \qquad \begin{cases} Z_C = Z_1+x_2Z_3+w_2Z_4 \\ Z_D = -Z_1+y_2Z_4+z_2Z_5\end{cases}$$

C-matrix is

$$C = \begin{pmatrix} 1 & x_1 & w_1 & 0 & 0 \\ -1 & 0 & y_1 & z_1 & 0 \\ 1 & 0 & x_2 & w_2 & 0 \\ -1 & 0 & 0 & y_2 & z_2 \end{pmatrix} \tag{A.1.22}$$

$$\begin{aligned}\langle ABCD\rangle &= (y_1w_2-z_1x_2)(\langle 1345\rangle z_2+\langle 1234\rangle x_1+\langle 2345\rangle x_1z_2) \\ &+ z_2(z_1+w_2)(\langle 1345\rangle w_1+\langle 1245\rangle x_1)+x_1(y_1+x_2)(\langle 1235\rangle z_2+\langle 1234\rangle y_2)\end{aligned} \tag{A.1.23}$$

In this case, from $\langle ABCD\rangle>0$,

$$y_1w_2-z_1x_2 > -\frac{z_2(z_1+w_2)(\langle 1345\rangle w_1+\langle 1245\rangle x_1)+x_1(y_1+x_2)(\langle 1235\rangle z_2+\langle 1234\rangle y_2)}{\langle 1345\rangle z_2+\langle 1234\rangle x_1+\langle 2345\rangle x_1z_2} = -a \tag{A.1.24}$$

However, from $x_1, x_2, w_1, w_2, y_1, y_2, z_1, z_2 > 0$, $a > 0$. Therefore

$$z_1 < \frac{w_2 y_1}{x_2} + \frac{a}{x_2} \tag{A.1.25}$$

Then the form is

$$\begin{aligned}
\Omega_{2334} &= \frac{dx_1 dx_2 dw_1 dw_2 dy_1 dy_2 dz_1 dz_2}{x_1 x_2 w_1 w_2 y_1 y_2 z_2} \left(\frac{1}{z_1} - \frac{1}{z_1 - \frac{w_2 y_1}{x_2} - \frac{a}{x_2}} \right) \\
&= \frac{dx_1 dx_2 dw_1 dw_2 dy_1 dy_2 dz_1 dz_2}{x_1 x_2 w_1 w_2 y_1 y_2 z_1 z_2} \frac{1}{\langle ABCD \rangle} \{ w_2 y_1 (\langle 1345 \rangle z_2 + \langle 1234 \rangle x_1 + \langle 2345 \rangle x_1 z_2) \\
&+ z_2 (z_1 + w_2)(\langle 1345 \rangle w_1 + \langle 1245 \rangle x_1) + x_1 (y_1 + x_2)(\langle 1235 \rangle z_2 + \langle 1234 \rangle y_2) \}
\end{aligned} \tag{A.1.26}$$

We can write it in the momentum twistor space,

$$\begin{aligned}
\Omega_{2334} = &\frac{\langle 123A_3 \rangle \langle 134C_4 \rangle \langle ABd^2A \rangle \langle ABd^2B \rangle \langle CDd^2C \rangle \langle CDd^2D \rangle}{\left\{ \begin{array}{c} \langle AB12 \rangle \langle AB13 \rangle^2 \langle AB14 \rangle \langle AB23 \rangle \langle AB34 \rangle \\ \times \langle ABCD \rangle \langle CD13 \rangle \langle CD14 \rangle^2 \langle CD15 \rangle \langle CD34 \rangle \langle CD45 \rangle \end{array} \right\}} \\
&\times \Bigg\{ \langle AB13 \rangle \langle 123C_4 \rangle \langle CD4A_3 \rangle - \langle AB13 \rangle \langle AB14 \rangle \langle CD13 \rangle \langle 234C_4 \rangle \\
&+ \langle CD14 \rangle \langle 145A_2 \rangle \langle CD3A_3 \rangle - \langle AB14 \rangle \langle AB23 \rangle \langle CD13 \rangle \langle CD14 \rangle \langle 1345 \rangle \Bigg\}
\end{aligned} \tag{A.1.27}$$

The case of $(2,4) \times (2,4)$,

$$\begin{cases} Z_A = Z_1 + x_1 Z_2 + w_1 Z_3 \\ Z_B = -Z_1 + y_1 Z_4 + z_1 Z_5 \end{cases} \quad \begin{cases} Z_C = Z_1 + x_2 Z_2 + w_2 Z_3 \\ Z_D = -Z_1 + y_2 Z_4 + z_2 Z_5 \end{cases}$$

C-matrix is

$$C = \begin{pmatrix} 1 & x_1 & w_1 & 0 & 0 \\ -1 & 0 & 0 & y_1 & z_1 \\ 1 & x_2 & w_2 & 0 & 0 \\ -1 & 0 & 0 & y_2 & z_2 \end{pmatrix} \tag{A.1.28}$$

$$\begin{aligned}
\langle ABCD \rangle &= (y_2 - y_1)(x_1 w_2 - x_2 w_1) \langle 1234 \rangle + (z_2 - z_1)(x_1 w_2 - x_2 w_1) \langle 1235 \rangle \\
&+ (z_1 y_2 - z_2 y_1) \{ \langle 1245 \rangle (x_2 - x_1) + \langle 1345 \rangle (w_2 - w_1) + \langle 2345 \rangle (x_1 w_2 - x_2 w_1) \}
\end{aligned} \tag{A.1.29}$$

When $(x_1w_2 - x_2w_1) > 0,\ (z_1y_2 - z_2y_1) > 0$, from $\langle ABCD\rangle$,

$$y_2 > y_1 - a \tag{A.1.30}$$

For

$$\begin{aligned} a = & \frac{(z_2 - z_1)(x_1w_2 - x_2w_1)\langle 1235\rangle}{(x_1w_2 - x_2w_1)\langle 1234\rangle} \\ + & \frac{(z_1y_2 - z_2y_1)\{\langle 1245\rangle(x_2 - x_1) + \langle 1345\rangle(w_2 - w_1) + \langle 2345\rangle(x_1w_2 - x_2w_1)}{(x_1w_2 - x_2w_1)\langle 1234\rangle} \end{aligned} \tag{A.1.31}$$

The region of y_2 is depends on the sign of a. When $a < 0$,

$$\begin{aligned} y_2 > y_1 - a\ ,\ z_2 &< z_1 - \frac{(z_1y_2 - z_2y_1)\{\langle 1245\rangle(x_2 - x_1) + \langle 1345\rangle(w_2 - w_1) + \langle 2345\rangle(x_1w_2 - x_2w_1)\}}{(x_1w_2 - x_2w_1)\langle 1235\rangle} \\ &= z_1 - b \end{aligned} \tag{A.1.32}$$

Similarly, the region of z_2 is depends on the sign of b. When $b < 0$,

$$z_2 < z_1 - b \text{ and } \ x_2 < x_1 - \frac{\langle 1345\rangle(w_2 - w_1) + \langle 2345\rangle(x_1w_2 - x_2w_1)}{\langle 1245\rangle} = x_1 - c \tag{A.1.33}$$

When $c < 0$,

$$x_2 < x_1 - c \text{ and } \ w_2 < w_1 - \frac{\langle 2345\rangle(x_1w_2 - x_2w_1)}{\langle 1345\rangle} = w_1 - d \tag{A.1.34}$$

From $w_2 > 0, (x_1w_2 - x_2w_1) > 0$, then $d > 0$ and there are 8 cases depending on the signs of a,b,c.

$$\begin{aligned} \Omega_1 &= \frac{1}{x_1 - \frac{w_1}{w_2}x_2}\left(\frac{1}{x_2} - \frac{1}{x_2 - x_1 + c}\right)\frac{1}{w_1 - w_2 - d}\frac{1}{w_2}\frac{1}{y_1}\frac{1}{y_2 - y_1 + a}\frac{1}{z_1 - \frac{y_1}{y_2}z_2}\left(\frac{1}{z_2} - \frac{1}{z_2 - z_1 + c}\right) \\ \Omega_2 &= \frac{1}{x_1 - x_2 - c}\left(\frac{1}{x_2} - \frac{1}{x_2 - \frac{w_2}{w_1}x_1}\right)\left(\frac{1}{w_1} - \frac{1}{w_1 - w_2 - d}\right)\frac{1}{w_2}\frac{1}{y_1}\frac{1}{y_2 - y_1 + a} \\ &\times \frac{1}{z_1 - \frac{y_1}{y_2}z_2}\left(\frac{1}{z_2} - \frac{1}{z_2 - z_1 + b}\right) \\ \Omega_3 &= \frac{1}{x_1 - \frac{w_1}{w_2}x_2}\frac{1}{x_2 - x_1 + c}\frac{1}{w_1 - w_2 - d}\frac{1}{w_2}\frac{1}{y_1}\frac{1}{y_2 - y_1 + a}\frac{1}{z_1 - z_2 - b}\left(\frac{1}{z_2} - \frac{1}{z_2 - \frac{y_2}{y_1}z_1}\right) \\ \Omega_4 &= \left(\frac{1}{x_1} - \frac{1}{x_1 - x_2 - c}\right)\left(\frac{1}{x_2} - \frac{1}{x_2 - \frac{w_2}{w_1}x_1}\right)\left(\frac{1}{w_1} - \frac{1}{w_1 - w_2 - d}\right)\frac{1}{w_2}\frac{1}{y_1}\frac{1}{y_2 - y_1 + a} \end{aligned}$$

$$
\begin{aligned}
&\times \quad \frac{1}{z_1 - z_2 - b}\left(\frac{1}{z_2} - \frac{1}{z_2 - \frac{y_2}{y_1} z_1}\right) \\
\Omega_5 \quad &= \quad \frac{1}{x_1 - \frac{w_1}{w_2} x_2}\left(\frac{1}{x_2} - \frac{1}{x_2 - x_1 + c}\right)\frac{1}{w_1 - w_2 - d}\frac{1}{w_2}\left(\frac{1}{y_1} - \frac{1}{y_1 - y_2 - a}\right)\frac{1}{y_2}\frac{1}{z_1 - \frac{y_1}{y_2} z_2}\frac{1}{z_2 - z_1 + b} \\
\Omega_6 \quad &= \quad \frac{1}{x_1 - x_2 - c}\left(\frac{1}{x_2} - \frac{1}{x_2 - \frac{w_2}{w_1} x_1}\right)\left(\frac{1}{w_1} - \frac{1}{w_1 - w_2 - d}\right)\frac{1}{w_2}\left(\frac{1}{y_1} - \frac{1}{y_1 - y_2 - a}\right)\frac{1}{y_2} \\
&\times \quad \frac{1}{z_1 - \frac{y_1}{y_2} z_2}\frac{1}{z_2 - z_1 + b} \\
\Omega_7 \quad &= \quad \frac{1}{x_1 - \frac{w_1}{w_2} x_2}\frac{1}{x_2 - x_1 + c}\frac{1}{w_1 - w_2 - d}\frac{1}{w_2}\left(\frac{1}{y_1} - \frac{1}{y_1 - y_2 - a}\right)\frac{1}{y_2} \\
&\times \quad \left(\frac{1}{z_1} - \frac{1}{z_1 - z_2 - b}\right)\left(\frac{1}{z_2} - \frac{1}{z_2 - \frac{y_2}{y_1} z_1}\right) \\
\Omega_8 \quad &= \quad \left(\frac{1}{x_1} - \frac{1}{x_1 - x_2 - c}\right)\left(\frac{1}{x_2} - \frac{1}{x_2 - \frac{w_2}{w_1} x_1}\right)\left(\frac{1}{w_1} - \frac{1}{w_1 - w_2 - d}\right)\frac{1}{w_2}\left(\frac{1}{y_1} - \frac{1}{y_1 - y_2 - a}\right)\frac{1}{y_2} \\
&\times \quad \left(\frac{1}{z_1} - \frac{1}{z_1 - z_2 - b}\right)\left(\frac{1}{z_2} - \frac{1}{z_2 - \frac{y_2}{y_1} z_1}\right) \qquad \text{(A.1.35)}
\end{aligned}
$$

For

$$
\begin{aligned}
a \quad &= \quad \frac{(z_2 - z_1)(x_1 w_2 - x_2 w_1)\langle 1235\rangle}{(x_1 w_2 - x_2 w_1)\langle 1234\rangle} \\
&+ \quad \frac{(z_1 y_2 - z_2 y_1)\{\langle 1245\rangle(x_2 - x_1) + \langle 1345\rangle(w_2 - w_1) + \langle 2345\rangle(x_1 w_2 - x_2 w_1)}{(x_1 w_2 - x_2 w_1)\langle 1234\rangle} \\
b \quad &= \quad \frac{(z_1 y_2 - z_2 y_1)\{\langle 1245\rangle(x_2 - x_1) + \langle 1345\rangle(w_2 - w_1) + \langle 2345\rangle(x_1 w_2 - x_2 w_1)\}}{(x_1 w_2 - x_2 w_1)\langle 1235\rangle} \\
c \quad &= \quad \frac{\langle 1345\rangle(w_2 - w_1) + \langle 2345\rangle(x_1 w_2 - x_2 w_1)}{\langle 1245\rangle} \\
d \quad &= \quad \frac{\langle 2345\rangle(x_1 w_2 - x_2 w_1)}{\langle 1345\rangle} \qquad \text{(A.1.36)}
\end{aligned}
$$

This is the case of $(x_1 w_2 - x_2 w_1) > 0,\ (z_1 y_2 - z_2 y_1) > 0$. Next we consider the case of $(x_1 w_2 - x_2 w_1) > 0,\ (z_1 y_2 - z_2 y_1) < 0$. Forms are obtained by replacement as follows.

$$
\begin{aligned}
\left(\frac{1}{x_1} - \frac{1}{x_1 - x_2 - c}\right) \quad &\leftrightarrow \quad \frac{1}{x_1 - x_2 - c} \\
\left(\frac{1}{x_2} - \frac{1}{x_2 - x_1 + c}\right) \quad &\leftrightarrow \quad \frac{1}{x_2 - x_1 + c} \\
\frac{1}{z_1 - \frac{y_1}{y_2} z_2} \quad &\rightarrow \quad \left(\frac{1}{z_1} - \frac{1}{z_1 - \frac{y_1}{y_2} z_2}\right)
\end{aligned}
$$

$$\left(\frac{1}{z_2}-\frac{1}{z_2-\frac{y_2}{y_1}z_1}\right) \quad\to\quad \frac{1}{z_2-\frac{y_2}{y_1}z_1} \tag{A.1.37}$$

Then the cases of $(x_1w_2-x_2w_1)<0,\ (z_1y_2-z_2y_1)<0$ and $(x_1w_2-x_2w_1)<0,\ (z_1y_2-z_2y_1)>0$ are obtained that swap $1\leftrightarrow 2$ for the case of $(x_1w_2-x_2w_1)>0,\ (z_1y_2-z_2y_1)>0$ and $(x_1w_2-x_2w_1)>0,\ (z_1y_2-z_2y_1)<0$. Then sum of these 32 forms is

$$\begin{aligned}\Omega_{2424}=&\frac{dx_1dx_2\cdots dz_1dz_2}{x_1x_2w_1w_2y_1y_2z_1z_2}\frac{1}{\langle ABCD\rangle}\\&\times\{\langle 1234\rangle(x_2w_1y_1+x_1w_2y_2)+y_2z_1(\langle 1345\rangle w_2+\langle 1245\rangle x_2+\langle 2345\rangle x_1w_2)\\&+y_1z_2(\langle 1345\rangle w_1+\langle 1245\rangle x_1+\langle 2345\rangle x_2w_1)+\langle 1235\rangle(x_2w_1z_1+x_1w_2z_2)\}.\end{aligned} \tag{A.1.38}$$

In the momentum twistor space,

$$\begin{aligned}\Omega_{2424}\;=\;&\frac{\langle 123A_4\rangle\langle 123C_4\rangle\langle ABd^2A\rangle\langle ABd^2B\rangle\langle CDd^2C\rangle\langle CDd^2D\rangle}{\left\{\begin{matrix}\langle AB12\rangle\langle AB13\rangle\langle AB14\rangle\langle AB15\rangle\langle AB23\rangle\langle AB45\rangle\langle ABCD\rangle\\ \times\langle CD12\rangle\langle CD13\rangle\langle CD14\rangle\langle CD15\rangle\langle CD23\rangle\langle CD45\rangle\end{matrix}\right\}}\\ \times\;&\Big\{\langle 123A_4\rangle(\langle AB12\rangle\langle CD13\rangle\langle CD45\rangle+\langle AB15\rangle\langle CD14\rangle\langle CD23\rangle)\\&+\langle 123C_4\rangle(\langle AB13\rangle\langle AB45\rangle\langle CD12\rangle+\langle AB14\rangle\langle AB23\rangle\langle CD15\rangle)\\&+\langle 2345\rangle(\langle AB12\rangle\langle AB15\rangle\langle CD13\rangle\langle CD14\rangle+\langle AB13\rangle\langle AB14\rangle\langle CD12\rangle\langle CD15\rangle)\Big\}\end{aligned} \tag{A.1.39}$$

The remaining patterns are $(3,4)\times(2,3),(2,4)\times(2,3),(3,4)\times(2,4)$. These forms can be obtained from $\Omega_{2334},\Omega_{2324},\Omega_{2434}$ that swap $AB\leftrightarrow CD$.

A.2 n-point case

First we consider the (1) case $i<k<l<j$,

$$\begin{aligned}&\langle ABCD\rangle=\\&x_1x_2y_2\langle 1ikl\rangle+x_1x_2z_2\langle 1ikl+1\rangle+w_2x_1y_2\langle 1ik+1l\rangle+w_2x_1z_2\langle 1ik+1l+1\rangle-x_1(x_2y_1\langle 1ikj\rangle\\&+x_2z_1\langle 1ikj+1\rangle+w_2y_1\langle 1ik+1j\rangle+w_2z_1\langle 1ik+1j+1\rangle+y_1y_2\langle 1ilj\rangle+y_2z_1\langle 1ilj+1\rangle\\&+y_1z_2\langle 1il+1j\rangle+z_1z_2\langle 1il+1j+1\rangle)+w_1x_2y_2\langle 1i+1kl\rangle+w_1x_2z_2\langle 1i+1kl+1\rangle\\&-w_2(w_1y_1\langle 1i+1k+1j\rangle+w_1z_1\langle 1i+1k+1j+1\rangle)+w_1w_2y_2\langle 1i+1k+1l\rangle\\&+w_1w_2z_2\langle 1i+1k+1l+1\rangle-w_1(x_2y_1\langle 1i+1kj\rangle+x_2z_1\langle 1i+1kj+1\rangle+y_1y_2\langle 1i+1lj\rangle\\&+y_2z_1\langle 1i+1lj+1\rangle+y_1z_2\langle 1i+1l+1j\rangle+z_1z_2\langle 1i+1l+1j+1\rangle)+x_2y_1y_2\langle 1klj\rangle\\&+x_2y_2z_1\langle 1klj+1\rangle+x_2y_1z_2\langle 1kl+1j\rangle+x_2z_1z_2\langle 1kl+1j+1\rangle+w_2y_1y_2\langle 1k+1lj\rangle\end{aligned}$$

$$+w_2y_2z_1\langle 1k+1lj+1\rangle + w_2y_1z_2\langle 1k+1l+1j\rangle + w_2z_1z_2\langle 1k+1l+1j+1\rangle + x_1x_2y_1y_2\langle iklj\rangle$$
$$+x_1x_2y_2z_1\langle iklj+1\rangle + x_1x_2y_1z_2\langle ikl+1j\rangle + x_1x_2z_1z_2\langle ikl+1j+1\rangle + w_2x_1y_1y_2\langle ik+1lj\rangle$$
$$+w_2x_1y_2z_1\langle ik+1lj+1\rangle + w_2x_1y_1z_2\langle ik+1l+1j\rangle + w_2x_1z_1z_2\langle ik+1l+1j+1\rangle$$
$$+w_1x_2y_1y_2\langle i+1klj\rangle + w_1x_2y_2z_1\langle i+1klj+1\rangle + w_1x_2y_1z_2\langle i+1kl+1j\rangle$$
$$+w_1x_2z_1z_2\langle i+1kl+1j+1\rangle + w_1w_2y_1y_2\langle i+1k+1lj\rangle + w_1w_2y_2z_1\langle i+1k+1lj+1\rangle$$
$$+w_1w_2y_1z_2\langle i+1k+1l+1j\rangle + w_1w_2z_1z_2\langle i+1k+1l+1j+1\rangle$$
$$= az_2 - bw_1 - cx_1 - dw_2 + ey_2$$

for

$$\begin{aligned}
a &= x_1x_2\langle 1ikl+1\rangle + w_2x_1\langle 1ik+1l+1\rangle + w_1x_2\langle 1i+1kl+1\rangle + w_1w_2\langle 1i+1k+1l+1\rangle \\
&\quad +x_2y_1\langle 1kl+1j\rangle + x_2z_1\langle 1kl+1j+1\rangle + w_2y_1\langle 1k+1l+1j\rangle + w_2z_1\langle 1k+1l+1j+1\rangle \\
&\quad +x_1x_2y_1\langle ikl+1j\rangle + x_1x_2z_1\langle ikl+1j+1\rangle + w_2x_1y_1\langle ik+1l+1j\rangle \\
&\quad +w_2x_1z_1\langle ik+1l+1j+1\rangle + w_1x_2y_1\langle i+1kl+1j\rangle + w_1x_2z_1\langle i+1kl+1j+1\rangle \\
&\quad +w_1w_2y_1\langle i+1k+1l+1j\rangle + w_1w_2z_1\langle i+1k+1l+1j+1\rangle \\
b &= x_2y_1\langle 1i+1kj\rangle + x_2z_1\langle 1i+1kj+1\rangle + y_1y_2\langle 1i+1lj\rangle + y_2z_1\langle 1i+1lj+1\rangle \\
&\quad +y_1z_2\langle 1i+1l+1j\rangle + z_1z_2\langle 1i+1l+1j+1\rangle \\
c &= x_2y_1\langle 1ikj\rangle + x_2z_1\langle 1ikj+1\rangle + w_2y_1\langle 1ik+1j\rangle + w_2z_1\langle 1ik+1j+1\rangle + y_1y_2\langle 1ilj\rangle \\
&\quad +y_2z_1\langle 1il+1j\rangle + y_1z_2\langle 1il+1j\rangle + z_1z_2\langle 1il+1j+1\rangle \\
d &= w_1y_1\langle 1i+1k+1j\rangle + w_1z_1\langle 1i+1k+1j+1\rangle \\
e &= x_1x_2\langle 1ikl\rangle + w_2x_1\langle 1ik+1l\rangle + w_1x_2\langle 1i+1kl\rangle + w_1w_2\langle 1i+1k+1l\rangle + x_2y_1\langle 1klj\rangle \\
&\quad +x_2z_1\langle 1klj+1\rangle + w_2y_1\langle 1k+1lj\rangle + w_2z_1\langle 1k+1lj+1\rangle + x_1x_2y_1\langle iklj\rangle + x_1x_2z_1\langle iklj+1\rangle \\
&\quad +w_2x_1y_1\langle ik+1lj\rangle + w_2x_1z_1\langle ik+1lj+1\rangle + w_1x_2y_1\langle i+1klj\rangle + w_1x_2z_1\langle i+1klj+1\rangle \\
&\quad +w_1w_2y_1\langle i+1k+1lj\rangle + w_1w_2z_1\langle i+1k+1lj+1\rangle
\end{aligned}$$

and $a, b, c, d, e > 0$. From $\langle ABCD\rangle > 0$,

$$z_2 > \frac{b}{a}w_1 + \frac{cx_1 + dw_2 - ey_2}{a}$$

In the case of $cx_1 + dw_2 - ey_2 > 0$,

$$z_2 > \frac{b}{a}w_1 + \frac{cx_1 + dw_2 - ey_2}{a}, \quad \text{and} \quad y_2 < \frac{cx_1 + dw_2}{e}$$

Then the form of this sign pattern is

$$\Omega_1 = \frac{1}{x_1}\frac{1}{x_2}\frac{1}{w_1}\frac{1}{w_2}\frac{1}{y_1}\left(\frac{1}{y_2} - \frac{1}{y_2 - \frac{cx_1+dw_2}{e}}\right)\frac{1}{z_1}\frac{1}{z_2 - (\frac{b}{a}w_1 + \frac{cx_1+dw_2-ey_2}{a})}$$

Another pattern is that $cx_1 + dw_2 - ey_2 < 0$,

$$w_1 < \frac{a}{b}z_2 - \frac{cx_1 + dw_2 - ey_2}{b}, \quad \text{and} \quad y_2 > \frac{cx_1 + dw_2}{e}$$

Then the form is

$$\Omega_2 = \frac{1}{x_1}\frac{1}{x_2}\left(\frac{1}{w_1} - \frac{1}{w_1 - (\frac{a}{b}z_2 - \frac{cx_1+dw_2-ey_2}{b})}\right)\frac{1}{w_2}\frac{1}{y_1}\frac{1}{y_2 - \frac{cx_1+dw_2}{e}}\frac{1}{z_1}\frac{1}{z_2}$$

The canonical form for this sign flip pattern is

$$(\Omega_1 + \Omega_2)dx_1dx_2\cdots dz_1dz_2 = \frac{dx_1dx_2\cdots dz_1dz_2}{x_1x_2w_1w_2y_1y_2z_1z_2}\frac{1}{(az_2 - bw_1 - cx_1 - dw_2 + ey_2)} \times \omega^1_{ijkl} \tag{A.2.1}$$

$$\begin{aligned}
\omega^1_{ijkl} &= \langle 1ikl\rangle x_1x_2y_2 + \langle 1ikl+1\rangle x_1x_2z_2 + \langle 1ik+1l\rangle w_2x_1y_2 + \langle 1ik+1l+1\rangle w_2x_1z_2 \\
&+ \langle 1i+1kl\rangle w_1x_2y_2 + \langle 1i+1kl+1\rangle w_1x_2z_2 + \langle 1i+1k+1l\rangle w_1w_2y_2 \\
&+ \langle 1i+1k+1l+1\rangle w_1w_2z_2 + \langle 1klj\rangle x_2y_1y_2 + \langle 1klj+1\rangle x_2y_2z_1 + \langle 1kl+1j\rangle x_2y_1z_2 \\
&+ \langle 1kl+1j+1\rangle x_2z_1z_2 \\
&+ \langle 1k+1lj\rangle w_2y_1y_2 + \langle 1k+1lj+1\rangle w_2y_2z_1 + \langle 1k+1l+1j\rangle w_2y_1z_2 \\
&+ \langle 1k+1l+1j+1\rangle w_2z_1z_2 + \langle iklj\rangle x_1x_2y_1y_2 + \langle iklj+1\rangle x_1x_2y_2z_1 + \langle ikl+1j\rangle x_1x_2y_1z_2 \\
&+ \langle ikl+1j+1\rangle x_1x_2z_1z_2 + \langle ik+1lj\rangle w_2x_1y_1y_2 + \langle ik+1lj+1\rangle w_2x_1z_1y_2 \\
&+ \langle ik+1l+1j\rangle w_2x_1y_1z_2 + \langle ik+1l+1j+1\rangle w_2x_1z_1z_2 + \langle i+1k+1lj\rangle w_2w_1y_1y_2 \\
&+ \langle i+1k+1lj+1\rangle w_2w_1z_1y_2 + \langle i+1k+1l+1j\rangle w_2w_1y_1z_2 + \langle i+1k+1l+1j+1\rangle w_2w_1z_1z_2 \\
&+ \langle i+1klj\rangle w_1x_2y_1y_2 + \langle i+1klj+1\rangle w_1x_2y_2z_1 + \langle i+1kl+1j\rangle w_1x_2y_1z_2 \\
&+ \langle i+1kl+1j+1\rangle w_1x_2z_1z_2
\end{aligned} \tag{A.2.2}$$

In the momentum twistor space,

$$\Omega^1_{ijkl} = \frac{\langle 1ii+1A_j\rangle\langle 1kk+1C_l\rangle\langle ABd^2A\rangle\langle ABd^2B\rangle\langle CDd^2C\rangle\langle CDd^2D\rangle}{\langle AB1i\rangle\langle AB1i+1\rangle\langle AB1j\rangle\langle AB1j+1\rangle\langle ABCD\rangle\langle CD1k\rangle\langle CD1k+1\rangle\langle CD1l\rangle\langle CD1l+1\rangle} \times \omega^{1'}_{ijkl}. \tag{A.2.3}$$

for

$$\omega^{1'}_{ijkl} = \frac{\langle ABii+1\rangle\langle A_jC_kC_l1\rangle + \langle A_iA_jC_kC_l\rangle}{\langle ABii+1\rangle\langle ABjj+1\rangle\langle CDkk+1\rangle\langle CDll+1\rangle}. \tag{A.2.4}$$

Another forms can be obtained similarly

$$\begin{aligned}
\omega^{2}_{ijkl} &= \langle 1ikl\rangle x_1x_2y_2 + \langle 1ikl+1\rangle x_1x_2z_2 + \langle 1ik+1l\rangle w_2x_1y_2 + \langle 1ik+1l+1\rangle w_2x_1z_2 \\
&+ \langle 1ijl\rangle x_1y_1y_2 + \langle 1ijl+1\rangle x_1y_1z_2 + \langle 1ij+1l\rangle z_1x_1y_2 + \langle 1ij+1l+1\rangle z_1x_1z_2 \\
&+ \langle 1i+1kl\rangle w_1x_2y_2 + \langle 1i+1kl+1\rangle w_1x_2z_2 + \langle 1i+1k+1l\rangle w_2w_1y_2 \\
&+ \langle 1i+1k+1l+1\rangle w_2w_1z_2 + \langle 1i+1jl\rangle w_1y_1y_2 + \langle 1i+1jl+1\rangle w_1y_1z_2 \\
&+ \langle 1i+1j+1l\rangle z_1w_1y_2 + \langle 1i+1j+1l+1\rangle z_1w_1z_2
\end{aligned} \tag{A.2.5}$$

$$\begin{aligned}
\omega^{3}_{ijkl} &= \langle 1ijk\rangle x_1x_2y_1 + \langle 1ijk+1\rangle w_2x_1y_1 + \langle 1ijl\rangle x_1y_1y_2 + \langle 1ijl+1\rangle x_1y_1z_2 + \langle 1ij+1k\rangle x_1x_2z_1 \\
&+ \langle 1ij+1k+1\rangle w_2x_1z_1 + \langle 1ij+1l\rangle x_1y_2z_1 + \langle 1ij+1l+1\rangle x_1z_1z_2 + \langle 1ikl\rangle x_1x_2y_2 \\
&+ \langle 1ikl+1\rangle x_1x_2z_2 + \langle 1ik+1l\rangle w_2x_1y_2 + \langle 1ik+1l+1\rangle w_2x_1z_2 + \langle 1i+1jk\rangle w_1x_2y_1 \\
&+ \langle 1i+1jk+1\rangle w_2w_1y_1 + \langle 1i+1jl\rangle w_1y_1y_2 + \langle 1i+1jl+1\rangle w_1y_1z_2 + \langle 1i+1j+1k\rangle w_1x_2z_1 \\
&+ \langle 1i+1j+1k+1\rangle w_2w_1z_1 + \langle 1i+1j+1l\rangle w_1y_2z_1 + \langle 1i+1j+1l+1\rangle w_1z_1z_2 \\
&+ \langle 1i+1kl\rangle w_1x_2y_2 + \langle 1i+1kl+1\rangle w_1x_2z_2 + \langle 1i+1k+1l\rangle w_2w_1y_2 \\
&+ \langle 1i+1k+1l+1\rangle w_2w_1z_2 + \langle 1jkl\rangle x_2y_1y_2 + \langle 1jkl+1\rangle x_2y_1z_2 + \langle 1jk+1l\rangle w_2y_1y_2 \\
&+ \langle 1jk+1l+1\rangle w_2y_1z_2 + \langle 1j+1kl\rangle x_2z_1y_2 + \langle 1j+1kl+1\rangle x_2z_1z_2 + \langle 1j+1k+1l\rangle w_2z_1y_2 \\
&+ \langle 1j+1k+1l+1\rangle w_2z_1z_2 + \langle ijkl\rangle x_1x_2y_1y_2 + \langle ijkl+1\rangle x_1x_2y_1z_2 + \langle ijk+1l\rangle w_2x_1y_1y_2 \\
&+ \langle ijk+1l+1\rangle w_2x_1y_1z_2 + \langle ij+1kl\rangle x_1x_2z_1y_2 + \langle ij+1kl+1\rangle x_1x_2z_1z_2 \\
&+ \langle ij+1k+1l\rangle w_2x_1z_1y_2 + \langle ij+1k+1l+1\rangle w_2x_1z_1z_2 + \langle i+1jkl\rangle w_1x_2y_1y_2 \\
&+ \langle i+1jkl+1\rangle w_1x_2y_1z_2 + \langle i+1jk+1l\rangle w_2w_1y_1y_2 + \langle i+1jk+1l+1\rangle w_2w_1y_1z_2 \\
&+ \langle i+1j+1kl\rangle w_1x_2z_1y_2 + \langle i+1j+1kl+1\rangle w_1x_2z_1z_2 + \langle i+1j+1k+1l\rangle w_2w_1z_1y_2 \\
&+ \langle i+1j+1k+1l+1\rangle w_2w_1z_1z_2
\end{aligned} \tag{A.2.6}$$

$$\begin{aligned}
\omega^{4}_{ijkl} &= \langle 1ii+1l\rangle w_2x_1y_2 + \langle 1ii+1l+1\rangle w_2x_1z_2 + \langle 1ii+1j\rangle w_1x_2y_1 + \langle 1ii+1j+1\rangle w_1x_2z_1 \\
&+ \langle 1ilj\rangle x_2y_1y_2 + \langle 1ilj+1\rangle x_2y_2z_1 + \langle 1il+1j\rangle x_2y_1z_2 + \langle 1il+1j+1\rangle x_2z_1z_2 \\
&+ \langle 1i+1lj\rangle w_2y_1y_2 + \langle 1i+1lj+1\rangle w_2y_2z_1 + \langle 1i+1l+1j\rangle w_2y_1z_2 \\
&+ \langle 1i+1l+1j+1\rangle w_2z_1z_2 + \langle ii+1lj\rangle w_2x_1y_1y_2 + \langle ii+1lj+1\rangle w_2x_1y_2z_1 \\
&+ \langle ii+1l+1j\rangle w_2x_1y_1z_2 + \langle ii+1l+1j+1\rangle w_2x_1z_1z_2
\end{aligned} \tag{A.2.7}$$

$$
\begin{aligned}
\omega^5_{ijkl} &= \langle 1ii+1j\rangle(w_1x_2y_1+w_2x_1y_2)+\langle 1ii+1j+1\rangle(w_1x_2z_1+w_2x_1z_2)\\
&+ \langle 1ijj+1\rangle(x_2y_2z_1+x_1y_1z_2)+\langle 1i+1jj+1\rangle(w_2y_2z_1+w_1y_1z_2)\\
&+ \langle ii+1jj+1\rangle(w_2x_1y_2z_1+w_1x_2y_1z_2)
\end{aligned} \tag{A.2.8}
$$

$$
\begin{aligned}
\omega^6_{ijkl} &= \langle 1ikj\rangle x_1x_2y_2+\langle 1ikj+1\rangle x_1x_2z_2+\langle 1ik+1j\rangle w_2x_1y_2+\langle 1ik+1j+1\rangle w_2x_1z_2\\
&+ \langle 1ijj+1\rangle x_1y_1z_2+\langle 1i+1kj\rangle w_1x_2y_2+\langle 1i+1kj+1\rangle w_1x_2z_2+\langle 1i+1k+1j\rangle w_2w_1y_2\\
&+ \langle 1i+1k+1j+1\rangle w_2w_1z_2+\langle 1i+1jj+1\rangle w_1y_1z_2+\langle 1kjj+1\rangle x_2y_2z_1\\
&+ \langle 1k+1jj+1\rangle w_2y_2z_1+\langle ikjj+1\rangle x_1x_2y_2z_1+\langle ik+1jj+1\rangle w_2x_1y_2z_1\\
&+ \langle i+1kjj+1\rangle w_1x_2y_2z_1+\langle i+1k+1jj+1\rangle w_1w_2y_2z_1
\end{aligned} \tag{A.2.9}
$$

$$
\begin{aligned}
\omega^7_{ijkl} &= \langle 1ijj+1\rangle w_2x_1y_1+\langle 1ijl\rangle(x_1x_2y_2+x_1y_1y_2)+\langle 1ijl+1\rangle(x_1x_2z_2+x_1y_1z_2)\\
&+ \langle 1ij+1l\rangle(w_2x_1y_2+x_1y_2z_1)+\langle 1ij+1l+1\rangle(w_2x_1z_2+x_1z_1z_2)+\langle 1i+1jj+1\rangle w_2w_1y_1\\
&+ \langle 1i+1jl\rangle(w_1x_2y_2+w_1y_1y_2)+\langle 1i+1jl+1\rangle(w_1x_2z_2+w_1y_1z_2)\\
&+ \langle 1i+1j+1l\rangle(w_2w_1y_2+w_1y_2z_1)+\langle 1i+1j+1l+1\rangle(w_2w_1z_2+w_1z_1z_2)\\
&+ \langle 1jj+1l\rangle w_2y_1y_2+\langle 1jj+1l+1\rangle w_2y_1z_2+\langle ijj+1l\rangle w_2x_1y_1y_2\\
&+ \langle ijj+1l+1\rangle w_2x_1y_1z_2+\langle i+1jj+1l\rangle w_1w_2y_1y_2+\langle i+1jj+1l+1\rangle w_1w_2y_1z_2
\end{aligned} \tag{A.2.10}
$$

$$
\begin{aligned}
&\omega^8_{ijkl}=\omega^4_{ijkl},\quad \omega^9_{ijkl}=\omega^7_{ijkl},\quad \omega^{10}_{ijkl}=\omega^2_{ijkl},\quad \omega^{11}_{ijkl}=\omega^6_{ijkl},\quad \omega^{12}_{ijkl}=\omega^1_{ijkl},\quad \omega^{13}_{ijkl}=\omega^3_{ijkl},\\
&(x_1,w_1,y_1,z_1)\leftrightarrow(x_2,w_2,y_2,z_2),(i,j)\leftrightarrow(k,l).
\end{aligned} \tag{A.2.11}
$$

Next, we rewrite these forms in the momentum twistor space

$$
\omega^{1'}_{ijkl}=\frac{\langle ABii+1\rangle\langle A_jC_kC_l1\rangle+\langle A_iA_jC_kC_l\rangle}{\langle ABii+1\rangle\langle ABjj+1\rangle\langle CDkk+1\rangle\langle CDll+1\rangle}. \tag{A.2.12}
$$

$$
\omega^{2'}_{ijkl}=\frac{-\langle ABjj+1\rangle\langle A_iC_kC_l1\rangle+\langle CDkk+1\rangle\langle A_iA_jC_l1\rangle}{\langle ABii+1\rangle\langle ABjj+1\rangle\langle CDkk+1\rangle\langle CDll+1\rangle} \tag{A.2.13}
$$

$$
\begin{aligned}
\omega^{3'}_{ijkl} &= \frac{1}{\langle ABii+1\rangle\langle ABjj+1\rangle\langle CDkk+1\rangle\langle CDll+1\rangle}\\
&\times \{\langle ABii+1\rangle\langle A_jC_kC_l1\rangle-\langle ABjj+1\rangle\langle A_iC_kC_l1\rangle-\langle AB1i\rangle\langle i+1A_jC_kC_l\rangle\\
&+ \langle CDkk+1\rangle\langle A_iA_jC_l1\rangle-\langle CDll+1\rangle\langle A_iA_jC_k1\rangle+\langle AB1i+1\rangle\langle iA_jC_kC_l\rangle\}
\end{aligned}
$$

(A.2.14)

$$
\begin{aligned}
\omega^{4'}_{ijkl} &= \frac{1}{\langle ABii+1\rangle\langle ABjj+1\rangle\langle CDii+1\rangle\langle CDll+1\rangle} \\
&\times \{\langle ABii+1\rangle\langle A_jC_kC_l1\rangle + \langle AB1i+1\rangle\langle CD1i\rangle\langle A_jC_lii+1\rangle \\
&+ \langle AB1i\rangle\langle CD1i+1\rangle\langle CDll+1\rangle\langle A_j1ii+1\rangle + \langle AB1i+1\rangle\langle CD1i\rangle\langle ABjj+1\rangle\langle C_l1ii+1\rangle\}
\end{aligned}
\tag{A.2.15}
$$

$$
\begin{aligned}
\omega^{5'}_{ijkl} &= \frac{1}{\langle ABii+1\rangle\langle ABjj+1\rangle\langle CDii+1\rangle\langle CDjj+1\rangle} \\
&\times \{\langle 1ii+1A_j\rangle(\langle AB1i\rangle\langle CD1k+1\rangle\langle CDll+1\rangle + \langle AB1j+1\rangle\langle CD1l\rangle\langle CDkk+1\rangle) \\
&+ \langle 1kk+1C_l\rangle(\langle AB1i+1\rangle\langle ABjj+1\rangle\langle CD1k\rangle + \langle AB1j\rangle\langle ABii+1\rangle\langle CD1l+1\rangle) \\
&+ \langle ii+1jj+1\rangle(\langle AB1i\rangle\langle AB1j+1\rangle\langle CD1k+1\rangle\langle CD1l\rangle \\
&+ \langle AB1i+1\rangle\langle AB1j\rangle\langle CD1k\rangle\langle CD1l+1\rangle)\}
\end{aligned}
\tag{A.2.16}
$$

$$
\begin{aligned}
\omega^{6'}_{ijkl} &= \frac{1}{\langle ABii+1\rangle\langle ABjj+1\rangle\langle CDkk+1\rangle\langle CDjj+1\rangle} \\
&\times \{\langle AB1j\rangle\langle ABii+1\rangle\langle CD1l+1\rangle\langle C_k1jj+1\rangle + \langle AB1j+1\rangle\langle CD1l\rangle\langle CDkk+1\rangle\langle A_i1jj+1\rangle \\
&+ \langle ABjj+1\rangle(\langle CD1l+1\rangle\langle A_iC_k1j\rangle - \langle CD1l\rangle\langle A_iC_k1j+1\rangle)\}
\end{aligned}
\tag{A.2.17}
$$

$$
\begin{aligned}
\omega^{7'}_{ijkl} &= \frac{1}{\langle ABii+1\rangle\langle ABjj+1\rangle\langle CDjj+1\rangle\langle CDll+1\rangle} \\
&\times \{\langle AB1j\rangle\langle CD1j+1\rangle\langle AB1i\rangle\langle i+1jj+1C_k\rangle - \langle AB1j\rangle\langle CD1j+1\rangle\langle AB1i+1\rangle\langle ijj+1C_k\rangle \\
&+ \langle AB1j+1\rangle\langle CD1j\rangle\langle CDkk+1\rangle\langle 1jj+1A_i\rangle + \langle AB1j\rangle\langle CD1j+1\rangle\langle ABii+1\rangle\langle 1jj+1C_k\rangle \\
&+ \langle ABjj+1\rangle\langle 1C_kC_jA_i\rangle\}
\end{aligned}
\tag{A.2.18}
$$

$$
\begin{aligned}
&\omega^{8'}_{ijkl} = \omega^{4'}_{ijkl}, \quad \omega^{9'}_{ijkl} = \omega^{7'}_{ijkl}, \quad \omega^{10'}_{ijkl} = \omega^{2'}_{ijkl}, \quad \omega^{11'}_{ijkl} = \omega^{6'}_{ijkl}, \quad \omega^{12'}_{ijkl} = \omega^{1'}_{ijkl}, \quad \omega^{13'}_{ijkl} = \omega^{3'}_{ijkl}, \\
&\qquad (AB) \leftrightarrow (CD), (i,j) \leftrightarrow (k,l)
\end{aligned}
\tag{A.2.19}
$$

A.3 n-point 2-loop MHV Log Amplituhedron

$$
\Omega[\log[\mathcal{A}^{n\text{-pt 2-loop}}_{\text{MHV}}]] = \sum_{\substack{i,j,k,l=2,3,\cdots,n-1 \\ i<k<l<j}} \Omega^1_{ijkl}[\log] + \sum_{i<k<j<l} \Omega^2_{ijkl}[\log] \cdots + \sum_{k<l<i<j} \Omega^{13}_{ijkl}[\log]
\tag{A.3.1}
$$

for

$$\Omega^m_{ijkl}[\log] = \frac{dx_1 dx_2 \cdots dz_1 dz_2}{x_1 x_2 w_1 w_2 y_1 y_2 z_1 z_2} \frac{-1}{(az_2 - bw_1 - cx_1 - dw_2 + ey_2)} \times \omega^m_{ijkl}[\log] \tag{A.3.2}$$

where

$$\begin{aligned}
\omega^1_{ijkl}[\log] &= x_2 y_1 \langle 1i+1kj\rangle + x_2 z_1 \langle 1i+1kj+1\rangle + y_1 y_2 \langle 1i+1lj\rangle + y_2 z_1 \langle 1i+1lj+1\rangle \\
&+ y_1 z_2 \langle 1i+1l+1j\rangle + z_1 z_2 \langle 1i+1l+1j+1\rangle + x_2 y_1 \langle 1ikj\rangle + x_2 z_1 \langle 1ikj+1\rangle \\
&+ w_2 y_1 \langle 1ik+1j\rangle + w_2 z_1 \langle 1ik+1j+1\rangle + y_1 y_2 \langle 1ilj\rangle + y_2 z_1 \langle 1il+1j\rangle + y_1 z_2 \langle 1il+1j\rangle \\
&+ z_1 z_2 \langle 1il+1j+1\rangle + w_1 y_1 \langle 1i+1k+1j\rangle + w_1 z_1 \langle 1i+1k+1j+1\rangle
\end{aligned} \tag{A.3.3}$$

$$\begin{aligned}
\omega^2_{ijkl}[\log] &= -x_1 x_2 y_1 \langle 1ikj\rangle - x_1 x_2 z_1 \langle 1ikj+1\rangle - w_2 x_1 y_1 \langle 1ik+1j\rangle \\
&- w_2 x_1 z_1 \langle 1ik+1j+1\rangle - w_1 x_2 y_1 \langle 1i+1kj\rangle - w_1 x_2 z_1 \langle 1i+1kj+1\rangle \\
&- w_1 w_2 y_1 \langle 1i+1k+1j\rangle - w_1 w_2 z_1 \langle 1i+1k+1j+1\rangle - x_2 y_1 y_2 \langle 1kjl\rangle \\
&- x_2 y_1 z_2 \langle 1kjl+1\rangle - x_2 y_2 z_1 \langle 1kj+1l\rangle - x_2 z_1 z_2 \langle 1kj+1l+1\rangle - w_2 y_1 y_2 \langle 1k+1jl\rangle \\
&- w_2 y_1 z_2 \langle 1k+1jl+1\rangle - w_2 y_2 z_1 \langle 1k+1j+1l\rangle - w_2 z_1 z_2 \langle 1k+1j+1l+1\rangle \\
&- x_1 x_2 y_1 y_2 \langle ikjl\rangle - x_1 x_2 y_1 z_2 \langle ikjl+1\rangle - x_1 x_2 y_2 z_1 \langle ikj+1l\rangle - x_1 x_2 z_1 z_2 \langle ikj+1l+1\rangle \\
&- w_2 x_1 y_1 y_2 \langle ik+1jl\rangle - w_2 x_1 y_1 z_2 \langle ik+1jl+1\rangle - w_2 x_1 y_2 z_1 \langle ik+1j+1l\rangle \\
&- w_2 x_1 z_1 z_2 \langle ik+1j+1l+1\rangle - w_1 x_2 y_1 y_2 \langle i+1kjl\rangle - w_1 x_2 y_1 z_2 \langle i+1kjl+1\rangle \\
&- w_1 x_2 y_2 z_1 \langle i+1kj+1l\rangle - w_1 x_2 z_1 z_2 \langle i+1kj+1l+1\rangle - w_1 w_2 y_1 y_2 \langle i+1k+1jl\rangle - \\
&- w_1 w_2 y_1 z_2 \langle i+1k+1jl+1\rangle - w_1 w_2 y_2 z_1 \langle i+1k+1j+1l\rangle \\
&- w_1 w_2 z_1 z_2 \langle i+1k+1j+1l+1\rangle
\end{aligned} \tag{A.3.4}$$

$$\omega^3_{ijkl}[\log] = 0 \tag{A.3.5}$$

$$\begin{aligned}
\omega^4_{ijkl}[\log] &= -w_1 x_2 y_2 \langle 1ii+1l\rangle - w_1 x_2 z_2 \langle 1ii+1l+1\rangle - w_2 x_1 y_1 \langle 1ii+1j\rangle - w_2 x_1 z_1 \langle 1ii+1j+1\rangle \\
&- x_1 y_1 y_2 \langle 1ilj\rangle - x_1 y_2 z_1 \langle 1ilj+1\rangle - x_1 y_1 z_2 \langle 1il+1j\rangle - x_1 z_1 z_2 \langle 1il+1j+1\rangle \\
&- w_1 y_1 y_2 \langle 1i+1lj\rangle - w_1 y_2 z_1 \langle 1i+1lj+1\rangle - w_1 y_1 z_2 \langle 1i+1l+1j\rangle \\
&- w_1 z_1 z_2 \langle 1i+1l+1j+1\rangle - w_1 x_2 y_1 y_2 \langle ii+1lj\rangle - w_1 x_2 y_2 z_1 \langle ii+1lj+1\rangle \\
&- w_1 x_2 y_1 z_2 \langle ii+1l+1j\rangle - w_1 x_2 z_1 z_2 \langle ii+1l+1j+1\rangle
\end{aligned} \tag{A.3.6}$$

$$
\begin{aligned}
\omega^5_{ijkl}[\log] \;&=\; -w_2x_1y_1\langle 1ii+1j\rangle - w_1x_2y_2\langle 1ii+1j\rangle - w_2x_1z_1\langle 1ii+1j+1\rangle - w_1x_2z_2\langle 1ii+1j+1\rangle \\
&-\; x_1y_2z_1\langle 1ijj+1\rangle - x_2y_1z_2\langle 1ijj+1\rangle - w_1y_2z_1\langle 1i+1jj+1\rangle - w_2y_1z_2\langle 1i+1jj+1\rangle \\
&-\; w_1x_2y_2z_1\langle ii+1jj+1\rangle - w_2x_1y_1z_2\langle ii+1jj+1\rangle
\end{aligned}
\tag{A.3.7}
$$

$$
\begin{aligned}
\omega^6_{ijkl}[\log] \;&=\; -w_2x_1y_1\langle 1ii+1j\rangle - w_2x_1z_1\langle 1ii+1j+1\rangle - w_1x_2y_2\langle 1ii+1l\rangle - w_1x_2z_2\langle 1ii+1l+1\rangle \\
&-\; x_2y_1y_2\langle 1ijl\rangle - x_2y_1z_2\langle 1ijl+1\rangle - x_2y_2z_1\langle 1ij+1l\rangle - x_2z_1z_2\langle 1ij+1l+1\rangle \\
&-\; w_2y_1y_2\langle 1i+1jl\rangle - w_2y_1z_2\langle 1i+1jl+1\rangle - w_2y_2z_1\langle 1i+1j+1l\rangle \\
&-\; w_2z_1z_2\langle 1i+1j+1l+1\rangle - w_2x_1y_1y_2\langle ii+1jl\rangle - w_2x_1y_1z_2\langle ii+1jl+1\rangle \\
&-\; w_2x_1y_2z_1\langle ii+1j+1l\rangle - w_2x_1z_1z_2\langle ii+1j+1l+1\rangle
\end{aligned}
\tag{A.3.8}
$$

$$
\begin{aligned}
\omega^7_{ijkl}[\log] \;&=\; -x_1x_2y_1\langle 1ikj\rangle - x_1x_2z_1\langle 1ikj+1\rangle - w_2x_1y_1\langle 1ik+1j\rangle - w_2x_1z_1\langle 1ik+1j+1\rangle \\
&-\; x_1y_2z_1\langle 1ijj+1\rangle - w_1x_2y_1\langle 1i+1kj\rangle - w_1x_2z_1\langle 1i+1kj+1\rangle - w_1w_2y_1\langle 1i+1k+1j\rangle \\
&-\; w_1w_2z_1\langle 1i+1k+1j+1\rangle - w_1y_2z_1\langle 1i+1jj+1\rangle - x_2y_1z_2\langle 1kjj+1\rangle \\
&-\; w_2y_1z_2\langle 1k+1jj+1\rangle - x_1x_2y_1z_2\langle ikjj+1\rangle - w_2x_1y_1z_2\langle ik+1jj+1\rangle \\
&-\; w_1x_2y_1z_2\langle i+1kjj+1\rangle - w_1w_2y_1z_2\langle i+1k+1jj+1\rangle
\end{aligned}
\tag{A.3.9}
$$

$$
\begin{gathered}
\omega^8_{ijkl}[\log] = \omega^4_{ijkl}[\log], \qquad \omega^9_{ijkl}[\log] = \omega^7_{ijkl}[\log], \quad \omega^{10}_{ijkl}[\log] = \omega^2_{ijkl}[\log], \quad \omega^{11}_{ijkl}\log] = \omega^6_{ijkl}[\log], \\
\omega^{12}_{ijkl}[\log] = \omega^1_{ijkl}[\log], \quad \omega^{13}_{ijkl}[\log] = 0, \\
(x_1,w_1,y_1,z_1) \leftrightarrow (x_2,w_2,y_2,z_2), (i,j) \leftrightarrow (k,l)
\end{gathered}
\tag{A.3.10}
$$

We can similarly write these forms in the momentum twistor language.

Appendix B

Explicit Calculation of the 1-loop NMHV Amplituhedron

B.1 6-2 Representation of the 6-pt case

The 6-pt case, there are four sign flip cells $\mathcal{A}_{234}, \mathcal{A}_{235}, \mathcal{A}_{245}, \mathcal{A}_{345}$. We have already obtained the intersecting polygon for the $\mathcal{A}_{234}$ cell. Then we consider the remain cells in this appendix. The vertices of the polygon which intersects with $\mathcal{A}^6_{235}$ cell are

$$(1,2,3),\ (2,3,4),\ (5,6,1) \tag{B.1.1}$$

and there are another vertices depending on the signs of other brackets as

<table>
<tr><th>(25)</th><th>(35)</th><th>(26)</th><th>(36)</th><th>(46)</th><th>vertices</th><th>pentagon</th></tr>
<tr><td>$+$</td><td>$-$</td><td>$+$</td><td>$+$</td><td>$-$</td><td rowspan="4">$(345),(456),(612)$</td><td rowspan="4">(4)</td></tr>
<tr><td>$-$</td><td>$-$</td><td>$+$</td><td>$+$</td><td>$-$</td></tr>
<tr><td>$+$</td><td>$-$</td><td>$+$</td><td>$-$</td><td>$-$</td></tr>
<tr><td>$-$</td><td>$-$</td><td>$+$</td><td>$-$</td><td>$-$</td></tr>
<tr><td>$+$</td><td>$-$</td><td>$+$</td><td>$-$</td><td>$+$</td><td rowspan="2">$(345),(346),(612),(461)$</td><td rowspan="2">(5)</td></tr>
<tr><td>$-$</td><td>$-$</td><td>$+$</td><td>$-$</td><td>$+$</td></tr>
<tr><td>$-$</td><td>$+$</td><td>$+$</td><td>$-$</td><td>$+$</td><td>$(235),(346),(612),(356),(461)$</td><td>(6)</td></tr>
<tr><td>$+$</td><td>$-$</td><td>$-$</td><td>$+$</td><td>$-$</td><td>$(236),(345),(456),(256)$</td><td>(7)</td></tr>
<tr><td>$-$</td><td>$+$</td><td>$+$</td><td>$-$</td><td>$-$</td><td>$(235),(456),(612),(356)$</td><td>(8)</td></tr>
</table>

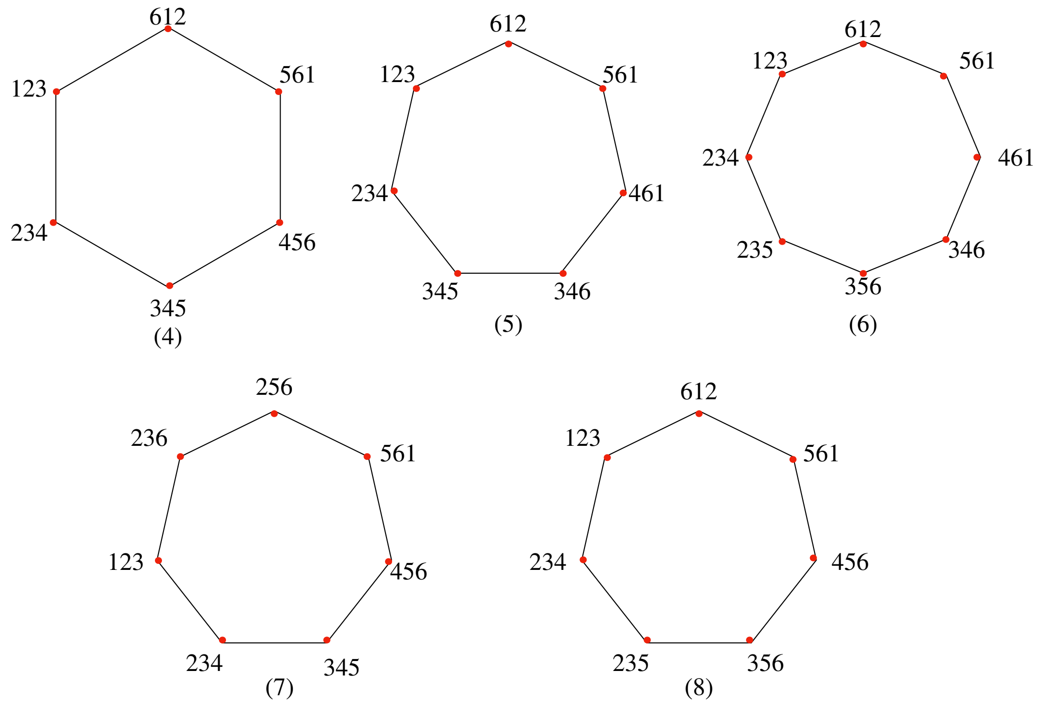

Figure B.1: Polygons for $\mathcal{A}_{235}$

where (ij) is $\langle Yij \rangle$ and the shape of the intersecting polygons are Figure B.1. Next, the vertices of the polygon which intersects with $\mathcal{A}_{245}^{6}$ cell are

$$(1,2,3),\ (4,5,6),\ (5,6,1) \tag{B.1.2}$$

and there are another vertices depending on the signs of other brackets as

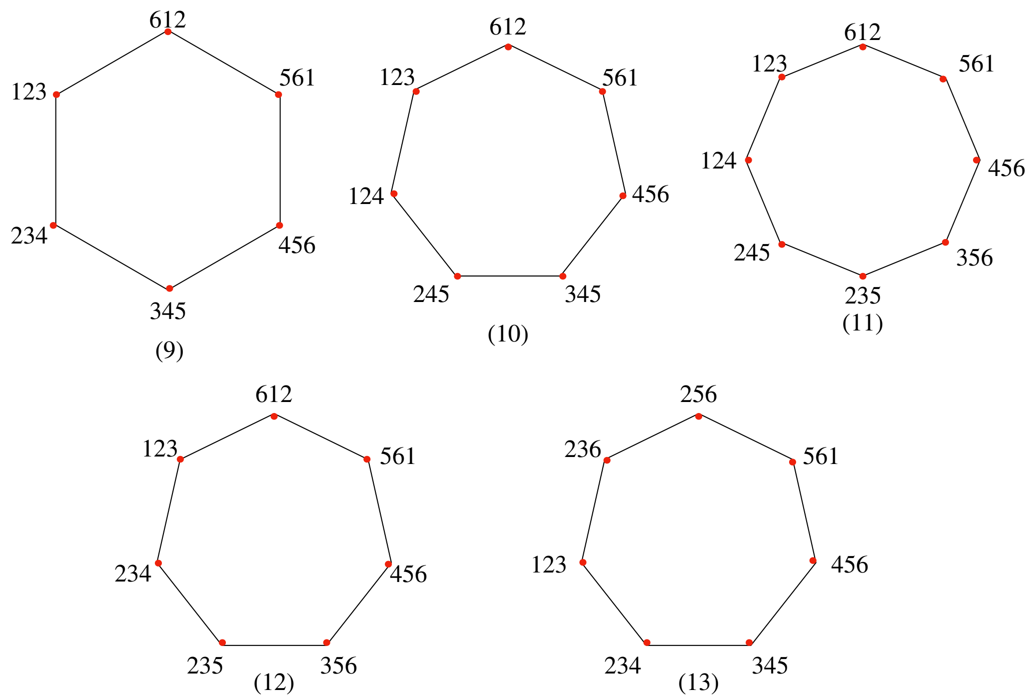

Figure B.2: Polygons for $\mathcal{A}_{245}$

(24)	(25)	(35)	(26)	(36)	vertices	pentagon
$+$	$+$	$-$	$+$	$+$	(234), (345), (612)	(9)
$-$	$-$	$-$	$+$	$+$		
$-$	$+$	$-$	$+$	$-$		
$-$	$-$	$-$	$+$	$-$		
$+$	$-$	$-$	$+$	$+$	(124), (345), (245), (612)	(10)
$+$	$-$	$-$	$+$	$-$		
$+$	$-$	$+$	$+$	$-$	(124), (235), (245), (612), (356)	(11)
$-$	$-$	$+$	$+$	$-$	(234), (235), (612), (356)	(12)
$-$	$+$	$-$	$-$	$+$	(234), (236), (345), (256)	(13)

and the shape of the intersecting polygons are Figure B.2. The vertices of the polygon which intersects with $\mathcal{A}^6_{345}$ cell are

$$(1,3,4),\ (3,4,5),\ (4,5,6),\ (5,6,1),\ (1,3,4),\ (1,3,6) \qquad \text{(B.1.3)}$$

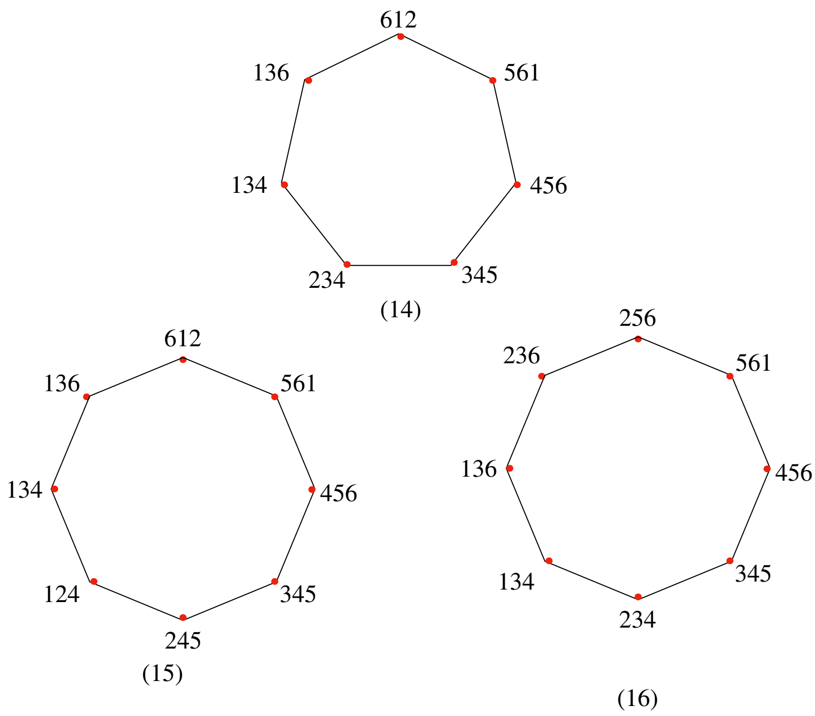

Figure B.3: Polygons for $\mathcal{A}_{345}$

and there are another vertices depending on the signs of other brackets as

(24)	(25)	(26)	vertices	pentagon
$-$	$+$	$+$	$(234), (612)$	(14)
$-$	$-$	$+$		
$+$	$-$	$+$	$(124), (612), (245)$	(15)
$-$	$+$	$-$	$(234), (236), (256)$	(16)

and the shape of the intersecting polygons are Figure B.3. Next, we consider the 6-2 representation of this amplituhedron. First, we can see that all of these polygons are related to the basic polygon P_6 which has the six vertices $(612), (123), (234), (345), (456), (561)$ as

$$(1) = P_6 - \Delta(561)(125)(145), \quad (2) = P_6 - \Delta(561)(125)(145) - \Delta(612)(236)(256),$$
$$(3) = P_6 - \Delta(561)(125)(145) - \Delta(456)(461)(346), \quad (5) = P_6 - \Delta(456)(461)(346),$$
$$(6) = P_6 - \Delta(345)(235)(356) - \Delta(456)(461)(346), \quad (7) = P_6 - \Delta(612)(236)(256),$$
$$(8) = P_6 - \Delta(345)(235)(356), \quad (9) = P_6 - \Delta(234)(124)(245),$$

$$(11) = P_6 - \Delta(234)(124)(245) - \Delta(345)(235)(356), \quad (12) = P_6 - \Delta(345)(235)(356),$$
$$(13) = P_6 - \Delta(612)(236)(256), \quad (14) = P_6 - \Delta(123)(136)(134),$$
$$(15) = P_6 - \Delta(123)(136)(134) - \Delta(234)(124)(245),$$
$$(16) = P_6 - \Delta(123)(136)(134) - \Delta(612)(236)(256). \tag{B.1.4}$$

where (i) is the pentagon (i) and $\Delta(i)(j)(k)$ is the triangle whose vertices are i, j, k. From this, the 6-2 representation of the 6-pt case is expressed as

$$\begin{aligned} \mathcal{A}^{6\times 2}_{\text{6-pt}} &= (\mathcal{A}_{234} + \mathcal{A}_{235} + \mathcal{A}_{245} + \mathcal{A}_{345}) \times P_6 + \mathcal{A}^1 \times \Delta(612)(236)(256) \\ &+ \mathcal{A}^2 \times \Delta(123)(136)(134) + \cdots + \mathcal{A}^6 \times \Delta(561)(125)(145). \end{aligned} \tag{B.1.5}$$

where $\mathcal{A}^i$ is the sum of the futher triangulated sign flip cells. For example,

$$\begin{aligned} \mathcal{A}^1 &= \mathcal{A}'_{234} + \mathcal{A}'_{235} + \mathcal{A}'_{245} + \mathcal{A}'_{345} \\ \mathcal{A}'_{234} &: \mathcal{A}_{234} \text{ with } \{(26),(36),(46)\} = \{-,+,-\} \\ \mathcal{A}'_{235} &: \mathcal{A}_{235} \text{ with } \{(25),(35),(26),(46)\} = \{+,-,-,-\} \\ \mathcal{A}'_{245} &: \mathcal{A}_{245} \text{ with } \{(25),(26),(36)\} = \{+,-,+\} \\ \mathcal{A}'_{345} &: \mathcal{A}_{345} \text{ with } \{(24),(25),(26)\} = \{-,+,-\}. \end{aligned} \tag{B.1.6}$$

Then the sign of the brackets $\langle YABij\rangle$ for this $\mathcal{A}^1$ is

$$\langle YABii+1\rangle > 0, \quad \{\langle YAB62\rangle, \langle YAB63\rangle, \langle YAB64\rangle, \langle YAB65\rangle\} = \{+,-,+,-\}. \tag{B.1.7}$$

This is the sign flip condition of the 5-pt $m = 2, k = 3$ amplituhedron $\mathcal{A}^{m=2,k=3}_{\text{5-pt}}(2,3,4,5,6)$. Similarly we can see that other $\mathcal{A}^i$ is 5-pt $m = 2, k = 3$ amplituhedron as

$$\begin{aligned} &\mathcal{A}^1 = \mathcal{A}^{m=2,k=3}_{\text{5-pt}}(2,3,4,5,6),\ \mathcal{A}^2 = \mathcal{A}^{m=2,k=3}_{\text{5-pt}}(3,4,5,6,1),\ \mathcal{A}^3 = \mathcal{A}^{m=2,k=3}_{\text{5-pt}}(4,5,6,1,2), \\ &\mathcal{A}^4 = \mathcal{A}^{m=2,k=3}_{\text{5-pt}}(5,6,1,2,3),\ \mathcal{A}^5 = \mathcal{A}^{m=2,k=3}_{\text{5-pt}}(6,1,2,3,4),\ \mathcal{A}^6 = \mathcal{A}^{m=2,k=3}_{\text{5-pt}}(1,2,3,4,5). \end{aligned} \tag{B.1.8}$$

From this, we can obtain the final result of the 6-2 representation of the 6-pt case (4.1.19).

Appendix C

Explicit Results of the 1-loop NMHV Amplituhedron

C.1 Canonical form of the 6-pt case

The explicit expression of the canonical of the 6-2 representation for the 6-pt 1-loop NMHV amplituhedron in the (YAB) space is

$$\begin{aligned}
\Omega^{6\times 2}_{\text{6-pt}} &= \langle Yd^4Y\rangle\langle YABd^2A\rangle\langle YABd^2B\rangle \\
&\times \left(\Omega'_{234}+\Omega'_{235}+\Omega'_{245}+\Omega'_{345}\right)\times([123]+[134]+[145]+[156]) \\
&+ \frac{\langle 12345\rangle^2\langle 12456\rangle^2}{\langle YAB12\rangle\langle YAB23\rangle\langle YAB34\rangle\langle YAB45\rangle\langle Y1245\rangle\langle Y1256\rangle\langle Y4561\rangle} \\
&+ \frac{\langle 13456\rangle^2\langle 12346\rangle^2}{\langle YAB34\rangle\langle YAB45\rangle\langle YAB56\rangle\langle YAB61\rangle\langle Y1234\rangle\langle Y3461\rangle\langle Y2361\rangle} \\
&+ \frac{\langle 12346\rangle^2\langle 13456\rangle^2}{\langle YAB12\rangle\langle YAB23\rangle\langle YAB34\rangle\langle YAB61\rangle\langle Y4561\rangle\langle Y3461\rangle\langle Y3456\rangle} \\
&+ \frac{\langle 12356\rangle^2\langle 23456\rangle^2}{\langle YAB12\rangle\langle YAB23\rangle\langle YAB56\rangle\langle YAB61\rangle\langle Y2356\rangle\langle Y2345\rangle\langle Y3456\rangle} \\
&+ \frac{\langle 12456\rangle^2\langle 12345\rangle^2}{\langle YAB12\rangle\langle YAB45\rangle\langle YAB56\rangle\langle YAB61\rangle\langle Y1234\rangle\langle Y2345\rangle\langle Y1245\rangle} \\
&+ \frac{\langle 23456\rangle^2\langle 12356\rangle^2}{\langle YAB23\rangle\langle YAB34\rangle\langle YAB45\rangle\langle YAB56\rangle\langle Y2361\rangle\langle Y2356\rangle\langle Y1256\rangle}
\end{aligned} \tag{C.1.1}$$

where

$$\Omega'_{ijk} = \frac{\begin{vmatrix} \langle YA1ii+1\rangle & \langle YA1jj+1\rangle & \langle YA1kk+1\rangle \\ \langle AB1ii+1\rangle & \langle AB1jj+1\rangle & \langle AB1kk+1\rangle \\ \langle BY1ii+1\rangle & \langle BY1jj+1\rangle & \langle BY1kk+1\rangle \end{vmatrix}^2}{\substack{\langle YAB1i\rangle\langle YAB1i{+}1\rangle\langle YABii{+}1\rangle\langle YAB1j\rangle\langle YAB1j{+}1\rangle\langle YABjj{+}1\rangle \\ \langle YAB1k\rangle\langle YAB1k{+}1\rangle\langle YABkk{+}1\rangle}} \tag{C.1.2}$$

and

$$[1ii+1] = \frac{\begin{vmatrix} \langle YAn12\rangle & \langle YAi-1ii+1\rangle & \langle YAii+1i+2\rangle \\ \langle ABn12\rangle & \langle ABi-1ii+1\rangle & \langle ABii+1i+2\rangle \\ \langle BYn12\rangle & \langle BYi-1ii+1\rangle & \langle BYii+1i+2\rangle \end{vmatrix}^2}{\langle YAB(n12)\cap(i{-}1ii{+}1)\rangle\langle YAB(i{-}1ii{+}1)\cap(ii{+}1i{+}2)\rangle\langle YAB(ii{+}1i{+}2)\cap(n12)\rangle}. \tag{C.1.3}$$

C.2 Super-Local Representation of the 6-pt case

$$\begin{aligned}
\Omega^{6\times 2}_{\text{6-pt}} &= \langle Yd^4Y\rangle\langle YABd^2A\rangle\langle YABd^2B\rangle \\
&\times \left(\frac{\langle 23456\rangle\langle 12345\rangle}{\langle YAB12\rangle\langle YAB23\rangle\langle YAB34\rangle\langle YAB45\rangle\langle YAB56\rangle}\right. \\
&+ \frac{\langle YAB(156)\cap(2345)\rangle\langle 12346\rangle}{\langle YAB12\rangle\langle YAB23\rangle\langle YAB34\rangle\langle YAB45\rangle\langle YAB56\rangle\langle YAB61\rangle} \\
&+ \frac{\langle YAB(234)\cap(4561)\rangle\langle 12356\rangle}{\langle YAB12\rangle\langle YAB23\rangle\langle YAB34\rangle\langle YAB45\rangle\langle YAB56\rangle\langle YAB61\rangle} \\
&+ \left.\frac{\langle 34561\rangle\langle 12456\rangle}{\langle YAB12\rangle\langle YAB34\rangle\langle YAB45\rangle\langle YAB56\rangle\langle YAB61\rangle}\right) \\
&\times \left(\frac{\langle YAB23\rangle\langle 12346\rangle\langle 12345\rangle}{\langle Y6123\rangle\langle Y1234\rangle\langle Y2345\rangle} + \frac{\langle YAB(345)\cap(1236)\rangle\langle 23456\rangle}{\langle Y6123\rangle\langle Y2345\rangle\langle Y3456\rangle}\right. \\
&+ \left.\frac{\langle YAB(456)\cap(1236)\rangle\langle 34561\rangle}{\langle Y6123\rangle\langle Y3456\rangle\langle Y4561\rangle} + \frac{\langle YAB61\rangle\langle 12356\rangle\langle 45612\rangle}{\langle Y6123\rangle\langle Y4561\rangle\langle Y5612\rangle}\right) \\
&+ \frac{\langle 12345\rangle^2\langle 12456\rangle^2}{\langle YAB12\rangle\langle YAB23\rangle\langle YAB34\rangle\langle YAB45\rangle\langle Y1245\rangle\langle Y1256\rangle\langle Y4561\rangle} \\
&+ \frac{\langle 13456\rangle^2\langle 12346\rangle^2}{\langle YAB34\rangle\langle YAB45\rangle\langle YAB56\rangle\langle YAB61\rangle\langle Y1234\rangle\langle Y3461\rangle\langle Y2361\rangle} \\
&+ \frac{\langle 12346\rangle^2\langle 13456\rangle^2}{\langle YAB12\rangle\langle YAB23\rangle\langle YAB34\rangle\langle YAB61\rangle\langle Y4561\rangle\langle Y3461\rangle\langle Y3456\rangle}
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{\langle 12356\rangle^2\langle 23456\rangle^2}{\langle YAB12\rangle\langle YAB23\rangle\langle YAB56\rangle\langle YAB61\rangle\langle Y2356\rangle\langle Y2345\rangle\langle Y3456\rangle} \\
&+ \quad \frac{\langle 12456\rangle^2\langle 12345\rangle^2}{\langle YAB12\rangle\langle YAB45\rangle\langle YAB56\rangle\langle YAB61\rangle\langle Y1234\rangle\langle Y2345\rangle\langle Y1245\rangle} \\
&+ \quad \frac{\langle 23456\rangle^2\langle 12356\rangle^2}{\langle YAB23\rangle\langle YAB34\rangle\langle YAB45\rangle\langle YAB56\rangle\langle Y2361\rangle\langle Y2356\rangle\langle Y1256\rangle}
\end{aligned}
\tag{C.2.1}
$$

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