Fibrational Theory of Behaviors and Observations: Bisimulation, Logic, and Games from Modalities

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Abstract

Mathematical modeling of computer systems is of fundamental importance in software verification. Some of such mathematical models focus on the possible states and transitions of the target system. Such “state-and-transition” models are unified, using the language of category theory, into the concept called coalgebra. It generalizes models in automata theory and process theory. One way to view coalgebra theory is that it is “a theory of observable behaviors”: Notions like bisimulation relation and simulation preorder are defined in such a way that they are invariant under coalgebra morphisms, which morally means that they depend only on “observable” information. These are generalized to fibrational coinduction, which is parametrized in fibrations and functor liftings. It gives rise to a wide range of examples, including quantitative ones like behavioral metrics. However, it lacks explicit modeling of “observation.” Bisimulation relation and behavioral distance are known to be tightly connected to games and modal logic through “observations” on the system, but in this fibrational framework, such connections are not understood well.

This thesis proposes a new framework in which observations are modeled as certain morphisms. The main point is to adopt codensity lifting, a known method to define functor lifting to use in fibrational coinduction. The resulting object generalizes bisimulation relation and simulation preorder, and we call it codensity bisimilarity. The definition of codensity lifting involves a set of morphisms, which represents the set of “observations.” This feature makes it possible to connect codensity bisimilarity to games and modal logic.

After introducing our framework, we show two main results. The first is a game characterizing codensity bisimilarity, which we call the codensity bisimilarity game or just the codensity game. Just like the conventional bisimilarity game, the codensity game is played by two players named Duplicator and Spoiler, and it may last indefinitely long, in which case Duplicator wins. The difference between the conventional game and our game is the players’ moves. Spoiler’s moves are “observations,” i.e., arrows from the state space object. Duplicator’s moves are predicates themselves: for example, in the codensity game for bisimilarity relation, Duplicator chooses an equivalence relation as a move each time. We show that the codensity game characterizes the codensity bisimilarity in general: both the statement and the proof are independent of the specific property of the behavior functor and the fibration. For some better-behaved fibrations, we construct another game called the trimmed codensity bisimilarity game. In this game, the set of Duplicator’s moves is smaller: for example, in the game for bisimilarity relation, Duplicator’s moves are restricted to pairs of states instead of equivalence relations on the state space. We instantiate the underlying fibrations of codensity games so that they can characterize bisimilarity relation and behavioral distance. We also consider

the fibration of topologies and devise a new concept called bisimulation topology with a game characterization.

The second main result concerns the adequacy and expressivity of modal logic for codensity bisimilarity. We define fibrational logical equivalence, a generalization of logical equivalence and logical distance, and show some adequacy and expressivity results. We adopt coalgebraic modal logic and fix a predicate on the "truth value" object, which we call an expressivity situation. We define two concepts from this: the fibrational logical equivalence and the codensity bisimilarity. The fibrational logical equivalence is a generalization of logical equivalence and is defined through modal logic. The codensity bisimilarity is defined through the codensity lifting, using part of the expressivity situation as the parameter of codensity lifting. We define adequacy and expressivity as the comparison between the fibrational logical equivalence and the codensity bisimilarity. Here adequacy turns out to be implied automatically. On the other hand, to prove expressivity, we need to investigate "observation" arrows in more depth. By abstracting from existing approximation arguments, we introduce approximating families and use the notion to formulate our expressivity result. We instantiate this to recover a few known expressivity results for bisimulation relations and behavioral distances. We also define a new kind of codensity bisimilarity called bisimulation uniformity. We use a known Stone-Weierstrass-type theorem for uniform spaces to derive an expressive modal logic for bisimulation uniformity.

Along with these main results, we show another technical result about codensity lifting itself. It concerns a technical condition called fiberedness, and we prove a sufficient condition for a codensity lifting to be fibered. To formulate and prove the result, we define the notion of a c-injective object. The result has a consequence also for codensity bisimilarities: it implies the reflection of the codensity bisimilarity by coalgebra morphisms. We identify c-injective objects in some cases, for example, complete lattices in the fibration of preorders and continuous lattices in the fibration of topologies. We also prove the fiberedness of several codensity lifting and, thus, the reflection of the corresponding codensity bisimilarities.

Contents

1. Introduction

1.1. Coalgebra: a Theory of Observable Behaviors

Computer programs work as we write them, not necessarily as we expect. One approach to overcome this gap is to *verify* the systems so that we can make sure that they meet our requirements. Abstract mathematical methods are often useful for the purpose, but before that, we have to *model* the target system by some mathematical structure.

Coalgebra [60] is one of such mathematical structure with a broad scope of application. It is defined in terms of the theory of *categories and functors*. Given a category $\mathbb{C}$ and an endofunctor $B\colon \mathbb{C} \to \mathbb{C}$, a B-coalgebra is defined as an arrow $c\colon X \to BX$ (Definition 2.1.1). This simple definition includes many kinds of state-transition systems as special cases, e.g., Kripke frame (and model), Markov chain (and process), and (deterministic and non-deterministic) automata.

A salient feature shared by all kinds of systems is the contrast between *observable* transitions and *hidden* states. In order to extract meaningful information from a coalgebra, we have to define construct mathematical structures so that they do not depend on the hidden, internal change of states. Technically, this requirement can be formulated using *coalgebra morphisms*. A coalgebra morphism from $c\colon X \to BX$ to $d\colon Y \to BY$ is an arrow $f\colon X \to Y$ such that $d \circ f = Bf \circ c$ (Definition 2.1.5). Since any coalgebra morphism preserves "observable" transitions, a meaningful behavioral structure should also be preserved by such a morphism.

As an example, let us focus on a question: which states behave the same? *Bisimilarity* [55, 57] is one of the notions to define such equivalence. (For an introduction, see, e.g., [62].) We sketch the idea in the case where $\mathbb{C} = \mathbf{Set}$ and $B = \Sigma \times (-)$. In this case, B-coalgebras are deterministic LTSs. Consider a coalgebra $c\colon X \to \Sigma \times X$ and define $l\colon X \to \Sigma$ and $n\colon X \to X$ by $(l(x), n(x)) = c(x)$. The point here is the following observation: if $x, y \in X$ behave the same, then $l(x) = l(y)$ must hold, and $n(x)$ and $n(y)$ must behave the same. This is almost the definition of bisimilarity: the bisimilarity relation is the greatest binary relation $\sim \subseteq X \times X$ that satisfies

$$x \sim y \implies l(x) = l(y) \wedge n(x) \sim n(y).$$

It is preserved by any coalgebra morphism.

For other functors B, the idea is roughly the same: in a coalgebra $c\colon X \to BX$, for $x, y \in X$ to behave the same, $c(x)$ and $c(y)$ must behave the same. To define bisimilarity precisely, however, we have to turn a relation $R \subseteq X \times X$ into $R' \subseteq BX \times BX$.

1.2. Fibrational Coinduction: Obtaining Information from Transition

An elegant way to formulate this is the following: bundle binary relations on all sets into one $\mathbf{CLat}_\sqcap$-*fibration* and use *functor lifting* as in [32]. We give ideas on them here. The precise definitions are in Sections 2.3 and 2.4.

First, we gather all pairs (X, R) of a set X and a binary relation $R \subseteq X \times X$ into one category $\mathbf{ERel}$ (Example 2.3.13). It comes with a forgetful functor $U: \mathbf{ERel} \to \mathbf{Set}$. (This is a *fibration*.) Any binary relation R on X is sent to X by U; placing the things vertically, R is "above" X. Now let us assume that there exists a functor $\dot{B}: \mathbf{ERel} \to \mathbf{ERel}$ satisfying $U \circ \dot{B} = B \circ U$. This means that any binary relation R on X is sent to one on BX:

$$\begin{tikzcd} \mathbf{ERel} \arrow[r, "\dot{B}"'] \arrow[d, "U"'] & \mathbf{ERel} \arrow[d, "U"'] \\ \mathbf{Set} \arrow[r, "FB"'] & \mathbf{Set} \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, mapsto] \arrow[d, dotted, no head] & \dot{B}R \arrow[d, dotted, no head] \\ X \arrow[r, mapsto] & BX \end{tikzcd}$$

(This means that the functor $\dot{B}$ is a *lifting* of B along U.) The functor $U: \mathbf{ERel} \to \mathbf{Set}$ has an important structure: for any $f: Y \to X$ and a relation R on X, we can obtain a relation f^*R on Y in a canonical way:

$$f^*R = \{(y, y') \in Y \times Y | (f(y), f(y')) \in R\}.$$

(This is called *reindexing* or *pullback*.) By using these, we can define the bisimulation relation on $c: X \to BX$ as the greatest fixed point of $f^* \circ \dot{B}$.

This procedure works for any $\mathbf{CLat}_\sqcap$-fibration $p: \mathbb{E} \to \mathbb{C}$ and the resulting object is preserved by any coalgebra morphism. Thus,

Another advantage of this approach is that we can readily generalize this to other "bisimilarity-like" notions. For example, by changing the fibration to $\mathbf{PMet}_\top \to \mathbf{Set}$ (Example 2.3.12), one can define a *behavioral (pseudo)metric* [5].

1.3. Codensity Lifting: Explicit Modelling of Observation

Now we know that a functor lifting induces a bisimilarity-like notion. Then, how can we obtain a functor lifting? Moreover, so far we have no "observation" modelled as a mathematical object, although we have been talking about "observable" behaviors.

In this thesis, we solve these by *codensity lifting*, a scheme to obtain a functor lifting. It is first introduced in [41] for monads using *codensity monad* construction [52]. It is later extended to general endofunctors by Sprunger et al. [66]. The construction is parametrized in a set of data called a *lifting parameter*. By changing lifting parameters, a broad class of functor liftings can be represented as codensity liftings.

As mentioned in the last section, we can define a bisimilarity-like notion using codensity lifting. It is called *codensity bisimilarity* in [44, Sections III and VI]. This is the main object in this thesis. Unlike the original framework of fibrational coinduction, our codensity framework explicitly models "observations" as arrows in the base category. This enables us to connect codensity bisimilarity to infinite games and modal logic systems. In this thesis, we will pursue this triality.

1.4. Thesis Outline

The following diagram shows the dependencies between the chapters and sections of this thesis:

$$\begin{tikzcd}
 & & \text{Section 3.3} \\
\text{Chapter 2} \arrow[r] & \text{Sections 3.1 and 3.2} \arrow[ur] \arrow[r] \arrow[dr] & \text{Chapter 4} \\
 & & \text{Chapter 5}
\end{tikzcd}$$

As can be seen from the diagram, Chapter 2 and Sections 3.1 and 3.2 lay down the technical ground and Section 3.3 and Chapters 4 and 5 develop on it. After those, we conclude with Chapter 6. Some brief explanations of Chapters 2 to 5 follow. This thesis is based on the published papers [44, 45, 47, 46][1]. In the following, the original paper of each part is also mentioned.

Chapter 2 Preliminaries In this chapter we review the technical preliminaries mentioned in Sections 1.1 and 1.2. Coalgebra is introduced in Section 2.1. After that, to lay down the ground for fibrational coinduction, some fixed-point theorems are reviewed in Section 2.2 and the basics of $\mathbf{CLat}_\sqcap$-fibrations are explained in Section 2.3. Using these, Section 2.4 shows the framework of fibrational coinduction. In the framework of fibrational coinduction, the input is summarized as follows:

$$\begin{tikzcd}
\mathbb{E} \arrow[r, "\dot{B}"] \arrow[d, "p"] & \mathbb{E} \arrow[d, "p"] \\
\mathbb{C} \arrow[r, "B"] & \mathbb{C}
\end{tikzcd} \qquad\qquad X \xrightarrow{c} BX \tag{1.1}$$

Here, the functor $B\colon \mathbb{C} \to \mathbb{C}$ is the type of system, the coalgebra $c\colon X \to BX$ is the model of the target system, the $\mathbf{CLat}_\sqcap$-fibration $p\colon \mathbb{E} \to \mathbb{C}$ is the "form of information" we need, and the lifting $\dot{B}\colon \mathbb{E} \to \mathbb{E}$ specifies "how to extract $\mathbb{E}$-information from B-behaviors". Using these, we can obtain the greatest fixed point $\nu(c^* \circ \dot{B})$ of $c^* \circ \dot{B}$, which can be thought of as "what we can know from the B-behaviors of c through $\dot{B}$". If we

[1]The paper [46] is the journal version of the conference paper [44].

let $P = \nu(c^* \circ \dot{B})$, the situation is as follows:

$$\begin{tikzcd} \mathbb{E} \arrow[r, "\dot{B}"] \arrow[d, "p"] & \mathbb{E} \arrow[d, "p"] \\ \mathbb{C} \arrow[r, "B"] & \mathbb{C} \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, "=_\nu"] \arrow[dr, dotted, no head] & c^*\dot{B}P \arrow[r, dotted] \arrow[d, dotted, no head] & \dot{B}P \arrow[d, dotted, no head] \\ & X \arrow[r, "c"] & BX \end{tikzcd} \tag{1.2}$$

Most of the contents of this chapter is taken from [45, 46], but some explanations are added.

Chapter 3 Codensity Lifting This chapter develops the theory of *codensity lifting* and *codensity bisimilarity*. It is divided into two parts: one is Sections 3.1 and 3.2 and the other is Section 3.3.

The first part defines codensity lifting (Definitions 3.1.1 and 3.3.9) and codensity bisimilarity (Definition 5.2.9) and shows some examples of them. Codensity lifting is a method to obtain a functor lifting. It is introduced in [41] for monads and extended to general endofunctors in [66]. The input of codensity lifting (one-parameter form, as defined in Section 3.1) can be summarized as follows:

$$\begin{tikzcd} \mathbb{E} \arrow[d, "p"] \\ \mathbb{C} \arrow[loop left, "B"] \end{tikzcd} \qquad \begin{tikzcd} & \mathbf{\Omega} \arrow[d, dotted, no head] \\ B\Omega \arrow[r, "\tau"] & \Omega \end{tikzcd} \tag{1.3}$$

Then codensity lifting gives a lifting $B^{\mathbf{\Omega},\tau} \colon \mathbb{E} \to \mathbb{E}$. For each $P \in \mathbb{E}_X$, the definition of $B^{\mathbf{\Omega},\tau}P$ depends on the arrows $k \colon P \to \mathbf{\Omega}$ in $\mathbb{E}$, which can be regarded as "observations" on P. The situation is depicted by the following:

$$\begin{tikzcd} \mathbb{E} \arrow[d, "p"] \\ \mathbb{C} \arrow[loop left, "B"] \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, "k"] \arrow[d, dotted, no head] & \mathbf{\Omega} \arrow[d, dotted, no head] \\ X \arrow[r, "p(k)"] & \Omega \end{tikzcd} \qquad \begin{tikzcd} \big(\tau \circ B\big(p(k)\big)\big)^*\mathbf{\Omega} \arrow[rr, dotted] \arrow[d, dotted, no head] & & \mathbf{\Omega} \arrow[d, dotted, no head] \\ BX \arrow[r, "B(p(k))"] & B\Omega \arrow[r, "\tau"] & \Omega. \end{tikzcd}$$

Using $\big(\tau \circ B\big(p(k)\big)\big)^*\mathbf{\Omega}$ for each k, the codensity lifting is defined (Definition 3.1.1). This explicit and "observation-based" construction enables us to seek connections to games and modal logic, which are the theme of Chapters 4 and 5. Plugging in the codensity lifting $B^{\mathbf{\Omega},\tau}$ to fibrational coinduction (1.1) gives the codensity bisimilarity $\nu(c^* \circ B^{\mathbf{\Omega},\tau})$. Letting $P = \nu(c^* \circ B^{\mathbf{\Omega},\tau})$ yields the following summary, which is an extension of (1.2):

$$\begin{tikzcd} \mathbb{E} \arrow[r, "B^{\mathbf{\Omega},\tau}"] \arrow[d, "p"] & \mathbb{E} \arrow[d, "p"] \\ \mathbb{C} \arrow[r, "B"] & \mathbb{C} \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, "=_\nu"] \arrow[dr, dotted, no head] & c^*B^{\mathbf{\Omega},\tau}P \arrow[r, dotted] \arrow[d, dotted, no head] & B^{\mathbf{\Omega},\tau}P \arrow[d, dotted, no head] \\ & X \arrow[r, "c"] & BX \end{tikzcd} \qquad \begin{tikzcd} & \mathbf{\Omega} \arrow[d, dotted, no head] \\ B\Omega \arrow[r, "\tau"] & \Omega \end{tikzcd} \tag{1.4}$$

Most content of Sections 3.1 and 3.2 is taken from [46], while others are taken from [45] or newly written. As has been mentioned, the technical developments in this part are from [41, 66].

The second part, Section 3.3, introduces the main contribution of [45]. Here, we focus on a technical condition, *fiberedness*, of codensity lifting. The condition is about the following situation, involving the pullback operations and $B^{\mathbf{\Omega},\tau}$ itself:

$$\begin{tikzcd}
\mathbb{E} \arrow[loop left, "B^{\mathbf{\Omega},\tau}"] \arrow[d, "p"] \\
\mathbb{C} \arrow[loop left, "B"]
\end{tikzcd}
\qquad
\begin{tikzcd}
f^*Q \arrow[r, dotted] \arrow[d, dash, dotted] & Q \arrow[d, dash, dotted] \\
X \arrow[r, "f"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
B^{\mathbf{\Omega},\tau}(f^*Q) \arrow[r, "\sqsubseteq"] \arrow[dr, dash, dotted] & (Bf)^*B^{\mathbf{\Omega},\tau}Q \arrow[r, dotted] \arrow[d, dash, dotted] & B^{\mathbf{\Omega},\tau}Q \arrow[d, dash, dotted] \\
& BX \arrow[r, "Bf"] & BY
\end{tikzcd}$$

If the inequality $B^{\mathbf{\Omega},\tau}(f^*Q) \sqsubseteq (Bf)^*B^{\mathbf{\Omega},\tau}Q$ is upgraded to an equality $B^{\mathbf{\Omega},\tau}(f^*Q) = (Bf)^*B^{\mathbf{\Omega},\tau}Q$ for all f and Q, then the lifting is called fibered.

Since codensity lifting is defined by using arrows to $\mathbf{\Omega}$, it is natural to think that its fiberedness can be seen from some condition on $\mathbf{\Omega}$. It turns out to be true; if $\mathbf{\Omega}$ is *c-injective*, then $B^{\mathbf{\Omega},\tau}$ is fibered. The term "c-injective" means "injective with respect to Cartesian arrows" and it is about the following situation:

$$\begin{tikzcd}
\mathbb{E} \arrow[d, "p"] \\
\mathbb{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
f^*Q \arrow[r, dotted, "\bar{f}"'] \arrow[rr, bend left, "g"] \arrow[d, dash, dotted] & Q \arrow[r, "\exists h"'] \arrow[d, dash, dotted] & \mathbf{\Omega} \arrow[d, dash, dotted] \\
X \arrow[r, "f"] & Y & \Omega
\end{tikzcd}$$

If, for each g in this diagram, there exists an h making the upper triangle commute, then $\mathbf{\Omega}$ is called c-injective. This result is not about coalgebra, so it is not in the main thread of this thesis, but it does reflect our central strategy in this thesis: investigate the "observations", and obtain a result on codensity lifting and codensity bisimilarity.

Summarizing, in Section 3.3,

- We define *c-injective objects* in $\mathbf{CLat}_\sqcap$-fibrations.
- We identified c-injective objects in several $\mathbf{CLat}_\sqcap$-fibrations and show some connections between known notions and c-injectiveness.
- We show that, if $\mathbf{\Omega}$ is c-injective, then the codensity lifting $B^{\mathbf{\Omega},\tau}$ is fibered.
- Using the result on codensity lifting, we show that several functor liftings are fibered.

The content of Section 3.3 is taken from [45].

Chapter 4 Codensity Games for Bisimilarity In this chapter we seek game characterizations of codensity bisimilarity. Concretely, we construct a *safety game*, which is a certain kind of two-player infinite game, from the parameters of codensity lifting.

Indeed, a safety game characterizing the conventional bisimilarity is well-known. It has two players which we call Duplicator and Spoiler. Duplicator's aim is to show that two states are bisimilar, while Spoiler wants to disprove it. Any infinite play is defined to won by Duplicator, which reflects the coinductive nature of bisimilarity.

Our codensity game follows the same framework, so the problem is to give appropriate game arena and moves. The point is that codensity bisimilarity is defined by using "observations". Thus, the moves of Spoiler should be "observations", which are, mathematically, arrows from the state space $X \to \Omega$.

Pursuing this line of idea, we obtain the following game for the situation of (1.4):

Table 4.6.: Untrimmed codensity bisimilarity game (repeated from page 55)

position	player	possible moves
$P \in \mathbb{E}_X$	Spoiler	$k \in \mathbb{C}(X, \Omega)$ s.t. $\tau \circ Fk \circ c : (X, P) \dot{\not\to} (\Omega, \mathbf{\Omega})$
$k \in \mathbb{C}(X, \Omega)$	Duplicator	$P' \in \mathbb{E}_X$ s.t. $k : (X, P') \dot{\not\to} (\Omega, \mathbf{\Omega})$

In the above game, Duplicator needs to choose an object of $\mathbb{E}_X$ every time. In many examples it is not very intuitive. For instance, in the game for the conventional bisimilarity, Duplicator has to choose one equivalence relation; intuitively it should be enough to choose a pair of states and claim "these are bisimilar". To solve this, we introduce a refinement called *trimmed codensity bisimilarity game*.

The technical point here is to formulate the following claim in fibrational terms: an equivalence relation is determined by which pairs of states are related. For this aim, we defined *fibered separator*. Using this terminology, the previous claim on equivalence relation can be rephrased to: the two-point set $2 \in \mathbf{Set}$ is a fibered separator of the $\mathbf{CLat}_\sqcap$-fibration $\mathbf{EqRel} \to \mathbf{Set}$ of equivalence relations. For a $\mathbf{CLat}_\sqcap$-fibration with a fibered separator, we defined *trimmed codensity bisimilarity game*.

Summarizing, in Chapter 4,

- Using the parameters of codensity lifting, we define a safety game called the (untrimmed) *codensity bisimilarity game* and show that it correctly characterizes the codensity bisimilarity.
- For $\mathbf{CLat}_\sqcap$-fibrations having a special kind of object called *fibered separator*, we define another game called the *trimmed codensity bisimilarity game*, which has a smaller game arena.
- We show several examples of codensity bisimilarities and its corresponding codensity bisimilarity games.

The contents of this chapter is taken from the joint work [46] with Shin-ya Katsumata, Nick Hu, Bartek Klin, Samuel Humeau, Clovis Eberhart, and Ichiro Hasuo.

Chapter 5 Expressivity of Modal Logic for Codensity Bisimilarity In this chapter we investigate connections between modal logic and codensity bisimilarity. For the conventional bisimilarity relation, it is known that, for finitely branching LTSs, the Hennessy–Milner modal logic is adequate; it means that bisimilar states satisfy exactly the same set of modal formulas. For finitely branching LTSs, it is also expressive; that is, two states are bisimilar if they agree on the truth values of all modal formulas. A similar adequacy and expressivity result is known for the behavioral distance on a Markov chain; in this case, the modal logic is real-valued and the distance of two states turns out to be equal to the supremum of the differences of the truth values, where all modal formulas are considered. Our aim here is to seek a joint generalization of these expressivity results.

To formulate such a generalization, we use the framework of fibrational coinduction (1.2) and added to it a variation of *coalgebraic modal logic*. On a fixed "truth values" object $\Omega \in \mathbb{C}$, the logic interprets propositional connectives and modal operators by arrows in the form of $f\colon \Omega^n \to \Omega$ and $\tau\colon B\Omega \to \Omega$, respectively. To each formula φ, an arrow $[\![\varphi]\!]_c\colon X \to \Omega$ from the state space to the truth values object is assigned as its interpretation. The behaviors of the target system $c\colon X \to BX$ are reflected in the interpretation of the modality: if $\heartsuit$ is the (syntactical) unary modality, the interpretation of $\heartsuit\varphi$ is given by:

$$\mathbb{C} \qquad \begin{tikzcd} X \arrow[r, "c"'] \arrow[rr, bend left, "[\![\heartsuit\varphi]\!]"] & BX \arrow[r, "B[\![\varphi]\!]"'] & B\Omega \arrow[r, "\tau"'] & \Omega. \end{tikzcd}$$

Connecting this with the $\mathbf{CLat}_\sqcap$-fibration $p\colon \mathbb{E} \to \mathbb{C}$, we can jointly generalize logical equivalence and logical distance; we fix an object $\boldsymbol{\Omega}$ above Ω and consider the following pullback for each formula φ:

$$\begin{tikzcd} \mathbb{E} \arrow[d, "p"] \\ \mathbb{C} \end{tikzcd} \qquad \begin{tikzcd} {[\![\varphi]\!]_c^*\boldsymbol{\Omega}} \arrow[r, dotted] \arrow[d, dash, dotted] & \boldsymbol{\Omega} \arrow[d, dash, dotted] \\ X \arrow[r, "{[\![\varphi]\!]_c}"] & \Omega. \end{tikzcd}$$

Then we form the meet $\mathsf{LE}(c) = \bigsqcap_\varphi [\![\varphi]\!]_c^*\boldsymbol{\Omega}$ and we call it *fibrational logical equivalence*. Using this, we can also define adequacy and expressivity by comparing $\mathsf{LE}(c)$ and the coinductive predicate $\nu(c^* \circ \dot{B})$.

To prove these generalized adequacy and expressivity, we have to investigate both the modal logic and the functor lifting $\dot{B}$. However, codensity lifting turns out to be a helpful gadget here: it constructs a functor lifting from the data used to define $\mathsf{LE}(c)$, namely, Ω, $\boldsymbol{\Omega}$, and τ. Thus, the input of our framework here is the following:

$$\begin{tikzcd} \mathbb{E} \arrow[d, "p"] \\ \mathbb{C} \arrow[loop left, "B"] \end{tikzcd} \qquad X \xrightarrow{c} BX \qquad \Omega^n \xrightarrow{f} \Omega \qquad \begin{tikzcd} & \boldsymbol{\Omega} \arrow[d, dash, dotted] \\ B\Omega \arrow[r, "\tau"] & \Omega \end{tikzcd}$$

From these the codensity bisimilarity $\nu(c^* \circ B^{\boldsymbol{\Omega},\tau})$ and the fibrational logical equivalence $\mathsf{LE}(c)$ are defined and compared. In this situation, adequacy is obtained automatically.

Expressivity is much harder to prove and we need to "approximate" the non-logical observations with the logical observations (i.e. interpretations of modal formulas). We abstract the essence of the approximation arguments from the literature and devised a notion of *approximating family*. Using this notion, we obtained expressivity theorems.

Summarizing, in Chapter 5,

- Adopting coalgebraic modal logic, we formulate general fibrational forms of adequacy and expressivity of modal logic, which are applicable both logical relations and logical distances.

- In that fibrational framework, we prove adequacy with respect to codensity bisimilarity without any restriction on the modal logic system.
- Abstracting from the existing expressivity arguments, we single out a notion that we call *approximating family* and we prove expressivity under certain conditions involving it.

The contents of this chapter is taken from the joint work [47] with Shin-ya Katsumata, Clemens Kupke, Jurriaan Rot, and Ichiro Hasuo. Helpful comments from Bart Jacobs are gratefully acknowledged.

2. Preliminaries

As mentioned in Chapter 1, we build our theory based on two preceding theories: *universal coalgebra* and *fibrational coinduction*. In this chapter, we recall these two to the extent that we need. For universal coalgebra, the part we need is rather small; we review it in Section 2.1. For fibrational coinduction, we have to go through a few more topics: *fixed-point theorems* (Section 2.2) and $\mathbf{CLat}_\sqcap$*-fibrations* (Section 2.3). After those, we proceed to fibrational coinduction in Section 2.4.

We assume some knowledge of *category theory*, but the full content of the standard reference [54] is not needed. The basic definitions and theorems, e.g., those in Leinster [53], are enough. Another thing we require is some basic notions of *order theory*, like the definitions of meet, join, and complete lattice.

2.1. Coalgebra

In this section we review the theory of *universal coalgebra*. It is a joint generalization of process theory and automata theory based on category theory. Here we just review the basic notions like coalgebra morphisms and behavioral equivalence. For a more thorough introduction to this topic, see the standard references like [37, 60].

In this theory, systems are modeled as *coalgebras*:

Definition 2.1.1 (B-coalgebra)**.** Let $\mathbb{C}$ be a category and $B\colon \mathbb{C} \to \mathbb{C}$ be a functor. A B*-coalgebra* is a pair (X, c) of an object $X \in \mathbb{C}$ and a $\mathbb{C}$-arrow $c\colon X \to BX$; this coalgebra is often denoted simply by $c\colon X \to BX$.

Intuitively, X is the space of states and $c\colon X \to BX$ describes the transition of each state. The role of the functor B here is a bit less obvious; it specifies the kind of transitions that we allow, but we need a few examples to make it clear.

Example 2.1.2 (Kripke frame)**.** Let **Set** be the category of sets and maps and $\mathcal{P}\colon \mathbf{Set} \to \mathbf{Set}$ be the (covariant) powerset functor. Then a $\mathcal{P}$-coalgebra $c\colon X \to \mathcal{P}X$ corresponds to a *Kripke frame*: indeed, for such a c, defining the accessibility relation by

$$R_c = \{(x, x') \mid x' \in c(x)\}$$

yields a Kripke frame (X, R_c) and any Kripke frame can be represented in this form.

Example 2.1.3 (nondeterministic automaton)**.** Let $2 = \{\top, \bot\}$ be the two-point set, Σ be any set, and $N_\Sigma\colon \mathbf{Set} \to \mathbf{Set}$ be the functor which sends each $X \in \mathbf{Set}$ to $2 \times (\mathcal{P}X)^\Sigma$. Then a N_Σ-coalgebra $c\colon X \to N_\Sigma X$ corresponds to a *nondeterministic automaton* for an alphabet Σ (without initial state): for each $x \in X$, if we decompose $c(x)$ as $c(x) =$

$(c_1(x), c_2(x))$ where $c_1(x) \in 2$ and $c_2(x) \in (\mathcal{P}X)^\Sigma$, we can interpret $c_1(x)$ as defining whether x is accepting and $c_2(x)$ as the transitions from x.

Example 2.1.4 (Markov chain)**.** Let $\mathcal{D}_{\leq 1} \colon \mathbf{Set} \to \mathbf{Set}$ be a discrete subdistribution functor defined by, for each $X \in \mathbf{Set}$, $\mathcal{D}_{\leq 1}(X) = \{p \colon X \to [0,1] \mid \sum_{x \in X} p(x) \leq 1\}$ and, for each $f \colon X \to Y$, $p \in \mathcal{D}_{\leq 1}(X)$, and $y \in Y$, $\mathcal{D}_{\leq 1}(f)(p)(y) = \sum_{f(x)=y} p(x)$. Then a $\mathcal{D}_{\leq 1}$-coalgebra $c \colon X \to \mathcal{D}_{\leq 1}X$ corresponds to a (time-homogeneous) *Markov chain* (with discrete time and discrete space): the transition probability from x to x' can be extracted as $c(x)(x') \in [0,1]$.

In Example 2.1.2 above, taking $B = \mathcal{P}$ in Definition 2.1.1 enables a B-coalgebra to have *nondeterministic* branching. On the other hand, in Example 2.1.4, we take $B = \mathcal{D}_{\leq 1}$ and a B-coalgebra has *probabilistic* transitions. This is the role that B plays in Definition 2.1.1: specifying a functor B means specifying possible transition patterns in B-coalgebras.

Note that, in Example 2.1.3, the information of accepting states and the external input of elements of Σ are also encoded in the functor $N_\Sigma = 2 \times (\mathcal{P}\_)^\Sigma$. This means that, in coalgebraic modelling, inputs and outputs of a system is also regarded as a part of "transition".

For a fixed functor B, the B-coalgebras form a category and its morphisms are defined as follows:

Definition 2.1.5 (coalgebra morphism)**.** Assume the setting of Definition 2.1.1. Let $c \colon X \to BX$ and $d \colon Y \to BY$ be B-coalgebras. A *coalgebra morphism* from c to d is a $\mathbb{C}$-arrow $f \colon X \to Y$ such that $d \circ f = Bf \circ c$.

Intuitively, a coalgebra morphism preserves any transition. Now recall that "transition" here includes all inputs and outputs. From this, we may speculate that a coalgebra morphism actually preserves any "observable" information. Indeed, in many cases it is true; an illustrative example is the following:

Fact 2.1.6. *Assume the setting of Example 2.1.3. Let $c \colon X \to N_\Sigma X$ and $d \colon Y \to N_\Sigma Y$ be nondeterministic automata regarded as N_Σ-coalgebras. Let $f \colon X \to Y$ be a coalgebra morphism from c to d. Then for each $x \in X$, the recognized language of c starting from x and that of d starting from $f(x)$ are equal.* □

One general equivalence notion, *behavioral equivalence*, is defined along this line:

Definition 2.1.7 (behavioral equivalence [64, Definition 1])**.** Let $B \colon \mathbf{Set} \to \mathbf{Set}$ be a functor and $c \colon X \to BX$ be a B-coalgebra. The states $x, x' \in X$ are *behaviorally equivalent* if there is another B-coalgebra $d \colon Y \to BY$ and a coalgebra morphism $f \colon X \to Y$ such that $f(x) = f(x')$.

Remark 2.1.8. Another central equivalence notion of coalgebras is *bisimilarity*. While behavioral equivalence is based on *cospans* of coalgebra morphisms, bisimilarity is defined by *spans* of such morphisms. For their detailed comparison, see [67] (where behavioral equivalence is often referred to as *kernel-bisimulation*).

These are among the fundamental notions making universal coalgebra "the theory of observable behaviors". One of its central concern is to extract "observable information" from a system modeled as a coalgebra. However, what are "information" and "observation" here? In this thesis, we see corresponding mathematical structures for "information" (in Sections 2.2 and 2.3) and "observation" (in Chapter 3).

2.2. Fixed-Point Theorems

In this section we briefly recall some fixed-point theorems for complete lattices. We will use $\sqsubseteq$ for a partial order, and accordingly, meet and join are denoted $\bigsqcap$ and $\bigsqcup$, respectively. Each complete lattice has its greatest and least elements; they are denoted $\top$ and $\bot$. We use fixed points for modelling the process of collecting information. Let L be a complete lattice of "pieces of information" and $f\colon L \to L$ be a map chosen so that, intuitively, "if we know $x \in L$ now, then in the next step we will know $f(x)$". Then all the information we can obtain is represented as the *greatest fixed point* of f.

For a complete lattice, it is guaranteed that we can always do the same:

Fact 2.2.1. *Let L be a complete lattice and $f\colon L \to L$ be a monotone map. Then there exists the greatest fixed point of f.* □

Remark 2.2.2. A similar viewpoint is also taken in *domain theory*. In domain theory, "knowing nothing" is represented by the *least* element $\bot$. In this thesis, however, such a situation with no available information is modeled by the *greatest* element $\top$. This is because of the convention we adopt in Definition 2.3.2. See also Remark 2.3.7.

Just having a greatest fixed point is, actually, not very helpful. We need some more explicit characterizations of the fixed point: the *Knaster–Tarski* and *Cousot–Cousot* theorems.

Fact 2.2.3 (Knaster–Tarski [68]). *Let L be a complete lattice and $f\colon L \to L$ be a monotone map. Then the greatest fixed point of f exists and it is the greatest* pre-fixpoint *of f, i.e., the greatest element $x \in L$ such that $f(x) \sqsubseteq x$.* □

Fact 2.2.4 (Cousot–Cousot [17]). *Let L be a complete lattice and $f\colon L \to L$ be a monotone map. Using transfinite induction, let us define a sequence $(f_\alpha)_\alpha$ (indexed by an ordinal α) by the following:*

$$f_\alpha = \bigsqcap_{\beta<\alpha} f(f_\beta).$$

Then there is an ordinal α such that $f_\alpha = f_{\alpha+1}$ and, for such α, f_α is the greatest fixed point of f. □

Note that, in the above,

1. $f_0 = \top$, the greatest element of L,

2. $f_{\alpha+1} = f(f_\alpha)$ for any ordinal α, and

3. $f_\alpha = \bigsqcap_{\beta<\alpha} f_\beta$ for any limit ordinal α.

We also use *Kleene theorem*, a simplification of Fact 2.2.4.

Fact 2.2.5 (Kleene)**.** *In the setting of Fact 2.2.4, if f preserves the meet*

$$f_\omega = \bigsqcap_{n\in\omega} f_n,$$

then f_ω is the greatest fixed point of f. □

2.3. $\mathbf{CLat}_\sqcap$-Fibrations

Here we introduce $\mathbf{CLat}_\sqcap$-*fibrations*, as defined in [44]. We use them to model various "forms of information" like preorder, equivalence relation, and pseudometric. Assuming full knowledge of the theory of fibrations, we could define them as poset fibrations with fibered small meets. Instead, we give an explicit definition below. This is mainly because we need the notion of *Cartesian arrow*. For a comprehensive account of the theory of fibrations, the reader can consult, e.g., a book by Jacobs [36] or Hermida's thesis [31], but in the following, we do not assume any knowledge of fibrations.

We first define a fiber of a functor over an object. Basically, this is only considered in the case where the functor is a fibration.

Definition 2.3.1 (fiber)**.** Let $p\colon \mathbb{E} \to \mathbb{C}$ be a functor and $X \in \mathbb{C}$ be an object. The *fiber over X* is the subcategory of $\mathbb{E}$

- whose objects are $P \in \mathbb{E}$ such that $pP = X$ and
- whose arrows are $f\colon P \to Q$ such that $pf = \mathrm{id}_X$.

We denote it by $\mathbb{E}_X$.

Note that, if p is faithful, then each fiber is a thin category, i.e., a preordered class. The following definition of poset fibration is a special case of that in [36].

Definition 2.3.2 (cartesian arrow and poset fibration)**.** Let $p\colon \mathbb{E} \to \mathbb{C}$ be a faithful functor.

An arrow $f\colon P \to Q$ in $\mathbb{E}$ is *Cartesian* if the following condition is satisfied:

- For each $R \in \mathbb{E}$ and $g\colon R \to Q$, there exists $h\colon R \to P$ such that $g = f \circ h$ if and only if there exists $h'\colon pR \to pP$ such that $pg = pf \circ h'$.

The functor p is called a *poset fibration* if the following are satisfied:

- For each $X \in \mathbb{C}$, the fiber $\mathbb{E}_X$ is a poset. The order is denoted by $\sqsubseteq$. We define the direction so that $P \sqsubseteq Q$ holds if and only if there is an arrow $P \to Q$ in $\mathbb{E}_X$.
- For each $Q \in \mathbb{E}$ and $f\colon X \to pQ$, there exists an object $f^*Q \in \mathbb{E}_X$ and a Cartesian arrow $\dot{f}\colon f^*Q \to Q$ such that $p\dot{f} = f$. (Such f^*Q and $\dot{f}$ are necessarily unique.)

The map $Q \mapsto f^*Q$ turns out to be a monotone map from $\mathbb{E}_Y$ to $\mathbb{E}_X$. We call it the *pullback functor* and denote it by $f^* \colon \mathbb{E}_Y \to \mathbb{E}_X$.

There are a few different series of intuitions for poset fibrations. One of them is from mathematical logic. Indeed, several "logical" examples of poset fibrations can be found in, e.g., [36]. In these examples, an object $P \in \mathbb{E}_X$ represents a "predicate" and pullback functors model substitutions. We adopt this as a wording convension:

Notation 2.3.3. Let $\mathbb{E} \xrightarrow{p} \mathbb{C}$ be a poset fibration. An object $P \in \mathbb{E}_X$ in the fiber category $\mathbb{E}_X$ is often called a *predicate* over X.

Another series of intuition is more puremath-oriented: in this perspective, $P \in \mathbb{E}_X$ is regarded as an additional structure on X. From this perspective, arrows in the total category $\mathbb{E}$ has a good intuition: they are structure-preserving maps. To make it more visible, we adopt the following notation convension:

Notation 2.3.4. Let $\mathbb{E} \xrightarrow{p} \mathbb{C}$ be a poset fibration. A predicate P over X (that is, $P \in \mathbb{E}_X$) shall also be denoted by $(X, P) \in \mathbb{E}_X$.

Our intuition here is a hybrid one: we regard a predicate $P \in \mathbb{E}_X$ as a "piece of information". An arrow $(X, P) \to (Y, Q)$ is intuitively a map that is "consistent" w.r.t. the information given by P and Q. We call such arrow *decent*:

Definition 2.3.5 (decent map)**.** Let $\mathbb{E} \xrightarrow{p} \mathbb{C}$ be a $\mathbf{CLat}_\sqcap$-fibration, $f \colon X \to Y$ be an arrow in $\mathbb{C}$, $(X, P) \in \mathbb{E}_X$ and $(Y, Q) \in \mathbb{E}_Y$ be objects in the fibers. We say that f is *decent* from P to Q, or (P, Q)-*decent*, if there exists a (necessarily unique) arrow $\dot{f} \colon P \to Q$ in $\mathbb{E}$ such that $p\dot{f} = f$. We write $f \colon (X, P) \dot{\to} (Y, Q)$ in this case. We write $f \colon (X, P) \dot{\not\to} (Y, Q)$ if f is *not* decent.

From this "informational" perspective, pullback f^* models the act of getting information from an "observation" f. Decency and pullback are interconnected:

Proposition 2.3.6. *Let $p \colon \mathbb{E} \to \mathbb{C}$ be a poset fibration, $f \colon X \to Y$ be an arrow in $\mathbb{C}$ and $P \in \mathbb{E}_X$ and $Q \in \mathbb{E}_Y$ be objects in $\mathbb{E}$. Then f is (P, Q)-decent if and only if $P \sqsubseteq f^*Q$. Moreover, such g is Cartesian if and only if $P = f^*Q$.* □

Remark 2.3.7. As mentioned in Remark 2.2.2, in our framework, "knowing nothing" is modelled by $\top$. This can now be explained as follows. If Q corresponds to "knowing nothing", any arrow to Q is "consistent". Thus, any P and f satisfies $\dot{f} \colon P \to Q$, and $P \sqsubseteq f^*Q$. For this to hold, Q must be the largest element in the fiber.

We mention another basic property of pullback:

Proposition 2.3.8. *Let $p \colon \mathbb{E} \to \mathbb{C}$ be a poset fibration.*

- *For each $X \in \mathbb{C}$, $(\mathrm{id}_X)^* \colon \mathbb{E}_X \to \mathbb{E}_X$ is the identity functor.*
- *For each composable pair of arrows $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathbb{C}$, $(g \circ f)^* = f^* \circ g^*$ holds.* □

The following example is a good illustration of this hybrid perspective. A pseudometric is a mathematical structure on a set, but it can also be regarded as giving some information: $d(x,y) > \varepsilon$ means that x and y are different to the extent of ε.

Example 2.3.9 (pseudometric)**.** Let $\top$ be a positive real number or $+\infty$. Define a category $\mathbf{PMet}_\top$ as follows:

- Each object is a pair (X,d) of a set X and a $[0,\top]$-valued pseudometric $d\colon X\times X \to [0,\top]$. (A pseudometric is a metric without the condition $d(x,y)=0 \implies x=y$.)
- Each arrow from (X,d_X) to (Y,d_Y) is a nonexpansive map $f\colon X \to Y$. (f is nonexpansive if, for all x and $x' \in X$, $d_X(x,x') \geq d_Y(f(x),f(x'))$.)

The obvious forgetful functor $\mathbf{PMet}_\top \to \mathbf{Set}$ is a poset fibration. For each $X \in \mathbf{Set}$, the objects of the fiber $(\mathbf{PMet}_\top)_X$ are the pseudometrics on X. However, the order is reversed: in our notation, the order is defined by

$$(X,d_1) \sqsubseteq (X,d_2) \Leftrightarrow \forall x,x' \in X, d_1(x,x') \geq d_2(x,x').$$

An arrow $f\colon (X,d_X) \to (Y,d_Y)$ is Cartesian if and only if it is an isometry, i.e., $d_X(x,x') = d_Y(f(x),f(x'))$ holds for all x,x'. For $(Y,d_Y) \in \mathbf{PMet}_\top$ and $f\colon X \to Y$, the pullback $f^*(Y,d_Y)$ is the set X with the pseudometric $(x,x') \mapsto d_Y(f(x),f(x'))$.

In many cases we can gather pieces of information to obtain one larger piece of information. We model such situation by a $\mathbf{CLat}_\sqcap$*-fibration.*

Definition 2.3.10 ($\mathbf{CLat}_\sqcap$-fibration)**.** A poset fibration $p\colon \mathbb{E} \to \mathbb{C}$ is a $\mathbf{CLat}_\sqcap$*-fibration* if the following conditions are satisfied:

- Each fiber $\mathbb{E}_X$ is small and has small meets, which we denote by $\bigsqcap$.
- Each pullback functor f^* preserves small meets.

Note that, in the situation above, each fiber $\mathbb{E}_X$ is a complete lattice: the small joins can be constructed using small meets.

Proposition 2.3.11. *Let $\mathbb{E} \xrightarrow{p} \mathbb{C}$ be a $\mathbf{CLat}_\sqcap$-fibration.*

1. *Each arrow $f\colon X \to Y$ has its* pushforward *$f_*\colon \mathbb{E}_X \to \mathbb{E}_Y$, so that an adjunction $f_* \dashv f^*$ is formed. This is a consequence of Freyd's adjoint functor theorem; it makes p a* bifibration *[36].*
2. *The change-of-base [36, Lemma 1.5.1] of p along any functor $H : \mathbb{D} \to \mathbb{C}$ is also a $\mathbf{CLat}_\sqcap$-fibration.*
3. *If $\mathbb{C}$ is (co)complete, then the total category $\mathbb{E}$ is also (co)complete. This follows from [36, Proposition 9.2.1].* □

Example 2.3.12 (pseudometric)**.** The poset fibration $\mathbf{PMet}_\top \to \mathbf{Set}$ in Example 2.3.9 is a $\mathbf{CLat}_\sqcap$-fibration. Indeed, meets can be defined by sups of pseudometrics: if we let $(X, d) = \bigsqcap_{a \in A}(X, d_a)$, then

$$d(x, x') = \sup_{a \in A} d_a(x, x')$$

holds.

Example 2.3.13 (binary relations)**.** Define a category $\mathbf{ERel}$ of sets with an endorelation as follows:

- Each object is a pair (X, R) of a set X and a binary relation $R \subseteq X \times X$.
- Each arrow from (X, R_X) to (Y, R_Y) is a map $f \colon X \to Y$ preserving the relations; that is, we require f to satisfy $(x, x') \in R_X \implies (f(x), f(x')) \in R_Y$.

The obvious forgetful functor $\mathbf{ERel} \to \mathbf{Set}$ is a $\mathbf{CLat}_\sqcap$-fibration. For each $X \in \mathbf{Set}$, the fiber $\mathbf{ERel}_X$ is the complete lattice of subsets of $X \times X$.

An arrow $f \colon (X, R_X) \to (Y, R_Y)$ is Cartesian if and only if it reflects the relations, i.e., $(x, x') \in R_X \Leftrightarrow (f(x), f(x')) \in R_Y$ holds for all x, x'. For $(Y, R_Y) \in \mathbf{ERel}$ and $f \colon X \to Y$, the pullback $f^*(Y, R_Y)$ is the set X with the relation $\{(x, x') \in X \times X | (f(x), f(x')) \in R_Y\}$.

Define the following full subcategories of $\mathbf{ERel}$:

- The category $\mathbf{Pre}$ of preordered sets and monotone maps.
- The category $\mathbf{EqRel}$ of sets with an equivalence relation and maps preserving them.

The forgetful functors $\mathbf{Pre} \to \mathbf{Set}$ and $\mathbf{EqRel} \to \mathbf{Set}$ are also $\mathbf{CLat}_\sqcap$-fibrations.

In the following, we write Eq_I for the diagonal (equality) relation over a set I. It follows that Eq_I is the least element of $\mathbf{EqRel}_I$.

$\mathbf{CLat}_\sqcap$-fibrations are not necessarily "relation-like". There also is an example with a much more "space-like" flavor.

Example 2.3.14. The forgetful functor $\mathbf{Top} \to \mathbf{Set}$ from the category $\mathbf{Top}$ of topological spaces and continuous maps is a $\mathbf{CLat}_\sqcap$-fibration.

Example 2.3.15. The forgetful functor $\mathbf{Meas} \to \mathbf{Set}$ from the category $\mathbf{Meas}$ of measurable spaces (sets with a σ-algebra) and measurable maps is a $\mathbf{CLat}_\sqcap$-fibration.

We also use a few $\mathbf{CLat}_\sqcap$-fibrations over categories other than $\mathbf{Set}$. One is "the fibration of binary relations":

Definition 2.3.16 ($\mathbf{BRel} \to \mathbf{Set}^2$)**.** We define the category $\mathbf{BRel}$ as follows:

- An object is a triple $(X, Y, R \subseteq X \times Y)$ of two sets and a relation between them.
- An arrow from (X, Y, R) to (Z, W, S) is a pair $(f \colon X \to Z, g \colon Y \to W)$ of functions such that $(x, y) \in R$ implies $(f(x), g(y)) \in S$.

The forgetful functor $\mathbf{BRel} \to \mathbf{Set}^2$ is then a $\mathbf{CLat}_\sqcap$-fibration.

This can be used for modeling bisimulations between two different systems. See Section 4.8.2.

The other is "the fibration of equivalence relations on measurable spaces":

Definition 2.3.17 ($\mathbf{EqRel}_{\mathbf{Meas}} \to \mathbf{Meas}$)**.** We define the category $\mathbf{EqRel}_{\mathbf{Meas}}$ as follows:

- An object is a triple $(X, \Sigma_X, R \subseteq X \times X)$ of a set, a σ-algebra, and an equivalence relation.
- An arrow from (X, Σ_X, R) to (Y, Σ_Y, S) is a map $f\colon X \to Y$ that is measurable with respect to (Σ_X, Σ_Y) and preserves the equivalence relations.

The forgetful functor $\mathbf{EqRel}_{\mathbf{Meas}} \to \mathbf{Meas}$ is then a $\mathbf{CLat}_\sqcap$-fibration.

Yet another class of examples is given as follows: for any well-powered category $\mathbb{B}$ admitting small limits, the subobject fibration $\mathbf{Sub}(\mathbb{B}) \to \mathbb{B}$ of $\mathbb{B}$ is a $\mathbf{CLat}_\sqcap$-fibration. All the algebraic categories over $\mathbf{Set}$ and Grothendieck toposes satisfy these conditions of $\mathbb{B}$. We note, however, that the forgetful functors from algebraic categories over $\mathbf{Set}$ are rarely ($\mathbf{CLat}_\sqcap$-)fibrations.

Remark 2.3.18. A $\mathbf{CLat}_\sqcap$-fibration can equivalently be defined as a *topological functor* [33] with small fibers. Some papers like [26] use this as an alternative framework. Topological functors are a well-studied topic, and many examples and results are available; a good summary is found in [2]. Here we prefer the fibrational presentation, following works on coinductive predicates [32, 12, 29, 66, 50, 45].

2.4. Lifting and Fibrational Coinduction

Now we proceed to obtain information, modelled on a $\mathbf{CLat}_\sqcap$-fibration, from a coalgebra. To do so, we need *functor lifting*. In Section 1.2 we have seen that it is used to define bisimilarity, or more generally bisimilarity-like notions, as a way to turn a relation (or pseudometric, etc.) on X into one on BX. Here we review the formal definition in a restricted form that only considers $\mathbf{CLat}_\sqcap$-fibration. (Note that, usually it is defined more generally, and there are indeed applications of such general definition.)

Definition 2.4.1 (lifting of endofunctor)**.** Let $p\colon \mathbb{E} \to \mathbb{C}$ be a $\mathbf{CLat}_\sqcap$-fibration and $B\colon \mathbb{C} \to \mathbb{C}$ be a functor. A *lifting* of B along p is a functor $\dot{B}\colon \mathbb{E} \to \mathbb{E}$ such that $p \circ \dot{B} = B \circ p$ holds:

$$\begin{tikzcd} \mathbb{E} \arrow[r, "\dot{B}"'] \arrow[d, "p"'] & \mathbb{E} \arrow[d, "p"'] \\ \mathbb{C} \arrow[r, "B"'] & \mathbb{C}. \end{tikzcd}$$

Table 2.1.: $\mathbf{CLat}_{\sqcap}$-fibrations over $\mathbf{Set}$

fibration	predicate	decent map	$P \sqsubseteq Q$	$\bigsqcap P_i$
$\mathbf{Top} \to \mathbf{Set}$	topology	continuous func.	$P \supseteq Q$	generated from $\bigcup P_i$
$\mathbf{Meas} \to \mathbf{Set}$	σ-algebra	measurable func.	$P \supseteq Q$	generated from $\bigcup P_i$
$\mathbf{PMet}_1 \to \mathbf{Set}$	pseudometric	non-expansive func.	$\forall x, y. P(x,y) \geq Q(x,y)$	$(x,y) \mapsto \sup_i P_i(x,y)$
$\mathbf{ERel} \to \mathbf{Set}$	endorelation	relation preserving func.	$P \subseteq Q$	$\bigcap P_i$
$\mathbf{Pre} \to \mathbf{Set}$	preorder	monotone func.	$P \subseteq Q$	$\bigcap P_i$
$\mathbf{EqRel} \to \mathbf{Set}$	equivalence relation	relation preserving func.	$P \subseteq Q$	$\bigcap P_i$

Intuitively, using a functor lifting $\dot{B}$ of B, we can obtain $\mathbb{E}$-information from B-behaviors. The formal definition of this procedure is *predicate transformer*.

Definition 2.4.2 ((fibrational) predicate transformer $x^* \circ \dot{B}$)**.** In the setting of Definition 2.4.1, let $\dot{B}$ be a lifting of B, and $x\colon X \to BX$ be a B-coalgebra. We then obtain an endofunctor

$$x^* \circ \dot{B}\colon \mathbb{E}_X \to \mathbb{E}_X \text{ by the composite } \mathbb{E}_X \xrightarrow{\dot{B}} \mathbb{E}_{BX} \xrightarrow{x^*} \mathbb{E}_X.$$

The functor $x^* \circ \dot{B}$ is called the *predicate transformer* induced by $\dot{B}$ over the B-coalgebra x.

Definition 2.4.3 (coinductive predicate $\nu(x^* \circ \dot{B}) \in \mathbb{E}_X$, and invariant)**.** In the setting of Definition 2.4.2, the carrier of the final $x^* \circ \dot{B}$-coalgebra (if it exists) is called the *$\dot{B}$-coinductive predicate* over x. It is denoted by $\nu(x^* \circ \dot{B}) \in \mathbb{E}_X$.

An $x^* \circ \dot{B}$-coalgebra is called a *$\dot{B}$-invariant* over x.

The names in the above definition reflect the common reasoning principle for gfp specifications (such as safety), namely that an invariant underapproximates (and thus witnesses) the gfp specification. Each $\overline{B}$-invariant indeed witnesses the $\overline{B}$-coinductive predicate, in the sense that there is a unique morphism from the former to the latter.

We now define a technical condition on a functor lifting: *fiberedness*. This means that the lifting interacts well with the pullback structure of the fibration, but we first give a definition focusing on Cartesian arrows. Here we define it in a slightly more general way so that we can use them later (Section 3.3.2).

Definition 2.4.4 (fibered functor [36, Definition 1.7.1])**.** Let $p\colon \mathbb{E} \to \mathbb{C}$ and $q\colon \mathbb{F} \to \mathbb{D}$ be $\mathbf{CLat}_\sqcap$-fibrations. A *fibered functor* from p to q is a functor $\dot{B}\colon \mathbb{E} \to \mathbb{F}$ such that there is another functor $B\colon \mathbb{C} \to \mathbb{D}$ satisfying $q \circ \dot{B} = B \circ p$ and $\dot{B}$ sends each Cartesian arrow to a Cartesian arrow.

Note that, in the situation above, such B is uniquely determined by p, q, and $\dot{B}$. Now we see a characterization of fiberedness by means of pullback.

Proposition 2.4.5. *Let $p\colon \mathbb{E} \to \mathbb{C}$ and $q\colon \mathbb{F} \to \mathbb{D}$ be $\mathbf{CLat}_\sqcap$-fibrations and $\dot{B}\colon \mathbb{E} \to \mathbb{F}$ and $B\colon \mathbb{C} \to \mathbb{D}$ be functors satisfying $q \circ \dot{B} = B \circ p$. $\dot{B}$ is a fibered functor if and only if, for any $f\colon X \to Y$ in $\mathbb{C}$ and $P \in \mathbb{E}_Y$, $\dot{B}(f^*P) = (Bf)^*(\dot{B}P)$ holds.* $\square$

Fiberedness implies a notable feature of the coinductive predicate:

Proposition 2.4.6 (stability of coinductive predicate)**.** *Assume the setting of Definition 2.4.3. Assume that $\dot{B}$ is a fibered functor. Then, the coinductive predicate is stable under coalgebra morphisms: for any morphism of coalgebras f from (X, c) to (Y, d), we have $\nu(c^* \circ \dot{B}) = f^*\left(\nu(d^* \circ \dot{B})\right)$.*

Proof. Let $\Phi_c = c^* \circ \dot{B}$ and $\Phi_d = d^* \circ \dot{B}$ be the predicate transformers. Define a transfinite sequence $(\nu_\alpha \Phi_c)_{\alpha \text{ is an ordinal}}$ of elements of $\mathbb{E}_X$ by the following:

$$\nu_\alpha \Phi_c = \bigsqcap_{\beta < \alpha} \Phi_c \left(\nu_\beta \Phi_c \right).$$

Define another transfinite sequence $(\nu_\alpha \Phi_d)_{\alpha \text{ is an ordinal}}$ by a similar manner. By Fact 2.2.4, there is an ordinal γ such that $\nu_\gamma \Phi_c = \nu \Phi_c$ and $\nu_\gamma \Phi_d = \nu \Phi_d$. [1] Thus, it suffices to show the following claim:

Claim. *For any ordinal α, we have $\nu_\alpha \Phi_c = f^* \left(\nu_\alpha \Phi_d \right)$.*

We show this by transfinite induction on α. Assume the claim holds for all $\beta < \alpha$.

Using the assumption that f is a morphism of coalgebras, the fiberedness of $\dot{B}$, and the functoriality of pullback (Proposition 2.3.8), we have $f^* \circ \Phi_d = \Phi_c \circ f^*$. It implies the claim for α

$$\begin{aligned} f^* \left(\nu_\alpha \Phi_d \right) &= f^* \left(\bigsqcap_{\beta < \alpha} \Phi_d \left(\nu_\beta \Phi_d \right) \right) = \bigsqcap_{\beta < \alpha} f^* \left(\Phi_d \left(\nu_\beta \Phi_d \right) \right) \\ &= \bigsqcap_{\beta < \alpha} \Phi_c \left(f^* \nu_\beta \Phi_d \right) = \bigsqcap_{\beta < \alpha} \Phi_c \left(\nu_\beta \Phi_c \right) \\ &= \nu_\alpha \Phi_c. \end{aligned}$$

□

[1] This formulation differs slightly from the conventional one where successor and limit ordinals are distinguished, but the result also holds under this definition.

3. Codensity Lifting

In Chapter 2, we reviewed two preceding frameworks: *universal coalgebra* (Section 2.1) and *fibrational coinduction* (Section 2.4). Using these, one can define and manipulate "behavioral" structures on systems (i.e. bisimilarity relation and behavioral distance) in a unified manner. However, the latter needs a functor lifting, which is rather a large data in many cases, and it is hard to give one in general. Moreover, while we are talking about "behavioral" data, which depends only on "observable behaviors", these do not include a mathematical gadget modelling "observations".

In this chapter we focus on *codensity lifting*. Technically, it is a method for constructing a functor lifting from a small data. This reduces the hard problem of giving a whole functor lifting. Intuitively, codensity lifting is a generic way to obtain a functor lifting using "observations". This enables us to model "observations" in our framework explicitly and connect the fixed-point-based formulation to games and modal logic.

The chapter consists of two sections: in Section 3.1 we review the definition of codensity lifting and in Section 3.3 we discuss fiberedness of codensity lifting.

3.1. Codensity Lifting and Codensity Bisimilarity

We introduce *codensity lifting* and *codensity bisimilarity* based on [66]. These turn out to subsume many bisimilarity-like notions in the literature. The technical contents in Sections 3.1.1 and 3.1.2 are largely from [66];

3.1.1. Codensity Lifting

Definition 3.1.1 (codensity lifting (as in [44])). Let

- $p\colon \mathbb{E} \to \mathbb{C}$ be a $\mathbf{CLat}_{\sqcap}$-fibration,
- $B\colon \mathbb{C} \to \mathbb{C}$ be a functor,
- $\mathbf{\Omega} \in \mathbb{E}$ be an object above $\Omega \in \mathbb{C}$, and
- $\tau\colon B\Omega \to \Omega$ be a B-algebra.

Define a functor $B^{\mathbf{\Omega},\tau}\colon \mathbb{E} \to \mathbb{E}$, which is a lifting of B along p, by

$$B^{\mathbf{\Omega},\tau} P = \bigsqcap_{f \in \mathbb{E}(P,\mathbf{\Omega})} (B(pf))^* \tau^* \mathbf{\Omega}$$

for each $P \in \mathbb{E}$. The functor $B^{\mathbf{\Omega},\tau}$ is called a *codensity lifting* of B. Note that, for each $P \in \mathbb{E}$ and $f\colon P \to \mathbf{\Omega}$, the situation is as follows:

$$\begin{tikzcd} & & \mathbf{\Omega} \arrow[d, dotted, no head] \\ BpP \arrow[r, "B(pf)"'] & B\Omega \arrow[r, "\tau"'] & \Omega \end{tikzcd}$$

and we can indeed obtain the pullback $(B(pf))^*\tau^*\mathbf{\Omega}$.

We have given only the object part of $B^{\mathbf{\Omega},\tau}$ above, but the arrow part, if it is well-defined, should be determined uniquely since p is faithful. We give a proof that it is indeed well-defined. For each $f\colon P \to Q$, we need another arrow $g\colon B^{\mathbf{\Omega},\tau}P \to B^{\mathbf{\Omega},\tau}Q$ such that $pg = B(pf)$. By Proposition 2.3.6, it suffices to show the following proposition:

Proposition 3.1.2. *For any $f\colon P \to Q$, $B^{\mathbf{\Omega},\tau}P \sqsubseteq (B(pf))^*\left(B^{\mathbf{\Omega},\tau}Q\right)$ holds.*

Proof. By definition, the l.h.s. satisfies

$$B^{\mathbf{\Omega},\tau}P = \bigsqcap_{g\in\mathbb{E}(P,\mathbf{\Omega})} (B(pg))^*\tau^*\mathbf{\Omega}.$$

On the other hand, the r.h.s. satisfies

$$\begin{aligned}(B(pf))^*\left(B^{\mathbf{\Omega},\tau}Q\right) &= (B(pf))^*\left(\bigsqcap_{h\in\mathbb{E}(Q,\mathbf{\Omega})} (B(ph))^*\tau^*\mathbf{\Omega}\right)\\ &= \bigsqcap_{h\in\mathbb{E}(Q,\mathbf{\Omega})} (B(pf))^*(B(ph))^*\tau^*\mathbf{\Omega}\\ &= \bigsqcap_{h\in\mathbb{E}(Q,\mathbf{\Omega})} (B(p(h\circ f)))^*\tau^*\mathbf{\Omega}.\end{aligned}$$

Here, since $\{g \in \mathbb{E}(P,\mathbf{\Omega})\} \supseteq \{h\circ f \mid h \in \mathbb{E}(Q,\mathbf{\Omega})\}$ holds, we have

$$\bigsqcap_{g\in\mathbb{E}(P,\mathbf{\Omega})} (B(pg))^*\tau^*\mathbf{\Omega} \sqsubseteq \bigsqcap_{h\in\mathbb{E}(Q,\mathbf{\Omega})} (B(p(h\circ f)))^*\tau^*\mathbf{\Omega}.$$

This means $B^{\mathbf{\Omega},\tau}P \sqsubseteq (B(pf))^*\left(B^{\mathbf{\Omega},\tau}Q\right)$. $\square$

Let us elaborate on the above definition. Let $P \in \mathbb{E}_X$ and $X = pP$. The point is to regard an arrow $X \to \Omega$ in $\mathbb{C}$ as an "observation" on X and an object $P \in \mathbb{E}_X$ as "information" on X. Our goal is to obtain "information" on BX from that on X.

We begin with taking some $k\colon P \to \mathbf{\Omega}$ in $\mathbb{E}$. For such k, $p(k)\colon X \to \Omega$ can be seen as an "observation" on the space $X \in \mathbb{C}$. Here, $p(k)$ has to be decent from P to $\mathbf{\Omega}$. Intuitively, this means that the resulting "information" $(p(k))^*\mathbf{\Omega} \in \mathbb{E}_X$ of the "observation" $p(k)$ must be consistent with the information P on X we already have. For example, in

Example 3.1.4, the arrow $p(k)\colon X \to 2$, intuitively an "observation," corresponds to a subset of X. The resulting "information" $(p(k))^*\mathrm{Eq}_2 \in \mathbf{EqRel}_X$ of $p(k)$ is the induced equivalence relation

$$(p(k))^*\mathrm{Eq}_2 = \{(x, y) \in X^2 \mid p(k)(x) = p(k)(y)\},$$

and it must be "consistent" with the given equivalence relation $(X, R) \in \mathbf{EqRel}$, that is, each equivalence class of $(p(k))^*\mathrm{Eq}_2$ must be R-closed.

The "observation" $p(k)\colon X \to \Omega$ is simply an arrow, so we can apply the given functor $B\colon \mathbb{C} \to \mathbb{C}$ to it. The result is $B(p(k))\colon BX \to B\Omega$. To obtain an "observation" on BX, we have to compose it with some modality $\tau\colon B\Omega \to \Omega$. In Example 3.1.4, this process gives an "observation" $\Diamond \circ \mathcal{P}(p(k))\colon \mathcal{P}X \to 2$ on $\mathcal{P}X$, and it satisfies the following for each $S \in \mathcal{P}X$:

$$(\Diamond \circ \mathcal{P}(p(k)))(S) = \top \iff \exists x \in S.\ p(k)(x) = \top.$$

Note the existential quantification $\exists$ above. It is the part where the modality $\Diamond$ comes up.

Now that we have an "observation" on BX, we obtain "information" on BX by pullback. The following diagram is the summary of this situation:

$$\begin{tikzcd}
\mathbb{E} \arrow[d, "p"] & \big(\tau \circ B\big(p(k)\big)\big)^*\mathbf{\Omega} \arrow[rr, dotted] & & \mathbf{\Omega} \\
\mathbb{C} & B(pP) \arrow[r, "B(p(k))"'] & B\Omega \arrow[r, "\tau"'] & \Omega.
\end{tikzcd}$$

Finally, gathering all the "information" $(\tau \circ B(p(k)))^*\mathbf{\Omega}$ leads to the definition (Definition 5.2.8). In the setting of Example 3.1.4, the result of this process is the equivalence relation on $\mathcal{P}X$, defined for each $S, T \subseteq X$ by

$$\begin{aligned}
\forall k\colon X \to 2.\ \big(&(\forall (x, y) \in R.\ k(x) = k(y)) \\
&\implies \big((\exists x \in S.\ k(x) = \top) \iff (\exists x \in T.\ k(x) = \top)\big)\big).
\end{aligned}$$

It is equivalent to another more familiar definition, as described in Example 3.1.4.

One might wonder how codensity lifting is related with *codensity monad* [54, Exercise X.7.3]. The following proposition exhibits the relationship.

Proposition 3.1.3. *Let $p : \mathbb{E} \to \mathbb{C}$ be a $\mathbf{CLat}_\sqcap$-fibration, $B : \mathbb{C} \to \mathbb{C}$ be a functor and $(\mathbf{\Omega}, \tau)$ be a parameter of codensity lifting of F along p. Moreover we assume that $\mathbb{E}$ has powers [54, Section III.4] and p preserves them. For any $P \in \mathbb{E}$, $B^{\mathbf{\Omega},\tau}P$ coincides with the vertex of the following pullback:*

$$\begin{tikzcd}
\mathbb{E} \arrow[d, "p"] & B^{\mathbf{\Omega},\tau}P \arrow[r, dotted] & \mathbf{\Omega}^{\mathbb{E}(P,\mathbf{\Omega})} \\
\mathbb{C} & B(pP) \arrow[r, "\alpha_P"'] & \Omega^{\mathbb{E}(P,\mathbf{\Omega})}
\end{tikzcd}$$

where $\alpha_P = \langle \tau \circ B(p(k)) \rangle_{k \in \mathbb{E}(P,\mathbf{\Omega})}$ is the morphism obtained by the tupling of the power of Ω in $\mathbb{C}$.

In fact, codensity lifting of monads is first defined in terms of the above pullback [41]. The name "codensity lifting" comes from the fact that the above pullback involves the codensity monad $\mathbf{\Omega}^{\mathbb{E}(-,\mathbf{\Omega})}$.

Table 3.1 lists concrete examples of codensity liftings, with various fibrations p, functors B, and parameters $(\mathbf{\Omega}, \tau)$. Some of them coincide with known notions. For example, the entry 5 of the table says that the functor $(\mathcal{D}_{\leq 1})^{\mathbf{\Omega},\tau}$, with the designated $\mathbf{\Omega}$ and τ, carries a metric space (X, d) to the set $\mathcal{D}_{\leq 1}X$ equipped with the well-known Kantorovich metric $\mathcal{K}(d)$ induced by d. See (4.1).

Besides the functors listed in the table, there are some natural ways to systematically lift polynomial functors, by defining $\tau \colon F\Omega \to \Omega$ in an inductive manner—see, e.g., [11].

Example 3.1.4. Let us take a close look at the entry 4 of Table 3.1. There we codensity-lift the covariant powerset functor $\mathcal{P}$ along the $\mathbf{CLat}_{\sqcap}$-fibration $\mathbf{EqRel} \to \mathbf{Set}$. We use the parameter $((2, \mathrm{Eq}_2), \Diamond)$, where $\Diamond : \mathcal{P}2 \to 2$ is the *may-modality* defined by $\Diamond S = \top$ if and only if $\top \in S$.

We shall abbreviate $(2, \mathrm{Eq}_2)$ by Eq_2—a notational convention that is used throughout the paper.

Then $\mathcal{P}^{\mathrm{Eq}_2,\Diamond}(X, R)$ relates $S, T \in \mathcal{P}X$ if and only if

$$\forall k \colon X \to 2.\ \Big((\forall (x, y) \in R.\ k(x) = k(y)) \\ \implies \big((\exists x \in S.\ k(x) = \top) \iff (\exists x \in T.\ k(x) = \top) \big) \Big).$$

Straightforward calculation shows that this is equivalent to

$$(\forall x \in S.\ \exists y \in T.\ (x, y) \in R) \wedge (\forall y \in T.\ \exists x \in S.\ (x, y) \in R).$$

This lifting is the restriction (to $\mathbf{EqRel}$) of the standard relational lifting of $\mathcal{P}$ along $\mathbf{ERel} \to \mathbf{Set}$, which is used for the usual bisimulation notion for Kripke frames [10].

Example 3.1.5. In the entry 3 of Table 3.1, we codensity-lift $\mathcal{P}$ along the $\mathbf{CLat}_{\sqcap}$-fibration $\mathbf{ERel} \to \mathbf{Set}$ (instead of $\mathbf{EqRel} \to \mathbf{Set}$) with the parameter $\big((2, \mathrm{Eq}_2), \Diamond\big)$.

The characterization of $\mathcal{P}^{\mathrm{Eq}_2,\Diamond}(X, R)$ is slightly involved. Its relation part relates $S, T \in \mathcal{P}X$ if and only if

$$(\forall x \in S.\ \exists y \in T.\ (x, y) \in R^{\mathrm{eq}}) \ \wedge \ (\forall y \in T.\ \exists x \in S.\ (x, y) \in R^{\mathrm{eq}}),$$

where R^{eq} denotes the equivalence closure of R.

It is not clear at this stage whether the codensity bisimilarities induced by the above liftings (Examples 3.1.4 and 3.1.5, i.e. the entries 4 and 3 of Table 3.1) coincide with the usual bisimilarity notion for Kripke frames. This is because of the involvement of mandatory equivalence closures—specifically by the use of $\mathbf{EqRel}$ in Example 3.1.4, and by the occurrence of $(\_)^{\mathrm{eq}}$ in Example 3.1.5. Later, in Example 4.7.4, we prove that both of the codensity bisimilarities indeed coincide with the usual bisimilarity notion. The proof relies crucially on transfer of codensity liftings via fibered functors.

Table 3.1.: Codensity lifting of functors

	fibration $\mathbb{E} \xrightarrow{p} \mathbb{C}$	functor $B\colon \mathbb{C} \to \mathbb{C}$	obs. dom. $\mathbf{\Omega}$	modality τ	lifting $B^{\mathbf{\Omega},\tau}$ of F
1	$\mathbf{Pre} \to \mathbf{Set}$	powerset $\mathcal{P}$	$(2, \leq)$	$\Diamond\colon \mathcal{P}2 \to 2$	lower preorder [41]
2	$\mathbf{Pre} \to \mathbf{Set}$	powerset $\mathcal{P}$	$(2, \geq)$	$\Diamond\colon \mathcal{P}2 \to 2$	upper preorder [41]
3	$\mathbf{ERel} \to \mathbf{Set}$	powerset $\mathcal{P}$	$(2, \mathrm{Eq}_2)$	$\Diamond\colon \mathcal{P}2 \to 2$	(see Ex. 3.1.5 & 4.7.4)
4	$\mathbf{EqRel} \to \mathbf{Set}$	powerset $\mathcal{P}$	$(2, \mathrm{Eq}_2)$	$\Diamond\colon \mathcal{P}2 \to 2$	(see Ex. 3.1.4 & 4.7.4)
5	$\mathbf{PMet}_1 \to \mathbf{Set}$	subdistrib. $\mathcal{D}_{\leq 1}$	$([0,1], d_{[0,1]})$	$e\colon \mathcal{D}_{\leq 1}[0,1] \to [0,1]$	Kantorovich metric [41]
6	$\mathbf{PMet}_1 \to \mathbf{Set}$	powerset $\mathcal{P}$	$([0,1], d_{[0,1]})$	$\inf\colon \mathcal{P}[0,1] \to [0,1]$	Hausdorff metric (Appx. A.2)
7	$U^*(\mathbf{PMet}_1) \to \mathbf{Meas}$	sub-Giry $\mathcal{G}_{\leq 1}$	$([0,1], d_{[0,1]})$	$e\colon \mathcal{G}_{\leq 1}[0,1] \to [0,1]$	Kantorovich metric [41]
8†	$\mathbf{Pre} \to \mathbf{Set}$	powerset $\mathcal{P}$	$(2, \leq), (2, \geq)$	$\Diamond\colon \mathcal{P}2 \to 2$	convex preorder [41]
9†	$\mathbf{EqRel} \to \mathbf{Set}$	subdistrib. $\mathcal{D}_{\leq 1}$	$(2, \mathrm{Eq}_2)$	$(\tau_r\colon \mathcal{D}_{\leq 1}2 \to 2)_{r \in [0,1]}$	(for prob. bisim., see Ex. 4.8.14)
10†	$\mathbf{Top} \to \mathbf{Set}$	$2 \times (\_)^{\Sigma}$	Sierpinski sp.	(see Ex. 4.6.9)	(for bisim. top., see Ex. 4.6.9)
11†	$\mathbf{BRel} \to \mathbf{Set}^2$	any functor	$((1,1), R_2)$	any family	(for Λ-bisim., see §4.8.2)

The fibration $U^*(\mathbf{PMet}_1) \to \mathbf{Meas}$ is introduced in §4.8.5. $d_{[0,1]}$ denotes the Euclidean metric on the unit interval $[0,1]$. The modality $\Diamond$ is introduced in Example 3.1.4. The functions $e\colon \mathcal{D}_{\leq 1}[0,1] \to [0,1]$ and $e\colon \mathcal{G}_{\leq 1}[0,1] \to [0,1]$ both return expected values. The lower, upper and convex preorders are known for powerdomains; see e.g. [69]. The function $\tau_r\colon \mathcal{D}_{\leq 1}2 \to 2$ is introduced in Example 4.8.14. The examples marked with † involve multiple modalities and observation domains (§4.6).

Example 3.1.6. Here we follow [66, Example 5.13] and show that codensity lifting generalizes a categorical construction introduced in [6], namely the *Kantorovich lifting* of functors. Take $\mathbf{PMet}_1 \to \mathbf{Set}$ as the $\mathbf{CLat}_\sqcap$-fibration p in Definition 5.2.8. As $\mathbf{\Omega}$, we take $\Omega = [0,1]$ with the usual Euclidean metric $d_{[0,1]}$. There is freedom in the choice of a modality $\tau \colon B\Omega \to \Omega$—this corresponds to what is called an *evaluation function* in [6]. This way we recover the Kantorovich lifting in [6] as $B^{\mathbf{\Omega},\tau}$.

3.1.2. Codensity Bisimilarity

In [66], *codensity bisimulation* and *bisimilarity* are introduced.

Definition 3.1.7 (codensity bisimulation)**.** Assume the setting of Definition 5.2.8. Let $c : X \to FX$ be an F-coalgebra. An object $P \in \mathbb{E}_X$ is a *$((\mathbf{\Omega},\tau)$-$)$codensity bisimulation* over c if $c : (X,P) \dot{\to} (FX, F^{\mathbf{\Omega},\tau}P)$; that is, c is decent with respect to the designated indistinguishability structures on X and FX.

We move on to the characterization of codensity bisimulations as post-fixpoints of suitable predicate transformers.

Definition 3.1.8 (predicate transformer $\Phi^{\mathbf{\Omega},\tau}$)**.** Assume the setting of Definition 3.1.7. We define a *predicate transformer* $\Phi_c^{\mathbf{\Omega},\tau} : \mathbb{E}_X \to \mathbb{E}_X$ with respect to c and $F^{\mathbf{\Omega},\tau}$ by

$$\Phi_c^{\mathbf{\Omega},\tau} P = c^*(F^{\mathbf{\Omega},\tau} P). \tag{3.1}$$

Since c^* is $\bigsqcap$-preserving, expanding the definition of $F^{\mathbf{\Omega},\tau}$ yields

$$\Phi_c^{\mathbf{\Omega},\tau} P = \bigsqcap_{k \in \mathbb{E}(P,\mathbf{\Omega})} \bigl(\tau \circ F(p(k)) \circ c\bigr)^* \mathbf{\Omega}.$$

Theorem 3.1.9. *Assume the setting of Definition 3.1.7. For any $P \in \mathbb{E}_X$, the following are equivalent.*

1. *$c : (X,P) \dot{\to} (FX, F^{\mathbf{\Omega},\tau}P)$; that is, P is a codensity bisimulation over c (Definition 3.1.7).*
2. *$P \sqsubseteq \Phi_c^{\mathbf{\Omega},\tau} P$.*
3. *For each $k \in \mathbb{C}(X,\Omega)$, $k : (X,P) \dot{\to} (\Omega,\mathbf{\Omega})$ implies $\tau \circ Fk \circ c : (X,P) \dot{\to} (\Omega,\mathbf{\Omega})$.*

Proof. The equivalence between the conditions (1) and (2) can be seen from the definitions of $\Phi_c^{\mathbf{\Omega},\tau}$ (Definition 3.1.8) and decency (Definition 2.3.5). Now we show (2) $\iff$ (3).

By using Definition 3.1.8, the condition (2) is equivalent to

$$P \sqsubseteq \bigsqcap_{k \in \mathbb{E}(P,\mathbf{\Omega})} \bigl(\tau \circ F(p(k)) \circ c\bigr)^* \mathbf{\Omega}.$$

The definition of meet implies that the above inequality is equivalent to the following:

$$\text{For each } k \in \mathbb{C}(X, \Omega),\ k : (X, P) \dot{\to} (\Omega, \mathbf{\Omega}) \text{ implies } P \sqsubseteq \left(\tau \circ F(p(k)) \circ c\right)^{*}\mathbf{\Omega}.$$

This is, in turn, equivalent to the condition (3), as can be seen from the definition of decency (Definition 2.3.5). □

The predicate transformer $\Phi_c^{\mathbf{\Omega},\tau}$ is a monotone map from the complete lattice $\mathbb{E}_X$ to itself. Therefore, by the Knaster–Tarski theorem (Fact 2.2.3), the greatest post-fixed point of $\Phi_c^{\mathbf{\Omega},\tau}$ exists and it is the greatest fixed point of $\Phi_c^{\mathbf{\Omega},\tau}$.

Definition 3.1.10 (codensity bisimilarity $\nu\Phi_c^{\mathbf{\Omega},\tau}$)**.** Assume the setting of Definition 3.1.7. The greatest codensity bisimulation, whose existence is guaranteed by the above arguments, is called the *codensity bisimilarity*. It is denoted by $\nu\Phi_c^{\mathbf{\Omega},\tau}$.

Some bisimilarity notions, including bisimilarity of deterministic automata (Example 4.8.10), are accommodated in the generalized framework with multiple observation domains—see Section 4.6.

Example 3.1.11 (bisimulation metric)**.** Consider the $\mathbf{CLat}_\sqcap$-fibration $\mathbf{PMet}_1 \to \mathbf{Set}$ and the subdistribution functor $\mathcal{D}_{\leq 1} \colon \mathbf{Set} \to \mathbf{Set}$. Recall that $\mathcal{D}_{\leq 1}(X) = \{p \colon X \to [0,1] \mid \sum_{x \in X} p(x) \leq 1\}$. As a parameter of codensity lifting, we take $(\mathbf{\Omega}, \tau) = \left(\left([0,1], d_{[0,1]}\right),\ e \colon \mathcal{D}_{\leq 1}[0,1] \to [0,1]\right)$, where e is the *expectation function* $e(p) = \sum_{r \in [0,1]} r \cdot p(r)$ and $d_{[0,1]}$ is the Euclidean metric. Let $c \colon X \to \mathcal{D}_{\leq 1}X$ be a coalgebra, identified with a Markov chain.

The codensity bisimilarity in this setting coincides with the bisimulation metric from [22] (see also Section 4.1.1). This fact is not hard to check directly; one can also derive the coincidence via Example 3.1.6 and the observations in [6].

3.2. Codensity Lifting with Multiple Parameters

We extend the theory so far and accommodate multiple observation domains and modalities. This extension is needed for some examples, such as those marked with † in Table 3.1.

We consider the class $\mathbf{Lift}(B, p)$ of liftings of an endofunctor $B : \mathbb{C} \to \mathbb{C}$ along a $\mathbf{CLat}_\sqcap$-fibration $\mathbb{E} \xrightarrow{p} \mathbb{C}$. It comes with a natural pointwise partial order:

$$G \sqsubseteq H \iff \forall X \in \mathbb{E}.\ GX \sqsubseteq HX \quad (G, H \in \mathbf{Lift}(F, p)), \tag{3.2}$$

and the partially ordered class $\mathbf{Lift}(F, p)$ admits meets of arbitrary size. As done in the original codensity lifting of endofunctors in [66] (and that of monads in [41]), we extend the codensity lifting so that it takes a family of parameters $\{(\mathbf{\Omega}_A, \tau_A)\}_{A \in \mathbb{A}}$, and returns the *intersection* of the codensity liftings of B with these parameters.

Definition 3.2.1 (codensity lifting of a functor with multiple parameters [66])**.** Let $B : \mathbb{C} \to \mathbb{C}$ be a functor, $\mathbb{E} \xrightarrow{p} \mathbb{C}$ be a $\mathbf{CLat}_\sqcap$-fibration, $\mathbb{A}$ be a class, and $\{(\mathbf{\Omega}_A, \tau_A)\}_{A \in \mathbb{A}}$ be an $\mathbb{A}$-indexed family of parameters (of the codensity lifting of B along p), which is denoted simply by $(\mathbf{\Omega}, \tau)$. The (multiple-parameter) *codensity lifting* of B with $(\mathbf{\Omega}, \tau)$ is the endofunctor $B^{\mathbf{\Omega},\tau} : \mathbb{E} \to \mathbb{E}$ defined by the intersection of the codensity liftings:

$$B^{\mathbf{\Omega},\tau} P = \bigsqcap_{A \in \mathbb{A}} B^{\mathbf{\Omega}_A, \tau_A} P, \quad \text{that is,} \quad \bigsqcap_{A \in \mathbb{A}, k \in \mathbb{E}(P, \mathbf{\Omega}_A)} \left(\tau_A \circ B(p(k))\right)^{*}(\mathbf{\Omega}_A).$$

Here we briefly list some corresponding definitions for multiple parameter cases. These play a role in, e.g., Section 4.6 in Chapter 4 and all of Chapter 5.

Definition 3.2.2 (codensity bisimulation and codensity bisimilarity)**.** Assume the setting of Definition 3.2.1. Let $c : X \to BX$ be a B-coalgebra. An object $P \in \mathbb{E}_X$ is a *codensity bisimulation* over c if $c : (X, P) \dot{\to} (BX, B^{\mathbf{\Omega},\tau} P)$; that is, $c\colon X \to BX$ is decent with respect to the designated indistinguishability structures.

The largest codensity bisimulation is called the *codensity bisimilarity* and denoted by $\nu\Phi_c^{\mathbf{\Omega},\tau}$.

Definition 3.2.3 (predicate transformer $\Phi^{\mathbf{\Omega},\tau}$)**.** Assume the setting of Definition 3.2.2. We define a *predicate transformer* $\Phi_c^{\mathbf{\Omega},\tau} : \mathbb{E}_X \to \mathbb{E}_X$ with respect to c and $B^{\mathbf{\Omega},\tau}$ by

$$\Phi_c^{\mathbf{\Omega},\tau} P = c^*(B^{\mathbf{\Omega},\tau} P).$$

Since c^* is $\bigsqcap$-preserving, expanding the definition of $B^{\mathbf{\Omega},\tau}$ yields

$$\Phi_c^{\mathbf{\Omega},\tau} P = \bigsqcap_{A \in \mathbb{A}, k \in \mathbb{E}(P, \mathbf{\Omega}_A)} \bigl(\tau_A \circ B(p(k)) \circ c\bigr)^* \mathbf{\Omega}_A.$$

3.3. C-injective Objects and Fiberedness of Codensity Lifting

3.3.1. C-injective Objects and Codensity Lifting

C-injective Object

In the proof of the functoriality of $B^{\mathbf{\Omega},\tau}$, ultimately we use the fact that, for any $f\colon P \to Q$, any "test" $k\colon Q \to \mathbf{\Omega}$ can be turned into another "test" $k \circ f\colon P \to \mathbf{\Omega}$. On the other hand, when we try to prove fiberedness of $B^{\mathbf{\Omega},\tau}$, we have to somehow lift a "test" $g\colon P \to \mathbf{\Omega}$ along a Cartesian arrow $f\colon P \to Q$ and obtain another "test" $h\colon Q \to \mathbf{\Omega}$. This observation leads us to the following definition of *c-injective object*. (The letter c here comes from *Cartesian*.)

Definition 3.3.1 (c-injective object)**.** Let $p\colon \mathbb{E} \to \mathbb{C}$ be a fibration. An object $\mathbf{\Omega} \in \mathbb{E}$ is a *c-injective object* if the functor $\mathbb{E}(-, \mathbf{\Omega})\colon \mathbb{E}^{\mathrm{op}} \to \mathbf{Set}$ sends every Cartesian arrow to a surjective map.

Equivalently, $\mathbf{\Omega} \in \mathbb{E}$ is a c-injective object if, for any Cartesian arrow $f\colon P \to Q$ in $\mathbb{E}$ and any (not necessarily Cartesian) arrow $g\colon P \to \mathbf{\Omega}$, there is a (not necessarily Cartesian) arrow $h\colon Q \to \mathbf{\Omega}$ satisfying $g = h \circ f$.

Some basic objects can be shown to be c-injective objects.

Example 3.3.2 (the two-point set)**.** In the fibration $\mathbf{EqRel} \to \mathbf{Set}$, $(2, =)$ is a c-injective object. Here, $2 = \{\bot, \top\}$ is the two-point set and $=$ means the equality relation. Indeed, for any Cartesian $f\colon (X, R_X) \to (Y, R_Y)$ and any $g\colon (X, R_X) \to (2, =)$, if we define $h\colon (Y, R_Y) \to (2, =)$ by

$$h(y) = \begin{cases} g(x) & \text{if } (y, f(x)) \in R_Y \\ \top & \text{otherwise,} \end{cases}$$

then this turns out to be well-defined and satisfies $h \circ f = g$.

Example 3.3.3 (the two-point poset of truth values)**.** In the fibration $\mathbf{Pre} \to \mathbf{Set}$, $(2, \leq)$ is a c-injective object. Here, $\leq$ is the unique partial order satisfying $\bot \leq \top$ and $\top \not\leq \bot$. Indeed, for any Cartesian arrow $f\colon (X, R_X) \to (Y, R_Y)$ and any $g\colon (X, R_X) \to (2, \leq)$, if we define $h\colon : (Y, R_Y) \to (2, \leq)$ by

$$h(y) = \begin{cases} \bot & \text{if } (y, f(x)) \in R_Y \text{ for some } x \text{ such that } g(x) = \bot \\ \top & \text{otherwise,} \end{cases}$$

then this turns out to be well-defined and satisfies $h \circ f = g$.

Example 3.3.4 (the unit interval as a pseudometric space [6, Theorem 5.8])**.** In the fibration $\mathbf{PMet}_\top \to \mathbf{Set}$, $[0, \top]$ is a c-injective object. Indeed, for any arrow $g\colon (X, d_X) \to ([0, \top], d_e)$ and any Cartesian arrow $f\colon (X, d_X) \to (Y, d_Y)$, we can show that the map $h\colon Y \to [0, \top]$ defined by $h(y) = \inf_{x \in X} (g(x) + d_Y(f(x), y))$ is nonexpansive from (Y, d_Y) to $([0, \top], d_e)$.

The following non-example shows that c-injectivity crucially depends on the fibration we consider.

Example 3.3.5 (non-example)**.** In contrast to Example 3.3.3, in the fibration $\mathbf{ERel} \to \mathbf{Set}$, $(2, \leq)$ is not c-injective, where $2 = \{\bot, \top\}$ is the two-point set and $\leq$ is the unique partial order satisfying $\bot \leq \top$ and $\top \not\leq \bot$.

This can be seen as follows. Let $X = \{a, b\}$, $Y = \{x, y, z\}$, $R_X = \emptyset$, and $R_Y = \{(x, z), (z, y)\}$. Then (X, R_X) and (Y, R_Y) are objects of $\mathbf{ERel}$. Consider the maps $f\colon (X, R_X) \to (Y, R_Y)$ and $g\colon (X, R_X) \to (2, \leq)$ defined by $f(a) = x$, $f(b) = y$, $g(a) = \top$, and $g(b) = \bot$. Note that f is Cartesian. However, there is no $h\colon (Y, R_Y) \to (2, \leq)$ such that $h \circ f = g$: such h would satisfy $\top = h(f(a)) = h(x) \leq h(z) \leq h(y) = h(f(b)) = \bot$, which contradict $\top \not\leq \bot$.

The same example can also be used to show that, in contrast to Example 3.3.2, $(2, =)$ is not c-injective, where $=$ means the equality relation.

Sufficient Condition for Fibered Codensity Lifting

Now we are prepared to state the following main theorem of the current paper. The strategy of the proof is roughly as mentioned earlier.

Theorem 3.3.6 (fiberedness from injective object)**.** *In the setting of Definition 3.1.1, if $\mathbf{\Omega}$ is a c-injective object, then $B^{\mathbf{\Omega},\tau}$ is fibered.*

Proof. Let $f\colon P \to Q$ be any Cartesian arrow. By Proposition 2.4.5, it suffices to show $B^{\mathbf{\Omega},\tau} P = (F(pf))^* \left(B^{\mathbf{\Omega},\tau} Q\right)$. Here, $B^{\mathbf{\Omega},\tau} P \sqsubseteq (F(pf))^* \left(B^{\mathbf{\Omega},\tau} Q\right)$ has already been proven. Thus, our goal is the inequality $B^{\mathbf{\Omega},\tau} P \sqsupseteq (F(pf))^* \left(B^{\mathbf{\Omega},\tau} Q\right)$.

Here, since $\mathbf{\Omega}$ is c-injective and f is Cartesian, the following inclusion holds:

$$\{g \in \mathbb{E}(P, \mathbf{\Omega})\} \subseteq \{h \circ f \mid h \in \mathbb{E}(Q, \mathbf{\Omega})\}.$$

By the definition of the meet, we have

$$\bigsqcap_{g\in\mathbb{E}(P,\mathbf{\Omega})} (F(pg))^*\tau^*\mathbf{\Omega} \sqsupseteq \bigsqcap_{h\in\mathbb{E}(Q,\mathbf{\Omega})} (F(p(h\circ f)))^*\tau^*\mathbf{\Omega}.$$

By the calculation in the proof of Proposition 3.1.2, this implies

$$B^{\mathbf{\Omega},\tau}P \sqsupseteq (F(pf))^* \left(B^{\mathbf{\Omega},\tau}Q\right). \qquad \square$$

Remark 3.3.7. A refinement of Theorem 3.3.6 to an if-and-only-if result seems hard. At least there is a simple counterexample to the most naive version of it: Consider a $\mathbf{CLat}_\sqcap$-fibration $\mathrm{Id}\colon \mathbb{C}\to\mathbb{C}$, an endofunctor $\mathrm{Id}\colon\mathbb{C}\to\mathbb{C}$, an object $C\in\mathbb{C}$, and an arrow $\tau\colon C\to C$. The codensity lifting $\mathrm{Id}^{C,\tau}$ is always equal to Id, which is fibered. However, since any arrow in $\mathbb{C}$ is a Cartesian arrow w.r.t. Id, it is not hard to find an example of C and $\mathbb{C}$ such that C is not c-injective w.r.t. Id.

Example 3.3.8 (Kantorovich lifting)**.** Baldan et al. [6, Theorem 5.8] have shown that any Kantorovich lifting preserves isometries. In terms of fibrations, this means that such functor is a fibered endofunctor on the fibration $\mathbf{PMet}_\top \to \mathbf{Set}$.

Since Kantorovich lifting is a special case of codensity lifting where $\mathbf{\Omega} = ([0,\top], d_\mathbb{R})$, Theorem 3.3.6 and Example 3.3.4 recover the same result. Actually, this has inspired Theorem 3.3.6 as a prototype.

The argument above also applies to situations with multiple parameters.

Definition 3.3.9 (codensity lifting with multiple parameters (as in [44]))**.** Let $\mathbb{E},\mathbb{C},p$, and F be as in Definition 3.1.1. Let A be a set. Assume that, for each $a\in A$, we are given $\mathbf{\Omega}_a\in\mathbb{E}$ above $\Omega_a\in\mathbb{C}$ and $\tau_a\colon F\Omega_a\to\Omega_a$. Define a functor $B^{\mathbf{\Omega},\tau}\colon\mathbb{E}\to\mathbb{E}$ by

$$B^{\mathbf{\Omega},\tau}P = \bigsqcap_{a\in A} B^{\mathbf{\Omega}_a,\tau_a}P$$

for each $P\in\mathbb{E}$.

Corollary 3.3.10. *In the setting of Definition 3.2.1, if, for each $a\in A$, $\mathbf{\Omega}_a$ is a c-injective object, then $B^{\mathbf{\Omega},\tau}$ is fibered.*

Proof. For any $P\in\mathbb{E}$ above $X\in\mathbb{C}$ and $f\colon Y\to X$ in $\mathbb{C}$, using Theorem 3.3.6, we can see

$$\begin{aligned}(Ff)^*B^{\mathbf{\Omega},\tau}P &= (Ff)^*\bigsqcap_{a\in A} B^{\mathbf{\Omega}_a,\tau_a}P &&= \bigsqcap_{a\in A}(Ff)^*B^{\mathbf{\Omega}_a,\tau_a}P\\ &= \bigsqcap_{a\in A} B^{\mathbf{\Omega}_a,\tau_a}f^*P &&= B^{\mathbf{\Omega},\tau}f^*P.\end{aligned}$$

$\square$

Example 3.3.11 (Kantorovich lifting with multiple parameters)**.** In [48], König and Mika-Michalski introduced a generalized version of Kantorovich lifting.

Since it is a special case of Definition 3.2.1 where p is the fibration $\mathbf{PMet}_\top\to\mathbf{Set}$ and $\mathbf{\Omega} = ([0,\top], d_\mathbb{R})$, Corollary 3.3.10 and Example 3.3.4 imply that such lifting always preserves isometries.

3.3.2. Results on C-injective Objects

Here we seek properties of c-injective objects, mainly to obtain more examples of them. We also see that, in a few fibrations, c-injective objects have been essentially identified by previous works.

$\mathcal{M}$-injective Objects

To connect c-injectivity with existing works, we consider a more general notion of $\mathcal{M}$-injective object. The following definition is found e.g. in [40, Section 9.5].

Definition 3.3.12. Let $\mathbb{C}$ be a category and $\mathcal{M}$ be a class of arrows in $\mathbb{C}$. An object $X \in \mathbb{C}$ is an $\mathcal{M}$*-injective object* if the functor $\mathbb{C}(-, X) \colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$ sends every arrow in $\mathcal{M}$ to a surjective map.

The definition of c-injective objects is a special case of the definition above where $\mathcal{M}$ is the class of all Cartesian arrows.

The following is a folklore result. The dual is found e.g. in [34, Proposition 10.2].

Proposition 3.3.13. *Let $\mathbb{C}, \mathbb{D}$ be categories, $\mathcal{M}_{\mathbb{C}}, \mathcal{M}_{\mathbb{D}}$ be classes of arrows, and $L \dashv R \colon \mathbb{C} \to \mathbb{D}$ be a pair of adjoint functors. Assume that L sends any arrow in $\mathcal{M}_{\mathbb{D}}$ to one in $\mathcal{M}_{\mathbb{C}}$. For any $\mathcal{M}_{\mathbb{C}}$-injective $C \in \mathbb{C}$, $RC \in \mathbb{D}$ is $\mathcal{M}_{\mathbb{D}}$-injective.*

Proof. It suffices to show that $\mathbb{D}(-, RC) \colon \mathbb{D}^{\mathrm{op}} \to \mathbf{Set}$ sends each arrow in $\mathcal{M}_{\mathbb{D}}$ to a surjective map. By the assumption, the functor above factorizes to $L \colon \mathbb{D} \to \mathbb{C}$ and $\mathbb{C}(-, C) \colon \mathbb{D}^{\mathrm{op}} \to \mathbf{Set}$. The former sends each arrow in $\mathcal{M}_{\mathbb{D}}$ to one in $\mathcal{M}_{\mathbb{C}}$ and the latter sends one in $\mathcal{M}_{\mathbb{C}}$ to a surjective map. Thus, the composition of these sends each arrow in $\mathcal{M}_{\mathbb{D}}$ to a surjective map. □

For epireflective subcategories, we have a sharper result:

Proposition 3.3.14. *In the setting of Proposition 3.3.13, assume, in addition,*

- *R is fully faithful,*
- *R sends each arrow in $\mathcal{M}_{\mathbb{C}}$ to one in $\mathcal{M}_{\mathbb{D}}$, and*
- *each component of the unit $\eta \colon \mathrm{Id} \to RL$ is an epimorphism in $\mathcal{M}_{\mathbb{D}}$.*

Then, $D \in \mathbb{D}$ is $\mathcal{M}_{\mathbb{D}}$-injective if and only if it is isomorphic to RC for some $\mathcal{M}_{\mathbb{C}}$-injective $C \in \mathbb{C}$.

Proof. The "if" part is Proposition 3.3.13. We show the "only if" part.

Let $D \in \mathbb{D}$ be any $\mathcal{M}_{\mathbb{D}}$-injective object. Since $\eta_D \colon D \to RLD$ is in $\mathcal{M}_{\mathbb{D}}$, we can use the $\mathcal{M}_{\mathbb{D}}$-injectiveness of D to obtain $f \colon RLD \to D$ such that $f \circ \eta_D = \mathrm{id}_D$. Here, $\eta_D \circ f \circ \eta_D = \eta_D$ and, by epi-ness of η_D, $\eta_D \circ f = \mathrm{id}_{RLD}$. Thus, η_D is an isomorphism.

Now we show that LD is $\mathcal{M}_{\mathbb{C}}$-injective. Let $f \colon C \to LD$ and $g \colon C \to C'$ be any arrow in $\mathbb{C}$ and assume that g is in $\mathcal{M}_{\mathbb{C}}$. Send these by R to $\mathbb{D}$ and consider Rf and Rg. By the assumption, Rg is in $\mathcal{M}_{\mathbb{D}}$. Since RLD is isomorphic to D, it is also $\mathcal{M}_{\mathbb{D}}$-injective.

Using these, we can obtain $h' \colon RC' \to RLD$ such that $h' \circ Rg = Rf$. Since R is full, there is $h \colon C' \to LD$ such that $Rh = h'$. The faithfulness of R implies $h \circ g = f$. Thus LD is $\mathcal{M}_{\mathbb{C}}$-injective. □

Using this result, we can identify c-injective objects in a few situations.

Example 3.3.15 (continuous lattices in **Top** $\to$ **Set** [65])**.** In the setting of Proposition 3.3.14, consider the case where $\mathbb{D} = \mathbf{Top}$, $\mathbb{C} = \mathbf{Top}_0$. Here $\mathbf{Top}_0$ is the full subcategory of **Top** of T_0 spaces. Let R be the inclusion. It has a left adjoint L, taking each space to its Kolmogorov quotient. Let $\mathcal{M}_{\mathbb{C}}$ be the class of topological embeddings (i.e. homeomorphisms to their images) and $\mathcal{M}_{\mathbb{D}}$ be the class of Cartesian arrows (w.r.t. the fibration **Top** $\to$ **Set**). Then the assumptions in Proposition 3.3.14 are satisfied and we can conclude that c-injective objects in **Top** are precisely injective objects in $\mathbf{Top}_0$ w.r.t. embeddings.

The latter has been identified by Scott [65]. According to his result, such objects are precisely *continuous lattices* with the Scott topology. Thus, we can see that c-injective objects in **Top** are precisely such spaces.

Example 3.3.16 (complete lattices in **Pre** $\to$ **Set** [8])**.** In the setting of Proposition 3.3.14, consider the case where $\mathbb{D} = \mathbf{Pre}$, $\mathbb{C} = \mathbf{Pos}$. Here **Pos** is the full subcategory of **Pre** of posets. Let R be the inclusion. It has a left adjoint L, taking each preordered set to its poset reflection. Let $\mathcal{M}_{\mathbb{C}}$ be the class of embeddings and $\mathcal{M}_{\mathbb{D}}$ be the class of Cartesian arrows (w.r.t. the fibration **Pre** $\to$ **Set**). Then the assumptions in Proposition 3.3.14 are satisfied and we can conclude that c-injective objects in **Pre** are precisely injective objects in **Pos** w.r.t. embeddings.

The latter has been identified by Banaschewski and Bruns [8] . According to their result, such objects are precisely complete lattices. Thus, we can see that c-injective objects in **Pre** are precisely complete lattices.

Results Specific to C-injective Objects

To develop the theory of c-injective objects further, we establish some preservation results for c-injectivity. Based on the two propositions of the last section, we show two propositions specific to fibrations and c-injective objects.

From Proposition 3.3.13, we can derive the following:

Proposition 3.3.17. *Let $p \colon \mathbb{E} \to \mathbb{C}, q \colon \mathbb{F} \to \mathbb{D}$ be $\mathbf{CLat}_{\sqcap}$-fibrations and $L \dashv R \colon \mathbb{E} \to \mathbb{F}$ be a pair of adjoint functors. If L is fibered (from q to p), then $RE \in \mathbb{F}$ is c-injective (in q) for each c-injective $E \in \mathbb{E}$.*

Proof. Let $\mathcal{M}_{\mathbb{E}}$ be the class of all arrows Cartesian w.r.t. p and $\mathcal{M}_{\mathbb{F}}$ be the class of all arrows Cartesian w.r.t. q. Then, use Proposition 3.3.13 to the pair $L \dashv R$ of adjoint functors. □

From Proposition 3.3.14, we can derive the following:

Proposition 3.3.18. *In the setting of Proposition 3.3.17, assume in addition that both L and R are fibered and that $\eta\colon \mathrm{Id} \to RL$ is componentwise epi. Then, $F \in \mathbb{F}$ is c-injective if and only if it is isomorphic to RE for some c-injective $E \in \mathbb{E}$.*

Proof. Use Proposition 3.3.14 in the same setting as the proof of Proposition 3.3.17. □

3.3.3. Examples

We list several examples of Theorem 3.3.6. Indeed, most of the examples listed in [44, Table VI] turn out to be fibered by Theorem 3.3.6. Since the conditions in Theorem 3.3.6 only refer to $p\colon \mathbb{E} \to \mathbb{C}$ and $\mathbf{\Omega}$, we sort the examples by these data.

We here recall some basic functors considered:

Definition 3.3.19. Let $\mathcal{P}\colon \mathbf{Set} \to \mathbf{Set}$ be the covariant powerset functor and $\mathcal{D}_{\leq 1}\colon \mathbf{Set} \to \mathbf{Set}$ be the subdistribution functor. Here, a subdistribution $p \in \mathcal{D}_{\leq 1}X$ is a measure on the σ-algebra of all subsets of X with total mass ≤ 1. We abbreviate $p(\{x\})$ to $p(x)$.

Kantorovich Lifting

In Example 3.3.4 we have seen that, in the fibration $\mathbf{PMet}_\top \to \mathbf{Set}$, the object $([0,\top], d_\mathbb{R})$ is c-injective. We gather examples of this case here. As mentioned in Example 3.3.8 and Example 3.3.11, this class of examples has been already studied and shown to be fibered in [6, 48].

Example 3.3.20 (Hausdorff pseudometric)**.** Let $\inf\colon \mathcal{P}[0,\top] \to [0,\top]$ be the map taking any set to its infimum. Then, the codensity lifting $\mathcal{P}^{([0,\top],d_\mathbb{R}),\inf}\colon \mathbf{PMet}_\top \to \mathbf{PMet}_\top$ turns out to induce the *Hausdorff distance*: for any $(X, d_X) \in \mathbf{PMet}_\top$, if we let $(\mathcal{P}X, d_{\mathcal{P}X}) = \mathcal{P}^{([0,\top],d_\mathbb{R}),\inf}(X, d_X)$, then

$$d_{\mathcal{P}X}(S,T) = \max\left(\sup_{x\in S}\inf_{y\in T} d_X(x,y), \sup_{y\in T}\inf_{x\in S} d_X(x,y)\right)$$

holds for any $S, T \in \mathcal{P}X$. By Theorem 3.3.6, this functor is fibered.

Example 3.3.21 (Kantorovich pseudometric)**.** Let $e\colon \mathcal{D}_{\leq 1}[0,\top] \to [0,\top]$ be the map taking any distribution to its expected value. Then, the codensity lifting

$$\mathcal{D}_{\leq 1}{}^{([0,\top],d_\mathbb{R}),e}\colon \mathbf{PMet}_\top \to \mathbf{PMet}_\top$$

turns out to induce the *Kantorovich distance*: for any $(X, d_X) \in \mathbf{PMet}_\top$, if we let $(\mathcal{D}_{\leq 1}X, d_{\mathcal{D}_{\leq 1}X}) = \mathcal{D}_{\leq 1}{}^{([0,\top],d_\mathbb{R}),e}(X, d_X)$, then

$$d_{\mathcal{D}_{\leq 1}X}(p,q) = \sup_{f\colon (X,d_X)\to([0,\top],d_\mathbb{R})\text{ nonexpansive}} \left|\sum_{x\in X} f(x)p(x) - \sum_{x\in X} f(x)q(x)\right|$$

holds for any $p, q \in \mathcal{D}_{\leq 1}X$. By Theorem 3.3.6, this functor is fibered.

Lower, Upper, and Convex Preorders

In Example 3.3.16, we have identified complete lattices as c-injective objects in the fibration $\mathbf{Pre} \to \mathbf{Set}$. In particular, the two-point set $(2, \leq)$ is a c-injective object (Example 3.3.3).

Katsumata and Sato [41, Section 4.1] used codensity lifting to recover the *lower*, *upper*, and *convex preorders* on powersets. Here we see that our result applies to them: all of the following liftings are fibered.

Example 3.3.22 (lower preorder)**.** Define $\Diamond\colon \mathcal{P}2 \to 2$ so that $\Diamond S = \top$ if and only if $\top \in S$. Then, the codensity lifting $\mathcal{P}^{(2,\leq),\Diamond}\colon \mathbf{Pre} \to \mathbf{Pre}$ turns out to induce the *lower preorder*: if we let $(\mathcal{P}X, \leq^{\Diamond}_{\mathcal{P}X}) = \mathcal{P}^{(2,\leq),\Diamond}(X, \leq_X)$, then, for any $S, T \in \mathcal{P}X$,

$$S \leq^{\Diamond}_{\mathcal{P}X} T \Leftrightarrow \forall x \in S, \exists y \in T, x \leq_X y.$$

Example 3.3.23 (upper preorder)**.** Define $\Box\colon \mathcal{P}2 \to 2$ so that $\Box S = \top$ if and only if $\bot \notin S$. Then, the codensity lifting $\mathcal{P}^{(2,\leq),\Box}\colon \mathbf{Pre} \to \mathbf{Pre}$ turns out to induce the *upper preorder*: if we let $(\mathcal{P}X, \leq^{\Box}_{\mathcal{P}X}) = \mathcal{P}^{(2,\leq),\Box}(X, \leq_X)$, then, for any $S, T \in \mathcal{P}X$,

$$S \leq^{\Box}_{\mathcal{P}X} T \Leftrightarrow \forall y \in T, \exists x \in S, x \leq_X y.$$

Example 3.3.24 (convex preorder)**.** Denote the family of the two lifting parameters above by $((2, \leq), \{\Diamond, \Box\})$. Then, the codensity lifting (with multiple parameters, Definition 3.2.1) $\mathcal{P}^{(2,\leq),\{\Diamond,\Box\}}\colon \mathbf{Pre} \to \mathbf{Pre}$ is simply the meet of $\mathcal{P}^{(2,\leq),\Diamond}$ and $\mathcal{P}^{(2,\leq),\Box}$. This is what is called the *convex preorder*.

Remark 3.3.25. The original formulation [41, Section 4.1] is based on codensity lifting of monads, so apparently different to ours. In our terms, they used the multiplication $\mu_1\colon \mathcal{P}\mathcal{P}1 \to \mathcal{P}1$ and two different preorders on $\mathcal{P}1$. Using two different bijections between $\mathcal{P}1$ and 2, it can be shown that their formulation is actually equivalent to ours.

Equivalence relations

In Example 3.3.2 we have seen that, in the fibration $\mathbf{EqRel} \to \mathbf{Set}$, the object $(2, =)$ is c-injective. We gather examples of this case here. All of the following liftings are fibered. Details on the following examples can be found in [44].

Example 3.3.26 (lifting for bisimilarity on Kripke frames)**.** Consider the codensity lifting $\mathcal{P}^{(2,=),\Diamond}\colon \mathbf{EqRel} \to \mathbf{EqRel}$, where $\Diamond$ is as defined in Example 3.3.22. This turns out to satisfy the following: if we let $(\mathcal{P}X, \sim_{\mathcal{P}X}) = \mathcal{P}^{(2,=),\Diamond}(X, \sim_X)$, then

$$S \sim_{\mathcal{P}X} T \Leftrightarrow (\forall x \in S, \exists y \in T, x \sim_X y) \wedge (\forall y \in T, \exists x \in S, x \sim_X y)$$

holds for any $S, T \in \mathcal{P}X$. This can be used to define (the conventional notion of) bisimilarity on Kripke frames ($\mathcal{P}$-coalgebras).

Example 3.3.27 (lifting for bisimilarity on Markov chains)**.** For each $r \in [0, 1]$, define a map $\mathrm{thr}_r\colon \mathcal{D}_{\leq 1}2 \to 2$ so that $\mathrm{thr}_r(p) = \top$ if and only if $p(\top) \geq r$. These define a $[0, 1]$-indexed family of lifting parameters $((2, =), \mathrm{thr}_r)_{r\in[0,1]}$. The codensity lifting ${\mathcal{D}_{\leq 1}}^{(2,=),\mathrm{thr}}$ defined by this family can be used to define probabilistic bisimilarity on Markov chains ($\mathcal{D}_{\leq 1}$-coalgebras).

Topologies

In Example 3.3.15, we have identified c-injective objects in the fibration $\mathbf{Top} \to \mathbf{Set}$. In particular, the *Sierpinski space*, defined as follows, is a c-injective object:

Definition 3.3.28 (Sierpinski space)**.** The *Sierpinski space* is a topological space $(2, \mathcal{O}_{\mathbb{O}})$ where $2 = \{\perp, \top\}$ and the family $\mathcal{O}_{\mathbb{O}}$ of open sets is $\{\emptyset, \{\top\}, 2\}$. We denote this space by $\mathbb{O}$.

The following liftings of $\mathcal{P}$ have appeared in [41, Section 4.2]. All of them are fibered: in other words, they send embeddings to embeddings.

Example 3.3.29 (lower Vietoris lifting)**.** Consider the codensity lifting $\mathcal{P}^{\mathbb{O},\Diamond}\colon \mathbf{Top} \to \mathbf{Top}$, where $\Diamond$ is as defined in Example 3.3.22. For each $(X, \mathcal{O}_X) \in \mathbf{Top}$, if we let $(\mathcal{P}X, \mathcal{O}^{\Diamond}_{\mathcal{P}X}) = \mathcal{P}^{\mathbb{O},\Diamond}(X, \mathcal{O}_X)$, then the topology $\mathcal{O}^{\Diamond}_{\mathcal{P}X}$ is the coarsest one such that, for each $U \in \mathcal{O}_X$, the set $\{V \subseteq X \mid V \cap U \neq \emptyset\}$ is open. This is called *lower Vietoris lifting* in [41].

Example 3.3.30 (upper Vietoris lifting)**.** Consider the codensity lifting $\mathcal{P}^{\mathbb{O},\Box}\colon \mathbf{Top} \to \mathbf{Top}$, where $\Box$ is as defined in Example 3.3.23. For each $(X, \mathcal{O}_X) \in \mathbf{Top}$, if we let $(\mathcal{P}X, \mathcal{O}^{\Box}_{\mathcal{P}X}) = \mathcal{P}^{\mathbb{O},\Box}(X, \mathcal{O}_X)$, then the topology $\mathcal{O}^{\Box}_{\mathcal{P}X}$ is the coarsest one such that, for each $U \in \mathcal{O}_X$, the set $\{V \subseteq X \mid V \subseteq U\}$ is open. This is called *upper Vietoris lifting* in [41].

Example 3.3.31 (Vietoris lifting)**.** Define the codensity lifting $\mathcal{P}^{\mathbb{O},\{\Diamond,\Box\}}\colon \mathbf{Top} \to \mathbf{Top}$ like one in Example 3.3.24. We call this *Vietoris lifting.*

This turns out to be connected to *Vietoris topology* [49] as follows. For each $(X, \mathcal{O}_X) \in \mathbf{Top}$, let $(\mathcal{P}X, \mathcal{O}^{\Diamond,\Box}_{\mathcal{P}X}) = \mathcal{P}^{\mathbb{O},\{\Diamond,\Box\}}(X, \mathcal{O}_X)$. The set $K(X, \mathcal{O}_X)$ of closed subsets of $(X, \mathcal{O}_X)$ is a subset of $\mathcal{P}X$. Here, the topology on $K(X, \mathcal{O}_X)$ induced from $\mathcal{O}^{\Diamond,\Box}_{\mathcal{P}X}$ is the same as the Vietoris topology.

This coincidence and the fiberedness of $\mathcal{P}^{\mathbb{O},\{\Diamond,\Box\}}$ implies that the *Vietoris functor* $\mathbb{V}\colon \mathbf{Stone} \to \mathbf{Stone}$, defined in [49], sends embeddings to embeddings.

In [44], we considered another lifting:

Example 3.3.32 (lifting for bisimulation topology)**.** Fix any set Σ. Let $A_\Sigma\colon \mathbf{Set} \to \mathbf{Set}$ be the functor defined by $A_\Sigma X = 2 \times X^\Sigma$. Define $\mathrm{acc}\colon A_\Sigma 2 \to 2$ by $\mathrm{acc}(t, \rho) = t$. For each $a \in \Sigma$, define $\langle a \rangle : A_\Sigma 2 \to 2$ by $\langle a \rangle (t, \rho) = \rho(a)$. Here, $(\mathbb{O}, \mathrm{acc})$ and $(\mathbb{O}, \langle a \rangle)$ for each $a \in \Sigma$ consist of a family of lifting parameters. The codensity lifting (with multiple parameters, Definition 3.2.1) ${A_\Sigma}^{\mathbb{O},\{\mathrm{acc}\}\cup\{\langle a \rangle \mid a \in \Sigma\}}\colon \mathbf{Top} \to \mathbf{Top}$ was used to define *bisimulation topology* for deterministic automata (A_Σ-coalgebras). This is fibered. This fact is used in Example 3.3.35, where we will look at bisimulation topology again.

3.3.4. Application to Codensity Bisimilarity

Fiberedness of codensity lifting has a consequence for codensity bisimilarity

Proposition 3.3.33 (stability of codensity bisimilarity)**.** *Assume the setting of Definition 3.2.1 (codensity lifting with multiple parameters). Assume also that each $\mathbf{\Omega}_a$ is a c-injective object. Then, codensity bisimilarity is stable under coalgebra morphisms: for any morphism of coalgebras f from (X, c) to (Y, d), we have $\nu\Phi_c^{\mathbf{\Omega},\tau} = f^*\left(\nu\Phi_d^{\mathbf{\Omega},\tau}\right)$.*

Proof. By Corollary 3.3.10, the codensity lifting $B^{\mathbf{\Omega},\tau}$ is fibered. Thus, we can use Proposition 2.4.6 to obtain the desired result. □

In particular, the codensity bisimilarity is determined by that on the final coalgebra:

Corollary 3.3.34. *Assume the setting of Proposition 3.3.33. Assume also that there exists a final F-coalgebra $z\colon Z \to FZ$. Then, for any F-coalgebra $c\colon X \to FX$, the unique coalgebra morphism $!_X\colon X \to Z$ satisfies $\nu\Phi_c^{\mathbf{\Omega},\tau} = (!_X)^*\left(\nu\Phi_z^{\mathbf{\Omega},\tau}\right)$.*

Example 3.3.35 (bisimulation topology for deterministic automata)**.** Recall Example 3.3.32. For any A_Σ-coalgebra $c\colon X \to A_\Sigma X$, we defined the codensity bisimilarity on X by $\nu\Phi_c^{\mathbb{O},\{\mathrm{acc}\}\cup\{\langle a\rangle | a\in\Sigma\}} \in \mathbf{Top}_X$ [44].

The functor A_Σ has a final coalgebra: the set 2^{Σ^*} of all languages on the alphabet Σ can be given an A_Σ-coalgebra structure and it is final. For an A_Σ-coalgebra $c\colon X \to A_\Sigma X$, the unique coalgebra morphism $l\colon X \to 2^{\Sigma^*}$ assigns to each state the recognized language when started from it.

Corollary 3.3.34 implies that this map l determines the bisimulation topology on X. We believe that this fact is new, and it supports our use of the term *language topology* in [44, §VIII-C].

4. Codensity Games for Bisimilarity

4.1. Overview

4.1.1. Bisimilarity Notions and Games

Since the seminal works by Park and Milner [57, 55], *bisimilarity* has played a central role in theoretical computer science. It is an equivalence notion between branching systems; it abstracts away internal states and stresses the black-box observation-oriented view on process semantics. Bisimilarity is usually defined as the largest *bisimulation*, which is a binary relation that satisfies a suitable mimicking condition. In fact, a bisimulation R can be characterized as a post-fixed point $R \subseteq \Phi(R)$ using a suitable relation transformer Φ; from this we obtain that bisimilarity is the greatest *fixed* point of Φ by the Knaster–Tarski theorem. This order-theoretic foundation is the basis of a variety of advanced techniques for reasoning about (or using) bisimilarity, such as bisimulation up-to—see, e.g., [63].

Bisimilarity is conventionally defined for state-based systems with nondeterministic branching. However, as the applications of computer systems become increasingly pervasive and diverse (such as cyber-physical systems), extension of bisimilarity to systems with other branching types has been energetically sought in the literature. One notable example is the bisimulation notion for probabilistic systems in [51]: it is a relation that witnesses that two states are indistinguishable in their behaviors henceforth. This qualitative notion has also been made quantitative, as the notion of *bisimulation metric* [22]. It replaces a relation with a metric that is induced by the probabilistic transition structure.

There is a body of literature (including [32, 29, 6, 12, 48, 11, 77]) that aims to identify the mathematical essences that are shared by this variety of bisimilarity, and to express the identified essences in a rigorous manner using *category theory*. Our particular interest is in the correspondence between bisimilarity notions and *(safety) games*; three examples of the latter are given below. This interest in bisimilarity games is shared by the recent work [48], and the comparison is discussed in Section 4.1.4.

Bisimilarity Games

It is well-known that the following game (summarized in Table 4.1) characterizes the conventional notion of bisimilarity between Kripke frames. Let $(X, \to)$ be a Kripke frame where $\to \subseteq X^2$; the game is played between Duplicator (D) and Spoiler (S). In a position (x, y), Spoiler challenges Duplicator's claim that x and y are bisimilar, by choosing one of the states (say x) and further choosing a transition $x \to x'$. Duplicator responds by choosing a transition $y \to y'$ from the other state, and the game is continued from (x', y'). Duplicator wins if Spoiler gets stuck, or the game continues infinitely long, and this witnesses that x and y are bisimilar.

Table 4.1.: The game for bisimilarity in a Kripke frame

position	player	possible moves
$(x, y) \in X^2$	Spoiler	$(1, x', y)$ s.t. $x \to x'$ or $(2, x, y')$ s.t. $y \to y'$
$(1, x', y) \in \{1\} \times X^2$	Duplicator	(x', y') s.t. $y \to y'$
$(2, x, y') \in \{2\} \times X^2$	Duplicator	(x', y') s.t. $x \to x'$

Games for Probabilistic Bisimilarity

A recent step forward in the topic of bisimilarity and games is the characterization of probabilistic bisimulation introduced in [16]. For simplicity, here we describe its discrete version.

Let (X, c) be a Markov chain, where X is a countable set of states, and $c\colon X \to \mathcal{D}_{\leq 1}X$ is a transition kernel that assigns to each state $x \in X$ a probability subdistribution $c(x) \in \mathcal{D}_{\leq 1}X$. Here $\mathcal{D}_{\leq 1}X = \{d\colon X \to [0, 1] \mid \sum_{x \in X} d(x) \leq 1\}$ denotes the set of probability subdistributions over X. For $Z \subseteq X$, let $c(x)(Z)$ denote the probability with which a successor of x is chosen from Z; that is, $c(x)(Z) = \sum_{x' \in Z} c(x)(x')$. Since $c(x)$ is only a *sub*-distribution over X, the probability $c(x)(X)$ is ≤ 1 rather than $= 1$. The remaining probability $1 - c(x)(X)$ can be thought of as the probability of x getting stuck.

Recall from [51] that an equivalence relation $R \subseteq X^2$ is a *(probabilistic) bisimulation* if, for any $(x, y) \in R$ and each R-closed subset $Z \subseteq X$, $c(x)(Z) = c(y)(Z)$ holds.

Table 4.2.: The game for probabilistic bisimilarity from [16]

position	player	possible moves
$(x, y) \in X^2$	Spoiler	$Z \subseteq X$ s.t. $c(x)(Z) \neq c(y)(Z)$
$Z \subseteq X$	Duplicator	$(x', y') \in X^2$ s.t. $x' \in Z \wedge y' \notin Z$

The game introduced in [16] is in Table 4.2. It is shown in [16] that Duplicator is winning in the game at (x, y) if and only if x and y are bisimilar, in the sense of [51] (recalled above). It is not hard to find an intuitive correspondence between the game in Table 4.2 and the definition of bisimulation [51]: Spoiler challenges the bisimilarity claim between x, y by exhibiting Z such that $c(x)(Z) = c(y)(Z)$ is violated; Duplicator makes a counterargument by claiming that Z is in fact not bisimilarity-closed, exhibiting a pair of states (x', y') that Duplicator claims are bisimilar.

Games for Probabilistic Bisimulation Metric

Our following observation marked the beginning of the current work: the game for (qualitative) bisimilarity for probabilistic systems (from [16], Table 4.2) can be almost literally adapted to (quantitative) *bisimulation metric* for probabilistic systems. This metric was first introduced in [22].

For simplicity we focus on the discrete setting; we also restrict to pseudometrics bounded by 1. Let (X, c) be a Markov chain with a countable state space X. The *bisimulation metric* $d_{(X,c)} : X^2 \to [0,1]$ is defined to be the smallest pseudometric (with respect to the pointwise order) that makes the transition kernel

$$c \colon (X,\, d_{(X,c)}) \longrightarrow \big(\mathcal{D}_{\le 1}X,\, \mathcal{K}(d_{(X,c)})\big)$$

non-expansive with respect to the specified pseudometrics. Here $\mathcal{K}(d_{(X,c)})$ is the so-called *Kantorovich metric* over $\mathcal{D}_{\le 1}X$ induced by the pseudometric $d_{(X,c)}$ over X. It is defined as follows. For $\mu, \nu \in \mathcal{D}_{\le 1}X$,

$$\mathcal{K}(d_{(X,c)})(\mu,\nu) = \sup_f \left| E_\mu[f] - E_\nu[f] \right|, \tag{4.1}$$

where in the above sup,

- f ranges over all non-expansive functions from $(X, d_{(X,c)})$ to $\big([0,1], d_{[0,1]}\big)$,
- $d_{[0,1]}$ denotes the usual Euclidean metric, and
- $E_\mu[f]$ is the expectation $\sum_{x\in X} f(x)\cdot\mu(x)$ of f with respect to μ.

Our observation is that the bisimulation metric $d_{(X,c)}$ is characterized by the game in Table 4.3: Duplicator is winning at (x, y, ε) if and only if $d_{(X,c)}(x,y) \le \varepsilon$. The game

Table 4.3.: The game for (probabilistic) bisimulation metric, adapting [16]

position	player	possible moves
(x, y, ε) $\in X^2 \times [0,1]$	Spoiler	$f \colon X \to [0,1]$ such that $\left\lvert E_{c(x)}[f] - E_{c(y)}[f] \right\rvert > \varepsilon$
$f \colon X \to [0,1]$	Duplicator	$(x', y', \varepsilon') \in X^2 \times [0,1]$ such that $\left\lvert\, f(x') - f(y') \,\right\rvert > \varepsilon'$

seems to be new, although its intuition is similar to the one for Table 4.2. Note that the formula (4.1) appears in the condition of Spoiler's moves. Spoiler challenges by exhibiting a "predicate" f that suggests violation of the non-expansiveness of c; and Duplicator makes a counterargument that f is in fact not non-expansive and thus invalid.

Towards a Unifying Framework

The last two games (Table 4.2 from [16] and Table 4.3 that seems new) motivate a general framework that embraces both. There are some clear analogies: the games are about *indistinguishability* of states x, y under a class of *observations* (Z and f respectively), and the *predicates* usable in those observations are subject to certain preservation properties (bisimilarity-closedness in the former, and non-expansiveness in the latter).

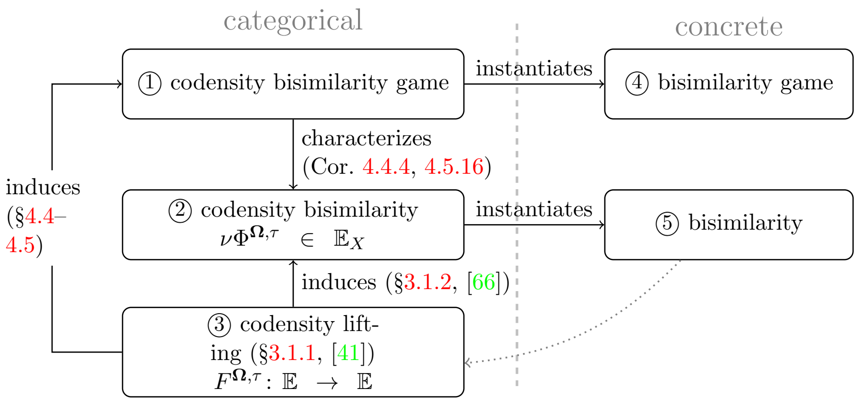

Figure 4.1.: Our codensity-based framework for bisimilarity and games

4.1.2. A Codensity-Based Framework for Bisimilarity and Games

The main contribution of the current paper is a categorical framework that derives a variety of bisimilarity notions and corresponding game notions. The correspondence is proved once and for all on the categorical level of generality. It covers the three examples introduced earlier in Section 4.1.1, much like the recent categorical framework in [48] does. However, our fibration-based formalization has another dimension of generality. For example, besides relations and metrics, our examples include an existing notion called *bisimulation seminorm* and a new one that we call *bisimulation topology*.

The overview of our categorical framework is in the left half of Fig. 4.1. We build on our previous works [41] and [66]. In [41] a general construction called *codensity lifting* is introduced (see ③): given a fibration $\mathbb{E} \xrightarrow{p} \mathbb{C}$ and parameters $(\mathbf{\Omega}, \tau)$ that embody the kind of *observations* we can make, a functor $F \colon \mathbb{C} \to \mathbb{C}$ is lifted to $F^{\mathbf{\Omega},\tau} \colon \mathbb{E} \to \mathbb{E}$. In [66], codensity lifting is used to introduce a generic family of bisimulation notions called *codensity bisimilarity*—see ②. In this paper, we extend these previous results by

- introducing the notion of *codensity bisimilarity game* (①) that comes in two variants (*untrimmed* (Section 4.4) and *trimmed* (Section 4.5)),
- establishing the correspondence between codensity bisimulations (②) and games (①) on a fibrational level of generality, and
- working out several concrete examples (④, ⑤).

In general, devising a game notion (④) directly from a bisimilarity notion (⑤) is far from trivial. Indeed, doing so for an individual bisimilarity notion has itself been deemed a scientific novelty [21, 16]. Our codensity-based framework (in the left half of Fig. 4.1) can automate *part of* this process in the following precise sense.

We derive concrete notions of bisimilarity (⑤) and bisimilarity game (④) as instances; then the correspondence between the two is guaranteed by the categorical general result between ① and ②.

We note, however, that this is no panacea. When one starts with a given concrete notion of bisimilarity (⑤), their next task would be to identify the right choice of the parameters $\mathbb{E} \xrightarrow{p} \mathbb{C}, \mathbf{\Omega}, \tau$ for the codensity lifting (③). This task is not easy in general: we needed to get our hands dirty working out the examples in this paper, in [41], and in [66]. Nevertheless, we believe that the required passage from ⑤ to ③ is much easier than the direct derivation from ⑤ to ④, with our categorical framework providing templates of bisimilarity games (see Tables 4.6, 4.8 and 4.9). After all, our framework identifies which part of the path from ⑤ to ④ can be automated, and which part remains to be done individually. This is much like what many other categorical frameworks offer, as meta-level theories.

As an additional benefit, our categorical framework can be used to *discover* new bisimilarity notions (⑤), starting from (choices of parameters for) ③. We believe those derived new bisimilarity notions are useful, since our categorical theory embodies sound intuitions about observation, predicate transformation, and indistinguishability—see e.g. Section 2.3.

4.1.3. Contributions

Our main technical contributions are as follows.

- We introduce a categorical framework that uniformly describes various bisimulation notions (including metrics, preorders and topologies) and the corresponding game notions (Fig. 4.1). The framework is based on coalgebras, fibrations, and codensity liftings in particular [41]. Our general game notion comes in two variants.
  - The first (the *untrimmed* codensity game in Section 4.4) arises naturally in a fibration, using its objects and arrows as possible moves. The untrimmed game is theoretically clean, but it tends to have a huge arena.
  - We therefore introduce a method that restricts these arenas, leading to the (*trimmed*) codensity bisimilarity game (Section 4.5). The reduction method is also described in general fibrational terms, specifically using fibered separators and join-dense subsets.
- From the general framework, we derive several concrete examples of bisimilarity and its related notions (④ and ⑤ in Fig. 4.1). They are listed in Table 3.1 and elaborated in Section 4.8. Among them, a few bisimilarity notions seem new (especially *bisimulation topology* in Section 4.8.3), and several game notions also seem new (especially that for Λ-*bisimulation* in Section 4.8.2).
- We discuss the *transfer of codensity bisimilarity* by suitable fibered functors (Section 4.7). As an example usage, we give an abstract proof of the fact that (usual) bisimilarity for Kripke frames is necessarily an equivalence (Example 4.7.4).

Additionally, we give a direct proof of the equivalence between our game for bisimulation metric (Table 4.3), obtained from our general framework, and another game notion for probabilistic bisimilarity, previously introduced in [21]. In the proof, we exhibit a mutual translation of winning strategies (Appendix A.1).

The current paper is an extended version of our previous paper [44]. The major additions are the following.

- We show a new transfer result in Section 4.7.2, which has a broader applicability than the result already presented in [44] (and in Section 4.7.1).
- In Section 4.8.1, we additionally present how to specialize codensity bisimilarity to recover another known notion of equivalence, namely *behavioral equivalence* (see [67] for its relation to bisimilarity). Some examples (already presented in [44]) are reorganized using the result.
- In Section 4.8.2, we show a new connection between our codensity bisimulation and an existing notion of Λ-*bisimulation* [4]. We also derive a general game characterization of some special cases of it, where all the modalities are unary.

We included some proofs that were omitted in [44], too.

4.1.4. Related Work

Besides the one in [16], another game characterization of probabilistic bisimulation has been given in [21]. It is described later in Section 4.2 (Table 4.4). The latter game has a bigger arena than the one in [16]: in [21] both players have to play a subset $Z \subseteq X$, while in [16] only Spoiler does so.

The work that is the closest to ours is the recent work [48] that studies bisimilarity games in a categorical setting. Their formalization uses (co)algebras (following the (co)algebraic generalization of the Kantorovich metric introduced in [6]), and therefore embraces a variety of different branching types. The major differences between the two works are as follows.

- Our current work is fibration-based (in particular $\mathbf{CLat}_\sqcap$-fibrations), while [48] is not. As a consequence, ours accommodates an additional dimension of generality by changing fibrations, which correspond to different indistinguishability notions (relation, metric, topology, preorder, measurable structures, etc.). In contrast, the works [48] and [6] deal exclusively with two settings: binary relations and pseudometrics.
- A relationship to *modal logic* is beautifully established in [48], while it is not done in this work. Some results connecting our codensity framework and modal logic are presented in [47].
- The categorical generalization [48] is based on the game notion in [21], while ours is based on that in [16]. Therefore, for some bisimulation notions (including the bisimulation metric), we obtain a game notion with a smaller arena. Compare Table 4.3 (an instance of ours) and Table 4.5 (an instance of [48]).

There are a number of categorical studies of bisimilarity notions; notable mentions include open map-based approaches [38] and coalgebraic ones [60, 37]. The fibrational approach we adopt also uses coalgebras; it was initiated in [32] and pursued, e.g., in [11, 29, 12], and [66]. For example, in the recent work [11], fibrational generality is exploited to study up-to techniques for bisimilarity metric. They use the *Wasserstein lifting* of functors introduced in [6] instead of the codensity lifting that we use (it generalizes the *Kantorovich lifting* in [6], see Example 3.1.6). It is known [6] that the Wasserstein and Kantorovich liftings can differ in general, while they coincide for some specific functors such as the distribution functor.

Some of our new examples are topological: we derive what we call *bisimulation topology* and a game notion that characterizes it. The relation between these notions and the existing works on bisimulation and topology (including [70, 19]) is left as future work.

In Section 4.5, we reduce the game arena by focusing on a join-dense subset. A game notion proposed in [7] uses a similar method. A major difference is that they restrict themselves to *continuous lattices*, while we only require each fiber to be a complete lattice. This condition plays a critical role in their framework, but it is a future work to seek consequences of the continuity assumption in our setting.

4.1.5. Organization

In Section 4.2, we present preliminaries on a general theory of games (we can restrict to *safety* games). We also make use of $\mathbf{CLat}_\sqcap$-*fibrations* as introduced in Chapter 2, and argue that they offer an appropriate categorical abstraction of sets equipped with indistinguishability structures. As mentioned in Chapters 1 and 3, an essential role is played by codensity lifting and codensity bisimilarity (②, ③ in Fig. 4.1). In Section 4.3, we introduce some auxiliary notions needed for the correspondence with games. Our first game notion (the *untrimmed* one) is introduced in Section 4.4; in Section 4.5, we cut down the arenas and obtain *trimmed* codensity bisimilarity game. The theory is further extended in Sections 4.6 and 4.7: in Section 4.6 we accommodate multiple observation domains, and in Section 4.7 we discuss the transfer of codensity bisimilarities by fibered functors preserving meets. These categorical observations give rise to the concrete examples in Section 4.8.

4.2. Safety Games

Here we recall some standard game-theoretic notions and results. In capturing bisimilarity-like notions, we can restrict ourselves to *safety games*—they have a simple winning condition where every infinite play is won by the same player (namely Duplicator). This winning condition reflects the characterization of bisimilarity-like notions by suitable *greatest* fixed points; the correspondence generalizes, for example, to the one between parity games and nested alternating fixed points—see [76]. The term "safety game" occurs, e.g., in [24, 9].

Safety games are played between two players; in this paper, they are called *Duplicator* (D) and *Spoiler* (S). We restrict to those games in which Duplicator and Spoiler alternate turns.

Definition 4.2.1 (safety game)**.** A *(safety game) arena* is a triple $\mathcal{G} = (Q_{\mathrm{D}}, Q_{\mathrm{S}}, E)$ of a set Q_{D} of *Duplicator's positions*, a set Q_{S} of *Spoiler's positions*, and a *transition relation* $E \subseteq (Q_{\mathrm{D}} \times Q_{\mathrm{S}}) \cup (Q_{\mathrm{S}} \times Q_{\mathrm{D}})$. Hence $\mathcal{G}$ is a bipartite graph. We require that Q_{D} and Q_{S} are disjoint, and that $Q_{\mathrm{D}} \cup Q_{\mathrm{S}} \neq \emptyset$. We write $Q = Q_{\mathrm{D}} \cup Q_{\mathrm{S}}$.

For a position $q \in Q$, an element of the set $\{q' \in Q \mid (q, q') \in E\}$ is called a *possible move* at q. Unlike some works, we allow positions that have no possible moves at them.

A *play* in an arena $\mathcal{G} = (Q_{\mathrm{D}}, Q_{\mathrm{S}}, E)$ is a (finite or infinite) sequence of positions $q_0 q_1 \dots$, such that $(q_{i-1}, q_i) \in E$ so long as q_i belongs to the sequence.

A play in $\mathcal{G}$ is *won* by either player, according to the following conditions: 1) a finite play $q_0 \dots q_n$ is won by Spoiler (or by Duplicator) if $q_n \in Q_{\mathrm{D}}$ (or $q_n \in Q_{\mathrm{S}}$ respectively); and 2) every infinite play $q_0 q_1 \dots$ is won by Duplicator.

Definition 4.2.2 (strategy, winning position)**.** In an arena $\mathcal{G} = (Q_{\mathrm{D}}, Q_{\mathrm{S}}, E)$, a *strategy* of Duplicator is a partial function $\sigma_{\mathrm{D}} \colon Q^* \times Q_{\mathrm{D}} \rightharpoonup Q_{\mathrm{S}}$; we require that $\sigma_{\mathrm{D}}(\vec{q}, q) = q'$ implies $(q, q') \in E$. A strategy of Duplicator σ_{D} is *positional* if $\sigma_{\mathrm{D}}(\vec{q}, q)$ depends only on q. A *strategy* of Spoiler is defined similarly, as a partial function $\sigma_{\mathrm{S}} \colon Q^* \times Q_{\mathrm{S}} \rightharpoonup Q_{\mathrm{D}}$ that returns a possible move at the last position in the history. It is *positional* if $\sigma_{\mathrm{S}}(\vec{q}, q)$ does not depend on $\vec{q}$.

Given an initial position $q \in Q$ and two strategies σ_{D} and σ_{S} for Duplicator and Spoiler respectively, the *play* from q induced by $(\sigma_{\mathrm{D}}, \sigma_{\mathrm{S}})$ is defined in a natural inductive manner. The induced play is denoted by $\pi^{\sigma_{\mathrm{D}}, \sigma_{\mathrm{S}}}(q)$.

A position $q \in Q$ is said to be *winning* for Duplicator if there exists a strategy σ_{D} of Duplicator such that, for any strategy σ_{S} of Spoiler, the induced play $\pi^{\sigma_{\mathrm{D}}, \sigma_{\mathrm{S}}}(q)$ is won by Duplicator.

In what follows, for simplicity, we restrict the initial position q of a play $\pi^{\sigma_{\mathrm{D}}, \sigma_{\mathrm{S}}}(q)$ to be in Q_{S}. (Note that Spoiler's position can be winning for Duplicator.)

Any position in a safety game is winning for one of the players. Moreover, the winning strategy can be taken to be positional one [76, Theorem 6]. Thus, we can focus on the winning positions of the players.

Winning positions of safety games are witnessed by *invariants* (Proposition 4.2.4). This is a well-known fact.

Definition 4.2.3 (invariant)**.** Let $\mathcal{G} = (Q_{\mathrm{D}}, Q_{\mathrm{S}}, E)$ be an arena. A subset $P \subseteq Q_{\mathrm{S}}$ is called an *invariant* for Duplicator if, for each $q \in P$ and any possible move $q' \in Q_{\mathrm{D}}$ at q, there exists a possible move q'' at q' that is in P. That is,

$$\forall q \in P. \forall q' \in Q_{\mathrm{D}}. \bigl((q, q') \in E \;\Rightarrow\; \exists q'' \in Q_{\mathrm{S}}.\, (q', q'') \in E \wedge q'' \in P \bigr).$$

Proposition 4.2.4.

1. *Any position $q \in P$ in an invariant P for Duplicator is winning for Duplicator.*
2. *Invariants are closed under arbitrary union. Therefore, there exists the largest invariant for Duplicator.*

3. *The largest invariant for Duplicator coincides with the set of winning positions for Duplicator in Q_S.*

Proof. 1. Turn P into a positional strategy of Duplicator that forces a play back in P.

2. Obvious.

3. It suffices to show that every position $q \in Q_{\mathrm{S}}$ winning for Duplicator lies in some invariant. Let $\sigma_{\mathrm{D}} \colon Q^* \times Q_{\mathrm{D}} \rightharpoonup Q_{\mathrm{S}}$ be a strategy of Duplicator ensuring that q is winning. Define $P \subseteq Q_{\mathrm{S}}$ as follows:

$$P = \{q' \in Q_{\mathrm{S}} \mid \exists \sigma_{\mathrm{S}}.\ q' \text{ is visited in } \pi^{\sigma_{\mathrm{D}},\sigma_{\mathrm{S}}}(q)\}.$$

Then P is an invariant because q is winning for Duplicator. □

Examples of safety games have been given in Tables 4.2 and 4.3. We present two other examples (Tables 4.4 and 4.5).

Example 4.2.5 (alternative games for probabilistic bisimilarity and bisimulation metric)**.** In [21], the notion of ε-*bisimulation* and a game notion characterizing it are introduced. In the case where ε is 0, ε-bisimulation coincides with (qualitative) probabilistic bisimilarity and thus the game characterizes it. The game in $\varepsilon = 0$ case is in Table 4.4, presented in a slightly adapted form. This game notion is categorically generalized

Table 4.4.: The game for probabilistic bisimilarity, from [21]

position	player	possible moves
$(1, x, y) \in \{1\} \times X^2$	Spoiler	$(2, s, t, Z) \in \{2\} \times X^2 \times \mathcal{P}X$ s.t. $\{s, t\} = \{x, y\}$
$(2, s, t, Z) \in$ $\{2\} \times X^2 \times \mathcal{P}X$	Duplicator	$(Z, Z') \in (\mathcal{P}X)^2$ s.t. $c(s)(Z) \leq c(t)(Z')$
$(Z, Z') \in (\mathcal{P}X)^2$	Spoiler	$(Z, y') \in \mathcal{P}X \times X$ s.t. $y' \in Z'$ or $(Z', y) \in \mathcal{P}X \times X$ s.t. $y \in Z$
$(Z, y') \in \mathcal{P}X \times X$	Duplicator	$(x', y') \in X^2$ s.t. $x' \in Z$

in [48]; the generalization has freedom in the choice of coalgebra functors (i.e. branching types), as well as in the choice between relations and metrics. The instance of this general game notion for bisimulation metric is shown in Table 4.5.

The two games (Tables 4.4 and 4.5) characterize the same bisimilarity-like notions as the games in Tables 4.2 and 4.3, respectively; so they are equivalent. We can go further and give a direct equivalence proof by mutually translating winning strategies. Such a proof is not totally trivial; we do so for the pair for probabilistic bisimilarity. See Appendix A.1.

We note that the game in Table 4.3 (an instance of our current framework) is simpler than Table 4.5 (an instance of [48]). Table 4.3 is not only structurally simpler (it has fewer rows), but its set of moves are smaller too, asking for functions $X \to [0, 1]$ only at one place.

Our categorical framework based on codensity liftings (presented in later sections) covers Tables 4.2 and 4.3 but not Tables 4.4 and 4.5.

Table 4.5.: The game for bisimulation metric, from [48]

position	player	possible moves
$(x, y, \varepsilon) \in X^2 \times [0,1]$	Spoiler	$(s, t, f, \varepsilon) \in X^2 \times [0,1]^X \times [0,1]$ s.t. $\{s, t\} = \{x, y\}$
$(s, t, f, \varepsilon) \in X^2 \times [0,1]^X \times [0,1]$	Duplicator	$(f, g, \varepsilon) \in ([0,1]^X)^2 \times [0,1]$ such that $\max\{0, E_{c(s)}[f] - E_{c(t)}[g]\} \leq \varepsilon$
$(f, g, \varepsilon) \in ([0,1]^X)^2 \times [0,1]$	Spoiler	$(x', i, j, \varepsilon) \in X \times ([0,1]^X)^2 \times [0,1]$ such that $\{i, j\} = \{f, g\}$
$(x', i, j, \varepsilon) \in X \times ([0,1]^X)^2 \times [0,1]$	Duplicator	$(x', y', \varepsilon') \in X^2 \times [0,1]$ such that $i(x') \leq j(y')$, and $\varepsilon' = j(y') - i(x')$

4.3. Joint Codensity Bisimulation

We introduce the notion of *joint codensity bisimulation*. This minor variation of codensity bisimulation becomes useful in the proof of soundness and completeness of our game notion (Section 4.4).

Definition 4.3.1 (joint codensity bisimulation)**.** Assume the setting of Definition 3.1.7. Let $\mathcal{V} \subseteq |\mathbb{E}_X|$; joins in $\mathbb{E}_X$ are denoted by $\bigsqcup$. We say that $\mathcal{V}$ is a *joint codensity bisimulation* over c if $\bigsqcup_{P \in \mathcal{V}} P$ is a codensity bisimulation over c.

For instance, the set of all codensity bisimulations is a joint codensity bisimulation because the join of all codensity bisimulations is the largest codensity bisimulation $\nu\Phi_c^{\mathbf{\Omega},\tau}$, as discussed just before Definition 3.1.10.

Lemma 4.3.2. *In the setting of Definition 3.1.7, the downset $\downarrow(\nu\Phi_c^{\mathbf{\Omega},\tau})$ is the largest joint codensity bisimulation (with respect to the inclusion order).*

Proof. The downset $\downarrow(\nu\Phi^{\mathbf{\Omega},\tau})$ is a joint codensity bisimulation, because the union of all elements of $\downarrow(\nu\Phi^{\mathbf{\Omega},\tau})$ is equal to a codensity bisimulation $\nu\Phi^{\mathbf{\Omega},\tau}$.

Let $\mathcal{V}$ be a joint codensity bisimulation. Then for any $P \in \mathcal{V}$, we have $P \sqsubseteq \nu\Phi^{\mathbf{\Omega},\tau}$, because $P \sqsubseteq \bigsqcup_{Q \in \mathcal{V}} Q \sqsubseteq \nu\Phi^{\mathbf{\Omega},\tau}$. □

4.4. Untrimmed Games for Codensity Bisimilarity

As the first main technical contribution, we introduce what we call the *untrimmed* version of codensity bisimilarity game. It is mathematically simple but its game arenas can become much bigger than necessary. The *trimmed* version of games—with smaller arenas—will be introduced later in Section 4.5, after developing necessary categorical infrastructure.

Definition 4.4.1 (untrimmed codensity bisimilarity game)**.** Assume the setting of Definition 3.1.7. The *untrimmed codensity bisimilarity game* is the safety game played by two players Duplicator and Spoiler, shown in Table 4.6.

Table 4.6.: Untrimmed codensity bisimilarity game

position	player	possible moves
$P \in \mathbb{E}_X$	Spoiler	$k \in \mathbb{C}(X, \Omega)$ s.t. $\tau \circ Fk \circ c : (X, P) \dot{\nrightarrow} (\Omega, \mathbf{\Omega})$
$k \in \mathbb{C}(X, \Omega)$	Duplicator	$P' \in \mathbb{E}_X$ s.t. $k : (X, P') \dot{\nrightarrow} (\Omega, \mathbf{\Omega})$

Lemma 4.4.2. *Assume the setting of Definition 3.1.7. Let $\mathcal{V} \subseteq |\mathbb{E}_X|$. The following are equivalent.*

1. *$\mathcal{V}$ is an invariant for Duplicator (Definition 4.2.3) in the untrimmed codensity bisimilarity game (Table 4.6).*
2. *$\mathcal{V}$ is a joint codensity bisimulation over c.*

Proof. We use the following logical equivalence:

$$
\begin{aligned}
1) \iff & \begin{pmatrix} \forall P \in \mathcal{V}, k \colon X \to \Omega. \\ \tau \circ Fk \circ c \colon (X, P) \dot{\nrightarrow} (\Omega, \mathbf{\Omega}) \implies \exists P' \in \mathcal{V}.\ k \colon (X, P') \dot{\nrightarrow} (\Omega, \mathbf{\Omega}) \end{pmatrix} \\
\iff & \begin{pmatrix} \forall P \in \mathcal{V}, k \colon X \to \Omega. \\ (\forall P' \in \mathcal{V}.\ k \colon (X, P') \dot{\to} (\Omega, \mathbf{\Omega})) \implies \tau \circ Fk \circ c \colon (X, P) \dot{\to} (\Omega, \mathbf{\Omega}) \end{pmatrix} \\
\iff & \begin{pmatrix} \forall k \colon X \to \Omega. \\ \quad (\forall P' \in \mathcal{V}.\ k \colon (X, P') \dot{\to} (\Omega, \mathbf{\Omega})) \\ \quad \implies \forall P \in \mathcal{V}.\ \tau \circ Fk \circ c \colon (X, P) \dot{\to} (\Omega, \mathbf{\Omega}) \end{pmatrix}.
\end{aligned}
$$

Here, since $k \colon (X, P') \dot{\to} (\Omega, \mathbf{\Omega})$ means $P' \sqsubseteq k^* \mathbf{\Omega}$, the condition

$$\forall P' \in \mathcal{V}.\ k \colon (X, P') \dot{\to} (\Omega, \mathbf{\Omega})$$

is equivalent to

$$k \colon (X, \textstyle\bigsqcup_{P' \in \mathcal{V}} P') \dot{\to} (\Omega, \mathbf{\Omega}).$$

Similarly, the condition

$$\forall P \in \mathcal{V}.\ \tau \circ Fk \circ c \colon (X, P) \dot{\to} (\Omega, \mathbf{\Omega})$$

is equivalent to

$$\tau \circ Fk \circ c \colon (X, \textstyle\bigsqcup_{P \in \mathcal{V}} P) \dot{\to} (\Omega, \mathbf{\Omega}).$$

These imply the following logical equivalence:

$$1) \iff \begin{pmatrix} \forall k \colon X \to \Omega. \\ \quad \left(k \colon \left(X, \bigsqcup_{P' \in \mathcal{V}} P'\right) \dot{\to} (\Omega, \mathbf{\Omega})\right) \\ \quad \implies \tau \circ Fk \circ c \colon (X, \bigsqcup_{P \in \mathcal{V}} P) \dot{\to} (\Omega, \mathbf{\Omega}) \end{pmatrix}.$$

By Theorem 3.1.9, the condition in the right-hand side is equivalent to

$$\bigsqcup_{P \in \mathcal{V}} P \sqsubseteq \Phi_c^{\mathbf{\Omega}, \tau} \left(\bigsqcup_{P \in \mathcal{V}} P \right).$$ □

Theorem 4.4.3. *Assume the setting of Definition 3.1.7. In the untrimmed codensity bisimilarity game (Table 4.8), the following coincide.*

1. *The set of all winning positions for Duplicator.*
2. *The downset $\downarrow(\nu\Phi_c^{\mathbf{\Omega},\tau})$ of the codensity bisimilarity.*

Proof. We use Lemma 4.4.2 to connect the game and the predicate transformer. By considering the largest set satisfying the condition in Lemma 4.4.2, it implies that the following two coincide if both exist:

1' the largest invariant for Duplicator in the game in Table 4.8 and

2' the largest joint codensity bisimulation over c.

By the general theory of safety games, in particular Proposition 4.2.4, the set (1') is equal to (1). On the other hand, by Lemma 4.3.2, the set (2') coincides with (2). Combining these proves the claim. □

We conclude that our game characterizes the codensity bisimilarity $\nu\Phi_c^{\mathbf{\Omega},\tau}$ (Definition 3.1.10).

Corollary 4.4.4 (soundness and completeness of untrimmed codensity games)**.** *In the untrimmed codensity bisimilarity game (Table 4.8), $P \in \mathbb{E}_X$ is a winning position for Duplicator if and only if $P \sqsubseteq \nu\Phi_c^{\mathbf{\Omega},\tau}$.* □

Example 4.4.5. Recall Example 3.1.11. Using the untrimmed codensity bisimilarity game, we can characterize the bisimulation metric from [22]. Our general definition (Definition 4.4.1) instantiates to the one in Table 4.7, which is however more complicated than the game we exhibited in the introduction (Table 4.3). For example, in Table 4.7, Duplicator's move is a pseudometric $d\colon X^2 \to [0,1]$ rather than a triple (x, y, ε).

Table 4.7.: Untrimmed codensity game for bisimulation metric

position	player	possible moves
$d \in (\mathbf{PMet}_1)_X$	Spoiler	$k \in \mathbf{Set}(X, [0,1])$ s.t. $e \circ Fk \circ c \notin \mathbf{PMet}_1(d, d_{[0,1]})$
$k \in \mathbf{Set}(X, [0,1])$	Duplicator	$d' \in (\mathbf{PMet}_1)_X$ s.t. $k \notin \mathbf{PMet}_1(d', d_{[0,1]})$

4.5. Trimmed Codensity Games for Bisimilarity

Our previous untrimmed game (Table 4.6) is pleasantly simple from a theoretical point of view. However, as we saw in Example 4.4.5, its instances tend to have a much bigger arena than some known game notions.

Here we push our theory a step further, and present a fibrational construction that allows us to *trim* our games. We note that our construction still remains on the fibrational level of abstraction.

4.5.1. Join-Dense Subsets of Fibers and Fibered Separators

Our approach to trim down the game arena is to restrict Spoiler's position to *approximants* of elements in the fiber complete lattice. In lattice theory, the collection of such approximants is specified by a *join-dense subset* [20], which we recall below.

Definition 4.5.1 (join-dense subset)**.** A subset $\mathcal{G}$ of a complete lattice L is *join-dense* if for any $P \in L$, there exists $\mathcal{A} \subseteq \mathcal{G}$ such that $P = \bigsqcup \mathcal{A}$.

Example 4.5.2. Consider the $\mathbf{CLat}_{\sqcap}$-fibration $\mathbf{EqRel} \to \mathbf{Set}$ and $X \in \mathbf{Set}$. For any $x, y \in X$, we define the equivalence relation $E_{x,y}$ to be the least one equating x, y, that is, $(z, w) \in E_{x,y}$ if and only if $(z = w \vee \{z, w\} = \{x, y\})$. Then the set $\mathcal{G} = \{E_{x,y} \mid x, y \in X\}$ of all such equivalence relations is a join-dense subset of the fiber $\mathbf{EqRel}_X$.

Example 4.5.3. Recall Example 3.1.11. For $x, y \in X$ $(x \neq y)$ and $r \in [0, 1]$, the pseudometric $d_{x,y,r}$ over X is defined by

$$d_{x,y,r}(z, w) = \begin{cases} 0 & z = w \\ r & \{z, w\} = \{x, y\} \\ 1 & \text{otherwise.} \end{cases}$$

Then the set of pseudometrics $\{d_{x,y,r} \mid x, y \in X, x \neq y, r \in [0, 1]\}$ is a join-dense subset of the fiber $(\mathbf{PMet}_1)_X$.

We use the following characterization of a join-dense subset.

Lemma 4.5.4. *For a subset $\mathcal{G}$ of a complete lattice L, the following are equivalent.*

- *$\mathcal{G}$ is join-dense.*
- *For any $P, Q \in L$,*

$$(\forall G \in \mathcal{G}.\ G \sqsubseteq P \implies G \sqsubseteq Q) \implies P \sqsubseteq Q$$

  holds.

Proof. Assume that $\mathcal{G}$ is join-dense. For any $P, Q \in L$, we show $(\forall G \in \mathcal{G}.\ G \sqsubseteq P \implies G \sqsubseteq Q) \implies P \sqsubseteq Q$. Since $\mathcal{G}$ is join-dense, there exists a subset $\mathcal{A} \subseteq \mathcal{G}$ such that $P = \bigsqcup \mathcal{A}$. If $(\forall G \in \mathcal{G}.\ G \sqsubseteq P \implies G \sqsubseteq Q)$ holds, then, for each $A \in \mathcal{A}$, we have $A \sqsubseteq P$, and thus $A \sqsubseteq Q$. This implies $P \sqsubseteq Q$.

Conversely, assume that, for any $P, Q \in L$, $(\forall G \in \mathcal{G}.\ G \sqsubseteq P \implies G \sqsubseteq Q) \implies P \sqsubseteq Q$ holds. We show that $\mathcal{G}$ is join-dense, that is, for any $P \in L$, there exists $\mathcal{A} \subseteq \mathcal{G}$ such that $P = \bigsqcup \mathcal{A}$. More concretely, we define

$$A^{\mathcal{G}}(P) = \{P' \in \mathcal{G} \mid P' \sqsubseteq P\}$$

for each $P \in L$ and we show $P = \bigsqcup A^{\mathcal{G}}(P)$. It suffices to show the following for each $Q \in L$:

$$(\forall P' \in A^{\mathcal{G}}(P).\ P' \sqsubseteq Q) \implies P \sqsubseteq Q.$$

By the definition of $A^{\mathcal{G}}(P)$, it is equivalent to the following:

$$(\forall P' \in \mathcal{G}.\ P' \sqsubseteq P \implies P' \sqsubseteq Q) \implies P \sqsubseteq Q.$$

This is nothing but our assumption. $\square$

We next consider the problem of equipping each fiber of a $\mathbf{CLat}_{\sqcap}$-fibration with a join-dense subset. One way to do so is to transfer a join-dense subset of the fiber over a special object called *fibered separator*, which we introduce below.

Definition 4.5.5 (fibered separator)**.** Let $\mathbb{E} \xrightarrow{p} \mathbb{C}$ be a $\mathbf{CLat}_{\sqcap}$-fibration. We say that $S \in \mathbb{C}$ is a *fibered separator* if, for any $X \in \mathbb{C}$ and $P, Q \in \mathbb{E}_X$, we have

$$(\forall f \in \mathbb{C}(S, X).\ f^*P = f^*Q) \implies P = Q.$$

Fibered separator can equivalently be defined using fiber order $\sqsubseteq$.

Lemma 4.5.6. *In the setting of Definition 4.5.5, the following are equivalent.*

- *$S \in \mathbb{C}$ is a fibered separator.*
- *For any $X \in \mathbb{C}$ and $P, Q \in \mathbb{E}_X$,*

  $$(\forall f \in \mathbb{C}(S, X).\ f^*P \sqsubseteq f^*Q) \implies P \sqsubseteq Q$$

  holds.

Proof. Assume that $S \in \mathbb{C}$ is a fibered separator. For each $X \in \mathbb{C}$ and $P, Q \in \mathbb{E}_X$, we show $(\forall f \in \mathbb{C}(S, X).\ f^*P \sqsubseteq f^*Q) \implies P \sqsubseteq Q$. Assume $(\forall f \in \mathbb{C}(S, X).\ f^*P \sqsubseteq f^*Q)$. Then for each $f \colon S \to X$ we have $f^*P = f^*P \sqcap f^*Q = f^*(P \sqcap Q)$ and, since S is a fibered separator, $P = P \sqcap Q$, that is, $P \sqsubseteq Q$. Thus we have $(\forall f \in \mathbb{C}(S, X).\ f^*P \sqsubseteq f^*Q) \implies P \sqsubseteq Q$.

Conversely, assume that $(\forall f \in \mathbb{C}(S, X).\ f^*P \sqsubseteq f^*Q) \implies P \sqsubseteq Q$ holds for any $X \in \mathbb{C}$ and $P, Q \in \mathbb{E}_X$. We show that $S \in \mathbb{C}$ is a fibered separator, that is, $(\forall f \in \mathbb{C}(S, X).\ f^*P = f^*Q) \implies P = Q$ for each $X \in \mathbb{C}$ and $P, Q \in \mathbb{E}_X$. Assume $(\forall f \in \mathbb{C}(S, X).\ f^*P = f^*Q)$. Then both $(\forall f \in \mathbb{C}(S, X).\ f^*P \sqsubseteq f^*Q)$ and $(\forall f \in \mathbb{C}(S, X).\ f^*P \sqsupseteq f^*Q)$ hold. By the assumption, we have both $P \sqsubseteq Q$ and $P \sqsupseteq Q$. Thus $P = Q$. $\square$

A join-dense subset of the fiber over a fibered separator induces one over any other fiber by the following theorem.

Theorem 4.5.7. *Let $S \in \mathbb{C}$ be a fibered separator of a $\mathbf{CLat}_{\sqcap}$-fibration $\mathbb{E} \xrightarrow{p} \mathbb{C}$, and $\mathcal{G}$ be a join-dense subset of $\mathbb{E}_S$. For any $X \in \mathbb{C}$, the following is a join-dense subset of $\mathbb{E}_X$ (below f_* denotes the pushforward along f; see Proposition 2.3.11):*

$$\{f_*G \mid G \in \mathcal{G}, f \in \mathbb{C}(S, X)\}.$$

Proof. Let $P, Q \in \mathbb{E}_X$. By Lemma 4.5.4, it suffices to show

$$(\forall G \in \mathcal{G}, f \in \mathbb{C}(S, X).\ f_* G \sqsubseteq P \implies f_* G \sqsubseteq Q) \implies P \sqsubseteq Q.$$

Since f_* is the left adjoint of f^* (Proposition 2.3.11), it is equivalent to

$$(\forall G \in \mathcal{G}, f \in \mathbb{C}(S, X).\ G \sqsubseteq f^* P \implies G \sqsubseteq f^* Q) \implies P \sqsubseteq Q.$$

Assume $(\forall G \in \mathcal{G}, f \in \mathbb{C}(S, X).\ G \sqsubseteq f^* P \implies G \sqsubseteq f^* Q)$. Since $\mathcal{G}$ is join-dense in $\mathbb{E}_S$, Lemma 4.5.4 implies $(\forall f \in \mathbb{C}(S, X).\ f^* P \sqsubseteq f^* Q)$. It in turn implies $P \sqsubseteq Q$ by Lemma 4.5.6. □

In fact, it is Theorem 4.5.7 that is behind Examples 4.5.2 and 4.5.3: in both cases, $2 \in \mathbf{Set}$ turns out to be a fibered separator for the fibrations in question ($\mathbf{EqRel} \to \mathbf{Set}$ and $\mathbf{PMet}_1 \to \mathbf{Set}$), and the presented generating sets are obtained via pushforward.

We next relate fibered separators and separators in a category $\mathbb{C}$. Recall that an object S in a category $\mathbb{C}$ is a *separator* [54, Section V.7] if for any parallel pair of morphisms $f, g : X \to Y$, if $f \circ x = g \circ x$ holds for any $x : S \to X$, then $f = g$.

Proposition 4.5.8 (fibered separator and separator)**.** *Let $\mathbb{E} \xrightarrow{p} \mathbb{C}$ be a $\mathbf{CLat}_\sqcap$-fibration. Let $V \in \mathbb{C}$ be an object such that there is a family of injections $\iota_X \colon |\mathbb{E}_X| \rightarrowtail \mathbb{C}(X, V)$ natural in $X \in \mathbb{C}$. If $S \in \mathbb{C}$ is a separator of $\mathbb{C}$, then it is also a fibered separator of p.*

Note that here we regard $|\mathbb{E}_{(\_)}|$ as a contravariant functor $\mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$ by the pullback operation.

Proof. Assume that $S \in \mathbb{C}$ is a separator of $\mathbb{C}$. Expanding the definition for $V \in \mathbb{C}$ yields the following:

$$\forall X \in \mathbb{C}, p, q \in \mathbb{C}(X, V).\ \big((\forall f \in \mathbb{C}(S, X).\ p \circ f = q \circ f) \implies p = q\big). \tag{4.2}$$

Now, let $X \in \mathbb{C}$ and $P, Q \in \mathbb{E}_X$. We show the implication in Definition 4.5.5, as follows.

$$\begin{aligned}
&(\forall f \in \mathbb{C}(S, X).\ f^* P = f^* Q) \\
&\implies (\forall f \in \mathbb{C}(S, X).\ \iota_S(f^* P) = \iota_S(f^* Q)) \\
&\implies (\forall f \in \mathbb{C}(S, X).\ \iota_S(P) \circ f = \iota_S(Q) \circ f) && \text{(by naturality)} \\
&\implies \iota_S(P) = \iota_S(Q) && \text{(by (4.2))} \\
&\implies P = Q. && \text{(since } \iota_X \text{ is injective)}
\end{aligned}$$

Thus $S \in \mathbb{C}$ is fibered separator. □

Any example of this is "unary," as can be seen in the following one.

Example 4.5.9 (fibered separator of $\mathbf{Pred} \to \mathbf{Set}$)**.** We define the category $\mathbf{Pred}$ as follows:

- An object is a triple $(X, R \subseteq X)$ of a set and a predicate on it.
- An arrow from (X, R) to (Y, S) is a function $f\colon X \to Y$ such that $x \in R$ implies $f(x) \in S$.

The forgetful functor $\mathbf{Pred} \to \mathbf{Set}$ is then a $\mathbf{CLat}_\sqcap$-fibration.

There exists a natural family of injections $\iota_X\colon |\mathbf{Pred}_X| \rightarrowtail \mathbf{Set}(X, 2)$, which sends $R \subseteq X$ to $\iota_X(R)\colon X \to 2$ defined as follows:

$$\iota_X(R)(x) = \begin{cases} \top & \text{if } x \in R \\ \bot & \text{otherwise} \end{cases}$$

Since 1 is a separator of $\mathbf{Set}$, we can conclude that it is also a fibered separator of $\mathbf{Pred} \to \mathbf{Set}$ by Proposition 4.5.8.

The following result is useful in finding fibered separators—see Section 4.8.5.

Proposition 4.5.10 (change-of-base and fibered separators)**.** *Let $\mathbb{E} \xrightarrow{p} \mathbb{C}$ be a $\mathbf{CLat}_\sqcap$-fibration, $R\colon \mathbb{D} \to \mathbb{C}$ be a functor with a left adjoint $L\colon \mathbb{C} \to \mathbb{D}$, and $S \in \mathbb{C}$ be a fibered separator for p. Then $LS \in \mathbb{D}$ is a fibered separator for the change-of-base fibration R^*p.*

$$\begin{tikzcd} R^*\mathbb{E} \arrow[r] \arrow[d, "R^*p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{E} \arrow[d, "p"] \\ \mathbb{D} \arrow[r, shift left, "R"] \arrow[r, phantom, "\top"] & \mathbb{C} \arrow[l, shift left, "L"] \end{tikzcd}$$

Proof. For any $X \in \mathbb{D}$, the mapping $f \mapsto Rf \circ \eta_S$ is a bijection of type $\mathbb{D}(LS, X) \to \mathbb{C}(S, RX)$. Thus, naturally identifying $(R^*\mathbb{E})_X$ and $\mathbb{E}_{RX}$, we have the following for any $P, Q \in (R^*\mathbb{E})_X$.

$$\begin{aligned} \forall f \in \mathbb{D}(LS, X).\ f^*P = f^*Q &\implies \forall f \in \mathbb{D}(LS, X).\ (Rf)^*P = (Rf)^*Q \\ &\implies \forall f \in \mathbb{D}(LS, X).\ (Rf \circ \eta_S)^*P = (Rf \circ \eta_S)^*Q \\ &\iff \forall g \in \mathbb{C}(S, RX).\ g^*P = g^*Q \\ &\iff P = Q \end{aligned}$$

□

4.5.2. $\mathcal{G}$-Joint Codensity Bisimulation

We use join-dense subsets to restrict moves in codensity games.

Definition 4.5.11. In the setting of Definition 3.1.7, let $\mathcal{G}$ be a join-dense subset of $\mathbb{E}_X$. A *$\mathcal{G}$-joint codensity bisimulation* over $c : X \to FX$ is a joint codensity bisimulation $\mathcal{V}$ over c such that $\mathcal{V} \subseteq \mathcal{G}$.

Lemma 4.5.12. *Assume the setting of Definition 3.1.7, and let $\mathcal{G}$ be a join-dense subset of $\mathbb{E}_X$. The intersection $\big(\downarrow(\nu\Phi_c^{\mathbf{\Omega},\tau})\big) \cap \mathcal{G}$ of the downset $\downarrow(\nu\Phi_c^{\mathbf{\Omega},\tau})$ and the join-dense subset $\mathcal{G}$ is the largest $\mathcal{G}$-joint codensity bisimulation.*

Proof. Since $\mathcal{G}$ is join-dense, the union of all elements of $\downarrow(\nu\Phi_c^{\mathbf{\Omega},\tau}) \cap \mathcal{G}$ is equal to $\nu\Phi_c^{\mathbf{\Omega},\tau}$. Thus, $\downarrow(\nu\Phi_c^{\mathbf{\Omega},\tau}) \cap \mathcal{G}$ is a $\mathcal{G}$-joint codensity bisimulation.

For any $\mathcal{G}$-joint codensity bisimulation $\mathcal{V}$, it is a joint codensity bisimulation, and we have already shown $\mathcal{V} \subseteq \downarrow(\nu\Phi_c^{\mathbf{\Omega},\tau})$ in the proof of Lemma 4.3.2. We also have $\mathcal{V} \subseteq \mathcal{G}$ by definition. These imply $\mathcal{V} \subseteq \downarrow(\nu\Phi_c^{\mathbf{\Omega},\tau}) \cap \mathcal{G}$. □

4.5.3. Trimmed Codensity Bisimilarity Games

The above structural results lead to our second game notion.

Definition 4.5.13 (trimmed codensity bisimilarity game)**.** Assume the setting of Definition 3.1.7, and that $\mathcal{G} \subseteq \mathbb{E}_X$ is a join-dense subset. The *(trimmed) codensity bisimilarity game* is the safety game played by two players Duplicator and Spoiler, shown in Table 4.8.

Table 4.8.: Trimmed codensity bisimilarity game

position	player	possible moves
$P \in \mathcal{G}$	Spoiler	$k \in \mathbb{C}(X,\Omega)$ s.t. $\tau \circ Fk \circ c : (X,P) \dot{\nrightarrow} (\Omega,\mathbf{\Omega})$
$k \in \mathbb{C}(X,\Omega)$	Duplicator	$P' \in \mathcal{G}$ s.t. $k : (X,P') \dot{\nrightarrow} (\Omega,\mathbf{\Omega})$

Lemma 4.5.14. *Assume the setting of Definition 4.5.13. Let $\mathcal{V} \subseteq |\mathbb{E}_X|$. The following are equivalent.*

1. *$\mathcal{V}$ is an invariant for Duplicator (Definition 4.2.3) in the trimmed codensity bisimilarity game (Table 4.8).*
2. *$\mathcal{V}$ is a $\mathcal{G}$-joint codensity bisimulation over c.*

Proof. For a subset $\mathcal{V} \subseteq |\mathbb{E}_X|$, the condition (1) is equivalent to the condition that the following both holds:

a For any $P \in \mathcal{V}$ and $k\colon X \to \Omega$ satisfying $\tau \circ Fk \circ c : (X,P) \dot{\nrightarrow} (\Omega,\mathbf{\Omega})$, there exists $P' \in \mathcal{V}$ such that $k : (X,P') \dot{\nrightarrow} (\Omega,\mathbf{\Omega})$ holds.

b $\mathcal{V} \subseteq \mathcal{G}$.

The above condition (a) is equivalent to "$\mathcal{V}$ is an invariant for Duplicator in the (untrimmed) codensity bisimilarity game (Table 4.6)." By Lemma 4.4.2, it is equivalent to "$\mathcal{V}$ is a joint codensity bisimulation over c."

Thus, the condition (1) is equivalent to "$\mathcal{V}$ is a joint codensity bisimulation c and it is a subset of $\mathcal{G}$." This is, by definition, equivalent to the condition (2). □

Theorem 4.5.15. *Assume the setting of Definition 4.5.13. The following sets coincide.*

1. *The set of winning positions for Duplicator in the trimmed codensity bisimilarity game (Table 4.8).*
2. *The intersection $\big(\downarrow(\nu\Phi_c^{\mathbf{\Omega},\tau})\big) \cap \mathcal{G}$ of the downset of the codensity bisimilarity over c and the join-dense subset $\mathcal{G}$.*

Proof. By Proposition 4.2.4, (1) is the largest invariant for Duplicator in the trimmed codensity bisimilarity game (Table 4.8). In turn, by Lemma 4.5.14, it is the largest $\mathcal{G}$-joint codensity bisimulation over c. By Lemma 4.5.12, it coincides with (2). $\square$

We conclude that our second game characterizes the codensity bisimilarity $\nu\Phi_c^{\mathbf{\Omega},\tau}$ (Definition 3.1.10) too.

Corollary 4.5.16 (soundness and completeness of trimmed codensity games)**.** *In Definition 4.5.13, $P \in \mathcal{G}$ is a winning position for Duplicator if and only if $P \sqsubseteq \nu\Phi_c^{\mathbf{\Omega},\tau}$.* $\square$

4.6. Multiple Observation Domains

Now we extend the framework using codensity lifting with multiple parameters (Section 3.2).

The theoretical development is completely parallel to the one in Sections 4.4 and 4.5. The difference is that we have to replace a single-parameter codensity lifting (Definition 5.2.8) by a multi-parameter one (Definition 3.2.1).

Theorem 4.6.1. *Assume the setting of Definition 3.2.2. For any $P \in \mathbb{E}_X$, the following are equivalent.*

1. *$c : (X, P) \dot{\to} (FX, F^{\mathbf{\Omega},\tau}P)$; that is, P is a codensity bisimulation over c (Definition 3.2.2).*
2. *$P \sqsubseteq \Phi_c^{\mathbf{\Omega},\tau} P$.*
3. *For each $A \in \mathbb{A}$ and $k \in \mathbb{C}(X, \Omega_A)$, $k : (X, P) \dot{\to} (\Omega_A, \mathbf{\Omega}_A)$ implies $\tau_A \circ Fk \circ c : (X, P) \dot{\to} (\Omega_A, \mathbf{\Omega}_A)$.*

Proof. The same as Theorem 3.1.9, except that we have multiple parameters here. $\square$

Definition 4.6.2 (joint codensity bisimulation)**.** Assume the setting of Definition 3.2.2. We say that $\mathcal{V} \subseteq |\mathbb{E}_X|$ is a *joint codensity bisimulation* over c if $\bigsqcup_{P\in\mathcal{V}} P$ is a codensity bisimulation over c.

Definition 4.6.3. In the setting of Definition 3.2.2, let $\mathcal{G}$ be a join-dense subset of $\mathbb{E}_X$. A *$\mathcal{G}$-joint codensity bisimulation* over $c : X \to FX$ is a joint codensity bisimulation $\mathcal{V}$ over c such that $\mathcal{V} \subseteq \mathcal{G}$.

Lemma 4.6.4. *Assume the setting of Definition 4.6.3. The intersection $\downarrow(\nu\Phi_c^{\mathbf{\Omega},\tau}) \cap \mathcal{G}$ of the join-dense subset $\mathcal{G}$ and the downset $\downarrow(\nu\Phi_c^{\mathbf{\Omega},\tau})$ is the largest $\mathcal{G}$-joint codensity bisimulation.*

Proof. The same as Lemma 4.5.12, except that we have multiple parameters here. $\square$

Definition 4.6.5 (codensity bisimilarity game)**.** In the setting of Definition 3.2.2, let $\mathcal{G}$ be a join-dense subset of $\mathbb{E}_X$. The *(trimmed) codensity bisimilarity game (with multiple observations)* is the safety game, played by two players D and S, shown in Table 4.9.

Table 4.9.: Trimmed codensity bisimilarity game with multiple observations

position	player	possible moves
$P \in \mathcal{G}$	Spoiler	$A \in \mathbb{A}$ and $k \in \mathbb{C}(X, \Omega_A)$ s.t. $\tau_A \circ Fk \circ c : (X, P) \dot{\nrightarrow} (\Omega_A, \mathbf{\Omega}_A)$
$A \in \mathbb{A}$ and $k \in \mathbb{C}(X, \Omega_A)$	Duplicator	$P' \in \mathcal{G}$ s.t. $k : (X, P') \dot{\nrightarrow} (\Omega_A, \mathbf{\Omega}_A)$

Lemma 4.6.6. *Assume the setting of Definition 4.6.3. Let $\mathcal{V} \subseteq |\mathbb{E}_X|$ be a set of objects. The following are equivalent.*

1. *$\mathcal{V}$ is an invariant for Duplicator in the trimmed codensity bisimilarity game with multiple observations (Table 4.9).*

2. *$\mathcal{V}$ is a $\mathcal{G}$-joint codensity bisimulation over c.*

Proof. We use the following logical equivalence:

$$
\begin{aligned}
1) \iff &\left(\begin{array}{l} \mathcal{V} \subseteq \mathcal{G} \text{ and} \\ \forall P \in \mathcal{V}, A \in \mathbb{A}, k \colon X \to \Omega_A. \\ \quad \tau_A \circ Fk \circ c \colon (X, P) \dot{\nrightarrow} (\Omega_A, \mathbf{\Omega}_A) \\ \quad \implies \exists P' \in \mathcal{V}.\ k \colon (X, P') \dot{\nrightarrow} (\Omega_A, \mathbf{\Omega}_A) \end{array}\right) \\
\iff &\left(\begin{array}{l} \mathcal{V} \subseteq \mathcal{G} \text{ and} \\ \forall P \in \mathcal{V}, A \in \mathbb{A}, k \colon X \to \Omega_A. \\ \quad (\forall P' \in \mathcal{V}.\ k \colon (X, P') \dot{\to} (\Omega_A, \mathbf{\Omega}_A)) \\ \quad \implies \tau_A \circ Fk \circ c \colon (X, P) \dot{\to} (\Omega_A, \mathbf{\Omega}_A) \end{array}\right) \\
\iff &\left(\begin{array}{l} \mathcal{V} \subseteq \mathcal{G} \text{ and} \\ \forall A \in \mathbb{A}, k \colon X \to \Omega_A. \\ \quad (\forall P' \in \mathcal{V}.\ k \colon (X, P') \dot{\to} (\Omega_A, \mathbf{\Omega}_A)) \\ \quad \implies \forall P \in \mathcal{V}.\ \tau_A \circ Fk \circ c \colon (X, P) \dot{\to} (\Omega_A, \mathbf{\Omega}_A) \end{array}\right).
\end{aligned}
$$

Here, since $k \colon (X, P') \dot{\to} (\Omega_A, \mathbf{\Omega}_A)$ means $P' \sqsubseteq k^* \mathbf{\Omega}_A$, the condition

$$\forall P' \in \mathcal{V}.\ k \colon (X, P') \dot{\to} (\Omega_A, \mathbf{\Omega}_A)$$

is equivalent to

$$k \colon (X, \textstyle\bigsqcup_{P' \in \mathcal{V}} P') \dot{\to} (\Omega_A, \mathbf{\Omega}_A).$$

Similarly, the condition

$$\forall P \in \mathcal{V}.\ \tau_A \circ Fk \circ c \colon (X, P) \dot{\to} (\Omega_A, \mathbf{\Omega}_A)$$

is equivalent to

$$\tau_A \circ Fk \circ c \colon (X, \textstyle\bigsqcup_{P \in \mathcal{V}} P) \dot{\to} (\Omega_A, \mathbf{\Omega}_A).$$

These imply the following logical equivalence:

$$1) \iff \begin{pmatrix} \mathcal{V} \subseteq \mathcal{G} \text{ and} \\ \forall A \in \mathbb{A}, k \colon X \to \Omega_A. \\ \quad \left(k \colon \left(X, \bigsqcup_{P' \in \mathcal{V}} P'\right) \dot{\to} (\Omega_A, \mathbf{\Omega}_A)\right) \\ \quad \implies \tau_A \circ Fk \circ c \colon (X, \bigsqcup_{P \in \mathcal{V}} P) \dot{\to} (\Omega_A, \mathbf{\Omega}_A) \end{pmatrix}.$$

By Theorem 4.6.1, the condition in the right-hand side is equivalent to the conjunction of $\mathcal{V} \subseteq \mathcal{G}$ and

$$\bigsqcup_{P \in \mathcal{V}} P \sqsubseteq \Phi_c^{\mathbf{\Omega},\tau} \left(\bigsqcup_{P \in \mathcal{V}} P \right). \qquad \square$$

Theorem 4.6.7. *Assume the setting of Definition 4.6.3. Let $\mathcal{G} \subseteq |\mathbb{E}_X|$ be a join-dense subset. The following sets coincide.*

1. *The set of winning positions for Duplicator in the game in Table 4.9.*
2. *The intersection $(\downarrow(\nu\Phi_c^{\mathbf{\Omega},\tau})) \cap \mathcal{G}$ of the downset $\downarrow(\nu\Phi_c^{\mathbf{\Omega},\tau})$ of the codensity bisimilarity over c and the join-dense subset $\mathcal{G}$.*

Proof. By Proposition 4.2.4, (1) is the largest invariant for Duplicator in the game in Table 4.8. In turn, by Lemma 4.6.6, it is the largest $\mathcal{G}$-joint codensity bisimulation over c. By Lemma 4.6.4, it coincides with (2). $\square$

Corollary 4.6.8 (soundness and completeness of codensity games)**.** *Assume the setting of Definition 4.6.5. In particular, let $\mathcal{G}$ be a join-dense subset of $\mathbb{E}_X$. $P \in \mathbb{E}_X$ is a winning position for Duplicator if and only if $P \sqsubseteq \nu\Phi_c^{\mathbf{\Omega},\tau}$.* $\square$

Example 4.6.9 (bisimulation topology for deterministic automata)**.** Here we describe the topological example in Table 2.1. Consider the $\mathbf{CLat}_\sqcap$-fibration $\mathbf{Top} \to \mathbf{Set}$ and the functor $A_\Sigma = 2 \times (\_)^\Sigma \colon \mathbf{Set} \to \mathbf{Set}$, where Σ is a fixed alphabet. Coalgebras for this functor are deterministic automata over Σ; see e.g. [37, 60].

We take the following data as a parameter of codensity lifting (cf. Definition 3.2.1): $\mathbb{A} = \{\varepsilon\} \cup \Sigma$, $\mathbf{\Omega}_\alpha$ is the Sierpinski space for each $\alpha \in \mathbb{A}$, and the modalities $\tau_\varepsilon, \tau_a \colon A_\Sigma 2 \to 2$ (where $a \in \Sigma$) are defined by

$$\tau_\varepsilon(t, \rho) = t \quad \text{and} \quad \tau_a(t, \rho) = \rho(a).$$

Recall that the Sierpinski space is the set $2 = \{\bot, \top\}$ with the topology $\{\emptyset, \{\top\}, 2\}$. Based on the slogan "Open sets are semi-decidable properties," which is explained in, e.g., [72], this observation domain models the situation where acceptance of a word is only *semi*-decidable, not decidable, in the sense of computability theory.

Let $c \colon X \to A_\Sigma X$ be a deterministic automata. The above choice of parameters leads to the following codensity bisimilarity: the state space X is equipped with the topology generated by the following family of open sets.

$$\{x \in X \mid w \text{ is accepted from } x\} \subseteq X, \quad \text{for each } w \in \Sigma^*$$

One can extract various information from this *bisimulation topology* via standard topological constructs. For example, the specialization order (see, e.g., [72, Chapter 7]) of this topology coincides with the language inclusion order.

For illustration by comparison, consider changing the observation domain from the Sierpinski space to the discrete 2-point set. The bisimulation topology over X is now generated by

$$\begin{aligned}&\{x \in X \mid w \text{ is accepted from } x\} \text{ and}\\ &\{x \in X \mid w \text{ is not accepted from } x\}, \quad \text{for each } w \in \Sigma^*.\end{aligned}$$

We can now observe rejection of a word, too, because $\{\bot\} \subseteq 2$ is open. The specialization order of this topology is the language equivalence, and it satisfies the R0 separation axiom (while the last Sierpinski example does not).

We take these examples of bisimulation topology as a process-semantical incarnation of the "observability via topology, computability via continuity" paradigm from domain theory. The definition of codensity bisimulation (cf. Definition 5.2.8) fits well with this intuition, too: a continuous map $k\colon (X, P) \dot{\to} \mathbf{\Omega}$ in Definition 5.2.8 is a "computable observation"; accordingly, an open set of the bisimulation topology is a property that is decided by finitely many of those computable observations.

4.7. Transfer of Codensity Bisimilarities

In our formulation, for the same endofunctor $F\colon \mathbb{C} \to \mathbb{C}$, we can use various $\mathbf{CLat}_\sqcap$-fibrations p and parameters $(\mathbf{\Omega}, \tau)$ to equip F-coalgebras with different bisimilarity-like notions. Some relations among those codensity bisimilarity notions can be categorically captured by general results. In this section we show two such results.

Definition 4.7.1. In this section, we consider the following situation:

$$\begin{tikzcd} \mathbb{E} \arrow[rr, "T"] \arrow[dr, "p"'] & & \mathbb{F} \arrow[dl, "q"] \\ & \mathbb{C} \arrow[loop right, "F"] & \end{tikzcd}.$$

Here, $p\colon \mathbb{E} \to \mathbb{C}$ and $q\colon \mathbb{F} \to \mathbb{C}$ are $\mathbf{CLat}_\sqcap$-fibrations. We assume that $q \circ T = p$ holds on the nose, and that T is "fibered": for $f\colon X \to Y$ in $\mathbb{C}$ and $E \in \mathbb{E}_Y$, $f^*(TE) = T(f^*E)$ holds.

4.7.1. Transfer Result for One Shared Family of Parameters

Firstly, we consider the case where the families of parameters are "shared" among two fibrations.

We use the following lemma.

Lemma 4.7.2 ([66, Proposition 6.2])**.** *In the setting of Definition 4.7.1, assume also that T preserves fiberwise meets. Let $\dot{F}\colon \mathbb{E} \to \mathbb{E}$ and $\ddot{F}\colon \mathbb{F} \to \mathbb{F}$ be liftings of F along p and q, respectively. Let $c\colon X \to FX$ be an F-coalgebra. If $T\dot{F}P = \ddot{F}TP$ holds for each $P \in \mathbb{E}$, then $T\nu(c^* \circ \dot{F}) = \nu(c^* \circ \ddot{F})$ holds.*

Proof. For each ordinal α, we define $\nu_\alpha(c^* \circ \dot{F})$ by

$$\nu_\alpha(c^* \circ \dot{F}) = \bigsqcap_{\beta<\alpha} c^* \dot{F}(\nu_\beta(c^* \circ \dot{F}))$$

using induction on α. We define $\nu_\alpha(c^* \circ \ddot{F})$ in the same way. By Fact 2.2.4, these converge to $\nu(c^* \circ \dot{F})$ and $\nu(c^* \circ \ddot{F})$, respectively. It suffices to show $T\nu_\alpha(c^* \circ \dot{F}) = \nu_\alpha(c^* \circ \ddot{F})$ by induction on α.

Assume that the above inequality holds for all ordinals smaller than α. Then we have

$$\begin{aligned}
T\nu_\alpha(c^* \circ \dot{F}) &= T \bigsqcap_{\beta<\alpha} c^* \dot{F}(\nu_\beta(c^* \circ \dot{F})) \\
&= \bigsqcap_{\beta<\alpha} Tc^* \dot{F}(\nu_\beta(c^* \circ \dot{F})) && \text{(since } T \text{ preserves meets)} \\
&= \bigsqcap_{\beta<\alpha} c^* T\dot{F}(\nu_\beta(c^* \circ \dot{F})) && \text{(since } T \text{ is fibered)} \\
&= \bigsqcap_{\beta<\alpha} c^* \ddot{F}T(\nu_\beta(c^* \circ \dot{F})) && \text{(by the assumption } T\dot{F} = \ddot{F}T\text{)} \\
&= \bigsqcap_{\beta<\alpha} c^* \ddot{F}(\nu_\beta(c^* \circ \ddot{F})) && \text{(by induction hypothesis)} \\
&= \nu_\alpha(c^* \circ \ddot{F}).
\end{aligned}$$

□

The following is the main result of Section 4.7.1. Note that the parameters $\{(T\mathbf{\Omega}_A, \tau_A)\}_{A\in\mathbb{A}}$ for $q\colon \mathbb{F} \to \mathbb{C}$ are "induced" from $\{(\mathbf{\Omega}_A, \tau_A)\}_{A\in\mathbb{A}}$ for $p\colon \mathbb{E} \to \mathbb{C}$.

Theorem 4.7.3 (transfer of codensity bisimilarity)**.** *In the setting of Definition 4.7.1, let $c\colon X \to FX$ be an F-coalgebra and $\{(\mathbf{\Omega}_A, \tau_A)\}_{A\in\mathbb{A}}$ be an $\mathbb{A}$-indexed family of parameters for codensity lifting of F along p (Definition 3.2.1). Assume that $T\colon \mathbb{E} \to \mathbb{F}$ is full and faithful, and that it preserves fiberwise meets. In this setting, $\{(T\mathbf{\Omega}_A, \tau_A)\}_{A\in\mathbb{A}}$ is an $\mathbb{A}$-indexed family of parameters for codensity lifting of F along q, and we have $\nu\Phi_c^{T\mathbf{\Omega},\tau} = T(\nu\Phi_c^{\mathbf{\Omega},\tau})$.*

Proof. For any $P \in \mathbb{E}_X$, we have $TF^{\mathbf{\Omega},\tau}P = F^{T\mathbf{\Omega},\tau}TP$ because the following hold:

$$
\begin{aligned}
& TF^{\mathbf{\Omega},\tau}P \\
&= T\left(\bigsqcap_{A\in\mathbb{A}}\ \bigsqcap_{k\in\mathbb{E}(P,\mathbf{\Omega}_A)} (\tau_A \circ F(pk))^*\mathbf{\Omega}_A\right) \\
&= \bigsqcap_{A\in\mathbb{A}}\ \bigsqcap_{k\in\mathbb{E}(P,\mathbf{\Omega}_A)} T(\tau_A \circ F(pk))^*\mathbf{\Omega}_A && \text{(since } T \text{ preserves meets)} \\
&= \bigsqcap_{A\in\mathbb{A}}\ \bigsqcap_{k\in\mathbb{E}(P,\mathbf{\Omega}_A)} (\tau_A \circ F(pk))^*T\mathbf{\Omega}_A && \text{(since } T \text{ is fibered)} \\
&= \bigsqcap_{A\in\mathbb{A}}\ \bigsqcap_{k\in\mathbb{E}(P,\mathbf{\Omega}_A)} (\tau_A \circ F(q(Tk)))^*T\mathbf{\Omega}_A && \text{(since } q\circ T = p\text{)} \\
&= \bigsqcap_{A\in\mathbb{A}}\ \bigsqcap_{l\in\mathbb{F}(TP,T\mathbf{\Omega}_A)} (\tau_A \circ F(ql))^*T\mathbf{\Omega}_A && \text{(since } T \text{ is full)} \\
&= F^{T\mathbf{\Omega},\tau}TP
\end{aligned}
$$

Considering this and the fact that T preserves meets, Lemma 4.7.2 implies $T(\nu\Phi^{\mathbf{\Omega},\tau}) = \nu\Phi^{T\mathbf{\Omega},\tau}$. □

Example 4.7.4. We show that the codensity bisimilarities in Examples 3.1.4 and 3.1.5 are indeed the usual bisimilarity notions for Kripke frames. Recall that they are built on the two $\mathbf{CLat}_\sqcap$-fibrations $\mathbf{EqRel} \to \mathbf{Set}$ and $\mathbf{ERel} \to \mathbf{Set}$.

We first note that the inclusion functor $i\colon \mathbf{EqRel} \to \mathbf{ERel}$ is a reflection, having the equivalence closure $(\_)^{\mathrm{eq}}\colon \mathbf{ERel} \to \mathbf{EqRel}$ as the left adjoint. It follows that i is meet-preserving. Moreover, i is fibered.

$$
\begin{tikzcd}
\mathbf{EqRel} \arrow[rr, shift left, "i"] \arrow[rr, phantom, "\top"] \arrow[dr, "p"'] & & \mathbf{ERel} \arrow[ll, shift left, "(\_)^{\mathrm{eq}}"] \arrow[dl, "q"] \\
& \mathbf{Set} \arrow[loop right, "\mathcal{P}"] &
\end{tikzcd}
$$

We introduce shorthands $\dot{\mathcal{P}}_2, \dot{\mathcal{P}}_3$ for the liftings in Examples 3.1.4 and 3.1.5:

$$
\begin{array}{lcll}
\dot{\mathcal{P}}_2 & = & \mathcal{P}^{\mathrm{Eq}_2,\Diamond} \ : \mathbf{EqRel} \to \mathbf{EqRel} & \text{(Example 3.1.4)}, \\
\dot{\mathcal{P}}_3 & = & \mathcal{P}^{\mathrm{Eq}_2,\Diamond} \ : \mathbf{ERel} \to \mathbf{ERel} & \text{(Example 3.1.5)}.
\end{array}
$$

Now, for the sake of our proof, let us introduce a relational lifting $\dot{\mathcal{P}}_1 : \mathbf{ERel} \to \mathbf{ERel}$ of $\mathcal{P}$ along $\mathbf{ERel} \to \mathbf{Set}$, for which it is obvious that the corresponding bisimilarity notion is the usual bisimilarity for Kripke frames. We do so in concrete terms, instead of as a codensity lifting:

$$
(S,T) \in \dot{\mathcal{P}}_1(R) \iff (\forall x \in S.\ \exists y \in T.\ (x,y) \in R)\ \wedge (\forall y \in T.\ \exists x \in S.\ (x,y) \in R).
$$

We note that $\dot{\mathcal{P}}_2$ is the restriction of $\dot{\mathcal{P}}_1$ from $\mathbf{ERel}$ to $\mathbf{EqRel}$ along i. This means $i \circ \dot{\mathcal{P}}_2 = \dot{\mathcal{P}}_1 \circ i$. Note also that $\dot{\mathcal{P}}_3 = \dot{\mathcal{P}}_1 \circ i \circ (\_)^{\mathrm{eq}}$.

Let $c : X \to \mathcal{P}X$ be a Kripke frame and $\Phi_i = c^* \circ \dot{\mathcal{P}}_i$ $(i = 1, 2, 3)$ be the predicate transformer corresponding to each lifting. Theorem 4.7.3 yields that $\nu\Phi_3 = i(\nu\Phi_2)$.

Furthermore, by $\dot{\mathcal{P}}_1 \sqsubseteq \dot{\mathcal{P}}_3$ (where $\sqsubseteq$ is the order in (3.2)), we have $\nu\Phi_1 \sqsubseteq \nu\Phi_3$. From $i \circ \dot{\mathcal{P}}_2 = \dot{\mathcal{P}}_1 \circ i$ and fiberedness of c, we can see that $i(\nu\Phi_2)$ is a fixed point of Φ_1:

$$\begin{aligned}\Phi_1(i(\nu\Phi_2)) &= c^*(\dot{\mathcal{P}}_1(i(\nu\Phi_2))) = c^*(i(\dot{\mathcal{P}}_2(\nu\Phi_2))) \\ &= i(c^*(\dot{\mathcal{P}}_2(\nu\Phi_2))) = i(\Phi_2(\nu\Phi_2)) = i(\nu\Phi_2).\end{aligned}$$

By this fact and the definition of $\nu\Phi_1$, $i(\nu\Phi_2) \sqsubseteq \nu\Phi_1$ holds. The three (in)equalities so far allow us to conclude $\nu\Phi_3 = i(\nu\Phi_2) = \nu\Phi_1$, stating that the conventional bisimilarity $\nu\Phi_1$ is equal to the codensity bisimilarities in Examples 3.1.4 and 3.1.5. As a consequence, the conventional bisimilarity $\nu\Phi_1$ is necessarily an equivalence relation.

4.7.2. Transfer Result for Two Different Families of Parameters

Consider the following situation again (Definition 4.7.1):

$$\begin{tikzcd} \mathbb{E} \arrow[rr, "T"] \arrow[dr, "p"'] & & \mathbb{F} \arrow[dl, "q"] \\ & \mathbb{C} \arrow[loop below, "F"] & \end{tikzcd} .$$

Now consider two families of parameters, $(\mathbf{\Omega}, \tau) = \{(\mathbf{\Omega}_A, \tau_A)\}_{A \in \mathring{\mathbb{A}}}$ for lifting F along p and $(\mathbf{\Psi}, \rho) = \{(\mathbf{\Psi}_B, \rho_B)\}_{B \in \mathbb{B}}$ for lifting F along q. Let $c\colon X \to FX$ be an F-coalgebra. In Section 4.7.2 we compare $T\nu\Phi_c^{\mathbf{\Omega},\tau}$ and $\nu\Phi_c^{\mathbf{\Psi},\rho}$ (both in $\mathbb{F}_X$).

Firstly, we show an "order-version" of Lemma 4.7.2. It reduces the comparison of $T\nu\Phi_c^{\mathbf{\Omega},\tau}$ and $\nu\Phi_c^{\mathbf{\Psi},\rho}$ to that of $TF^{\mathbf{\Omega},\tau}$ and $F^{\mathbf{\Psi},\rho}T$:

Proposition 4.7.5. *In the setting of Definition 4.7.1, assume also that T preserves fiberwise meets. Let $\dot{F}\colon \mathbb{E} \to \mathbb{E}$ and $\ddot{F}\colon \mathbb{F} \to \mathbb{F}$ be liftings of F along p and q, respectively. Let $c\colon X \to FX$ be an F-coalgebra. If $T\dot{F}P \sqsupseteq \ddot{F}TP$ holds for each $P \in \mathbb{E}$, then $T\nu(c^* \circ \dot{F}) \sqsupseteq \nu(c^* \circ \ddot{F})$ holds.*

Proof. For each ordinal α, we define $\nu_\alpha(c^* \circ \dot{F})$ by

$$\nu_\alpha(c^* \circ \dot{F}) = \bigsqcap_{\beta < \alpha} c^* \dot{F}(\nu_\beta(c^* \circ \dot{F}))$$

using induction on α. We define $\nu_\alpha(c^* \circ \ddot{F})$ in the same way. By Fact 2.2.4, these converge to $\nu(c^* \circ \dot{F})$ and $\nu(c^* \circ \ddot{F})$, respectively. It suffices to show $T\nu_\alpha(c^* \circ \dot{F}) \sqsupseteq \nu_\alpha(c^* \circ \ddot{F})$ by induction on α.

Assume that the above inequality holds for all ordinals smaller than α. Then we have

$$
\begin{aligned}
T\nu_\alpha(c^* \circ \dot{F}) &= T \bigsqcap_{\beta<\alpha} c^*\dot{F}(\nu_\beta(c^* \circ \dot{F})) \\
&= \bigsqcap_{\beta<\alpha} Tc^*\dot{F}(\nu_\beta(c^* \circ \dot{F})) && \text{(since } T \text{ preserves meets)} \\
&= \bigsqcap_{\beta<\alpha} c^*T\dot{F}(\nu_\beta(c^* \circ \dot{F})) && \text{(since } T \text{ is fibered)} \\
&\sqsupseteq \bigsqcap_{\beta<\alpha} c^*\ddot{F}T(\nu_\beta(c^* \circ \dot{F})) && \text{(by the assumption } T\dot{F} \sqsupseteq \ddot{F}T\text{)} \\
&\sqsupseteq \bigsqcap_{\beta<\alpha} c^*\ddot{F}(\nu_\beta(c^* \circ \ddot{F})) && \text{(by induction hypothesis)} \\
&= \nu_\alpha(c^* \circ \ddot{F}). && \square
\end{aligned}
$$

The following is the main result of Section 4.7.2. It says that, if we have a certain data connecting two families of parameters $\{(\mathbf{\Omega}_A, \tau_A)\}_{A\in\mathring{\mathbb{A}}}$ and $\{(\mathbf{\Psi}_B, \rho_B)\}_{B\in\mathbb{B}}$, then the inequality $TF^{\mathbf{\Omega},\tau} \sqsupseteq F^{\mathbf{\Psi},\rho}T$ holds:

Proposition 4.7.6. *In the setting of Definition 4.7.1, assume also that T preserves fiber-wise meets. Let $(I_{A,B})_{A\in\mathring{\mathbb{A}},B\in\mathbb{B}}$ be some family of sets and $(t_{A,B,i}\colon T\mathbf{\Omega}_A \to \mathbf{\Psi}_B)_{A\in\mathring{\mathbb{A}},B\in\mathbb{B},i\in I_{A,B}}$ be a family of $\mathbb{F}$-arrows such that*

$$\tau_A^* T\mathbf{\Omega}_A \sqsupseteq \bigsqcap_{B\in\mathbb{B},i\in I_{A,B}} (F(q(t_{A,B,i})))^* \rho_B^* \mathbf{\Psi}_B$$

holds for each $A \in \mathring{\mathbb{A}}$. Then $TF^{\mathbf{\Omega},\tau}P \sqsupseteq F^{\mathbf{\Psi},\rho}TP$ holds for each $P \in \mathbb{E}$.

Proof. Let $X = pP$. Since

$$
\begin{aligned}
TF^{\mathbf{\Omega},\tau}P &= T\left(\bigsqcap_{A\in\mathring{\mathbb{A}},f:P\to\mathbf{\Omega}_A} (F(pf))^*\tau_A^*\mathbf{\Omega}_A\right) \\
&= \bigsqcap_{A\in\mathring{\mathbb{A}},f:P\to\mathbf{\Omega}_A} (F(pf))^*\tau_A^* T\mathbf{\Omega}_A
\end{aligned}
$$

and

$$F^{\mathbf{\Psi},\rho}TP = \bigsqcap_{B\in\mathbb{B},g\colon TP\to\mathbf{\Psi}_B} (F(qg))^*\rho_B^*\mathbf{\Psi}_B,$$

it suffices to show that, for each $f\colon P \to \mathbf{\Omega}$ and $A \in \mathring{\mathbb{A}}$,

$$(F(pf))^*\tau_A^*T\mathbf{\Omega}_A \sqsupseteq \bigsqcap_{B\in\mathbb{B},g\colon TP\to\mathbf{\Psi}_B} (F(qg))^*\rho_B^*\mathbf{\Psi}_B$$

holds.

Let $A \in \mathring{\mathbb{A}}$ and $f\colon P \to \mathbf{\Omega}_A$. For each $B \in \mathbb{B}$ and $i \in I_{A,B}$, consider $g = TP \xrightarrow{Tf} T\mathbf{\Omega}_A \xrightarrow{t_{A,B,i}} \mathbf{\Psi}_B$. Then $F(qg) = FX \xrightarrow{F(pf)} F\Omega_A \xrightarrow{Fqt_{A,B,i}} Fq\Psi_B$. Thus $(F(qg))^*\rho_B^*\mathbf{\Psi}_B = (F(pf))^*(F(q(t_{A,B,i})))^*\rho_B^*\mathbf{\Psi}_B$ holds. Restricting the range of g to the class considered above, it suffices to show

$$\tau_A^* T\mathbf{\Omega}_A \sqsupseteq \bigsqcap_{B\in\mathbb{B}, i\in I_{A,B}} (F(q(t_{A,B,i})))^*\rho_B^*\mathbf{\Psi}_B.$$

This is nothing but our assumption. $\square$

Remark 4.7.7. As a special case, if each $I_{A,B}$ is the singleton $\{\bullet\}$ and the diagram

$$\begin{tikzcd} F\Omega_A \arrow[r, "F(q(t_{A,B,\bullet}))"'] \arrow[d, "\tau_A"'] & F\Psi_B \arrow[d, "\rho_B"] \\ \Omega_A \arrow[r, "q(t_{A,B,\bullet})"'] & \Psi_B \end{tikzcd}$$

commutes, then the condition in Proposition 4.7.6 holds. However, it seems that such cases are rather special. The condition in Proposition 4.7.6 can be regarded as a weakening of it: we use multiple $t_{A,B,i}$ to obtain as much information as given by one arrow making the above diagram commute.

Example 4.7.8. Consider the following situation:

$$\begin{tikzcd} \mathbf{PMet}_1 \arrow[rr, "T"] \arrow[dr, "U"'] & & \mathbf{EqRel} \arrow[dl, "U"] \\ & \mathbf{Set} \arrow[loop right, "\mathcal{D}_{\le 1}"] & \end{tikzcd}.$$

Here, T is defined by $T(X,d) = (X, R_d)$ and

$$(x,y) \in R_d \iff d(x,y) = 0.$$

This is a fibered lifting of $\mathrm{Id}_{\mathbb{C}}$ and preserves fibered meets.

We describe the parameter for $\mathbb{E} = \mathbf{PMet}_1$: $\mathring{\mathbb{A}} = \{\bullet\}$ and $\mathbf{\Omega}_\bullet = ([0,1], d_e)$, where d_e is the Euclidean metric. The modality $\tau_\bullet\colon \mathcal{D}_{\le 1}[0,1] \to [0,1]$ is given by the expected value function. In this setting, for each coalgebra $c\colon X \to \mathcal{D}_{\le 1}X$, the codensity bisimilarity $\nu\Phi_c^{\mathbf{\Omega},\tau}$ coincides with the bisimulation metric (Examples 3.1.11, 4.4.5 and 4.5.3).

We move on to the parameter for $\mathbb{F} = \mathbf{EqRel}$: $\mathbb{B} = [0,1]$ and $\mathbf{\Psi}_r = (2, \mathrm{Eq}_2)$ for all $r \in [0,1]$. The modality $\rho_r\colon \mathcal{D}_{\le 1}2 \to 2$ is the threshold modality defined by

$$\rho_r(p) = \top \iff p(\top) \ge r.$$

For each coalgebra $c\colon X \to \mathcal{D}_{\le 1}X$, the codensity bisimilarity $\nu\Phi^{\mathbf{\Psi},\rho}$ coincides with the probabilistic bisimilarity (Example 4.8.14).

For each $r \in [0,1]$, let $I_{\bullet,r} = [0,1]$. For each $r, s \in [0,1]$, we define an **EqRel**-arrow $t_{\bullet,r,s} \colon T([0,1], d_e) \to (2, \mathrm{Eq}_2)$ by

$$t_{\bullet,r,s}(u) = \top \iff u \geq s.$$

In this setting the condition in Proposition 4.7.6 is satisfied. Let $\mu, \nu \in \mathcal{D}_{\leq 1}[0,1]$; if $\mu(\{x \mid x \geq s\}) \geq r \iff \nu(\{x \mid x \geq s\}) \geq r$ holds for all $r, s \in [0,1]$, then their expected values coincide.

Using Propositions 4.7.5 and 4.7.6, we can conclude that, for any $\mathcal{D}_{\leq 1}$-coalgebra $c \colon X \to \mathcal{D}_{\leq 1} X$, $T(\nu \Phi^{\mathbf{\Omega},\tau}) \sqsupseteq \nu \Phi^{\mathbf{\Psi},\rho}$ holds. This means that, if two states are bisimilar, then the bisimulation metric between them is 0.

On the other hand, the converse inequality $T(\nu \Phi^{\mathbf{\Omega},\tau}) \sqsubseteq \nu \Phi^{\mathbf{\Psi},\rho}$ cannot be derived from the above general theory. It is known to hold [22, Theorem 5.2], but the proof involves a real-valued modal logic. Purely fibrational proof of this fact is a future work.

Note that this example does not make the diagram in Remark 4.7.7 commute.

4.8. Examples

In this section we list examples of our framework. We group them by the fibrations they rely upon: **EqRel** $\to$ **Set** in Section 4.8.1, **BRel** $\to$ **Set**^{2} in Section 4.8.2, **Top** $\to$ **Set** in Section 4.8.3, and **PMet**$_1$ $\to$ **Set** in Section 4.8.4. In Section 4.8.5, we use a fibration $U^*(\mathbf{PMet}_1) \to \mathbf{Meas}$ that is newly defined there.

4.8.1. Set-coalgebras and Behavioral Equivalence

In Section 4.8.1, we show that behavioral equivalence for coalgebras in **Set** can also be defined in terms of fibrations (Proposition 4.8.3), and that they can be characterized by codensity games (Theorem 4.8.7) in the cases where the functor admits a *separating family* (Definition 4.8.5).

We start with the standard definition of behavioral equivalence. The intuition here is that a coalgebra morphism is "behavior preserving." See [37].

This can be modeled fibrationally by the fibration **EqRel** $\to$ **Set**. We use a functor lifting, which is essentially the same as the one defined in [42, Section 4].

Definition 4.8.1 (the lifting $\overline{F}_{\mathrm{BE}} \colon \mathbf{EqRel} \to \mathbf{EqRel}$)**.** Let $F \colon \mathbf{Set} \to \mathbf{Set}$ be a functor. We define a lifting $\overline{F}_{\mathrm{BE}} \colon \mathbf{EqRel} \to \mathbf{EqRel}$ by the following: for $(X, R) \in \mathbf{EqRel}$, let $q \colon X \twoheadrightarrow X/R$ be the canonical surjection. Then $\overline{F}_{\mathrm{BE}}(X, R)$ is defined as the kernel of $Fq \colon FX \to F(X/R)$, that is,

$$\overline{F}_{\mathrm{BE}}(X, R) = (FX, \{(z, z') \in (FX)^2 \mid (Fq)(z) = (Fq)(z')\}).$$

Proposition 4.8.2. *The assignment $\overline{F}_{\mathrm{BE}}$ above indeed specifies a functor, i.e., for any decent morphism $f \colon (X, R) \to (Y, S)$, Ff is decent from $\overline{F}_{\mathrm{BE}}(X, R)$ to $\overline{F}_{\mathrm{BE}}(Y, S)$.*

Proof. Let $q\colon X \twoheadrightarrow X/R$ and $r\colon Y \twoheadrightarrow Y/S$ be the canonical surjections. Let us fix $z, z' \in FX$ and assume $(Fq)(z) = (Fq)(z')$. It suffices to show $(Fr)((Ff)(z)) = (Fr)((Ff)(z'))$.

Since $f\colon (X,R) \to (Y,S)$ is decent, $R \sqsubseteq f^*S$ holds. Therefore there exists a map $g\colon X/R \to Y/S$ which makes the diagram

$$\begin{tikzcd} X \arrow[r, "f"'] \arrow[d, two heads, "q"'] & Y \arrow[d, two heads, "r"'] \\ X/R \arrow[r, "g"'] & Y/S \end{tikzcd}$$

commute. Using this we see

$$(Fr)((Ff)(z)) = (F(r \circ f))(z) = (F(g \circ q))(z) = (Fg)((Fq)(z)).$$

For the same reason $(Fr)((Ff)(z')) = (Fg)((Fq)(z'))$ holds, and the assumption $(Fq)(z) = (Fq)(z')$ now implies $(Fr)((Ff)(z)) = (Fr)((Ff)(z'))$. ◻

The lifting $\overline{F}_{\mathrm{BE}}$ indeed captures behavioral equivalence, provided that F preserves monos.

Proposition 4.8.3. *Let $F\colon \mathbf{Set} \to \mathbf{Set}$ be a functor and $c\colon X \to FX$ be an F-coalgebra. Assume that F preserves monos. The states $x, x' \in X$ are behaviorally equivalent if and only if there is an equivalence relation R on X such that $(X,R) \sqsubseteq c^*\overline{F}_{\mathrm{BE}}(X,R)$.*

Proof. Let $q\colon X \twoheadrightarrow X/R$ be the canonical surjection. Then $c^*\overline{F}_{\mathrm{BE}}(X,R)$ can be concretely presented by

$$c^*\overline{F}_{\mathrm{BE}}(X,R) = (X, \{(x,x') \in X^2 \mid (Fq)(c(x)) = (Fq)(c(x'))\}).$$

Let $x, x' \in X$. Firstly, we show that if x and x' are behaviorally equivalent, there exists some R such that $(x,x') \in R$ and $(X,R) \sqsubseteq c^*\overline{F}_{\mathrm{BE}}(X,R)$ hold. Assume x and x' are behaviorally equivalent. There is another F-coalgebra $d\colon Y \to FY$ and a coalgebra morphism $f\colon X \to Y$ such that $f(x) = f(x')$. Let $R \subseteq X \times X$ be

$$R = \{(x_1, x_2) \in X^2 \mid f(x_1) = f(x_2)\}.$$

Then $(X,R) \in \mathbf{EqRel}$ and, by the definition, $(x,x') \in R$. Let $q\colon X \twoheadrightarrow X/R$ be the canonical surjection. By the definition of R, there exists a monomorphism $m\colon X/R \rightarrowtail Y$ such that $f = m \circ q$. Since f is a coalgebra morphism, the outer square of the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, two heads, "q"'] \arrow[d, "c"'] & X/R \arrow[r, tail, "m"'] \arrow[d, dashed, "e"'] & Y \arrow[d, "d"'] \\ FX \arrow[r, "Fq"'] & F(X/R) \arrow[r, tail, "Fm"'] & FY. \end{tikzcd}$$

In this diagram, q is epic and, since m is monic, Fm is also monic. Therefore, there exists a unique $e\colon X/R \to F(X/R)$ making the two squares in the above diagram commute (the *diagonalization property* of a factorization system—see [2]). Now we prove $(X, R) \sqsubseteq c^*\overline{F}_{\mathrm{BE}}(X, R)$. Assume $(x_1, x_2) \in R$. Since $(Fq)(c(x_1)) = e(q(x_1)) = e(q(x_2)) = (Fq)(c(x_2))$, $(x_1, x_2) \in c^*\overline{F}_{\mathrm{BE}}(X, R)$ holds.

Secondly, for R satisfying $(X, R) \sqsubseteq c^*\overline{F}_{\mathrm{BE}}(X, R)$, we show that any pair $(x, x') \in R$ is behaviorally equivalent. Assume that there exists R such that $(x, x') \in R$ and $(X, R) \sqsubseteq c^*\overline{F}_{\mathrm{BE}}(X, R)$ hold. The second condition means that, for each $(x_1, x_2) \in R$, $(Fq \circ c)(x_1) = (Fq \circ c)(x_2)$ holds. Thus there is a (unique) $d\colon X/R \to F(X/R)$ making the following diagram commute:

$$\begin{tikzcd} X \arrow[r, two heads, "q"'] \arrow[d, "c"'] & X/R \arrow[d, "d"'] \\ FX \arrow[r, "Fq"'] & F(X/R). \end{tikzcd}$$

Now q is a coalgebra morphism from $c\colon X \to FX$ to $d\colon X/R \to F(X/R)$. Since $q(x) = q(x')$, x and x' are behaviorally equivalent. □

Remark 4.8.4 (on preservation of monomorphisms). In Proposition 4.8.3, F is assumed to preserve monos. However, this is not very restricting: If $X \in \mathbf{Set}$ is nonempty, then any monomorphism $f\colon X \rightarrowtail Y$ splits, and Ff is also a split mono. Therefore, we only have to check that, for $f\colon 0 \to Y$, Ff is injective. See [1] for details.

Now we move on to representing $\overline{F}_{\mathrm{BE}}$ as a codensity lifting. The key notion here is *separation*. It is mainly used in coalgebraic modal logic literature like [58, 64]. While it is standard to define it for *predicate liftings* like in [64, Definition 7], we adapt it for F-algebras.

Definition 4.8.5 (separating family of F-algebras)**.** Let $X \in \mathbf{Set}$ and $F\colon \mathbf{Set} \to \mathbf{Set}$. An $\mathring{\mathbb{A}}$-indexed family $(\tau_A\colon F2 \to 2)_{A \in \mathring{\mathbb{A}}}$ of F-algebras is *separating for* X if each $z \in FX$ is uniquely determined by the values of $\tau_A((Ff)(z))$ for $A \in \mathring{\mathbb{A}}$ and $f\colon X \to 2$, that is, for each pair $z, z' \in FX$, if $\tau_A((Ff)(z)) = \tau_A((Ff)(z'))$ holds for all $A \in \mathring{\mathbb{A}}$ and $f\colon X \to 2$, then $z = z'$.

For an $\mathring{\mathbb{A}}$-indexed family $(\tau_A\colon F2 \to 2)_{A \in \mathring{\mathbb{A}}}$ of F-algebras, note that $\{(\mathrm{Eq}_2, \tau_A)\}_{A \in \mathring{\mathbb{A}}}$ is an $\mathbb{A}$-indexed family of lifting parameters and we can define the codensity lifting $F^{\mathrm{Eq}_2,\tau}$. This turns out to coincide with $\overline{F}_{\mathrm{BE}}$ if the family is separating.

Proposition 4.8.6. *Let (X, R) be an object in* **EqRel**, *$F\colon$* **Set** $\to$ **Set** *be a functor, and $(\tau_A\colon F2 \to 2)_{A \in \mathring{\mathbb{A}}}$ be an $\mathring{\mathbb{A}}$-indexed family of F-algebras. If $(\tau_A\colon F2 \to 2)_{A \in \mathring{\mathbb{A}}}$ is separating for X/R, then*

$$F^{\mathrm{Eq}_2,\tau}(X, R) = \overline{F}_{\mathrm{BE}}(X, R)$$

holds.

Proof. Firstly, we show $F^{\mathrm{Eq}_2,\tau}(X, R) \sqsupseteq \overline{F}_{\mathrm{BE}}(X, R)$. Let $(z, z') \in (FX)^2$, $f\colon (X, R) \to (2, \mathrm{Eq}_2)$ and $A \in \mathring{\mathbb{A}}$. Let $q\colon X \twoheadrightarrow X/R$ be the canonical surjection and assume that

$(Fq)(z) = (Fq)(z')$. It suffices to show $\tau_A((Ff)(z)) = \tau_A((Ff)(z'))$. Since $f\colon (X,R) \to (2, \mathrm{Eq}_2)$ is decent, there is a (unique) map $g\colon X/R \to 2$ making the left one of the following diagrams commute:

$$
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "q"'] & & 2 \\
& X/R \arrow[ur, "g"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
FX \arrow[rr, "Ff"] \arrow[dr, "Fq"'] & & F2. \\
& F(X/R) \arrow[ur, "Fg"'] &
\end{tikzcd}
$$

By the functoriality of F, the right one also commutes. Thus, we have $(Ff)(z) = (Fg)((Fq)(z)) = (Fg)((Fq)(z')) = (Ff)(z')$. This implies $\tau_A((Ff)(z)) = \tau_A((Ff)(z'))$.

Secondly, we show $F^{\mathrm{Eq}_2,\tau}(X,R) \sqsubseteq \overline{F}_{\mathrm{BE}}(X,R)$. Let $(z,z') \in (FX)^2$ and assume that, for each $A \in \mathring{A}$ and $f\colon (X,R) \to (2, \mathrm{Eq}_2)$, $\tau_A((Ff)(z)) = \tau_A((Ff)(z'))$ holds. Let $q\colon X \twoheadrightarrow X/R$ be the canonical surjection. It suffices to show $(Fq)(z) = (Fq)(z')$. Let $g\colon X/R \to 2$ be any arrow. Then $g \circ q$ is decent from (X,R) to $(2, \mathrm{Eq}_2)$. By the assumption, $\tau_A((Fg)((Fq)(z))) = \tau_A((Ff)(z)) = \tau_A((Ff)(z')) = \tau_A((Fg)((Fq)(z')))$ holds for each $A \in \mathring{A}$. Since g is arbitrary and $(\tau_A\colon F2 \to 2)_{A\in\mathring{A}}$ is separating for X/R, $(Fq)(z) = (Fq)(z')$ holds. □

In such case the codensity bisimilarity (Definition 3.2.2) coincides with the behavioral equivalence (Definition 2.1.7).

Theorem 4.8.7. *Let $F\colon \mathbf{Set} \to \mathbf{Set}$ be a functor, $(\tau_A\colon F2 \to 2)_{A\in\mathring{A}}$ be an $\mathring{A}$-indexed family of F-algebras, and $c\colon X \to FX$ be an F-coalgebra. Assume that F preserves monos. If $(\tau_A\colon F2 \to 2)_{A\in\mathring{A}}$ is separating for every set Y, then the behavioral equivalence of c coincides with the codensity bisimilarity $\nu\Phi_c^{\mathrm{Eq}_2,\tau}$.*

Proof. By Proposition 4.8.3, the behavioral equivalence is the greatest fixed point of $c^* \circ \overline{F}_{\mathrm{BE}}$. Moreover, this coincides with $\nu\Phi_c^{\mathrm{Eq}_2,\tau}$ by Proposition 4.8.6. □

Theorem 4.8.7 characterizes the behavioral equivalence of F-coalgebras by codensity games, when F preserves monos and has separating family of F-algebras. In the following, we use the join-dense subset described in Example 4.5.2 to trim games.

Example 4.8.8 (Kripke frames)**.** Consider the powerset functor $\mathcal{P}\colon \mathbf{Set} \to \mathbf{Set}$. Since $\mathcal{P}0 \simeq 1$, for any $f\colon 0 \to Y$ in $\mathbf{Set}$, $\mathcal{P}f\colon \mathcal{P}0 \to \mathcal{P}Y$ is monic. Thus it preserves monos by Remark 4.8.4. A $\mathcal{P}$-coalgebra $c\colon X \to \mathcal{P}X$ is nothing but a Kripke frame.

The one-member family $(\Diamond\colon \mathcal{P}2 \to 2)$ (used in Example 3.1.4) is separating for any set X. Indeed, if we define $f_x\colon X \to 2$ by $f_x(x') = \top \iff x = x'$, then for $S \in \mathcal{P}X$, $x \in S$ if and only if $\Diamond((\mathcal{P}f_x)(S)) = \top$.

By Theorem 4.8.7, the behavioral equivalence (Definition 2.1.7) for a Kripke frame $c\colon X \to \mathcal{P}X$ coincides with the codensity bisimilarity $\nu\Phi_c^{\mathrm{Eq}_2,\Diamond}$. Thus, by Corollary 4.5.16, it is characterized by the codensity game (Table 4.8) specialized to this situation. The game in this case is shown in Table 4.10. It is trimmed by the join-dense subset in Example 4.5.2.

Table 4.10.: Codensity bisimilarity game for conventional bisimilarity

position	player	possible moves
$(x, y) \in X \times X$	Spoiler	$k \in \mathbf{Set}(X, 2)$ such that exactly one of $\exists x' \in c(x).\ k(x') = \top$ and $\exists y' \in c(y).\ k(y') = \top$ holds
$k \in \mathbf{Set}(X, 2)$	Duplicator	(x'', y'') s.t. $k(x'') \neq k(y'')$

Theorem 4.8.9. *Let $c\colon X \to \mathcal{P}X$ be a Kripke frame. The position $(x, y) \in X \times X$ in the game in Table 4.10 is winning for Duplicator if and only if $(x, y) \in \nu\Phi_c^{\mathrm{Eq}_2,\Diamond}$, if and only if x and y are behaviorally equivalent.* □

As shown in Example 4.7.4, the codensity bisimilarity $\nu\Phi_c^{\mathrm{Eq}_2,\Diamond}$ (which is $\nu\Phi_2$ in Example 4.7.4) also coincides with the conventional bisimilarity on the Kripke frame c. Therefore, we also see that the conventional bisimilarity and the behavioral equivalence are equal for Kripke frames.

Example 4.8.10 (deterministic automata)**.** Consider the functor $A_\Sigma\colon \mathbf{Set} \to \mathbf{Set}$ from Example 4.6.9, for which a coalgebra is a deterministic automaton. Since $A_\Sigma 0 \simeq 0$, for any $f\colon 0 \to Y$ in $\mathbf{Set}$, $A_\Sigma f\colon A_\Sigma 0 \to A_\Sigma Y$ is monic. Thus it preserves monos by Remark 4.8.4.

The family $\{\tau_\varepsilon\} \cup \{\tau_a \mid a \in \Sigma\}$ introduced in Example 4.6.9 is separating for every set X. Indeed, if we define $f_x\colon X \to 2$ by $f_x(x') = \top \iff x = x'$, then for $y = (t, \rho) \in A_\Sigma X$ (where $t \in 2$ and $\rho\colon \Sigma \to X$), $t = \top$ if and only if $\tau_\varepsilon((A_\Sigma f_x)(y)) = \top$, and $\rho(a) = x$ if and only if $\tau_a((A_\Sigma f_x)(y)) = \top$.

By Theorem 4.8.7, the behavioral equivalence (Definition 2.1.7) for a deterministic automaton $c\colon X \to A_\Sigma X$ coincides with the codensity bisimilarity $\nu\Phi_c^{\mathrm{Eq}_2,\tau}$. Thus, by Corollary 4.6.8, it is characterized by the codensity game (Table 4.9) specialized to this situation. The game in this case is shown in Table 4.11. It is trimmed by the join-dense subset in Example 4.5.2. It is also simplified in the case where the position $(x, y) \in X \times X$ satisfies $c_1(x) \neq c_1(y)$: strictly in such case, Spoiler can play any constant map from X to 2 and any $a \in \Sigma$, and then Duplicator cannot play any longer.

Table 4.11.: Codensity bisimilarity game for deterministic automata and their language equivalence. The arrows $c_1\colon X \to 2$ and $c_2\colon X \to X^\Sigma$ are the first and second projections of $c\colon X \to A_\Sigma X = 2 \times X^\Sigma$, respectively.

position	player	possible moves
$(x, y) \in X \times X$	Spoiler	If $c_1(x) \neq c_1(y)$ then Spoiler wins If $c_1(x) = c_1(y)$ then $a \in \Sigma$ and $k \in \mathbf{Set}(X, 2)$ such that $k(c_2(x)(a)) \neq k(c_2(y)(a))$
$a \in \Sigma$ and $k \in \mathbf{Set}(X, 2)$	Duplicator	$(x'', y'') \in X \times X$ s.t. $k(x'') \neq k(y'')$

Theorem 4.8.11. *Let $c\colon X \to A_\Sigma X$ be a deterministic automaton. The position $(x, y) \in X \times X$ in the game in Table 4.11 is winning for Duplicator if and only if $(x, y) \in \nu\Phi_c^{\mathrm{Eq}_2,\tau}$, if and only if x and y are behaviorally equivalent.* □

Since we are considering deterministic automata here, the language equivalence coincides with the behavioral equivalence. Thus the game in Table 4.11 also characterizes the language equivalence.

Example 4.8.12 (nondeterministic automata)**.** Let us now turn to nondeterministic automata, that is, N_Σ-coalgebras for the functor $N_\Sigma = 2 \times (\mathcal{P}\_)^\Sigma$. Since $N_\Sigma = A_\Sigma \circ \mathcal{P}$ and both A_Σ and $\mathcal{P}$ preserve monos (Examples 4.8.8 and 4.8.10), N_Σ preserves monos.

Consider the family $\{\tau_\varepsilon\} \cup \{\tau_a \mid a \in \Sigma\}$ of maps from $N_\Sigma 2$ to 2 defined as follows:

$$\tau_\varepsilon(t, \rho) = t, \;\; \tau_a(t, \rho) = \Diamond(\rho(a)).$$

This family is separating for every set X. Indeed, if we define $f_x\colon X \to 2$ by $f_x(x') = \top \iff x = x'$, then for $y = (t, \rho) \in N_\Sigma X$ (where $t \in 2$ and $\rho\colon \Sigma \to \mathcal{P}X$), $t = \top$ if and only if $\tau_\varepsilon((N_\Sigma f_x)(y)) = \top$, and $x \in \rho(a)$ if and only if $\tau_a((N_\Sigma f_x)(y)) = \top$.

By Theorem 4.8.7, the behavioral equivalence (Definition 2.1.7) for a nondeterministic automaton $c\colon X \to N_\Sigma X$ coincides with the codensity bisimilarity $\nu\Phi_c^{\mathrm{Eq}_2,\tau}$. Thus, by Corollary 4.6.8, it is characterized by the codensity game (Table 4.9) specialized to this situation. The game in this case is shown in Table 4.12. It is trimmed by the join-dense subset in Example 4.5.2. It is also simplified in the case where the position $(x, y) \in X \times X$ satisfies $c_1(x) \neq c_1(y)$: strictly in such case, Spoiler can play any constant map from X to 2 and any $a \in \Sigma$, and then Duplicator cannot play any longer.

Table 4.12.: Codensity bisimilarity game for nondeterministic automata and their behavioral equivalence. The arrows $c_1\colon X \to 2$ and $c_2\colon X \to (\mathcal{P}X)^\Sigma$ are the first and second projections of $c\colon X \to N_\Sigma X = 2 \times (\mathcal{P}X)^\Sigma$, respectively.

position	player	possible moves
$(x, y) \in X \times X$	Spoiler	If $c_1(x) \neq c_1(y)$ then Spoiler wins If $c_1(x) = c_1(y)$ then $a \in \Sigma$ and $k \in \mathbf{Set}(X, 2)$ such that $\exists x' \in c_2(x)(a).\ k(x') = \top$ $\not\Leftrightarrow \exists y' \in c_2(y)(a).\ k(y') = \top$
$a \in \Sigma$ and $k \in \mathbf{Set}(X, 2)$	Duplicator	$(x'', y'') \in X \times X$ s.t. $k(x'') \neq k(y'')$

Theorem 4.8.13. *Let $c\colon X \to N_\Sigma X$ be a nondeterministic automaton. The position $(x, y) \in X \times X$ in the game in Table 4.12 is winning for Duplicator if and only if $(x, y) \in \nu\Phi_c^{\mathrm{Eq}_2,\tau}$, if and only if x and y are behaviorally equivalent.* □

Example 4.8.14 (Markov chains)**.** Consider the functor $\mathcal{D}_{\leq 1}\colon \mathbf{Set} \to \mathbf{Set}$ (introduced in Section 4.1.1), for which a coalgebra is a Markov chain. Since $\mathcal{D}_{\leq 1}0 \simeq 1$, for any $f\colon 0 \to Y$ in $\mathbf{Set}$, $\mathcal{D}_{\leq 1}f\colon \mathcal{D}_{\leq 1}0 \to \mathcal{D}_{\leq 1}Y$ is monic. Thus it preserves monos by Remark 4.8.4.

For each real number $r \in [0, 1]$, define a *threshold modality* $\tau_r\colon \mathcal{D}_{\leq 1}2 \to 2$ by $\tau_r(p) = \top$ if and only if $p(\top) \geq r$. Then the family $\{\tau_r \mid r \in [0, 1]\}$ is separating for every set X. Indeed, if we define $f_x\colon X \to 2$ by $f_x(x') = \top \iff x = x'$, then for $d \in \mathcal{D}_{\leq 1}X$, $d(x) = \sup\{r \in [0, 1] \mid \tau_r((\mathcal{D}_{\leq 1}f_x)(d)) = \top\}$ holds.

By Theorem 4.8.7, the behavioral equivalence (Definition 2.1.7) for a Markov chain $c\colon X \to N_\Sigma X$ coincides with the codensity bisimilarity $\nu\Phi_c^{\mathrm{Eq}_2,\tau}$. Thus, by Corollary 4.6.8, it is characterized by the codensity game (Table 4.9) specialized to this situation. The game in this case is shown in Table 4.13. It is trimmed by the join-dense subset in Example 4.5.2. It is essentially the same as Table 4.2 (arising from [16]). The difference is that r is additionally present in Table 4.13; it is easy to realize that r plays no role in the game.

Table 4.13.: Codensity bisimilarity game for probabilistic bisimilarity

position	player	possible moves
$(x, y) \in X \times X$	Spoiler	$r \in [0,1]$ and $k \in \mathbf{Set}(X, 2)$ s.t. $c(x)(k^{-1}(\top)) \geq r > c(y)(k^{-1}(\top))$, or $c(y)(k^{-1}(\top)) \geq r > c(x)(k^{-1}(\top))$
$r \in [0,1]$ and $k \in \mathbf{Set}(X, 2)$	Duplicator	(x'', y'') s.t. $k(x'') \neq k(y'')$

Theorem 4.8.15. *Let $c\colon X \to \mathcal{D}_{\leq 1}X$ be a Markov chain. The position $(x, y) \in X \times X$ in the game in Table 4.13 is winning for Duplicator if and only if $(x, y) \in \nu\Phi_c^{\mathrm{Eq}_2,\tau}$, if and only if x and y are behaviorally equivalent.* □

Concretely, for any $R \in \mathbf{EqRel}_X$, the relation part of the codensity lifting $\mathcal{D}_{\leq 1}^{\mathbf{\Omega},\tau}(X, R)$ relates $p, q \in \mathcal{D}_{\leq 1}(X)$ if and only if the following holds:

$$\begin{aligned}&\forall r \in [0,1].\ \forall k\colon X \to 2.\ \big((\forall (x,y) \in R.\ k(x) = k(y))\\ &\quad\Longrightarrow \big(\textstyle\sum_{x \in k^{-1}(\top)} p(x) \geq r \Leftrightarrow \sum_{x \in k^{-1}(\top)} q(x) \geq r\big)\big).\end{aligned}$$

From this, it is not hard to see that the resulting codensity bisimilarity also coincides with probabilistic bisimilarity in [51]. Note, for example, that a relation-preserving map $k\colon (X, R) \dot{\to} (2, \mathrm{Eq}_2)$ coincides with an R-closed subset of X.

4.8.2. Set-coalgebras and Λ-bisimulation

In [4], a bisimulation notion called Λ-*bisimulation* is introduced. Their intention is to start from a behavior functor and a modal logic, and construct a corresponding notion of bisimulation. The special cases include *precocongruence* for neighborhood frames, *rel-Δ-bisimulation* for Kripke frames, and *nbh-Δ-bisimulation* for neighborhood frames [4, Examples 14–16], and the latter two examples are related to *contingency logic*.

In Section 4.8.2 we see how their definition and our codensity bisimilarity overlap. Specifically, when all of the given modalities are unary, the induced Λ-bisimulation turns out to be a special case of codensity bisimulation (Proposition 4.8.20). Using this overlap, we also derive a game characterization of such Λ-bisimulations (Corollary 4.8.23).

Definition 4.8.16 (from [4, Section 2])**.** A *similarity type* is a set of modal operators with finite arities. For a similarity type Λ, a Λ-*structure* $(F, (\llbracket\heartsuit\rrbracket)_{\heartsuit\in\Lambda})$ is a pair of a functor $F\colon \mathbf{Set} \to \mathbf{Set}$ and a family of *predicate liftings* $\llbracket\heartsuit\rrbracket\colon \mathbf{Set}(\_, 2)^n \Rightarrow \mathbf{Set}(F\_, 2)$, where n is the arity of the modal operator $\heartsuit \in \Lambda$.

Note that, by the Yoneda lemma, a predicate lifting $[\![\heartsuit]\!]\colon \mathbf{Set}(\_, 2)^n \Rightarrow \mathbf{Set}(F\_, 2)$ can be equivalently represented by an arrow $\tau_\heartsuit\colon F(2^n) \to 2$. Concretely, from $[\![\heartsuit]\!]$, we can obtain $\tau_\heartsuit$ by $[\![\heartsuit]\!]_{2^n}(\pi_1, \dots, \pi_n)$; and from $\tau_\heartsuit$, we can recover $[\![\heartsuit]\!]$ by $[\![\heartsuit]\!]_X(f_1, \dots, f_n) = \tau_\heartsuit \circ F(\langle f_1, \dots, f_n \rangle)$.

Since Λ-bisimulations include not only endorelations but also binary relations between two different sets, we use the $\mathbf{CLat}_\sqcap$-fibration $\mathbf{BRel} \to \mathbf{Set}^2$ (Definition 2.3.16) here. One key notion in [4] is *R-coherence*.

Definition 4.8.17 (R-coherent pairs [4, Definition 8, Lemma 9 (b)])**.** Let $(X, Y, R) \in \mathbf{BRel}$, $U \subseteq X$, and $V \subseteq Y$. The pair (U, V) is *R-coherent* if both of the following hold:

- $(x, y) \in R \wedge x \in U \implies y \in V$.
- $(x, y) \in R \wedge y \in V \implies x \in U$.

Equivalently, the pair (U, V) is R-coherent if and only if, for each $(x, y) \in R$, $x \in U \iff y \in V$ holds.

The notion of R-coherence turns out to be expressible in terms of the fibration $\mathbf{BRel} \to \mathbf{Set}^2$.

Proposition 4.8.18 (coherence as decency)**.** *Let $(X, Y, R) \in \mathbf{BRel}$, $f\colon X \to 2$, and $g\colon Y \to 2$. Let $\mathrm{Eq}_2 \subseteq 2 \times 2$ be the diagonal relation (Example 2.3.13). Then the pair $(f^{-1}(\top), g^{-1}(\top))$ is R-coherent if and only if the arrow (f, g) in $\mathbf{Set}^2$ is decent from (X, Y, R) to $(2, 2, \mathrm{Eq}_2)$.*

Proof. By Definition 4.8.17, the pair $(f^{-1}(\top), g^{-1}(\top))$ is R-coherent if and only if, for each $(x, y) \in R$, $f(x) = \top \iff g(y) = \top$ holds. Here, the condition $f(x) = \top \iff g(y) = \top$ is equivalent to $(f(x), g(y)) \in \mathrm{Eq}_2$. The claim follows from Definition 2.3.16. □

From now on, we consider a similarity type Λ with only unary modal operators. It turns out that, in such cases, a Λ-bisimulation is the same thing as a codensity bisimulation with an appropriate family of lifting parameters.

Let us fix a Λ-structure $(F, ([\![\heartsuit]\!])_{\heartsuit \in \Lambda})$. For each $\heartsuit \in \Lambda$, let $\tau_\heartsuit\colon F2 \to 2$ be the arrow corresponding to $[\![\heartsuit]\!]\colon \mathbf{Set}(\_, 2) \Rightarrow \mathbf{Set}(F\_, 2)$.

Definition 4.8.19 (Λ-bisimulation [4, Definition 11])**.** Let $c\colon X \to FX$ and $d\colon Y \to FY$ be F-coalgebras. A relation $Z \subseteq X \times Y$ is a *Λ-bisimulation* if, for every pair $(x, y) \in Z$, modal operator $\heartsuit \in \Lambda$, and Z-coherent pair (U, V),

$$c(x) \in [\![\heartsuit]\!]_X(U) \iff d(y) \in [\![\heartsuit]\!]_X(V)$$

holds.

This definition can be characterized using codensity lifting. We use the lifting of $F^2\colon \mathbf{Set}^2 \to \mathbf{Set}^2$ by the family of parameters $\{((2, 2, \mathrm{Eq}_2), \tau_\heartsuit)_{\heartsuit \in \Lambda}\}$.

Proposition 4.8.20. *Let $c\colon X \to FX$ and $d\colon Y \to FY$ be F-coalgebras. A Λ-bisimulation is nothing but a codensity bisimulation for the family of lifting parameters $((2, 2, \mathrm{Eq}_2), \tau) = \{((2, 2, \mathrm{Eq}_2), \tau_\heartsuit)_{\heartsuit\in\Lambda}\}$, that is, $Z \subseteq X \times Y$ is a Λ-bisimulation if and only if $(X, Y, Z) \sqsubseteq (c, d)^*(F^2)^{(2,2,\mathrm{Eq}_2),\tau}(X, Y, Z)$ holds.*

Proof. Assume $(X, Y, Z) \sqsubseteq (c, d)^*(F^2)^{(2,2,\mathrm{Eq}_2),\tau}(X, Y, Z)$. Expanding the definitions, the following holds:

> If $(x, y) \in Z$, for each $(f, g)\colon (X, Y, Z) \to (2, 2, \mathrm{Eq}_2)$ and each $\heartsuit \in \Lambda$, $\tau_\heartsuit((Ff)(c(x))) = \tau_\heartsuit((Fg)(d(y)))$ holds.

Let (U, V) be any Z-coherent pair. We define $f\colon X \to 2$ and $g\colon Y \to 2$ by $f(x) = \top \iff x \in U$ and $g(y) = \top \iff y \in V$. By Proposition 4.8.18, $(f, g)\colon (X, Y, Z) \to (2, 2, \mathrm{Eq}_2)$ is decent. Thus, for each $\heartsuit \in \Lambda$, $\tau_\heartsuit((Ff)(c(x))) = \tau_\heartsuit((Fg)(d(y)))$ holds. By the definition of $\tau_\heartsuit$, this means

$$c(x) \in [\![\heartsuit]\!]_X(U) \iff d(y) \in [\![\heartsuit]\!]_X(V).$$

Since (U, V) is arbitrary, Z is a Λ-bisimulation.

Conversely, assume $Z \subseteq X \times Y$ is a Λ-bisimulation. For every pair $(x, y) \in Z$, modal operator $\heartsuit \in \Lambda$, and Z-coherent pair (U, V),

$$c(x) \in [\![\heartsuit]\!]_X(U) \iff d(y) \in [\![\heartsuit]\!]_X(V)$$

holds. Now, for each decent arrow $(f, g)\colon (X, Y, Z) \to (2, 2, \mathrm{Eq}_2)$, $(f^{-1}(\top), g^{-1}(\top))$ is Z-coherent by Proposition 4.8.18. Thus for every pair $(x, y) \in Z$ and modal operator $\heartsuit \in \Lambda$,

$$c(x) \in [\![\heartsuit]\!]_X(f^{-1}(\top)) \iff d(y) \in [\![\heartsuit]\!]_X(g^{-1}(\top))$$

holds. By the definition of $\tau_\heartsuit$, this is equivalent to $\tau_\heartsuit((Ff)(c(x))) = \tau_\heartsuit((Ff)(c(y)))$. Since this holds for any decent $(f, g)\colon (X, Y, Z) \to (2, 2, \mathrm{Eq}_2)$, $(X, Y, Z) \sqsubseteq (c, d)^*(F^2)^{(2,2,\mathrm{Eq}_2),\tau}(X, Y, Z)$ holds. $\square$

Corollary 4.8.21. *Let $c\colon X \to FX$ and $d\colon Y \to FY$ be F-coalgebras. The codensity bisimilarity $\nu\Phi^{(2,2,\mathrm{Eq}_2),\tau}$ is the largest Λ-bisimulation.* $\square$

In the case where the modal operators are all unary, we can derive a game characterization of Λ-bisimulation from our general framework. Let us first note the following fact:

Proposition 4.8.22. *The object $(1, 1) \in \mathbf{Set}^2$ is a fibered separator (Definition 4.5.5) of $\mathbf{BRel} \to \mathbf{Set}^2$.*

Proof. Let $(X, Y) \in \mathbf{Set}^2$ and $B_1, B_2 \in \mathbf{BRel}_{(X,Y)}$. Assume $B_1 \neq B_2$. There exists a pair $(x, y) \in X \times Y$ such that exactly one of $(x, y) \in B_1$ and $(x, y) \in B_2$ holds. Consider the arrow $(x, y)\colon (1, 1) \to (X, Y)$ in $\mathbf{Set}^2$. Then $(x, y)^*B_1 \neq (x, y)^*B_2$ holds. This concludes the proof. $\square$

By Corollary 4.6.8 and suppressing $\heartsuit$ (which does not affect Duplicator's moves), we obtain the following game characterization.

Corollary 4.8.23. *Let $c\colon X \to FX$ and $d\colon Y \to FY$ be F-coalgebras. For a pair of states $(x, y) \in X \times Y$, there exists a Λ-bisimulation containing (x, y) if and only if the position $(x, y) \in X \times Y$ in the game in Table 4.14 is winning for Duplicator.* □

Table 4.14.: Codensity bisimilarity game for Λ-bisimulation

position	player	possible moves
$(x, y) \in X \times Y$	Spoiler	f and g such that, for some $\heartsuit \in \Lambda$, exactly one of $\tau_\heartsuit((Ff)(c(x))) = \top$ and $\tau_\heartsuit((Fg)(d(x))) = \top$ holds
$f\colon X \to 2$ and $g\colon Y \to 2$	Duplicator	(x', y') such that exactly one of $f(x') = \top$ and $g(y') = \top$ holds

This in turn yields game characterizations of many bisimulation notions, e.g., those listed in [4, Example 13–16].

4.8.3. Deterministic Automata and the Language Topology

We introduced two versions of *bisimulation topology* for deterministic automata in Example 4.6.9. They are in close correspondences with accepted languages; therefore we call them *language topologies*.

For the first topology in Example 4.6.9 (where $\mathbf{\Omega}$ is the Sierpinski space, modeling the situation where acceptance is only semi-decidable), the corresponding (untrimmed) codensity game is shown in Table 4.15. It follows from our general results that the game notion is sound and complete.

Table 4.15.: Codensity bisimilarity game for deterministic automata and the bisimulation topology

position	player	possible moves
$\mathcal{O} \in \mathbf{Top}_X$	Spoiler	$a \in \{\varepsilon\} \cup \Sigma$ and $k \in \mathbf{Set}(X, 2)$ such that $\tau_a \circ (A_\Sigma k) \circ c\colon X \to 2$ is not continuous from $(X, \mathcal{O})$ to $(2, \mathbf{\Omega}_a)$
$a \in \{\varepsilon\} \cup \Sigma$ and $k \in \mathbf{Set}(X, 2)$	Duplicator	$\mathcal{O}' \in \mathbf{Top}_X$ such that $k\colon X \to 2$ is not continuous from $(X, \mathcal{O}')$ to $(2, \mathbf{\Omega}_a)$

We have not yet found a good way (e.g. join-dense subsets) of trimming the game arena. This is left as future work.

4.8.4. Markov Chains and Bisimulation Metric

Recall Examples 3.1.11, 4.4.5 and 4.5.3. Markov chains are $\mathcal{D}_{\leq 1}$-coalgebras. We use the $\mathbf{CLat}_\sqcap$-fibration $\mathbf{PMet}_1 \to \mathbf{Set}$ (Example 2.3.9), taking pseudometrics as a notion of indistinguishability. With the lifting parameter we described in Example 3.1.11, we get the bisimulation metric as the codensity bisimilarity. We can use the join-dense subset described in Example 4.5.3 to obtain a trimmed codensity game; the resulting game coincides with the one in Table 4.3 in the introduction. Therefore, Corollary 4.5.16 gives an abstract proof for the correctness of the game.

4.8.5. Continuous State Markov Chains and Bisimulation Metric

In order to accommodate continuous state Markov chains (for which measurable structures are essential), we consider an example that involves **Meas**. Continuing Section 4.8.4, by the change-of-base along the forgetful functor $U\colon \mathbf{Meas} \to \mathbf{Set}$, we get another $\mathbf{CLat}_\sqcap$-fibration $U^*(\mathbf{PMet}_1) \to \mathbf{Meas}$. A continuous state Markov chain is a coalgebra $X \to \mathcal{G}_{\leq 1}X$ of the so-called *sub-Giry* functor $\mathcal{G}_{\leq 1}\colon \mathbf{Meas} \to \mathbf{Meas}$—see, e.g., [28].

As a parameter of codensity lifting, we take roughly the same thing as used in Example 3.1.11. The major difference is that we have to equip $[0,1]$ with some σ-algebra. We use the σ-algebra of *Borel sets* $\mathcal{B}([0,1])$. Let us abuse the notation $[0,1]$ to mean the object $([0,1], \mathcal{B}([0,1])) \in \mathbf{Meas}$. Then the parameter of codensity lifting we use is

$$(\boldsymbol{\Omega}, \tau) = \big(([0,1], d_{[0,1]}), \, e\colon \mathcal{G}_{\leq 1}[0,1] \to [0,1] \big),$$

where e is the *expectation function* $e(\mu) = \int r d\mu(r)$, and $d_{[0,1]}$ is the Euclidean metric.

Let us expand the definition of the codensity lifting $\mathcal{G}_{\leq 1}^{\boldsymbol{\Omega},\tau}\colon U^*(\mathbf{PMet}_1) \to U^*(\mathbf{PMet}_1)$. For $X \in \mathbf{Meas}$ and $(X,d) \in U^*(\mathbf{PMet}_1)$, $\mathcal{G}_{\leq 1}^{\boldsymbol{\Omega},\tau}(X,d) = (\mathcal{G}_{\leq 1}X, \mathcal{K}(d))$ holds. Here, $\mathcal{K}(d)$ is a variation of Kantorovich metric. For $\mu, \nu \in \mathcal{G}_{\leq 1}X$,

$$\mathcal{K}(d)(\mu,\nu) = \sup_f \left| e((\mathcal{G}_{\leq 1}f)(\mu)) - e((\mathcal{G}_{\leq 1}f)(\nu)) \right|,$$

where f ranges over all non-expansive and measurable functions from (X,d) to $([0,1], d_{[0,1]})$. Note the similarity with the equation (4.1). The corresponding codensity bisimilarity $\nu\Phi_c^{\boldsymbol{\Omega},\tau} \in U^*(\mathbf{PMet}_1)$ (Definition 3.1.10) is a variation of the bisimulation metric from [22] for continuous state Markov chains.

Since the forgetful functor $\mathbf{Meas} \to \mathbf{Set}$ has a left adjoint, Proposition 4.5.10 gives us a fibered separator for $U^*(\mathbf{PMet}_1) \to \mathbf{Meas}$: concretely, the two-point set with the powerset σ-algebra $(2, \mathcal{P}2) \in \mathbf{Meas}$ is a fibered separator for $U^*(\mathbf{PMet}_1) \to \mathbf{Meas}$.

By Corollary 4.5.16, the codensity bisimilarity $\nu\Phi_c^{\boldsymbol{\Omega},\tau} \in U^*(\mathbf{PMet}_1)$ is characterized by the codensity game (Table 4.8) specialized in this situation. The game in this case is shown in Table 4.16.

4.9. Conclusions and Future Work

Motivated by some recent works [16, 48, 11, 6], and especially by the similarity of the two games in Tables 4.2 and 4.3, we introduced a fibrational framework that uniformly describes the correspondence between various bisimilarity notions and games. The fibrational abstraction allows us to accommodate new games for several known examples

Table 4.16.: Codensity bisimilarity game for (probabilistic) bisimulation metric for a continuous state Markov chain

position	player	possible moves
(x, y, ε) $\in X^2 \times [0,1]$	Spoiler	measurable $f\colon X \to [0,1]$ such that $\lvert e((\mathcal{G}_{\leq 1} f)(c(x))) - e((\mathcal{G}_{\leq 1} f)(c(y)))\rvert > \varepsilon$
measurable $f\colon X \to [0,1]$	Duplicator	$(x', y', \varepsilon') \in X^2 \times [0,1]$ such that $\big\lvert f(x') - f(y') \big\rvert > \varepsilon'$

(such as Λ-bisimulation in Section 4.8.2 and bisimulation metric in Section 4.8.4) and a new example (bisimulation topology in Section 4.8.3). Moreover, the structural theory developed in Sections 4.6 and 4.7 provides new insights to the nature of bisimilarity, identifying the crucial role of observation maps ($k\colon X \to \Omega$ in Definition 5.2.8) in bisimulation notions.

As future work, we are interested in using games with more complex winning conditions (e.g. parity); they have been used for (bi)simulation notions for Büchi and parity automata [25]. In addition, we will pursue the algorithmic use of the current results.

5. Expressivity of Modal Logic for Codensity Bisimilarity

5.1. Overview

(Quantitative) Modal Logics and Their Coalgebraic Unification The role of different kinds of *modal logics* is pervasive in computer science. Their principal functionality is to specify and reason about behaviors of state-transition systems. With the growing diversity of target systems (probabilistic, cyber-physical, etc.), the use of *quantitative* modal logics—where truth values and logical connectives can involve real numbers—is increasingly common. For such logics, however, providing the necessary theoretical foundations takes a significant effort and is often done individually for each variant.

It is therefore desirable to establish unifying and abstract foundations once and for all, which readily instantiate to individual modal logics. This is the goal pursued by the study of *coalgebraic modal logic* [56, 64, 58, 13, 14, 59, 43], which builds on the general categorical modeling of state-transition systems as *coalgebras* [37, 60].

Expressivity of Modal Logics When using a concrete modal logic, there are several important properties that we expect its metatheory to address, such as *soundness* and *completeness* of its proof system. In this paper, we are interested in the *adequacy* and *expressivity* properties of the logic. These properties are about comparison between 1) the expressive power of the logic, and 2) some notion of *indistinguishability* that is inherent in the target state-transition systems.

A prototypical example of such notions of indistinguishability is *bisimilarity* [55]. Expressivity with respect to bisimilarity—that modal logic formulas can distinguish non-bisimilar states—is the classic result by Hennessy and Milner [30]. Adequacy, the opposite of expressivity, means that semantics of modal formulas is invariant under bisimilarity, and holds in most modal logics. In contrast, expressivity is a desired property but not always true. Expressivity, when it holds, relies on a delicate balance between the choice of modal operators, the underlying propositional connectives, and the "size" of (branching of) the target state-transition systems.

Quantitative Expressivity The aforementioned interests in quantitative modal logics have sparked research efforts for *quantitative expressivity*. In quantitative settings, the inherent indistinguishability notion in target systems is quantitative, too, typically formulated in terms of a *bisimulation pseudometric* ("how much apart the two states are") that refines the quantitative notion of bisimilarity ("if the two states are indistinguishable or not") [27, 22]. In the expressivity problem, such an indistinguishability notion is compared against the quantitative truth values of logical formulas.

Recent works that study quantitative expressivity include [22, 71, 16, 74, 48, 73]; they often involve coalgebraic generalization, too, since quantitative modal logics often have immediate variations. Their quantitative expressivity proofs are much more mathematically involved compared to qualitative expressivity proofs. This is because the aforementioned balance between syntax and semantic equivalences is much more delicate. Specifically, target systems are quantitative and thus exhibit *continuity* of behaviors, while logical syntax is inherently *disconnected*, in the sense that each logical formula is an inductively defined and thus finitary entity. Expressivity needs to bridge these two seemingly incompatible worlds.

In order to do so, each of the expressivity proofs in [71, 16, 74, 48, 73] uses some kind of "approximation." However, each of these arguments has a specialized, tailor-made flavor: Stone–Weierstrass-like arguments for metric spaces [48], the unique structure theorem for analytic spaces [16], and so on. It does not seem easy to distill the essence that is common to different quantitative expressivity proofs. Indeed, there has not been a coalgebraic framework that unifies them.

Categorical Unification of Quantitative Expressivity via Codensity and Approximation
We present the first categorical framework that uniformly axiomatizes different approximation arguments—it uses a fibrational notion of *approximating family*—and unifies different quantitative expressivity results.

Our framework hinges on the construction called the *codensity lifting* [66, 44]; it is a general method for modeling a variety of bisimilarity-like notions (bisimilarity, probabilistic bisimilarity, bisimulation metric, etc.). The codensity lifting uses not only coalgebras (for unifying different state-transition systems) but also *fibrations* for different *observation modes*; the latter include Boolean predicates, quantitative/fuzzy predicates, equivalence relations, pseudometrics, topologies, etc. This use of fibrations provides flexibility to accommodate a variety of quantitative bisimilarity-like notions.

The codensity lifting, while defined in abstract categorical terms, has clear observational intuition (see Section 5.2.3). It also gives a class of *codensity bisimilarity games* that characterize a variety of (qualitative and quantitative) bisimilarity notions [44] (see also Section 5.2.3).

Our key contribution of a categorical formalization of approximation is enabled by the formalization of the codensity lifting. It has a similar observational intuition, too: see Section 5.3.1, where we characterize an approximating family of observations as a "winnable" set of moves in a suitable sense.

On top of our fibrational notion of approximating family, we establish a general expressivity framework, which is the first to unify existing quantitative expressivity results including [16, 74, 48]. In our unified framework, we have two proof principles for expressivity—Knaster–Tarski (Theorem 5.3.4) and Kleene (Theorem 5.3.6)—that mirror two classic characterizations of greatest fixed points. Our general framework is presented in terms of predicate lifting [64, 58]. This is mostly for presentation purposes (showing concrete syntax is easier this way). A more abstract and fully fibrational recap of our framework—where a modal logic is formalized with a dual adjunction [13, 14, 59, 43]—is found in [47].

We demonstrate our general framework with three examples: expressivity for the Kantorovich bisimulation metric (from [48], Section 5.4); that for Markov process bisimilarity (from [16], Section 5.5); and that for the so-called *bisimulation uniformity* (Section 5.6). Both the Knaster–Tarski and Kleene principles are used for proofs. See Table 5.1. The last is a new expressivity result that is not previously found in the literature.

We note that the role of the notion of approximating family is as a useful axiomatization: it tells us what key lemma to prove in an expressivity proof, but it does not tell how to prove the key lemma. The proof of this key lemma is where the technical hardcore lies in existing expressivity proofs (by a Stone–Weierstrass-like result in [48], by the unique structure theorem in [16], etc.). For the new instance of bisimulation uniformity (Section 5.6), the general axiomatization of approximating family allowed us to discover a result we need in a paper [18] that is seemingly unrelated to modal logic. The same result guided us in the design of modal logic, too, especially in the choice of propositional connectives.

Contributions We summarize our contributions.

- The notion of approximating family, whose instances occur in the key steps of existing quantitative expressivity proofs. It is built on top of the codensity lifting.
- We use it in a unified categorical expressivity framework. It offers two proof principes (Knaster–Tarski and Kleene) that have different applicability (Table 5.1).
- The framework is instantiated to two known expressivity results [48, 16] and one new result (Section 5.6).

Related Work Here we list related work considering quantitative expressivity.

Our framework is parameterized both in the kind of coalgebra and in the observation mode. To our knowledge, the only existing work with this generality is [50] which combines coalgebras and fibrations to provide a general setting for proving expressivity. However, that approach does not accommodate approximation arguments, therefore failing to cover any of the aforementioned quantitative expressivity proofs [71, 16, 74, 48, 73]. Compared to [50], our main novelty is the accommodation of approximation arguments and thus quantitative expressivity results, as we already discussed.

The idea of behavioral metrics was first proposed in [27]. In the setting of category theory, the *behavioral pseudometric* is introduced in [71] in terms of coalgebras in the category $\mathbf{PMet}_1$ of 1-bounded pseudometric spaces, and a corresponding expressivity result is established. Many other formulations of quantitative bisimilarity are based on *fibrational coinduction* [32]. The work [6] discusses general behavioral metrics (but not modal logics); expressivity w.r.t. these metrics is studied in [48] for general **Set**-coalgebras. The line of work on *codensity bisimilarity*—including [66, 44] and the current work—follows this fibrational tradition, too.

Table 5.1.: Examples of expressivity situations. The non-shaded rows describe data in expressivity situations, and the shaded ones describe the resulting bisimilarity notions and modal logics, i.e. two constructs compared in the problem of expressivity.

parameter	[48] (Section 5.4) by Kleene	[16] (Section 5.5) by Knaster–Tarski	Section 5.6 by Knaster–Tarski
$\mathbb{C}$ category of *spaces*	**Set** sets	**Meas** measurable sets	**Set** sets
B behavior functor	B (arbitrary)	$(\mathcal{G}_{\leq 1}\_)^A$ continuous-space Markov processes	B (arbitrary)
$p\colon \mathbb{E} \to \mathbb{C}$ observation mode	$\mathbf{PMet}_1 \to \mathbf{Set}$ 1-bounded pseudometrics	$\mathbf{EqRel}_{\mathbf{Meas}} \to \mathbf{Meas}$ equiv. relations	$\mathbf{Unif} \to \mathbf{Set}$ uniformity
$\Omega \in \mathbb{C}$ truth-value domain	$[0,1]$ the unit interval	$2 = \{0,1\}$ with the discrete σ-algebra	$[0,1]$ the unit interval
$\mathbf{\Omega} \in \mathbb{E}_\Omega$ observation predicate	$([0,1], d_e)$ Euclidean metric	$(2, =)$ equality	$([0,1], \mathcal{U}_e)$ metric uniformity
$(\tau_\lambda\colon B\Omega \to \Omega)_{\lambda\in\Lambda}$ observation modality	arbitrary but must satisfy Assumption 5.4.3	$\Lambda = A \times (\mathbb{Q} \cap [0,1])$ $\tau_{a,r}((\mu_a)_{a\in A}) = 1$ iff $\mu_a(1) > r$	arbitrary but must satisfy Assumption 5.6.6
resulting bisimilarity-like notion	B-bisimulation metric [48]	probabilistic bisimilarity	bisimulation uniformity
Σ propositional connectives	$\top, \neg, \min$, and $(\ominus q)$ for $q \in \mathbb{Q} \cap [0,1]$	$\top, \wedge$	$1, \min$, and $(r+), (r\times)$ for $r \in \mathbb{R}$
$(f_\sigma)_{\sigma\in\Sigma}$ propositional structure	Zadeh logic connectives on $[0,1]$	meet- semilattice with $0 \sqsubseteq 1$	affine lattice structure on $\mathbb{R}$
resulting modal logic **(modal operators are** $(\heartsuit_\lambda)_{\lambda\in\Lambda}$**)**	The logic in [48], generalization of Zadeh fuzzy modal logic in [74]	$\mathrm{PML}_\wedge$ [16]	A new modal logic

A recent work [73] uses a different formulation of bisimilarity-like notions: it does not use fibrations or functor liftings, but uses so-called *fuzzy lax extensions* of functors. This approach is a descendant of *relators* [61]; seeking the connection to these works is future work.

Organization We axiomatize the data under which we study expressivity—it is called an *expressivity situation*—in Section 5.2. In Section 5.3 we define our key notion of approximating family, from which we derive the Knaster–Tarski and Kleene proof principles for expressivity. §5.4–5.6 present instances of our framework: two known [48, 16] and one new (Section 5.6).

Most proofs are deferred to the appendix.

5.2. Expressivity Situation

On top of the above preliminaries, we fix the format of categorical data under which we study expressivity. It is called an expressivity situation. While it may seem overwhelming, we show that the data arises naturally, with clear intuition from the viewpoints of modal logics and observations (§5.2.2–5.2.3).

5.2.1. Definition

Definition 5.2.1. An *expressivity situation* $\mathscr{S} = (p, B, \Omega, \mathbf{\Omega}, \Sigma, \Lambda, (f_\sigma)_{\sigma\in\Sigma}, (\tau_\lambda)_{\lambda\in\Lambda})$ is given by the following.

- A $\mathbf{CLat}_\sqcap$-fibration $p\colon \mathbb{E} \to \mathbb{C}$.
- A functor $B\colon \mathbb{C} \to \mathbb{C}$ (a *behavior functor*).
- An object $\Omega \in \mathbb{C}$ (a *truth-value object*) equipped with finite powers ($\Omega^n \in \mathbb{C}$ for $n \in \mathbb{N}$), and another object $\mathbf{\Omega}$ (an *observation predicate*) above it. It follows that $\mathbf{\Omega}$ also has finite powers [36, Prop. 9.2.1].
- A ranked alphabet Σ of *propositional connectives* and a family of arrows $\big(f_\sigma\colon \Omega^{\mathrm{rank}(\sigma)} \to \Omega\big)_{\sigma\in\Sigma}$ (a *propositional structure*). Moreover, we require that each $f_\sigma\colon \Omega^{\mathrm{rank}(\sigma)} \to \Omega$ has a *lifting* $g_\sigma\colon \mathbf{\Omega}^{\mathrm{rank}(\sigma)} \to \mathbf{\Omega}$ (in $\mathbb{E}$) such that $pg_\sigma = f_\sigma$.
- A set Λ of *modality indices* and a family of algebras $(\tau_\lambda\colon B\Omega \to \Omega)_{\lambda\in\Lambda}$ (*observation modalities*).

Roughly speaking, Σ and Λ are used for modal logic *syntax*, and $\mathbb{C}$, B, Ω, $(f_\sigma)_{\sigma\in\Sigma}$, and $(\tau_\lambda)_{\lambda\in\Lambda}$ are used for modal logic *semantics*. The other constructs (p and $\mathbf{\Omega}$) are there for defining a bisimilarity-like notion.

In what follows, we formulate the expressivity problem on top of Definition 5.2.1, explaining the role of each piece of data in an expressivity situation $\mathscr{S}$. More specifically, we let $\mathscr{S}$ induce the following constructs: 1) the modal logic $L_{\mathscr{S}}$ (Definitions 5.2.2 and 5.2.4); 2) the *fibrational logical equivalence* $\mathsf{LE}_{\mathscr{S}}(x)$ induced by $L_{\mathscr{S}}$ (Definition 5.2.6);

and 3) the bisimilarity-like notion $\mathsf{Bisim}^{\mathbf{\Omega},\tau}(x)$ as a *codensity bisimilarity* (Definition 5.2.9). Comparison of the last two is the problem of expressivity. As an illustrating example, we use an expressivity situation $\mathscr{S}_{\mathrm{KMM}}$ that arises from the real-valued logic $\mathcal{M}(\Lambda)$ in [48] (see also Section 5.4).

5.2.2. Syntax and Semantics of Our Logic $L_{\mathscr{S}}$

The syntax of modal logic is specified by the propositional connectives in Σ and the modality indices in Λ.

Definition 5.2.2 ($L_{\mathscr{S}}$)**.** Let $\mathscr{S}$ be an expressivity situation in Definition 5.2.1. The modal logic $L_{\mathscr{S}}$ has the following syntax.

$$\begin{aligned} \varphi, \varphi_1, \dots, \varphi_n \ &::= \sigma(\varphi_1, \dots, \varphi_{\mathrm{rank}(\sigma)}) && (\sigma \in \Sigma) \\ &\mid \heartsuit_\lambda \varphi && (\lambda \in \Lambda) \end{aligned}$$

We also let $L_{\mathscr{S}}$ denote the set of all formulas.

Example 5.2.3. Let Λ be a set. To model the modal logic $\mathcal{M}(\Lambda)$ in [48], we let $\Sigma = \{\top^0, \min^2, \neg^1\} \cup \{(\ominus q)^1 \mid q \in \mathbb{Q} \cap [0,1]\}$. Then the syntax is given by

$$\begin{aligned} \varphi, \varphi_1, \dots, \varphi_n \ ::= \ &\top \mid \neg(\varphi) \mid \min(\varphi_1, \varphi_2) \\ &\mid (\ominus q)\varphi \ \ (q \in \mathbb{Q} \cap [0,1]) \ \mid \ \heartsuit_\lambda \varphi \ \ (\lambda \in \Lambda). \end{aligned}$$

Identifying $\heartsuit_\lambda$ with $[\lambda]$ in the original notation, this recovers the syntax of $\mathcal{M}(\Lambda)$ in [48].

Given a coalgebra $x \colon X \to BX$ of the behavior functor B, the semantics of each formula is a $\mathbb{C}$-arrow from the state space X to the truth-value object Ω, inductively defined as follows.

Definition 5.2.4. Let $\mathscr{S}$ be an expressivity situation in Definition 5.2.1 and let $x \colon X \to BX$ be a B-coalgebra. For each $\varphi \in L_{\mathscr{S}}$, the *interpretation* $[\![\varphi]\!]_x \colon X \to \Omega$ of φ with respect to x is defined inductively as follows:

$$\begin{aligned} [\![\sigma(\varphi_1, \dots, \varphi_{\mathrm{rank}(\sigma)})]\!] &= f_\sigma \circ \langle [\![\varphi_1]\!], \dots, [\![\varphi_{\mathrm{rank}(\sigma)}]\!] \rangle, && (\sigma \in \Sigma) \\ [\![\heartsuit_\lambda \varphi]\!] &= \tau_\lambda \circ (B[\![\varphi]\!]) \circ x. && (\lambda \in \Lambda) \end{aligned}$$

Example 5.2.5. Recall Example 5.2.3. Let $B \colon \mathbf{Set} \to \mathbf{Set}$ be an endofunctor, and Ω be the unit interval $[0,1]$. We specify the propositional structure $(f_\sigma \colon [0,1]^{\mathrm{rank}(\sigma)} \to [0,1])_{\sigma \in \Sigma}$ by:

$$\begin{aligned} f_\top() &= 1, & f_{\min}(x,y) &= \min(x,y), \\ f_\neg(x) &= 1 - x, & f_{\ominus q}(x) &= \max(x - q, 0). \end{aligned}$$

Here min plays the role of conjunction. Let $(\tau_\lambda \colon B[0,1] \to [0,1])$ be a family of observation modalities, and $x \colon X \to BX$ be a B-coalgebra. Then, the semantics $[\![\varphi]\!]_x$ of each formula φ in Definition 5.2.4 coincides with the definition in [48, §3.2].

The following definition generalizes, in fibrational terms, the conventional definition that two states are logically equivalent if each formula's truth values coincide.

Definition 5.2.6 (fibrational logical equivalence $\mathsf{LE}_{\mathscr{S}}(x)$)**.** Let $\mathscr{S}$ be an expressivity situation in Definition 5.2.1 and let $x\colon X \to BX$ be a B-coalgebra. The *fibrational logical equivalence* $\mathsf{LE}_{\mathscr{S}}(x)$ with respect to x is a predicate above X defined by

$$\mathsf{LE}_{\mathscr{S}}(x) = \textstyle\bigsqcap_{\varphi \in L_{\mathscr{S}}} [\![\varphi]\!]_x^* \mathbf{\Omega}, \quad \text{where} \quad \begin{array}{c} \mathbb{E} \\ \downarrow p \\ \mathbb{C} \end{array} \qquad \begin{array}{ccc} [\![\varphi]\!]_x^* \mathbf{\Omega} & \dashrightarrow & \mathbf{\Omega} \\ X & \xrightarrow{[\![\varphi]\!]_x} & \Omega. \end{array}$$

Example 5.2.7. Recall Example 5.2.5. To define a logical distance, we let p be the $\mathbf{CLat}_{\sqcap}$-fibration $\mathbf{PMet}_1 \to \mathbf{Set}$ (Example 2.3.9), and $\mathbf{\Omega}$ be the usual Euclidean metric d_e on $[0,1]$.

Then, for each B-coalgebra $x\colon X \to BX$, the pseudometric $d^L_x := \mathsf{LE}_{\mathscr{S}_{\mathrm{KMM}}}(x)$ is equivalently described by

$$d^L_x(s,t) = \sup_{\varphi} d_e\big([\![\varphi]\!]_x(s), [\![\varphi]\!]_x(t)\big),$$

where φ ranges over the modal formulas. Thus Definition 5.2.6 coincides with the notion of logical distance in [48, Def. 25].

5.2.3. Codensity Bisimilarity for Expressivity Situations

We unify different quantitative bisimilarity notions—such as probabilistic bisimilarity and bisimulation metric—using *codensity bisimilarity* introduced in Chapter 3. This is what is compared with the fibrational logical equivalence (Definition 5.2.6).

Concretely, we adopt the definitions for multiple parameters (Section 3.2). For an expressivity situation, we define as follows:

Definition 5.2.8 (codensity lifting)**.** Let $\mathscr{S}$ be an expressivity situation in Definition 5.2.1. The *codensity lifting* of B with respect to $\mathbf{\Omega}$ and $(\tau_\lambda)_{\lambda\in\Lambda}$ is the functor $B^{\mathbf{\Omega},\tau}\colon \mathbb{E} \to \mathbb{E}$, defined by

$$B^{\mathbf{\Omega},\tau} P = \textstyle\bigsqcap_{\lambda\in\Lambda, h\in\mathbb{E}(P,\mathbf{\Omega})} (\tau_\lambda \circ B(ph))^* \mathbf{\Omega}. \tag{5.1}$$

Definition 5.2.9 (codensity bisimilarity $\mathsf{Bisim}^{\mathbf{\Omega},\tau}(x)$)**.** Let $\mathscr{S}$ be an expressivity situation in Definition 5.2.1 and let $x\colon X \to BX$ be a B-coalgebra. The *codensity bisimilarity* $\mathsf{Bisim}^{\mathbf{\Omega},\tau}(x)$ of x is the $B^{\mathbf{\Omega},\tau}$-coinductive predicate (Definition 2.4.3), i.e., the greatest fixed point of the map $x^* \circ B^{\mathbf{\Omega},\tau}\colon \mathbb{E}_X \to \mathbb{E}_X$:

$$\mathsf{Bisim}^{\mathbf{\Omega},\tau}(x) = \nu(x^* \circ B^{\mathbf{\Omega},\tau}) \in \mathbb{E}_X.$$

5.2.4. Adequacy and Expressivity

We are ready to formulate adequacy and expressivity. Recall that $P \sqsubseteq Q$ in a fiber means that P is more discriminating.

Definition 5.2.10. Let $\mathscr{S}$ be an expressivity situation (Definition 5.2.1) and $x\colon X \to BX$ be a B-coalgebra.

- $\mathscr{S}$ is *expressive* for x if $\mathsf{Bisim}^{\mathbf{\Omega},\tau}(x) \sqsupseteq \mathsf{LE}_{\mathscr{S}}(x)$ holds.
- $\mathscr{S}$ is *adequate* for x if $\mathsf{Bisim}^{\mathbf{\Omega},\tau}(x) \sqsubseteq \mathsf{LE}_{\mathscr{S}}(x)$ holds.

$\mathscr{S}$ is *expressive* (or *adequate*) if it is expressive (or adequate, respectively) for any B-coalgebra x.

The following result justifies our axiomatization in Definition 5.2.1: adequacy, a property that is a prerequisite in most usage scenarios of modal logics, follows easily from the axiomatization itself.

Proposition 5.2.11. *Any expressivity situation $\mathscr{S}$ in Definition 5.2.1 is adequate.*

Proof. Fix a B-coalgebra $x\colon X \to BX$. It suffices to show that, for any $\varphi \in L_{\mathscr{S}}$, $\nu(x^* \circ B^{\mathbf{\Omega},\tau}) \sqsubseteq [\![\varphi]\!]_x^*\mathbf{\Omega}$ holds. We show this by structural induction on φ.

Assume $\varphi = \sigma(\varphi_1, \dots, \varphi_{\mathrm{rank}(\sigma)})$ where $\sigma \in \Sigma$. By IH, for each $i = 1, \dots, \mathrm{rank}(\sigma)$, $\nu(x^* \circ B^{\mathbf{\Omega},\tau}) \sqsubseteq [\![\varphi_i]\!]_x^*\mathbf{\Omega}$ holds, and thus there exists an arrow $h_i\colon \nu(x^* \circ B^{\mathbf{\Omega},\tau}) \to \mathbf{\Omega}$ in $\mathbb{E}$ that satisfies $ph_i = [\![\varphi_i]\!]_x$. Take an arrow $g_\sigma\colon \mathbf{\Omega}^{\mathrm{rank}(\sigma)} \to \mathbf{\Omega}$ satisfying $pg_\sigma = f_\sigma$ (its existence is required in Definition 5.2.1). Consider the arrow $g_\sigma \circ \langle h_1, \dots, h_{\mathrm{rank}(\sigma)}\rangle\colon \nu(x^* \circ B^{\mathbf{\Omega},\tau}) \to \mathbf{\Omega}$. Since p sends this arrow to $f_\sigma \circ \langle [\![\varphi_1]\!]_x, \dots, [\![\varphi_{\mathrm{rank}(\sigma)}]\!]\rangle = [\![\varphi]\!]_x$, $\nu(x^* \circ B^{\mathbf{\Omega},\tau}) \sqsubseteq [\![\varphi]\!]_x^*\mathbf{\Omega}$ holds.

Assume $\varphi = \heartsuit_\lambda \varphi'$ where $\lambda \in \Lambda$. By IH, $\nu(x^* \circ B^{\mathbf{\Omega},\tau}) \sqsubseteq [\![\varphi']\!]_x^*\mathbf{\Omega}$ holds. Applying $B^{\mathbf{\Omega},\tau}$ to both sides yields

$$\begin{aligned} B^{\mathbf{\Omega},\tau}\nu(x^* \circ B^{\mathbf{\Omega},\tau}) &\sqsubseteq B^{\mathbf{\Omega},\tau}[\![\varphi']\!]_x^*\mathbf{\Omega} \\ &= \bigsqcap_{\lambda' \in \Lambda, h\colon [\![\varphi']\!]_x^*\mathbf{\Omega} \to \mathbf{\Omega}} (B(ph))^*\tau_{\lambda'}^*\mathbf{\Omega} \\ &\sqsubseteq (B[\![\varphi']\!]_x)^*\tau_\lambda^*\mathbf{\Omega}. \end{aligned}$$

Then by applying x^* to both sides we obtain the claim:

$$\nu(x^* \circ B^{\mathbf{\Omega},\tau}) = x^* B^{\mathbf{\Omega},\tau}\nu(x^* \circ B^{\mathbf{\Omega},\tau}) \sqsubseteq x^*(B[\![\varphi']\!]_x)^*\tau_\lambda^*\mathbf{\Omega} = [\![\varphi]\!]_x^*\mathbf{\Omega}.$$

This concludes the induction. □

Example 5.2.12. Recall Example 5.2.7. In this case we can see that the codensity lifting coincides with the *Kantorovich lifting*; see Section 5.4 for details. Thus the codensity bisimilarity coincides with the behavioral distance defined in [48, Def. 22].

Expressivity of this expressivity situation, that we call $\mathscr{S}_{\mathrm{KMM}}$, means that, for each $x\colon X \to BX$ and each pair $(s,t) \in X^2$ of states, the inequality $d_x(s,t) \le d_x^L(s,t)$ holds between the behavioral and logical distances ("d_x^L is more discriminating"). Adequacy means that, for each x and (s,t), $d_x(s,t) \ge d_x^L(s,t)$ holds.

5.3. Approximation in Quantitative Expressivity

In this section, based on the axiomatization in Section 5.2, we present a fibrational notion of *approximating family of observations*. The notion axiomatizes and unifies the "approximation" properties that are key steps in many recent quantitative expressivity proofs, such as in [48, 16, 74, 75].

We then proceed to present two proof principles for expressivity—*Knaster–Tarski* and *Kleene*—that mirror two classic characterizations of greatest fixed points recalled in Section 2.2. These proof principles make a large part of an expressivity proof routine. The remaining technical challenges are 1) choosing a suitable propositional signature and 2) identifying suitable approximating families; our general framework singles out these technical challenges and thus eases the efforts for addressing them.

5.3.1. Approximating Family of Observations

Our categorical notion of *approximating family of observations* designates a "good" subset $S \subseteq \mathbb{C}(X, \Omega)$ of Ω-valued observations in a suitable sense. We will be asking if the set $\{[\![\varphi]\!]_x : X \to \Omega \mid \varphi \in L'\}$ of "logical observations" is approximating or not, where L' is some set of modal formulas.

Definition 5.3.1 (approximating family)**.** Let $\mathscr{S}$ be an expressivity situation in Definition 5.2.1 and X be an object of $\mathbb{C}$. A subset $S \subseteq \mathbb{C}(X, \Omega)$ is an *approximating family of observations*, or simply *approximating*, if, for every morphism

$$h\colon \left(\textstyle\prod_{k\in S} k^*\mathbf{\Omega}\right) \longrightarrow \mathbf{\Omega} \tag{5.2}$$

of $\mathbb{E}$ and every $\lambda \in \Lambda$, the following inequality holds:

$$\textstyle\bigsqcap_{k'\in S, \lambda'\in\Lambda} (\tau_{\lambda'} \circ Bk')^*\mathbf{\Omega} \;\sqsubseteq\; (\tau_\lambda \circ B(ph))^*\mathbf{\Omega}. \tag{5.3}$$

Note that $k\colon X \to \Omega$ is a $\mathbb{C}$-arrow while h is an $\mathbb{E}$-arrow.

Some explanation is in order. Intuitively, in the definition above, the set S is a set of "logical" observations. Each h as in Eq. (5.2) is a "non-logical" legitimate observation. For such h, the r.h.s. of Eq. (5.3) is the information obtained from the observation h. (Note the way h is used: it is not $h^*\mathbf{\Omega}$, but $(\tau_\lambda \circ B(ph))^*\mathbf{\Omega}$. This corresponds to Eq. (5.1).) On the other hand, the l.h.s. of Eq. (5.3) is the information from "logical" observations. Thus, an intuitive meaning of the definition above is that no "non-logical" observation gives any additional information. In many cases, the "logical" observations in S approximate each "non-logical" ones h. See Remark 5.3.2 for details.

Another intuition is given in terms of the codensity bisimilarity game (see Chapter 4). Roughly, S being an approximating family says that Spoiler may restrict its moves to $S \subseteq \mathbb{C}(X, \Omega)$.

Remark 5.3.2. In many examples, S being an approximating family is proved in the following two steps: 1) showing that ph can be approximated by observations in S; and 2) this approximation is preserved along the lifting $k \mapsto \tau_\lambda \circ Bk$ of observations over X

to those over BX. The former step is usually the harder one, and proved via arguments specific to the current situation (pseudometric spaces, measurable spaces, etc.).

Example 5.3.3. Recall Example 5.2.12. Let X be a set and $S \subseteq \mathbf{Set}(X, [0,1])$. In this case, $\bigsqcap_{k \in S} k^* \mathbf{\Omega} \in (\mathbf{PMet}_1)_X$ is the pseudometric d_S given by $d_S(x,y) = \sup_{k \in S} d_e(k(x), k(y))$. Therefore, in order to show S being an approximating family, we have to recover the pseudometric induced by $h\colon (X, d_S) \to ([0,1], d_e)$ from observations in S, for each h.

In Proposition 5.4.6 later, it will turn out that S is approximating if the following hold (under Assumption 5.4.3):

- S is closed under the four operations $\top$, $\min$, $\neg$, and $\ominus q$ for every $q \in \mathbb{Q} \cap [0,1]$.
- (X, d_S) is totally bounded.

In this case, any $h\colon (X, d_S) \to ([0,1], d_e)$ can be uniformly approximated by a countable sequence of arrows in S. Moreover, this approximation is preserved by the lifting $k \mapsto \tau_\lambda \circ Bk$ (this is what we require in Assumption 5.4.3). These two facts establish that S is approximating (cf. Remark 5.3.2). See Proposition 5.4.6 for details.

5.3.2. The Knaster–Tarski Proof Principle for Expressivity

From the Knaster–Tarski theorem (Fact 2.2.3), we can derive the following simple expressivity proof principle. Its proof is by showing that the logical equivalence $\mathsf{LE}_{\mathscr{S}}(x)$ is a suitable invariant and thus underapproximates the codensity bisimilarity.

Theorem 5.3.4 (the Knaster–Tarski proof principle)**.** *Let $\mathscr{S}$ be an expressivity situation in Definition 5.2.1 and $x\colon X \to BX$ be a B-coalgebra. If $\{[\![\varphi]\!]_x \mid \varphi \in L_{\mathscr{S}}\} \subseteq \mathbb{C}(X, \Omega)$ is approximating, then $\mathscr{S}$ is expressive for x.*

Proof. We show $\nu(x^* \circ B^{\mathbf{\Omega}, \tau}) \sqsupseteq \bigsqcap_{\varphi \in L_{\mathscr{S}}} [\![\varphi]\!]_x^* \mathbf{\Omega}$. By the Knaster–Tarski theorem (Fact 2.2.3), it suffices to show

$$x^* B^{\mathbf{\Omega}, \tau} \left(\bigsqcap_{\varphi \in L_{\mathscr{S}}} [\![\varphi]\!]_x^* \mathbf{\Omega} \right) \sqsupseteq \bigsqcap_{\varphi \in L_{\mathscr{S}}} [\![\varphi]\!]_x^* \mathbf{\Omega}.$$

Since the l.h.s. is equal to

$$\bigsqcap_{\lambda \in \Lambda, h\colon \bigsqcap_{\varphi \in L_{\mathscr{S}}} [\![\varphi]\!]_x^* \mathbf{\Omega} \to \mathbf{\Omega}} x^* (B(ph))^* \tau_\lambda^* \mathbf{\Omega},$$

it suffices to show

$$x^* (B(ph))^* \tau_\lambda^* \mathbf{\Omega} \sqsupseteq \bigsqcap_{\varphi \in L_{\mathscr{S}}} [\![\varphi]\!]_x^* \mathbf{\Omega}.$$

for each $\lambda \in \Lambda$ and $h\colon \bigsqcap_{\varphi \in L_{\mathscr{S}}} [\![\varphi]\!]_x^* \mathbf{\Omega} \to \mathbf{\Omega}$.

The set $\{[\![\varphi]\!]_x \mid \varphi \in L_{\mathscr{S}}\} \subseteq \mathbb{C}(X, \Omega)$ being approximating implies the following lower bound of the l.h.s.:

$$\begin{aligned} x^*(B(ph))^*\tau_\lambda^*\mathbf{\Omega} &\sqsupseteq \bigsqcap_{\varphi' \in L_{\mathscr{S}}, \lambda' \in \Lambda} x^*(\tau_{\lambda'} \circ B[\![\varphi']\!]_x)^*\mathbf{\Omega} \\ &= \bigsqcap_{\varphi' \in L_{\mathscr{S}}, \lambda' \in \Lambda} [\![\heartsuit_{\lambda'}\varphi']\!]_x^*\mathbf{\Omega} \\ &\sqsupseteq \bigsqcap_{\varphi \in L_{\mathscr{S}}} [\![\varphi]\!]_x^*\mathbf{\Omega}. \end{aligned}$$

This concludes the proof. □

The theorem's applicability hinges on whether we can show that the set $\{[\![\varphi]\!]_x \mid \varphi \in L_{\mathscr{S}}\} \subseteq \mathbb{C}(X, \Omega)$ is an approximating family (where φ ranges over all formulas). We use the theorem for the examples in §5.5 & 5.6.

5.3.3. The Kleene Proof Principle for Expressivity

To make use of Kleene theorem, we have to consider

$$\top \ \sqsupseteq\ (x^* \circ B^{\mathbf{\Omega},\tau})(\top)\ \sqsupseteq\ (x^* \circ B^{\mathbf{\Omega},\tau})^2(\top)\ \sqsupseteq\ \cdots \tag{5.4}$$

where the functor $x^* \circ B^{\mathbf{\Omega},\tau}$ is from Definition 5.2.9. We also have to assume that this sequence *stabilizes after ω steps*, i.e., $\bigsqcap_{i<\omega}(x^* \circ B^{\mathbf{\Omega},\tau})^i(\top)$ is a fixed point of $x^* \circ B^{\mathbf{\Omega},\tau}$.

We stratify $L_{\mathscr{S}}$ corresponding to the sequence Eq. (5.4).

Definition 5.3.5 (depth)**.** Let $\mathscr{S}$ be an expressivity situation in Definition 5.2.1. For each $\varphi \in L_{\mathscr{S}}$, the *depth of* φ $\mathrm{depth}(\varphi)$ is a natural number defined inductively as follows:

$$\begin{aligned} &\mathrm{depth}(\sigma(\varphi_1, \ldots, \varphi_{\mathrm{rank}(\sigma)})) \\ &\quad = \max(\mathrm{depth}(\varphi_1), \ldots, \mathrm{depth}(\varphi_{\mathrm{rank}(\sigma)})) && (\sigma \in \Sigma) \\ &\mathrm{depth}(\heartsuit_\lambda\varphi) = \mathrm{depth}(\varphi) + 1 && (\lambda \in \Lambda) \end{aligned}$$

For $\sigma \in \Sigma$ with $\mathrm{rank}(\sigma) = 0$, $\mathrm{depth}(\sigma())$ is defined to be 0.

We formulate the following proof principle. Unlike Knaster–Tarski (Fact 2.2.3), it uses an explicit induction on the depth i. Its proof is therefore more involved but not much more.

Theorem 5.3.6 (the Kleene proof principle)**.** *Let $\mathscr{S}$ be an expressivity situation as in Definition 5.2.1 and $x\colon X \to BX$ be a B-coalgebra. Assume that the chain Eq. (5.4) in $\mathbb{E}_X$ stabilizes after ω steps. If the set $\{[\![\varphi]\!]_x \mid \varphi \in L_{\mathscr{S}}, \mathrm{depth}(\varphi) \leq i\} \subseteq \mathbb{C}(X, \Omega)$ is approximating for each i, then $\mathscr{S}$ is expressive for x.*

Proof. By the Kleene theorem (Fact 2.2.5), it suffices to show $(x^* \circ B^{\mathbf{\Omega},\tau})^i(\top) \sqsupseteq \bigsqcap_{\varphi \in L_{\mathscr{S}}} [\![\varphi]\!]_x^* \mathbf{\Omega}$ for each i. We show

$$(x^* \circ B^{\mathbf{\Omega},\tau})^i(\top) \sqsupseteq \bigsqcap_{\varphi \in L_{\mathscr{S}}, \mathrm{depth}(\varphi) \leq i} [\![\varphi]\!]_x^* \mathbf{\Omega} \tag{5.5}$$

by induction on i.

For $i = 0$, Eq. (5.5) is trivial.

Assume that Eq. (5.5) holds for $i = j$ and we show it also holds for $i = j + 1$. Start with Eq. (5.5) for $i = j$. Applying $x^* \circ B^{\mathbf{\Omega},\tau}$ to both sides of it we obtain

$$(x^* \circ B^{\mathbf{\Omega},\tau})^{j+1}(\top) \sqsupseteq (x^* \circ B^{\mathbf{\Omega},\tau}) \bigsqcap_{\varphi \in L_{\mathscr{S}}, \mathrm{depth}(\varphi) \leq j} [\![\varphi]\!]_x^* \mathbf{\Omega}.$$

Here, expanding the definition of the r.h.s. we get

$$(x^* \circ B^{\mathbf{\Omega},\tau}) \bigsqcap_{\varphi \in L_{\mathscr{S}}, \mathrm{depth}(\varphi) \leq j} [\![\varphi]\!]_x^* \mathbf{\Omega} = \bigsqcap_{\lambda \in \Lambda, h \colon \bigsqcap_{\varphi \in L_{\mathscr{S}}, \mathrm{depth}(\varphi) \leq j} [\![\varphi]\!]_x^* \mathbf{\Omega} \to \mathbf{\Omega}} x^*(B(ph))^* \tau_\lambda^* \mathbf{\Omega}.$$

Now let $\lambda \in \Lambda$ and $h \colon \bigsqcap_{\varphi \in L_{\mathscr{S}}, \mathrm{depth}(\varphi) \leq j} [\![\varphi]\!]_x^* \mathbf{\Omega} \to \mathbf{\Omega}$. That $\{[\![\varphi]\!]_x \mid \varphi \in L_{\mathscr{S}}, \mathrm{depth}(\varphi) \leq j\} \subseteq \mathbb{C}(X, \Omega)$ is an approximating family yields

$$\begin{aligned}
x^*(B(ph))^* \tau_\lambda^* \mathbf{\Omega} &\sqsupseteq \bigsqcap_{\lambda' \in \Lambda, \varphi \in L_{\mathscr{S}}, \mathrm{depth}(\varphi) \leq j} x^*(B[\![\varphi]\!]_x)^* \tau_{\lambda'}^* \mathbf{\Omega} \\
&= \bigsqcap_{\lambda' \in \Lambda, \varphi \in L_{\mathscr{S}}, \mathrm{depth}(\varphi) \leq j} [\![\heartsuit_{\lambda'} \varphi]\!]_x^* \mathbf{\Omega} \\
&\sqsupseteq \bigsqcap_{\varphi' \in L_{\mathscr{S}}, \mathrm{depth}(\varphi') \leq j+1} [\![\varphi']\!]_x^* \mathbf{\Omega}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
(x^* \circ B^{\mathbf{\Omega},\tau})^{j+1}(\top) &\sqsupseteq (x^* \circ B^{\mathbf{\Omega},\tau}) \bigsqcap_{\varphi \in L_{\mathscr{S}}, \mathrm{depth}(\varphi) \leq j} [\![\varphi]\!]_x^* \mathbf{\Omega} \\
&= \bigsqcap_{\lambda \in \Lambda, h \colon \bigsqcap_{\varphi \in L_{\mathscr{S}}, \mathrm{depth}(\varphi) \leq j} [\![\varphi]\!]_x^* \mathbf{\Omega} \to \mathbf{\Omega}} x^*(B(ph))^* \tau_\lambda^* \mathbf{\Omega} \\
&\sqsupseteq \bigsqcap_{\varphi' \in L_{\mathscr{S}}, \mathrm{depth}(\varphi') \leq j+1} [\![\varphi']\!]_x^* \mathbf{\Omega}.
\end{aligned}$$

This concludes the induction. $\square$

In Theorem 5.3.6, we require that $\{[\![\varphi]\!]_x \mid \varphi \in L_{\mathscr{S}}, \mathrm{depth}(\varphi) \leq i\}$ is approximating for each depth i; this is often easier than the case where φ ranges over all formulas (as in Theorem 5.3.4). We use the theorem for the example in Section 5.4.

Example 5.3.7. Sufficient conditions for being an approximating family were given in Example 5.3.3. Combined with Theorem 5.3.6, it yields expressivity (Corollary 5.4.9), one of the main results of [48].

Remark 5.3.8. In Theorem 5.3.6, we assumed the stabilization of the chain Eq. (5.4) at length ω. This assumption turns out to be benign, essentially because our modal formulas all have a finite depth (Section 5.2.2). Specifically we can show the following: if the logic $L_{\mathscr{S}}$ is expressive for $x\colon X \to BX$, then the chain Eq. (5.4) stabilizes after ω steps. See Appendix B.1.

5.4. Expressivity for the Kantorovich Bisimulation Metrics

This section shows how one of the main results in [48], expressivity of a real-valued logic w.r.t. bisimulation metric, is proved by our Kleene proof principle (Theorem 5.3.6). See also Examples 5.2.3, 5.2.5 and 5.2.7.

Definition 5.4.1. Define an expressivity situation $\mathscr{S}_{\mathrm{KMM}}$ by:

- Its fibration is $\mathbf{PMet}_1 \to \mathbf{Set}$ (Example 2.3.9).
- Its truth-value object is $[0,1]$ and its observation predicate is d_e, the usual Euclidean metric on $[0,1]$.
- The ranked alphabet of its propositional connectives is $\Sigma = \{\top^0, \min^2, \neg^1\} \cup \{(\ominus q)^1 \mid q \in \mathbb{Q} \cap [0,1]\}$. Its propositional structure $(f_\sigma\colon [0,1]^{\mathrm{rank}(\sigma)} \to [0,1])_{\sigma\in\Sigma}$ is specified by:

$$f_\top() = 1 \qquad\qquad f_{\min}(x,y) = \min(x,y)$$
$$f_\neg(x) = 1 - x \qquad\qquad f_{\ominus q}(x) = \max(x-q, 0)$$

- The behavior functor $B\colon \mathbf{Set} \to \mathbf{Set}$, the set of its modality indices Λ, and its observation modalities $(\tau_\lambda\colon B[0,1] \to [0,1])_{\lambda\in\Lambda}$ are arbitrary.

The modal logic $L_{\mathscr{S}_{\mathrm{KMM}}}$ is the same as the logic $\mathcal{M}(\Lambda)$ in [48, Table 1]. What they call an *evaluation map* $\gamma \in \Gamma$ corresponds to an observation modality $\tau_\lambda (\lambda \in \Lambda)$ in our framework. Thus, the fibrational logical equivalence $\mathsf{LE}_{\mathscr{S}_{\mathrm{KMM}}}(\alpha)$ (Definition 5.2.6) coincides with the *logical distance* d^L_α [48, Def. 25] for a coalgebra $\alpha\colon X \to BX$.

Moreover, the codensity lifting $B^{d_e,\tau}$ specializes to the *Kantorovich lifting* by [6]:

$$\begin{aligned} &B^{d_e,\tau}(X,d) = (BX, d_B) \quad \text{where} \\ &\quad d_B(t_1,t_2) = \sup\nolimits_{\lambda,h} d_e(\tau_\lambda((Bh)(t_1)), \tau_\lambda((Bh)(t_2))). \end{aligned}$$

In the above sup, λ, h ranges over Λ and $\mathbf{PMet}_1((X,d),([0,1],d_e))$, respectively. Thus, the codensity bisimilarity $\mathsf{Bisim}^{d_e,\tau}(\alpha)$ (Definition 5.2.9) recovers the definition of the *behavioral distance* d_α [48, Def. 22] for a coalgebra $\alpha\colon X \to BX$.

From Proposition 5.2.11 we obtain:

Corollary 5.4.2. *For $\alpha\colon X \to BX$, $d_\alpha \geq d^L_\alpha$ holds.* $\square$

As mentioned in [48], it is harder to prove expressivity. We use the Kleene proof principle (Theorem 5.3.6) here. For this argument to work, we have to make further assumptions.

Assumption 5.4.3. For $\mathscr{S}_{\mathrm{KMM}}$, assume the following:

1. Λ is finite.
2. If a sequence k_i of functions of type $X \to [0,1]$ uniformly converges into l, then $\tau_\lambda \circ Bk_i \colon BX \to [0,1]$ uniformly converges into $\tau_\lambda \circ Bl$ for each $\lambda \in \Lambda$.

In particular, condition 2 above is satisfied if each τ_λ *induces a non-expansive predicate lifting* [48, Def. 17].

The notion of *total boundedness* below is pivotal in [48].

Definition 5.4.4 (from [48, Def. 28])**.** $(X, d) \in \mathbf{PMet}_1$ is *totally bounded* if, for any $\varepsilon > 0$, there is a finite set $F_\varepsilon \subseteq X$ satisfying the following: for each $x \in X$, there is $y \in F_\varepsilon$ such that $d(x, y) < \varepsilon$.

A critical step in their proof used a Stone–Weierstrass-like property of totally bounded spaces.

Proposition 5.4.5. *Let (X, d) be a totally bounded pseudometric space. A subset $S \subseteq \mathbf{PMet}_1((X, d), ([0,1], d_e))$ is dense in the topology of uniform convergence if the following are satisfied:*

1. *S is closed under the four operations $\top$, $\min$, $\neg$ and $\ominus q$ for every $q \in \mathbb{Q} \cap [0,1]$;*
2. *for every $h \in \mathbf{PMet}_1((X, d), ([0,1], d_e))$, and every $x, y \in X$, we have*
$$d_e(h(x), h(y)) \le \sup_{g \in S} d_e(g(x), g(y))\,. \qquad \square$$

Proof. By [74, Lemma 5.8], it suffices to show that, for each $h \in \mathbf{PMet}_1((X, d), ([0,1], d_e))$, each $\delta > 0$, and each pair of points $x, y \in X$, there is $g \in S$ such that $d_e(h(x), g(x)) \le \delta$ and $d_e(h(y), g(y)) \le \delta$ hold.

Without loss of generality, we can assume $h(x) \ge h(y)$. Let $\gamma = h(x) - h(y)$. Since $\gamma \ge 0$, $\gamma = d_e(h(x), h(y))$. By the second assumption, there is f such that $\gamma - \delta \le d_e(f(x), f(y))$. Since S is closed under $\neg$, we can assume that $f(x) \ge f(y)$ without loss of generality. This implies $\gamma - \delta \le f(x) - f(y)$.

Now, we do a case analysis.

Firstly, assume $f(y) \ge h(y)$. Take $r, s \in \mathbb{Q} \cap [0,1]$ satisfying $f(y) - h(y) - \delta \le r \le f(y) - h(y)$ and $h(x) \le s \le h(x) + \delta$. Then $g = \min(f \ominus r, s)$ is what we want.

Secondly, assume $f(y) < h(y)$. Take $r, s \in \mathbb{Q} \cap [0,1]$ satisfying $h(y) - f(y) - \delta \le r \le h(y) - f(y)$ and $h(x) \le s \le h(x) + \delta$. Then $g = \min(\neg((\neg f) \ominus r), s)$ is what we want. $\square$

In our framework, this can be stated in the following form:

Proposition 5.4.6. *Assume the setting of Definition 5.4.1. Let $X \in \mathbf{Set}$. Under Assumption 5.4.3, a subset $S \subseteq \mathbf{Set}(X, [0,1])$ is approximating if the following hold:*

- *S is closed under the four operations $\top$, $\min$, $\neg$ and $\ominus q$ for every $q \in \mathbb{Q} \cap [0,1]$.*
- *(X, d_S) is totally bounded, where $d_S(x,y) = \sup_{k \in S} d_e(k(x), k(y))$.*

Proof. Fix $h\colon (X, d_S) \to ([0,1], d_e)$, $\lambda \in \Lambda$, and $(z,w) \in (BX)^2$. It suffices to show the following:

$$\sup_{k \in S, \lambda' \in \Lambda} d_e(\tau_{\lambda'}((Bk)(z)), \tau_{\lambda'}((Bk)(w))) \geq d_e(\tau_\lambda((Bh)(z)), \tau_\lambda((Bh)(w))). \tag{5.6}$$

Use Proposition 5.4.5 for $d = d_S$. This ensures the existence of a sequence $(k_n\colon X \to [0,1])_{n=1,2,\ldots}$ that uniformly converges to ph as $n \to \infty$.

Fix $\varepsilon > 0$. By Assumption 5.4.3, the sequence $(\tau_\lambda \circ k_n)_{n=1,2,\ldots}$ also uniformly converges to $\tau_\lambda \circ (ph)$ as $n \to \infty$. Thus, we can fix n so that $d_e(\tau_\lambda((Bk)(z)), \tau_\lambda((Bk_n)(z))) < \varepsilon$ and $d_e(\tau_\lambda((Bh)(w)), \tau_\lambda((Bk_n)(w))) < \varepsilon$ both hold. From the triangle inequality, we obtain $d_e(\tau_\lambda((Bk_n)(z)), \tau_\lambda((Bk_n)(w))) \geq d_e(\tau_\lambda((Bh)(z)), \tau_\lambda((Bh)(w))) + 2\varepsilon$.

Since ε is arbitrary, we have Eq. (5.6). □

From now we use some facts on totally bounded space. Using the variation of Arzelà–Ascoli theorem [74, Lemma 5.6] for totally bounded spaces, we can show the following:

Fact 5.4.7. *Under Assumption 5.4.3, if $(X, d) \in \mathbf{PMet}_1$ is totally bounded, then*

- *$B^{d_e,\tau}(X, d)$ is also totally bounded.* [1]
- *If $(X,d) \sqsubseteq (X,d')$, (X,d') is also totally bounded.*

These enable us to use Theorem 5.3.6:

Proposition 5.4.8. *Let $x\colon X \to BX$ be a coalgebra. Under Assumption 5.4.3, for each i, $\{[\![\varphi]\!] \mid \varphi \in L_{\mathscr{S}_{\mathrm{KMM}}}, \mathrm{depth}(\varphi) \leq i\} \subseteq \mathbf{Set}(X, [0,1])$ is approximating.*

Proof. By induction, for each i, $(B^{d_e,\tau})^i(\top) \in (\mathbf{PMet}_1)_X$ is totally bounded. By the stepwise adequacy (Remark 5.3.8 & Appendix B.1) and Fact 5.4.7, for each i, $\bigsqcap_{\varphi \in L_{\mathscr{S}_{\mathrm{KMM}}}, \mathrm{rank}(\varphi) \leq i} [\![\varphi]\!]_x^* d_e$ is also totally bounded. From this and Proposition 5.4.6, we can show that the desired set is approximating. □

Corollary 5.4.9 (from [48, Thm. 32])**.** *Let $\alpha\colon X \to BX$ be a coalgebra. Assume that the sequence $\top \sqsupseteq (x^* B^{d_e,\tau})(\top) \sqsupseteq (x^* B^{d_e,\tau})^2(\top) \sqsupseteq \cdots$ stabilizes after ω steps (as in Theorem 5.3.6). Then, under Assumption 5.4.3, $d_\alpha \leq d_\alpha^L$ holds. In particular, d_α is characterized as the greatest pseudometric that makes all $[\![\varphi]\!]_\alpha$ nonexpansive.*

Proof. Use Theorem 5.3.6. The premises are satisfied by Proposition 5.4.8. □

[1] Here the finiteness of the number of modalities is crucial. When Λ is infinite, the Kantorovich lifting does not preserve total boundedness. See Appendix B.2.

5.5. Expressivity for Markov Process Bisimilarity

This section shows how one of the main results in [16], expressivity of probabilistic modal logic w.r.t. bisimilarity of labelled Markov process, is proved by our Knaster–Tarski proof principle (Theorem 5.3.4).

Throughout this section, fix a set A of labels.

Definition 5.5.1. Define an expressivity situation $\mathscr{S}_{\mathrm{CFKP}}$ by:

- Its fibration is $\mathbf{EqRel}_{\mathbf{Meas}} \to \mathbf{Meas}$ (Definition 2.3.17).
- Its behavior functor $B\colon \mathbf{Meas} \to \mathbf{Meas}$ is $BX = (\mathcal{G}_{\leq 1}X)^A$, where $\mathcal{G}_{\leq 1}$ is the variation of Giry functor, which sends each measurable space to its space of sub-distributions.
- Its truth-value object is $2 = \{0, 1\}$ with all subsets measurable and its observation predicate is the equality relation Eq_2 on 2.
- The ranked alphabet of its propositional connectives is $\Sigma = \{\top^0, \wedge^2\}$. Its propositional structure $(f_\top, f_\wedge)$ is specified as the usual boolean operations.
- The set of its modality indices is $A \times (\mathbb{Q} \cap [0,1])$. For each $(a, r) \in A \times (\mathbb{Q} \cap [0,1])$, the observation modality $\tau_{a,r}\colon (\mathcal{G}_{\leq 1}2)^A \to 2$ is defined by
$$\tau_{a,r}((\mu_a)_{a\in A}) = \mathrm{thr}_r(\mu_a(\{1\})),$$
where $\mathrm{thr}_r(s) = 1$ if and only if $s > r$.

Note that a *labelled Markov process (LMP) with label set A* [16, Definition 5.1] is the same as B-coalgebra. The modal logic $L_{\mathscr{S}_{\mathrm{CFKP}}}$ (Definition 5.2.2) has the following syntax:
$$\varphi_1, \varphi_2 \ ::= \ \top \mid \wedge(\varphi_1, \varphi_2) \mid \ \heartsuit_{a,r}\varphi_1 \ ((a, r) \in A \times (\mathbb{Q} \cap [0,1]))$$
So if we identify $\heartsuit_{a,r}$ with $\langle a\rangle_r$ in the original notation, this recovers the syntax of $\mathrm{PML}_\wedge$ defined in [16, Def. 2.3]. Under this identification, the semantics (Definition 5.2.4) is also essentially the same as the original logic: $[\![\varphi]\!]_x(s) = 1 \iff s \vDash \varphi$ holds for any LMP $x\colon X \to (\mathcal{G}_{\leq 1}X)^A$, any point $s \in X$, and any formula φ. The fibrational logical equivalence (Definition 5.2.6) can be concretely represented as
$$\mathsf{LE}_{\mathscr{S}_{\mathrm{CFKP}}}(x) = \{(s, t) \mid \forall\varphi \in L_{\mathscr{S}_{\mathrm{CFKP}}}, s \vDash \varphi \iff t \vDash \varphi\}.$$

By expanding the definition of the codensity lifting (Definition 5.2.8) of $(\mathcal{G}_{\leq 1}\_)^A$, we can see that it coincides with the one used to define *probabilistic bisimulation*:

Proposition 5.5.2. *The codensity lifting $(\mathcal{G}_{\leq 1}\_)^{A^{\mathrm{Eq}_2,\tau}}$ satisfies the following: for each $(\mu_a)_{a\in A}, (\nu_a)_{a\in A} \in (\mathcal{G}_{\leq 1}X)^A$, they are equivalent in $(\mathcal{G}_{\leq 1}\_)^{A^{\mathrm{Eq}_2,\tau}}(X, R)$ if and only if, for each $a \in A$ and each R-closed measurable set $S \subseteq X$, $\mu_a(S) = \nu_a(S)$ holds.* □

Thus the codensity bisimilarity $\mathsf{Bisim}^{\mathrm{Eq}_2,\tau}(x)$ (Definition 5.2.9) coincides with the probabilistic bisimilarity used in [16].

From Proposition 5.2.11, we readily obtain the following:

Corollary 5.5.3. *Let $x\colon X \to (\mathcal{G}_{\leq 1}X)^A$ be an LMP. If $s, t \in X$ are probabilistically bisimilar, for any $\varphi \in L_{\mathscr{S}_{\mathrm{CFKP}}}$, $s \vDash \varphi \iff t \vDash \varphi$ holds.* □

To show expressivity, we first have to review some mathematical key facts. In the rest of this section, we write $\sigma(\mathscr{E})$ for the σ-algebra generated by a family of sets $\mathscr{E}$.

Definition 5.5.4. A *Polish space* is a separable topological space which is metrizable by a complete metric. For any continuous map $f\colon X \to Y$ between Polish spaces X and Y, the image of f is called an *analytic topological space*. For an analytic topological space $(X, \mathcal{O}_X)$, the measurable space $(X, \sigma(\mathcal{O}_X))$ is called an *analytic measurable space*.

Let us review the two key facts they used in [16]. The first one is the following "elegant Borel space analogue of the Stone–Weierstrass theorem" [3].

Fact 5.5.5 (Unique Structure Theorem [3, Thm. 3.3.5])**.** *Let $X \in \mathbf{Meas}$ be an analytic measurable space and $\mathscr{E}$ be an (at most) countable family of measurable subsets of X such that $X \in \mathscr{E}$. Define an equivalence relation $\equiv_{\mathscr{E}}$ by*

$$x \equiv_{\mathscr{E}} y \iff \forall S \in \mathscr{E}, (x \in S \iff y \in S).$$

If $S \subseteq X$ is measurable and $\equiv_{\mathscr{E}}$-closed, then $S \in \sigma(\mathscr{E})$.

In the fact above, we use the operations of σ-algebras to construct S. The second key fact is about "decomposing" those operations into two parts.

Definition 5.5.6. Let X be a set. A family of subsets of X is called a *π-system* if it is closed under finite intersections. A family of subsets of X is a *λ-system* if it is closed under complement and countable disjoint unions.

Intuitively, π-systems correspond to the propositional connectives of $\mathscr{S}_{\mathrm{CFKP}}$ and λ-systems correspond to "approximation." These two operations are enough to recover all σ-algebra operations:

Fact 5.5.7 (π-λ Theorem [23])**.** *If Π is a π-system, Λ is a λ-system, and $\Pi \subseteq \Lambda$, then $\sigma(\Pi) \subseteq \Lambda$.*

Using Facts 5.5.5 and 5.5.7, we obtain a sufficient condition for being an approximating family. The proof follows the two steps outlined in Remark 5.3.2: 1) we can approximate a given $h\colon X \to 2$ by σ-algebra operations (Fact 5.5.5), which can be reduced to λ-system operations (Fact 5.5.7); and 2) λ-system operations are in some sense "preserved" by the modalities (since measures are σ-additive).

Proposition 5.5.8. *Assume the setting of Definition 5.5.1. Let $X \in \mathbf{Meas}$. A subset $S \subseteq \mathbf{Meas}(X, 2)$ is approximating if the following hold:*

- *X is an analytic measurable space.*
- *S is at most countable.*
- *For $k, l \in S$, $\top$ and $k \wedge l$ are also included in S.*

Proof. Fix any $h\colon \prod_{k \in S} k^*\mathrm{Eq}_2 \to \mathrm{Eq}_2$, any $a \in A$, and any $r \in \mathbb{Q} \cap [0,1]$. By definition, it suffices to show

$$\prod_{k \in S, a' \in A, r' \in \mathbb{Q}\cap[0,1]} (\tau_{a',r'} \circ Bk)^*\mathrm{Eq}_2 \sqsubseteq (\tau_{a,r} \circ B(ph))^*\mathrm{Eq}_2.$$

First we concretize these formulas. Let $R = \prod_{k \in S, a' \in A, r' \in \mathbb{Q}\cap[0,1]} (\tau_{a',r'} \circ Bk)^*\mathrm{Eq}_2$. Using the definition of the arrow part of the functor $B = (\mathcal{G}_{\leq 1}-)^A$, the relation R on $(\mathcal{G}_{\leq 1}X)^A$ can be rephrased as

$$\begin{aligned}
&((\mu_a)_{a \in A}, (\nu_a)_{a \in A}) \in R \\
&\iff \forall k, a', r', (\mu_{a'}(k^{-1}(\{1\})) > r' \iff \nu_{a'}(k^{-1}(\{1\})) > r') \\
&\iff \forall k, a', r', (\mu_{a'}(k^{-1}(\{1\})) = \nu_{a'}(k^{-1}(\{1\}))),
\end{aligned}$$

where $k \in S$, $a' \in A$, and $r' \in \mathbb{Q} \cap [0,1]$. In the same way, we can concretely describe $R' = (\tau_{a,r} \circ B(ph))^*\mathrm{Eq}_2$ as

$$\begin{aligned}
&((\mu_a)_{a \in A}, (\nu_a)_{a \in A}) \in R' \\
&\iff (\mu_a((ph)^{-1}(\{1\})) > r \iff \nu_a((ph)^{-1}(\{1\})) > r).
\end{aligned}$$

Thus, it suffices to show that the set $Y' = \{(\mu_a)_{a \in A} \mid \mu_a((ph)^{-1}(\{1\})) > r\} \subseteq (\mathcal{G}_{\leq 1}X)^A$ is R-closed. Let $X' = (ph)^{-1}(\{1\}) \subseteq X$. Now $Y' = \{(\mu_a)_{a \in A} \mid \mu_a(X') > r\}$.

The set X' corresponds to h, and we will “approximate” this by sets corresponding to the elements of S. Let $\mathscr{E} = \{k^{-1}(\{1\}) \mid k \in S\}$ and define an equivalence relation $\equiv_{\mathscr{E}}$ by

$$x \equiv_{\mathscr{E}} y \iff \forall E \in \mathscr{E}, (x \in E \iff y \in E).$$

Since $\equiv_{\mathscr{E}}$ coincides with the meet $\prod_{k \in S} k^*\mathrm{Eq}_2 \in (\mathbf{EqRel_{Meas}})_X$, X' is $\equiv_{\mathscr{E}}$-closed. Since X is analytic and S is at most countable, we can apply Fact 5.5.5 and show $X' \in \sigma(\mathscr{E})$.

Now we show that Y' is R-closed. In this step, intuitively, we use the fact that the modalities “preserve” the “approximation” by the operation of Λ-system. Assume $((\mu_a)_{a \in A}, (\nu_a)_{a \in A}) \in R$ and $(\mu_a)_{a \in A} \in Y'$. Define a family Λ of measurable subsets of X by

$$E \in \Lambda \iff \forall a \in A, \mu_a(E) = \nu_a(E).$$

Since S is closed under $\top$ and $\wedge$, $\mathscr{E}$ is a π-system. On the other hand, by the definition of measure, Λ is a λ-system. Since $((\mu_a)_{a \in A}, (\nu_a)_{a \in A}) \in R$, $\mathscr{E} \subseteq \Lambda$. Fact 5.5.7 implies $\sigma(\mathscr{E}) \subseteq \Lambda$. In particular, $X' \in \Lambda$. This and $(\mu_a)_{a \in A} \in Y'$ imply $(\nu_a)_{a \in A} \in Y'$. ☐

From this proposition and Theorem 5.3.4, we obtain the following expressivity result:

Corollary 5.5.9. *Let $x\colon X \to (\mathcal{G}_{\leq 1}X)^A$ be an LMP and $s, t \in X$ its states. Assume that the label set A is at most countable and that X is an analytic measurable space.*

Then $\mathscr{S}_{\mathrm{CFKP}}$ is expressive for x (Definition 5.2.10): that is, If $s \vDash \varphi \iff t \vDash \varphi$ holds for every $\varphi \in L_{\mathscr{S}_{\mathrm{CFKP}}}$, then s and t are probabilistically bisimilar.

Proof. Since A is at most countable, $\{[\![\varphi]\!] \mid \varphi \in L_{\mathscr{S}_{\mathrm{CFKP}}}\} \subseteq \mathbf{Meas}(X, 2)$ is also at most countable. Moreover, since the logic has $\top$ and $\wedge$, $\{[\![\varphi]\!] \mid \varphi \in L_{\mathscr{S}_{\mathrm{CFKP}}}\} \subseteq \mathbf{Meas}(X, 2)$ is closed under these operations. Thus we can use Proposition 5.5.8 and Theorem 5.3.4. $\square$

5.6. Expressivity for the Bisimulation Uniformity

In this section, we introduce *bisimulation uniformity* as a coinductive predicate in a fibration and a logic for it. By using our main results and a known mathematical result analogous to the Stone–Weierstrass theorem, the logic is readily proved to be adequate and expressive w.r.t. bisimulation uniformity. This example shows how our abstract framework can help to explore new bisimilarity-like notions.

5.6.1. Uniform Structure as Fibrational Predicate

Topological space can be regarded as an abstraction of (pseudo-)metric spaces w.r.t. continuous maps. In much the same way, *uniform space* [15] is an abstraction of (pseudo-)metric spaces w.r.t. uniformly continuous maps.

Definition 5.6.1 (from [15, Def. 1])**.** A *uniform structure*, or *uniformity*, on a set X is a nonempty family $\mathscr{U} \subseteq \mathcal{P}(X \times X)$ of subsets of $X \times X$ satisfying the following:

- If $V \in \mathscr{U}$ and $V \subseteq V' \subseteq X \times X$, then $V' \in \mathscr{U}$.
- If $V, W \in \mathscr{U}$, then $V \cap W \in \mathscr{U}$.
- If $V \in \mathscr{U}$, then $\{(x, x) \mid x \in X\} \subseteq V$.
- If $V \in \mathscr{U}$, then $\{(y, x) \mid (x, y) \in V\} \in \mathscr{U}$.
- If $V \in \mathscr{U}$, then there exists $W \in \mathscr{U}$ such that $\{(x, z) \mid \exists y\ (x, y) \in W \wedge (y, z) \in W\} \subseteq V$.

Here each element $V \in \mathscr{U}$ is called an *entourage*. A pair $(X, \mathscr{U})$ of a set and a uniformity on it is called a *uniform space*.

A function $f\colon X \to Y$ is a *uniformly continuous map from $(X, \mathscr{U}_X)$ to $(Y, \mathscr{U}_Y)$* if, for each entourage $V \in \mathscr{U}_Y$, $\{(x, x') \mid (f(x), f(x')) \in V\} \subseteq X \times X$ is an enrourage of $(X, \mathscr{U}_X)$. The category of uniform spaces and uniformly continuous maps is denoted **Unif**.

Each entourage represents some degree of "closeness." The following example is an archetypal one:

Example 5.6.2. Let (X, d) be a pseudometric space. Define a family $\mathscr{U} \subseteq \mathcal{P}(X \times X)$ as the set of all relations of the form $\{(x, x') \mid d(x, x') < \varepsilon\}$ for $\varepsilon > 0$ and their supersets. Then $(X, \mathscr{U})$ is a uniform space.

Some of the concepts considered for metric spaces, like completion, total boundedness, and characterization of compactness, can be lifted to uniform spaces. For us, the most important fact is that they form a $\mathbf{CLat}_\sqcap$-fibration:

Proposition 5.6.3 (from [15, Propositions 4 and 5])**.** *The forgetful functor* $\mathbf{Unif} \to \mathbf{Set}$ *is a* $\mathbf{CLat}_\sqcap$*-fibration.* □

Thus we can use uniform structures as a sort of indistinguishability structure. A uniform structure on a finite set is essentially the same as an equivalence relation. For infinite sets, however, it can be a helpful way to analyze coalgebras that is more quantitative than an equivalence relation and more robust than a pseudometric.

5.6.2. Expressivity Situation for Bisimulation Uniformity

Definition 5.6.4. Define an expressivity situation $\mathscr{S}_{\mathrm{BU}}$ by:

- Its fibration is $\mathbf{Unif} \to \mathbf{Set}$ (Proposition 5.6.3).
- Its truth-value object is $\mathbb{R}$ and its observation predicate is $\mathscr{U}_e$, the uniformity defined using the usual Euclidean metric as in Example 5.6.2.
- The ranked alphabet of its propositional connectives is $\Sigma = \{1^0, \min^2\} \cup \{(r+)^1, (r\times)^1 \mid r \in \mathbb{R}\}$. Its propositional structure $(f_\sigma \colon \mathbb{R}^{\mathrm{rank}(\sigma)} \to \mathbb{R})_{\sigma\in\Sigma}$ is specified by:

$$f_1() = 1 \qquad f_{\min}(x, y) = \min(x, y)$$
$$f_{r+}(x) = r + x \qquad f_{r\times}(x) = rx$$

- The behavior functor $B \colon \mathbf{Set} \to \mathbf{Set}$, the set of its modality indices Λ, and its observation modalities $(\tau_\lambda \colon B\mathbb{R} \to \mathbb{R})_{\lambda\in\Lambda}$ are arbitrary.

For a B-coalgebra $x \colon X \to BX$, the codensity lifting $B^{\mathscr{U}_e,\tau}$ (Definition 5.2.8) yields the codensity bisimilarity $\mathsf{Bisim}^{\mathscr{U}_e,\tau}(x) \in \mathbf{Unif}_X$ (Definition 5.2.9), which is a uniformity on the set X. We call it the *bisimulation uniformity of* x.

On the other hand, the logic $L_{\mathscr{S}_{\mathrm{BU}}}$ induces the fibrational logical equivalence $\mathsf{LE}_{\mathscr{S}_{\mathrm{BU}}}(x)$ (Definition 5.2.6) for each $x \colon X \to BX$. We call this the *logical uniformity of* x.

By Proposition 5.2.11, we obtain the following:

Proposition 5.6.5. *Assume the setting of Definition 5.6.4. Let* $x \colon X \to BX$ *be a* B*-coalgebra. Any entourage of the logical uniformity is also an entourage of the bisimulation uniformity. In particular, for any* $\varphi \in L_{\mathscr{S}_{\mathrm{BU}}}$, $[\![\varphi]\!]_x \colon X \to \mathbb{R}$ *is uniformly continuous w.r.t. the bisimulation uniformity.* □

To prove expressivity, we have to make further assumptions.

Assumption 5.6.6. For $\mathscr{S}_{\mathrm{BU}}$, assume the following:

1. If $k\colon X \to \mathbb{R}$ is bounded, then $\tau_\lambda \circ Bk\colon BX \to \mathbb{R}$ is also bounded for each $\lambda \in \Lambda$.
2. If a sequence k_i of functions of type $X \to \mathbb{R}$ uniformly converges into h, then $\tau_\lambda \circ Bk_i\colon BX \to \mathbb{R}$ uniformly converges into $\tau_\lambda \circ Bh$ for each $\lambda \in \Lambda$.

The key in the expressivity proof is the following known Stone–Weierstrass-like result:

Fact 5.6.7 (from [18, Thm. 1])**.** *Let X be a set and $\Gamma \subseteq \mathbb{R}$ a set of real numbers unbounded both from above and below. Assume a family Φ of bounded real-valued function satisfies the following:*

1. *Every constant is in Φ.*
2. *For $f \in \Phi$ and $r \in \Gamma$, $rf \in \Phi$ holds.*
3. *For $f \in \Phi$ and $r \in \mathbb{R}$, $r + f \in \Phi$ holds.*
4. *For $f, g \in \Phi$, $\min(f, g), \max(f, g) \in \Phi$ holds.*

Let $\mathscr{U}_\Phi$ be the coarsest uniformity on X that makes every function in Φ uniformly continuous. Then any real-valued function uniformly continuous w.r.t. $\mathscr{U}_\Phi$ is the limit of a uniformly convergent sequence of elements of Φ.

By using this, we can show that a suitable set is approximating. In its proof, we follow the two steps discussed in Remark 5.3.2.

Proposition 5.6.8. *Assume the setting of Definition 5.6.4. Let $X \in$ **Set**. Under Assumption 5.6.6, a subset $S \subseteq \mathbf{Set}(X, \mathbb{R})$ is approximating if the following hold:*

- *Every function in S is bounded.*
- *$1 \in S$.*
- *S is closed under the three operations $\min$, $(r+)$, and $(r\times)$ for every $r \in \mathbb{R}$.*

Proof. Define a uniformity $\mathscr{U}_S$ as the coarsest uniformity that every $k \in S$ is a uniformly continuous map $(X, \mathscr{U}_S) \to (\mathbb{R}, \mathscr{U}_e)$. Fix $h\colon (X, \mathscr{U}_S) \to (\mathbb{R}, \mathscr{U}_e)$ and $\lambda \in \Lambda$. We show that, in the fiber $\mathbf{Unif}_X$,

$$\bigsqcap_{k \in S, \lambda' \in \Lambda} (\tau_{\lambda'} \circ Bk)^* \mathscr{U}_e \sqsubseteq (\tau_\lambda \circ B(ph))^* \mathscr{U}_e$$

holds. Since $\{\{(x, y) \in \mathbb{R}^2 \mid d_e(x, y) < \varepsilon\} \mid \varepsilon > 0\}$ is a fundamental system of entourages of $\mathscr{U}_e$, the family $\{\{(x, y) \in X^2 \mid d_e((\tau_\lambda \circ B(ph))(x), (\tau_\lambda \circ B(ph))(y)) < \varepsilon\} \mid \varepsilon > 0\}$ is a fundamental system of entourages of $(\tau_\lambda \circ B(ph))^* \mathscr{U}_e$. So it suffices to show that each relation in the family is an entourage of $\bigsqcap_{k \in S, \lambda' \in \Lambda} (\tau_{\lambda'} \circ Bk)^* \mathscr{U}_e$.

We show the following stronger claim:

Claim. *For any $\varepsilon > 0$, there exists $k \in S$ such that $\{(x, y) \in X^2 \mid d_e((\tau_\lambda \circ Bk)(x), (\tau_\lambda \circ Bk)(y)) < \varepsilon/3\}$ is a subset of $\{(x, y) \in X^2 \mid d_e((\tau_\lambda \circ B(ph))(x), (\tau_\lambda \circ B(ph))(y)) < \varepsilon\}$.*

Use Fact 5.6.7 for $\Gamma = \mathbb{R}$ and $\Phi = S$. The condition (1) is satisfied because every constant is a multiple of 1, and the condition (4) is satisfied because $\max(x, y) = -\min(-x, -y)$. This ensures the existence of a sequence $(k_n : X \to \mathbb{R})_{n=1,2,\dots}$ that uniformly converges to ph as $n \to \infty$.

Fix $\varepsilon > 0$. By the assumption (2), the sequence $(\tau_\lambda \circ Bk_n)_{n=1,2,\dots}$ also uniformly converges to $\tau_\lambda \circ B(ph)$ as $n \to \infty$. Thus, we can fix n so that, for each $x \in X$, $d_e(\tau_\lambda((B(ph))(x)), \tau_\lambda((Bk_n)(x))) < \varepsilon/3$ holds. From the triangle inequality, we can take $k = pk_n$ for the claim above to hold. □

From this, we can obtain expressivity:

Corollary 5.6.9. *Assume the setting of Definition 5.6.4. Let $x : X \to BX$ be a B-coalgebra. Under Assumption 5.6.6, the bisimulation uniformity coincides with the logical uniformity, i.e., the former is characterized as the coarsest uniformity making every $[\![\varphi]\!]_x : X \to \mathbb{R}$ uniformly continuous.*

Proof. Use Theorem 5.3.4. Indeed, by the assumption (1), $[\![\varphi]\!]_x$ is bounded for every $\varphi \in L_{\mathscr{S}_{\mathrm{BU}}}$. □

5.7. Conclusions and Future Work

We introduced a categorical framework to study expressivity of quantitative modal logics, based on the novel notion of approximating family. This enabled us to cover not only existing examples (Section 5.4 and Section 5.5) but also a new one (Section 5.6). We conclude with some future research directions.

Making Use of Size Restrictions on Functors Many existing expressivity results make use of size restriction condition on the behavior functor B. For example, [30] required *image-finiteness*, [64] used *κ-accessibility*, and [73] was based on a quantitative notion, *finitary separability*. Importing these size restrictions is future work. A starting point can be [29], which successfully connected the finitarity of the behavior functor and the length of the final chain in a fiber.

Study of Bisimulation Uniformity We defined bisimulation uniformity in Section 5.6, but there are many topics left to study. One primary subject is the connection to bisimilarity and bisimulation metric. It is also important to see if it is robust under parameter changes of the target system.

Seeking Stone–Weierstrass-like Theorems To use our framework to show expressivity, one has to obtain a sufficient condition for being an approximating family. In many cases, this is reduced to finding an appropriate "Stone–Weierstrass-like" theorem. Concretely find ones and apply them to modal logics (other than those we have mentioned) is future work. Another research direction is to a seek connection to [35], where "Stone–Weierstrass-like" theorems are formulated in another way.

6. Conclusions and Future Topics

6.1. Conclusions

In the thesis, we developed a "theory of behaviors and observations" using coalgebra, fibrational coinduction, and codensity lifting. The resulting framework showed both the generality of coalgebra and fibrational coinduction and the concreteness from codensity lifting. While coalgebra accommodates various kinds of target systems and fibrational coinduction covers diverse forms of information, codensity lifting gave us explicit modelling of "observations" as arrows from the state space object. This in turn enabled us to obtain the following results on codensity bisimilarity:

- A sufficient condition for fiberedness of codensity lifting, as shown in Section 3.3, which in turn implies the reflection of codensity bisimilarity by coalgebra morphisms. Here, the "observation-based" definition of codensity lifting paved the way to the notion of c-injectivity, which is an "observation-extension" condition.

- A game characterization of codensity bisimilarity, as shown in Chapter 4. Here, "observations" became the moves of Spoiler, which suggests regarding codensity bisimilarity as describing all the possible outcomes of "repeated observations".

- An adequacy and expressivity results for modal logic with respect to codensity bisimilarity, as shown in Chapter 5. Here, the interpretations of modal formulas are also regarded as "observations" and the comparison of logical and non-logical "observations" led us to the main results.

6.2. Future Topics

In Chapters 4 and 5, some future research topics are listed. Here we focus on the future research directions applicable to all of the contents of this thesis.

Λ-Kantorovich Functors and their Codensity Counterpart In this thesis we based ourselves on codensity lifting, which is a generalization of Kantorovich lifting. In a recent paper on expressivity of modal logic, Forster et al. [26] introduced another generalization of Kantorovich lifting, Λ-*Kantorovich functor*. It is not a construction of a lifting, but a property that an endofunctor on the total category. Indeed, they found some important functors that are not liftings but Λ-Kantorovich. A Kantorovich lifting is exactly a functor lifting which is Λ-Kantorovich as a functor.

They defined Λ-Kantorovich functors by carefully removing the dependence to the base functor in the definition of Kantorovich lifting. Mimicking them, a joint generalization of codensity lifting and Λ-Kantorovich functor would be like:

Definition 6.2.1. Let

- $p\colon \mathbb{E} \to \mathbb{C}$ be a $\mathbf{CLat}_{\sqcap}$-fibration,
- $\dot{B}\colon \mathbb{E} \to \mathbb{E}$ be a functor,
- $\mathbf{\Omega} \in \mathbb{E}$ be an object, and
- $\dot{\tau}\colon \dot{B}\mathbf{\Omega} \to \mathbf{\Omega}$ be a $\dot{B}$-algebra.

Then $\dot{B}$ is $(\mathbf{\Omega}, \dot{\tau})$-*codensity functor* if, for each $P \in \mathbb{E}$, the following hold:

$$\dot{B}P = \prod_{f \in \mathbb{E}(P, \mathbf{\Omega})} (p(\dot{B}f))^*(p\dot{\tau})^*\mathbf{\Omega}.$$

Comparing this with Definition 3.1.1 may be helpful in grasping the idea here. Note that, for a functor lifting $p\dot{B}P = BpP$ holds, while in the above definition $p\dot{B}P$ may not be determined solely by BpP.

The topic we would like to mention here is a possible theory of this kind of "non-lifting" endofunctors:

Question 6.2.2. *Which part of the theory in this thesis can be generalized to codensity functors as defined in Definition 6.2.1? In particular, can the expressivity results in [26] be rephrased in the codensity terminology?*

General Fibrational Theory of Processes and Memorylessness as Property In this thesis, we fixed the target system model to be a coalgebra for an endofunctor. Any coalgebra is "memoryless", in that the behavior is determined solely by the current state. How can we deal with memoryful systems with this framework? A simple answer would be extending the state space to cover the "memory" needed to determine the behavior.

A probabilistic point of view gives another possible answer. From such a viewpoint, coalgebra is a generalization of (discrete-time) *Markov process*. However, in many cases, a (general) stochastic process is defined first, and then a Markov process is defined as a stochastic process that is "memoryless" (see, e.g., [39]). If we can generalize this to other kinds of coalgebras, then we can naturally accommodate memoryful systems.

How could we do that? In defining a stochastic process, the notion of *filtration* plays a pivotal role. A filtration on a set X is, in our terminology, a chain in $\mathbf{Meas}_X$ decreasing in the fiber order $\sqsubseteq$. It is used to keep track the amount of information one has in each moment. Thus, generalizing this to any $\mathbf{CLat}_{\sqcap}$-fibration could result in a general theory of memoryful systems. This is a possible future research topic:

Question 6.2.3. *Can we generalize the theory of stochastic process so that*

- *A* $\mathbf{CLat}_{\sqcap}$*-fibration* $p\colon \mathbb{E} \to \mathbb{C}$ *plays the role of* $\mathbf{Meas} \to \mathbf{Set}$*, and*
- *A memoryless process corresponds to a* B*-coalgebra?*

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A. Appendices for Codensity Games

A.1. Direct Proof of Equivalence of the Two Game Notions Characterizing Probabilistic Bisimilarity (Tables 4.2 and 4.4)

A.1.1. Table 4.4 $\rightsquigarrow$ Table 4.2

Assume that Duplicator wins Table 4.4 from (x, y), and let Spoiler play some Z in Table 4.2 . There are two cases to consider which are essentially identical, but we write them down separately for reference.

- If $\tau(x, Z) > \tau(y, Z)$ then we make Spoiler select $s = x$ and play Z in Table 4.4. To this Duplicator responds with some $Z' \supseteq Z$ such that $\tau(x, Z) \leq \tau(y, Z')$, which implies that $Z' \neq Z$. Pick any $y' \in Z' \setminus Z$ and play it as Spoiler in Table 4.4; when Duplicator responds with some $x' \in Z$, play the pair x' and y' as Duplicator in Table 4.2.
- If $\tau(x, Z) < \tau(y, Z)$ then we make Spoiler select $s = y$ and play Z in Table 4.4. To this Duplicator responds with some $Z' \supseteq Z$ such that $\tau(y, Z) \leq \tau(x, Z')$, which implies that $Z' \neq Z$. Pick any $y' \in Z' \setminus Z$ and play it as Spoiler in Table 4.4; when Duplicator responds with some $x' \in Z$, play the pair x' and y' as Duplicator in Table 4.2.

A.1.2. Table 4.2 $\rightsquigarrow$ Table 4.4

This is a less straightforward implication. A winning strategy for Duplicator in Table 4.4 is built not from a single strategy in Table 4.2, but rather from an entire collection of winning positions.

Formally, assume that Duplicator wins Table 4.2 from (x, y), and let Spoiler choose $s \in \{x, y\}$ and play some Z in Table 4.4. We define

$$\bar{Z} = \{w \in X \mid \exists v \in Z \text{ such that Duplicator wins Table 4.2 from } (v, w)\}.$$

One basic observation is that $Z \subseteq \bar{Z}$, since Duplicator wins from all positions of the form (w, w). As a result, we have

$$\tau(x, Z) \leq \tau(x, \bar{Z}) \qquad \text{and} \qquad \tau(y, Z) \leq \tau(y, \bar{Z}). \tag{A.1}$$

Another observation is that Spoiler wins Table 4.2 from the position $\bar{Z}$. To see this, consider any Duplicator's response $x' \in \bar{Z}$, $y' \notin \bar{Z}$. Then there is some $v \in Z$ such that

Duplicator wins Table 4.2 from (v, x'). If Duplicator could win Table 4.2 from (x', y') then she could win from (v, y') as well, which contradicts the assumption that $y' \notin \bar{Z}$.

Since we assume that Duplicator wins Table 4.2 from (x, y), $\bar{Z}$ cannot be a legal move for Spoiler from (x, y), hence

$$\tau(x, \bar{Z}) = \tau(y, \bar{Z}).$$

Together with (A.1) this implies that

$$\tau(x, Z) \leq \tau(y, \bar{Z}) \qquad \text{and} \qquad \tau(y, Z) \leq \tau(x, \bar{Z}),$$

so $Z' = \bar{Z}$ is a legal move for Duplicator in the stage (ii) of Table 4.4, no matter if Spoiler chose $s = x$ or $s = y$ in the stage (i). To this, in the stage (iii) Spoiler replies with some $y' \in \bar{Z} \setminus Z$. By the definition of $\bar{Z}$, there is some $v \in Z$ such that Duplicator wins Table 4.2 from (v, y'), so Duplicator can respond with $x' = v$.

A.2. Codensity Characterization of Hausdorff pseudometric

Proposition A.2.1. *Let (X, d) be a pseudometric space. For any $S, T \subseteq X$, we define two functions*

$$d_H(S, T) = \max\left(\sup_{x \in S} \inf_{y \in T} d(x, y), \sup_{y \in T} \inf_{x \in S} d(x, y)\right)$$

and

$$d_c(S, T) = \sup_{k \in \mathbf{PMet}_1((X,d),([0,1],d_{\mathbb{R}}))} d_{\mathbb{R}}\left(\inf_{x \in S} k(x), \inf_{y \in T} k(y)\right).$$

The values of two functions coincide.

Proof. First, we show $d_c(S, T) \geq d_H(S, T)$ by contradiction.

Suppose it does not hold. Then, by definition, at least one of

$$\sup_{x \in S} \inf_{y \in T} d(x, y)$$

and

$$\sup_{y \in T} \inf_{x \in S} d(x, y)$$

is greater than $d_c(S, T)$. We can assume the former is greater than $d_c(S, T)$ w.l.o.g.

Therefore, for some $x_0 \in S$,

$$d_c(S, T) < \inf_{y \in T} d(x_0, y)$$

holds.

Now, since $d(x_0, \_)$ is a non-expansive function by the triangle inequality, we have

$$d_c(S, T) \geq d_{\mathbb{R}}\left(\inf_{x \in S} d(x_0, x), \inf_{y \in T} d(x_0, y)\right).$$

However, since $\inf_{x\in S} d(x_0, x) = 0$, we have $d_c(S, T) \geq \inf_{y\in T} d(x_0, y)$, which is a contradiction.

Next, we show $d_c(S, T) \leq d_H(S, T)$ by contradiction.

Suppose $d_c(S, T) > d_H(S, T) + \varepsilon$ for some $\varepsilon > 0$. Then, for some non-expansive $k\colon X \to [0, 1]$,

$$d_{\mathbb{R}}\left(\inf_{x\in S} k(x), \inf_{y\in T} k(y)\right) > d_H(S, T) + \varepsilon$$

holds.

W.l.o.g. we can assume $\inf_{x\in S} k(x) \leq \inf_{y\in T} k(y)$.

Thus, for some $x_0 \in S$ and $y_0 \in T$ satisfying $k(x_0) \leq \inf_{x\in S} k(x) + \varepsilon/5$ and $k(y_0) \leq \inf_{y\in T} k(y) + \varepsilon/5$,

$$d_{\mathbb{R}}(k(x_0), k(y_0)) > d_H(S, T) + 3\varepsilon/5$$

holds. Since

$$d_H(S, T) \geq \sup_{x\in S} \inf_{y\in T} d(x, y),$$

there exists some $y_1 \in T$ satisfying

$$d_H(S, T) \geq d(x_0, y_1) \geq d_{\mathbb{R}}(k(x_0), k(y_1)).$$

However, we have $k(x_0) \leq k(y_0) + \varepsilon/5 \leq k(y_1) + 2\varepsilon/5$, so

$$d_{\mathbb{R}}(k(x_0), k(y_1) + \varepsilon/5) \geq d_{\mathbb{R}}(k(x_0), k(y_0) + 2\varepsilon/5)$$

and

$$d_{\mathbb{R}}(k(x_0), k(y_1)) + 3\varepsilon/5 \geq d_{\mathbb{R}}(k(x_0), k(y_0))$$

holds.

Then,

$$\begin{aligned}
&d_{\mathbb{R}}(k(x_0), k(y_0)) \\
&\leq d_{\mathbb{R}}(k(x_0), k(y_1)) + 3\varepsilon/5 \\
&\leq d_H(S, T) + 3\varepsilon/5 \\
&< d_{\mathbb{R}}(k(x_0), k(y_0))
\end{aligned}$$

holds, which is a contradiction. □

B. Appendices for Modal Logic

B.1. Further on Remark 5.3.8

Assume that $L_{\mathscr{S}}$ is expressive for $x\colon X \to BX$. First, in much the same way as Proposition 5.2.11, we can show "stepwise adequacy":

$$(x^* \circ B^{\mathbf{\Omega},\tau})^n(\top) \sqsubseteq \bigsqcap_{\varphi \in L_{\mathscr{S}}, \mathrm{depth}(\varphi) \leq n} [\![\varphi]\!]_x^* \mathbf{\Omega}$$

holds for $n \in \omega$. Taking the meets of both sides for $n \in \omega$ shows

$$\bigsqcap_{n \in \omega} (x^* \circ B^{\mathbf{\Omega},\tau})^n(\top) \sqsubseteq \bigsqcap_{\varphi \in L_{\mathscr{S}}} [\![\varphi]\!]_x^* \mathbf{\Omega} = \mathsf{LE}_{\mathscr{S}}(x) \sqsubseteq \mathsf{Bisim}^{\mathbf{\Omega},\tau}(x).$$

B.2. More on Total Boundedness

For the arguments in Section 5.4, finiteness of Λ is crucial, which was not very obvious in [48]. Here we consider a handy counterexample against Fact 5.4.7 where Λ is infinite. It turns out that the counterexample also affects our approximation argument.

Define $d_2\colon 2 \times 2 \to [0,1]$ by

$$d_2(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}.$$

Note that $(2, d_2)$ is totally bounded.

Consider an instance of the situation of Definition 5.4.1 where

- the behavior functor B is defined by $BX = X^\omega$,
- the set of modality indices Λ is $\omega = \{0,1,2,\dots\}$, and
- the observation modality $\tau_i\colon [0,1]^\omega \to [0,1]$ for a modality index $i \in \omega$ is defined as the projection $\tau_i((x_j)_{j\in\omega}) = x_i$.

Then the codensity lifting $(-)^{\omega d_e,\tau}$ does not preserve total boundedness. In fact, the pseudometric space $(-)^{\omega d_e,\tau}(2, d_2)$ is not totally bounded. First we show a lemma. Let $(2^\omega, d_{2^\omega}) = (-)^{\omega d_e,\tau}(2, d_2)$.

Lemma B.2.1. *The distance function d_{2^ω} satisfies*

$$d_{2^\omega}(x,y) = \begin{cases} 0 & \textit{if } x = y \\ 1 & \textit{otherwise} \end{cases}.$$

Proof. It suffices to show that, for $x \neq y$, $d_{2^\omega}(x, y) = 1$. Let $x = (x_i)_{i\in\omega}$ and $y = (y_i)_{i\in\omega}$. Take $i \in \omega$ so that $x_i \neq y_i$. Define $f : 2 \to [0, 1]$ by $f(0) = 0$ and $f(1) = 1$. Then f is a nonexpansive map from $(2, d_2)$ to $([0, 1], d_e)$.

Using these data, we can see

$$\begin{aligned} d_{2^\omega}(x, y) &= \sup_{g\colon (2,d_2)\to([0,1],d_e), j\in\omega} d_e(\tau_j(((g)^\omega)(x)), \tau_j(((g)^\omega)(y))) \\ &\geq d_e(\tau_i(((f)^\omega)(x)), \tau_i(((f)^\omega)(y))) \\ &= d_e(f(x_i), f(y_i)) \\ &= 1. \end{aligned}$$

$\square$

Proposition B.2.2. *The pseudometric space $(2^\omega, d_{2^\omega})$ is not totally bounded.*

Proof. For any given $0 < \varepsilon < 1$, each disc of radius ε covers only one point. This implies that finitely many such discs cannot cover the space $(2^\omega, d_{2^\omega})$, which means it is not totally bounded. $\square$

This space $(2^\omega, d_{2^\omega})$ also shows that we cannot simply remove total boundedness in Proposition 5.4.5. Let $\mathcal{F} = \mathbf{PMet}_1((2^\omega, d_{2^\omega}), ([0, 1], d_e))$. Define $\mathcal{G} \subseteq \mathcal{F}$ as the set of all functions that depend only on finitely many components. Then this $\mathcal{G}$ satisfies the two conditions in Proposition 5.4.5. However, this is not dense in $\mathcal{F}$:

Proposition B.2.3. *Under the topology of uniform convergence, $\mathcal{G}$ is not dense in $\mathcal{F}$.*

Proof. Define $h : (2^\omega, d_{2^\omega}) \to ([0, 1], d_e)$ by:

$$h((x_i)_{i\in\omega}) = \begin{cases} 1 & \text{if there is infinitely many } i\text{'s s.t. } x_i = 1 \\ 0 & \text{otherwise} \end{cases}.$$

By the lemma this is indeed nonexpansive.

Fix any $g \in \mathcal{G}$. By the definition of $\mathcal{G}$, we can take $n \in \omega$ such that g only depends on the first n components. Let $x = (0, 0, \dots) \in 2^\omega$. Define $y \in 2^\omega$ so that the first n components of y are 0 and all the others are 1. Then $h(x) = 0$, $h(y) = 1$ and $g(x) = g(y)$ holds. This implies that $d_e(h(x), g(x)) \geq 1/2$ or $d_e(h(y), g(y)) \geq 1/2$ holds. In particular, the uniform distance between h and g is at least $1/2$. Since g is arbitrary, $\mathcal{G}$ is not dense in $\mathcal{F}$. $\square$