New Types of Matrix Models

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Abstract

We investigate new types of matrix models based on the simply connected *compact* exceptional Lie group E_6 ([1]) and the super Lie algebras ([8]). In the former case, a matrix Chern-Simons theory is directly derived from the invariant on E_6. It is stated that the similar argument as Smolin [9] which derives an effective action of the matrix string type can also be held in our model. An important difference is that our model has twice as many degrees of freedom as Smolin's model has. One way to introduce the cosmological term is the compactification on directions. It is of great interest that the properties of the product space $\mathfrak{J}^c \times \mathcal{G}$, in which the degrees of freedom of our model live, are very similar to those of the physical Hilbert space. In the latter case, we investigate three super Lie algebras, $osp(1|32;\mathbf{R})$, $u(1|16,16)$, and $gl(1|32;\mathbf{R})$. In paticular, we study the supersymmetry structures of these models and discuss possible reductions to the IKKT model. In addition to those, one view on the diffeomorphism in the matrix model, a different $u(1|16,16)$ model from Smolin's, and some kind of *topological* effective action derived using Wigner-Inönü contraction are also discussed.

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Chapter 1

Introduction

Today particle physics except for gravity is well described by the standard model. However, gravity cannot be quantized in the same method because we cannot renormalize it. Therefore the main problem of current particle physics is to establish a consistent quantum theory which contains both the standard model and gravity. Under these circumstances, the most hopeful and popular candidate is the string theory.

The reason to favor the string theory is its wonderful nature. We can give as concrete examples that the theory has no ultraviolet divergence and includes gravitational field as well as matter and gauge fields automatically. However, due to the infinite ground states, this theory has no capability to predict; therefore we cannot answer why the standard model emerges. On the other hand it is possible to consider that this problem is the problem in the framework of perturbative formulation of the theory, because the completed region of the string theory is only the perturbative region. So if the non-perturbative formulation of the theory is accomplished, it is quite likely that this problem is resolved. Of course, it is pure speculation, but it seems quite probable that the non-perturbative effects turn infinite ground states into single one. In recent years, some kinds of non-perturbative effects of the string theory were investigated using the concepts like D-brane, duality, and M-theory. However these are not constructive definitions of the string theory as yet, but attempts to understand the non-perturbative effects along the line of the perturbation theory.

What must not be forgotten is that one theory never finish before the non-perturbative formulation is completed. One of candidates for the non-perturbative formulation of the string theory at present is the string field theory. Although a considerable number of studies have been conducted on these theories, the only successful string field theories so far are the ones formulated in the light-cone gauge. So it is not clear whether we can extract some essential information of the non-perturbative effects. Another candidate is what is called the matrix model. With the advent of the BFSS model [3][1] as a starter,

[1] The action itself has been introduced before by B.de Wit, J.Hoppe and H.Nicolai [2].

many proposals (e.g.[3],[4],[5],[6],[7],[8],[9],$\cdots$) have been being made since. The common idea of these models is that they reproduce sting or membrane theory in the large-N limit. In a sense the matrix model is similar to the lattice gauge theory, which is the non-perturbative formulation of the field theory, in that they can be analyzed using numerical simulation. Therefore it is reasonable to suppose that we will develop current matrix models a little further and find the true model.

A virtue of the matrix model is that it has a possibility of putting an interpretation on the space-time itself. Matrix models can describe both space-time and matter, i.e. fluctuation around a classical background, in the same footing. (Such a unification seems only possible in the case of gauge theory where bosonic fields have the same indices as the space-time.) However, some important questions such as "what would be the real mechanism to realize the 4-dimensional world from the 10(or 11)-dimensional universe", "how we can describe a curved space-time in matrix models" and "how is the diffeomorphism introduced into the theory" remain unsettled. One of them is the question of background independence. Consider the IKKT model [4] for example. This model has an $SO(10) \times SU(N)$ symmetry, and this is just a symmetry like *some theory* was expanded around the flat background. Therefore we cannot deny the existence of different matrix model whose expansion around a special background gets the IKKT model. On this point Smolin proposed a new type of matrix model [6] in which the action is cubic in matrices. Matrices are built from the super Lie algebra $osp(1|32;\mathbf{R})$, and one multiplet is pushed into a single supermatrix. Smolin's conjecture is that the expansions around different backgrounds of the $osp(1|32;\mathbf{R})$ matrix model will reduce to the BFSS or IKKT model. However, as far as the IKKT model is concerned, the theory made from Smolin's way dose not reproduce the supersymmetry of the IKKT model. That is, indeed the 10-dimensionality is realized, but the *half* of supersymmetry required by the IKKT model cannot be held. Anyway, the model described by a single matrix alone is very attractive, and Smolin's courageous attempt demonstrated one concrete possibility. In this paper, "how 10-dimensional IKKT model can be embedded in 11-dimensional Smolin's models" are reconsidered. These arguments are proceeded using group theoretical study on the supersymmetry of the IKKT model. In addition to those, one view on the diffeomorphism in the matrix model, a different $u(1|16,16)$ model from Smolin's, and some kind of *topological* effective action derived using Wigner-Inönü contraction are also discussed.

Moreover, as Smolin's $u(1|16,16)$ model [7] has demonstrated, the matrix models are not irrelevant to the *loop quantum gravity* which is another approach to the Theory of Everything. Furthermore, it was pointed out in [9] that the matrix string theory [5] has a connection with the matrix model based on the exceptional Jordan algebra $\mathfrak{J}$, while B.Kim and A.Schwarz have discussed in [10][2] a tie-in between the IKKT model and the Jordan algebra $\mathfrak{j}$ with its spinor representation. For these reasons, doing research on extended matrix model is very interesting and important. Over and above, we should not overlook the fact that several approaches which are very similar to the matrix model

[2]The author learned of the existence of this study from Smolin's paper [9].

have been pursued by other fields. We can take [11] from the non-commutative geometry, [12] from the fuzzy sphere, and [13] from the simplicial lattice for example. It might be inferred from these circumstantial evidence that the attempt to renounce the space-time as a *continuum* holds one important key to the future progress of physics. It seems at least that there is no need to relate the matrix model to the string theory alone.

For these purposes, we consider new types of matrix models based on the simply connected *compact* exceptional Lie group E_6 ([1]) and the super Lie algebras ([8]) in this paper. This paper is organized as follows.

In the next chapter we briefly review IKKT matrix model, and Smolin's matrix model based on the groups of type F_4.

After that, in chapter 3, a matrix model based on the *compact* E_6 group is presented. The action of the model is constructed from the *cubic form* which is the *invariant* on E_6 mapping. This action is an essentially complex action. Of course if one wants, one may take up only real part of the action; however, there are some circumstantial evidence where it is essential for the theory including gravity to employ complex variables. Ashtekar variables make the constraints of the canonical formalism of general relativity quite easy ([14],[15]). Actually the chiral action is a complex action whose real part agrees with Palatini action. Besides, the $u(1|16,16)$ model have also complex action because of the $33th$ component which is pure imaginary. As in the loop quantum gravity, we may be able to impose the *reality condition* on our model. In addition, it is possible that the action expanded around the vacuum, which is a specific background of the model, gets on-shell real. In this paper, therefore, we do not restrict the action to real part only. We proceed with arguments using complex action. Moreover, we discuss symmetries which our model possesses. Supersymmetry has a deep connection with *cycle mapping* $\mathcal{P}$. Furthermore, we investigate constraints which are imposed on our model. The reason why the model is a constrained system is that the group we consider here is the *compact* E_6 group of the groups of type E_6. The resulting conditions are quite similar to what we postulate as fundamental properties of inner products of the physical Hilbert space. In addition, we investigate one of the classical solutions of our model, and compactify the theory on the three directions. Our model has twice as many degrees of freedom as Smolin's model has because we are considering E_6 instead of F_4. However, we can have the similar argument as Smolin [9] which derives an effective action of the matrix string type. In appendix, the author put together an elementary knowledge about the complex Graves-Cayley algebra $\mathfrak{C}^c$ and the complex exceptional Jordan algebra $\mathfrak{J}^c$ which are needed for this paper. However, if the reader has a great interest in the exceptional linear Lie groups, see, in particular, a series of excellent reports presented by I.Yokota ([16],[17],[18]) one time.

In chapter 4, matrix models based on super Lie algebras are presented. We grope for the extended model with larger symmetries that reproduces IKKT model after gauge fixing and integrating irrelevant fields. In order that the model may describe a curved space-time, a spin connection term containing the γ-matrix of rank 3 must be included or generated in the fermionic action. Therefore we search for models with higher rank tensor fields coupled to fermions through γ-matrices. Another guiding principle to construct a model is a sufficient number of supersymmetries. In order to reproduce the IKKT model we need at least 10-dimensional $\mathcal{N} = 2$ supersymmetry. These requirements can be satisfied by considering matrix models based on super Lie algebra $osp(1|32;\mathbf{R})$. Construction of matrix models based on this superalgebra was proposed by Smolin [6] first. In this paper, we investigate such models, especially from the point of supersymmetries. In the first place, we consider a model based on $osp(1|32;\mathbf{R})$ super Lie algebra. Bosonic fields in this model can be expanded in terms of 11-dimensional γ-matrices of rank 1, 2 and 5. They are real fields. Fermionic fields are composed of 11-dimensional Majorana fermion. Hence, reduced to d=10, this model becomes vector-like and we have to integrate out a right(or left)-handed sector in order to reproduce IKKT model. The symmetry of this model is a direct product of $OSp(1|32;\mathbf{R})$ and $U(N)$. The $OSp(1|32;\mathbf{R})$ group is a generalization of $SO(9,1)$ in IKKT model. The model is also invariant under constant shifts of fields and we show that the $OSp(1|32;\mathbf{R})$ symmetry and the constant shift of fields are combined to form space-time algebras including space-time supersymmetry. We discuss a possibility to obtain IKKT model by integrating some of the fields. Moreover, we study how diffeomorphism invariance is hidden in IKKT model. After a brief summary of the relation between matrix models and gauge theories on noncommutative space, we show that the unitary gauge transformation is much larger in noncommutative space than that in ordinary commutative space. Even local coordinate transformations are generated by the unitary transformations. It is also pointed out that this diffeomorphism invariance is independent of the global $SO(9,1)$ invariance and it is difficult to extend the global $SO(9,1)$ to local symmetry of the model. Furthermore, to search for extended models with local Lorentz invariance, we then construct matrix models with local $SO(9,1)$ symmetry. In particular, we investigate matrix models based on $u(1|16,16)$ or $gl(1|32;\mathbf{R})$ super Lie algebra. These models are invariant under coupled symmetries of $U(1|16,16)$ (or $GL(1|32;\mathbf{R})$) and $U(N)$. Since $U(1|16,16)$ (or $GL(1|32;\mathbf{R})$) is an extension of $SO(9,1)$, this model has local (i.e. $U(N)$-dependent) Lorentz invariance. At the cost of this enhanced gauge symmetries, these models break invariance under constant shifts of fields and we need another interpretation of space-time translation. We make use of the Wigner-Inönü contraction and identify generators of $SO(10,1)$ (which is a subgroup of $U(1|16,16)$ and $GL(1|32;\mathbf{R})$) with generators of $SO(9,1)$ rotations and translations in 10-dimensional space-time. In this way, we can obtain 10-dimensional space-time picture. We also determine how to scale the fields to obtain the correct 10-dimensional theory.

The final chapter is devoted to conclusion and discussion.

Chapter 2

Short reviews of matrix models

2.1 IKKT model

IKKT model is a large N reduced model of 10-dimentional $\mathcal{N}=1$ super Yang-Mills theory to 0 dimension. This model has been proposed as a nonperturbative formulation of type IIB superstring theory. The action of the model is given by

$$S = -\frac{1}{g^2}Tr(\frac{1}{4}[A_\mu, A_\nu][A^\mu, A^\nu] + \frac{1}{2}\bar{\psi}\Gamma^\mu[A_\mu, \psi]), \tag{2.1.1}$$

where ψ is a 10-dimensional Majorana-Weyl spinor field, and A_μ and ψ are $N \times N$ Hermitian matrices. This action is formulated in a manifestly covariant way which enables us to study the nonperturbative issues of superstring theory. The claim of this model is that the space-time, gauge filed, gravity, and matter are all generated by solving this model.

There are several reasons to speculate that this model is a constructive definition of type IIB superstring, and one of them is that this model is the same as the matrix reguralization of the Schild action of type IIB superstring. Let us introduce Green-Schwarz action of IIB superstring.

$$S_{GS} = -T\int d^2\sigma[\sqrt{-M} + i\epsilon^{ab}\partial_a X^i(\bar{\theta}^1\Gamma_i\partial_b\theta_1 + \bar{\theta}^2\Gamma_i\partial_b\theta_2) + \epsilon^{ab}(\bar{\theta}^1\Gamma^i\partial_a\theta^i)(\bar{\theta}^2\Gamma_i\partial_b\theta^2)]. \tag{2.1.2}$$

$$M = \det(\epsilon^{ab}\Pi^i_a\Pi^j_b) \tag{2.1.3}$$

$$\Pi^i_a = \partial_a X^i - i\bar{\theta}^1\Gamma^i\partial_a\theta^1 + i\bar{\theta}^2\Gamma^i\partial_a\theta^2 \tag{2.1.4}$$

This action possesses the $N=2$ space-time supersymmetry.

$$\delta_S X^i = i\bar{\epsilon}^1\Gamma^i\theta^1 - i\bar{\epsilon}^2\Gamma^i\theta^2, \ \ \delta_S\theta^{1,2} = \epsilon^{1,2} \tag{2.1.5}$$

Then, it has another symmetry called κ *symmetry*

$$\delta_\kappa X^i = i\bar{\theta}^1\Gamma^i\alpha^1 - i\bar{\theta}^2\Gamma^i\alpha^2,\ \ \delta_\kappa\theta^{1,2} = \alpha^{1,2}, \tag{2.1.6}$$

where

$$\alpha^1 = (1+\tilde{\Gamma})\kappa_1,\ \ \alpha_2 = (1-\tilde{\Gamma})\kappa^2,\ \ \tilde{\Gamma} = \frac{1}{2!\sqrt{-M}}\epsilon^{ab}\Pi^i_a\Pi^j_b\Gamma_{ij} \tag{2.1.7}$$

$$\epsilon_{01} = +1 \text{ (hence } \epsilon^{01} = -1 \text{)} \tag{2.1.8}$$

This κ *symmetry* can be gauge-fixed by imposing the condition $\theta^1 = \theta^2 = \psi$. It leads to the action

$$S_{GS} = -T\int d^2\sigma(\sqrt{-m} + 2i\epsilon^{ab}\partial_a X^i\bar{\psi}\Gamma_i\partial_b\psi), \tag{2.1.9}$$

where $m = \det(m_{ab}) = \det(\partial_a X^i\partial_b X_i)$. This action has the new $\mathcal{N}=2$ SUSY as follows,

$$\delta^{(1)}_\epsilon\psi = \frac{1}{2}\sqrt{-m}m_{ij}\Gamma^{ij}\epsilon,\ \ \delta^{(1)}_\epsilon X^i = 4i\bar{\epsilon}\Gamma^i\psi, \tag{2.1.10}$$

$$\delta^{(2)}_\xi\psi = \xi,\ \ \delta^{(2)}_\xi X^i = 0, \tag{2.1.11}$$

where $m_{ij} = \epsilon^{ab}\partial_a X_i\partial_b X_j$. Our next job is to rewrite this Green-Schwarz action into Schild form. Defining g_{ab} as the metric of the worldsheet, and introducing the Poisson bracket as $\{X,Y\} = \frac{1}{\sqrt{g}}\epsilon^{ab}\partial_a X\partial_b Y$, this action is rewritten as

$$S_{Sh} = \int d^2\sigma[\sqrt{g}\alpha(\frac{1}{4}\{X^i,X^i\}^2 - \frac{i}{2}\bar{\psi}\Gamma^i\{X_i,\psi\}) + \beta\sqrt{g}]. \tag{2.1.12}$$

We obtain S_{GS} from S_{Sh} after eliminating $\sqrt{g}$. The $\mathcal{N}=2$ SUSY of this Schild action is

$$\delta^{(1)}_\epsilon\psi = -\frac{1}{2}m_{ij}\Gamma^{ij}\epsilon,\ \ \delta^{(1)}X^i = i\bar{\epsilon}\Gamma^i\psi, \tag{2.1.13}$$

$$\delta^{(2)}_\xi\psi = \xi,\ \ \delta^{(2)}_\xi X^i = 0. \tag{2.1.14}$$

We perform a procedure called 'matrix regularization'. This procedure is a mapping from the Poisson bracket to the commutator of large N matrices

$$-i[\ ,\ ] \leftrightarrow \{\ ,\ \}\ ,\ Tr \leftrightarrow \int d^2\sigma\sqrt{g}. \tag{2.1.15}$$

The functions X^i are now mapped into the $N\times N$ matrices A^i. And we obtain the action similar to the original proposal of IKKT model :

$$S = -\alpha(\frac{1}{4}Tr[A_i,A_j][A^i,A^j] + \frac{1}{2}Tr(\bar{\psi}\Gamma^i[A_i,\psi])) + \beta Tr\mathbf{1}. \tag{2.1.16}$$

By dropping the term $\beta Tr\mathbf{1}$ and setting $\alpha = \frac{1}{g^2}$, we get the action of IKKT model.

Although one have not yet obtained the complete interpretation of this model as the theory of gravity, the following arguments on supersymmetry lead us to interpret distributed eigenvalues as the extent of space-time. In addition to the original supersymmetry of the $\mathcal{N}=1$ super Yang-Mills

$$\delta^{(1)}\psi = \frac{i}{2}[A_\mu, A_\nu]\Gamma^{\mu\nu}\epsilon \tag{2.1.17}$$

$$\delta^{(1)}A_\mu = i\bar{\epsilon}\Gamma^\mu\psi, \tag{2.1.18}$$

this action is invariant under the following shift of the fermionic variables

$$\delta^{(2)}\psi = \xi \tag{2.1.19}$$

$$\delta^{(2)}A_\mu = 0. \tag{2.1.20}$$

The action is balanced between $Tr_{N\times N}[A_\mu, A_\nu]^2$ and $Tr_{N\times N}\bar{\psi}\gamma^\mu[A_\mu, \psi]$ under above homogeneous supersymmetry since the transformation for ψ contains two bosonic fields. Since the original 10-dimensional spacetime is reduced to the 0 dimension, the repetitions of the first transformations no longer reproduce the spacetime translation. It can be set to be zero by the $SU(N)$ gauge transformation. However, if we take a linear combination of $\delta^{(1)}$ and $\delta^{(2)}$ as

$$\tilde{\delta}^{(1)} = \delta^{(1)} + \delta^{(2)} \tag{2.1.21}$$

$$\tilde{\delta}^{(2)} = i(\delta^{(1)} - \delta^{(2)}), \tag{2.1.22}$$

we obtain an enhanced $\mathcal{N}=2$ supersymmetry algebra

$$(\tilde{\delta}^{(i)}_\epsilon\tilde{\delta}^{(j)}_\xi - \tilde{\delta}^{(j)}_\xi\tilde{\delta}^{(i)}_\epsilon)\psi = 0 \tag{2.1.23}$$

$$(\tilde{\delta}^{(i)}_\epsilon\tilde{\delta}^{(j)}_\xi - \tilde{\delta}^{(j)}_\xi\tilde{\delta}^{(i)}_\epsilon)A_\mu = -2i\bar{\epsilon}\Gamma^\mu\xi\delta_{ij}. \tag{2.1.24}$$

The r.h.s. is a shift of the diagonal part of the bosonic variables.

$$\delta A^\mu = c^\mu\mathbf{1} \tag{2.1.25}$$

The reduced model action is, of course, invariant under this shift. Therefore, if we interpret the diagonal part of the bosonic variables as the spacetime coordinates, the above $\mathcal{N}=2$ supersymmetry generates translation in the new spacetime.

Another reason to consider the bosonic variables A^μ as the space-time comes from the relation between matrix models and field theory on noncommutative geometry. Matrix models can be rewritten as gauge theories on noncommutative space by expanding the bosonic variables A^μ around the noncommutative background $\hat{x}_\mu$ satisfying

$$[\hat{x}^\mu, \hat{x}^\nu] = -i\theta^{\mu\nu}, \tag{2.1.26}$$

where $\theta^{\mu\nu}$ are c-numbers. We assume the rank of $\theta^{\mu\nu}$ to be $\tilde{d}$ and define its inverse $\beta_{\mu\nu}$ in $\tilde{d}$-dimensional subspace. $\hat{x}^\mu$ satisfy the canonical commutation relations and they span the

$\tilde{d}$-dimensional phase space. The semiclassical correspondence shows that the volume of the phase space (measured in the coordinate space of x^μ) is $V = N(2\pi)^{\tilde{d}/2}\sqrt{\det\theta}$. Distribution of eigenvalues of $\hat{x}$ is therefore interpreted as space-time. In noncommutative space, space-time translation can be generated by unitary transformation of A_μ. Furthermore, the dynamical generation of space-time implies that the fluctuation of space-time is also dynamical and graviton will be hidden in IKKT model or equivalently in noncommutative gauge theory. Investigations of noncommutative gauge theories have indeed clarified that they can contain much larger degrees of freedom than those in the ordinary commutative field theories. For example, noncommutative plane waves are interpreted as bi-local rather than local [25]. After they are expanded in terms of local operators it is expected that higher spin fields will appear, even if we start from Yang-Mills theory. From this point of view, we expect that noncommutative Yang-Mills can contain graviton. A possible interpretation of diffeomorphism invariance in noncommutative Yang-Mills is given in this paper. In the flat $\tilde{d}$-dimensional space like this case, we can identify some of the $SO(9,1)$ indices with the indices of $SO(\tilde{d})$ isometries of the background. But this cannot be expected for more general curved backgrounds whose isometries cannot be embedded in $SO(9,1)$. If the IKKT model is a background independent model, general coordinate transformations will be hidden and the $SO(9,1)$ symmetries should be rather considered as a gauge fixed local Lorentz symmetry instead of isometry of 10-dimensional flat space-time.

2.2 Smolin's matrix model based on the groups of type F_4

We also briefly review Smolin's matrix model based on the groups of type F_4 [9].

The action of Smolin's model is given by

$$S \;=\; \frac{k}{4\pi}\; tr\Big(\; J^A\;,\; \mathcal{P}(J^B)\;,\; \mathcal{P}^2(J^C)\;\Big)\; f_{ABC}\;, \tag{2.2.1}$$

where J^A are elements of real exceptional Jordan algebra $\mathfrak{J}$, $\mathcal{P}(J)$ is the cycle mapping of J, $tr(A,B,C)$ $(A,B,C \in \mathfrak{J})$ is the *trilinear form*, and f_{ABC} are the structure constants of $\mathcal{G}$ which is a Lie algebra. Therefore the degrees of freedom of this model live in $\mathfrak{J}\times\mathcal{G}$.

The specific components of $J^A \in \mathfrak{J}$ are written as

$$J^A \;=\; \begin{pmatrix} z_1{}^A & \mathcal{O}_0{}^A & \bar{\mathcal{O}}_2{}^A \\ \bar{\mathcal{O}}_0{}^A & z_2{}^A & \mathcal{O}_1{}^A \\ \mathcal{O}_2{}^A & \bar{\mathcal{O}}_1{}^A & z_0{}^A \end{pmatrix} \tag{2.2.2}$$
$$z_I{}^A \in \mathbf{R} \qquad \mathcal{O}_I{}^A \in \mathfrak{C} \qquad (I=0,1,2)\;,$$

where $z_I{}^A$ are real numbers, and $\mathcal{O}_I{}^A$ are elements of Graves-Cayley algebra.

In this model, following change of variables are required in order to make the theory Chern-Simons type,

$$x_0 \equiv z_1 + z_2 \quad , \quad x_1 \equiv z_2 + z_0 \quad , \quad x_2 \equiv z_0 + z_1 \tag{2.2.3}$$

and the effective theory of T^3 compactified theory of this model is expected to reproduce, at the one loop level, a theory related to the matrix string theory.

Chapter 3

Matrix model based on the compact E_6 group

We consider a new matrix model based on the simply connected *compact* exceptional Lie group E_6 ([1]). A matrix Chern-Simons theory is directly derived from the invariant on E_6. It is stated that the similar argument as Smolin [9] which derives an effective action of the matrix string type can also be held in our model. An important difference is that our model has twice as many degrees of freedom as Smolin's model has. One way to introduce the cosmological term is the compactification on directions. It is of great interest that the properties of the product space $\mathfrak{J}^c \times \mathcal{G}$, in which the degrees of freedom of our model live, are very similar to those of the physical Hilbert space.

3.1 Why E_6 ?

As a beginning of this chapter, let us examine the reason why E_6 is better than the groups of type F_4. Although, as early as 1983, T.Kugo and P.Townsend referred to the relevance between physics and division algebras [19], the reason why the concern with F_4 has been growing is that the critical dimension of the string theory is 10 dimensions. The exceptional Jordan algebra $\mathfrak{J}$ is a 27-dimensional $\mathbf{R}$-vector space. This space can be classified into three main parts. One is the Jordan algebra $\mathfrak{j}$ which is a 10-dimensional $\mathbf{R}$-vector space. The others are the part of 16 dimensions which is related to the spinors and the extra 1 dimension. In brief, what are expected as the degrees of freedom of the Theory of Everything are all involved in this $\mathfrak{J}$. The extra 1 dimension may account for the M-theory. Moreover, an important point to emphasize is the fact that the groups of type F_4 have some definite geometrical interpretations. For example, F_4 is a subgroup of projective transformation group of Graves-Cayley projective geometry $\mathfrak{C}P$ which corresponds to the *elliptic* non-Euclidean geometry; and $F_{4(-20)}$ is a subgroup of projective transformation group of Graves-Cayley projective geometry which corresponds to the *hyperbolic* non-

Euclidean geometry. For these reasons, the groups of type F_4 are very attractive to us, but there is one flaw in these groups. That is the fact that elements of $\mathfrak{J}$ do not have an imaginary unit 'i'. It is true the matrix model is one of candidates for the unified theory. Now may be the state of affairs in the model building, but we will have to account for the standard model someday. Fermions which appear in the standard model are Weyl spinors, namely *complex* spinors. If the standard model were described by using Majorana spinors only, F_4 might be the underlying symmetry of the universe. However, the actual world requires *complex* fermions without doubt. In the usual string theory, the real γ-matrices in 10 dimensions can be decomposed by direct products of Pauli matrices, so that the appropriate compactification can change 10-dimensional Majorana fermions into 4-dimensional Weyl fermions. In the approaches using Jordan type algebras, however, it needs care because the elements are non-associative which have no matrix representations. We cannot use above usual trick. Let us take one concrete example. In the case of $osp(1|32;\mathbf{R})$ supermatrix models the bosonic part can be expanded in terms of 11-dimensional γ-matrices with rank 1,2 and 5, and in the case of $u(1|16,16)$ models the bosonic part can be expanded in terms of 11-dimensional γ-matrices with rank 0,1,2,3,4 and 5. These super Lie algebras contain the γ-matrices clearly, but the Jordan type algebras do not have the usual (associative) Clifford algebra trivially. Therefore, the complex structure needs to be introduced into the theory from the beginning.

This is the reason why we consider E_6 is better than the simply connected compact exceptional Lie group F_4. Of course, the group $F_4{}^c$ is still available, but the model based on $F_4{}^c$ is merely the complexification of the Smolin's model. So we attempt to use the *compact* E_6 group. The complex exceptional Jordan algebra $\mathfrak{J}^c$ is the complexification of the exceptional Jordan algebra $\mathfrak{J}$. One must not confuse the complexification of the exceptional Lie group with that of the exceptional Jordan algebra. Writing down some definitions of the exceptional Lie groups is very informative for the comparison.

$$\begin{aligned}
F_4 &= \{\alpha \in Iso_R(\mathfrak{J},\mathfrak{J}) \mid tr(\alpha A, \alpha B, \alpha C) = tr(A,B,C)\,,\ (\alpha A, \alpha B) = (A,B)\} \\
F_4{}^c &= \{\alpha \in Iso_C(\mathfrak{J}^c,\mathfrak{J}^c) \mid tr(\alpha X, \alpha Y, \alpha Z) = tr(X,Y,Z)\,,\ (\alpha X, \alpha Y) = (X,Y)\}
\end{aligned}$$

$$\begin{aligned}
E_{6(-26)} &= \{\alpha \in Iso_R(\mathfrak{J},\mathfrak{J}) \mid (\alpha A, \alpha B, \alpha C) = (A,B,C)\} \\
E_6 &= \{\alpha \in Iso_C(\mathfrak{J}^c,\mathfrak{J}^c) \mid (\alpha X, \alpha Y, \alpha Z) = (X,Y,Z)\,,\ \langle \alpha X, \alpha Y\rangle = \langle X,Y\rangle\} \\
E_6{}^c &= \{\alpha \in Iso_C(\mathfrak{J}^c,\mathfrak{J}^c) \mid (\alpha X, \alpha Y, \alpha Z) = (X,Y,Z)\}
\end{aligned}$$

Here, $tr(*,*,*)$ is the *trilinear form*, and $(*,*,*)$ is the *cubic form*. The cubic form is very different from the trilinear form, and their concrete forms are given in appendix.

Another virtue is that E_6 contains $Spin(10)$. This is never achieved by type F_4, so this is a very good point of our model. It is quite likely that other matrix models are reproduced via expansions around specific backgrounds of our model. In addition, E_6 is also interesting from the viewpoint of phenomenology.

On the other hand, the problem now arises: Naturally if we deal with *compact* E_6, the degrees of freedom of the theory double because of $\mathfrak{J}^c$. Although this problem always follows us as long as we handle *compact* E_6, even so, it seems that the benefit of the fact that we can introduce complex structure into the theory exceeds this trouble. Therefore we have to prepare a mechanism separately, which reduces the number of degrees of freedom by half. This is a future problem which needs to be asked.

3.2 The model

We adopt a following definition for the simply connected *compact* exceptional Lie group E_6.

$$E_6 = \{\alpha \in Iso_C(\mathfrak{J}^c, \mathfrak{J}^c) \mid (\alpha X, \alpha Y, \alpha Z) = (X, Y, Z)\ ,\ \langle \alpha X, \alpha Y\rangle = \langle X, Y\rangle\} \quad (3.2.1)$$

The second condition $\langle \alpha X, \alpha Y\rangle = \langle X, Y\rangle$ is added to the definition to make the group *compact*.

We then define our theory by the following action,

$$S \;=\; \Big(\mathcal{P}^2(\mathcal{M}^{[A}) \ ,\ \mathcal{P}(\mathcal{M}^B) \ ,\ \mathcal{M}^{C]} \Big)\ f_{ABC} \quad , \qquad (3.2.2)$$

where $\mathcal{M}^A$ are elements of complex exceptional Jordan algebra $\mathfrak{J}^c$, $\mathcal{P}(\mathcal{M})$ is the cycle mapping of $\mathcal{M}$, (X, Y, Z) $(X, Y, Z \in \mathfrak{J}^c)$ is the *cubic form*, and f_{ABC} are the structure constants of $\mathcal{G}$ which is a Lie algebra whose Lie group is *compact*. $[\cdots]$ denotes the *weight*-1 anti-symmetrization on indices.

The cubic form cannot be coupled to the structure constants as it is, because it is symmetric with respect to the interchange of fields. The cycle mapping is introduced in order to combine the cubic form with f_{ABC}. The way of constructing this action is basically the same as Smolin's case, but the *invariant* is different because the group we adopt is different from Smolin's. The *cubic form* is employed instead of the *trilinear form*. Of course, the exceptional Jordan algebra is not real ($\mathfrak{J}$), but complex ($\mathfrak{J}^c$). Therefore the degrees of freedom of our model live in $\mathfrak{J}^c \times \mathcal{G}$. One of the methods to introduce the cosmological term or the coupling constant into the theory is the compactification on directions. The physical dimensions can also be introduced via this compactification. Up till then, only the units of angle are in existence.

A beauty of this model is that there is no need to change variables intentionally in order to get a Chern-Simons type action.

The specific components of $\mathcal{M}^A \in \mathfrak{J}^c$ are written as follows,

$$\mathcal{M}^A \;=\; \begin{pmatrix} Q_1{}^A & \phi_3{}^A & \bar{\phi}_2{}^A \\ \bar{\phi}_3{}^A & Q_2{}^A & \phi_1{}^A \\ \phi_2{}^A & \bar{\phi}_1{}^A & Q_3{}^A \end{pmatrix} + i \begin{pmatrix} P_1{}^A & \pi_3{}^A & \bar{\pi}_2{}^A \\ \bar{\pi}_3{}^A & P_2{}^A & \pi_1{}^A \\ \pi_2{}^A & \bar{\pi}_1{}^A & P_3{}^A \end{pmatrix} \qquad (3.2.3)$$

$$Q_I{}^A, P_I{}^A \in \mathbf{R} \qquad \phi_I{}^A, \pi_I{}^A \in \mathfrak{C} \qquad (I = 1, 2, 3)$$

$$= \begin{pmatrix} \mathcal{A}_1{}^A & \Phi_3{}^A & \bar{\Phi}_2{}^A \\ \bar{\Phi}_3{}^A & \mathcal{A}_2{}^A & \Phi_1{}^A \\ \Phi_2{}^A & \bar{\Phi}_1{}^A & \mathcal{A}_3{}^A \end{pmatrix} \tag{3.2.4}$$

$$\mathcal{A}_I{}^A \in \mathbf{C} \qquad \Phi_I{}^A \in \mathfrak{C}^c \qquad (I = 1, 2, 3)$$

$$\equiv \mathcal{M}^A(\mathcal{A}, \Phi) , \tag{3.2.5}$$

where $\mathcal{A}_I{}^A$ are complex numbers, and $\Phi_I{}^A$ are elements of complex Graves-Cayley algebra.

In order to analyze the theory, it is necessary to decompose the action in terms of the variables defined in (3.2.4). This gives us

$$\begin{aligned} S &= \frac{1}{4} f_{ABC}\, \epsilon^{IJK}\, \mathcal{A}_I{}^A \mathcal{A}_J{}^B \mathcal{A}_K{}^C + \frac{3}{2} f_{ABC}\, \epsilon^{IJK}\, \mathcal{A}_I{}^A (\Phi_J{}^B, \Phi_K{}^C) \\ &\quad - 3\, f_{ABC}\, \boldsymbol{Re}^c(\Phi_3{}^A \Phi_2{}^B \Phi_1{}^C) + f_{ABC} \sum_{I=1}^{3} (\ \boldsymbol{Re}^c(\Phi_I{}^A \Phi_I{}^B \Phi_I{}^C)\ ) \qquad (3.2.6) \\ &= \frac{1}{4} f_{ABC}\, \epsilon^{IJK}\, \mathcal{A}_I{}^A \mathcal{A}_J{}^B \mathcal{A}_K{}^C + \frac{3}{4} f_{ABC}\, \epsilon^{IJK}\, \mathcal{A}_I{}^A (\bar{\Phi}_J{}^B \Phi_K{}^C + \bar{\Phi}_K{}^C \Phi_J{}^B) \\ &\quad - \frac{3}{2} f_{ABC}\, (\Phi_3{}^A (\Phi_2{}^B \Phi_1{}^C) + (\bar{\Phi}_1{}^C \bar{\Phi}_2{}^B) \bar{\Phi}_3{}^A) \\ &\quad + \frac{1}{2} f_{ABC} \sum_{I=1}^{3} (\Phi_I{}^A (\Phi_I{}^B \Phi_I{}^C) + (\bar{\Phi}_I{}^C \bar{\Phi}_I{}^B) \bar{\Phi}_I{}^A) , \qquad (3.2.7) \end{aligned}$$

and depending on circumstances, it is helpful to rewrite these expressions using following relations.

$$\begin{aligned} (\Phi_J{}^B, \Phi_K{}^C) &= \boldsymbol{Re}^c(\bar{\Phi}_J{}^B \Phi_K{}^C) \qquad (3.2.8) \\ f_{ABC}\, \boldsymbol{Re}^c(\Phi_3{}^A \Phi_2{}^B \Phi_1{}^C) &= -\frac{1}{6} f_{ABC}\, \epsilon^{IJK}\, \boldsymbol{Re}^c(\Phi_I{}^A \Phi_J{}^B \Phi_K{}^C) \\ &\quad - f_{ABC}\, \sigma_{ijk}\, \Phi_{i1}{}^A \Phi_{j2}{}^B \Phi_{k3}{}^C \qquad (3.2.9) \\ f_{ABC} \sum_{I=1}^{3} (\ \boldsymbol{Re}^c(\Phi_I{}^A \Phi_I{}^B \Phi_I{}^C)\ ) &= -f_{ABC}\, \sigma_{ijk} \sum_{I=1}^{3} (\Phi_{iI}{}^A \Phi_{jI}{}^B \Phi_{kI}{}^C) \qquad (3.2.10) \end{aligned}$$

Of course there is no kinetic term because we are considering the matrix model, but the first term of this action reproduces the Chern-Simons term. What has to be noticed is that this Chern-Simons type action is directly derived from the invariant on E_6. Therefore it is quite likely that one quality of the theory based on E_6 is *topological*.

Now, in order to get the equations of motion of this theory, we have to vary with respect to all fields. We first vary the action with respect to $\mathcal{A}_L{}^D$, and the resulting equations are the following.

$$\begin{aligned} \frac{\delta S}{\delta \mathcal{A}_L{}^D} &= \frac{3}{2} f_{ABD}\, \epsilon^{IJL} \left(\frac{1}{2} \mathcal{A}_I{}^A \mathcal{A}_J{}^B + (\Phi_I{}^A, \Phi_J{}^B) \right) \\ &= \frac{3}{2} f_{ABD}\, \epsilon^{IJL} \left(\frac{1}{2} \mathcal{A}_I{}^A \mathcal{A}_J{}^B + \Phi_{0I}{}^A \Phi_{0J}{}^B + \Phi_{iI}{}^A \Phi_{iJ}{}^B \right) \\ &= 0 \qquad (3.2.11) \end{aligned}$$

Naturally, the equations of motion are obtained by differentiations with respect to the fields only, because there is no derivative term in the action. It is often convenient to rewrite these equations in terms of matrix form.

$$
\begin{aligned}
\frac{\delta S}{\delta \mathcal{A}_L{}^D}\, \boldsymbol{\tau}^L \mathbf{T}_D &= -\frac{3}{2}\left(\frac{1}{2}\mathcal{A}_I{}^A \mathcal{A}_J{}^B + \Phi_{0I}{}^A \Phi_{0J}{}^B + \Phi_{iI}{}^A \Phi_{iJ}{}^B \right) [\boldsymbol{\tau}^I, \boldsymbol{\tau}^J]\, [\mathbf{T}_A, \mathbf{T}_B] \\
&= 0
\end{aligned} \tag{3.2.12}
$$

The variations with respect to $\Phi_{0L}{}^D$ give us

$$
\begin{aligned}
\frac{\delta S}{\delta \Phi_{0L}{}^D} &= \frac{3}{2}\, f_{ABD}\, \epsilon^{IJL} \left(2\mathcal{A}_I{}^A \Phi_{0J}{}^B + \boldsymbol{Re}^c(\Phi_I{}^A \Phi_J{}^B) \right) \\
&= \frac{3}{2}\, f_{ABD}\, \epsilon^{IJL} \left(2\mathcal{A}_I{}^A \Phi_{0J}{}^B + \Phi_{0I}{}^A \Phi_{0J}{}^B - \Phi_{iI}{}^A \Phi_{iJ}{}^B \right) \\
&= 0\,.
\end{aligned} \tag{3.2.13}
$$

Similarly, we read off the following equations of motion from the variations with respect to $\Phi_{lL}{}^D$ $(l = 1, \cdots, 7)$,

$$
\begin{aligned}
\frac{\delta S}{\delta \Phi_{lL}{}^D} &= 3\, f_{ABD}\, \epsilon^{IJL} \left(\mathcal{A}_I{}^A \Phi_{lJ}{}^B - \Phi_{0I}{}^A \Phi_{lJ}{}^B \right) \\
&\quad + 3\, f_{ABD}\, \sigma_{ijl} \left(\Phi_{i(L+1)}{}^A \Phi_{j(L+2)}{}^B - \Phi_{iL}{}^A \Phi_{jL}{}^B \right) \\
&= 0\,,
\end{aligned} \tag{3.2.14}
$$

where the index L is $\boldsymbol{mod}$ **3**, and the summation convention is not used about this L.

We now introduce the following notations.

$$
\boldsymbol{\mathcal{A}}_I = \mathcal{A}_I{}^A\, \mathbf{T}_A \quad , \quad \boldsymbol{\Phi}_{0I} = \Phi_{0I}{}^A\, \mathbf{T}_A \quad , \quad \boldsymbol{\Phi}_{iI} = \Phi_{iI}{}^A\, \mathbf{T}_A \tag{3.2.15}
$$

These enable us to write preceding equations of motion as

$$
\left\{
\begin{aligned}
\epsilon^{IJL} \left(\tfrac{1}{2}\, [\boldsymbol{\mathcal{A}}_I, \boldsymbol{\mathcal{A}}_J] + [\boldsymbol{\Phi}_{0I}, \boldsymbol{\Phi}_{0J}] + [\boldsymbol{\Phi}_{iI}, \boldsymbol{\Phi}_{iJ}] \right) &= 0 \\
\epsilon^{IJL} \left(2\, [\boldsymbol{\mathcal{A}}_I, \boldsymbol{\Phi}_{0J}] + [\boldsymbol{\Phi}_{0I}, \boldsymbol{\Phi}_{0J}] - [\boldsymbol{\Phi}_{iI}, \boldsymbol{\Phi}_{iJ}] \right) &= 0 \\
\epsilon^{IJL} \left([\boldsymbol{\mathcal{A}}_I, \boldsymbol{\Phi}_{lJ}] - [\boldsymbol{\Phi}_{0I}, \boldsymbol{\Phi}_{lJ}] \right) + \sigma_{ijl} \left([\boldsymbol{\Phi}_{i(L+1)}, \boldsymbol{\Phi}_{j(L+2)}] - [\boldsymbol{\Phi}_{iL}, \boldsymbol{\Phi}_{jL}] \right) &= 0
\end{aligned}
\right. \tag{3.2.16}
$$

Likewise, we obtain the following form for the action,

$$
\begin{aligned}
S &= -\frac{3i}{2}\, \epsilon^{IJK}\, tr\Big(\frac{1}{3}\boldsymbol{\mathcal{A}}_I[\boldsymbol{\mathcal{A}}_J, \boldsymbol{\mathcal{A}}_K] + 2\boldsymbol{\Phi}_{\tilde{i}I}[\boldsymbol{\mathcal{A}}_J, \boldsymbol{\Phi}_{\tilde{i}K}] \Big) \\
&\quad + 6i\; tr\Big(\boldsymbol{\Phi}_{01}[\boldsymbol{\Phi}_{03}, \boldsymbol{\Phi}_{02}] - \boldsymbol{\Phi}_{i1}[\boldsymbol{\Phi}_{03}, \boldsymbol{\Phi}_{i2}] - \boldsymbol{\Phi}_{i1}[\boldsymbol{\Phi}_{i3}, \boldsymbol{\Phi}_{02}] - \boldsymbol{\Phi}_{01}[\boldsymbol{\Phi}_{i3}, \boldsymbol{\Phi}_{i2}] \\
&\qquad - \sigma_{ijk}\, (\boldsymbol{\Phi}_{i1}[\boldsymbol{\Phi}_{j3}, \boldsymbol{\Phi}_{k2}]) + \frac{1}{3}\sigma_{ijk} \sum_{I=1}^{3} (\boldsymbol{\Phi}_{iI}[\boldsymbol{\Phi}_{jI}, \boldsymbol{\Phi}_{kI}]) \Big)\,,
\end{aligned} \tag{3.2.17}
$$

where $(\tilde{i} = 0, \cdots, 7)$ and $(i = 1, \cdots, 7)$.

3.3 Symmetries of the model

In this section, the symmetries of our model are described. First of all, the action is invariant under the following E_6 mapping as a result of our definition for the simply connected compact exceptional Lie group E_6.

$$
\begin{aligned}
&\Big(\alpha\, \mathcal{P}^2(\mathcal{M}^{[A}) \,,\, \alpha\, \mathcal{P}(\mathcal{M}^{B}) \,,\, \alpha\, \mathcal{M}^{C]} \Big) \; f_{ABC} \\
&\quad = \; \Big(\mathcal{P}^2(\mathcal{M}^{[A}) \,,\, \mathcal{P}(\mathcal{M}^{B}) \,,\, \mathcal{M}^{C]} \Big) \; f_{ABC} \qquad (3.3.1)\\
&\quad = \; S
\end{aligned}
$$

Then, we have the gauge symmetry for the compact Lie group because the structure constants f_{ABC} of $\boldsymbol{\mathcal{G}}$ are coupled on to this action. We can take $\boldsymbol{\mathcal{G}} = u(N)$ for example.

Moreover, as we can clearly see in (3.2.17), there exist matrix translation symmetries with respect to the diagonal parts of the fields.

In addition to these, there is a particular symmetry which we call *cycle mapping* $\mathcal{P}$. This mapping is defined by the cyclic permutation with respect to the indices $I = 1, 2, 3$, and probably it belongs to the F_4. The action is invariant under the transformation $\mathcal{M}^A \mapsto \mathcal{P}(\mathcal{M}^A)$.

$$
\begin{aligned}
&\Big(\mathcal{P}^2(\mathcal{P}(\mathcal{M}^{[A})) \,,\, \mathcal{P}(\mathcal{P}(\mathcal{M}^{B})) \,,\, \mathcal{P}(\mathcal{M}^{C]}) \Big) \; f_{ABC} \\
&\quad = \; \Big(\mathcal{P}^3(\mathcal{M}^{[A}) \,,\, \mathcal{P}^2(\mathcal{M}^{B}) \,,\, \mathcal{P}(\mathcal{M}^{C]}) \Big) \; f_{ABC} \qquad (3.3.2)\\
&\quad = \; \Big(\mathcal{M}^{[A} \,,\, \mathcal{P}^2(\mathcal{M}^{B}) \,,\, \mathcal{P}(\mathcal{M}^{C]}) \Big) \; f_{ABC} \qquad (3.3.3)\\
&\quad = \; \Big(\mathcal{P}^2(\mathcal{M}^{[B}) \,,\, \mathcal{P}(\mathcal{M}^{C}) \,,\, \mathcal{M}^{A]} \Big) \; f_{BCA} \qquad (3.3.4)\\
&\quad = \; S
\end{aligned}
$$

The reason why the cycle mapping is important is that the invariance of the action under this mapping has a deep connection with the *supersymmetry*. Basically, we would like to think that the specific components of $\mathcal{M}^A \in \mathfrak{J}^c$ are divided as follows,

$$
\mathcal{M}^A = \begin{pmatrix} \mathcal{A}_1{}^A & \Phi_3{}^A & \bar{\Phi}_2{}^A \\ \bar{\Phi}_3{}^A & \mathcal{A}_2{}^A & \Phi_1{}^A \\ \Phi_2{}^A & \bar{\Phi}_1{}^A & \mathcal{A}_3{}^A \end{pmatrix} \equiv \begin{pmatrix} \mathcal{W}^A & \Psi^A \\ \Psi^{\ddagger A} & v^A \end{pmatrix} \; , \qquad (3.3.5)
$$

where $\mathcal{W}^A$, Ψ^A, and v^A are defined by

$$
\mathcal{W}^A = \begin{pmatrix} \mathcal{A}_1{}^A & \Phi_3{}^A \\ \bar{\Phi}_3{}^A & \mathcal{A}_2{}^A \end{pmatrix} \quad , \quad \Psi^A = \begin{pmatrix} \bar{\Phi}_2{}^A \\ \Phi_1{}^A \end{pmatrix} \quad ; \quad v^A = \mathcal{A}_3{}^A \, . \qquad (3.3.6)
$$

We would like to consider that $\mathcal{W}^A$ and v^A are bosonic fields, and Ψ^A are fermionic fields in the long run. Unfortunately, however, Ψ^A are still bosonic fields at this stage. This is a problem common to all theories which are based on the exceptional Jordan algebra $\mathfrak{J}$ or the complex exceptional Jordan algebra $\mathfrak{J}^c$. Of course, if one would like to avoid this problem, one may consider the theory based on the Jordan algebra $\mathfrak{j}$ or the complex Jordan algebra $\mathfrak{j}^c$ and may prepare fermions made up of Grassmann variables separately. However, the groups like F_4 or E_6 are very attractive, and the idea that one multiplet is pushed into a single matrix, which is an element of $\mathfrak{J}$ or $\mathfrak{J}^c$, is mathematically beautiful. Therefore we have to prepare a mechanism separately, which introduces an anticommuting factor into the theory. An example of the way is that we impose different boundary conditions when we compactify the theory. It is a future problem whether other mechanisms exist. This is very interesting, because there is a close resemblance between this situation and that of the fractional quantum hall effects of condensed matter systems. It is necessary to keep in mind that it is possible that our definition of spinors itself is still incomplete. For another, this situation may have a connection with 'Bosonic M Theory' [20]. Anyway, the expression (3.3.5) is the reason why we would like to relate the cycle mapping to the supersymmetry.

3.4 Constraints of the model

We next study the constraints of our model. If this algebraically defined theory has a deep connection with some geometry, it must be essentially the *non-associative geometry*. In particular, it is quite likely that this theory has its geometrical interpretation in the *projective geometry* because the group we consider here is the compact E_6 group. For example, the non-compact group $E_{6(-26)}$, which is also one of the groups of type E_6, is the projective transformation group of Graves-Cayley projective plane $\mathfrak{C}P_2$ ([21]). Of course, little is known about the geometrical interpretation of our model at this stage. However, in order to pursue the geometrical interpretation of the theory in due course, it is important to refer to the fact that this model is a constrained system first.

The reason why additional conditions are imposed on our model is that the group we consider here is especially the *compact* E_6 group of type E_6. The constraints result from the following condition.

$$\begin{aligned} \langle \alpha X, \alpha Y\rangle &= \langle X, Y\rangle \\ & \quad (X, Y \in \mathfrak{J}^c) \end{aligned} \tag{3.4.1}$$

Thanks to this condition, one invariant under the E_6 mapping is introduced into the theory.

$$\begin{cases} invariant_{E_6} = \langle \mathcal{M}^A, \mathcal{M}^B\rangle \quad \in \quad \mathbf{C} \\ \langle \mathcal{M}^B, \mathcal{M}^A\rangle = \langle \mathcal{M}^A, \mathcal{M}^B\rangle^* \end{cases} \tag{3.4.2}$$

Clearly, we can couple another invariant, which is the δ-term concerning $\mathcal{G}$, to this. If we consider $\mathcal{G} = u(N)$ for example, summing up indices leads us to the following expression.

$$\begin{aligned} invariant_{E_6 \times U(N)} &= 2 \langle \mathcal{M}^A, \mathcal{M}^B \rangle \; tr(\mathbf{T}_A \mathbf{T}_B) && (3.4.3) \\ &= \langle \mathcal{M}^A, \mathcal{M}^A \rangle \quad \in \quad \mathbf{R} && (3.4.4) \\ &\geq 0 && (3.4.5) \end{aligned}$$

The crucial point to observe here is that *this invariant satisfies a positivity condition:* in general $\langle \mathcal{M}^A, \mathcal{M}^A \rangle \geq 0$, and vanishes if and only if $\mathcal{M}^A = 0$. Although this invariant is the quantity defined on the product space $\mathfrak{J}^c \times \mathcal{G}$, the resulting structure is, in a sense, quite similar to that of the physical Hilbert space. Since there is a cycle mapping, this space is, as it were, the Hilbert space with the spinor structure. In passing, we can rewrite above quantity with emphasis on $U(N)$,

$$\begin{aligned} invariant_{E_6 \times U(N)} &= \sum_{I=1}^{3} (\mathcal{A}_I^{*A} \mathcal{A}_I{}^B + 2 \langle \Phi_I{}^A, \Phi_I{}^B \rangle) \; tr(\mathbf{T}_A \mathbf{T}_B) && (3.4.6) \\ &= tr_{N \times N} \Big(\; (\mathcal{A}_I^{*A} \mathcal{A}_I{}^B + 2 \Phi_{\bar{i}I}^{*}{}^A \Phi_{\bar{i}I}{}^B) \; \mathbf{T}_A \mathbf{T}_B \; \Big) && (3.4.7) \\ &\geq 0 \, , && (3.4.8) \end{aligned}$$

where $N \to \infty$, or finite but boundlessly large.

Furthermore, we can consider the following combinations too.

$$\begin{aligned} \langle \alpha \, \mathcal{P}^2(\mathcal{M}^A), \alpha \, \mathcal{P}(\mathcal{M}^B) \rangle &= \langle \mathcal{P}^2(\mathcal{M}^A), \mathcal{P}(\mathcal{M}^B) \rangle && (3.4.9) \\ \langle \alpha \, \mathcal{P}(\mathcal{M}^A), \alpha \, \mathcal{M}^B \rangle &= \langle \mathcal{P}(\mathcal{M}^A), \mathcal{M}^B \rangle && (3.4.10) \\ \langle \alpha \, \mathcal{M}^A, \alpha \, \mathcal{P}^2(\mathcal{M}^B) \rangle &= \langle \mathcal{M}^A, \mathcal{P}^2(\mathcal{M}^B) \rangle && (3.4.11) \end{aligned}$$

Next, let us consider more general case. Two $\mathcal{M}$'s can be independent this time. In this case, the invariant is as follows.

$$\left\{ \begin{array}{l} invariant_{E_6} = \langle \mathcal{M}^A, \mathcal{M}'^B \rangle \quad \in \quad \mathbf{C} \\ \\ \langle \mathcal{M}'^B, \mathcal{M}^A \rangle = \langle \mathcal{M}^A, \mathcal{M}'^B \rangle^* \end{array} \right. \qquad (3.4.12)$$

Therefore we can couple the δ-term concerning $\mathcal{G}$ to this quantity in the same way as previous case.

$$\left\{ \begin{array}{l} invariant_{E_6 \times U(N)} = \langle \mathcal{M}^A, \mathcal{M}'^A \rangle \quad \in \quad \mathbf{C} \\ \\ \langle \mathcal{M}'^A, \mathcal{M}^A \rangle = \langle \mathcal{M}^A, \mathcal{M}'^A \rangle^* \end{array} \right. \qquad (3.4.13)$$

This invariant is, in general, a complex number. The second expression of (3.4.13) indicates that $\langle \mathcal{M}'^A, \mathcal{M}^A \rangle$ and $\langle \mathcal{M}^A, \mathcal{M}'^A \rangle$ are complex conjugates of each other.

Of course, we may also consider the following combinations as before.

$$\langle \alpha\, \mathcal{P}^2(\mathcal{M}^A), \alpha\, \mathcal{P}(\mathcal{M}'^B)\rangle = \langle \mathcal{P}^2(\mathcal{M}^A), \mathcal{P}(\mathcal{M}'^B)\rangle \tag{3.4.14}$$
$$\langle \alpha\, \mathcal{P}(\mathcal{M}^A), \alpha\, \mathcal{M}'^B\rangle = \langle \mathcal{P}(\mathcal{M}^A), \mathcal{M}'^B\rangle \tag{3.4.15}$$
$$\langle \alpha\, \mathcal{M}^A, \alpha\, \mathcal{P}^2(\mathcal{M}'^B)\rangle = \langle \mathcal{M}^A, \mathcal{P}^2(\mathcal{M}'^B)\rangle \tag{3.4.16}$$

The author does not know the geometrical interpretation of the simply connected compact exceptional Lie group E_6. The attempt to relate the model based on the *compact* group E_6 to some geometry is a very exciting theme. It is quite likely that the theory based on E_6 is closely connected with the topological theory, because although the action was originally constructed from the *algebraic* invariant on E_6 mapping (i.e. cubic form), it has a form which is similar to the Chern-Simons theory automatically. To take a hypothetical example, if we can give the Freudenthal multiplication $X \times Y$ a definite interpretation such as the *outer product* in the space, we might be able to give the action a geometrical interpretation such as some kind of *volume*. What seems to be lacking is the knowledge of the projective geometry or the non-associative geometry. Therefore, we get the feeling that we had better pay attention to the advancement of these fields.

3.5 One of the classical solutions and T^3 compactification

One way to study the dynamics of the theory is the compactification on directions. In this section, we follow Smolin's arguments. We investigate one of the classical solutions of our model, and compactify the theory on the three directions. As a result, we can have the same argument as Smolin [9] which derives an effective action similar to the matrix string theory. The main difference is that our model has twice as many degrees of freedom as Smolin's model has because we are considering E_6 instead of F_4.

To begin with, we represent the matrix elements $(\boldsymbol{\mathcal{A}}_I)^P{}_Q$, where P stands for the index of 'row' and Q stands for the index of 'column', of $N \times N$ square matrices $\boldsymbol{\mathcal{A}}_I$ as $\mathcal{A}_I{}^P_Q$. Then, let us view $\boldsymbol{\mathcal{G}}$ as a product space which is made up of four parts. Accordingly, we can give the one-to-one correspondence between P, Q and $(p_1p_2p_3\tilde{P}), (q_1q_2q_3\tilde{Q})$,

$$\mathcal{A}_I{}^P_Q \equiv \mathcal{A}_I{}^{p_1\,p_2\,p_3\,\tilde{P}}_{q_1\,q_2\,q_3\,\tilde{Q}}\ , \tag{3.5.1}$$

where $(p_I, q_I = -L_I, \cdots, 0, \cdots, L_I)_{\,(I=1,2,3)}$, $(\tilde{P}, \tilde{Q} = 1, \cdots, M)$, so that

$$N = (\prod_{I=1}^{3}(2L_I + 1))\ M\ . \tag{3.5.2}$$

Next, let us focus on one of the classical solutions of our model, given by

$$\left\{ \begin{array}{c} \mathcal{A}_I{}^{p_1\,p_2\,p_3\,\tilde{P}}_{q_1\,q_2\,q_3\,\tilde{Q}} = P_I{}^{p_1\,p_2\,p_3}_{q_1\,q_2\,q_3}\ \delta^{\tilde{P}}_{\tilde{Q}} \\ P_I{}^{p_1\,p_2\,p_3}_{q_1\,q_2\,q_3} = p_I \delta^{p_1}_{q_1} \delta^{p_2}_{q_2} \delta^{p_3}_{q_3} \end{array} \right. \tag{3.5.3}$$

with other fields vanishing. This solution satisfies the following relations.

$$[(\boldsymbol{P}_I \otimes \mathbf{1}_{M\times M}), (\boldsymbol{P}_J \otimes \mathbf{1}_{M\times M})] \;=\; 0 \tag{3.5.4}$$

Now, we expand the theory around this classical solution,

$$\mathcal{A}_I{}^{p_1\,p_2\,p_3\,\tilde{P}}_{q_1\,q_2\,q_3\,\tilde{Q}} \;=\; P_I{}^{p_1\,p_2\,p_3}_{q_1\,q_2\,q_3}\ \delta^{\tilde{P}}_{\tilde{Q}} \;+\; a_I{}^{p_1\,p_2\,p_3\,\tilde{P}}_{q_1\,q_2\,q_3\,\tilde{Q}}\ , \tag{3.5.5}$$

and then consider the mapping into the space of functional by using the usual matrix compactification procedure based on the complex Fourier series expansion,

$$tr_{N\times N}\Big(F[P,G]\Big) \;=\; \frac{1}{T}\oint dt\ tr_{M\times M}\Big(F(t)(-i\frac{\partial G(t)}{\partial t})\Big)\ . \tag{3.5.6}$$

At this time, while the fields $a_I{}^{p_1\,p_2\,p_3\,\tilde{P}}_{q_1\,q_2\,q_3\,\tilde{Q}}$ and $\Phi_{\bar{i}3}{}^{p_1\,p_2\,p_3\,\tilde{P}}_{q_1\,q_2\,q_3\,\tilde{Q}}$ are compactified as bosonic fields, the fields $\Phi_{\bar{i}\alpha}{}^{p_1\,p_2\,p_3\,\tilde{P}}_{q_1\,q_2\,q_3\,\tilde{Q}}\ {}_{(\alpha=1,2)}$ are compactified as fermionic fields.

$$a_I{}^{(p_1+(2L_1+1))\,p_2\,p_3\,\tilde{P}}_{(q_1+(2L_1+1))\,q_2\,q_3\,\tilde{Q}} \;=\; +a_I{}^{p_1\,p_2\,p_3\,\tilde{P}}_{q_1\,q_2\,q_3\,\tilde{Q}} \tag{3.5.7}$$

$$\Phi_{\bar{i}3}{}^{(p_1+(2L_1+1))\,p_2\,p_3\,\tilde{P}}_{(q_1+(2L_1+1))\,q_2\,q_3\,\tilde{Q}} \;=\; +\Phi_{\bar{i}3}{}^{p_1\,p_2\,p_3\,\tilde{P}}_{q_1\,q_2\,q_3\,\tilde{Q}} \tag{3.5.8}$$

$$\Phi_{\bar{i}\alpha}{}^{(p_1+(2L_1+1))\,p_2\,p_3\,\tilde{P}}_{(q_1+(2L_1+1))\,q_2\,q_3\,\tilde{Q}} \;=\; -\Phi_{\bar{i}\alpha}{}^{p_1\,p_2\,p_3\,\tilde{P}}_{q_1\,q_2\,q_3\,\tilde{Q}} \tag{3.5.9}$$

The x^2 and x^3 directions are also compactified with the same signs.

Under these conditions, the action of the theory becomes

$$\begin{aligned} S \;=\; & -\frac{3}{2(T_1T_2T_3)}\oint_{T^3} d^3x\ tr_{M\times M}\Big(\ \epsilon^{IJK}\ (\ \boldsymbol{a}_I\partial_J\boldsymbol{a}_K + \frac{2i}{3}\boldsymbol{a}_I\boldsymbol{a}_J\boldsymbol{a}_K\ ) \\ & \qquad + 2\ (\ -\boldsymbol{\Phi}_{\bar{i}1}[\mathcal{D}_3, \boldsymbol{\Phi}_{\bar{i}2}] + \boldsymbol{\Phi}_{\bar{i}2}[\mathcal{D}_3, \boldsymbol{\Phi}_{\bar{i}1}]\ ) \\ & \qquad - 4i\ (\ \boldsymbol{\Phi}_{01}[\boldsymbol{\Phi}_{03}, \boldsymbol{\Phi}_{02}] - \boldsymbol{\Phi}_{i1}[\boldsymbol{\Phi}_{03}, \boldsymbol{\Phi}_{i2}] - \boldsymbol{\Phi}_{i1}[\boldsymbol{\Phi}_{i3}, \boldsymbol{\Phi}_{02}] - \boldsymbol{\Phi}_{01}[\boldsymbol{\Phi}_{i3}, \boldsymbol{\Phi}_{i2}] \\ & \qquad\qquad - \sigma_{ijk}\boldsymbol{\Phi}_{i1}[\boldsymbol{\Phi}_{j3}, \boldsymbol{\Phi}_{k2}] + \frac{1}{3}\sigma_{ijk}\boldsymbol{\Phi}_{i3}[\boldsymbol{\Phi}_{j3}, \boldsymbol{\Phi}_{k3}]\ )\ \Big) \end{aligned} \tag{3.5.10}$$

$$\begin{aligned} \;=\; & -\frac{3}{2\Lambda}\oint_{T^3} d^3x\ tr_{M\times M}\Big(\ \epsilon^{IJK}\ (\ \boldsymbol{a}_I\partial_J\boldsymbol{a}_K + \frac{2i}{3}\boldsymbol{a}_I\boldsymbol{a}_J\boldsymbol{a}_K\ ) \\ & \qquad\qquad - 2\ \epsilon^{\alpha\beta}\ \boldsymbol{\Phi}_{\bar{i}\alpha}[\mathcal{D}_3, \boldsymbol{\Phi}_{\bar{i}\beta}] - \frac{8i}{3}\sigma_{ijk}\boldsymbol{\Phi}_{i3}\boldsymbol{\Phi}_{j3}\boldsymbol{\Phi}_{k3}\ \Big) \\ & \qquad - 4i\ \boldsymbol{Re}^c(\Phi_3{}^{\tilde{P}}_{\tilde{Q}}\Phi_2{}^{\tilde{Q}}_{\tilde{R}}\Phi_1{}^{\tilde{R}}_{\tilde{P}}) + 4i\ \boldsymbol{Re}^c(\Phi_1{}^{\tilde{P}}_{\tilde{Q}}\Phi_2{}^{\tilde{Q}}_{\tilde{R}}\Phi_3{}^{\tilde{R}}_{\tilde{P}}) \end{aligned} \tag{3.5.11}$$

in the $L_I \to \infty$ limits, where $T_I = l^{(I)}(2L_I+1)$, with T_I held fixed and $l^{(I)} \to 0$. The dimensional scales $l^{(I)}$ are introduced in order to adjust the physical dimensions. The point to observe here is that the coefficient $\frac{1}{(T_1 T_2 T_3)}$ resulting from the compactification on directions fills the role of a cosmological term $\frac{1}{\Lambda}$.

Incidentally, the expressions in terms of continuous functions like (3.5.6), (3.5.10) and (3.5.11) are symbolical notations. It would be rather essentially proper to think that there exists an l_{Planck}, which is sufficiently small but finite, such that $l^{(I)} \geq l_{Planck}$. In brief, this is to say that a minimum length is introduced into the space-time itself. Accordingly L_I are also very large but finite. In consequence, such concept as *universality class* seems to be unacceptable because we cannot achieve the genuine continuum limit. Applying the concept of universality class to the matrix model implies that the matrix model drops its position down to a mere regularization of the continuum theory. We do not take the position that we consider the matrix model to be the regularization of the continuum theory. We take a view that it is the expression in terms of matrices that is rather proper descriptive language. The reason why Wilson's method achieved a great success is that the field theory itself was a low-energy effective theory. Of course it is possible that such concepts as *nonstandard number* and *nonstandard analysis* are not irrelevant to the future physics. However, what needs to be emphasized here is that we must not handle the micro-world as an object of 'regulation' in order to suit macro-phenomena to our own convenience. Clearly, the macro-world is in existence as a consequence of phenomena of the micro-world. The research worker of the elementary particle should pursue the structure of minute world to the bitter end, and now, it would be wise for us to abandon the concept of the space-time *continuum* as a product of illusion.

Let us now return to our main concern. One process leading to (3.5.10) is that we eliminate the terms which comprise the odd degree with respect to the fermionic fields $\Phi_{\tilde{i}\alpha\, q_1\, q_2\, q_3\, \tilde{Q}}^{\ \ \ p_1\, p_2\, p_3\, \tilde{P}}$ $(\alpha=1,2)$ from the compactified action because they contradict the boundary conditions. It is now easy to show that what is true for Smolin's model is true for our model as well (See [9], Section 5.). If we take the limit $T_3 \to T_{Planck}$ and then drop terms in ∂_3 and a_3 intentionally, aside from the coefficient of each term we can expect that from analyses of symmetries and power counting the resulting 2-dimensional effective action results in what is similar to the action of the matrix string theory at the one loop level.

$$
\begin{aligned}
I_{eff} \;=\; \oint_{T^2} d^2x \; tr_{M\times M}\Big(& \boldsymbol{\Phi}_{\tilde{i}\alpha}\sigma^{\mu\alpha\beta}{}_{\tilde{i}\tilde{j}}[\mathcal{D}_\mu, \boldsymbol{\Phi}_{\tilde{j}\beta}] + \boldsymbol{\Phi}_{\tilde{i}1}\gamma^{\tilde{k}}{}_{\tilde{i}\tilde{j}}[\boldsymbol{\Phi}_{\tilde{k}3}, \boldsymbol{\Phi}_{\tilde{j}2}] + \sigma_{ijk}\boldsymbol{\Phi}_{i3}\boldsymbol{\Phi}_{j3}\boldsymbol{\Phi}_{k3} \\
& + (\boldsymbol{f}_{\mu\nu})^2 + (\mathcal{D}_\mu\boldsymbol{\Phi}_{\tilde{i}3})^2 + \rho_{ijkl}[\boldsymbol{\Phi}_{i3}, \boldsymbol{\Phi}_{j3}][\boldsymbol{\Phi}_{k3}, \boldsymbol{\Phi}_{l3}] \Big) \qquad (3.5.12)
\end{aligned}
$$

One of the differences between this action and that of the matrix string theory is the form of the four-matrix interaction terms as has been pointed out by Smolin. This is, so to speak, a matrix string-like theory based on ${G_2}^{\mathbf{c}}$. The other crucial point is that our model has twice as many degrees of freedom as Smolin's model has because we are considering E_6 instead of F_4. This trouble always follows us as long as we handle E_6. We will take the argument about this matter up some other time.

Lastly, the author would like to mention the possibility that the 3-dimensional compactified action (3.5.11) might be associated with not only the constructive formulation of the string theory but also the loop quantum gravity which is another hopeful approach to the unified theory. Let us now exponentiate the action (3.5.11) and suppose the following quantity.

$$\begin{aligned}
\boldsymbol{\Psi}_\Lambda[a,\Phi] &\equiv e^{iS} && (3.5.13)\\
&= e^{-i\frac{3}{2\Lambda}\left(I_{cs}(a)+I(\Phi)\right)} && (3.5.14)\\
I_{cs}(a) &= \oint_{T^3} d^3x\ \epsilon^{IJK}\ tr_{M\times M}\Big(\ \boldsymbol{a}_I\partial_J\boldsymbol{a}_K+\frac{2i}{3}\boldsymbol{a}_I\boldsymbol{a}_J\boldsymbol{a}_K\ \Big) && (3.5.15)
\end{aligned}$$

Except for the facts that this quantity contains the fermionic terms and $U(M)$ takes the place of $SU(2)$, it is similar to Kodama's wave function ([22],[23],[24]) which is an exact solution to all the constraints of the loop quantum gravity in the case of non-zero cosmological constant. In fact, the overall factor is $-i\frac{3}{2\Lambda}$ which is the same as Kodama's wave function. We did not use any approximation, nevertheless the same factor is directly derived from our model. Therefore, it seems quite probable that there exists something like an $U(M)$ *generalized* loop quantum gravity whose physical state is $\boldsymbol{\Psi}_\Lambda[a,\Phi]$.

Chapter 4

Matrix models based on super Lie algebras

We investigate several matrix models based on super Lie algebras, $osp(1|32, R)$, $u(1|16, 16)$ and $gl(1|32, R)$ ([8]). In particular, we study the supersymmetry structures of these models and discuss possible reductions to IKKT model. We also point out that diffeomorphism invariance is hidden in gauge theories on noncommutative space which are derived from matrix models. This symmetry is independent of the global $SO(9, 1)$ invariance in IKKT model and we report our trial to extend the global Lorentz invariance to local symmetry by introducing $u(1|16, 16)$ or $gl(1|32, R)$ super Lie algebras.

4.1 $osp(1|32, R)$ Cubic Matrix Model

Smolin proposed a matrix model based on the super Lie algebra $osp(1|32, R)$ [6] as an M-theory matrix model. The action is constructed from $osp(1|32, R)$ matrix M whose components are also $N \times N$ matrices. The bosonic part of this model can be expanded in terms of 11-dimensional γ-matrices with rank 1, 2 and 5. Therefore it is a natural extension of ordinary matrix models containing only vector field with rank 1. Furthermore it has a simple cubic form in terms of matrix M and is reminiscent of Witten's string field theory. Before going into detailed analysis of the model, we first give the definitions of $osp(1|32, R)$ super Lie algebra.

4.1.1 Definition of $osp(1|32, R)$ supermatrix

$osp(1|32, R)$ super matrix is a 33×33 real supermatrix satisfying the following conditions:

$$ {}^{T}MG + GM = 0 \ \text{ for } \ G = \begin{pmatrix} \Gamma^0 & 0 \\ 0 & i \end{pmatrix}, $$

$$M^* = M. \tag{4.1.1}$$

Γ^0 is a 32×32 11-dimensional γ-matrix in Majorana basis which is real and satisfies $(\Gamma^0)^2 = -1$. For conventions of a super matrix, see Appendix A. An element of $OSp(1|32, R)$ group is written as $U = \exp(M)$ and satisfies

$$^T U G U = G. \tag{4.1.2}$$

The above conditions (4.1.1) restrict the matrix M to be

$$M = \begin{pmatrix} m & \psi \\ i\bar{\psi} & 0 \end{pmatrix}, \tag{4.1.3}$$

where ψ is a Majorana spinor with 32 components and $\bar{\psi} = {}^T\psi\Gamma^0$. m is a real 32×32 bosonic matrix satisfying

$$^T m \Gamma^0 + \Gamma^0 m = 0. \tag{4.1.4}$$

The bosonic part m is an element of $sp(32, R)$ algebra. It can be expanded in terms of 11-dimensional γ-matrices as

$$m = u_{A_1}\Gamma^{A_1} + \frac{1}{2!}u_{A_1A_2}\Gamma^{A_1A_2} + \frac{1}{5!}u_{A_1\cdots A_5}\Gamma^{A_1\cdots A_5}, \tag{4.1.5}$$

and contains $528 = 11 + 55 + 462$ degrees of freedom. A_i are denoted as 11-dimensional indices and run from 0 to 10. We are working in Majorana basis, where all γ-matrices are real. Therefore the real condition means that all the coefficients u_{A_1}, $u_{A_1A_2}$ and $u_{A_1\cdots A_5}$ are real.

4.1.2 Action and symmetries

In considering the action of a large N reduced model, we regard each of the coefficients u_{A_1}, $u_{A_1A_2}$ and $u_{A_1\cdots A_5}$ and each component of ψ as an $N \times N$ hermitian matrix. $M_P{}^Q$, each component of the supermatrix M, is thus an $N \times N$ hermitian matrix. We further introduce N^2 $osp(1|32, R)$ supermatrices M^a as the coefficients of M expanded in terms of the Gell-Mann matrices t^a:

$$M = \sum_{a=1}^{N^2} t^a M^a. \tag{4.1.6}$$

The action proposed by Smolin is

$$\begin{aligned} I &= \frac{i}{g^2} Tr_{N\times N} \sum_{Q,R=1}^{33} \left(\left(\sum_{p=1}^{32} M_p{}^Q [M_Q{}^R, M_R{}^p]\right) - M_{33}{}^Q [M_Q{}^R, M_R{}^{33}] \right) \\ &= \frac{i}{g^2} \sum_{a,b,c=1}^{N^2} Str_{33\times 33}(M^a M^b M^c) Tr_{N\times N}(t^a [t^b, t^c]), \end{aligned} \tag{4.1.7}$$

where $p = 1, \cdots, 32$, $P, Q, R = 1, 2, \cdots 33$ and a, b, c are indices for $U(N)$. To avoid confusions, we note here that we use Tr as a trace of $N \times N$ matrices while tr (or Str) as a trace of 32×32 (33×33 super) matrices. This action can be rewritten as

$$\begin{aligned} I &= -\frac{f_{abc}}{2g^2}(tr_{32\times 32}(m^a m^b m^c) - 3i\bar{\psi}^a m^b \psi^c) \\ &= \frac{i}{g^2} Tr_{N\times N}(m_p{}^q[m_q{}^r, m_r{}^p]) - 3i\bar{\psi}^p[m_p{}^q, \psi_q]). \end{aligned} \tag{4.1.8}$$

f_{abc} are structure constants defined by $[t_a, t_b] = if_{abc}t_c$. The fermionic term has the same form as that of IKKT model but the bosonic part is cubic and different. This difference is related to the difference in supersymmetry. That is, in the IKKT case the supersymmetry transformation for ψ is proportional to a commutator of the bosonic field $[A_\mu, A_\nu]$ while here all (homogeneous) transformations are linear in fields as we will see soon. Another big difference is that this model contains 32 component Majorana fermion compared to 16 in IKKT model. Due to this doubling, we need to integrate out half of fermions in order to show the equivalence to IKKT model.

In spite of these differences, this model possesses several similarities. First it has no free parameter since the coupling constant is always absorbed by a field redefinition of matrix M. Hence g gives the only dimensionful parameter in the model. Symmetries of the model also have similar structures to IKKT model. If we write the matrix M as a tensor product of $osp(1|32, R)$ and $N \times N$ matrix, the action is invariant under

$$M \to (U \otimes 1_{N\times N})\ M\ (U \otimes 1_{N\times N})^{-1}, \tag{4.1.9}$$

where U is an element of $OSp(1|32, R)$ group. For an infinitesimal transformation,

$$\delta M = [H, M] = [\begin{pmatrix} h & \chi \\ i\bar{\chi} & 0 \end{pmatrix}, \begin{pmatrix} m & \psi \\ i\bar{\psi} & 0 \end{pmatrix}] = \delta_h M + \delta_\chi^{(1)} M. \tag{4.1.10}$$

The bosonic part is identified with $sp(32, R)$ rotations:

$$\delta_h M = \begin{pmatrix} [h, m] & h\psi \\ i\bar{\psi}h & 0 \end{pmatrix}. \tag{4.1.11}$$

The fermionic part, supersymmetry transformation, is given by

$$\delta_\chi^{(1)} M = \begin{pmatrix} i(\chi\bar{\psi} - \psi\bar{\chi}) & -m\chi \\ i\bar{\chi}m & 0 \end{pmatrix}. \tag{4.1.12}$$

The bosonic part is a natural extension of $SO(9, 1)$ rotation in IKKT model and indeed includes $SO(10, 1)$ symmetry. Besides this $SO(10, 1)$ symmetry generated by $\Gamma^{A_1 A_2}$, there are bosonic symmetries generated by γ-matrices with rank 1 and 5. These transformations mix bosonic fields with a different number of 11-dimensional indices. The fermionic part is a generalization of homogeneous supersymmetry in IKKT model. As already mentioned, this homogeneous supersymmetry is linear in all fields and the action is invariant under this supersymmetry among terms with the same number of fields. This is different from IKKT model where the action is balanced between $Tr_{N\times N}[A_\mu, A_\nu]^2$ and $Tr_{N\times N}\bar{\psi}\gamma^\mu[A_\mu, \psi]$ under supersymmetry since the transformation for the fermion contains two bosonic fields. We expect that, by integrating some of the fields, the supersymmetry structure of IKKT model may be reproduced from this $osp(1|32, R)$ model.

The action is also invariant under $U(N)$ symmetry

$$M \to (1_{33\times 33} \otimes U)\ M\ (1_{33\times 33} \otimes U)^{-1}, \tag{4.1.13}$$

where U is an element of $U(N)$ group. All the $osp(1|32, R)$ fields must be transformed simultaneously. The symmetry of our model is therefore a direct product of these two Lie groups $OSp(1|32, R) \times U(N)$.

Another symmetry of the model is a trivial shift of the supermatrix M:

$$M_P{}^Q \to M_P{}^Q + c_P{}^Q 1_{N\times N}. \tag{4.1.14}$$

This shift contains both bosonic and fermionic inhomogeneous transformations. Some of the bosonic shifts are identified with space-time translations while the fermionic shifts form space-time supersymmetry together with the fermionic part of the homogeneous $osp(1|32, R)$ transformations. We write down the fermionic part explicitly for later convenience:

$$\delta_\epsilon^{(2)} m = 0, \quad \delta_\epsilon^{(2)} \psi = \epsilon. \tag{4.1.15}$$

Summarizing the three kinds of symmetries, the bosonic invariance of the model contains $Sp(32, R)$ rotation with 528 generators, constant shifts for each 528 $sp(32, R)$ fields and $U(N)$ gauge symmetry. The fermionic invariance, i.e. supersymmetry, is generated by homogeneous supersymmetry transformations (4.1.12) with real 32 components and inhomogeneous transformations (4.1.15) with the same number of components.

We then study the algebraic structures of these symmetries . The commutation relations among the homogeneous supersymmetries (4.1.12) are, of course, written by $Sp(32, R)$ rotations:

$$[\delta_\chi^{(1)}, \delta_\epsilon^{(1)}]m = i[(\chi\bar{\epsilon} - \epsilon\bar{\chi}), m], \tag{4.1.16}$$

$$[\delta_\chi^{(1)}, \delta_\epsilon^{(1)}]\psi = i(\chi\bar{\epsilon} - \epsilon\bar{\chi})\psi. \tag{4.1.17}$$

$h = i(\chi\bar{\epsilon} - \epsilon\bar{\chi})$ is an element of $sp(32, R)$ and can be expanded as

$$h = h_A\Gamma^A + \frac{1}{2!}h_{A_1A_2}\Gamma^{A_1A_2} + \frac{1}{5!}h_{A_1\cdots A_5}\Gamma^{A_1\cdots A_5}, \tag{4.1.18}$$

where

$$h_A = \frac{1}{32}tr(h\Gamma_A), \quad h_{A_1A_2} = -\frac{1}{32}tr(h\Gamma_{A_1A_2}), \quad h_{A_1\cdots A_5} = \frac{1}{32}tr(h\Gamma_{A_1\cdots A_5}). \tag{4.1.19}$$

It has the same algebraic structure as the 11-dimensional space-time supersymmetry with central charges of rank 2 and 5. But this algebra itself can no longer be interpreted as space-time supersymmetry since transformations generated by Γ^A are not the translation of space-time. The situation is the same as in IKKT model. If we interpret eigenvalues of some bosonic variables as our space-time coordinates, space-time translation should be identified with the constant shift of bosonic fields. A difference is that, in IKKT model, this type of commutation relation vanishes up to a field dependent $U(N)$ gauge transformation while here we have $sp(32, R)$ rotations.

Commutation relations between the homogeneous and inhomogeneous supersymmetry transformations are, on the other hand, given by

$$[\delta_\chi^{(1)}, \delta_\epsilon^{(2)}]m = -i(\chi\bar{\epsilon} - \epsilon\bar{\chi}), \quad [\delta_\chi^{(1)}, \delta_\epsilon^{(2)}]\psi = 0, \tag{4.1.20}$$

and generate a constant shift of bosonic fields. Commutators between inhomogeneous transformations trivially vanish.

By taking linear combinations as

$$\begin{aligned} \tilde{\delta}^{(1)} &= \delta^{(1)} + \delta^{(2)}, \\ \tilde{\delta}^{(2)} &= i(\delta^{(1)} - \delta^{(2)}), \end{aligned} \tag{4.1.21}$$

we obtain an enhanced 'space-time' supersymmetry algebra

$$\begin{aligned} [\tilde{\delta}_\chi^{(i)}, \tilde{\delta}_\epsilon^{(j)}]m &= -2i(\chi\bar{\epsilon} - \epsilon\bar{\chi})\delta_{ij}, \\ [\tilde{\delta}_\chi^{(i)}, \tilde{\delta}_\epsilon^{(j)}]\psi &= 0, \end{aligned} \tag{4.1.22}$$

up to $sp(32, R)$ rotations. As far as the supersymmetry algebra is concerned, $sp(32, R)$ transformations are more appropriately interpreted as a kind of gauge symmetries.

4.1.3 Reduction to $d = 10$

So far we have studied the model from 11-dimensional point of view. In this subsection, we investigate it from 10-dimensional point of view by specializing the 10th direction. For this purpose, we first introduce the following new variables

$$\begin{aligned} &W = u_\sharp, \quad A_\mu^{(\pm)} = u_\mu \pm u_{\mu\sharp}, \quad C_{\mu_1\mu_2} = u_{\mu_1\mu_2}, \\ &H_{\mu_1\cdots\mu_4} = u_{\mu_1\cdots\mu_4\sharp}, \quad I^{(\pm)}_{\mu_1\cdots\mu_5} = \frac{1}{2}(u_{\mu_1\cdots\mu_5} \pm \tilde{u}_{\mu_1\cdots\mu_5}). \end{aligned} \tag{4.1.23}$$

Here we use the indices $\mu_1, \mu_2, \cdots$ running from 0 to 9. $\sharp$ denotes the 10th direction. The quantity $\tilde{u}_{\mu_1\cdots\mu_5}$ denotes the dual of $u_{\mu_1\cdots\mu_5}$:

$$\tilde{u}_{\mu_1\cdots\mu_5} = \frac{-1}{5!} u_{\mu_6\cdots\mu_{10}} \epsilon^{\mu_1\cdots\mu_{10}\sharp}. \tag{4.1.24}$$

$I^{(+)}_{\mu_1\cdots\mu_5}$ and $I^{(-)}_{\mu_1\cdots\mu_5}$ are self-dual and anti-self-dual respectively:

$$I^{(\pm)}_{\mu_1\cdots\mu_5} = \pm\tilde{I}^{(\pm)}_{\mu_1\cdots\mu_5}. \tag{4.1.25}$$

Looking at these fields, we have two set of fields $A^{(\pm)}_\mu$ that can be identified with A_μ field in IKKT model. This is in accord with the doubling of fermions. These doublings cannot be avoided since we start from a 11-dimensional model. It is now convenient to define an even rank bosonic field and two odd rank fields $m_o{}^{(\pm)}$ by

$$\begin{aligned} m_e &= W\Gamma^\sharp + \frac{1}{2}C_{\mu\nu}\Gamma^{\mu\nu} + \frac{1}{4!}H_{\mu_1\cdots\mu_4}\Gamma^{\mu_1\cdots\mu_4}, \\ m_o^{(+)} &= (\frac{1}{2}A^{(+)}_\mu\Gamma^\mu + \frac{1}{5!}I^{(+)}_{\mu_1\cdots\mu_5}\Gamma^{\mu_1\cdots\mu_5})(1+\Gamma^\sharp), \\ m_o^{(-)} &= (\frac{1}{2}A^{(-)}_\mu\Gamma^\mu + \frac{1}{5!}I^{(-)}_{\mu_1\cdots\mu_5}\Gamma^{\mu_1\cdots\mu_5})(1-\Gamma^\sharp). \end{aligned} \tag{4.1.26}$$

Fermions are also decomposed into left and right handed chiralities

$$\psi_L = \frac{1+\Gamma^\sharp}{2}\psi, \qquad \psi_R = \frac{1-\Gamma^\sharp}{2}\psi. \tag{4.1.27}$$

Here we note the following useful identities:

$$m_o^{(+)}\psi_R = m_o^{(-)}\psi_L = 0, \quad \bar{\psi}_R m_o^{(+)} = \bar{\psi}_L m_o^{(-)} = 0. \tag{4.1.28}$$

If we denote sets of the fields $m_e, m_o^{(\pm)}$ by $\mathcal{M}_e, \mathcal{M}_o^{(\pm)}$, we also have the relations

$$\begin{aligned} &\chi_L\bar{\epsilon}_L \in \mathcal{M}_o^{(-)},\ \chi_R\bar{\epsilon}_R \in \mathcal{M}_o^{(+)},\ \chi_L\bar{\epsilon}_R \in \mathcal{M}_e,\ \chi_R\bar{\epsilon}_L \in \mathcal{M}_e, \\ &[\mathcal{M}_e, \mathcal{M}_e] \in \mathcal{M}_e,\ [\mathcal{M}_e, \mathcal{M}_o^{(\pm)}] \in \mathcal{M}_o^{(\pm)},\ [\mathcal{M}_o^{(+)}, \mathcal{M}_o^{(-)}] \in \mathcal{M}_e, \\ &\mathcal{M}_o^{(+)}\mathcal{M}_o^{(+)} = 0,\ \mathcal{M}_o^{(-)}\mathcal{M}_o^{(-)} = 0. \end{aligned} \tag{4.1.29}$$

Then the action $I = I_b + I_f$ becomes

$$\begin{aligned} I_b &= \frac{-f_{abc}}{2g^2}tr_{32\times 32}[m_e^a m_e^b m_e^c + 3m_e^a m_o^{(+)b} m_o^{(-)c} + 3m_e^a m_o^{(-)b} m_o^{(+)c}], \\ I_f &= \frac{3if_{abc}}{2g^2}[2\bar{\psi}_L^a m_e^b \psi_R^c + \bar{\psi}_L^a m_o^{(+)b}\psi_L^c + \bar{\psi}_R^a m_o^{(-)b}\psi_R^c]. \end{aligned} \tag{4.1.30}$$

The structure is very simple. There are two sectors (ψ_L and $m_o^{(+)}$) and (ψ_R and $m_o^{(-)}$) which are coupled through m_e fields. We can then expect to obtain IKKT like model if we succeed in integrating one sector. The situation is unfortunately more complicated as we will see in the following discussions of supersymmetries. Here we write down the action in terms of 10-dimensional components for later purpose:

$$I_b = \frac{i}{g^2}Tr_{N\times N}(-96[A^{(+)}_{\mu_1}, A^{(-)}_{\mu_2}]C^{\mu_1\mu_2} - 96W[A^{(+)\mu}, A^{(-)}_\mu] + \frac{4}{5}W[I^{(+)}_{\mu_1\cdots\mu_5}, I^{(-)\mu_1\cdots\mu_5}]$$

$$
\begin{aligned}
&- \quad 4H_{\mu_1\cdots\mu_4}([A^{(+)}_{\mu_5}, I^{(-)\mu_1\cdots\mu_5}] - [A^{(-)}_{\mu_5}, I^{(+)\mu_1\cdots\mu_5}]) - 8C_{\mu_1\mu_2}[I^{(+)\mu_1}{}_{\mu_3\cdots\mu_6}, I^{(-)\mu_2\cdots\mu_6}] \\
&+ \quad \frac{8}{3}H^{\nu\lambda}{}_{\mu_1\mu_2}([I^{(+)}{}_{\nu\lambda\mu_3\mu_4\mu_5}, I^{(-)\mu_1\cdots\mu_5}] - [I^{(-)}{}_{\nu\lambda\mu_3\mu_4\mu_5}, I^{(+)\mu_1\cdots\mu_5}]) \\
&+ \quad 32[C^{\mu_1}{}_{\mu_2}, C_{\mu_1\mu_3}]C^{\mu_2\mu_3} - 16C_{\mu_1\mu_2}[H^{\mu_1}{}_{\mu_3\mu_4\mu_5}, H^{\mu_2\cdots\mu_5}] \qquad (4.1.31) \\
&+ \quad \frac{1}{27}H_{\mu_1\cdots\mu_4}[H^{\nu}{}_{\mu_5\cdots\mu_7}, H_{\nu\mu_8\mu_9\mu_{10}}]\epsilon^{\mu_1\cdots\mu_{10}\sharp}),
\end{aligned}
$$

$$
\begin{aligned}
I_f \;=\; & \frac{i}{g^2}Tr_{N\times N}(-3i(-\bar{\psi}_L[W,\psi_R] + \bar{\psi}_R[W,\psi_L]) \\
&- 3i(\bar{\psi}_L\Gamma^i[A^{(+)}_\mu, \psi_L] + \bar{\psi}_R\Gamma^\mu[A^{(-)}_\mu, \psi_R]) \\
&- \frac{3i}{2!}(\bar{\psi}_L\Gamma^{\mu_1\mu_2}[C_{\mu_1\mu_2}, \psi_R] + \bar{\psi}_R\Gamma^{\mu_1\mu_2}[C_{\mu_1\mu_2}, \psi_L]) \\
&- \frac{3i}{4!}(-\bar{\psi}_L\Gamma^{\mu_1\mu_2\mu_3\mu_4}[H_{\mu_1\mu_2\mu_3\mu_4}, \psi_R] + \bar{\psi}_R\Gamma^{\mu_1\mu_2\mu_3\mu_4}[H_{\mu_1\mu_2\mu_3\mu_4}, \psi_L]) \\
&- \frac{3i}{5!}(2\bar{\psi}_L\Gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5}[I^{(+)}{}_{\mu_1\mu_2\mu_3\mu_4\mu_5}, \psi_L] + 2\bar{\psi}_R\Gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5}[I^{(-)}{}_{\mu_1\mu_2\mu_3\mu_4\mu_5}, \psi_R])).
\end{aligned}
\tag{4.1.32}
$$

We then investigate the symmetry structures, especially the structures of supersymmetry, instead of explicitly integrating out some fields in order to see a possibility to induce IKKT model. We first perform chiral decomposition of both homogeneous and inhomogeneous supersymmetries. 64 supersymmetries are decomposed into 16 left(right)-handed homogeneous (inhomogeneous) supersymmetries:

$$
\delta^{(1)}_{\epsilon_{L,R}}, \quad \delta^{(2)}_{\epsilon_{L,R}}. \tag{4.1.33}
$$

Under the homogeneous supersymmetries, the fields transform as

$$
\begin{aligned}
\delta^{(1)}_\chi m_e &= i(\chi_L\bar{\psi}_R - \psi_L\bar{\chi}_R + \chi_R\bar{\psi}_L - \psi_R\bar{\chi}_L), \\
\delta^{(1)}_\chi m_o^{(+)} &= i(\chi_R\bar{\psi}_R - \psi_R\bar{\chi}_R), \qquad \delta^{(1)}_\chi\psi_R = -m_e\chi_R - m_o^{(+)}\chi_L, \\
\delta^{(1)}_\chi m_o^{(-)} &= i(\chi_L\bar{\psi}_L - \psi_L\bar{\chi}_L), \qquad \delta^{(1)}_\chi\psi_L = -m_e\chi_L - m_o^{(-)}\chi_R.
\end{aligned}
\tag{4.1.34}
$$

Here a natural pairing is

$$
m_o^{(+)} \leftrightarrow \psi_R, \qquad m_o^{(-)} \leftrightarrow \psi_L, \tag{4.1.35}
$$

which is different from the pairing which appears in the action (4.1.30). The inhomogeneous supersymmetry transformations are trivial

$$
\begin{aligned}
\delta^{(2)}_\epsilon\psi_{L(R)} &= \epsilon_{L(R)}, \\
\delta^{(2)}_\epsilon m &= 0.
\end{aligned}
\tag{4.1.36}
$$

The commutation relations between the homogeneous supersymmetry transformations are written in terms of even and odd fields as

$$
[\delta^{(1)}_\chi, \delta^{(1)}_\epsilon]m_e = i[(\chi_L\bar{\epsilon}_R - \epsilon_L\bar{\chi}_R + \chi_R\bar{\epsilon}_L - \epsilon_R\bar{\chi}_L), m_e]
$$

$$
\begin{aligned}
& +i[\chi_L\bar{\epsilon}_L - \epsilon_L\bar{\chi}_L, m_o^{(+)}] + i[\chi_R\bar{\epsilon}_R - \epsilon_R\bar{\chi}_R, m_o^{(-)}], \\
[\delta_\chi^{(1)}, \delta_\epsilon^{(1)}]m_o^{(+)} &= i[(\chi_R\bar{\epsilon}_R - \epsilon_R\bar{\chi}_R), m_\epsilon] + i(\chi_R\bar{\epsilon}_L - \epsilon_R\bar{\chi}_L)m_o^{(+)} - im_o^{(+)}(\chi_L\bar{\epsilon}_R - \epsilon_L\bar{\chi}_R), \\
[\delta_\chi^{(1)}, \delta_\epsilon^{(1)}]m_o^{(-)} &= i[(\chi_L\bar{\epsilon}_L - \epsilon_L\bar{\chi}_L), m_\epsilon] + +i(\chi_L\bar{\epsilon}_R - \epsilon_L\bar{\chi}_R)m_o^{(-)} - im_o^{(-)}(\chi_R\bar{\epsilon}_L - \epsilon_R\bar{\chi}_L).
\end{aligned}
\tag{4.1.37}
$$

The commutation relations between the homogeneous and inhomogeneous supersymmetry transformations are similarly written as

$$
\begin{aligned}
[\delta_\chi^{(1)}, \delta_\epsilon^{(2)}]m_\epsilon &= -i(\chi_L\bar{\epsilon}_R - \epsilon_L\bar{\chi}_R + \chi_R\bar{\epsilon}_L - \epsilon_R\bar{\chi}_L), \\
[\delta_\chi^{(1)}, \delta_\epsilon^{(2)}]m_o^{(+)} &= -i(\chi_R\bar{\epsilon}_R - \epsilon_R\bar{\chi}_R), \\
[\delta_\chi^{(1)}, \delta_\epsilon^{(2)}]m_o^{(-)} &= -i(\chi_L\bar{\epsilon}_L - \epsilon_L\bar{\chi}_L).
\end{aligned}
\tag{4.1.38}
$$

These commutation relations (4.1.38) show that constant shifts of the $+(-)$ fields are generated by the right(left)-handed supersymmetries. If we neglect the m_ϵ fields, the two sectors ($m_o^{(+)}$ and ψ_R) and ($m_o^{(-)}$ and ψ_L) are completely decoupled. If we can successfully integrate out m_ϵ, $m_o^{(-)}$ and ψ_L fields, we expect to obtain a IKKT-like matrix model. As we see from (4.1.34), if we simply neglect these fields, the right-handed homogeneous supersymmetry transformations for the remaining fields $(m_o^{(+)}, \psi_R)$ become

$$
\delta_{\chi_R}^{(1)} m_o^{(+)} = i(\chi_R\bar{\psi}_R - \psi_R\bar{\chi}_R), \qquad \delta_{\chi_R}^{(1)}\psi_R = -\langle m_\epsilon\rangle\chi_R. \tag{4.1.39}
$$

Here $\langle m_\epsilon\rangle$ should be understood as the vacuum expectation value expressed in terms of $m_o^{(+)}$. Hence if $C_{\mu\nu}$ field in $\langle m_\epsilon\rangle$ is replaced by $[A_\mu^{(+)}, A_\nu^{(+)}]$, the transformation law can be identified with the homogeneous supersymmetry in IKKT model.

Let us look at these transformations explicitly for the rank 1 field $A_\mu^{(\pm)}$. Under the homogeneous supersymmetry transformation, they transform as

$$
\delta_\chi^{(1)} A_\mu^{(+)} = \frac{i}{8}\bar{\chi}_R\Gamma_\mu\psi_R, \qquad \delta_\chi^{(1)} A_\mu^{(-)} = \frac{i}{8}\bar{\chi}_L\Gamma_\mu\psi_L. \tag{4.1.40}
$$

We next consider the commutation relations among supersymmetry acting on $A_\mu^{(\pm)}$ fields. Extracting the specific chirality of the supersymmetry parameters, the commutators become

$$
\begin{aligned}
&[\delta_{\chi_R}^{(1)}, \delta_{\epsilon_R}^{(2)}]A_\mu^{(-)} = 0, \qquad && [\delta_{\chi_R}^{(1)}, \delta_{\epsilon_R}^{(2)}]A_\mu^{(+)} = \frac{i}{8}\bar{\epsilon}_R\Gamma_\mu\chi_R, \\
&[\delta_{\chi_L}^{(1)}, \delta_{\epsilon_L}^{(2)}]A_\mu^{(-)} = \frac{i}{8}\bar{\epsilon}_L\Gamma_\mu\chi_L, && [\delta_{\chi_L}^{(1)}, \delta_{\epsilon_L}^{(2)}]A_\mu^{(+)} = 0.
\end{aligned}
\tag{4.1.41}
$$

This is consistent with the above identification of pairs: $A_\mu^{(-)}$ ($A_\mu^{(+)}$) field is paired with the left (right) chirality. Commutators of two supersymmetry parameters with different chiralities vanish when they act on $A_\mu^{(\pm)}$:

$$
[\delta_{\chi_L}^{(1)}, \delta_{\epsilon_R}^{(2)}]A_\mu^{(\pm)} = [\delta_{\chi_R}^{(1)}, \delta_{\epsilon_L}^{(2)}]A_\mu^{(\pm)} = 0. \tag{4.1.42}
$$

We then look at the commutation relations between homogeneous supersymmetries. We are interested in which generators of $sp(32, R)$ rotations appear in the commutator. The commutators between the same chirality

$$[\delta^{(1)}_{\chi_R}, \delta^{(1)}_{\epsilon_R}]A^{(+)}_\mu = \frac{i}{8}(\bar{\chi}_R[m, \Gamma_\mu]\epsilon_R) \tag{4.1.43}$$

survive only for the fields of even rank m_e (W, $C_{i_1 i_2}$ and $H_{i_1 \cdots i_4}$) in m in the r.h.s. Since these fields are integrated out at last, we do not mind the appearance. On the other hand, commutators between different chiralities

$$[\delta^{(1)}_{\chi_L}, \delta^{(1)}_{\epsilon_R}]A^{(+)}_\mu = \frac{i}{16}(\bar{\chi}_L m\Gamma_\mu\epsilon_R + \bar{\epsilon}_R\Gamma_\mu m\chi_L) \tag{4.1.44}$$

survive for odd rank fields, $A^{(+)}_\mu$ and $I^{(+)}_{\mu_1 \cdots \mu_5}$, and contains the field $A^{(+)}_\mu$ itself as

$$[\delta^{(1)}_{\chi_L}, \delta^{(1)}_{\epsilon_R}]A^{(+)}_\mu = \frac{-i}{8}\bar{\chi}_L A^{(+)}_\nu \Gamma_\mu{}^\nu \epsilon_R + \cdots \tag{4.1.45}$$

The r.h.s. is generated by $SO(9, 1)$ rotation. Hence, if we interpret the eigenvalue distribution of $A^{(+)}_\mu$ or $A^{(-)}_\mu$ as space-time extension, we need to perform $SO(9, 1)$ rotation to obtain the correct space-time supersymmetry simultaneously with supersymmetries. In this sense, $SO(9, 1)$ symmetry should be more appropriately considered as a kind of gauge symmetry.

The above symmetry arguments support our expectation that the $osp(1|32, R)$ matrix model becomes IKKT model after integrating out some fields. However, there are no terms in the action consistent with the pairing expected from the symmetry arguments (4.1.35). In the next subsection we discuss a possibility and also a difficulty to obtain the correct coupling between fermion and boson by integrating out unnecessary fields.

4.1.4 Integrating out m_e, $m_o^{(-)}$ and ψ_L fields

In order to show that the correct coupling between ψ_R and $A^{(+)}_\mu$ can be generated, we need to integrate out the unnecessary fields (m_e, $m_o^{(-)}$ and ψ_L). For this purpose, these unnecessary fields need to have quadratic terms which may be generated by giving vacuum expectation values to some fields. The action (4.1.32) indicates that, if W acquires a vacuum expectation value, quadratic terms do not appear for m_e fields. On the other hand, if $A^{(+)}_\mu$ acquires VEV such as the noncommutative $\hat{p}_\mu$, all the unnecessary fields can get quadratic terms. Hence in the following we expand $A^{(+)}_\mu$ fields around the noncommutative classical background

$$A^{(+)}_\mu = \hat{p}_\mu + a^{(+)}_\mu, \tag{4.1.46}$$

where $\hat{p}_\mu$ satisfy $[\hat{p}_\mu, \hat{p}_\mu] = i\beta_{\mu\nu}$ and each $\beta_{\mu\nu}$ is a c-number. Applying to the action the mapping rule from matrices to functions (briefly reviewed in the next section), we obtain the following action. In the following expression, all products are the so-called star products and $Tr_{N\times N}$ should be understood as an integral over noncommutative space. The quadratic terms are given by

$$\begin{aligned}
I_b^{(2)} &= \frac{1}{g^2}Tr_{N\times N}(-96(\partial_{\mu_1}A^{(-)}_{\mu_2})C^{\mu_1\mu_2} + 96(\partial_\mu W)A^{(-)\mu} + 4(\partial_{\mu_1}H_{\mu_2\cdots\mu_5})I^{(-)\mu_1\cdots\mu_5}, \\
I_f^{(2)} &= \frac{1}{g^2}Tr_{N\times N}(-3i\bar{\psi}_L\Gamma^\mu\partial_\mu\psi_L).
\end{aligned} \tag{4.1.47}$$

The cubic interaction terms are given by

$$\begin{aligned}
I_b^{(3)} &= \frac{i}{g^2}Tr_{N\times N}(-96[a^{(+)}_{\mu_1}, A^{(-)}_{\mu_2}]C^{\mu_1\mu_2} - 96W[a^{(+)\mu}, A^{(-)}_\mu] + \frac{4}{5}W[I^{(+)}_{\mu_1\cdots\mu_5}, I^{(-)\mu_1\cdots\mu_5}] \\
&+ 4([a^{(+)}_{\mu_1}, H_{\mu_2\cdots\mu_5}]I^{(-)\mu_1\cdots\mu_5} - [A^{(-)}_{\mu_1}, H_{\mu_2\cdots\mu_5}]I^{(+)\mu_1\cdots\mu_5}) - 8C_{\mu_1\mu_2}[I^{(+)\mu_1}{}_{\mu_3\cdots\mu_6}, I^{(-)\mu_2\cdots\mu_6}] \\
&+ \frac{8}{3}H^{\nu\lambda}{}_{\mu_1\mu_2}([I^{(+)}{}_{\nu\lambda\mu_3\mu_4\mu_5}, I^{(-)\mu_1\cdots\mu_5}] - [I^{(-)}{}_{\nu\lambda\mu_3\mu_4\mu_5}, I^{(+)\mu_1\cdots\mu_5}]) \\
&+ 32[C^{\mu_1}{}_{\mu_2}, C_{\mu_1\mu_3}]C^{\mu_2\mu_3} - 16C_{\mu_1\mu_2}[H^{\mu_1}{}_{\mu_3\mu_4\mu_5}, H^{\mu_2\cdots\mu_5}] \\
&+ \frac{1}{27}H_{\mu_1\cdots\mu_4}[H^{\nu}{}_{\mu_5\cdots\mu_7}, H_{\nu\mu_8\mu_9\mu_{10}}]\epsilon^{\mu_1\cdots\mu_{10}\sharp}),
\end{aligned} \tag{4.1.48}$$

$$\begin{aligned}
I_f^{(3)} &= \frac{i}{g^2}Tr_{N\times N}(-3i(-\bar{\psi}_L[W, \psi_R] + \bar{\psi}_R[W, \psi_L]) - 3i(\bar{\psi}_L\Gamma^\mu[a^{(+)}_\mu, \psi_L] + \bar{\psi}_R\Gamma^\mu[A^{(-)}_\mu, \psi_R]) \\
&- \frac{3i}{2!}(\bar{\psi}_L\Gamma^{\mu_1\mu_2}[C_{\mu_1\mu_2}, \psi_R] + \bar{\psi}_R\Gamma^{\mu_1\mu_2}[C_{\mu_1\mu_2}, \psi_L]) \\
&- \frac{3i}{4!}(-\bar{\psi}_L\Gamma^{\mu_1\mu_2\mu_3\mu_4}[H_{\mu_1\mu_2\mu_3\mu_4}, \psi_R] + \bar{\psi}_R\Gamma^{\mu_1\mu_2\mu_3\mu_4}[H_{\mu_1\mu_2\mu_3\mu_4}, \psi_L]) \\
&- \frac{3i}{5!}(2\bar{\psi}_L\Gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5}[I^{(+)}{}_{\mu_1\mu_2\mu_3\mu_4\mu_5}, \psi_L] + 2\bar{\psi}_R\Gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5}[I^{(-)}{}_{\mu_1\mu_2\mu_3\mu_4\mu_5}, \psi_R])).
\end{aligned} \tag{4.1.49}$$

Vertices

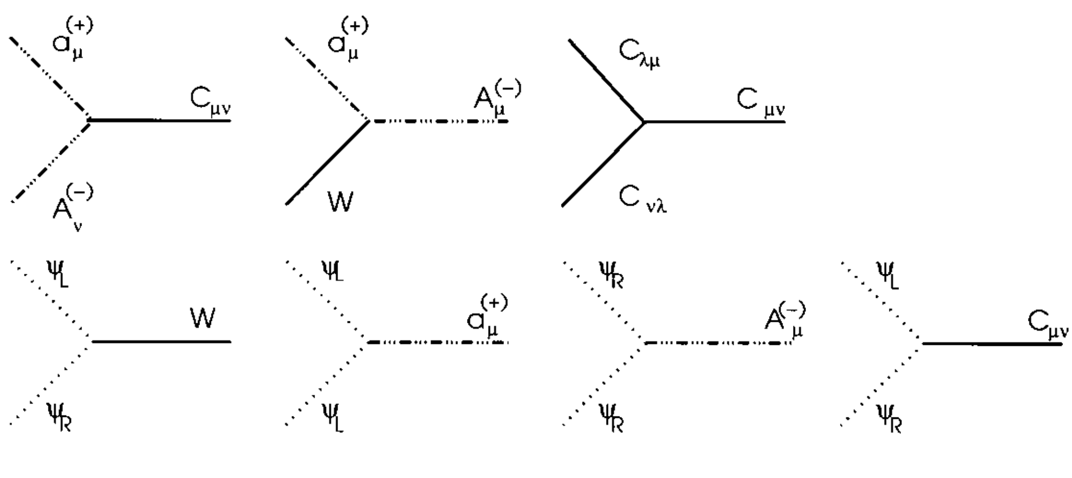

Figure 4.1: Typical vertices of the $osp(1|32, R)$ matrix model

We draw some typical vertices in fig. 4.1.

If we neglect the interaction terms first, an integration over $C_{\mu\nu}$ field gives a constraint on $A^{(-)}$ as

$$\partial_\mu A_\nu^{(-)} - \partial_\nu A_\mu^{(-)} = 0, \tag{4.1.50}$$

and we can solve it in terms of a scalar as $A_\mu^{(-)} = \partial_\mu \lambda$. Inserting this into the quadratic term, we have a propagator connecting two scalars, λ and W. Then we can integrate unnecessary fields $A_\mu^{(-)}, W$ $I^{(-)}, H$ and ψ_L. An issue here is whether we can generate the IKKT-like terms such as $Tr_{N\times N}[A_\mu^{(+)}, A_\nu^{(+)}]^2$ or $Tr_{N\times N}\bar{\psi}_R\Gamma^\mu[A_\mu^{(+)}, \psi_R]$. $U(N)$ gauge symmetry assures the existence of these terms if we can show that quadratic kinetic terms for $a_\mu^{(+)}$ and ψ_R, that is, $(\partial_\mu a_\nu^{(+)} - \partial_\nu a_\mu^{(+)})^2$ and $\bar{\psi}_R\Gamma^\mu\partial_\mu\psi_R$, are generated. First, the kinetic term for $a_\mu^{(+)}$ is easily generated by integrating out ψ_L as in fig. 4.2, since there is a vertex $\bar{\psi}_L\Gamma^\mu[a_\mu^{(+)}, \psi_L]$.

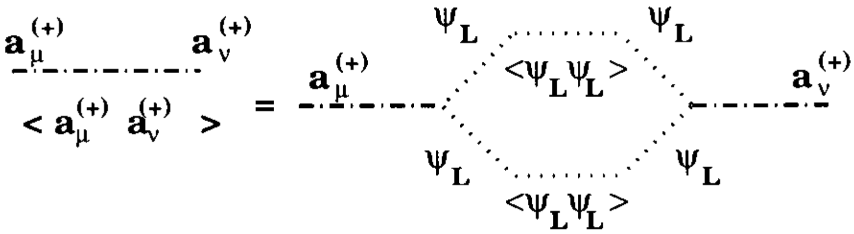

Figure 4.2: A propagator $\langle a_\mu^{(+)} a_\nu^{(+)}\rangle$ is induced by one-loop effect.

The kinetic term for ψ_R is more difficult to generate. As is seen from fig. 4.3, one way to generate such a term is to connect two $\bar{\psi}_L[W, \psi_R]$ vertices by propagators of ψ_L and W.

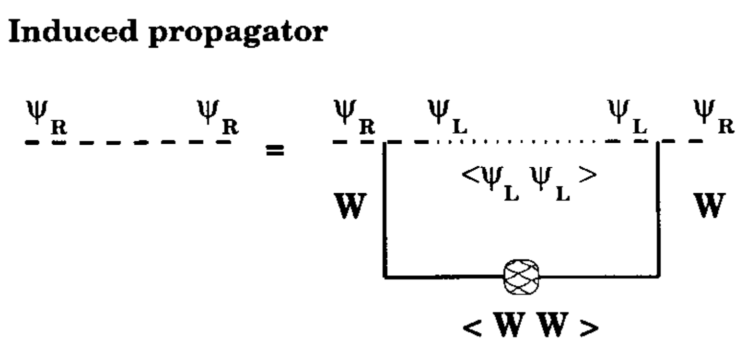

Figure 4.3: A propagator for ψ_R is induced if W can acquire a propagator.

Propagators for W and λ fields cannot be generated perturbatively as we prove in Appendix B. But since there is no symmetry prohibiting such terms it does not exclude a nonperturbative generation. The existence of the propagator connecting these two scalar fields rather indicate that both of these two propagators can be generated self-consistently. We do not discuss more details here, but it is probable that W acquires a propagator and the above mentioned kinetic term for ψ_R will be also generated. In this way, we expect that IKKT model is induced from $osp(1|32, R)$ model.

4.2 Diffeomorphism in noncommutative Yang-Mills

In this section, we first review how noncommutative Yang-Mills is obtained from matrix models and then investigate the special properties regarding the local gauge transformations in noncommutative background. Especially we study special types of gauge transformations which can be interpreted as local coordinate transformations.

First we give a brief review on a matrix model description of noncommutative field theories. The noncommutative background $\hat{x}$ satisfying

$$[\hat{x}^\mu, \hat{x}^\nu] = -i\theta^{\mu\nu} \tag{4.2.1}$$

with a c-number $\theta^{\mu\nu}$ is a classical solution of IKKT model. We assume the rank of $\theta^{\mu\nu}$ to be $\tilde{d}$ and define its inverse $\beta_{\mu\nu}$ in $\tilde{d}$-dimensional subspace. This expression is formal and only valid for infinite N. $\hat{x}^\mu$ satisfy the canonical commutation relations and they span the $\tilde{d}$-dimensional phase space. Therefore the momentum operators are proportional to the coordinates as

$$\hat{p}_\mu = \beta_{\mu\nu}\hat{x}^\nu. \tag{4.2.2}$$

The semiclassical correspondence shows that the volume of the phase space (measured in the coordinate space of x^μ) is $V = N(2\pi)^{\tilde{d}/2}\sqrt{\det\theta}$. We expand the bosonic matrices A^μ around $\hat{x}^\mu = \theta^{\mu\nu}\hat{p}_\nu$ as

$$A^\mu = \theta^{\mu\nu}(\hat{p}_\nu + \hat{a}_\nu). \tag{4.2.3}$$

If we assume that all fields can be expanded in terms of noncommutative plane wave $\exp(ik\cdot\hat{x})$, we obtain a map from a matrix

$$\hat{a} = \sum_k \tilde{a}(k)\exp(ik\cdot\hat{x}) \tag{4.2.4}$$

to a function

$$a(x) = \sum_k \tilde{a}(k)\exp(ik\cdot x) \tag{4.2.5}$$

in the $\tilde{d}$-dimensional noncommutative plane. By this construction, a product of matrices is mapped to the $\star$ product of functions

$$\begin{aligned} \hat{a}\hat{b} &\to a(x)\star b(x), \\ a(x)\star b(x) &\equiv \exp\left(\frac{\theta^{\mu\nu}}{2i}\frac{\partial^2}{\partial\xi^\mu\partial\eta^\nu}\right)a(x+\xi)b(x+\eta)|_{\xi=\eta=0} \end{aligned} \tag{4.2.6}$$

and the operation Tr over matrices can be exactly mapped onto the integration over functions as

$$Tr[\hat{a}] = \sqrt{\det\beta}(\frac{1}{2\pi})^{\frac{\tilde{d}}{2}} \int d^{\tilde{d}}x \; a(x). \tag{4.2.7}$$

The reduced model can be shown to be equivalent to noncommutative Yang-Mills by the following map from matrices onto functions

$$\begin{aligned} \hat{a} &\rightarrow a(x), \\ \hat{a}\hat{b} &\rightarrow a(x) \star b(x), \\ Tr &\rightarrow \sqrt{\det\beta}(\frac{1}{2\pi})^{\frac{\tilde{d}}{2}} \int d^{\tilde{d}}x. \end{aligned} \tag{4.2.8}$$

Applying the rule eq.(4.2.8), we can obtain $U(1)$ gauge theory on $\tilde{d}$-dimensional noncommutative space. (Noncommutative $U(m)$ gauge theory can be similarly obtained by expanding around $x^{\mu} \otimes 1_m$.)

The extension of $\hat{x}^{\mu}$ can be interpreted as the space-time and the space-time translation is realized by the following unitary operator:

$$\exp(i\hat{p}\cdot\epsilon)\hat{x}^{\mu}\exp(-i\hat{p}\cdot\epsilon) = \hat{x}^{\mu} + \epsilon^{\mu}. \tag{4.2.9}$$

It is amusing that the translation in the noncommutative space is realized by $U(N)$ gauge transformations in matrix models. This realization has been known as Parisi prescription in the old reduced models [26]. The local gauge symmetry of noncommutative Yang-Mills is originated in the invariance under $U(N)$ invariance of IKKT model

$$A_{\mu} \rightarrow UA_{\mu}U^{\dagger}. \tag{4.2.10}$$

Indeed, if we expand $U = \exp(i\hat{\lambda})$ and parameterize $\hat{\lambda}$ as

$$\hat{\lambda} = \sum_k \tilde{\lambda}(k)\exp(ik\cdot\hat{x}), \tag{4.2.11}$$

we find that the fluctuating field of A_{μ} around the fixed noncommutative background transforms as

$$\hat{a}_{\mu} \rightarrow \hat{a}_{\mu} + i[\hat{p}_{\mu}, \hat{\lambda}] - i[\hat{a}_{\mu}, \hat{\lambda}]. \tag{4.2.12}$$

After mapping the transformation onto functions, we have

$$\begin{aligned} a_{\alpha}(x) &\rightarrow a_{\alpha}(x) + \frac{\partial}{\partial x^{\alpha}}\lambda(x) - i[a_{\alpha}(x), \lambda(x)]_{\star}, \\ a_i &\rightarrow a_i - i[a_i(x), \lambda(x)]_{\star}, \\ \psi &\rightarrow \psi - i[\psi(x), \lambda(x)]_{\star}, \end{aligned} \tag{4.2.13}$$

where $1 \leq \alpha \leq \tilde{d}$ and $i > \tilde{d}$. If we take λ as in (4.2.9),

$$\hat{\lambda} = \epsilon^{\alpha}\hat{p}_{\alpha}, \tag{4.2.14}$$

the transformations (4.2.13) become translation in the noncommutative space up to a constant shift of the gauge field:

$$a_\alpha(x) \to a_\alpha(x) - \beta_{\alpha\beta}\epsilon^\beta + \epsilon^\beta \partial_\beta a_\alpha(x),$$
$$a_i(x) \to a_i(x) + \epsilon^\beta \partial_\beta a_i(x),$$
$$\psi(x) \to \psi(x) + \epsilon^\beta \partial_\beta \psi(x). \tag{4.2.15}$$

As the above example shows, local gauge symmetries in noncommutative gauge theories are very different from those in the ordinary gauge theories and even space-time translation is generated. Hence all gauge invariant operators are invariant under the space-time translation and should be constructed by integrating over space-time, which is reminiscent of the theory of gravity. We are, therefore, tempted to generalize the above discussion to local transformations. The reason why the gauge transformations in noncommutative space-time are much larger than those in commutative space is that gauge transformation parameters λ contain not only ordinary functions but differential operators in the semi-classical limit in the following sense. Functions in noncommutative space are expanded in terms of noncommutative plane waves as in (4.2.11). For finite N, the momenta of plane waves take values of

$$k_n = \sqrt{2\pi\beta} N^{-1/\tilde{d}}\ n, \tag{4.2.16}$$

where $n = 0, 1, \cdots, N^{2/\tilde{d}}$. This can be seen from the explicit construction of the plane waves in terms of 't Hooft matrices U and V satisfying $UV = \exp(2\pi i/N)VU$. The number of independent plane waves is N^2, which is the same as the number of degrees of freedom of a hermitian matrix. Since the natural cut off scale in the noncommutative plane (2.1.26) is given by $l_0 = \sqrt{2\pi\theta} = \sqrt{2\pi/\beta}$, the natural cut off of momenta should be $2\pi/l_0 = \sqrt{2\pi\beta}$. However, some plane waves with momenta (4.2.16) exceed this natural bound and they become very nonlocal objects since such high momentum plane waves generate translation in space with $l_0 N^{-1/\tilde{d}} n$. Hence only N out of N^2 plane waves whose momenta are smaller than $\sqrt{2\pi\beta}$ can be interpreted as ordinary plane waves in the semiclassical limit and others should be interpreted as differential operators that can generate translation in noncommutative direction in space-time. From a matrix model point of view, such nonlocal waves correspond to off-diagonal elements while local ones to diagonal elements.

Now let us consider the gauge transformations related to local coordinate transformations. A natural generalization of the global translation (4.2.14) will be

$$\lambda = \frac{1}{2}(\hat{p}_\alpha \hat{\epsilon}^\alpha + \hat{\epsilon}^\alpha \hat{p}_\alpha), \tag{4.2.17}$$

or, if we want to include both of the gauge transformations and local coordinate transformations in the semiclassical limit, we can expand λ as

$$\lambda = \hat{\lambda}_0 + \frac{1}{2}(\hat{p}_\alpha \hat{\epsilon}^\alpha + \hat{\epsilon}^\alpha \hat{p}_\alpha). \tag{4.2.18}$$

Similarly we expand bosonic matrices as

$$\tilde{A}_\mu = \beta_{\mu\nu} A^\nu = \hat{a}_\mu + \frac{1}{2}(\hat{p}_\alpha \hat{e}^\alpha_\mu + \hat{e}^\alpha_\mu \hat{p}_\alpha), \tag{4.2.19}$$

and assume that the field $\hat{e}^{\alpha}_{\mu}$ is close to δ^{α}_{μ}:

$$\hat{e}^{\alpha}_{\mu} = \delta^{\alpha}_{\mu} + h^{\alpha}_{\mu}. \tag{4.2.20}$$

This is a natural generalization of the expansion (4.2.3). Applying the unitary transformation generated by (4.2.17) to the bosonic field expanded as above, we have the following transformation law:

$$\begin{aligned} \delta h^{\alpha}_{\mu}(x) &= -e^{\beta}_{\mu}(x)\partial_{\beta}\epsilon^{\alpha} + \epsilon^{\beta}\partial_{\beta}h^{\alpha}_{\mu}, \\ \delta a_{\mu}(x) &= \epsilon^{\beta}\partial_{\beta}a_{\mu}(x). \end{aligned} \tag{4.2.21}$$

Here we have assumed that all of $\epsilon^{\alpha}, a_{\mu}$ and e^{α}_{μ} are slowly varying and dropped higher derivative terms. In addition to the transformations of the fields, we need to transform the background as

$$\delta\hat{p}_{\alpha} = \beta_{\alpha\beta}\hat{\epsilon}^{\beta} \tag{4.2.22}$$

or $\delta\hat{x}^{\mu} = \epsilon^{\mu}$ in terms of $\hat{x}^{\mu}$. Accordingly, the commutation relations between $\hat{p}_{\mu}$ change as

$$[\hat{p}_{\alpha} + \beta_{\alpha\beta}\hat{\epsilon}^{\beta}, \hat{p}_{\alpha'} + \beta_{\alpha'\beta'}\hat{\epsilon}^{\beta'}] = i(\beta_{\alpha\alpha'} + \beta_{\beta\alpha'}\partial_{\alpha}\epsilon^{\beta} + \beta_{\alpha\beta}\partial_{\alpha'}\epsilon^{\beta}). \tag{4.2.23}$$

Therefore the shift of the background can be interpreted as the transformation of $\beta_{\alpha\alpha'}$.

The transformation (4.2.21) indicates that local coordinate transformations can be realized by gauge transformations in noncommutative space. But as the transformations (4.2.21) show, the gauge field and the fermion field transform as a scalar and $h_{\mu}{}^{\alpha}$ as a vector field. In other words, the $SO(9,1)$ index has nothing to do with this coordinate transformation as it should be since $SU(N)$ transformations and $SO(9,1)$ transformations are independent from the beginning. IKKT model is not explicitly invariant under local $SO(9,1)$. Local Lorentz transformations might be realized in a complicated way in IKKT model and we expect that there is an extended model with obvious local Lorentz symmetry, which becomes IKKT model after gauge fixing. In the next section, we search for such models.

4.3 Gauged matrix models

In this section we investigate another type of matrix models with larger *local* symmetries. The model studied earlier has an extension of $SO(9,1)$ symmetry, that is, $OSp(1|32, R)$ symmetry. But this symmetry is decoupled from $U(N)$ gauge symmetry. As we have seen in the previous section, space-time is realized as eigenvalues of bosonic matrices and consequently some $U(N)$ symmetry is identified with space-time translation. Hence if local Lorentz symmetry exists it should be $U(N)$ dependent $SO(9,1)$ symmetry and we need to unify decoupled $SO(9,1)$ and $U(N)$ invariance in IKKT model.

Let us first try to gauge the global $SO(9,1)$ symmetry. A convenient way to write $SO(9,1)$ is to use γ-matrices. Defining

$$m = \Gamma^{\mu} A_{\mu}, \quad h = \frac{1}{2}\zeta_{\mu\nu}\Gamma^{\mu\nu}, \tag{4.3.1}$$

$SO(9,1)$ rotations of A_μ and ψ are given by

$$\delta m = [h, m], \quad \delta\psi = h\psi, \quad \delta\bar{\psi} = -\bar{\psi}h. \tag{4.3.2}$$

The rotation angle $\zeta_{\mu\nu}$ is a c-number. One way to gauge global symmetries in matrix models is to make transformation parameters $U(N)$ dependent. A big difference here from local gauge symmetries in ordinary commutative space-time is that $U(N)$-dependent matrices are generally not commutative while x-dependent local parameters are of course commutative. Therefore, the algebra does not close within the original transformations. In our case of $SO(9,1)$, since

$$[\hat{h}, \hat{h}'] = [\frac{1}{2}\hat{\zeta}_{\mu\nu}\Gamma^{\mu\nu}, \frac{1}{2}\hat{\zeta}_{\mu'\nu'}\Gamma^{\mu'\nu'}] = \frac{1}{8}([\Gamma^{\mu\nu}, \Gamma^{\mu'\nu'}]\{\hat{\zeta}_{\mu\nu}, \hat{\zeta}_{\mu'\nu'}\} + \{\Gamma^{\mu\nu}, \Gamma^{\mu'\nu'}\}[\hat{\zeta}_{\mu\nu}, \hat{\zeta}_{\mu'\nu'}]), \tag{4.3.3}$$

and the commutator between $\hat{\zeta}_{\mu\nu}$ does not vanish, we need to include transformations generated by the anti-commutators of $\Gamma^{\mu\nu}$, that is, 1 and $\Gamma^{\mu_1\mu_2\mu_3\mu_4}$. Repeating this procedure, the algebra finally closes in the γ-matrices with even rank, 1, $\Gamma^{\sharp}$, $\Gamma^{\mu\nu}$, $\Gamma^{\mu\nu\sharp}$, $\Gamma^{\mu_1\mu_2\mu_3\mu_4}$ and $\Gamma^{\mu_1\mu_2\mu_3\mu_4\sharp}$. There are 512 bosonic generators. The coefficients $\zeta_{\mu\nu}$ must be extended to complex matrices. Since these transformations can be restricted to chiral sectors of fermions, we can obtain closed gauged algebra acting on Weyl fermions of IKKT type. Generalizing this bosonic algebra by including supersymmetries, we obtain $gl(1|16, C)$ super Lie algebra. As far as the algebras are concerned, $gl(1|16, C)$ is a minimal gauged extension of $so(9,1)$. As for dynamical fields, if we start from a vector boson with rank 1 γ-matrix, $gl(16, C)$ bosonic transformations generate fields with other odd rank γ-matrices and we have to include $A_{\mu_1\mu_2\mu_3}$ and $A_{\mu_1\cdots\mu_5}$ in addition to A_μ. There are 256 bosonic fields. A model based on this $gl(1|16, C)$ super Lie algebra is an interesting possibility, but it turns out difficult to find an invariant action. In the following we instead investigate a model with local $osp(1|32, R)$ gauge symmetry. That is, we demand that the model should be invariant under $U(N)$ dependent $osp(1|32, R)$ symmetry. This model has larger symmetries and more fields than the $gl(16, C)$ model, but the invariant action can be easily constructed in terms of supermatrices as we show below. We must extend the chiral fermions to include both chiralities. We also have to extend the bosonic degrees of freedom m by including fields with all ranks in 11 dimensions.

In constructing an invariant action of the gauged matrix model, it is generally difficult to keep both the gauge symmetries and invariance under a constant shift of fields. If the fields transform as in (4.3.2) after gauging, the action of the type $Tr_{N\times N}(\bar{\psi}m\psi)$ or $Tr_{N\times N}(m^3)$ are invariant. However, the action such as $Tr_{N\times N}(\bar{\psi}\Gamma^{\mu}[A_\mu, \psi])$ is not invariant under gauge transformations and it is difficult to keep both invariances. In this paper we abandon the latter invariance and consider the action

$$I = \frac{1}{g^2} Tr_{N\times N} Str_{33\times 33}(M^3). \tag{4.3.4}$$

This action was also proposed by Smolin. We call it a gauged model because it is invariant under local $osp(1|32,R)$ symmetry, that is, a tensor product of two gauge symmetries $osp(1|32,R)$ and $u(N)$. Instead of this enhancement of the symmetries, this action is not invariant under a constant shift of field. This looks troubling since, as we have seen, commutators between the homogeneous and the inhomogeneous supersymmetries generate a space-time translation, a constant shift of bosonic field, and if we lose inhomogeneous translational invariance of bosons and fermions we may also lose space-time interpretation of supersymmetries. However, this problem can be resolved by identifying some generators of $osp(1|32,R)$ (or its extension $u(1|16,16)$) with space-time translation generators using the Wigner-Inönü contraction.

There are two ways to gauge the $osp(1|32,R)$ model. One way proposed by Smolin is to use $u(1|16,16)$ super Lie algebra, a complexification of $osp(1|32,R)$. He conjectured that this gauged model describes loop quantum gravity [7]. Another way to gauge is to use $gl(1|32,R)$ super Lie algebra, an analytic continuation of $u(1|16,16)$. We next see the definitions of these super Lie algebras and also see why they are gauged symmetries of the global $osp(1|32,R)$.

4.3.1 Definitions of $u(1|16,16)$ and $gl(1|32,R)$

An element M of $u(1|16,16)$ super Lie algebra satisfies

$$M^\dagger G + GM = 0 \ \text{ for } \ G = \begin{pmatrix} \Gamma^0 & 0 \\ 0 & i \end{pmatrix}. \tag{4.3.5}$$

The reality condition is not imposed in this case. The above definition restricts the 33×33 matrix form of M as

$$M = \begin{pmatrix} m & \psi \\ i\bar{\psi} & v \end{pmatrix}, \tag{4.3.6}$$

where v is pure imaginary, ψ is a general complex spinor and $\bar{\psi} = \psi^\dagger \Gamma^0$. The bosonic part m can be expanded in terms of 11-dimensional γ-matrices,

$$\begin{aligned} m &= u\mathbf{1} + u_{A_1}\Gamma^{A_1} + \frac{1}{2!}u_{A_1A_2}\Gamma^{A_1A_2} + \frac{1}{3!}u_{A_1A_2A_3}\Gamma^{A_1A_2A_3} \\ &+ \frac{1}{4!}u_{A_1\cdots A_4}\Gamma^{A_1\cdots A_4} + \frac{1}{5!}u_{A_1\cdots A_5}\Gamma^{A_1\cdots A_5}, \end{aligned} \tag{4.3.7}$$

where u_{A_1}, $u_{A_1A_2}$ and $u_{A_1\cdots A_5}$ are real, while u, $u_{A_1A_2A_3}$ and $u_{A_1\cdots A_4}$ are pure imaginary. Pure imaginary valued coefficients u, $u_{A_1A_2A_3}$ and $u_{A_1\cdots A_4}$ are new compared to

$osp(1|32, R)$. Fermions are also doubled since we do not impose the Majorana condition. This matrix can be decomposed into two matrices

$$M = H + A', \quad H = \begin{pmatrix} m_h & \psi_h \\ i\bar{\psi}_h & 0 \end{pmatrix}, \quad A' = \begin{pmatrix} m_a & i\psi_a \\ \bar{\psi}_a & iv \end{pmatrix}, \tag{4.3.8}$$

where

$$\begin{aligned} m_h &= u_{A_1}\Gamma^{A_1} + \frac{1}{2!}u_{A_1A_2}\Gamma^{A_1A_2} + \frac{1}{5!}u_{A_1\cdots A_5}\Gamma^{A_1\cdots A_5}, \\ m_a &= u + \frac{1}{3!}u_{A_1A_2A_3}\Gamma^{A_1A_2A_3} + \frac{1}{4!}u_{A_1\cdots A_4}\Gamma^{A_1\cdots A_4}, \end{aligned} \tag{4.3.9}$$

and ψ_h and ψ_a are real fermions. They satisfy the following relations

$${}^T HG + GH = 0, \quad {}^T A'G - GA' = 0. \tag{4.3.10}$$

The matrix H forms $osp(1|32, R)$ super Lie subalgebra of $u(1|16, 16)$ algebra but A' does not form an algebra by themselves. We denote the former set of matrices by $\mathcal{H}$ and the latter by $\mathcal{A}'$. Then the following commutation and anti-commutation structures are satisfied

$$\begin{aligned} &[\mathcal{H}, \mathcal{H}] \in \mathcal{H}, \ [\mathcal{H}, \mathcal{A}'] \in \mathcal{A}', \ [\mathcal{A}', \mathcal{A}'] \in \mathcal{H}, \\ &\{\mathcal{H}, \mathcal{H}\} \in \mathcal{A}', \ \{\mathcal{H}, \mathcal{A}'\} \in \mathcal{H}, \ \{\mathcal{A}', \mathcal{A}'\} \in \mathcal{A}'. \end{aligned} \tag{4.3.11}$$

We can see that A' is another representation of $osp(1|32, R)$.

The definition of $gl(1|32, R)$ super Lie algebra is simply given by the following form of 33×33 supermatrix

$$M = \begin{pmatrix} m & \psi \\ i\bar{\phi} & v \end{pmatrix}, \tag{4.3.12}$$

where all components are real. The boson m can be expanded similarly in terms of 11-dimensional γ-matrices as in (4.3.7) but all the coefficients $u, \cdots, u_{A_1\cdots A_5}$ are real. Two fermions ψ and ϕ are also real. This matrix is decomposed into two parts as

$$M = H + A, \tag{4.3.13}$$

where

$$\begin{aligned} H &= \begin{pmatrix} m_h & \psi_1 \\ i\bar{\psi}_1 & 0 \end{pmatrix}, \\ A &= \begin{pmatrix} m_a & \psi_2 \\ -i\bar{\psi}_2 & v \end{pmatrix}. \end{aligned} \tag{4.3.14}$$

Here we have defined ψ_i by

$$\psi = \psi_1 + \psi_2, \quad \phi = \psi_1 - \psi_2 \tag{4.3.15}$$

and m_h and m_a are given in (4.3.9) with real coefficients. H is again an element of $osp(1|32, R)$ generators and A is its representation.

These two super Lie algebras $u(1|16, 16)$ and $gl(1|32, R)$ are related as follows. A matrix $M = H + A'$ in $u(1|16, 16)$ is mapped to a matrix in $gl(1|32, R)$ by $N = H + A$ where $A = iA'$ and vice versa. Hence these two algebras are related by an analytic continuation.

We now promote each real element of matrices to an $N \times N$ hermitian matrix to make our model invariant under local $osp(1|32, R)$ symmetry. If we start from a set of $osp(1|32, R)$ matrices $\mathcal{H}$ and make a tensor product with $u(N)$, the algebra does not close within them because of the following relation:

$$\begin{aligned} [(\mathcal{H} \otimes \mathbf{H}), (\mathcal{H} \otimes \mathbf{H})] &= (\{\mathcal{H}, \mathcal{H}\} \otimes [\mathbf{H}, \mathbf{H}]) + ([\mathcal{H}, \mathcal{H}] \otimes \{\mathbf{H}, \mathbf{H}\}) \\ &= (\mathcal{A}' \otimes \mathbf{A}) + (\mathcal{H} \otimes \mathbf{H}). \end{aligned} \tag{4.3.16}$$

Here we have used (4.3.11) and denoted $\mathbf{H}$ and $\mathbf{A}$ as hermitian and anti-hermitian matrices. In order for the algebra to close, it is necessary to include $\mathcal{A}' \otimes \mathbf{A}$. From the following relation

$$\begin{aligned} [(\mathcal{A}' \otimes \mathbf{A}), (\mathcal{A}' \otimes \mathbf{A})] &= (\{\mathcal{A}', \mathcal{A}'\} \otimes [\mathbf{A}, \mathbf{A}]) + ([\mathcal{A}', \mathcal{A}'] \otimes \{\mathbf{A}, \mathbf{A}\}) \\ &= (\mathcal{A}' \otimes \mathbf{A}) + (\mathcal{H} \otimes \mathbf{H}), \end{aligned} \tag{4.3.17}$$

we can form a closed algebra by combining $(\mathcal{H} \otimes \mathbf{H})$ and $(\mathcal{A}' \otimes \mathbf{A})$ together. For $N = 1$ case, $\mathbf{H}$ and $\mathbf{A}$ are replaced by 1 and i respectively and it is nothing but $u(1|16, 16)$ algebra we discussed before. This is a reason why we need to enlarge $osp(1|32, R)$ to $u(1|16, 16)$ for gauging the $osp(1|32, R)$ symmetry.

Instead of promoting each element to a hermitian matrix, we can make a closed algebra by restricting them to real matrices. Since real matrices are closed under commutators and anti-commutations, it is clear that $(\mathcal{H} + \mathcal{A}) \otimes gl(N, R) = gl(1|32, R) \otimes gl(N, R)$ forms another closed algebra. In this case, we have to embed the space-time into real matrices instead of hermitian matrices.

4.3.2 Action and symmetries

The action we consider is

$$\begin{aligned} I &= \frac{1}{g^2} Tr_{N\times N} \sum_{Q,R=1}^{33} ((\sum_{p=1}^{32} M_p{}^Q M_Q{}^R M_R{}^p) - M_{33}{}^Q M_Q{}^R M_R{}^{33}) = \frac{1}{g^2} Tr_{N\times N}(Str_{33\times 33} M^3) \\ &= \frac{1}{g^2} \sum_{a,b,c=1}^{N^2} Str_{33\times 33}(M^a M^b M^c) Tr_{N\times N}(t^a t^b t^c). \end{aligned} \tag{4.3.18}$$

where $p = 1, \cdots, 32$ and $Q, R = 1, \cdots 33$. M is a supermatrix belonging to $u(1|16, 16)$ or $gl(1|32, R)$ super Lie algebra. Each component ${M_Q}^R$ of the 33×33 supermatrix M is promoted to an $N \times N$ matrix and can be expanded in terms of Gell-Mann matrices:

$$ {M_Q}^R = \sum_{a=1}^{N^2} t^a {(M^a)_Q}^R. \tag{4.3.19} $$

This action (4.3.18) is invariant under a tensor product of two gauge groups

$$ M \Rightarrow M + [u, M], \tag{4.3.20} $$

where

$$ u \in gl(1|32, R) \otimes gl(N, R) \text{ or } u(1|16, 16) \otimes u(N). \tag{4.3.21} $$

Hence the action is invariant under local (or gauged) $u(1|16, 16)$ or $gl(1|32, R)$ symmetry. That is, the $u(1|16, 16)$ symmetry and $u(N)$ symmetry (or $gl(1|32, R)$ and $gl(N, R)$) are coupled. Not only the bosonic but the fermionic symmetries are also gauged. In this sense this action is considered as a matrix regularization of 11-dimensional supergravity if we can successfully treat this model. In terms of the components of $M = \begin{pmatrix} m & \psi \\ i\bar{\phi} & v \end{pmatrix}$, the action becomes

$$ I = \frac{1}{g^2} Tr_{N \times N} (tr_{32 \times 32}(m^3) - 3i\bar{\phi} m \psi - 3i\bar{\phi}\psi v - v^3). \tag{4.3.22} $$

In both cases of $u(1|16, 16)$ model and $gl(1|32, R)$ model, there are $64 = 32 + 32$ (real) supercharges. The action is not invariant under the space-time translation which was identified with a constant shift of bosonic fields in the case of IKKT or $osp(1|32, R)$ model, and there are no inhomogeneous supersymmetry in this gauged model. To extract space-time translation, we need another interpretation different from the non-gauged model. Here we adopt the Wigner-Inönü contraction of the $SO(10, 1)$ symmetry and identify $SO(9, 1)$ rotation and space-time translation generators with the $SO(10, 1)$ generators. In other words, we zoom in around the north pole of a 10-dimensional sphere on which $SO(10, 1)$ rotations are generated by Γ^{AB}.

First let us consider a $gl(1|32, R)$-case. To perform the Wigner-Inönü contraction systematically, it is convenient to add another term to the action

$$ I = \frac{1}{3} Tr_{N \times N} Str_{33 \times 33}(M^3) - R^2 Tr_{N \times N} Str_{33 \times 33} M. \tag{4.3.23} $$

This action has a classical solution

$$ \langle M \rangle = \begin{pmatrix} R\Gamma^\sharp \otimes \mathbf{1}_{N \times N} & 0 \\ 0 & R \otimes \mathbf{1}_{N \times N} \end{pmatrix}. \tag{4.3.24} $$

We take a large R limit, which is equivalent to zooming in around the north pole.

Similarly in the case of $u(1|16,16)$, we need to consider a quintic action $I_{U(1|16,16)} = \frac{1}{5}Str(M^5) - R^4 Str M$ in order to have the classical solution,

$$\langle M\rangle = \begin{pmatrix} R\Gamma^\sharp \otimes \mathbf{1}_{N\times N} & 0 \\ 0 & iR\otimes \mathbf{1}_{N\times N} \end{pmatrix}. \tag{4.3.25}$$

We focus on the $gl(1|32,R)$ type in this section, but the following discussions of the Wigner-Inönü contraction are essentially the same as in the $u(1|16,16)$ case.

We expand the matrix M around the above classical solution $\langle M\rangle$:

$$M = \begin{pmatrix} m & \psi \\ i\bar{\phi} & v \end{pmatrix} = \langle M\rangle + \tilde{M} = \begin{pmatrix} R\Gamma^\sharp & 0 \\ 0 & R \end{pmatrix} + \begin{pmatrix} \tilde{m} & \psi \\ i\bar{\phi} & \tilde{v} \end{pmatrix}. \tag{4.3.26}$$

The action becomes

$$I = R(tr_{32\times 32}(\tilde{m}^2\Gamma^\sharp) - \tilde{v}^2 - i\bar{\phi}(1+\Gamma^\sharp)\psi) + \frac{1}{3}tr_{32\times 32}(\tilde{m}^3) - \frac{\tilde{v}^3}{3} - i(\bar{\phi}\tilde{m}\psi + \tilde{v}\bar{\phi}\psi) \tag{4.3.27}$$

up to a constant. In the next subsection, we investigate the model in the large R limit.

4.3.3 Wigner-Inönü contraction and supersymmetry

In the background proportional to $\Gamma^\sharp$, it is convenient to decompose bosonic fields into even and odd rank fields with respect to 10-dimensional γ-matrices:

$$\tilde{m} = m_e + m_o, \tag{4.3.28}$$

where m_e is given by

$$\begin{aligned} m_e &= Z\mathbf{1} + W\Gamma^\sharp + \frac{1}{2}(C_{\mu_1\mu_2}\Gamma^{\mu_1\mu_2} + D_{\mu_1\mu_2}\Gamma^{\mu_1\mu_2\sharp}) \\ &+ \frac{1}{4!}(G_{\mu_1\cdots\mu_4}\Gamma^{\mu_1\cdots\mu_4} + H_{\mu_1\cdots\mu_4}\Gamma^{\mu_1\cdots\mu_4\sharp}), \end{aligned} \tag{4.3.29}$$

and m_o by

$$\begin{aligned} m_o &= \frac{1}{2}(A^{(+)}_\mu\Gamma^\mu(1+\Gamma^\sharp) + A^{(-)}_\mu\Gamma^\mu(1-\Gamma^\sharp)) \\ &+ \frac{1}{2\times 3!}(E^{(+)}_{\mu_1\mu_2\mu_3}\Gamma^{\mu_1\mu_2\mu_3}(1+\Gamma^\sharp) + E^{(-)}_{\mu_1\mu_2\mu_3}\Gamma^{\mu_1\mu_2\mu_3}(1-\Gamma^\sharp)) \\ &+ \frac{1}{5!}(I^{(+)}_{\mu_1\cdots\mu_5}\Gamma^{\mu_1\cdots\mu_5}(1+\Gamma^\sharp) + I^{(-)}_{\mu_1\cdots\mu_5}\Gamma^{\mu_1\cdots\mu_5}(1-\Gamma^\sharp)). \end{aligned} \tag{4.3.30}$$

We further decompose the m_o into $m_o^{(\pm)}$ according to the $(\pm)$ in the above decomposition: $m_o = m_o^{(+)} + m_o^{(-)}$. Fermionic fields are also decomposed according to their chiralities:

$\psi_{L,R} = \frac{1\pm\Gamma^\sharp}{2}\psi$. The action then becomes

$$\begin{aligned} I &= R(tr_{32\times 32}(m_e^2\Gamma^\sharp) - \tilde{v}^2 - 2i\bar{\phi}_R\psi_L) + tr_{32\times 32}(\frac{1}{3}m_e^3 + m_e m_o^2) \\ &- i(\bar{\phi}_R(m_e+\tilde{v})\psi_L + \bar{\phi}_L(m_e+\tilde{v})\psi_R + \bar{\phi}_L m_o\psi_L + \bar{\phi}_R m_o\psi_R) - \frac{1}{3}\tilde{v}^3. \quad (4.3.31) \end{aligned}$$

Since the quadratic term is proportional to R, we first rescale $\tilde{v}, m_e, \phi_R$ and ψ_L as $R^{-1/2}$. Then, in order to make terms containing other fields such as $tr_{32\times 32}(m_e m_o^2)$ finite in the large R limit, we have to rescale the other fields as $R^{1/4}$. The rescalings are summarized as

$$\begin{aligned} & m = R\Gamma^\sharp + \tilde{m} = R\Gamma^\sharp + R^{-\frac{1}{2}}m'_e + R^{\frac{1}{4}}m'_o, \quad v = R + \tilde{v} = R + R^{-\frac{1}{2}}v', \\ & \psi = \psi_L + \psi_R = R^{-\frac{1}{2}}\psi'_L + R^{\frac{1}{4}}\psi'_R, \quad \bar{\phi} = \bar{\phi}_L + \bar{\phi}_R = R^{\frac{1}{4}}\bar{\phi}'_L + R^{-\frac{1}{2}}\bar{\phi}'_R. \quad (4.3.32) \end{aligned}$$

In terms of these rescaled fields, we can rewrite the action, by dropping terms with a negative power of R, as

$$\begin{aligned} I = & \; (tr_{32\times 32}(m_e'^2\Gamma^\sharp) - v'^2 + tr_{32\times 32}(m'_e m_o'^2)) \\ & + i(-2\bar{\phi}'_R\psi'_L + \bar{\phi}'_L(m'_e + v')\psi'_R + \bar{\psi}'_L m'_o\psi'_L + \bar{\phi}'_R m'_o\psi'_R). \quad (4.3.33) \end{aligned}$$

Since only the fields v', m'_e, ϕ'_R and ψ'_L have quadratic terms, we may integrate them and obtain an effective action for the other fields. Before performing the integration, let us first look at the supersymmetry structure in order to see how we can obtain space-time supersymmetry in our model. We can also see the above scalings are consistent with supersymmetries.

The 10-dimensional space-time translation around the north pole is generated by $\Gamma_{\mu\sharp}$. Since R is interpreted as the radius of S^{10}, space-time translation generator should be identified with $P_\mu = \frac{1}{R}\Gamma_{\mu\sharp}$. On the other hand, a commutator of two supercharges $Q_{\chi\epsilon} = \begin{pmatrix} 0 & \chi \\ i\bar{\epsilon} & 0 \end{pmatrix}$ and $Q_{\rho\eta} = \begin{pmatrix} 0 & \rho \\ i\bar{\eta} & 0 \end{pmatrix}$ becomes

$$[Q_{\chi\epsilon}, Q_{\rho\eta}] = \begin{pmatrix} i(\chi\bar{\eta} - \rho\bar{\epsilon}) & 0 \\ 0 & i(\bar{\epsilon}\rho - \bar{\eta}\chi) \end{pmatrix}, \quad (4.3.34)$$

which contains the translation P_μ besides other $gl(32, R)$ bosonic generators. In this way, the homogeneous supersymmetry in gauged models is considered as 10-dimensional space-time supersymmetry.

In addition to scaling the fields as above, we need to scale gauge parameters of $gl(1|32, R)$. Writing the gauge parameter h by

$$h = \begin{pmatrix} a & \chi \\ i\bar{\epsilon} & b \end{pmatrix}, \quad (4.3.35)$$

the field M is transformed as

$$\begin{aligned}
\delta\tilde{M} &= [h, M] = [h, \langle M\rangle + \tilde{M}] \\
&= \begin{pmatrix} [a, \tilde{m} + R\Gamma^\sharp] + i(\chi\bar{\phi} - \psi\bar{\epsilon}) & -(\tilde{m} + R\Gamma^\sharp)\chi + a\psi - b\psi + \chi\tilde{v} \\ i\bar{\epsilon}(\tilde{m} + R\Gamma^\sharp) - i(\bar{\phi}a + \tilde{v}\bar{\epsilon}) + ib\bar{\phi} & i(\bar{\epsilon}\psi - \bar{\phi}\chi) + [b, \tilde{v}] \end{pmatrix},
\end{aligned} \tag{4.3.36}$$

where inhomogeneous terms come from $[h, \langle M\rangle]$. The inhomogeneous term for m_o should survive after taking the large R limit, since space-time translations for $A_\mu^{(\pm)}$ are included there. Decomposing the bosonic gauge parameter a into a_o and a_e similarly to (4.3.28), the inhomogeneous part of δm_o is given by $\delta m_o = [a_o, R\Gamma^\sharp]$. Since we have rescaled $m_o = R^{1/4}m'_o$, we should rescale a_o as $R^{-3/4}$ so as to make this inhomogeneous term finite in the large R limit. On the other hand, $SO(9,1)$ rotation generated by $\Gamma_{\mu\nu}$ is included in a_e and it transforms even (odd) rank fields into themselves. The gauge parameter a_e is, therefore, not necessary to be rescaled. Similar arguments can be applied to supersymmetries and we finally obtain the following rescalings

$$h = \begin{pmatrix} a'_e + R^{-\frac{3}{4}}a'_o & \chi'_L + R^{-\frac{3}{4}}\chi'_R \\ i(R^{-\frac{3}{4}}\bar{\epsilon}'_L + \bar{\epsilon}'_R) & b' \end{pmatrix}. \tag{4.3.37}$$

Under this gauge transformation, each field transforms in the large R limit as

$$\begin{aligned}
\delta m'_o &= \underline{[a_o, \Gamma^\sharp]} + [a'_e, m'_o] + i(\chi'_L\bar{\phi}'_L - \psi'_R\bar{\epsilon}'_R), && (4.3.38)\\
\delta\psi'_R &= \underline{2\chi'_R} + (a'_e\psi'_R - b\psi'_R - (m'_o\chi'_L)), && (4.3.39)\\
\delta\bar{\phi}'_L &= \underline{-2\bar{\epsilon}'_L} + ((\bar{\epsilon}'_R m'_o) + b'\bar{\phi}'_L - \bar{\phi}'_L a'_e), && (4.3.40)\\
\delta m'_e &= [a_e, m'_e] + [a_o, m'_o] + i(\chi'_L\bar{\epsilon}'_R + \chi'_R\bar{\epsilon}'_L) - i(\psi'_L\bar{\epsilon}'_R + \chi'_R\bar{\epsilon}'_L), && (4.3.41)\\
\delta\psi'_L &= -(m'_o\chi'_R + m'_e\chi_L) + (a_o\psi'_R + a_e\psi'_L) - b'\psi'_L + v'\chi'_L, && (4.3.42)\\
\delta\bar{\phi}'_R &= (\bar{\epsilon}'_R m'_e + \bar{\epsilon}'_L m'_o) + b'\bar{\phi}'_R - (\bar{\phi}'_L a'_o + \bar{\phi}'_R a'_e) - v'\bar{\epsilon}'_R, && (4.3.43)\\
\delta v' &= i(\bar{\epsilon}'_R\psi'_L + \bar{\epsilon}'_L m'_o) - i(\bar{\phi}'_R\chi'_L + \bar{\phi}'_L\chi'_R) + [b', v']. && (4.3.44)
\end{aligned}$$

The underlined terms are inhomogeneous transformations. The other transformations are homogeneous and linear in fields. As we have seen in the action, the fields that do not receive inhomogeneous transformations, that is, m'_e, v', ψ'_L and ϕ'_R contain quadratic terms and can be integrated out by Gaussian integration. An important point is that the transformations of the first three fields m'_o, ψ'_R and ϕ'_L do not include the other fields in the right hand side. This means that these transformation rules are not changed after integrating out the other fields, m'_e, ψ'_L, ϕ'_R and v'.

Now let us obtain the effective action by integrating out v', m'_e, ψ'_L and ϕ'_R. The integration can be easily done and the effective action vanishes!

$$W = -\frac{1}{4}tr_{32\times 32}(\Gamma^\sharp\{m_o'^2 + i(\psi'_R\bar{\phi}'_L)\}^2) - \frac{1}{4}(\bar{\phi}'_L\psi_R)^2 + \frac{i}{2}(\bar{\phi}'_L m_o'^2\psi'_R) = 0. \tag{4.3.45}$$

Here we have used

$$tr_{32\times 32}(\Gamma^{\sharp} m'^{\,4}_{o}) = 0. \tag{4.3.46}$$

The reason for the vanishment of the effective action can be understood from the symmetry point of view. Since the transformations for m'_o, ψ'_R and ϕ'_L do not include the other integrated fields, they are not changed after integration. Therefore the effective action must be invariant under the same transformations, which include $U(N)$ dependent shifts of them, not restricted to constant shifts. The only action invariant under such transformations is a trivial one. In this sense, this model is a topological model of fields m'_o, ψ'_R and ϕ'_L.

Although the action vanishes, we go on to investigate the supersymmetry structures. If we decompose the boson fields m'_o into $m'^{(\pm)}_o$, we obtain the following transformations

$$\begin{aligned} &\delta m'^{(+)}_o = -i(\psi'_R \bar{\epsilon}_R), \quad \delta m'^{(-)}_o = i(\chi'_L \bar{\phi}'_L), \\ &\delta\psi'_R = 2\chi'_R - (m'^{(+)}_o \chi'_L), \quad \delta\bar{\phi}'_L = -2\bar{\epsilon}'_L + (\bar{\epsilon}'_R m'^{(-)}_o), \end{aligned} \tag{4.3.47}$$

and we can see that left and right handed fields are decoupled. We have two pairs of fields, ($m_o^{(+)}$ and ψ_R) and ($m_o^{(-)}$ and ϕ_L). This pairing is the same as that of $osp(1|32,R)$ model. In this case, since the effective action vanishes we do not have the problem of compatibility with the pairing in the action. More explicitly in terms of $A_\mu{}^{(\pm)}$ fields, the homogeneous supersymmetry transformations become

$$\delta^{(1)}_{\chi_L} A'^{(-)}_i = -\frac{i}{16}\bar{\phi}'_L \Gamma_i \chi_L, \qquad \delta^{(1)}_{\epsilon_R} A'^{(+)}_i = \frac{i}{16}\bar{\epsilon}_R \Gamma_i \psi'_R, \tag{4.3.48}$$

which are the same as those of IKKT model. On the other hand, transformations for fermions are different from those of IKKT model. Instead of the commutators $[A^{(\pm)}_\mu, A^{(\pm)}_\nu]$, they are proportional to a single $A^{(\pm)}_\mu$ and accordingly supersymmetry parameters with opposite chirality. Because of this reason, it seems difficult to interpret IKKT model as a gauge fixed version of the gauged matrix model investigated here. But such gauged models are interesting from various points of view, especially the existence of local Lorentz invariance, and it is worth further investigations. More analysis will be reported elsewhere.

Chapter 5

Conclusion and Discussion

In this paper, we have investigated new types of matrix models based on the complex exceptional Jordan algebra and the super Lie algebras. In the former case, we constructed a new matrix model which has a *compact* E_6 symmetry and a Chern-Simons like structure. The definition of E_6 itself (i.e. the *cubic form*) was adopted for the construction of the action. The resulting theory has a Chern-Simons term in the action as in the case of type F_4 (i.e. the *trilinear form*) [9]. The *compactness* of E_6 derives *the postulate of positive definite metric* of our model. In the latter case, we investigated three super Lie algebras, $osp(1|32;\mathbf{R})$, $u(1|16,16)$, and $gl(1|32;\mathbf{R})$. In paticular, we studied the supersymmetry structures of these models and discussed possible reductions to the IKKT model. In addition to those, a different $u(1|16,16)$ model from Smolin's, and some kind of *topological* effective action derived using Wigner-Inönü contraction were also discussed.

In the former part, we have introduced a new matrix model based on the simply connected compact exceptional Lie group E_6, and discovered that the Chern-Simons like term is also derived from the *cubic form*. This theory modeled itself on Smolin's approach based on the groups of type F_4 [9]. We have adopted the *cubic form* in place of the *trilinear form*. The *cubic form* is a quite different cubic linear form from the *trilinear form*. Our model has twice as many degrees of freedom as Smolin's model has because we consider *compact* E_6 instead of F_4. That point aside, we can have the same argument as Smolin which derives an effective action similar to the matrix string theory. An important point to emphasize is that it is characteristic of E_6 and F_4 to derive the Chern-Simons type action using this method, because G_2, E_7 and E_8 have no cubic linear forms which are made up of the pure exceptional Jordan algebra alone just like the *trilinear form* on F_4 or the *cubic form* on E_6. One way to introduce the cosmological term or the coupling constant into the theory is the compactification on directions. Of course, what we have reported here is just the first step in the analysis of the model, and many things need to be investigated. However, as we have seen, this model has several very interesting characteristics. Therefore it is quite likely that this theory will evolve in the future.

The problem to be specially considered is the quantization. It may be difficult to path-integrate as usual because the action of the theory is an essentially complex action. Consequently, it seems that it is necessary to reexamine the *canonical formalism* such as the loop quantum gravity. The author is very interested in the relation between the exponentiated quantity of the action (3.2.2) and some kind of quantity like a bi-local expression of Dirac equation. If one point is decided as a fiducial point, another point might be indicated using twistor-like method.

Besides, because this model is written in 'cubic' terms with respect to the fields, there cannot exist the term such as $R_{\mu\nu}{}^{\mu\nu}$ essentially. In consequence, we are inevitably obliged to take the position of the *induced gravity*. To put it the other way round, however, thanks to being cubic, the theory becomes what is close to the topological theory; and that is also one of the interesting properties of this model. Namely, there is a strong possibility that this theory is defined as a background independent theory from the very beginning. Therefore it is quite likely that other matrix models such as the BFSS model and the IKKT model are reproduced via expansions around specific backgrounds of this model.

Moreover, it seems quite probable that this algebraically defined model has a geometrical interpretation. The existence of the projective geometry can be seen off and on behind the Freudenthal multiplication or the cubic form. Although, depending on how things go, we might have to introduce even the Freudenthal manifold $\mathfrak{M}$ which is needed to understand the groups of type E_7 and type E_8, the attempt to relate this theory to some geometry is one of the most exciting subjects.

Furthermore, as discussed in chapter 3, the author would like to emphasize that there is considerable validity in considering the physical Hilbert space to be a *product space* composed of two parts $\mathfrak{J}^c$ and $\mathcal{G}$. Because there exists a cycle mapping, the resulting product space has a structure such that the concept of the spinor is introduced into the infinitely dimensional Hilbert space itself. The author does not believe it is a coincidence. It is likely that $\mathfrak{J}^c$ describes some degrees of freedom belonging to some internal structure in the each point of the space, and $\mathcal{G}$ plays a role of the network to the each point of the space. What is called *the postulate of positive definite metric* is, from the viewpoint of our model, merely an immediate consequence of the fact that our universe is *compact*.

By the way, we have considered thus far only the combination of the cubic form and the structure constant as an action of the E_6 matrix model. From the viewpoint of the construction of invariants, however, we can take not only f-coupling but also d-coupling into account,

$$\begin{aligned} S \;=\; & \alpha \left(\mathcal{P}^2(\mathcal{M}^{[A}) \,,\, \mathcal{P}(\mathcal{M}^{B}) \,,\, \mathcal{M}^{C]} \right) f_{ABC} \\ & + \beta \left(\mathcal{P}^2(\mathcal{M}^{(A}) \,,\, \mathcal{P}(\mathcal{M}^{B}) \,,\, \mathcal{M}^{C)} \right) d_{ABC} \end{aligned} \tag{5.0.1}$$

where $(\cdots)$ denotes the *weight*-1 symmetrization on indices, and d_{ABC} is defined as follows.

$$\{\mathbf{T}_A, \mathbf{T}_B\} \;=\; d_{ABC}\, \mathbf{T}_C \tag{5.0.2}$$

$$d_{ABC} \;=\; 2\, tr(\mathbf{T}_A\{\mathbf{T}_B, \mathbf{T}_C\}) \tag{5.0.3}$$

Although it is not yet clear whether considering such an action is invaluable, it is possible that some interesting physics concerned with anomalies are gained because the third-rank symmetric tensor is introduced into the theory. If some kinds of anomalies are given rise to in our model, the cause of those seems to be $\boldsymbol{\mathcal{G}}$. Because E_6 itself is a safe group.

Incidentally, the theories expressed by such actions as (3.2.2) and (5.0.1) are the *global* E_6 matrix models. Consequently, we naturally hit upon an idea that we might be able to localize this symmetry. What is called the *local* E_6 matrix model is the theory which has an invariant action under the mixed transformation on $\mathfrak{e}_6$ and $\boldsymbol{\mathcal{G}}$. To use Smolin's words, we can call this type of matrix model the 'gauged' matrix model. In fact, if we localize Smolin's $osp(1|32;\mathbf{R})$ matrix model, the resulting theory is just the $u(1|16,16)$ matrix model. This $u(1|16,16)$ matrix model is a very beautiful model. Therefore, we get the feeling that we would like to carry out the same thing as this to the global E_6 matrix model. However, the attempt to construct the local E_6 matrix model has not been succeeded so far because of some mathematical difficulties.

Finally, there is a further question which needs to be asked: Our model has twice as many degrees of freedom as Smolin's model has. This trouble always follows us as long as we handle E_6, and it is a serious problem which cannot be avoided. This problem will be discussed on another occasion.

In the latter part, we have investigated the cubic matrix model whose global symmetry is the super Lie algebra $osp(1|32;\mathbf{R})$. $osp(1|32;\mathbf{R})$ cubic matrix model possesses a two-fold structure of the $\mathcal{N}=2$ SUSY of IKKT model. IKKT model is induced from this model by the multi-loop effect. And we have investigated the $gl(1|32;\mathbf{R}) \otimes gl(N;\mathbf{R})$ gauged matrix model as an extension. The space-time translation is introduced by means of the Wigner-Inönü contraction. The effective action vanishes, therefore this model may be related to a topological matrix model.

Firstly, we have studied $osp(1|32;\mathbf{R})$ matrix model. This model is considered as an 11-dimensional model and contains twice as many fermionic degrees of freedom as IKKT model. The model is invariant under global $osp(1|32;\mathbf{R})$ symmetry and $u(N)$ gauge symmetry. It is also invariant under a constant shift of fields. Combining the $osp(1|32;\mathbf{R})$ and the constant shifts, we obtain space-time algebras including space-time supersymmetries. In this sense, this model is a natural generalization of IKKT model. Since this model has twice as many fermions, we need to integrate half of the degrees of freedom. We have given an identification of the fields in this model with the fields in IKKT model from the view point of the supersymmetry structures. We have also discussed a possibility to induce IKKT model by integrating out the unnecessary fields.

Next, we have studied the gauged matrix models with local Lorentz symmetry. First we have shown that the unitary transformations in noncommutative gauge theories contain much larger symmetries than the ordinary gauge transformations. Especially, local coordinate transformations can be described within this gauge transformations. This is understandable from the D-brane point of view. If a noncommutative gauge theory is considered as an effective low-energy action for D-branes, the action should be invariant under coordinate transformations on the brane. Under this transformation, all the fields ψ and A_μ in gauge theory transform as scalars. More interestingly we have shown that if we expand those fields, not only in terms of $\exp(i\hat{p}_\mu k_\mu)$ but as a power series of $\hat{p}_\mu$, we can obtain higher rank fields which transform as tensors under this coordinate transformations. However, the original $SO(9,1)$ indices are completely decoupled from the internal diffeomorphism.

Lastly, we have considered a model with local $SO(9,1)$ symmetry by extending the $osp(1|32;\mathbf{R})$ algebra to $u(1|16,16)$ or $gl(1|32;\mathbf{R})$ super Lie algebras. We have enhanced the global $osp(1|32;\mathbf{R})$ to local symmetries, but lost the invariance under constant shifts of fields and we need a different interpretation of space-time translation. We have adopted the Wigner-Inönü contraction and extracted 10-dimensional space-time translation from $SO(10,1)$ rotations. We have then identified how to scale the fields in order to obtain the correct 10-dimensional theory in the large radius limit. Since this model contains four times as many fermionic fields as IKKT model, we need to integrate out half of them first and then restrict the fermions further to be halved. But after integrating the first half of the fields, the effective action was shown to vanish. This is because the resultant action should be invariant under an arbitrary shift of the fields, not restricted to a constant shift. We can interpret the final model as a topological model. This kind of topological model was studied in [27]. It is also interesting to investigate such a possibility from the gauged matrix model point of view.

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Appendix A

Convention

In this paper, repeated indices are generally summed unless otherwise indicated.

A.1 Convention of E_6 matrix model

A.1.1 (Anti-)symmetrization

$[\cdots]$ denotes the *weight*-1 anti-symmetrization and $(\cdots)$ denotes the *weight*-1 symmetrization on indices as follows.

$$\begin{aligned} X^{[A}Y^{B}Z^{C]} &= \frac{1}{6}(X^AY^BZ^C + X^BY^CZ^A + X^CY^AZ^B \\ &\quad - X^CY^BZ^A - X^BY^AZ^C - X^AY^CZ^B) \qquad (A.1.1) \\ X^{(A}Y^{B}Z^{C)} &= \frac{1}{6}(X^AY^BZ^C + X^BY^CZ^A + X^CY^AZ^B \\ &\quad + X^CY^BZ^A + X^BY^AZ^C + X^AY^CZ^B) \qquad (A.1.2) \end{aligned}$$

Therefore, contracting indices with the totally anti-symmetric tensor A_{ABC} or the totally symmetric tensor S_{ABC} results in the following ordinary summation.

$$\begin{aligned} X^{[A}Y^{B}Z^{C]}\, A_{ABC} &= X^AY^BZ^C\, A_{ABC} \qquad (A.1.3) \\ X^{(A}Y^{B}Z^{C)}\, S_{ABC} &= X^AY^BZ^C\, S_{ABC} \qquad (A.1.4) \end{aligned}$$

A.1.2 Levi-Civita tensor in 2 dimensions

The 2-dimensional Levi-Civita tensor is defined by

$$(\epsilon^{\alpha\beta}) = (\epsilon_{\alpha\beta}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \qquad (A.1.5)$$

A.1.3 Fundamental representation of $su(2)$

The matrices which generate $su(2)$ are given by

$$\begin{aligned} \boldsymbol{\tau}^1 &= \frac{1}{2}\boldsymbol{\sigma}^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} && \text{(A.1.6)} \\ \boldsymbol{\tau}^2 &= \frac{1}{2}\boldsymbol{\sigma}^2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} && \text{(A.1.7)} \\ \boldsymbol{\tau}^3 &= \frac{1}{2}\boldsymbol{\sigma}^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . && \text{(A.1.8)} \end{aligned}$$

These satisfy

$$\begin{aligned} [\boldsymbol{\tau}^I, \boldsymbol{\tau}^J] &= i\epsilon^{IJK}\, \boldsymbol{\tau}^K && \text{(A.1.9)} \\ tr(\boldsymbol{\tau}^I\boldsymbol{\tau}^J) &= \frac{1}{2}\delta^{IJ} && \text{(A.1.10)} \\ \epsilon^{IJK} &= -2i\; tr(\boldsymbol{\tau}^I[\boldsymbol{\tau}^J, \boldsymbol{\tau}^K]) . && \text{(A.1.11)} \end{aligned}$$

Therefore, the 3-dimensional Levi-Civita tensor ϵ^{IJK} $(I = 1, 2, 3)$ is defined with

$$\epsilon^{123} = \epsilon_{123} = +1 , \qquad \text{(A.1.12)}$$

in which the exchange of the position between the upper suffix and the downstairs suffix does not make any sense.

A.1.4 Fundamental representation of the Lie algebra $\mathcal{G}$

The matrices which generate $\mathcal{G}$ satisfy

$$\begin{aligned} [\mathbf{T}_A, \mathbf{T}_B] &= i f_{ABC}\, \mathbf{T}_C && \text{(A.1.13)} \\ tr(\mathbf{T}_A\mathbf{T}_B) &= \frac{1}{2}\delta_{AB} && \text{(A.1.14)} \\ f_{ABC} &= -2i\; tr(\mathbf{T}_A[\mathbf{T}_B, \mathbf{T}_C]) . && \text{(A.1.15)} \end{aligned}$$

A.2 Convention of supermatrix models

A.2.1 Invariant tensor

We use the negative signature with respect to the time-component of the metric about the 11-dimensional Poincaré algebra $SO(10,1)$.

$$\eta_{\mathsf{ab}} = diag\overbrace{(-1, 1, 1, \cdots, 1, 1)}^{(11)} \qquad \text{(A.2.1)}$$

(In the viewpoint of the 10 dimensionality, this can be seen as the metric about the 10-dimensional de Sitter algebra (dS_{10+1}).)

Higher rank ε-symbol is defined as follows

$$\varepsilon_{012\cdots 9\sharp} = -\varepsilon^{012\cdots 9\sharp} = 1 \ , \tag{A.2.2}$$

where we define $\sharp := 10$.

A.2.2 Indices

$$\begin{aligned} \mathbf{A} &= A^A{}_{a\mu}\, dx^\mu \Gamma^a \mathbf{T}_A && \text{(A.2.3)} \\ &= A^A{}_a\, \Gamma^a \mathbf{T}_A && \text{(A.2.4)} \\ &= A^A\, \mathbf{T}_A && \text{(A.2.5)} \end{aligned}$$

A.2.3 γ-matrix

Clifford algebra is defined by

$$\{\Gamma^{\mathbf{a}}, \Gamma^{\mathbf{b}}\} = 2\eta^{\mathbf{ab}} \ . \tag{A.2.6}$$

We use the following representation of Γ-matrix, where γ^I ($I = 1, \cdots, 9$) denote the $SO(9)$ gamma-matrices.

$$\Gamma^0 = \begin{pmatrix} 0 & \mathbf{1}_{16} \\ -\mathbf{1}_{16} & 0 \end{pmatrix} \ , \ \Gamma^I = \begin{pmatrix} 0 & \gamma^I \\ \gamma^I & 0 \end{pmatrix} \ , \ \Gamma^{10} = \Gamma^\sharp = \begin{pmatrix} \mathbf{1}_{16} & 0 \\ 0 & -\mathbf{1}_{16} \end{pmatrix} \tag{A.2.7}$$

Of course, if one wants one may use the following representations.

$$\tilde{\Gamma}^0 = \begin{pmatrix} 0 & -\mathbf{1}_{16} \\ \mathbf{1}_{16} & 0 \end{pmatrix} \ , \ \tilde{\Gamma}^I = \begin{pmatrix} \gamma^I & 0 \\ 0 & -\gamma^I \end{pmatrix} \ , \ \tilde{\Gamma}^{10} = \tilde{\Gamma}^\sharp = \begin{pmatrix} 0 & \mathbf{1}_{16} \\ \mathbf{1}_{16} & 0 \end{pmatrix} \tag{A.2.8}$$

The relations between these two representations are the following.

$$\begin{aligned} \Gamma^{\mathbf{a}} &= R\tilde{\Gamma}^{\mathbf{a}} R^T && \text{(A.2.9)} \\ R &= \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1}_{16} & \mathbf{1}_{16} \\ \mathbf{1}_{16} & -\mathbf{1}_{16} \end{pmatrix} && \text{(A.2.10)} \end{aligned}$$

A.2.4 dS_{d+1} and AdS_{d+1} space-time

dS_{d+1} space-time

$$-(X^0)^2 + \sum_{I=1}^{d-1}(X^I)^2 + (X^d)^2 \;=\; R^2 \tag{A.2.11}$$

$$\eta_{\mathbf{ab}} = diag\overbrace{(-1,1,1,\cdots,1,1)}^{(d+1)} \tag{A.2.12}$$

$$\begin{aligned}
[J^{\mathbf{ab}}, J^{\mathbf{cd}}] &= 4i\eta^{[\mathbf{a}[\mathbf{c}}J^{\mathbf{b}]\mathbf{d}]} && \text{(A.2.13)}\\
&= 2i(\eta^{[\mathbf{a}^{\mathbf{c}}}J^{\mathbf{b}]^{\mathbf{d}}} - \eta^{[\mathbf{a}^{\mathbf{d}}}J^{\mathbf{b}]^{\mathbf{c}}}) && \text{(A.2.14)}\\
&= i(\eta^{\mathbf{ac}}J^{\mathbf{bd}} - \eta^{\mathbf{bc}}J^{\mathbf{ad}} - \eta^{\mathbf{ad}}J^{\mathbf{bc}} + \eta^{\mathbf{bd}}J^{\mathbf{ac}}) && \text{(A.2.15)}
\end{aligned}$$

AdS_{d+1} space-time

$$-(X^0)^2 + \sum_{I=1}^{d-1}(X^I)^2 - (X^d)^2 \;=\; -R^2 \tag{A.2.16}$$

$$\eta_{\mathbf{ab}} = diag\overbrace{(-1,1,1,\cdots,1,-1)}^{(d+1)} \tag{A.2.17}$$

$$\begin{aligned}
[J^{\mathbf{ab}}, J^{\mathbf{cd}}] &= 4i\eta^{[\mathbf{a}[\mathbf{c}}J^{\mathbf{b}]\mathbf{d}]} && \text{(A.2.18)}\\
&= 2i(\eta^{[\mathbf{a}^{\mathbf{c}}}J^{\mathbf{b}]^{\mathbf{d}}} - \eta^{[\mathbf{a}^{\mathbf{d}}}J^{\mathbf{b}]^{\mathbf{c}}}) && \text{(A.2.19)}\\
&= i(\eta^{\mathbf{ac}}J^{\mathbf{bd}} - \eta^{\mathbf{bc}}J^{\mathbf{ad}} - \eta^{\mathbf{ad}}J^{\mathbf{bc}} + \eta^{\mathbf{bd}}J^{\mathbf{ac}}) && \text{(A.2.20)}
\end{aligned}$$

Wigner Inönü Contraction

$$R \;\longrightarrow\; \infty \tag{A.2.21}$$

A.2.5 $u(N)$ Lie algebra

f-term

We define the structure constants of $u(N)$ algebra by

$$[\mathbf{T}_A, \mathbf{T}_B] = i f_{AB}{}^{C} \mathbf{T}_C \ . \tag{A.2.22}$$

If the generators satisfy the orthonormal relation

$$\begin{aligned} Tr_{\underline{N}}(\mathbf{T}^A \mathbf{T}_B) &= C_{\mathbf{T}}\ \delta^A_B && \text{(A.2.23)} \\ &= \frac{1}{2}\ \delta^A_B\ , && \text{(A.2.24)} \end{aligned}$$

the totally anti-symmetric constants f_{ABC} is expressed as follows.

$$f_{ABC} = \frac{1}{iC_{\mathbf{T}}} Tr_{\underline{N}}(\mathbf{T}_A[\mathbf{T}_B, \mathbf{T}_C]) \tag{A.2.25}$$

d-term

As the counter of f_{ABC}, we define the totally symmetric constants d_{ABC} by

$$\{\mathbf{T}_A, \mathbf{T}_B\} = d_{AB}{}^{C} \mathbf{T}_C \ . \tag{A.2.26}$$

Therefore we have

$$d_{ABC} = \frac{1}{C_{\mathbf{T}}} Tr_{\underline{N}}(\mathbf{T}_A\{\mathbf{T}_B, \mathbf{T}_C\}) \ . \tag{A.2.27}$$

$(if + d)$-term

$$if_{ABC} + d_{ABC} = \frac{2}{C_{\mathbf{T}}} Tr_{\underline{N}}(\mathbf{T}_A \mathbf{T}_B \mathbf{T}_C) \tag{A.2.28}$$

A.2.6 $gl(N; \mathbf{R})$ algebra

f-term

We define the structure constants of $gl(N; \mathbf{R})$ algebra by

$$[\mathbf{T}_A, \mathbf{T}_B] = f_{AB}{}^{C} \mathbf{T}_C \ . \tag{A.2.29}$$

If the generators satisfy the orthonormal relation

$$Tr_{\underline{N}}(\mathbf{T}^A\mathbf{T}_B) = C_{\mathbf{T}}\ \delta^A_B\ , \tag{A.2.30}$$

the totally anti-symmetric constants f_{ABC} is expressed as follows.

$$f_{ABC} = \frac{1}{C_{\mathbf{T}}} Tr_{\underline{N}}(\mathbf{T}_A[\mathbf{T}_B, \mathbf{T}_C]) \tag{A.2.31}$$

d-term

As the counter of f_{ABC}, we define the totally symmetric constants d_{ABC} by

$$\{\mathbf{T}_A, \mathbf{T}_B\} = d_{AB}{}^C\mathbf{T}_C\ . \tag{A.2.32}$$

Therefore we have

$$d_{ABC} = \frac{1}{C_{\mathbf{T}}} Tr_{\underline{N}}(\mathbf{T}_A\{\mathbf{T}_B, \mathbf{T}_C\})\ . \tag{A.2.33}$$

$(f + d)$-term

$$f_{ABC} + d_{ABC} = \frac{2}{C_{\mathbf{T}}} Tr_{\underline{N}}(\mathbf{T}_A\mathbf{T}_B\mathbf{T}_C) \tag{A.2.34}$$

A.2.7 Convention of Supermatrix

Complex conjugation of Grassmann number

As a complex conjugation of the anti-commuting c-number, we use the following convention.

$$(\alpha\beta)^* = \beta^*\alpha^* \tag{A.2.35}$$

(Therefore if $\alpha^* = \alpha$ and $\beta^* = \beta$, $(i\alpha\beta)$ fills the role of real number.)

Supermatrix

Let B be the $m \times m$ and b be the $n \times n$ matrices whose elements are Grassmann even numbers. Let F be the $m \times n$ and f be the $n \times m$ matrices whose elements are Grassmann odd numbers. And we construct the following $(m + n) \times (m + n)$ matrix $\mathcal{M}$.

$$\mathcal{M} = \begin{pmatrix} B & F \\ f & b \end{pmatrix} \tag{A.2.36}$$

This type of matrix $\mathcal{M}$ is called "supermatrix".

Construction of superspace (vector space)

Column vector

We denote the column vector space, whose linear transformation is expressed by the supermatrix $\mathcal{M}$, V. Any element x of V has the column vector ϕ whose elements are Grassmann odd at the upper part, and the column vector r whose elements are Grassmann even at the downstairs part respectively.

$$x = \begin{pmatrix} \phi \\ r \end{pmatrix} \tag{A.2.37}$$

The transposition and hermitian adjoint of the column vector x is defined as follows.

$$x^T := \begin{pmatrix} \phi^T & r^T \end{pmatrix} \quad , \quad x^\dagger := \begin{pmatrix} \phi^\dagger & r^\dagger \end{pmatrix} \tag{A.2.38}$$

Row vector

The transposition or hermitian adjoint of $x \in V$ constructs the row vector space. Any element y of the row vector space has the row vector ψ whose elements are Grassmann odd at the left part, and the row vector s whose elements are Grassmann even at the right part respectively.

$$y = \begin{pmatrix} \psi & s \end{pmatrix} \tag{A.2.39}$$

The transposition and hermitian adjoint of the row vector y is defined as follows.

$$y^T := \begin{pmatrix} -\psi^T \\ s^T \end{pmatrix} \quad ; \quad y^\dagger := \begin{pmatrix} \psi^\dagger \\ s^\dagger \end{pmatrix} \tag{A.2.40}$$

Complex conjugate of the vector

The complex conjugate of the column and row vector are respectively defined as follows.

$$x^* := (x^T)^\dagger = \begin{pmatrix} \phi^* \\ r^* \end{pmatrix} \quad , \quad y^* := (y^T)^\dagger = \begin{pmatrix} -\psi^* & s^* \end{pmatrix} \tag{A.2.41}$$

Attention concerning the transposition of the vector

The transposition of the vector has a cycle of 4 times.

(However, hermitian adjoint and complex conjugate had a cycle of twice as usual.)

$$x = \begin{pmatrix} \phi \\ r \end{pmatrix} \implies (x^T)^T = \begin{pmatrix} -\phi \\ r \end{pmatrix} \; , \; ((x^T)^T)^T = \begin{pmatrix} -\phi^T & r^T \end{pmatrix} \; , \; (((x^T)^T)^T)^T = x \tag{A.2.42}$$

$$\implies (x^\dagger)^\dagger = x \tag{A.2.43}$$

$$\Longrightarrow \quad (x^*)^* = x \tag{A.2.44}$$

$$y = \begin{pmatrix} \psi & s \end{pmatrix} \quad \Longrightarrow \quad (y^T)^T = \begin{pmatrix} -\psi & s \end{pmatrix} \; , \quad ((y^T)^T)^T = \begin{pmatrix} \psi^T \\ s^T \end{pmatrix} \; , \quad (((y^T)^T)^T)^T = y \tag{A.2.45}$$

$$\Longrightarrow \quad (y^\dagger)^\dagger = y \tag{A.2.46}$$

$$\Longrightarrow \quad (y^*)^* = y \tag{A.2.47}$$

⟨Transposition⟩

The transposition of the supermatrix $\mathcal{M}$ is defined by

$$(\mathcal{M}x)^T =: x^T \mathcal{M}^T \; . \tag{A.2.48}$$

Therefore we have

$$\mathcal{M} = \begin{pmatrix} B & F \\ f & b \end{pmatrix} \quad \Longrightarrow \quad \mathcal{M}^T = \begin{pmatrix} B^T & -f^T \\ F^T & b^T \end{pmatrix} \; . \tag{A.2.49}$$

For two supermatrices $\mathcal{M}_1$ and $\mathcal{M}_2$, we have the identities $(\mathcal{M}_1\mathcal{M}_2)^T = \mathcal{M}_2{}^T\mathcal{M}_1{}^T$.

⟨Hermitian adjoint⟩

The hermitian adjoint of the supermatrix $\mathcal{M}$ is defined by

$$(\mathcal{M}x)^\dagger =: x^\dagger \mathcal{M}^\dagger \; . \tag{A.2.50}$$

Therefore we have

$$\mathcal{M} = \begin{pmatrix} B & F \\ f & b \end{pmatrix} \quad \Longrightarrow \quad \mathcal{M}^\dagger = \begin{pmatrix} B^\dagger & f^\dagger \\ F^\dagger & b^\dagger \end{pmatrix} \; . \tag{A.2.51}$$

For two supermatrices $\mathcal{M}_1$ and $\mathcal{M}_2$, we have the identities $(\mathcal{M}_1\mathcal{M}_2)^\dagger = \mathcal{M}_2{}^\dagger\mathcal{M}_1{}^\dagger$.

⟨Complex conjugate⟩

The complex conjugate of the supermatrix $\mathcal{M}$ is defined by

$$\mathcal{M}^* := (\mathcal{M}^T)^\dagger \; . \tag{A.2.52}$$

Therefore we have

$$\mathcal{M} = \begin{pmatrix} B & F \\ f & b \end{pmatrix} \quad \Longrightarrow \quad \mathcal{M}^* = \begin{pmatrix} B^* & F^* \\ -f^* & b^* \end{pmatrix} \; . \tag{A.2.53}$$

For two supermatrices $\mathcal{M}_1$ and $\mathcal{M}_2$, we have the identities $(\mathcal{M}_1\mathcal{M}_2)^* = \mathcal{M}_1{}^*\mathcal{M}_2{}^*$.

Attention concerning the transposition of the supermatrix

The transposition of the supermatrix has a cycle of 4 times.

(However, hermitian adjoint and complex conjugate had a cycle of twice as usual.)

$$\mathcal{M} = \begin{pmatrix} B & F \\ f & b \end{pmatrix} \tag{A.2.54}$$

$$\Longrightarrow \quad (\mathcal{M}^T)^T = \begin{pmatrix} B & -F \\ -f & b \end{pmatrix} \; ; \; ((\mathcal{M}^T)^T)^T = \begin{pmatrix} B^T & f^T \\ -F^T & b^T \end{pmatrix} \; ; \; (((\mathcal{M}^T)^T)^T)^T = \mathcal{M} \tag{A.2.55}$$

$$\Longrightarrow \quad (\mathcal{M}^\dagger)^\dagger = \mathcal{M} \tag{A.2.56}$$

$$\Longrightarrow \quad (\mathcal{M}^*)^* = \mathcal{M} \tag{A.2.57}$$

Relations between transposition, hermitian adjoint, complex conjugation and column vector, row vector

$$(\mathcal{M}x)^T \;=:\; x^T \mathcal{M}^T \tag{A.2.58}$$

$$(\mathcal{M}x)^\dagger \;=:\; x^\dagger \mathcal{M}^\dagger \tag{A.2.59}$$

$$(\mathcal{M}x)^* \;=\; \mathcal{M}^* x^* \tag{A.2.60}$$

$$(y\mathcal{M})^T \;=\; \mathcal{M}^T y^T \tag{A.2.61}$$

$$(y\mathcal{M})^\dagger \;=\; \mathcal{M}^\dagger y^\dagger \tag{A.2.62}$$

$$(y\mathcal{M})^* \;=\; y^* \mathcal{M}^* \tag{A.2.63}$$

Relations among transposition, hermitian adjoint, complex conjugation

$$\mathcal{M}^T = (\mathcal{M}^*)^\dagger \quad \neq \; (\mathcal{M}^\dagger)^* \tag{A.2.64}$$

$$\mathcal{M}^\dagger = (\mathcal{M}^*)^T \quad \neq \; (\mathcal{M}^T)^* \tag{A.2.65}$$

$$\mathcal{M}^* = (\mathcal{M}^T)^\dagger \quad \neq \; (\mathcal{M}^\dagger)^T \tag{A.2.66}$$

Supertrace

The supertrace ($STr_{\underline{m+n}}(\mathcal{M})$) of the supermatrix $\mathcal{M}$ is defined by

$$STr_{\underline{m+n}}(\mathcal{M}) \;:=\; Tr_{\underline{m}}(B) - Tr_{\underline{n}}(b) \; . \tag{A.2.67}$$

$\boxed{\text{Supertrace of the product matrix}}$

Now we consider the product matrix of the $(m+n)\times(m+n)$ supermatrix $\mathcal{M}$ and $N\times N$ usual matrix $\mathbf{T}$.

In this case, the supertrace of this product matrix can be expressed as follows.

$$STr_{\underline{(m+n)N}}(\mathcal{M}\otimes\mathbf{T}) \;=\; STr_{\underline{(m+n)}}(\mathcal{M})\ Tr_{\underline{N}}(\mathbf{T}) \tag{A.2.68}$$

Therefore, we obtain the cubic expression which is divided into f-term and d-term, by considering the direct product with the generators and taking the supertrace of the product cubic expression.

$u(N)$ **case**

$$\begin{aligned}
&STr_{\underline{(m+n)N}}((\mathcal{M}^A\otimes\mathbf{T}_A)(\mathcal{M}^B\otimes\mathbf{T}_B)(\mathcal{M}^C\otimes\mathbf{T}_C))\\
&\quad= STr_{\underline{(m+n)N}}(\mathcal{M}^A\mathcal{M}^B\mathcal{M}^C\otimes\mathbf{T}_A\mathbf{T}_B\mathbf{T}_C)\\
&\quad= STr_{\underline{(m+n)}}(\mathcal{M}^A\mathcal{M}^B\mathcal{M}^C)\ Tr_{\underline{N}}(\mathbf{T}_A\mathbf{T}_B\mathbf{T}_C)\\
&\quad= \frac{C_{\mathbf{T}}}{2}(if_{ABC}+d_{ABC})\ STr_{\underline{(m+n)}}(\mathcal{M}^A\mathcal{M}^B\mathcal{M}^C)\\
&\quad= i\frac{C_{\mathbf{T}}}{4}f_{ABC}\ STr_{\underline{(m+n)}}(\mathcal{M}^A[\mathcal{M}^B,\mathcal{M}^C])+\frac{C_{\mathbf{T}}}{4}d_{ABC}\ STr_{\underline{(m+n)}}(\mathcal{M}^A\{\mathcal{M}^B,\mathcal{M}^C\})
\end{aligned} \tag{A.2.69}$$

$gl(N;\mathbf{R})$ **case**

$$\begin{aligned}
&STr_{\underline{(m+n)N}}((\mathcal{M}^A\otimes\mathbf{T}_A)(\mathcal{M}^B\otimes\mathbf{T}_B)(\mathcal{M}^C\otimes\mathbf{T}_C))\\
&\quad= STr_{\underline{(m+n)N}}(\mathcal{M}^A\mathcal{M}^B\mathcal{M}^C\otimes\mathbf{T}_A\mathbf{T}_B\mathbf{T}_C)\\
&\quad= STr_{\underline{(m+n)}}(\mathcal{M}^A\mathcal{M}^B\mathcal{M}^C)\ Tr_{\underline{N}}(\mathbf{T}_A\mathbf{T}_B\mathbf{T}_C)\\
&\quad= \frac{C_{\mathbf{T}}}{2}(f_{ABC}+d_{ABC})\ STr_{\underline{(m+n)}}(\mathcal{M}^A\mathcal{M}^B\mathcal{M}^C)\\
&\quad= \frac{C_{\mathbf{T}}}{4}f_{ABC}\ STr_{\underline{(m+n)}}(\mathcal{M}^A[\mathcal{M}^B,\mathcal{M}^C])+\frac{C_{\mathbf{T}}}{4}d_{ABC}\ STr_{\underline{(m+n)}}(\mathcal{M}^A\{\mathcal{M}^B,\mathcal{M}^C\})
\end{aligned} \tag{A.2.70}$$

Superdeterminant

The superdeterminant $(SDet(\mathcal{M}))$ of the supermatrix $\mathcal{M}$ is defined by

$$SDet(\mathcal{M}) \;:=\; e^{STr(\ln\,\mathcal{M})}\ . \tag{A.2.71}$$

Real Supermatrix

The "real" supermatrix $\mathcal{M}$ is defined by the following condition.

$$\mathcal{M}^* = \mathcal{M} \tag{A.2.72}$$

Therefore, in the case that all the elements of $\tilde{B}, \tilde{b}/\tilde{F}, \tilde{f}$ are "real" Grassmann even/odd number, the specific form is given by

$$\mathcal{M} = \begin{pmatrix} \tilde{B} & \tilde{F} \\ i\tilde{f} & \tilde{b} \end{pmatrix} . \tag{A.2.73}$$

Appendix B

Definitions of the super Lie algebras

B.1 Definition of $osp(1|32;\mathbf{R})$ Super Lie Algebra

$osp(1|32;\mathbf{R})$ super Lie algebra is defined by the following relation.

$$
\begin{aligned}
osp(1|32;\mathbf{R}) &= \{\mathcal{M} \in \boldsymbol{\mathcal{M}}(32|1;\mathbf{R}) \mid \mathcal{M}^T\mathcal{G} + \mathcal{G}\mathcal{M} = 0\ \} && \text{(B.1.1)}\\
\mathcal{G} &= \begin{pmatrix} 0 & \mathbf{1}_{16} & 0 \\ -\mathbf{1}_{16} & 0 & 0 \\ 0 & 0 & i \end{pmatrix} = \begin{pmatrix} \Gamma^0 & 0 \\ 0 & i \end{pmatrix} && \text{(B.1.2)}\\
\mathcal{M}^* &= \mathcal{M} && \text{(B.1.3)}
\end{aligned}
$$

Γ^0 can be identified as a 32×32 11-dimensional γ-matrix in Majorana basis which is real and satisfies $(\Gamma^0)^2 = -1$. The specific components of $\mathcal{M}$ are written as

$$
\mathcal{M} = \begin{pmatrix} s & \phi \\ i\bar{\phi} & 0 \end{pmatrix}, \tag{B.1.4}
$$

where ϕ is a Majorana spinor with 32 components and $\bar{\phi} := \phi^\dagger\Gamma^0 \equiv \phi^T\Gamma^0$. The bosonic part s is a real 32×32 bosonic matrix satisfying

$$
s^T\Gamma^0 + \Gamma^0 s = 0, \tag{B.1.5}
$$

so that s is an element of $sp(32;R)$ algebra. We can expand this s using 11-dimensional γ-matrices as follows.

$$
\begin{aligned}
s &= \Gamma^{\langle\mathbf{1}\rangle}s_{\langle\mathbf{1}\rangle} + \Gamma^{\langle\mathbf{2}\rangle}s_{\langle\mathbf{2}\rangle} + \Gamma^{\langle\mathbf{5}\rangle}s_{\langle\mathbf{5}\rangle}\\
&= \Gamma^{\mathbf{a}}u_{\mathbf{a}} + \frac{1}{2}\Gamma^{\mathbf{ab}}u_{\mathbf{ab}} + \frac{1}{5!}\Gamma^{\mathbf{abcde}}u_{\mathbf{abcde}} && \text{(B.1.6)}
\end{aligned}
$$

Here we use the following notations about $\mathbf{n} = \mathbf{1}, \mathbf{2}, \mathbf{5}$.

$$
\begin{aligned}
\Gamma^{\langle\mathbf{n}\rangle} &:= \Gamma^{\mathbf{a}_1\cdots\mathbf{a}_n} && \text{(B.1.7)}\\
s_{\langle\mathbf{n}\rangle} &:= \frac{1}{n!}u_{\mathbf{a}_1\cdots\mathbf{a}_n} && \text{(B.1.8)}
\end{aligned}
$$

s contains $528 = 11 + 55 + 462$ degrees of freedom. $u_{\mathbf{a}}$ are denoted as 11-dimensional indices and run from 0 to 10.

$\boxed{\textbf{Inner product of the vector space}}$

Next we show the inner product of the vector space about the $osp(1|32;\mathbf{R})$. For any two elements x_1, x_2 in real column vector space V, a scalar quantity is defined as follows.

$$x_1 = \begin{pmatrix} \phi_1 \\ r_1 \end{pmatrix} \in V \quad , \quad x_2 = \begin{pmatrix} \phi_2 \\ r_2 \end{pmatrix} \in V \tag{B.1.9}$$

$$\begin{aligned} (x_1, x_2) &:= x_1^T(-i\mathcal{G})x_2 && \text{(B.1.10)} \\ &= -i\phi_1^T\Gamma^0\phi_2 + r_1r_2 && \text{(B.1.11)} \\ &= (x_2, x_1) && \text{(B.1.12)} \end{aligned}$$

B.2 Two representations of $osp(1|32;\mathbf{R})$

B.2.1 $\boldsymbol{\mathcal{S}}$ representation

The element $(\mathcal{S})$ of $\boldsymbol{\mathcal{S}}$ representation is the supermatrix which satisfies the following expression.

$$\mathcal{S}^T\mathcal{G} + \mathcal{G}\mathcal{S} = 0 \tag{B.2.1}$$

The specific component is given by

$$\mathcal{S} = \begin{pmatrix} s & \phi \\ i\bar{\phi} & 0 \end{pmatrix} , \tag{B.2.2}$$

where $\bar{\phi} := \phi^\dagger\Gamma^0 \equiv \phi^T\Gamma^0$.

Of course this is the representation of $osp(1|32;\mathbf{R})$.

$$[\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{S}}] = \boldsymbol{\mathcal{S}} \tag{B.2.3}$$

This s can be expanded as follows.

$$\begin{aligned} s &= \Gamma^{\langle \mathbf{1}\rangle}s_{\langle \mathbf{1}\rangle} + \Gamma^{\langle \mathbf{2}\rangle}s_{\langle \mathbf{2}\rangle} + \Gamma^{\langle \mathbf{5}\rangle}s_{\langle \mathbf{5}\rangle} \\ &= \Gamma^{\mathbf{a}}u_{\mathbf{a}} + \frac{1}{2}\Gamma^{\mathbf{ab}}u_{\mathbf{ab}} + \frac{1}{5!}\Gamma^{\mathbf{abcde}}u_{\mathbf{abcde}} \end{aligned} \tag{B.2.4}$$

Here, for $\mathbf{n} = \mathbf{1}, \mathbf{2}, \mathbf{5}$, we use the following notations.

$$\begin{aligned} \Gamma^{\langle \mathbf{n}\rangle} &:= \Gamma^{\mathbf{a}_1\cdots\mathbf{a}_n} && \text{(B.2.5)} \\ s_{\langle \mathbf{n}\rangle} &:= \frac{1}{n!}u_{\mathbf{a}_1\cdots\mathbf{a}_n} && \text{(B.2.6)} \end{aligned}$$

B.2.2 $\mathcal{A}$ representation

The element $(\mathcal{A})$ of $\boldsymbol{\mathcal{A}}$ representation is the supermatrix which satisfies the following expression.

$$-\mathcal{A}^T\mathcal{G} + \mathcal{G}\mathcal{A} = 0 \tag{B.2.7}$$

The specific component is given by

$$\mathcal{A} = \begin{pmatrix} a & \chi \\ -i\bar{\chi} & v \end{pmatrix} , \tag{B.2.8}$$

where $\bar{\chi} := \chi^\dagger\Gamma^0 \equiv \chi^T\Gamma^0$.

This is the representation of $osp(1|32;\mathbf{R})$ too.

$$[\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{A}}] \;=\; \boldsymbol{\mathcal{A}} \tag{B.2.9}$$

This a can be expanded as follows.

$$\begin{aligned} a &= \Gamma^{\langle 0\rangle}a_{\langle 0\rangle} + \Gamma^{\langle 3\rangle}a_{\langle 3\rangle} + \Gamma^{\langle 4\rangle}a_{\langle 4\rangle} \\ &= \mathbf{1}_{\underline{32}}u + \frac{1}{3!}\Gamma^{\mathbf{abc}}u_{\mathbf{abc}} + \frac{1}{4!}\Gamma^{\mathbf{abcd}}u_{\mathbf{abcd}} \end{aligned} \tag{B.2.10}$$

Here, for $\mathbf{n} = \mathbf{0}$, we put

$$\Gamma^{\langle 0\rangle} := \mathbf{1}_{\underline{32}} \tag{B.2.11}$$

$$a_{\langle 0\rangle} := u , \tag{B.2.12}$$

and for $\mathbf{n} = \mathbf{3}, \mathbf{4}$, we use the following notations.

$$\Gamma^{\langle \mathbf{n}\rangle} := \Gamma^{\mathbf{a}_1\cdots\mathbf{a}_n} \tag{B.2.13}$$

$$a_{\langle \mathbf{n}\rangle} := \frac{1}{n!}u_{\mathbf{a}_1\cdots\mathbf{a}_n} \tag{B.2.14}$$

B.2.3 (Anti-)commutation relation between $\boldsymbol{\mathcal{S}}$ and $\boldsymbol{\mathcal{A}}$

$$\underline{[\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{S}}] = \boldsymbol{\mathcal{S}}} \; , \; \{\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{S}}\} = \boldsymbol{\mathcal{A}} \tag{B.2.15}$$

$$[\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{A}}] = \boldsymbol{\mathcal{A}} \; , \; \{\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{A}}\} = \boldsymbol{\mathcal{S}} \tag{B.2.16}$$

$$[\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{A}}] = \boldsymbol{\mathcal{S}} \; , \; \underline{\{\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{A}}\} = \boldsymbol{\mathcal{A}}} \tag{B.2.17}$$

B.3 Definition of $u(16,16|1)$ Super Lie Algebra

B.3.1 $u(16,16|1)$

$u(16,16|1)$ super Lie algebra is defined by

$$
\begin{aligned}
u(16,16|1) &= \{\tilde{\mathcal{M}} \in \boldsymbol{\mathcal{M}}(32|1;\mathbf{C}) \mid \tilde{\mathcal{M}}^\dagger \tilde{\mathcal{G}} + \tilde{\mathcal{G}}\tilde{\mathcal{M}} = 0\ \} && \text{(B.3.1)}\\
\tilde{\mathcal{G}} &= \begin{pmatrix} \mathbf{1}_{16} & 0 & 0 \\ 0 & -\mathbf{1}_{16} & 0 \\ 0 & 0 & 1 \end{pmatrix} . && \text{(B.3.2)}
\end{aligned}
$$

And let us consider the different definition too.

If we use the matrix $\mathcal{U}$ which satisfy $\mathcal{U}^{-1} \equiv \mathcal{U}^\dagger$, we get the following expression.

$$
(\mathcal{U}\tilde{\mathcal{M}}^\dagger\mathcal{U}^\dagger)(i\mathcal{U}\tilde{\mathcal{G}}\mathcal{U}^\dagger) + (i\mathcal{U}\tilde{\mathcal{G}}\mathcal{U}^\dagger)(\mathcal{U}\tilde{\mathcal{M}}\mathcal{U}^\dagger) = 0 \tag{B.3.3}
$$

Now, we put

$$
\mathcal{M} := \mathcal{U}\tilde{\mathcal{M}}\mathcal{U}^\dagger\ ,\ \mathcal{G} := i\mathcal{U}\tilde{\mathcal{G}}\mathcal{U}^\dagger\ , \tag{B.3.4}
$$

and, for example, we adopt the following matrix as $\mathcal{U}$.

$$
\mathcal{U} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}\ ,\ \mathcal{U}^\dagger = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 \\ \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{B.3.5}
$$

This gives us

$$
\begin{aligned}
u(16,16|1) &= \{\mathcal{M} \in \boldsymbol{\mathcal{M}}(32|1;\mathbf{C}) \mid \mathcal{M}^\dagger \mathcal{G} + \mathcal{G}\mathcal{M} = 0\ \} && \text{(B.3.6)}\\
\mathcal{G} &= \begin{pmatrix} 0 & \mathbf{1}_{16} & 0 \\ -\mathbf{1}_{16} & 0 & 0 \\ 0 & 0 & i \end{pmatrix} = \begin{pmatrix} \Gamma^0 & 0 \\ 0 & i \end{pmatrix} . && \text{(B.3.7)}
\end{aligned}
$$

The specific components of $\mathcal{M}$ is given by

$$
\mathcal{M} = \begin{pmatrix} M & \Psi \\ i\bar{\Psi} & V \end{pmatrix} , \tag{B.3.8}
$$

where $\bar{\Psi} := \Psi^\dagger\Gamma^0$ and V is a pure imaginary number.

M can be expanded by γ-matrices as follows.

$$
\begin{aligned}
M &= i\Gamma^{\langle 0\rangle}M_{\langle 0\rangle} + \Gamma^{\langle 1\rangle}M_{\langle 1\rangle} + \Gamma^{\langle 2\rangle}M_{\langle 2\rangle} + i\Gamma^{\langle 3\rangle}M_{\langle 3\rangle} + i\Gamma^{\langle 4\rangle}M_{\langle 4\rangle} + \Gamma^{\langle 5\rangle}M_{\langle 5\rangle} \\
(&= (\Gamma^{\langle 1\rangle}M_{\langle 1\rangle} + \Gamma^{\langle 2\rangle}M_{\langle 2\rangle} + \Gamma^{\langle 5\rangle}M_{\langle 5\rangle}) + i(\Gamma^{\langle 0\rangle}M_{\langle 0\rangle} + \Gamma^{\langle 3\rangle}M_{\langle 3\rangle} + \Gamma^{\langle 4\rangle}M_{\langle 4\rangle}) \qquad) \\
&= i\mathbf{1}_{\underline{32}}u + \Gamma^{\mathbf{a}}u_{\mathbf{a}} + \frac{1}{2}\Gamma^{\mathbf{ab}}u_{\mathbf{ab}} + \frac{i}{3!}\Gamma^{\mathbf{abc}}u_{\mathbf{abc}} + \frac{i}{4!}\Gamma^{\mathbf{abcd}}u_{\mathbf{abcd}} + \frac{1}{5!}\Gamma^{\mathbf{abcde}}u_{\mathbf{abcde}} \\
(&= (\Gamma^{\mathbf{a}}u_{\mathbf{a}} + \frac{1}{2}\Gamma^{\mathbf{ab}}u_{\mathbf{ab}} + \frac{1}{5!}\Gamma^{\mathbf{abcde}}u_{\mathbf{abcde}}) + i(\mathbf{1}_{\underline{32}}u + \frac{1}{3!}\Gamma^{\mathbf{abc}}u_{\mathbf{abc}} + \frac{1}{4!}\Gamma^{\mathbf{abcd}}u_{\mathbf{abcd}}) \qquad)
\end{aligned} \tag{B.3.9}
$$

Here, for $\mathbf{n} = \mathbf{0}$, we put

$$\Gamma^{\langle 0 \rangle} := \mathbf{1}_{\underline{32}} \tag{B.3.10}$$

$$M_{\langle 0 \rangle} := u \ , \tag{B.3.11}$$

and for $\mathbf{n} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}$, we use the following notations.

$$\Gamma^{\langle \mathbf{n} \rangle} := \Gamma^{\mathbf{a}_1 \cdots \mathbf{a}_n} \tag{B.3.12}$$

$$M_{\langle \mathbf{n} \rangle} := \frac{1}{n!} u_{\mathbf{a}_1 \cdots \mathbf{a}_n} \tag{B.3.13}$$

Moreover, this $u(16, 16|1)$ can be seen the direct sum of $\boldsymbol{\mathcal{S}}$ and $i\boldsymbol{\mathcal{A}}$.

Inner product of the vector space

Next we show the inner product of the vector space about the $u(16, 16|1)$. For any two elements x_1, x_2 in complex column vector space V, a scalar quantity is defined as follows.

$$x_1 = \begin{pmatrix} \phi_1 \\ r_1 \end{pmatrix} \in V \quad , \quad x_2 = \begin{pmatrix} \phi_2 \\ r_2 \end{pmatrix} \in V \tag{B.3.14}$$

$$(x_1, x_2) := x_1^\dagger (-i\mathcal{G}) x_2 \tag{B.3.15}$$

$$= -i\phi_1^\dagger \Gamma^0 \phi_2 + r_1^* r_2 \tag{B.3.16}$$

$$= (x_2, x_1)^* \tag{B.3.17}$$

B.3.2 $gl(32|1; \mathbf{R})$

$gl(32|1; \mathbf{R})$ is the set of "real" supermatrix.

The specific components of $\mathcal{M}$ is given by

$$\mathcal{M} = \begin{pmatrix} m & \psi^{(+)} \\ i\bar{\psi}^{(-)} & v \end{pmatrix} , \tag{B.3.18}$$

where $\psi^{(+)}$ and $\psi^{(-)}$ is independent fermions respectively, and $\bar{\psi}^{(-)} := \psi^{(-)\dagger}\Gamma^0 \equiv \psi^{(-)T}\Gamma^0$.

m can be expanded by γ-matrices as follows.

$$\begin{aligned} m &= \sum_{\mathbf{n}=\mathbf{0}}^{\mathbf{5}} \Gamma^{\langle \mathbf{n} \rangle} m_{\langle \mathbf{n} \rangle} \\ &= \Gamma^{\langle 0 \rangle} m_{\langle 0 \rangle} + \Gamma^{\langle 1 \rangle} m_{\langle 1 \rangle} + \Gamma^{\langle 2 \rangle} m_{\langle 2 \rangle} + \Gamma^{\langle 3 \rangle} m_{\langle 3 \rangle} + \Gamma^{\langle 4 \rangle} m_{\langle 4 \rangle} + \Gamma^{\langle 5 \rangle} m_{\langle 5 \rangle} \\ (&= (\Gamma^{\langle 1 \rangle} m_{\langle 1 \rangle} + \Gamma^{\langle 2 \rangle} m_{\langle 2 \rangle} + \Gamma^{\langle 5 \rangle} m_{\langle 5 \rangle}) + (\Gamma^{\langle 0 \rangle} m_{\langle 0 \rangle} + \Gamma^{\langle 3 \rangle} m_{\langle 3 \rangle} + \Gamma^{\langle 4 \rangle} m_{\langle 4 \rangle}) \qquad) \end{aligned}$$

$$= \mathbf{1}_{\underline{32}}u + \Gamma^{\mathbf{a}}u_{\mathbf{a}} + \frac{1}{2}\Gamma^{\mathbf{ab}}u_{\mathbf{ab}} + \frac{1}{3!}\Gamma^{\mathbf{abc}}u_{\mathbf{abc}} + \frac{1}{4!}\Gamma^{\mathbf{abcd}}u_{\mathbf{abcd}} + \frac{1}{5!}\Gamma^{\mathbf{abcde}}u_{\mathbf{abcde}}$$

$$\left(= (\Gamma^{\mathbf{a}}u_{\mathbf{a}} + \frac{1}{2}\Gamma^{\mathbf{ab}}u_{\mathbf{ab}} + \frac{1}{5!}\Gamma^{\mathbf{abcde}}u_{\mathbf{abcde}}) + (\mathbf{1}_{\underline{32}}u + \frac{1}{3!}\Gamma^{\mathbf{abc}}u_{\mathbf{abc}} + \frac{1}{4!}\Gamma^{\mathbf{abcd}}u_{\mathbf{abcd}}) \right) \tag{B.3.19}$$

Here, in the same way as $u(16,16|1)$ case, for $\mathbf{n} = \mathbf{0}$, we put

$$\Gamma^{\langle \mathbf{0} \rangle} := \mathbf{1}_{\underline{32}} \tag{B.3.20}$$

$$m_{\langle \mathbf{0} \rangle} := u \; , \tag{B.3.21}$$

and for $\mathbf{n} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}$, we use the following notations.

$$\Gamma^{\langle \mathbf{n} \rangle} := \Gamma^{\mathbf{a}_1 \cdots \mathbf{a}_n} \tag{B.3.22}$$

$$m_{\langle \mathbf{n} \rangle} := \frac{1}{n!}u_{\mathbf{a}_1 \cdots \mathbf{a}_n} \tag{B.3.23}$$

Moreover, this $gl(32|1;\mathbf{R})$ can be seen the direct sum of $\boldsymbol{\mathcal{S}}$ and $\boldsymbol{\mathcal{A}}$.

B.3.3 Decomposition of $u(16,16|1)$ into $\boldsymbol{\mathcal{S}}$ and $\boldsymbol{\mathcal{A}}$

The elements of $u(16,16|1)$ can be uniquely decomposed into the sum of "the element($\mathcal{S}$) of $\boldsymbol{\mathcal{S}}$" and "i times the element($\mathcal{A}$) of $\boldsymbol{\mathcal{A}}$".

$$(\mathcal{S}^T - i\mathcal{A}^T)\mathcal{G} + \mathcal{G}(\mathcal{S} + i\mathcal{A}) = 0 \tag{B.3.24}$$

$$\left(\iff (\mathcal{S} + i\mathcal{A})^\dagger\mathcal{G} + \mathcal{G}(\mathcal{S} + i\mathcal{A}) = 0 \right) \tag{B.3.25}$$

B.3.4 Decomposition of $gl(32|1;\mathbf{R})$ into $\boldsymbol{\mathcal{S}}$ and $\boldsymbol{\mathcal{A}}$

The elements of $gl(32|1;\mathbf{R})$ can be uniquely decomposed into the sum of "the element($\mathcal{S}$) of $\boldsymbol{\mathcal{S}}$" and "the element($\mathcal{A}$) of $\boldsymbol{\mathcal{A}}$".

$$(\mathcal{S}^T - \mathcal{A}^T)\mathcal{G} + \mathcal{G}(\mathcal{S} + \mathcal{A}) = 0 \tag{B.3.26}$$

($\iff$ There is no expression which corresponds to (B.3.25).)

B.3.5 Relation between $u(16,16|1)$ and $gl(32|1;\mathbf{R})$

One can expresses the elements of $u(16,16|1)$ as $(\mathcal{S} + i\mathcal{A})$, and can adapt this to the elements of $gl(32|1;\mathbf{R})$, $(\mathcal{S} + \mathcal{A})$.

In a sense, this is a kind of analytic continuation.

Appendix C

Proof on propagators

C.1 Proof against a perturbative generation for W and $A_\mu^{(-)}$ propagators

In this appendix we prove that it is impossible to generate propagators for W and $A_\mu^{(-)}$ fields through perturbative calculations in $osp(1|32, R)$ matrix model discussed in section 2. Of course, this proof does not exclude nonperturbative appearance of propagators for them. First we assign charges $(1, 0, -1)$ to the bosonic fields $(m_e, m_o^{(+)}, m_o^{(-)})$ and $(0, 1/2)$ to (ψ_L, ψ_R). As we can see from the action (4.1.30), every three point vertex has charge $3, 3/2$ or 0. Similarly in the background of (4.1.46), propagators appear at tree level for $\langle m_e m_o^{(-)} \rangle$ and $\langle \psi_L \bar{\psi}_L \rangle$ and they have charges 1 and 0 respectively. On the other hand, two point function $\langle WW \rangle$ which is included in $\langle m_e m_e \rangle$ or $\langle A_\mu^{(-)} A_\nu^{(-)} \rangle$ in $\langle m_o^{(-)} m_o^{(-)} \rangle$ has charge 2 or -2 respectively. Hence it is clearly impossible to generate these two point functions perturbatively no matter how we combine the above vertices and tree level propagators.

Appendix D

Complex Graves-Cayley algebra $\mathfrak{C}^C$

D.1 Graves-Cayley algebra $\mathfrak{C}$

Let $\mathfrak{C} = \sum_{i=0}^{7} \mathbf{R}\, e_i$ be the Graves-Cayley algebra: $\mathfrak{C}$ is an 8-dimensional $\mathbf{R}$-vector space with the multiplication such that $e_0 = 1$ is the unit, ${e_i}^2 = -1$ $_{(i=1,\cdots,7)}$, $e_i e_j = -e_j e_i$ $_{(1 \leq i \neq j \leq 7)}$ and $e_1 e_2 = e_3$, $e_1 e_4 = e_5$, $e_2 e_5 = e_7$, *etc.*. The element 'a' of $\mathfrak{C}$ is called *octonion* (or *Graves-Cayley number*). The multiplication rule among the bases of $\mathfrak{C}$ can be represented in a diagram as Figure D.1. So if we take e_1, e_2, e_3 for example,

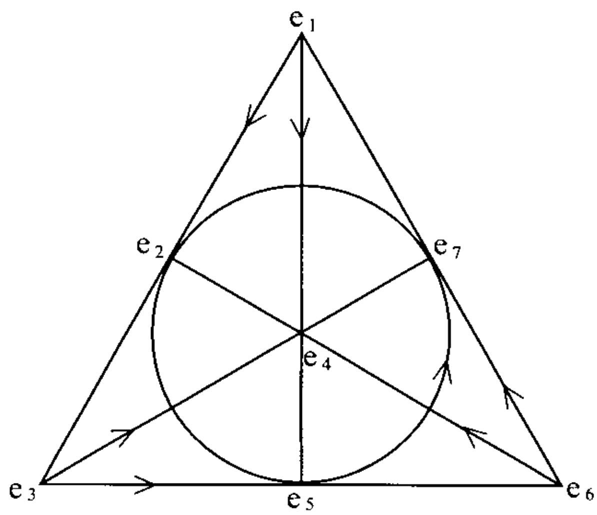

Figure D.1: Multiplication diagram for octonion

$$\begin{aligned} e_1 e_2 = e_3 \ , \ e_2 e_3 = e_1 \ , \ e_3 e_1 = e_2 \\ e_3 e_2 = -e_1 \ , \ e_2 e_1 = -e_3 \ , \ e_1 e_3 = -e_2 \ . \end{aligned} \tag{D.1.1}$$

The same things can be said of other six lines. What has to be noticed is that this algebra is *non-associative* as well as non-commutative. It is often very useful to introduce the following notations,

$$e_i e_j = -\delta_{ij} + \sum_{k=1}^{7} \sigma_{ijk} e_k \ , \qquad (i,j,k = 1,\cdots,7) \tag{D.1.2}$$

where the σ_{ijk} are totally anti-symmetric in indices, with values 1, 0, -1. For instance, $\sigma_{ijk} = +1$ for $ijk = 123,\ 356,\ 671,\ 145,\ 347,\ 642,\ 257$.

In $\mathfrak{C}$, the conjugate $\bar{a}$ and the real part $\boldsymbol{Re}(a)$ are defined respectively as follows.

$$a \equiv a_0 + \sum_{i=1}^{7} a_i e_i \tag{D.1.3}$$

$$\bar{a} = \overline{a_0 + \sum_{i=1}^{7} a_i e_i} \tag{D.1.4}$$

$$\equiv a_0 - \sum_{i=1}^{7} a_i e_i \tag{D.1.5}$$

$$\boldsymbol{Re}(a) \equiv \frac{1}{2}(a + \bar{a}) \qquad \in \mathbf{R} \tag{D.1.6}$$

$$= a_0 \tag{D.1.7}$$

$$= \boldsymbol{Re}(\bar{a}) \tag{D.1.8}$$

Moreover, the inner product (a,b) $(a,b \in \mathfrak{C})$ is defined by

$$(a,b) \equiv a_0 b_0 + \sum_{i=1}^{7} a_i b_i \quad \in \mathbf{R} \tag{D.1.9}$$

$$= (b,a) \ . \tag{D.1.10}$$

Therefore we have

$$(a,a) = (a_0)^2 + \sum_{i=1}^{7} (a_i)^2 \quad \in \mathbf{R} \tag{D.1.11}$$

$$\geq 0 \ . \tag{D.1.12}$$

D.2 C and H in $\mathfrak{C}$

D.2.1 Complex number field in $\mathfrak{C}$

The Graves-Cayley algebra $\mathfrak{C}$ contains the field of complex numbers $\mathbf{C}$.

$$\mathbf{C} \;=\; \{r_0 + r_4 e_4 \mid r_0, r_4 \in \mathbf{R}\} \tag{D.2.1}$$

$$\begin{aligned} a &= a_0 + \sum_{i=1}^{7} a_i e_i && \text{(D.2.2)}\\ &= (a_0 + a_4 e_4) + (a_1 - a_5 e_4)e_1 + (a_2 + a_6 e_4)e_2 + (a_3 - a_7 e_4)e_3 && \text{(D.2.3)}\\ &= c_0 + c_1 e_1 + c_2 e_2 + c_3 e_3 && \text{(D.2.4)}\\ & \qquad c_k \in \mathbf{C} \qquad (k = 0, 1, 2, 3) \end{aligned}$$

It must be noted that these complex numbers which have an imaginary unit 'e_4' are independent of what are introduced in the following subsection whose imaginary unit is 'i'.

D.2.2 Quaternion field in $\mathfrak{C}$

Furthermore, the Graves-Cayley algebra $\mathfrak{C}$ contains the field of quaternions $\mathbf{H}$ as well.

$$\mathbf{H} \;=\; \{r_0 + r_1 e_1 + r_2 e_2 + r_3 e_3 \mid r_0, r_1, r_2, r_3 \in \mathbf{R}\} \tag{D.2.5}$$

$$\begin{aligned} a &= a_0 + \sum_{i=1}^{7} a_i e_i && \text{(D.2.6)}\\ &= (a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3) + (a_4 + a_5 e_1 - a_6 e_2 + a_7 e_3)e_4 && \text{(D.2.7)}\\ &= q_0 + q_4 e_4 && \text{(D.2.8)}\\ & \qquad q_k \in \mathbf{H} \qquad (k = 0, 4) \end{aligned}$$

D.3 Complex Graves-Cayley algebra $\mathfrak{C}^c$

Let $\mathfrak{C}^c$, called the complex Graves-Cayley algebra, be the complexification of $\mathfrak{C}$.

$$\mathfrak{C}^c = \{a + ib \mid a, b \in \mathfrak{C}\ ,\ i^2 = -1\} \tag{D.3.1}$$

Here, we should notice that 'i' is introduced as a different imaginary unit from 'e_4' which is that of the complex number field embedded in $\mathfrak{C}$ as mentioned in the previous subsection. This 'i' commutes to all the $e_{\tilde{i}}$'s $_{(\tilde{i}=0,\cdots,7)}$.

In $\mathfrak{C}^c$, the conjugate $\bar{x}$ and the real part $\boldsymbol{Re}^c(x)$ are defined respectively as follows.

$$\begin{aligned} x &= (a_0 + \sum_{i=1}^{7} a_i e_i) + i(b_0 + \sum_{i=1}^{7} b_i e_i) \\ &= (a_0 + ib_0) + \sum_{i=1}^{7}(a_i + ib_i)e_i \\ &\equiv x_0 + \sum_{i=1}^{7} x_i e_i \qquad \text{(D.3.2)} \\ \bar{x} &= \overline{(a_0 + \sum_{i=1}^{7} a_i e_i)} + i\overline{(b_0 + \sum_{i=1}^{7} b_i e_i)} \\ &= (a_0 + ib_0) - \sum_{i=1}^{7}(a_i + ib_i)e_i \\ &\equiv x_0 - \sum_{i=1}^{7} x_i e_i \qquad \text{(D.3.3)} \\ \boldsymbol{Re}^c(x) &\equiv \frac{1}{2}(x + \bar{x}) \qquad \in \mathbf{C} \qquad \text{(D.3.4)} \\ &= x_0 \qquad \text{(D.3.5)} \\ &= a_0 + ib_0 \qquad \text{(D.3.6)} \\ &= \boldsymbol{Re}^c(\bar{x}) \qquad \text{(D.3.7)} \end{aligned}$$

Moreover, for any two elements $x = a + ib$ and $y = c + id$ of $\mathfrak{C}^c$, the inner product (x, y) is defined by

$$\begin{aligned} (x, y) &\equiv x_0 y_0 + \sum_{i=1}^{7} x_i y_i \quad \in \mathbf{C} \qquad \text{(D.3.8)} \\ &= (a_0 + ib_0)(c_0 + id_0) + \sum_{i=1}^{7}(a_i + ib_i)(c_i + id_i) \qquad \text{(D.3.9)} \\ &= (y, x) \ . \qquad \text{(D.3.10)} \end{aligned}$$

Therefore we have

$$\begin{aligned} (x, x) &= (x_0)^2 + \sum_{i=1}^{7}(x_i)^2 \quad \in \mathbf{C} \qquad \text{(D.3.11)} \\ &= \bar{x}x = x\bar{x} \ . \qquad \text{(D.3.12)} \end{aligned}$$

Furthermore, in $\mathfrak{C}^c$, the hermitian product $\langle x, y \rangle$ $(x, y \in \mathfrak{C}^c)$ is defined by

$$\langle x, y \rangle \equiv (x^*, y) \quad \in \mathbf{C} \qquad \text{(D.3.13)}$$

$$= \quad (a_0 - ib_0)(c_0 + id_0) + \sum_{i=1}^{7}(a_i - ib_i)(c_i + id_i) \; , \qquad \text{(D.3.14)}$$

where $(\cdots)^*$, called the complex conjugation with respect to 'i', is defined by the following mapping.

$$(a + ib)^* \;\equiv\; a - ib \qquad \text{(D.3.15)}$$
$$a, b \in \mathfrak{C}$$

Therefore we have

$$x^* \;=\; x_0^* + \sum_{i=1}^{7} x_i^* e_i \; . \qquad \text{(D.3.16)}$$

Naturally, we must not confuse this complex conjugation $(\cdots)^*$ with the octonionic conjugation $\overline{(\cdots)}$. An example is

$$\boldsymbol{Re}^c(x) = \boldsymbol{Re}^c(\bar{x}) \neq \boldsymbol{Re}^c(x^*) \; . \qquad \text{(D.3.17)}$$

Consequently, for any element $x = a + ib$ of $\mathfrak{C}^c$, we have

$$\begin{aligned}
\langle x, x \rangle \;&=\; \Big((a_0)^2 + \sum_{i=1}^{7}(a_i)^2\Big) + \Big((b_0)^2 + \sum_{i=1}^{7}(b_i)^2\Big) && \text{(D.3.18)}\\
&=\; (a, a) + (b, b) \quad \in \mathbf{R} && \text{(D.3.19)}\\
&\geq\; 0 \; . && \text{(D.3.20)}
\end{aligned}$$

D.4 Some helpful formulas on elements of $\mathfrak{C}^c$

We can use the following formulas for any $w, x, y, z \in \mathfrak{C}^c$.

$$\begin{aligned}
(x^*)^* \;&=\; x && \text{(D.4.1)}\\
(x + y)^* \;&=\; x^* + y^* && \text{(D.4.2)}\\
(xy)^* \;&=\; x^* y^* && \text{(D.4.3)}\\
\overline{(\bar{x})} \;&=\; x && \text{(D.4.4)}\\
\overline{(x + y)} \;&=\; \bar{x} + \bar{y} && \text{(D.4.5)}\\
\overline{(xy)} \;&=\; \bar{y}\bar{x} && \text{(D.4.6)}\\
(x, y) \;&=\; \frac{1}{2}(\bar{x}y + \bar{y}x) \;=\; \frac{1}{2}(x\bar{y} + y\bar{x}) && \text{(D.4.7)}\\
&=\; \boldsymbol{Re}^c(\bar{x}y) \;=\; \boldsymbol{Re}^c(x\bar{y}) && \text{(D.4.8)}\\
(x, yz) \;&=\; (y, x\bar{z}) \;=\; (z, \bar{y}x) && \text{(D.4.9)}
\end{aligned}$$

$$
\begin{aligned}
(w,x)(y,z) &= \frac{1}{2}\{(wy,xz)+(xy,wz)\} && \text{(D.4.10)}\\
&= \frac{1}{2}\{(yw,zx)+(yx,zw)\} && \text{(D.4.11)}\\
\boldsymbol{Re}^c(xy) &= x_0y_0 - x_iy_i && \text{(D.4.12)}\\
&= \frac{1}{2}(xy+\bar{y}\bar{x}) \;=\; \frac{1}{2}(\bar{x}\bar{y}+yx) && \text{(D.4.13)}\\
&= \boldsymbol{Re}^c(yx) && \text{(D.4.14)}\\
\boldsymbol{Re}^c(xyz) &\equiv \boldsymbol{Re}^c(x(yz)) \;=\; \boldsymbol{Re}^c((xy)z) && \text{(D.4.15)}\\
&= x_0y_0z_0 - x_0y_iz_i - x_iy_0z_i - x_iy_iz_0 - x_iy_jz_k\sigma_{ijk} && \text{(D.4.16)}\\
&= \frac{1}{2}(x(yz)+(\bar{z}\bar{y})\bar{x}) \;=\; \frac{1}{2}((xy)z+\bar{z}(\bar{y}\bar{x})) && \text{(D.4.17)}\\
&= \boldsymbol{Re}^c(yzx) \;=\; \boldsymbol{Re}^c(zxy) && \text{(D.4.18)}\\
&= \boldsymbol{Re}^c(zyx) - 2x_iy_jz_k\sigma_{ijk} && \text{(D.4.19)}\\
& && (i,j,k=1,\cdots,7)
\end{aligned}
$$

$$
\begin{aligned}
(xy)z &= \boldsymbol{Re}^c(xyz)\\
&\quad + \Big(x_0y_0z_l + x_0y_lz_0 + x_ly_0z_0 - x_iy_iz_l\\
&\quad + x_0y_iz_j\sigma_{ijl} + x_iy_0z_j\sigma_{ijl} + x_iy_jz_0\sigma_{ijl}\\
&\quad + x_iy_jz_k\sigma_{ijm}\sigma_{klm} \Big)e_l && \text{(D.4.20)}\\
x(yz) &= \boldsymbol{Re}^c(xyz)\\
&\quad + \Big(x_0y_0z_l + x_0y_lz_0 + x_ly_0z_0 - x_ly_iz_i\\
&\quad + x_0y_iz_j\sigma_{ijl} + x_iy_0z_j\sigma_{ijl} + x_iy_jz_0\sigma_{ijl}\\
&\quad - x_iy_jz_k\sigma_{jkm}\sigma_{ilm} \Big)e_l && \text{(D.4.21)}\\
(xy)\bar{y} &= x(y\bar{y}) \;=\; (y\bar{y})x \;=\; y(\bar{y}x) && \text{(D.4.22)}\\
(xy)\bar{x} &= x(y\bar{x}) \quad , \quad (xy)x \;=\; x(yx) && \text{(D.4.23)}\\
(xx)y &= x(xy) \quad , \quad x(yy) \;=\; (xy)y && \text{(D.4.24)}\\
(x,y)z &= \frac{1}{2}\{\bar{x}(yz)+\bar{y}(xz)\} && \text{(D.4.25)}\\
&= \frac{1}{2}\{(zy)\bar{x}+(zx)\bar{y}\} && \text{(D.4.26)}\\
[x,y,z] &\equiv (xy)z - x(yz) && \text{(D.4.27)}\\
&= x_iy_jz_k\left(\sigma_{ijm}\sigma_{klm}+\sigma_{jkm}\sigma_{ilm}+\delta_{kj}\delta_{il}-\delta_{kl}\delta_{ij}\right)e_l && \text{(D.4.28)}\\
&= x_iy_jz_k\left(\sigma_{ijm}\sigma_{klm}+\sigma_{jkm}\sigma_{ilm}+\sigma_{kim}\sigma_{jlm}\right)e_l && \text{(D.4.29)}\\
&\equiv x_iy_jz_k\left(\rho_{ijkl}\right)e_l && \text{(D.4.30)}\\
& && (\rho_{ijkl}:\ \text{completely antisymmetric})\\
[x,y,z] &= [y,z,x] \;=\; [z,x,y] && \text{(D.4.31)}\\
&= -[z,y,x] \;=\; -[y,x,z] \;=\; -[x,z,y] && \text{(D.4.32)}
\end{aligned}
$$

$$
\begin{aligned}
&= -[x, y, \bar{z}] \;=\; [\bar{z}, \bar{y}, \bar{x}] \;=\; -\overline{[x, y, z]} && \text{(D.4.33)}\\
\boldsymbol{Re}^{c}(\,[x, y, z]\,) &= 0 && \text{(D.4.34)}\\
(xy)z + \overline{x(yz)} &= 2\boldsymbol{Re}^{c}(xyz) + [x, y, z] && \text{(D.4.35)}\\
\overline{(xy)z} + x(yz) &= 2\boldsymbol{Re}^{c}(xyz) - [x, y, z] && \text{(D.4.36)}\\
x(yz) + (yz)x &= (xy)z + y(zx) && \text{(D.4.37)}\\
x(yz) + x(zy) &= (xy)z + (xz)y && \text{(D.4.38)}\\
(xy)z + (yx)z &= x(yz) + y(xz) && \text{(D.4.39)}
\end{aligned}
$$

Appendix E

Complex exceptional Jordan algebra $\mathfrak{J}^C$

E.1 Jordan algebra j

We define $\mathfrak{j}$ as the Jordan algebra consisting of all 2×2 hermitian matrices A with entries in the Graves-Cayley algebra $\mathfrak{C}$.

$$\mathfrak{j} = \{A \in M(2, \mathfrak{C}) \mid A^\ddagger = A\} \qquad (A^\ddagger \equiv (\bar{A})^T) \tag{E.1.1}$$

The specific components of A can be written as follows.

$$A = \begin{pmatrix} Q_1 & \phi_3 \\ \bar{\phi}_3 & Q_2 \end{pmatrix} \tag{E.1.2}$$
$$Q_I \in \mathbf{R} \qquad \phi_3 \in \mathfrak{C} \qquad (I = 1, 2)$$

Therefore, $\mathfrak{j}$ is a 10-dimensional $\mathbf{R}$-vector space.

E.2 Exceptional Jordan algebra $\mathfrak{J}$

We define $\mathfrak{J}$ as the exceptional Jordan algebra consisting of all 3×3 hermitian matrices A with entries in the Graves-Cayley algebra $\mathfrak{C}$.

$$\mathfrak{J} = \{A \in M(3, \mathfrak{C}) \mid A^\ddagger = A\} \qquad (A^\ddagger \equiv (\bar{A})^T) \tag{E.2.1}$$

The specific components of A can be written as follows.

$$A = \begin{pmatrix} Q_1 & \phi_3 & \bar{\phi}_2 \\ \bar{\phi}_3 & Q_2 & \phi_1 \\ \phi_2 & \bar{\phi}_1 & Q_3 \end{pmatrix} \tag{E.2.2}$$
$$Q_I \in \mathbf{R} \qquad \phi_I \in \mathfrak{C} \qquad (I = 1, 2, 3)$$

Therefore, $\mathfrak{J}$ is a 27-dimensional $\mathbf{R}$-vector space.

E.3 Complex exceptional Jordan algebra $\mathfrak{J}^c$

Let $\mathfrak{J}^c$, called the complex exceptional Jordan algebra, be the complexification of $\mathfrak{J}$.

$$\mathfrak{J}^c = \{A + iB \mid A, B \in \mathfrak{J}\ ,\ i^2 = -1\} \tag{E.3.1}$$

Therefore, the specific components of $X \in \mathfrak{J}^c$ can be written as follows.

$$\begin{aligned}
X &= \begin{pmatrix} Q_1 & \phi_3 & \bar{\phi}_2 \\ \bar{\phi}_3 & Q_2 & \phi_1 \\ \phi_2 & \bar{\phi}_1 & Q_3 \end{pmatrix} + i \begin{pmatrix} P_1 & \pi_3 & \bar{\pi}_2 \\ \bar{\pi}_3 & P_2 & \pi_1 \\ \pi_2 & \bar{\pi}_1 & P_3 \end{pmatrix} && \text{(E.3.2)} \\
&\qquad Q_I, P_I \in \mathbf{R} \qquad \phi_I, \pi_I \in \mathfrak{C} \qquad (I = 1, 2, 3) \\
&= \begin{pmatrix} x_1 & \xi_3 & \bar{\xi}_2 \\ \bar{\xi}_3 & x_2 & \xi_1 \\ \xi_2 & \bar{\xi}_1 & x_3 \end{pmatrix} && \text{(E.3.3)} \\
&\qquad x_I \in \mathbf{C} \qquad \xi_I \in \mathfrak{C}^c \qquad (I = 1, 2, 3) \\
&\equiv X(x, \xi) && \text{(E.3.4)} \\
&\qquad \begin{cases} x_I = Q_I + iP_I \\ \xi_I = \phi_I + i\pi_I \\ \bar{\xi}_I = \bar{\phi}_I + i\bar{\pi}_I \end{cases}
\end{aligned}$$

Accordingly, we can also define this $\mathfrak{J}^c$ as

$$\mathfrak{J}^c = \{X \in M(3, \mathfrak{C}^c) \mid X^\ddagger = X\} \qquad (X^\ddagger \equiv (\bar{X})^T)\ . \tag{E.3.5}$$

E.4 Two kinds of hermitian adjoints

In $\mathfrak{C}^c$, there exist two conjugations: One is the complex conjugation $(\cdots)^*$, another is the octonionic conjugation $\overline{(\cdots)}$. As a result, in $\mathfrak{J}^c$, there exist two kinds of hermitian adjoints.

$$X^\dagger \equiv (X^*)^T \tag{E.4.1}$$
$$X^\ddagger \equiv (\bar{X})^T \tag{E.4.2}$$

E.5 Operations

Now, for any $X, Y, Z \in \mathfrak{J}^c$, each of the operations is defined as follows.

$$X = \begin{pmatrix} x_1 & \xi_3 & \bar{\xi}_2 \\ \bar{\xi}_3 & x_2 & \xi_1 \\ \xi_2 & \bar{\xi}_1 & x_3 \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 & \eta_3 & \bar{\eta}_2 \\ \bar{\eta}_3 & y_2 & \eta_1 \\ \eta_2 & \bar{\eta}_1 & y_3 \end{pmatrix}, \quad Z = \begin{pmatrix} z_1 & \zeta_3 & \bar{\zeta}_2 \\ \bar{\zeta}_3 & z_2 & \zeta_1 \\ \zeta_2 & \bar{\zeta}_1 & z_3 \end{pmatrix} \qquad \text{(E.5.1)}$$

trace $tr(X)$

$$tr(X) \equiv x_1 + x_2 + x_3 \qquad \text{(E.5.2)}$$

Jordan multiplication $X \circ Y$

$$X \circ Y \equiv \frac{1}{2}(XY + YX) \qquad \text{(E.5.3)}$$
$$= \frac{1}{2}\{X, Y\} \qquad \text{(E.5.4)}$$
$$= Y \circ X \qquad \text{(E.5.5)}$$

inner product $(X, Y) \in \mathbf{C}$

$$(X, Y) \equiv tr(X \circ Y) \qquad \text{(E.5.6)}$$
$$= \frac{1}{2}tr(XY) + \frac{1}{2}tr(YX) \qquad \text{(E.5.7)}$$
$$= (Y, X) \qquad \text{(E.5.8)}$$

hermitian product $\langle X, Y \rangle \in \mathbf{C}$

$$\langle X, Y \rangle \equiv (X^*, Y) \qquad \text{(E.5.9)}$$

$$\left(\; 0 \;\leq\; \langle X, X \rangle \;\in\; \mathbf{R} \; \right) \qquad \text{(E.5.10)}$$

Freudenthal multiplication $X \times Y$

$$X \times Y \equiv X \circ Y - \frac{1}{2}tr(X)Y - \frac{1}{2}tr(Y)X + \frac{1}{2}tr(X)tr(Y)I - \frac{1}{2}(X, Y)I \qquad \text{(E.5.11)}$$

$$= Y \times X \qquad \text{(E.5.12)}$$

I : unit matrix

trilinear form $tr(X,Y,Z) \ \in \mathbf{C}$

$$\begin{aligned}
tr(X,Y,Z) &\equiv (X, Y \circ Z) && \text{(E.5.13)}\\
&= tr(X \circ (Y \circ Z)) && \text{(E.5.14)}\\
&= \frac{1}{4}tr(X(YZ)) + \frac{1}{4}tr(X(ZY)) \\
&\quad + \frac{1}{4}tr((YZ)X) + \frac{1}{4}tr((ZY)X) && \text{(E.5.15)}\\
&= \frac{1}{2}(X, YZ) + \frac{1}{2}(X, ZY) && \text{(E.5.16)}\\
&= tr(Y,Z,X) \ = \ tr(Z,X,Y) && \text{(E.5.17)}\\
&= tr(Z,Y,X) \ = \ tr(Y,X,Z) \ = \ tr(X,Z,Y) && \text{(E.5.18)}\\
&= (X \circ Y, Z) && \text{(E.5.19)}
\end{aligned}$$

cubic form $(X,Y,Z) \ \in \mathbf{C}$

$$\begin{aligned}
(X,Y,Z) &\equiv (X, Y \times Z) && \text{(E.5.20)}\\
&= tr(X \circ (Y \times Z)) && \text{(E.5.21)}\\
&= tr(X,Y,Z) \\
&\quad - \frac{1}{2}tr(X)\ (Y,Z) - \frac{1}{2}tr(Y)\ (Z,X) - \frac{1}{2}tr(Z)\ (X,Y) \\
&\quad + \frac{1}{2}tr(X)\ tr(Y)\ tr(Z) && \text{(E.5.22)}\\
&= (Y,Z,X) \ = \ (Z,X,Y) && \text{(E.5.23)}\\
&= (Z,Y,X) \ = \ (Y,X,Z) \ = \ (X,Z,Y) && \text{(E.5.24)}\\
&= (X \times Y, Z) && \text{(E.5.25)}
\end{aligned}$$

determinant $det(X) \ \in \mathbf{C}$

$$\begin{aligned}
det(X) &\equiv \frac{1}{3}(X,X,X) && \text{(E.5.26)}\\
&= \frac{1}{6}tr(X(XX)) + \frac{1}{6}tr((XX)X) - \frac{1}{2}tr(X^2)tr(X) + \frac{1}{6}tr(X)^3 && \text{(E.5.27)}
\end{aligned}$$

cycle mapping $\mathcal{P}(X)$

$$X = \begin{pmatrix} x_1 & \xi_3 & \bar{\xi}_2 \\ \bar{\xi}_3 & x_2 & \xi_1 \\ \xi_2 & \bar{\xi}_1 & x_3 \end{pmatrix} \qquad \text{(E.5.28)}$$

For any $X \in \mathfrak{J}^c$, the cycle mapping $\mathcal{P}(X)$ is defined by

$$\mathcal{P}(X) \equiv \begin{pmatrix} x_2 & \xi_1 & \bar{\xi}_3 \\ \bar{\xi}_1 & x_3 & \xi_2 \\ \xi_3 & \bar{\xi}_2 & x_1 \end{pmatrix} . \tag{E.5.29}$$

Namely, the cycle mapping is the cyclic permutation with respect to the indices $I = 1, 2, 3$. Therefore, we have

$$\mathcal{P}^2(X) = \begin{pmatrix} x_3 & \xi_2 & \bar{\xi}_1 \\ \bar{\xi}_2 & x_1 & \xi_3 \\ \xi_1 & \bar{\xi}_3 & x_2 \end{pmatrix} , \tag{E.5.30}$$

$$\mathcal{P}^3(X) = 1 \cdot X = X . \tag{E.5.31}$$

E.6 Some helpful formulas on elements of $\mathfrak{J}^c$

We can use the following formulas for any $X, Y, Z \in \mathfrak{J}^c$.

$$I \circ I = I \tag{E.6.1}$$
$$I \circ X = X \tag{E.6.2}$$
$$I \times I = I \tag{E.6.3}$$
$$I \times X = \frac{1}{2}(tr(X)I - X) \tag{E.6.4}$$
$$(I, I) = 3 \tag{E.6.5}$$
$$(X, I) = tr(X, I, I) = (X, I, I) = tr(X) \tag{E.6.6}$$
$$(X, Y) = tr(X, Y, I) \tag{E.6.7}$$
$$(X, YZ) = (Y, ZX) = (Z, XY) \tag{E.6.8}$$
$$tr(X \times Y) = \frac{1}{2}tr(X)tr(Y) - \frac{1}{2}(X, Y) \tag{E.6.9}$$
$$X \circ (X \times X) = det(X)\ I \tag{E.6.10}$$

I : unit matrix

E.7 Indication by components

$$X \circ X = \begin{pmatrix} (x_1)^2 + \xi_2\bar{\xi}_2 + \xi_3\bar{\xi}_3 & \overline{\xi_1\xi_2} + (x_1 + x_2)\xi_3 & \xi_3\xi_1 + (x_3 + x_1)\bar{\xi}_2 \\ \xi_1\xi_2 + (x_1 + x_2)\bar{\xi}_3 & (x_2)^2 + \xi_3\bar{\xi}_3 + \xi_1\bar{\xi}_1 & \overline{\xi_2\xi_3} + (x_2 + x_3)\xi_1 \\ \overline{\xi_3\xi_1} + (x_3 + x_1)\xi_2 & \xi_2\xi_3 + (x_2 + x_3)\bar{\xi}_1 & (x_3)^2 + \xi_1\bar{\xi}_1 + \xi_2\bar{\xi}_2 \end{pmatrix} \tag{E.7.1}$$

$$= \quad XX \tag{E.7.2}$$

$$X \times X = \begin{pmatrix} x_2x_3 - \xi_1\bar{\xi}_1 & \overline{\xi_1\xi_2} - x_3\xi_3 & \xi_3\xi_1 - x_2\bar{\xi}_2 \\ \xi_1\xi_2 - x_3\bar{\xi}_3 & x_3x_1 - \xi_2\bar{\xi}_2 & \overline{\xi_2\xi_3} - x_1\xi_1 \\ \overline{\xi_3\xi_1} - x_2\xi_2 & \xi_2\xi_3 - x_1\bar{\xi}_1 & x_1x_2 - \xi_3\bar{\xi}_3 \end{pmatrix} \tag{E.7.3}$$

$$tr(XY) = \sum_{I=1}^{3}\Big(x_Iy_I + (\bar{\xi}_I\eta_I + \xi_I\bar{\eta}_I)\Big) \tag{E.7.4}$$

$$\begin{aligned} tr(X(YZ)) = \sum_{I=1}^{3}\Big(& x_Iy_Iz_I + x_I((\bar{\eta}_{I+1}\zeta_{I+1}) + (\eta_{I+2}\bar{\zeta}_{I+2})) \\ & + y_I((\xi_{I+1}\bar{\zeta}_{I+1}) + (\bar{\xi}_{I+2}\zeta_{I+2})) + z_I((\bar{\xi}_{I+1}\eta_{I+1}) + (\xi_{I+2}\bar{\eta}_{I+2})) \\ & + (\xi_I(\eta_{I+1}\zeta_{I+2}) + \overline{(\zeta_{I+1}\eta_{I+2})\xi_I}) \Big) \end{aligned} \tag{E.7.5}$$

$$\begin{aligned} tr((XY)Z) = \sum_{I=1}^{3}\Big(& x_Iy_Iz_I + x_I((\bar{\eta}_{I+1}\zeta_{I+1}) + (\eta_{I+2}\bar{\zeta}_{I+2})) \\ & + y_I((\xi_{I+1}\bar{\zeta}_{I+1}) + (\bar{\xi}_{I+2}\zeta_{I+2})) + z_I((\bar{\xi}_{I+1}\eta_{I+1}) + (\xi_{I+2}\bar{\eta}_{I+2})) \\ & + ((\xi_I\eta_{I+1})\zeta_{I+2} + \overline{\zeta_{I+1}(\eta_{I+2}\xi_I)}) \Big) \end{aligned} \tag{E.7.6}$$

$$(X,Y) = \sum_{I=1}^{3}\Big(x_Iy_I + 2(\xi_I,\eta_I)\Big) \tag{E.7.7}$$

$$\langle X,Y\rangle = \sum_{I=1}^{3}\Big(x_I^*y_I + 2\langle\xi_I,\eta_I\rangle\Big) \tag{E.7.8}$$

$$\begin{aligned} tr(X,Y,Z) = \sum_{I=1}^{3}\Big(& x_Iy_Iz_I + x_I((\eta_{I+1},\zeta_{I+1}) + (\eta_{I+2},\zeta_{I+2})) \\ & + y_I((\zeta_{I+1},\xi_{I+1}) + (\zeta_{I+2},\xi_{I+2})) + z_I((\xi_{I+1},\eta_{I+1}) + (\xi_{I+2},\eta_{I+2})) \\ & + \boldsymbol{Re}^c(\xi_I\eta_{I+1}\zeta_{I+2} + \xi_I\zeta_{I+1}\eta_{I+2}) \Big) \end{aligned} \tag{E.7.9}$$

$$\begin{aligned} (X,Y,Z) = \sum_{I=1}^{3}\Big(& \frac{1}{2}(x_Iy_{I+1}z_{I+2} + x_Iy_{I+2}z_{I+1}) - (x_I(\eta_I,\zeta_I) + y_I(\zeta_I,\xi_I) + z_I(\xi_I,\eta_I)) \\ & + \boldsymbol{Re}^c(\xi_I\eta_{I+1}\zeta_{I+2} + \xi_I\zeta_{I+1}\eta_{I+2}) \Big) \end{aligned} \tag{E.7.10}$$

Here, the index I is $\boldsymbol{mod}$ **3**.

Therefore, we have

$$det(X) = x_1x_2x_3 - x_1\xi_1\bar{\xi}_1 - x_2\xi_2\bar{\xi}_2 - x_3\xi_3\bar{\xi}_3 + 2\boldsymbol{Re}^c(\xi_1\xi_2\xi_3) \ . \tag{E.7.11}$$

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