Doctor Thesis

Supergravity backgrounds in matrix models

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Abstract

In this doctoral thesis, we discuss how the type IIB supergravity is induced in IKKT matrix model. After the establishment of the Standard Model of particle physics, the quantization of gravity is one of the most impotant problems. Matrix models which propose non-perturbative formulation of superstring theory might become the clue which solves this problem.

IKKT matrix model has the picture that the spacetime is dynamically generated as a discrete object made of N D-instantons (D(-1)). It is the impotant problem what fills the role of the background when we deal with gravity in the matrix model. In this paper, we introduce $(N+1) \times (N+1)$ matrices and regard the extra 1×1 block as a background. We may expect that the effective action for N D-instantons is modified by backgrounds so that they live in a curved space-time. This is analogous to a thermodynamic system. In a canonical ensemble, a subsystem in a heat bath is characterized by several thermodynamic quantities like temperature and pressure. Similarly a subsystem of N D-instantons in a "matrix bath" can be characterized by several "thermodynamic quantities". We call this extra D-instanton a "*mean field D-instanton*".

We construct wave functions and vertex operators for N D-instantons by expanding a supersymmetric Wilson loop operator. They form a massless multiplet of the $\mathcal{N} = 2$ type IIB supergravity and automatically satisfy conservation laws. The emergence of conservation laws seems to be a sign of the local symmetry.

Next, we discuss the condensation of supergravity modes with the analogy between thermodynamics and the multi-particle system of N D-instantons. The condensation of a mean field D-instanton with an appropriate wave function $f_k(y, \xi)$ represents the background with various terms corresponding to the choice of the wave functions by integrating over off-diagonal blocks of the one-loop effective action. In particular, a Chern-Simons-like term is induced when the mean-field D-instanton has a wave function of the antisymmetric tensor field. A fuzzy sphere becomes a classical solution to the equation of motion for the effective action.

Contents

1 Introduction

The Standard Model of particle physics which is a great accomplishment of modern physics describes the three forces — the electromagnetic, weak and strong forces — as the gauge theories of quantum electro-weak and quantum chromodynamics (QCD). These quantum field theories describe the forces between particles by exchanging field quanta which are the spin-1 gauge bosons: the photon for the electromagnetic force, the W and Z bosons for the weak force, and the gluon for the strong force.

The gravitational forces can be also described as one of the gauge theories. The gauge boson for the gravitational forces is the spin-2 graviton. The earliest attempts to unify gravity and quantum mechanics were based on trying to fit general theory of relativity into a quantum field theory like electrodynamics (QED). The aim was to find a set of rules for calculating scattering amplitudes in which the photons of QED are replaced by the gravitons of the gravitational field. In the framework of the quantum field theory, however, the gravitational forces become increasingly strong at the higher order for the perturbative calculation as the energy of the participating quanta increases wildly, and the theory turned out to be violently out of control. Attempting to treat the graviton just as "a point particle" gave rise to non-renomalizable divergence at short distances.

Superstring theory is the theory which considers extended 1D elastic strings as essential elements of nature instead of point particles. The typical size of a string is very small. Its scale is of the order of the Planck length which is 1.6×10^{-35}m. Thus, in the far distance, the string will effectively appear as a point-like object. Moreover, the spectrum of a closed string contains a spin-2 massless particle that is the graviton. Superstring theory is our most promising candidate to be such a high energy theory for gravity, as it regularizes the divergences found in a quantum field theory of gravity.

Superstring theory is mathematically beautiful and consistent. But it offers us many challenges to be a real framework for particle physics and cosmology, and has been studied aggressively. The most important discovery in recent years is "D-branes". D-branes are surfaces that exist in superstring theory, and they have various dimensions. For example, D0-branes are like particle and D1-branes are like string. D2-branes are two-dimensional and can also be called membranes. And moreover, higher-dimensional objects can exist.

These non-perturbative objects which were found by J. Polchinski in 1995[1] had renewed the understanding of superstring theory. Until then, we had thought that closed and open strings were different objects, and there were five different superstring theories — $SO(32)$ type I, type IIA, type IIB, $SO(32)$ heterotic and $E_8 \times E_8$ heterotic superstring theories.But the discovery of D-branes enables to introduce dualities into five superstring theories, and interpret that five 10D string theories are themselves a Kaluza-Klein compactification of an 11D theory which became known as "M-theory".

M-theory appears to be basic for all superstring theories. The five different versions of superstring theory are just different ways of compactifying the extra-dimensions. But M-theory is not itself a string theory. It has no strings, and the argument that string is only the fundamental object becomes weak. Indeed, some attempts to treat D-brane rather than string as a fundamental object were suggested, like BFSS matrix model[2].

It can be seen from the fact that Einstein's theory is based on Riemannian geometry, the gravitation is closely related with spacetime geometry. Therefore the quantization of gravity is the same as the quantization of spacetime geometry. The problem is what is the quantized geometry instead of Riemannian.

Spacetime needs to be quantize at the region that the gravity becames strong such as the beginning of universe. At that time, the spacetime described by Einstein's gravitation theory is crushed and becomes a point, and Riemannian geometry becomes nonsense. If it was true that universe came from "nothing" at first, then spacetime should be itself semi-classically generated by the dynamics of multi-D-branes system. Therefore the correct theory quantizing spacetime would be the supersymmetric matrix model which has "D-instantons(D(-1))" — all of the spacetime degrees of freedom are jammed into matrices. That is just IKKT matrix model, which was proposed by N. Ishibashi, H. Kawai, Y. Kitazawa, A. Tsuchiya in 1996[3]. This model is beautiful in the sense that it has high symmetry such as the Lorentz symmetry, the $\mathcal{N} = 2$ type IIB supersymmetry and the $SO(9,1)$ rotational symmetry. The action of IKKT matrix model is the same as the effective action for N D-instantons [4]. This correspondence suggests a possibility that D-instantons(D(-1)) can be considered as fundamental objects to generate both of the space-time and the dynamical fields (or strings) on the space-time. The bosonic matrices represent noncommutative coordinates of D(-1)'s and the distribution of eigenvalues of the bosonic matrices A_μ is interpreted to form space-time. Although there are still many issues to be resolved, IKKT matrix model is solving fundamental and philosophical problems of quantum gravity, for example, why spacetime is just four dimension[5, 6].

In this paper, as a threshold for the quantization of gravity from IKKT matrix model, we consider the matrix model to be an effective theory for D-instanton multiparticle system, and induce 10D supergravity multiplet using an analogy with "thermodynamics". Namely, we think that several geometrical quantity, such as the metric, in the multi-D-instanton system can correspond with several thermodynamical quantities, such as the temperature, in the heat bath.

The contents are as follows. In chapter 2, as a first step, we review IKKT matrix model. At the beginning, we propose IKKT matrix model as the Green-Schwarz action of type IIB string in the Schild gauge. Then we discuss the classical solution for the static D-strings in the matrix model and consider the one loop effective action of the model. And, we examine interactions between diagonal blocks and cluster property. Finally, conjecturing a background as "heat bath" of matrices, we discuss the thermodynamic analogy to multi-D-instantone. In chapter 3, we discuss what role $\mathcal{N} = 2$ supersymmetry plays in IKKT matrix model with the heat bath background, and we construct a supersymmetric Wilson loop operator which is proposed by K. J. Hamada[7]. In chapter 4 and chaper 5, We construct wave functions and vertex operators corresponding to 10D supergravity multiplet for the N D-instantons by using the supersymmetry transformations and expanding a supersymmetric Wilson loop operator. They automatically form a supersymmetry multiplet and satisfy conservation laws [8]. Finally, in chapter 6, we discuss the graviton condensation of the type IIB supergravity multiplets using the perturbative expansion of matrices.

2 IKKT matrix model

In this section, we review IKKT matrix model [3]. BFSS matrix model naturally describes ten dimensional type IIA superstring. In analogy with BFSS in which the Lagrangian is expressed in terms of D0-branes, we might expect that type IIB superstring is described in terms of D-instanton variables. Namely, IKKT matrix model was proposed another matrix model associated with type IIB superstring.

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2 IKKT matrix model

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2.1 The IKKT action from Schild form

The starting point in the IKKT approach is the Green–Schwartz action of type IIB superstring theory in Schild gauge:

$$S_{\text{Schild}} = \int d^2\sigma \left[\sqrt{g}\alpha \left(\frac{1}{4}\{X^\mu, X^\nu\}^2 - \frac{i}{2}\bar{\Psi}\Gamma^\mu\{X^\mu, \Psi\} \right) + \beta\sqrt{g} \right], \tag{2.1}$$

where the vector index μ of X^μ runs from 0 to 9 and the spinor index α of Ψ_α runs from 1 to 32. The fermion Ψ is a Majorana-Weyl spinor in 10D which satisfies the condition $\Gamma_{11}\Psi = \Psi$, so that only 16 components effectively remain, and $\sqrt{g}$ is positive definite scalar density defind on the world sheet which can be identified wih $\sqrt{\det(g_{ab})}$ made from a worldsheet metric g_{ab}, and the Poisson bracket is defined by

$$\{X, Y\} \equiv \frac{1}{\sqrt{g}}\varepsilon^{ab}\partial_a X \partial_b Y \, . \tag{2.2}$$

The equation of motion for $\sqrt{g}$ in (2.1) becomes

$$-\frac{1}{4}\alpha\frac{1}{(\sqrt{g})^2}(\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu)^2 + \beta = 0 \, . \tag{2.3}$$

The action (2.1) have the $N = 2$ supersymmetry as follows;

$$\begin{aligned} \delta^{(1)}\Psi &= -\frac{1}{2}\epsilon^{ab}\partial_a X_\mu \partial_b X_\nu \Gamma^{\mu\nu}\epsilon, \\ \delta^{(1)}X^\mu &= i\bar{\epsilon}\Gamma^\mu\Psi, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \delta^{(2)}\Psi &= \xi, \\ \delta^{(2)}X^\mu &= 0. \end{aligned} \tag{2.5}$$

The path integral over the positive definite function $\sqrt{g}$, the bosonic matrices X^μ and the fermionic Ψ_α

$$\mathcal{Z}_{\text{Schild}} = \int \mathcal{D}\sqrt{g}\,\mathcal{D}X\,\mathcal{D}\Psi\; e^{-S_{\text{Schild}}} \tag{2.6}$$

can be interpreted as the definition of the "partition function" in the Schild formulation of type IIB superstring at least in the classical limit. Eq. (2.1) and (2.6) represent type IIB superstring in the Schild formalism with fixed κ-symmetry[3].

The IKKT matrix model can be obtained from the representation (2.6) of type IIB superstring in the Schild formulation by replacing

$$X_\mu \Longrightarrow A_\mu^{ab} , \tag{2.7}$$

$$\Psi_\alpha \Longrightarrow \psi_\alpha^{ab} , \tag{2.8}$$

where A_μ^{ab} and ψ_α^{ab} are hermitian $n \times n$ bosonic and fermionic matrices, respectively.

The IKKT matrix model is defined by the partition function

$$Z = \sum_{n=0}^{\infty} \int dA \, d\psi \; \mathrm{e}^{-S} , \tag{2.9}$$

with the action

$$S = \alpha \left(-\frac{1}{4} \mathrm{tr} \, [A_\mu, A_\nu]^2 - \frac{1}{2} \mathrm{tr} \, \left(\bar{\psi} \Gamma^\mu [A_\mu, \psi] \right) \right) + \beta \, \mathrm{tr} \, \mathbf{1}_n . \tag{2.10}$$

The summation over the matrix size n in Eq. (2.9) implies that n is a dynamical variable (an analog of $\sqrt{g}$ in Eq. (2.6)). The action (2.10) closely resembles the Schild action (2.1) in the sense of forms respectively. Indeed, when the matrix size n gets large value in (2.9) and the distributions of eigenvalues for A_μ and ψ_α becomes smooth enough, we expect that the commutator and the trace can be replaced with the Poisson bracket and the integration, respectively:

$$\begin{aligned} -i[\ ,\ ] \ &\Rightarrow \ \{\ ,\ \} , \\ \mathrm{tr} \ &\Rightarrow \ \int d^2\sigma \sqrt{g} \, . \end{aligned} \tag{2.11}$$

This is the same as the ordinary correspondence between the quantum and classical mechanics. The similarity between the Eguchi-Kawai large-N reduced model, which is the original spirit of IKKT, and string theory in the Schild gauge was investigated by Bars some time ago for bosonic string case[9].

Furthermore we can easily check that the $\mathcal{N} = 2$ supersymmetry (2.4) and (2.5) is directly translated into the symmetry of the action (2.10) as

$$\begin{aligned} \delta^{(1)} \psi &= \frac{i}{2} [A_\mu, A_\nu] \Gamma^{\mu\nu} \epsilon , \\ \delta^{(1)} A_\mu &= i \bar{\epsilon} \Gamma_\mu \psi \, , \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} \delta^{(2)}\psi &= \xi, \\ \delta^{(2)}A_\mu &= 0\ . \end{aligned} \tag{2.13}$$

So, if we interpret the eigenvalues of the matrices A_μ as "space-time coordinates", then the above symmetries can be regarded as 10D $\mathcal{N}=2$ space-time supersymmetry.

2.2 Classical solutions of static D-strings

The classical equations of motion for the Schild action (2.1) read

$$\{X^\mu\,,\{X_\mu\,,\,X_\nu\}\} = 0\,, \qquad \left\{X^\mu\,,(\Gamma_\mu\Psi)_\alpha\right\} = 0\,. \tag{2.14}$$

Their matrix model counterparts are

$$[A^\mu,\,[A_\mu, A_\nu]] = 0\,, \qquad [A^\mu\,,(\Gamma_\mu\psi)_\alpha] = 0\,, \tag{2.15}$$

which are got by being solved for $n\times n$ matrices A_μ at infinite n.

We can easily construct a solution of Eq. (2.14), which represents a static D-string extending straightly in the X^1 direction:

$$X^{\rm cl}_\mu = \left(T\tau, \frac{L}{2\pi}\sigma, 0, \ldots, 0\right), \qquad \Psi^{\rm cl}_\alpha = 0\,, \tag{2.16}$$

where T and $L/2\pi$ are large enough compactification radii and

$$0\le\tau\le 1\ ,\ \ 0\le\sigma\le 2\pi\ . \tag{2.17}$$

Considering the relation between the commutator and the Poisson bracket, we obtain a solution of Eq. (2.15) corresponding to the above one as follows:

$$A^{\rm cl}_\mu = \left(\frac{T}{\sqrt{2\pi n}}q, \frac{L}{\sqrt{2\pi n}}p, 0, \ldots, 0\right), \qquad \psi^{\rm cl}_\alpha = 0\,, \tag{2.18}$$

where the (infinite) $n\times n$ matrices p and q obey the canonical commutation relation;

$$[q,p] = i\ , \tag{2.19}$$

and

$$0 \le q \le \sqrt{2\pi n}\ , \quad 0 \le p \le \sqrt{2\pi n}\ . \tag{2.20}$$

while $T/\sqrt{2\pi n}$ and $L/\sqrt{2\pi n}$ are (large enough) compactification radii. They are *operator-like* solutions because such p and q can not exist for finite n but Eq. (2.19) can be approximately satisfied as the canonical commutation relation for large n. An argument which supports identification of the classical solution (2.18) with static D-string is that the interaction between two D-strings is reproduced at large distances [3].

2.3 One-loop effective action

The calculation of the one-loop effective action in the IKKT matrix model at fixed n can be performed for an arbitrary background, $A_\mu^{\rm cl}$ and $\psi_\alpha^{\rm cl}$, obeying the classical equations of motion (2.15).

To expand around the classical solution, we decompose matrices,

$$\begin{aligned} A_\mu &= A_\mu^{\rm cl} + a_\mu, & (2.21)\\ \psi_\alpha &= \psi_\alpha^{\rm cl} + \varphi_\alpha, & (2.22)\end{aligned}$$

where $A_\mu^{\rm cl}$ and $\psi_\alpha^{\rm cl}$ are (classical) backgrounds and a_μ and φ_α are quantum fluctuations. We expand the action (2.10) ($\alpha = 1$ and $\beta = 0$) up to the second order for the quantum fluctuations as follows;

$$S = S_0 + S_1 + S_2 + \cdots\ , \tag{2.23}$$

where

$$S_0 = -\frac{1}{4}{\rm tr}\ [A_\mu^{\rm cl}, A_\nu^{\rm cl}]^2 - \frac{1}{2}{\rm tr}\ \bar{\psi}^{\rm cl}\Gamma^\mu[X_\mu, \psi^{\rm cl}]\,, \tag{2.24}$$

$$S_1 = {\rm tr}\left(-[A_\mu^{\rm cl}, A_\nu^{\rm cl}][A^{{\rm cl}\mu}, a^\nu] - \frac{1}{2}\bar{\psi}^{\rm cl}\Gamma^\mu[a_\mu, \psi^{\rm cl}] - \frac{1}{2}\bar{\varphi}\Gamma^\mu[A_\mu^{\rm cl}, \psi^{\rm cl}] - \frac{1}{2}\bar{\psi}^{\rm cl}\Gamma^\mu[A_\mu^{\rm cl}, \varphi]\right)\,, \tag{2.25}$$

and

$$\begin{aligned} S_2 \;=\; {\rm tr}\Big(&-\frac{1}{2}[A_\mu^{\rm cl}, A_\nu^{\rm cl}][a^\mu, a^\nu] - \frac{1}{2}[A^{{\rm cl}\mu}, a^\nu][A_\mu^{\rm cl}, a_\nu] - \frac{1}{2}[A^{{\rm cl}\mu}, a^\nu][a_\mu, A_\nu^{\rm cl}] \\ &- \frac{1}{2}\bar{\varphi}\Gamma^\mu[A_\mu^{\rm cl}, \varphi] - \bar{\psi}^{\rm cl}\Gamma^\mu[a_\mu, \varphi]\Big)\,. \end{aligned} \tag{2.26}$$

S_1 vanishes due to the equations of motions, or is dropped in the back ground field method. To fix the gauge invariance, we add the gauge fixing and the Faddeev-Popov term

$$S_{g.f.+F.P.} = -\,\mathrm{tr}\left(\frac{1}{2}[A^{\mathrm{cl}}_\mu, a_\mu]^2 + [A^{\mathrm{cl}}_\mu, b][A^{\mathrm{cl}\mu}, c]\right), \tag{2.27}$$

where c and b are ghosts and anti-ghosts. In the following, we set $\psi^{\mathrm{cl}} = 0$. And when we drop the first order term of the quantum fluctuations, we obtain

$$\begin{aligned} \tilde{S}_2 &= S_2 + S_{g.f+F.P.} \\ &= -\,\mathrm{tr}\left(\frac{1}{2}[A^{\mathrm{cl}}_\mu, a_\nu]^2 + [A^{\mathrm{cl}}_\mu, A^{\mathrm{cl}}_\nu][a^\mu, a^\nu] + \frac{1}{2}\bar{\varphi}\Gamma^\mu[A^{\mathrm{cl}}_\mu, \varphi] + [A^{\mathrm{cl}}_\mu, b][A^{\mathrm{cl}\mu}, c]\right). \end{aligned} \tag{2.28}$$

We rewrite the action (2.28) as

$$\tilde{S}_2 = \mathcal{T}r\left(\frac{1}{2}a_\mu(P^2\eta_{\mu\nu} - 2i\mathcal{F}_{\mu\nu})a_\nu - \frac{1}{2}\bar{\varphi}\Gamma^\mu P_\mu \varphi - \bar{\Phi}\Gamma_\mu\,[a_\mu, \varphi] + bP^2 c\right), \tag{2.29}$$

where the *adjoint* operators P_μ and $\mathcal{F}_{\mu\nu}$ are defined on the space of matrices by

$$\begin{aligned} P_\mu &= \left[A^{\mathrm{cl}}_\mu, \cdot\,\right], \\ \mathcal{F}_{\mu\nu} &= \left[f^{\mathrm{cl}}_{\mu\nu}, \cdot\,\right] = i\left[\left[A^{\mathrm{cl}}_\mu, A^{\mathrm{cl}}_\nu\right], \cdot\,\right], \\ \Phi_\alpha &= \left[\psi^{\mathrm{cl}}_\alpha, \cdot\,\right], \end{aligned} \tag{2.30}$$

and $\mathcal{T}r$ is the trace for the adjoint operators. From (2.29), the one-loop effective action W is evaluated as

$$W = W_b + W_f = -\ln\int dad\varphi dcdb\, e^{-\tilde{S}_2} \tag{2.31}$$

where

$$\begin{aligned} W_b &= \frac{1}{2}\mathcal{T}r\ln(P^2\eta_{\mu\nu} - 2i\mathcal{F}_{\mu\nu}) - \frac{1}{4}\mathcal{T}r\ln\left(\left(P^2 + \frac{i}{2}\mathcal{F}_{\mu\nu}\Gamma^{\mu\nu}\right)\left(\frac{1+\Gamma_{11}}{2}\right)\right) \\ &\quad -\mathcal{T}r\ln P^2, \end{aligned} \tag{2.32}$$

$$W_f = \frac{1}{2}\mathcal{T}r\ln\left(\eta_{\mu\nu} + \left(\frac{1}{P^2 + 2\mathcal{F}}\right)_{\mu\lambda}\bar{\Phi}\,\Gamma_\lambda \frac{1}{\Gamma\cdot P}\Gamma_\nu\,\Phi\right). \tag{2.33}$$

The first term on the right hand side of Eq. (2.32) comes from the quantum fluctuations of A_μ, the second and third terms which come from fermions and ghosts have the minus sign for this reason. The same thing is applicable also about the terms of Eq. (2.33). The extra factor 1/2 in W_b and W_f is because the matrices A and ψ are hermitian.

If $A_\mu^{\rm cl}$ is diagonal

$$A_\mu^{\rm cl} = {\rm diag}\left(p_\mu^{(1)}, \dots, p_\mu^{(n)}\right), \qquad \psi_\alpha^{\rm cl} = 0\,, \tag{2.34}$$

which is a solution of Eq. (2.15) associated with the flat space-time, then $F_{\mu\nu} = 0$ and

$$W_b = \left(\frac{1}{2}\cdot 10 - \frac{1}{4}\cdot 16 - 1\right)\ {\rm tr}\ \log(P^2) = 0\ ,\quad W_f = 0\,. \tag{2.35}$$

The plane vacuum is a BPS state. Namely, half of the supersymmetry is preserved in these backgrounds.

The same thing is proper (to all loops) for any $A_\mu^{\rm cl}$ whose commutator is diagonal:

$$\left[A_\mu^{\rm cl}, A_\nu^{\rm cl}\right] = c_{\mu\nu}\mathbf{1}_n\,, \tag{2.36}$$

where $c_{\mu\nu}$ are c-numbers rather than matrices. Such solutions preserve [3] a half of SUSY and are BPS states.

2.4 Long range interactions without fermionic backgrounds

In this subsection, we investigate the interactions between diagonal blocks for the one-loop effective action (2.31). By considering such interaction, the gravitational interactions can be observed and type IIB supergravity is expected to be reproduced. Graviton and dilaton exchange are culculated in [3, 10].

At the beginning, we consider backgrounds having a block-diagonal form;

$$A_\mu = p_\mu = \begin{pmatrix} p_\mu^{(1)} & & \\ & p_\mu^{(2)} & \\ & & \ddots \end{pmatrix}, \tag{2.37}$$

where $p_\mu^{(i)}$ $(i = 1, 2, \cdots)$ are $n^{(i)} \times n^{(i)}$ matrices. We can interpret each $p_\mu^{(i)}$ as a D-object occupying some region of space-time. We use a term D-object to represent D-instantons, D-strings, D3-branes, $\cdots$, and their composites. We decompose $p_\mu^{(i)}$ into diagonal and off-diagonal parts as follows;

$$\begin{aligned} p_\mu^{(i)} &= x_\mu^{(i)}\mathbf{1}_{n^{(i)}} + \tilde{p}_\mu^{(i)}, \\ {\rm tr}\,\tilde{p}_\mu^{(i)} &= 0\,, \end{aligned} \tag{2.38}$$

where $x_\mu^{(i)}$ is a real number and represents the center of mass coordinate of the i-th block. We assume that blocks are respectively separated far enough from each other, that is, $1/\sqrt{(x^{(i)}-x^{(j)})^2}$ are so small for all of i and j that we can expand the bosonic term of the one-loop effective action (2.32) perturbatively.

Actually, we expand the bosonic term of the one-loop effective action (2.32). We can take traces of the Γ-matrices after expanding the logarithm in Eq. (2.32). Due to the supersymmetry, contributions of bosons and fermions cancel each other to the third order in $F_{\mu\nu}$, and we have,

$$\begin{aligned} W_b &= -\mathcal{T}r\left(\frac{1}{P^2}\mathcal{F}_{\mu\nu}\frac{1}{P^2}\mathcal{F}_{\nu\lambda}\frac{1}{P^2}\mathcal{F}_{\lambda\rho}\frac{1}{P^2}\mathcal{F}_{\rho\mu}\right) - 2\mathcal{T}r\left(\frac{1}{P^2}\mathcal{F}_{\mu\nu}\frac{1}{P^2}\mathcal{F}_{\lambda\rho}\frac{1}{P^2}\mathcal{F}_{\mu\rho}\frac{1}{P^2}\mathcal{F}_{\lambda\nu}\right) \\ &\quad +\frac{1}{2}\mathcal{T}r\left(\frac{1}{P^2}\mathcal{F}_{\mu\nu}\frac{1}{P^2}\mathcal{F}_{\mu\nu}\frac{1}{P^2}\mathcal{F}_{\lambda\rho}\frac{1}{P^2}\mathcal{F}_{\lambda\rho}\right) + \frac{1}{4}\mathcal{T}r\left(\frac{1}{P^2}\mathcal{F}_{\mu\nu}\frac{1}{P^2}\mathcal{F}_{\lambda\rho}\frac{1}{P^2}\mathcal{F}_{\mu\nu}\frac{1}{P^2}\mathcal{F}_{\lambda\rho}\right) \\ &\quad +O\left((\mathcal{F}_{\mu\nu})^5\right). \end{aligned} \tag{2.39}$$

P_μ and $F_{\mu\nu}$ operate on each (i,j) block independently, therefore the one-loop effective action W_b is expressed as the sum of contributions of the (i,j) blocks $W_b^{(i,j)}$, and $W_b^{(i,j)}$ can be recognized as the interaction between the i-th and j-th blocks. We can easily calculate $W_b^{(i,j)}$ to the leading order of $1/r$ (where $r=\sqrt{(x^{(i)}-x^{(j)})^2}$.) as

$$\begin{aligned} W_b^{(i,j)} &= \frac{1}{r^8}\Big\{-\mathcal{T}r^{(i,j)}(\mathcal{F}_{\mu\nu}\mathcal{F}_{\nu\lambda}\mathcal{F}_{\lambda\rho}\mathcal{F}_{\rho\mu}) - 2\mathcal{T}r^{(i,j)}(\mathcal{F}_{\mu\nu}\mathcal{F}_{\lambda\rho}\mathcal{F}_{\mu\rho}\mathcal{F}_{\lambda\nu}) \\ &\qquad + \frac{1}{2}\mathcal{T}r^{(i,j)}(\mathcal{F}_{\mu\nu}\mathcal{F}_{\mu\nu}\mathcal{F}_{\lambda\rho}\mathcal{F}_{\lambda\rho}) + \frac{1}{4}\mathcal{T}r^{(i,j)}(\mathcal{F}_{\mu\nu}\mathcal{F}_{\lambda\rho}\mathcal{F}_{\mu\nu}\mathcal{F}_{\lambda\rho})\Big\} \\ &= \frac{3}{2r^8}\Big\{-n_j\tilde{b}_8(f^{(i)}) - n_i\tilde{b}_8(f^{(j)}) \\ &\qquad - 8\,\mathrm{tr}\,(f^{(i)}_{\mu\nu}f^{(i)}_{\nu\sigma})\,\mathrm{tr}\,(f^{(j)}_{\mu\rho}f^{(j)}_{\rho\sigma}) + \mathrm{tr}\,(f^{(i)}_{\mu\nu}f^{(i)}_{\mu\nu})\,\mathrm{tr}\,(f^{(j)}_{\rho\sigma}f^{(j)}_{\rho\sigma})\Big\}, \end{aligned} \tag{2.40}$$

where

$$\begin{aligned} \tilde{b}_8(f) &= \frac{2}{3}\Big\{\mathrm{tr}\,(f_{\mu\nu}f_{\nu\lambda}f_{\lambda\rho}f_{\rho\mu}) + 2\,\mathrm{tr}\,(f_{\mu\nu}f_{\lambda\rho}f_{\mu\rho}f_{\lambda\nu}) \\ &\qquad - \frac{1}{2}\,\mathrm{tr}\,(f_{\mu\nu}f_{\mu\nu}f_{\lambda\rho}f_{\lambda\rho}) - \frac{1}{4}\,\mathrm{tr}\,(f_{\mu\nu}f_{\lambda\rho}f_{\mu\nu}f_{\lambda\rho})\Big\}. \end{aligned} \tag{2.41}$$

In order to investigate the gravitational interaction, we consider the "photon-photon scattering amplitude" on the brane as in [10], which amplitude corresponds to nonplanar diagrams in noncommutative gauge theory. If we impose the forward scattering limit on the amplitude, then we have

$$-12(\frac{1}{2\pi})^{\tilde{d}} C^{\tilde{d}-8} \int d^{\tilde{d}}x \int d^{\tilde{d}}y \frac{1}{(x-y)^8} \,\mathrm{tr}\,(f_{\mu\rho}(x) f^{\rho\nu}(x)) \,\mathrm{tr}\,(f_{\mu\sigma}(y) f^{\sigma\nu}(y)) \,, \tag{2.42}$$

where C is the determinant of the c-number $c_{\mu\nu}$ of Eq. (2.36). This is expected to be just graviton exchange.

2.5 Thermodynamic analogy to multi-D-instantone

As in the previous subsection, the interactions between two blocks are weaker than or equal to $1/r^8$. Therefore, when D object is fully located in the distance from each other, they can exist independently and a system owns the *cluster property*. Indeed, Eq. (2.42) consists of the product of x- and y-systems. With the cluster property, the trace parts of diagonal-blocks become collective coordinates. Moreover the blocks obtain the physical meaning as the centers of mass of the D-objects. In other words, space time coordinate is dynamically generated as a trace parts.

Now, IKKT matrix model is also considered as an effective theory for N D-instantons (D(−1)) [4]. Therefore D-instantons could be considered as fundamental objects to generate both the spacetime and the dynamical fields. That is to say, the space-time is constructed by distribution of D-instantons.

If we take the above interpretation, how can we interpret the $SO(9,1)$ rotational symmetry of the matrix model action? This symmetry can be interpreted in the sense of mean field. Namely we can consider that the system of N D-instantons are embedded in larger size $(N+M)\times(N+M)$ matrices as

$$\left(\begin{array}{c|c} N\,\mathrm{D}(-1) & \\ \hline & \\ & \text{“heat bath” of } M\,\mathrm{D}(-1) \\ & \end{array}\right) , \tag{2.43}$$

and consider the action (2.10) as an effective action in the background where the rest M eigenvalues distribute uniformly in 10 dimensions. If the M eigenvalues distribute inhomogeneously, we may expect that the effective action for N D-instantons is modified so that they live in a curved space-time. This is analogous to a thermodynamic system. In a canonical ensemble, a subsystem in a heat bath of matrices is characterized by several thermodynamic quantities like temperature. Similarly a subsystem of N D-instantons in a "heat bath" can be characterized by several thermodynamic quantities. Under such thermodynamic picture, what does the $\mathcal{N}=2$ type IIB supersymmetry, which is the other feature of IKKT matrix model, fill the role of? We would talk about the interpretation and the role of supersymmetry of IKKT matrix model in the next section.

3 Supersymmetric Wilson loops in IKKT matrix model

In this section, we discuss the $\mathcal{N}=2$ type IIB supersymmetry which IKKT matrix model has, and supersymmetric Wilson loop operators. By virtue of this supersymmetry, we expect that the configuration of the M D-instantons can describe condensation of massless fields of the type IIB supergravity and the thermodynamic quantities of the heat bath of matrices are characterized by the values of the condensations. And the supersymmetry Wilson loop operator is necessary for introducing the type IIB supergravity multiplets and the corresponding vertex operators.

3.1 $\mathcal{N}=2$ supersymmetry and mean field D-instanton

As we have seen in Eq. (2.12) and Eq. (2.13), IKKT matrix model has the $\mathcal{N}=2$ type IIB supersymmetry

$$\begin{cases} \delta A_\mu = & i\bar{\varepsilon}\Gamma_\mu\psi, \\ \delta\psi = & -\frac{i}{2}[A_\mu, A_\nu]\Gamma^{\mu\nu}\epsilon + \epsilon' 1_N, \end{cases} \tag{3.1}$$

we expect that the configuration of the M D-instantons can describe condensation of massless fields of the type IIB supergravity and the thermodynamic quantities of the heat bath of matrices are characterized by the values of the condensations.

In order to discuss which type of configurations for M D-instantons correspond to the condensation of massless type IIB supergravity multiplet, we consider the supersymmetry transformations (3.1) in the system of $N+M$ D-instantons (2.43). In particular, from inhomogeneously, we may expect that the effective action for N D-instantons is modified so that they live in a curved space-time. This is analogous to a thermodynamic system. In a canonical ensemble, a subsystem in a heat bath of matrices is characterized by several thermodynamic quantities like temperature. Similarly a subsystem of N D-instantons in a "heat bath" can be characterized by several thermodynamic quantities. Under such thermodynamic picture, what does the $\mathcal{N}=2$ type IIB supersymmetry, which is the other feature of IKKT matrix model, fill the role of? We would talk about the interpretation and the role of supersymmetry of IKKT matrix model in the next section.

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In order to discuss which type of configurations for M D-instantons correspond to the condensation of massless type IIB supergravity multiplet, we consider the supersymmetry transformations (3.1) in the system of $N+M$ D-instantons (2.43). In particular, from here, we consider the simplest case that the background is represented by one D-instanton (namely $M = 1$). This simplification can be considered as a mean field approximation that the configuration of M D-instantons is represented by a mean field described by a single D-instanton. We call this extra D-instanton a **mean field D-instanton**. This kind of idea was first discussed by Yoneya in [11]. We hence embed $N \times N$ matrices into $(N+1) \times (N+1)$ matrices as

$$A'_\mu = \begin{pmatrix} A_\mu & a_\mu \\ a_\mu^\dagger & y_\mu \end{pmatrix}, \qquad \psi' = \begin{pmatrix} \psi & \varphi \\ \varphi^\dagger & \xi \end{pmatrix}. \tag{3.2}$$

Here we use A'_μ, ψ' for $(N+1) \times (N+1)$ matrices and A_μ, ψ for $N \times N$ parts of the matrices. (y, ξ) is the coordinate of the mean field D-instanton and its configuration (or the wave function) $f(y, \xi)$ specifies a certain background of the massless type IIB supergravity multiplet. The supersymmetry transformation (3.1) for (A'_μ, ψ') can be rewritten as

$$\begin{cases} \delta A_\mu &= i\bar{\epsilon}\Gamma_\mu \psi, \\ \delta y_\mu &= i\bar{\epsilon}\Gamma_\mu \xi, \\ \delta a_\mu &= i\bar{\epsilon}\Gamma_\mu \varphi, \end{cases} \tag{3.3}$$

and

$$\begin{cases} \delta\psi &= -\frac{i}{2}(F_{\mu\nu} + a_\mu a_\nu^\dagger - a_\nu a_\mu^\dagger)\Gamma^{\mu\nu}\epsilon + \epsilon' 1_N, \\ \delta\xi &= -\frac{i}{2}(a_\mu^\dagger a_\nu - a_\nu^\dagger a_\mu)\Gamma^{\mu\nu}\epsilon + \epsilon', \\ \delta\varphi &= -\frac{i}{2}\left\{(A_\mu - y_\mu)a_\nu - (A_\nu - y_\nu)a_\mu\right\}\Gamma^{\mu\nu}\epsilon, \end{cases} \tag{3.4}$$

where $F_{\mu\nu} = [A_\mu, A_\nu]$. We can obtain an effective action for the diagonal blocks by integrating the off-diagonal parts a_μ, ϕ. In the leading order of the perturbation, we can neglect terms depending on the off-diagonal fields and the susy transformations are given by

$$\begin{cases} \delta A_\mu = i\bar{\epsilon}\Gamma_\mu \psi, \\ \delta\psi = -\frac{i}{2}F_{\mu\nu}\Gamma^{\mu\nu}\epsilon + \epsilon', \end{cases} \tag{3.5}$$

$$\begin{cases} \delta y_\mu &= i\bar{\epsilon}\Gamma_\mu \xi, \\ \delta\xi &= \epsilon'. \end{cases} \tag{3.6}$$

The first transformations for N D-instantons are the same as the original susy transformations, eq. (3.1). The second ones are $\mathcal{N}=2$ supersymmetry transformations for the single mean field D-instanton. The generators of the former are given by

$$\bar{\epsilon}Q_1 = i(\bar{\epsilon}\Gamma_\mu\psi)\frac{\delta}{\delta A_\mu} - \frac{i}{2}F_{\mu\nu}\Gamma^{\mu\nu}\epsilon\frac{\delta}{\delta\psi}, \tag{3.7}$$

$$\bar{\epsilon}Q_2 = \epsilon\frac{\delta}{\delta\psi}, \tag{3.8}$$

while those of the latter are given by

$$\bar{\epsilon}q_1 = i\bar{\epsilon}\Gamma_\mu\xi\frac{\partial}{\partial y_\mu}\ , \tag{3.9}$$

$$\bar{\epsilon}'q_2 = \epsilon'\frac{\partial}{\partial\xi}. \tag{3.10}$$

In order to obtain the correct wave functions $f(y,\xi)$ corresponding to the massless supergravity multiplet, we need to obtain the multiplet of wave functions that transform correctly under the supersymmetry transformation (3.6).

When the supersymmetry transformations (3.6) act on wave functions of the form $e^{-ik\cdot y}f(\xi)$, they become

$$\begin{aligned}\bar{\epsilon}q_1 f(\xi)e^{-ik\cdot y} &= (\bar{\epsilon}\not{k}\xi)\ f(\xi)e^{-ik\cdot y},\\ \bar{\epsilon}'q_2 f(\xi)e^{-ik\cdot y} &= \epsilon'\frac{\partial}{\partial\xi}\ f(\xi)e^{-ik\cdot y}.\end{aligned} \tag{3.11}$$

3.2 Supersymmetric Wilson loop

In order to construct wave functions $f_A(\xi)e^{-ik\cdot y}$ and vertex operators $V_A(A^\mu,\psi;k)$ that transform covariantly under supersymmetries (3.6) and (3.5) respectively (A denotes a field of a massless $\mathcal{N}=2$ supergravity multiplet), we first consider a supersymmetric Wilson loop operator first introduced in [7] for the IIB matrix model;

$$w(C) = \mathrm{tr}\prod_{j=1} e^{\bar{\lambda}_j Q_1} e^{-i\epsilon k_j^\mu A_\mu} e^{-\bar{\lambda}_j Q_1}. \tag{3.12}$$

Since we are interested in the massless multiplet, we here consider the following simplest Wilson loop operator

$$\omega(\lambda,k) = e^{\bar{\lambda}Q_1}\ \mathrm{tr}e^{ik\cdot A}\ e^{-\bar{\lambda}Q_1}. \tag{3.13}$$

We will then show that by expanding the operator $\omega(\lambda, k)$ we can obtain a set of wave functions and vertex operators. Hereafter, we assume that the $N \times N$ matrices A_μ and ψ satisfy the equations of motion,

$$[A^\nu, [A_\mu, A_\nu]] - \frac{1}{2}(\Gamma_0 \Gamma_\mu)_{\alpha\beta}\{\psi_\alpha, \psi_\beta\} = 0, \tag{3.14}$$

$$\Gamma^\mu [A_\mu, \psi] = 0. \tag{3.15}$$

First we show that $\omega(\lambda, k)$ is invariant under simultaneous supersymmetry transformations for $N \times N$ matrices A^μ, ψ and the parameter (λ, k). When we act supersymmetry transformation $e^{\bar{\epsilon}Q_1}$ on $\omega(\lambda, k)$, it becomes

$$\begin{aligned} e^{\bar{\epsilon}Q_1}\omega(\lambda, k)e^{-\bar{\epsilon}Q_1} &= e^{\bar{\epsilon}Q_1}e^{\bar{\lambda}Q_1}\ \mathrm{tr}e^{ik\cdot A}\ e^{-\bar{\lambda}Q_1}e^{-\bar{\epsilon}Q_1} \\ &= e^{(A^\mu \bar{\epsilon}\Gamma_\mu \lambda)G}e^{(\bar{\epsilon}+\bar{\lambda})Q_1}\ \mathrm{tr}e^{ik\cdot A}\ e^{-(\bar{\lambda}+\bar{\epsilon})Q_1}e^{(A^\nu \bar{\epsilon}\Gamma_\nu \lambda)G} \\ &= \omega(\epsilon + \lambda, k). \end{aligned} \tag{3.16}$$

Here G is the generator of $U(N)$ transformation and we have used the commutation relation

$$\begin{aligned} [\bar{\epsilon}_1 Q_1, \bar{\epsilon}_2 Q_1] &= 2A^\mu \bar{\epsilon}_1 \Gamma_\mu \epsilon_2\ G \\ &\quad + \left(-\frac{7}{8}(\bar{\epsilon}_1 \Gamma^\mu \epsilon_2)\Gamma_\mu + \frac{1}{16\cdot 5!}(\bar{\epsilon}_1 \Gamma^{\mu_1\cdots\mu_5}\epsilon_2)\Gamma_{\mu_1\cdots\mu_5}\right)\Gamma_\lambda [A^\lambda, \psi]\frac{\delta}{\delta\psi}. \end{aligned} \tag{3.17}$$

The second term on the right hand side vanishes due to the equation of motion (3.15). Similarly for the other supersymmetry transformation $e^{\bar{\epsilon}Q_2}$, the Wilson loop operator transforms as

$$\begin{aligned} e^{\bar{\epsilon}Q_2}\omega(\lambda, k)e^{-\bar{\epsilon}Q_2} &= e^{\bar{\epsilon}Q_2}e^{\bar{\lambda}Q_1}\ \mathrm{tr}e^{ik\cdot A}\ e^{-\bar{\lambda}Q_1}e^{-\bar{\epsilon}Q_2} \\ &= e^{\bar{\lambda}Q_1}e^{\bar{\epsilon}Q_2}e^{i(\bar{\lambda}\Gamma_\mu \epsilon)\frac{\delta}{\delta A_\mu}}\mathrm{tr}e^{ik\cdot A}e^{-i(\bar{\lambda}\Gamma_\mu \epsilon)\frac{\delta}{\delta A_\mu}}e^{-\bar{\epsilon}Q_2}e^{-\bar{\lambda}Q_1} \\ &= e^{-(\bar{\lambda}\not{k}\epsilon)}\omega(\lambda, k), \end{aligned} \tag{3.18}$$

where we have used the commutation relation

$$[\bar{\epsilon}_1 Q_1, \bar{\epsilon}_2 Q_2] = -i\ (\bar{\epsilon}_1 \Gamma^\mu \epsilon_2)\frac{\partial}{\partial A^\mu}. \tag{3.19}$$

From (3.16) and (3.18), the following two relations for the supersymmetric Wilson loop operator are obtained;

$$[\bar{\epsilon}Q_1, \omega(\lambda, k)] - \epsilon\frac{\partial}{\partial\lambda}\omega(\lambda, k) = 0, \tag{3.20}$$

$$[\bar{\epsilon}Q_2, \omega(\lambda, k)] + (\bar{\lambda}\not{k}\epsilon)\omega(\lambda, k) = 0. \tag{3.21}$$

These relations mean that the supersymmetric Wilson loop operator is invariant if we perform supersymmetry transformations (3.5) simultaneously with the supersymmetry transformations for (λ, k). By expanding $\omega(\lambda, k)$ in terms of an appropriate basis of wave functions for λ as

$$\omega(\lambda, k) = \sum_A f_A(\xi)\ V_A(A_\mu, \psi;\ k), \tag{3.22}$$

we can define supersymmetry transformations for the wave functions by

$$\begin{aligned} \delta^{(1)} f(\lambda, k) &= \epsilon \frac{\partial}{\partial \lambda} f(\lambda, k), & (3.23) \\ \delta^{(2)} f(\lambda, k) &= (\bar{\epsilon} \not{k} \lambda) f(\lambda, k). & (3.24) \end{aligned}$$

These transformations are the same as (3.11) except that these two supersymmetries are interchanged. As we explain later, the interchanging can be realized by a charge conjugation operation.

The Majorana-Weyl fermion λ contains 16 degrees of freedom and there are 2^{16} independent wave functions for λ. To reduce the number, we impose massless condition for the momenta k. Then since $\not{k}\lambda$ has only 8 independent degrees of freedom the supersymmetry can generate only $2^8 = 256$ independent wave functions for λ. They form a massless type IIB supergravity multiplet containing a complex dilaton Φ, a complex dilatino $\tilde{\Phi}$, a complex antisymmetric tensor $B_{\mu\nu}$, a complex gravitino Ψ_μ, a real graviton $h_{\mu\nu}$ and a real 4-rank antisymmetric tensor $A_{\mu\nu\rho\sigma}$.

We now define a charge conjugation operation on the massless wave functions $f(\lambda, k)$. The charge conjugation is an operation to interchange a wave function with $p\,(\leq 8)$ λ's and that with $(8-p)$ λ's. It is defined by

$$(\hat{C} f)(\zeta, k) = f^c(\zeta, k) \equiv \int [d\lambda]\ \ e^{\bar{\zeta}\not{k}\lambda} f(\lambda, k), \tag{3.25}$$

where the integration of λ is performed with respect to eight λ's included in $\not{k}\lambda$. The integral measure is normalized so that $\hat{C}^2 = 1$. Acting $\hat{C}^2 = 1$ on a wave function, we get

$$\begin{aligned} (\hat{C}^2 f)(\lambda', k) &= \int [d\zeta][d\lambda] e^{\bar{\zeta}\not{k}(\lambda - \lambda')} f(\lambda, k) = \int [d\zeta][d\lambda] e^{\bar{\zeta}\not{k}\lambda} f(\lambda + \lambda', k) \\ &= \int [d\zeta][d\lambda] \frac{1}{8!} (\bar{\zeta}\not{k}\lambda)^8 f(\lambda + \lambda'). & (3.26) \end{aligned}$$

If we take a special momentum $k^\mu = (E, 0 \cdots 0, E)$ and use the Gamma matrices given in appendix, we have

$$\frac{1}{8!}(\bar{\zeta}\not{k}\lambda)^8 = (2E)^8(\zeta_9\zeta_{10}\cdots\zeta_{16})(\lambda_9\cdots\lambda_{16}), \tag{3.27}$$

and the normalization of the integration is given by

$$\int [d\zeta] \quad (2E)^4(\zeta_9\cdots\zeta_{16}) = 1. \tag{3.28}$$

It is easy to show that supersymmetry transformations for the charge conjugated fields are interchanged between $\delta^{(1)}$ and $\delta^{(2)}$;

$$(\delta^{(1)}f)^c(\zeta, k) \quad = \quad (\bar{\epsilon}\not{k}\zeta)f^c(\zeta) = \delta^{(2)}f^c(\zeta), \tag{3.29}$$

$$(\delta^{(2)}f)^c(\zeta, k) \quad = \quad \epsilon\frac{\partial}{\partial\zeta}\, f^c(\zeta) = \delta^{(1)}f^c(\zeta). \tag{3.30}$$

4 Wave functions corresponding to type IIB supergravity multiplets

In this section we derive wave functions $f(\lambda, k)$ for a massless supergravity multiplet by using the transformations (3.23) and (3.24). It can be seen that these wave functions satisfy the susy transformations of the type IIB supergravity. We refer to the works of J. H. Schwarz and P. C. West [12] and of Y. Kitazawa [13] for the type IIB supergravity transformation.

4.1 Dilaton Φ and dilatino $\tilde{\Phi}$

We start with the simplest wave function which can be interpreted as a dilaton field Φ in the type IIB supergravity multiplet;

$$\Phi(\lambda, k) = 1. \tag{4.1}$$

Dilatino wave function $\tilde{\Phi}$ can be generated from the dilaton wave function Φ by supersymmetry $\delta^{(2)}$ as

$$\delta^{(2)}\Phi(\lambda, k) = \bar{\epsilon}\not{k}\lambda \equiv \bar{\epsilon}\tilde{\Phi}(\lambda, k). \tag{4.2}$$

If we take a special momentum $k^\mu = (E, 0 \cdots 0, E)$ and use the Gamma matrices given in appendix, we have

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$$\delta^{(2)}\Phi(\lambda, k) = \bar{\epsilon}\not{k}\lambda \equiv \bar{\epsilon}\tilde{\Phi}(\lambda, k). \tag{4.2}$$

Hence the dilatino wave function is given by

$$\tilde{\Phi}(\lambda, k) = \not{k}\lambda. \tag{4.3}$$

The dilatino wave function automatically satisfies the equation of motion

$$\not{k}\tilde{\Phi} = 0, \tag{4.4}$$

because of the massless condition $k^2 = 0$. Then we can show the supersymmetry transformation between the dilaton and the dilatino;

$$\delta^{(1)}\tilde{\Phi} = \not{k}\epsilon = \Gamma^\mu \epsilon(-i\partial_\mu \Phi). \tag{4.5}$$

4.2 Antisymmetric tensor field $B_{\mu\nu}$

The wave function of the next field, an antisymmetric tensor field contains two λ's and can be generated from the dilatino wave function by $\delta^{(2)}$ transformation as

$$\delta^{(2)}\tilde{\Phi}(\lambda, k) = -\frac{1}{16}\Gamma^{\mu\nu\rho}\epsilon\ k_\mu \left(k_\sigma \bar{\lambda}\Gamma_{\nu\rho\sigma}\lambda\right) \equiv -\frac{i}{24}\Gamma^{\mu\nu\rho}\epsilon H_{\mu\nu\rho}, \tag{4.6}$$

We identify $H_{\mu\nu\rho}$ as the field strength of the antisymmetric tensor $B_{\mu\nu}(\lambda, k)$,

$$H_{\mu\nu\rho} = i(k_\mu B_{\nu\rho} + k_\nu B_{\rho\mu} + k_\rho B_{\mu\nu}). \tag{4.7}$$

Then the wave function $B_{\mu\nu}$ is given by

$$B_{\mu\nu}(\lambda, k) = -\frac{1}{2}b_{\mu\nu} + (k_\mu v_\nu - k_\nu v_\mu) \equiv -\frac{1}{2}b_{\mu\nu} + k_{[\mu}v_{\nu]}, \tag{4.8}$$

where v_μ represents gauge degrees of freedom corresponding to the two form gauge field $B_{\mu\nu}$. Here we have defined an antisymmetric bilinear of λ by

$$b_{\mu\nu}(\lambda) \equiv k^\rho(\bar{\lambda}\Gamma_{\mu\nu\rho}\lambda). \tag{4.9}$$

They are the only independent bilinear forms constructed from 8 independent massless spinors (namely nonzero component of $\not{k}\lambda$) and there are ${}_8C_2 = 28$ degrees of freedom. This number can be understood as follows. $b_{\mu\nu}$ satisfies two relations

$$k^\mu b_{\mu\nu} = 0, \tag{4.10}$$

$$b_{\mu\nu}\Gamma^{\mu\nu}\lambda = 0, \tag{4.11}$$

and an independent number of each relation is 9 and 8. Hence the number of independent $b_{\mu\nu}$ is ${}_{10}C_2 - 9 - 8 = 28$. The proof of the second relation (4.11) is given in the appendix.

For simplicity we fix the gauge degrees of freedom as $v_\mu = 0$. For the wave function (4.8), the equation of motion for the antisymmetric tensor is satisfied,

$$k^\mu H_{\mu\nu\rho} = 0, \tag{4.12}$$

because of $k^2 = 0$ and

$$k^\mu B_{\mu\nu} = 0. \tag{4.13}$$

A variation under the other supersymmetry $\delta^{(1)}$ of the wave function $B_{\mu\nu}(\lambda, k)$ is calculated as

$$\delta^{(1)} B_{\mu\nu} = -\bar{\epsilon}\Gamma_{\mu\nu}\tilde{\Phi}. \tag{4.14}$$

4.3 Gravitino Ψ_μ

A gravitino wave function contains three λ's and can be generated from $B_{\mu\nu}$ through $\delta^{(2)}$ supersymmetry transformation. It is defined through the susy transformation

$$\delta^{(2)} B_{\mu\nu} = 2i(\bar{\epsilon}\Gamma_{[\mu}\Psi_{\nu]} + k_{[\mu}\Lambda_{\nu]}). \tag{4.15}$$

Λ^μ is a gauge transformation parameter. Since the left hand side of (4.15) becomes

$$\delta^{(2)} B_{\mu\nu} = (\bar{\epsilon}\not{k}\lambda) B_{\mu\nu}(\lambda, k) = -\frac{1}{2}(\bar{\epsilon}\not{k}\lambda) b_{\mu\nu}, \tag{4.16}$$

we can identify the wave function

$$\Psi_\mu(\lambda, k) = -\frac{i}{24}(k_\rho \Gamma^{\nu\rho}\lambda) b_{\mu\nu}, \tag{4.17}$$

and the gauge transformation parameter

$$\Lambda_\mu(\lambda, k) = -\frac{i}{12}(\bar{\epsilon}\Gamma^\nu\lambda) b_{\mu\nu}. \tag{4.18}$$

The wave function (4.17) automatically satisfies the equation of motion

$$k_\nu \Gamma^{\mu\nu\rho}\Psi_\rho = 0. \tag{4.19}$$

With the gauge choice in (4.17), this equation of motion is equivalent to

$$\not{k}\Psi_\mu = 0, \tag{4.20}$$

because of

$$\Gamma^\mu \Psi_\mu = k^\mu \Psi_\mu = 0. \tag{4.21}$$

The supersymmetry transformation $\delta^{(1)}$ for the gravitino wave function is given by

$$\begin{aligned} \delta^{(1)}\Psi_\mu(\lambda, k) &= -\frac{i}{24}\left[(\Gamma^\nu \not{k}\epsilon)b_{\mu\nu} + 2(\Gamma^\nu \not{k}\lambda)(\bar{\epsilon}\Gamma_{\mu\nu\rho}\lambda)k^\rho\right] \\ &= \frac{1}{24\cdot 4}\left[9\Gamma^{\nu\rho}\epsilon H_{\mu\nu\rho} - \Gamma_{\mu\nu\rho\sigma}\epsilon H^{\nu\rho\sigma}\right] + (\text{gauge tr.}). \end{aligned} \tag{4.22}$$

4.4 Graviton $h_{\mu\nu}$ and 4-rank antisymmetric tensor $A_{\mu\nu\rho\sigma}$

In the wave functions containing four λ's there are two fields, graviton $h_{\mu\nu}$ and 4-rank antisymmetric tensor field $A_{\mu\nu\rho\sigma}$. These wave functions can be read from the supersymmetry transformations of the gravitino field as

$$\delta^{(2)}\Psi_\mu(\lambda, k) = \frac{i}{2}\Gamma^{\lambda\rho}k_\rho h_{\mu\lambda}\epsilon + \frac{i}{4\cdot 5!}\Gamma^{\rho_1\cdots\rho_5}\Gamma_\mu\epsilon F_{\rho_1\cdots\rho_5} + (\text{gauge tr.}). \tag{4.23}$$

Here the field strength $F_{\mu\nu\rho\sigma\tau}(\lambda, k)$ is defined by

$$F_{\mu\nu\rho\sigma\tau} = ik_\mu A_{\nu\rho\sigma\tau} + (\text{antisymmetrization}) = ik_{[\mu}A_{\nu\rho\sigma\tau]}\ . \tag{4.24}$$

Since the left hand side becomes

$$\delta^{(2)}\Psi_\mu(\lambda, k) = (\bar{\epsilon}\not{k}\lambda)\Psi_\mu \tag{4.25}$$

$$\begin{aligned} &= -\frac{i}{24}(\bar{\epsilon}\not{k}\lambda)(\Gamma^\nu \not{k}\lambda)b_{\mu\nu} \\ &= \frac{i}{12\cdot 16}\Gamma^{\nu\rho}\epsilon k_\rho b_\mu{}^\sigma b_{\sigma\nu} + \frac{i}{24\cdot 16}\left[\frac{1}{5!}\Gamma^{\rho_1\rho_2\rho_3\rho_4}\epsilon k_{[\mu}b_{\rho_1\rho_2}b_{\rho_3\rho_4]} - (\text{gauge tr.})\right], \end{aligned} \tag{4.26}$$

we have the graviton wave function $h_{\mu\nu}$ as

$$h_{\mu\nu}(\lambda, k) = \frac{1}{96}b_\mu{}^\rho b_{\rho\nu}. \tag{4.27}$$

Because of the identity $b_{\mu\nu}b^{\mu\nu} = 0$, the graviton wave function is traceless. By using the self-duality of $F_{\mu\nu\rho\sigma\tau}$,

$$\Gamma^{\rho_1\cdots\rho_5}\Gamma_\mu F_{\rho_1\cdots\rho_5} = \Gamma^{\rho_1\cdots\rho_5}{}_\mu F_{\rho_1\cdots\rho_5} + 5\Gamma^{\rho_1\cdots\rho_4}F_{\rho_1\cdots\rho_4\mu} \tag{4.28}$$

$$= 10\Gamma^{\rho_1\cdots\rho_4}F_{\rho_1\cdots\rho_4\mu}\ , \tag{4.29}$$

we can also obtain the wave function for the field strength as

$$F_{\rho_1\cdots\rho_4\mu} = \frac{1}{32\cdot 4!}k_{[\mu}b_{\rho_1\rho_2}b_{\rho_3\rho_4]}. \tag{4.30}$$

and hence for the 4-rank antisymmetric tensor $A_{\rho_1\cdots\rho_4}$ as

$$A_{\rho_1\cdots\rho_4}(\lambda, k) = -\frac{i}{32(4!)^2}b_{[\rho_1\rho_2}b_{\rho_3\rho_4]}, \tag{4.31}$$

up to gauge transformations. It can be checked directly that the field strength $F_{\mu\nu\rho\sigma\tau}$ is self-dual with this wave function.

Under the other susy transformation $\delta^{(1)}$, these wave functions transform as follows,

$$\delta^{(1)}h_{\mu\nu} = -\frac{i}{2}\bar{\epsilon}\Gamma_{(\mu}\Psi_{\nu)} + (\text{gauge tr.}), \tag{4.32}$$

$$\delta^{(1)}A_{\mu\nu\rho\sigma} = -\frac{1}{32\cdot 4!}\bar{\epsilon}\Gamma_{[\mu\nu\rho}\Psi_{\sigma]} + (\text{gauge tr.}), \tag{4.33}$$

where a round bracket for indices means symmetrization with a weight 1.

4.5 Charge conjugation and the other wave functions

The other wave functions in the massless supergravity multiplet can be similarly constructed by using the supersymmetry transformations. In the following we instead make use of the charge conjugation operation (3.25) to obtain the other wave functions.

First the charge conjugation of the dilaton field is given by

$$\Phi^c(\zeta, k) = \int [d\lambda]\ \ e^{\bar{\zeta}\not{k}\lambda} = (2E)^4(\zeta_9\cdots\zeta_{16}) = \frac{1}{8\cdot 8!}b_\mu{}^\nu b_\nu{}^\lambda b_\lambda{}^\sigma b_\sigma{}^\mu(\zeta). \tag{4.34}$$

The determination of the coefficient is straightforward but not easy to obtain. We have determined the coefficient by using a computer and verified that it is consistent with the susy transformations of the wave functions.

The charge conjugated dilatino wave function becomes

$$\tilde{\Phi}^c(\zeta, k) = \int [d\lambda] \;\; e^{\bar{\zeta}\not{k}\lambda} \tilde{\Phi}(\lambda, k) = \frac{1}{8!} k_\alpha \Gamma^{\mu\nu\alpha} \lambda b_{\nu\rho} b^{\rho\sigma} b_{\sigma\mu}. \tag{4.35}$$

It also satisfies the same equation of motion as the dilatino field

$$\not{k}\tilde{\Phi}^c = 0. \tag{4.36}$$

By taking the charge conjugation of the transformation (4.2) and (4.5), we have

$$\delta^{(1)}\Phi^c(\zeta, k) = \bar{\epsilon}\tilde{\Phi}^c(\zeta, k), \tag{4.37}$$

$$\delta^{(2)}\tilde{\Phi}^c(\zeta, k) = \Gamma^\mu \epsilon(-i\partial_\mu \Phi^c). \tag{4.38}$$

The wave function for the charge conjugated antisymmetric tensor field is given by

$$B^c_{\mu\nu}(\zeta, k) = \int [d\lambda] \; e^{\bar{\zeta}\not{k}\lambda} B_{\mu\nu}(\lambda, k) = -\frac{1}{6!} b_{\mu\rho} b^{\rho\sigma} b_{\sigma\nu}. \tag{4.39}$$

From transformations (4.6) and (4.14), we have supersymmetry transformations for the charge conjugated field as

$$\delta^{(1)}\tilde{\Phi}^c(\zeta, k) \;\; = \;\; -\frac{i}{24}\Gamma^{\mu\nu\rho}\epsilon(H_{\mu\nu\rho})^c, \tag{4.40}$$

$$\delta^{(2)} B^c_{\mu\nu} \;\; = \;\; -\bar{\epsilon}\Gamma_{\mu\nu}\tilde{\Phi}^c. \tag{4.41}$$

Finally the charge conjugated gravitino wave function becomes

$$\Psi^c_\mu(\zeta, k) = \int [d\lambda] \;\; e^{\bar{\zeta}\not{k}\lambda} \Psi_\mu(\lambda, k) = -\frac{i}{4 \cdot 5!} k^\rho \Gamma_{\rho\lambda} \lambda b^{\lambda\sigma} b_{\mu\sigma}, \tag{4.42}$$

and its supersymmetry transformation is given by

$$\delta^{(1)} B^c_{\mu\nu} \;\; = \;\; 2i(\bar{\epsilon}\Gamma_{[\mu}\Psi^c_{\nu]} + k_{[\mu}\Lambda^c_{\nu]}), \tag{4.43}$$

$$\delta^{(2)}\Psi^c_\mu(\zeta, k) \;\; = \;\; \frac{1}{24 \cdot 4}\left[9\Gamma^{\nu\rho}\epsilon(H_{\mu\nu\rho})^c - \Gamma_{\mu\nu\rho\sigma}\epsilon(H^{\nu\rho\sigma})^c\right] + (\text{gauge tr.}). \tag{4.44}$$

Graviton and 4-rank antisymmetric tensor field are invariant under the charge conjugation:

$$h^c_{\mu\nu} = h_{\mu\nu} \;, \quad A^c_{\mu\nu\rho\sigma} = A_{\mu\nu\rho\sigma} \;. \tag{4.45}$$

Therefore we have the charge conjugated supersymmetry transformation as

$$\delta^{(1)}\Psi^c_\mu = \frac{i}{2}\Gamma^{\nu\rho}k_\rho h_{\mu\nu}\epsilon + \frac{i}{4\cdot 5!}\Gamma^{\rho_1\cdots\rho_5}\Gamma_\mu\epsilon F_{\rho_1\cdots\rho_5} + (\text{gauge tr.}), \tag{4.46}$$

$$\delta^{(2)}h_{\mu\nu} = -\frac{i}{2}\bar{\epsilon}\Gamma_{(\mu}\Psi^c_{\nu)} + (\text{gauge tr.}), \tag{4.47}$$

$$\delta^{(2)}A_{\mu\nu\rho\sigma} = -\frac{1}{32\cdot 4!}\bar{\epsilon}\Gamma_{[\mu\nu\rho}\Psi^c_{\sigma]} + (\text{gauge tr.}). \tag{4.48}$$

We summarize the wave functions for the massless multiplet and their supersymmetry transformations in appendix.

5 Vertex operators in IKKT matrix model

In this section, we construct the vertex operators in IKKT matrix model.The construction can be done systematically by expanding the supersymmetric Wilson loop operator in terms of the wave functions $f_A(\lambda)$ constructed in the previous section. Such vertex operators were obtained up to leading order of k_μ by Kitazawa[13] for the type IIB matrix model by using the supersymmetry transformations. But, in this section, we give the vertex operators with more higher order of k_μ. And we see that they automatically form a supersymmetry multiplet and satisfy conservation laws.

In BFSS matrix model, such vertex operators corresponding to the supergravity multiplets were also constructed by one-loop calculations in fermionic backgrounds[14] or by using supersymmetry transformations of the Wilson loop operator[15]. Vertex operators for matrix strings were also constructed in [16].

5.1 expansion of supersymmetric Wilson loop operator

First we rewrite the Wilson loop operator (3.13) in terms of the supersymmetry transformations of $(ik\cdot A)$ as follows,

$$\begin{aligned}\omega(\lambda,k) &= e^{\bar{\lambda}Q_1}\ \mathrm{tr}e^{ik\cdot A}\ e^{-\bar{\lambda}Q_1} \\ &\equiv \mathrm{tr}\ e^{G},\end{aligned} \tag{5.1}$$

Therefore we have the charge conjugated supersymmetry transformation as

$$\delta^{(1)}\Psi^c_\mu = \frac{i}{2}\Gamma^{\nu\rho}k_\rho h_{\mu\nu}\epsilon + \frac{i}{4\cdot 5!}\Gamma^{\rho_1\cdots\rho_5}\Gamma_\mu\epsilon F_{\rho_1\cdots\rho_5} + (\text{gauge tr.}), \tag{4.46}$$

$$\delta^{(2)}h_{\mu\nu} = -\frac{i}{2}\bar{\epsilon}\Gamma_{(\mu}\Psi^c_{\nu)} + (\text{gauge tr.}), \tag{4.47}$$

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where G is given as a finite sum

$$\begin{aligned} G &= ik\cdot A + [\bar{\lambda}Q_1, ik\cdot A] + \frac{1}{2}[\bar{\lambda}Q_1,[\bar{\lambda}Q_1, ik\cdot A]] + \cdots \\ &\quad + \frac{1}{n!}[\bar{\lambda}Q_1,\cdots,[\bar{\lambda}Q_1, e^{ik\cdot A}] + \cdots \\ &\equiv \sum_{i=0}^{8} G_i. \end{aligned} \tag{5.2}$$

Note that the sum terminates at $i = 8$ because there are only 8 independent λ's for on-shell ($k^2 = 0$) Wilson loop operator. Each term can be evaluated as follows;

$$G_0 = ik\cdot A, \tag{5.3}$$

$$G_1 = -(\bar{\lambda}\not{k}\psi), \tag{5.4}$$

$$G_2 = \frac{i}{4}b^{\mu\nu}[A_\mu, A_\nu], \tag{5.5}$$

$$G_3 = -\frac{1}{3!}b^{\mu\nu}[\bar{\lambda}\Gamma_\mu\psi, A_\nu], \tag{5.6}$$

$$G_4 = \frac{1}{4!}\left\{\frac{i}{2}b^{\mu\nu}(\bar{\lambda}\Gamma_{\mu\rho\sigma}\lambda)[[A^\rho, A^\sigma], A_\nu] - ib^{\mu\nu}[\bar{\lambda}\Gamma_\mu\psi, \bar{\lambda}\Gamma_\nu\psi]\right\}, \tag{5.7}$$

$$G_5 = -\frac{1}{5!}\left\{b^{\mu\nu}(\bar{\lambda}\Gamma_{\mu\rho\sigma}\lambda)[[\bar{\lambda}\Gamma^\rho\psi, A^\sigma], A_\nu] + \frac{3}{2}b^{\mu\nu}(\bar{\lambda}\Gamma_{\mu\rho\sigma}\lambda)[[A^\rho, A^\sigma], \bar{\lambda}\Gamma_\nu\psi]\right\}. \tag{5.8}$$

$$\vdots$$

Note that G_n contains n λ's. In order to obtain a vertex operator of each field, we need to expand $\omega(\lambda, k)$ and collect all terms with the same number of λ as

$$\begin{aligned} \omega(\lambda, k) &= \text{tr}\ (e^{ik\cdot A + \sum_{i=1}^{16} G_i}) \\ &= \text{Str}\ e^{ik\cdot A}\Bigg[1 + G_1 + \left(\frac{1}{2}G_1\cdot G_1 + G_2\right) + \left(\frac{(G_1^3).}{3!} + G_1\cdot G_2 + G_3\right) \\ &\quad + \left(\frac{(G_1^4).}{4!} + \frac{1}{2}(G_1^2).\cdot G_2 + \frac{1}{2}(G_2^2). + G_1\cdot G_3 + G_4\right) \\ &\quad + \left(\frac{(G_1^5).}{5!} + \frac{1}{3!}(G_1^3).\cdot G_2 + \frac{1}{2}G_1\cdot(G_2^2). + \frac{1}{2}(G_1^2).\cdot G_3 + G_2\cdot G_3 \right.\\ &\quad \left. + G_1\cdot G_4 + G_5\right) \\ &\quad + \cdots\Bigg]. \end{aligned} \tag{5.9}$$

Here "Str" means a symmetrized trace which is defined by

$$\begin{aligned}\mathrm{Str}\ e^{ik\cdot A}B_1\cdot B_2\cdots B_n &= \int_0^1 dt_1\int_{t_1}^1 dt_2\cdots\int_{t_{n-2}}^1 dt_{n-1}\\ &\quad\times\mathrm{tr}\ e^{ik\cdot At_1}B_1e^{ik\cdot A(t_2-t_1)}B_2\cdots e^{ik\cdot A(t_{n-1}-t_{n-2})}B_{n-1}e^{ik\cdot A(1-t_{n-1})}B_n\\ &\quad+(\text{ permutations of } B_i\text{'s } (i=2,3,\cdots,n)\ ).\end{aligned} \tag{5.10}$$

The center-dot on the left hand side means that the operators B_i are symmetrized. We denoted $\underbrace{G_k\cdot G_k\cdots G_k}_{n}$ as (G_k^n).. Various properties of the symmetrized trace is given in the appendix. For notational simplicity we sometimes use Str with a single operator like Str $(e^{ik\cdot A}B)$ which is equivalent to an ordinary trace. If we set $k=0$, the symmetrized trace becomes

$$\mathrm{Str}\,(B_1\cdot B_2\cdots B_n)=\frac{1}{n!}\sum_{perm.}\mathrm{tr}\,(B_{i_1}B_{i_2}\cdots B_{i_n})\,. \tag{5.11}$$

5.2 Dilaton Φ and dilatino $\tilde{\Phi}$

Dilaton vertex operator V^{Φ} is given by the leading order of λ, namely a term without λ,

$$V^{\Phi}=\mathrm{tr}\ e^{ik\cdot A}. \tag{5.12}$$

Dilatino vertex operator $V^{\tilde{\Phi}}$ is read from the term with a single λ. This is also easily obtained as

$$\mathrm{tr}\ e^{ik\cdot A}G_1=\mathrm{tr}\ e^{ik\cdot A}(-\bar{\lambda}\not{k}\psi)=(\mathrm{tr}\ e^{ik\cdot A}\bar{\psi})\cdot(\not{k}\lambda), \tag{5.13}$$

$$V^{\tilde{\Phi}}=\mathrm{tr}\ e^{ik\cdot A}\bar{\psi}. \tag{5.14}$$

5.3 Antisymmetric tensor field $B_{\mu\nu}$

The vertex operator for the antisymmetric tensor $B_{\mu\nu}$ can be obtained from the terms with two λ's;

$$\begin{aligned}\mathrm{Str}\ e^{ik\cdot A}\left(\frac{1}{2}G_1\cdot G_1+G_2\right) &= \mathrm{Str}\ e^{ik\cdot A}\left(\frac{1}{2}(\bar{\lambda}\not{k}\psi)\cdot(\bar{\lambda}\not{k}\psi)+\frac{i}{4}b^{\mu\nu}[A_\mu,A_\nu]\right)\\ &= \mathrm{Str}\ e^{ik\cdot A}\left(-\frac{1}{32}k^{\rho}(\bar{\psi}\cdot\Gamma_{\mu\nu\rho}\psi)+\frac{i}{4}[A_\mu,A_\nu]\right)\ b^{\mu\nu}.\end{aligned} \tag{5.15}$$

Hence the vertex operator for the antisymmetric tensor field is given by

$$V^B_{\mu\nu} \;=\; \mathrm{Str}\; e^{ik\cdot A}\left(\frac{1}{16}k^\rho(\bar{\psi}\cdot\Gamma_{\mu\nu\rho}\psi) - \frac{i}{2}[A_\mu, A_\nu]\right). \tag{5.16}$$

This vertex operator satisfies

$$k^\mu V^B_{\mu\nu} = 0, \tag{5.17}$$

which assures the gauge invariance of the coupling with the wave function obtained in the previous section, $B^{\mu\nu}(\lambda)V^B_{\mu\nu}$.

5.4 Gravitino Ψ_μ

The 3rd order terms give the gravitino Ψ_μ vertex operator as

$$\begin{aligned}
&\mathrm{Str}\; e^{ik\cdot A}\left(\frac{1}{3!}G_1\cdot G_1\cdot G_1 + G_1\cdot G_2 + G_3\right)\\
&= \mathrm{Str}\; e^{ik\cdot A}\left(-\frac{1}{6}(\bar{\lambda}\not{k}\psi)^3_{\cdot} - \frac{i}{4}(\bar{\lambda}\not{k}\psi)b^{\mu\nu}\cdot[A_\mu, A_\nu] - \frac{1}{6}b^{\mu\nu}[A_\mu, \bar{\lambda}\Gamma_\nu\psi]\right).
\end{aligned} \tag{5.18}$$

Here the following relation is useful,

$$b_{\mu\nu}(\lambda)\left(\bar{\lambda}\not{k}\psi\right) = \frac{1}{4}\left\{b_\mu{}^\sigma k^\rho(\bar{\lambda}\Gamma_{\sigma\nu\rho}\psi) - b_\nu{}^\sigma k^\rho(\bar{\lambda}\Gamma_{\sigma\mu\rho}\psi) - k_\mu b_\nu{}^\sigma(\bar{\lambda}\Gamma_\sigma\psi) + k_\nu b_\mu{}^\sigma(\bar{\lambda}\Gamma_\sigma\psi)\right\}. \tag{5.19}$$

Using this relation, the first term on the right hand side of (5.18) becomes

$$\left[-\frac{i}{12}\mathrm{Str}\; e^{ik\cdot A}k^\rho\left(\bar{\psi}\cdot\Gamma_{\mu\nu\rho}\psi\right)\cdot\bar{\psi}\Gamma^\nu\right]\Psi^\mu(\lambda), \tag{5.20}$$

where $\Psi^\mu(\lambda)$ is the wave function of the gravitino (4.17). Similarly the second term on the right hand side of (5.18) is rewritten as

$$\mathrm{Str}\; e^{ik\cdot A}\left[-\frac{i}{12}k^\rho b_\mu{}^\sigma\left(\bar{\psi}\Gamma_\nu\Gamma_{\rho\sigma}\lambda\right)\cdot[A_\mu, A_\nu] - \frac{1}{6}b^{\mu\nu}\left(\bar{\lambda}\Gamma_\nu\psi\right)\cdot[A_\mu, ik\cdot A]\right]. \tag{5.21}$$

By using the relation (E.3) in the appendix, it is easily understood that the last term cancels the third term of (5.18). Therefore the terms with three λ's become

$$\begin{aligned}
&\mathrm{Str}\; e^{ik\cdot A}\left(\frac{1}{3!}(G_1^3)_{\cdot} + G_1\cdot G_2 + G_3\right)\\
=\;&\mathrm{Str}\; e^{ik\cdot A}\left(-\frac{i}{12}k^\rho(\bar{\psi}\cdot\Gamma_{\mu\nu\rho}\psi) - 2[A_\mu, A_\nu]\right)\cdot\bar{\psi}\Gamma^\nu\times\Psi^\mu(\lambda),
\end{aligned} \tag{5.22}$$

and thus we have the vertex operator for the gravitino

$$
\begin{aligned}
V_\mu^\Psi &= \mathrm{Str}\ e^{ik\cdot A}\left(-\frac{i}{12}k^\rho(\bar\psi\cdot\Gamma_{\mu\nu\rho}\psi)-2[A_\mu,A_\nu]\right)\cdot\bar\psi\Gamma^\nu. \\
&= -\frac{i}{12}\mathrm{tr}\int_0^1 dt_1\int_{t_1}^1 dt_2\ e^{ik\cdot At_1}\bar\psi e^{ik\cdot A(t_2-t_1)}k^\rho\Gamma_{\mu\nu\rho}\psi e^{ik\cdot A(1-t_2)}\bar\psi\Gamma^\nu \\
&\quad -2\mathrm{tr}\int_0^1 dt\ e^{ik\cdot At}[A_\mu,A_\nu]e^{ik\cdot A(1-t)}\bar\psi\Gamma^\nu. \qquad (5.23)
\end{aligned}
$$

The second term is a matrix regularization of the supercurrent $\bar J_\mu=\{X_\mu,X_\nu\}\bar\psi\gamma^\nu$ associated with the supersymmetry $\delta\psi=\epsilon'$ of the Schild action. Here $\{\ \ \}$ is Poisson bracket on the world sheet.

This gravitino vertex operator is shown to satisfy

$$
k^\mu V_\mu^\Psi=0. \qquad (5.24)
$$

The first term of V_μ^Ψ trivially satisfies this relation and the second term is calculated as follows;

$$
\begin{aligned}
k^\mu\left(\text{the 2nd term of } V_\mu^\Psi\right) &= -2\mathrm{tr}\int_0^1 dt\ e^{ik\cdot At}[k\cdot A,A_\mu]e^{ik\cdot A(1-t)}\bar\psi\Gamma^\mu \\
&= 2i\mathrm{tr}\int_0^1 dt\ \frac{d}{dt}\left(e^{ik\cdot At}A_\mu e^{ik\cdot(1-t)}\right)\bar\psi\Gamma^\mu \\
&= 2i\mathrm{tr}\ [e^{ik\cdot A},A_\mu]\bar\psi\Gamma^\mu \\
&= 2i\mathrm{tr}\ e^{ik\cdot A}[A_\mu,\bar\psi]\Gamma^\mu \\
&= 0. \qquad (5.25)
\end{aligned}
$$

In the last line, we used the equation of motion for the fermion, $\Gamma^\lambda[A_\lambda,\psi]=0$. (5.24) assures the gauge invariance of the coupling with gravitino wave function

$$
V_\mu^\Psi\Psi^\mu. \qquad (5.26)
$$

5.5 Graviton and 4-rank antisymmetric tensor field

The next terms with four λ's give the vertex operators for the graviton $h_{\mu\nu}$ and the 4-rank antisymmetric tensor $A_{\mu\nu\rho\sigma}$. The calculation becomes more complicated and we need to use various identities involving fermions. Here we only write down the final results:

$$
\begin{aligned}
&\mathrm{Str}\ e^{ik\cdot A}\left(\frac{(G_1^4).}{4!}+\frac{1}{2}G_1\cdot G_1\cdot G_2+\frac{1}{2}G_2\cdot G_2+G_1\cdot G_3+G_4\right)\\
=\ &\mathrm{Str}\ e^{ik\cdot A}\bigg(\frac{1}{4!}(\bar\lambda \not k\psi)^4_{\cdot}+\frac{i}{8}(\bar\lambda\not k\psi)^2_{\cdot}b^{\mu\nu}\cdot[A_\mu,A_\nu]-\frac{1}{32}b^{\mu\nu}b^{\alpha\beta}[A_\mu,A_\nu]\cdot[A_\alpha,A_\beta]\\
&\qquad+\frac{1}{6}(\bar\lambda\not k\psi)b^{\mu\nu}\cdot[\bar\lambda\Gamma_\mu\psi,A_\nu]-\frac{i}{24}b^{\mu\nu}[\bar\lambda\Gamma_\mu\psi,\bar\lambda\Gamma_\nu\psi]+\frac{i}{48}b_{\mu\nu}(\bar\lambda\Gamma^{\alpha\beta\mu}\lambda)[[A_\alpha,A_\beta],A_\nu]\bigg)\\
=\ &\frac{1}{48}b^{\mu}{}_{a}b^{a\nu}\mathrm{Str}\ e^{ik\cdot A}\bigg\{[A_\mu,A^\rho]\cdot[A_\nu,A_\rho]+\frac{1}{2}\bar\psi\cdot\Gamma_\mu[A_\nu,\psi]\\
&\qquad+\frac{i}{4}k^\lambda(\bar\psi\cdot\Gamma_{\mu\lambda\sigma}\psi)\cdot[A^\sigma,A_\nu]-\frac{1}{8\cdot 4!}k^\lambda k^\tau(\bar\psi\cdot\Gamma_{\mu\lambda}{}^{\sigma}\psi)\cdot(\bar\psi\cdot\Gamma_{\nu\tau\sigma}\psi)\bigg\}\\
&+\frac{1}{3}\cdot\left(-\frac{1}{32}\right)(b^{\mu\nu}b^{\rho\sigma}+b^{\mu\rho}b^{\sigma\nu}+b^{\mu\sigma}b^{\nu\rho})\mathrm{Str}\ e^{ik\cdot A}\\
&\times\bigg\{[A_\mu,A_\nu]\cdot[A_\rho,A_\sigma]+C\bar\psi\cdot\Gamma_{\mu\nu\rho}[A_\sigma,\psi]-\frac{3i}{4}Ck^\lambda(\bar\psi\cdot\Gamma_{\mu\nu\lambda}\psi)\cdot[A_\rho,A_\sigma]\\
&\quad-\frac{1}{8\cdot 4!}k^\lambda k^\tau(\bar\psi\cdot\Gamma_{\mu\nu\lambda}\psi)\cdot(\bar\psi\cdot\Gamma_{\rho\sigma\tau}\psi)\bigg\},
\end{aligned}
\tag{5.27}
$$

where C is a numerical constant which we could not determine in this approach of the calculation. But we can instead make use of another information of the block-block interaction briefly explained in the next subsection and determine it to be $C=-1/3$. Therefore we have the vertex operators for the graviton and the 4-rank antisymmetric tensor field respectively,

$$
\begin{aligned}
V^h_{\mu\nu}\ =\ &2\ \mathrm{Str}\ e^{ik\cdot A}\bigg\{[A_\mu,A^\rho]\cdot[A_\nu,A_\rho]+\frac{1}{4}\bar\psi\cdot\Gamma_{(\mu}[A_{\nu)},\psi]-\frac{i}{8}k^\rho\bar\psi\cdot\Gamma_{\rho\sigma(\mu}\psi\cdot[A_{\nu)},A^\sigma]\\
&-\frac{1}{8\cdot 4!}k^\lambda k^\tau(\bar\psi\cdot\Gamma_{\mu\lambda}{}^{\sigma}\psi)\cdot(\bar\psi\cdot\Gamma_{\nu\tau\sigma}\psi)\bigg\}-(\text{trace part}),
\end{aligned}
\tag{5.28}
$$

$$
\begin{aligned}
V^A_{\mu\nu\rho\sigma}\ =\ &-i\ \mathrm{Str}\ e^{ik\cdot A}\bigg\{F_{[\mu\nu}\cdot F_{\rho\sigma]}+C\bar\psi\cdot\Gamma_{[\mu\nu\rho}[A_{\sigma]},\psi]-\frac{3i}{4}Ck^\lambda\bar\psi\cdot\Gamma_{\lambda[\mu\nu}\psi\cdot F_{\rho\sigma]}\\
&-\frac{1}{8\cdot 4!}k^\lambda k^\tau(\bar\psi\cdot\Gamma_{\lambda[\mu\nu}\psi)\cdot(\bar\psi\cdot\Gamma_{\rho\sigma]\tau}\psi)\bigg\},
\end{aligned}
\tag{5.29}
$$

where $F_{\mu\nu}=[A_\mu,A_\nu]$. These vertex operators satisfy the conservation laws by similar calculations as (5.25),

$$
k^\nu V^h_{\mu\nu}=0,\qquad k^\sigma V^A_{\mu\nu\rho\sigma}=0,
\tag{5.30}
$$

if we use the equations of motion (3.14) and (3.15). In the vertex operator of the graviton, while the fourth term trivially satisfies this equation, the first three terms multiplied by k_μ are combined to terms proportional to the equations of motion. In the case of the 4-rank antisymmetric tensor, by multiplying k_μ, the forth term trivially vanishes and so does the first term due to the Jacobi identity. The second and the third terms satisfy the conservation law because of properties of the symmetrized trace.

Thus the couplings with the graviton and the 4-rank antisymmetric tensor wave functions

$$h^{\mu\nu} V^h_{\mu\nu}, \qquad A^{\mu\nu\rho\sigma} V^A_{\mu\nu\rho\sigma}, \tag{5.31}$$

are respectively gauge invariant.

5.6 Other vertex operators

The other vertex operators are obtained from the terms containing more λ's and the calculations of them become exponentially difficult. Therefore we do not proceed with this calculation here and give a part of the vertex operators by using other approaches.

The IIB matrix model can be regarded as a matrix regularization of the Schild type action for the IIB superstring. The supercurrent of the Schild action associated with the homogeneous supersymmetry (3.1) is given by

$$J^{(2)}_\mu = \{X_\mu, X_\nu\}\{X_\rho, X_\sigma\}\Gamma^{\rho\sigma}\Gamma^\nu\psi - \frac{2i}{3}\left(\bar{\psi}\Gamma^\nu\{X_\mu, \psi\}\right)\Gamma_\nu\psi. \tag{5.32}$$

It is expected that the vertex operator for the charge conjugation of the gravitino includes a term which is a matrix regularization of the above supercurrent of the Schild action. Hence we have

$$V^{\Psi^c}_\mu = \mathrm{Str}\ e^{ik\cdot A}\left([A_\mu, A_\nu]\cdot[A_\rho, A_\sigma]\cdot\Gamma^{\rho\sigma}\Gamma^\nu\psi + \frac{2}{3}\bar{\psi}\cdot\Gamma_\nu[A_\mu, \psi]\cdot\Gamma^\nu\psi\right). \tag{5.33}$$

This satisfies the relation $k^\mu V^{\Psi^c}_\mu = 0$ up to the equations of motion. Of course, the vertex operator will also contain other terms which include more fermions and momentum k_μ.

In the IIB matrix model, the interactions between supergravity modes can be obtained from the one-loop calculation by integrating out off-diagonal components of the matrices. These interaction terms are interpreted as exchange of massless supergravity particles between vertex operators for the diagonal-blocks of the matrices. Exchange of the graviton, dilaton and antisymmetric tensor field is identified in [3] by calculation of one-loop effective action without fermionic backgrounds. With fermionic backgrounds we can also identify exchange of the fermionic fields such as gravitinos and dilatinos. Moreover we can also read off other terms of the bosonic vertex operators containing even number of fermion fields such as the second term of the graviton vertex operator (5.28) or the coefficient C in the 4-th rank antisymmetric tensor field. The one-loop effective action expanded with respect to the inverse powers of the relative distance between two blocks was given in [14, 17, 18];

$$\begin{aligned} W^{(i,j)} &= -3S\mathcal{T}r^{(i,j)}(\mathcal{F}_{\mu\nu}\mathcal{F}_{\nu\sigma}\mathcal{F}_{\sigma\tau}\mathcal{F}_{\tau\mu} - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}_{\mu\nu}\mathcal{F}_{\tau\sigma}\mathcal{F}_{\tau\sigma})\frac{1}{(x^{(i)}-x^{(j)})^8} \\ &\quad -3S\mathcal{T}r^{(i,j)}(\bar{\Phi}\Gamma^\mu\Gamma^\nu\Gamma^\rho\mathcal{F}_{\sigma\mu}\mathcal{F}_{\nu\rho}[P_\sigma,\Phi])\frac{1}{(x^{(i)}-x^{(j)})^8} \\ &\quad +W^{(i,j)}_{\Phi^4} + O\left(\frac{1}{(x^{(i)}-x^{(j)})^9}\right), \end{aligned} \tag{5.34}$$

where $W^{(i,j)}$ is the (i,j) block of the one loop effective action such as Eq. (2.31) and expresses the interaction between the i-th block and j-th block and $S\mathcal{T}r$ is the symmetrized trace of the adjoint operators. W_{Ψ^4} denotes terms including four Ψ's. The terms up to $O(r^{-7})$ cancel each other when backgrounds are restricted to satisfy the matrix model equations of motion. From the above result we can identify some terms in the vertex operators.

In the case of the vertex operator for the charge conjugation of the antisymmetric tensor $V^{B^c}_{\mu\nu}$, the leading term with the least number of fermion fields can be read from the calculations of the block-block interaction as,

$$\mathrm{Str}\ e^{ik\cdot A}\left([A_\mu,A_\rho]\cdot[A^\rho,A^\sigma]\cdot[A_\sigma,A_\nu] - \frac{1}{4}[A_\mu,A_\nu]\cdot[A^\rho,A^\sigma]\cdot[A_\sigma,A_\rho]\right). \tag{5.35}$$

Requiring the current conservation, $k^\mu V^{B^c}_{\mu\nu} = 0$, it can be understood that the vertex operator should include the following terms,

$$\begin{aligned} V^{B^c}_{\mu\nu} &= \mathrm{Str}\ e^{ik\cdot A}\Big([A_\mu,A_\rho]\cdot[A^\rho,A^\sigma]\cdot[A_\sigma,A_\nu] - \frac{1}{4}[A_\mu,A_\nu]\cdot[A^\rho,A^\sigma]\cdot[A_\sigma,A_\rho] \\ &\quad -\frac{1}{4}\bar{\psi}\cdot\Gamma_{(\mu}[A_{\rho)},\psi]\cdot[A^\rho,A_\nu] + \frac{1}{4}\bar{\psi}\cdot\gamma_{(\nu}[A_{\rho)},\psi]\cdot[A^\rho,A_\mu] \\ &\quad +\frac{1}{16}\bar{\psi}\cdot\Gamma_{\rho\sigma[\mu}\psi\cdot\left[A_{\nu]},[A^\rho,A^\sigma]\right] - \frac{i}{8}k_\lambda\bar{\psi}\cdot\Gamma^{\lambda\rho\sigma}\psi\cdot[A_\mu,A_\rho]\cdot[A_\nu,A_\sigma]\Big). \end{aligned} \tag{5.36}$$

For the charge conjugations of the dilaton, the leading terms of the vertex operators can be similarly read from block-block interactions as,

$$
\begin{aligned}
V^{\Phi^c} &= \mathrm{Str}\, e^{ik\cdot A}\Big\{[A_\mu, A_\nu]\cdot[A^\nu, A^\rho]\cdot[A_\rho, A_\sigma]\cdot[A^\sigma, A^\mu] \\
&\quad -\frac{1}{4}[A_\mu, A_\nu]\cdot[A^\nu, A^\mu]\cdot[A_\rho, A_\sigma]\cdot[A^\sigma, A^\rho] \\
&\quad +[A_\sigma, A_\mu]\cdot[A_\nu, A_\rho]\cdot\bar{\psi}\Gamma^\mu\Gamma^{\nu\rho}\cdot[A_\sigma, \psi]\Big\}. \qquad (5.37)
\end{aligned}
$$

The charge conjugated dilatino vertex operator can be obtained from this charge conjugated dilaton vertex operator by supersymmetry transformations. The leading order term is proportional to

$$
\begin{aligned}
V^{\Phi^c} &= \mathrm{Str}\, e^{ik\cdot A}\left\{\left([A_\mu, A_\rho]\cdot[A^\rho, A^\sigma]\cdot[A_\sigma, A_\nu] - \frac{1}{4}[A_\mu, A_\nu]\cdot[A^\rho, A^\sigma]\cdot[A_\sigma, A_\rho]\right)\cdot\Gamma^{\mu\nu}\psi\right. \\
&\quad \left. +\frac{1}{24}[A_\mu, A_\nu]\cdot[A_\rho, A_\sigma]\cdot[A_\lambda, A_\tau]\cdot\Gamma^{\mu\nu\rho\sigma\lambda\tau}\psi\right\}. \qquad (5.38)
\end{aligned}
$$

In order to obtain complete forms of the vertex operators, we need to accomplish the calculation which we performed in the previous section. The calculation is very complicated and tough. As we briefly explained above, we can instead determine the leading order terms of the vertex operators from the calculations of block-block interactions with bosonic and fermionic backgrounds.

6 Condensation of type IIB supergravity multiplets

In this section, we would like to present a considerasion on the treatment of graviton condensation in IKKT matrix model as the final topic of this paper. Using supersymmetry transformation for the Wilson loop operator, we have already seen that the matrix model has the 10D supergravity multiplets and the corresponding vertex operators in the previous two sections. It is clear, however, that we do not have considered the quantum and dynamical picture which might explain the emergence of gravity from the matrix model. we here discuss that the condensation of mean field D-instanton with certain wave function represents the background for N D-instantons by integrating over off-diagonal blocks of matrices.

For the charge conjugations of the dilaton, the leading terms of the vertex operators can be similarly read from block-block interactions as,

$$\begin{aligned} V^{\Phi^c} &= \text{Str } e^{ik\cdot A}\Big\{[A_\mu, A_\nu]\cdot[A^\nu, A^\rho]\cdot[A_\rho, A_\sigma]\cdot[A^\sigma, A^\mu] \\ &\quad -\frac{1}{4}[A_\mu, A_\nu]\cdot[A^\nu, A^\mu]\cdot[A_\rho, A_\sigma]\cdot[A^\sigma, A^\rho] \\ &\quad +[A_\sigma, A_\mu]\cdot[A_\nu, A_\rho]\cdot\bar{\psi}\Gamma^\mu\Gamma^{\nu\rho}\cdot[A_\sigma, \psi]\Big\}. \end{aligned} \tag{5.37}$$

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$$\begin{aligned} V^{\Phi^c} &= \text{Str } e^{ik\cdot A}\left\{\left([A_\mu, A_\rho]\cdot[A^\rho, A^\sigma]\cdot[A_\sigma, A_\nu] - \frac{1}{4}[A_\mu, A_\nu]\cdot[A^\rho, A^\sigma]\cdot[A_\sigma, A_\rho]\right)\cdot\Gamma^{\mu\nu}\psi \right. \\ &\quad \left. +\frac{1}{24}[A_\mu, A_\nu]\cdot[A_\rho, A_\sigma]\cdot[A_\lambda, A_\tau]\cdot\Gamma^{\mu\nu\rho\sigma\lambda\tau}\psi\right\}. \end{aligned} \tag{5.38}$$

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6 Condensation of type IIB supergravity multiplets

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6.1 Preliminary

To begin with, we consider the action of the IKKT matrix model with $(N+1)\times(N+1)$ matrices in Eq. (3.2) as follows;

$$S_{\rm IKKT}(A',\psi') = -\frac{1}{4}{\rm tr}\ [A'_\mu, A'_\nu]^2 - \frac{1}{2}{\rm tr}\ \bar{\psi}'\Gamma_\mu[A'_\mu,\psi']\,. \tag{6.1}$$

We decompose the matrices A' and ψ' to the backgroundss and the quantum fluctuations,

$$A'_\mu = X_\mu + a_\mu\,, \tag{6.2}$$

$$\psi' = \Psi + \varphi \tag{6.3}$$

where X_μ and Ψ are backgrounds and a_μ and φ quantum fluctuations. We find this decomposition is the same as Eq. (2.22) except for the size of matrices. Therefore, when we expand the action (6.1) around the above quantum fluctuations, we get the same one-loop effective action as Eq. (2.31);

$$W = W_b + W_f = -\ln\int dad\varphi dcdb\, e^{-\tilde{S}_2} \tag{6.4}$$

and

$$\begin{aligned} W_b &= \frac{1}{2}\mathcal{T}r\ln(P^2\eta_{\mu\nu} - 2i\mathcal{F}_{\mu\nu}) - \frac{1}{4}\mathcal{T}r\ln\left(\left(P^2+\frac{i}{2}\mathcal{F}_{\mu\nu}\Gamma^{\mu\nu}\right)\left(\frac{1+\Gamma_{11}}{2}\right)\right) \\ &\quad -\mathcal{T}r\ln P^2, \end{aligned} \tag{6.5}$$

$$W_f = \frac{1}{2}\mathcal{T}r\ln\left(\eta_{\mu\nu} + \left(\frac{1}{P^2+2\mathcal{F}}\right)_{\mu\lambda}\bar{\Phi}\ \Gamma_\lambda\frac{1}{\Gamma\cdot P}\Gamma_\nu\ \Phi\right)\,. \tag{6.6}$$

We have already seen the expansion of Eq. (6.5) as in Eq. (2.39). In this section, we would expand Eq. (6.6) with respect to the inverse powers of P. Now, we set matrices (6.2) and (6.3) as follows;

$$X_\mu = \left(\begin{array}{c|c} \tilde{A}_\mu & 0 \\ \hline 0 & y_\mu \end{array}\right) \begin{array}{l}\updownarrow N \\ \updownarrow\ 1\end{array}\quad,\quad \Psi = \left(\begin{array}{c|c} \psi & 0 \\ \hline 0 & \xi \end{array}\right)\,, \tag{6.7}$$

and

$$a_\mu = \left(\begin{array}{c|c} 0 & a_\mu \\ \hline a_\mu^\dagger & 0 \end{array}\right)\quad,\quad \varphi = \left(\begin{array}{c|c} 0 & \varphi \\ \hline \varphi^\dagger & 0 \end{array}\right)\,, \tag{6.8}$$

where

$$\tilde{A}_\mu = x_\mu + A_\mu. \tag{6.9}$$

x_μ is a real and diagonal matrix which represents the corrective coordinate of $\tilde{A}_\mu$ and A_μ is the off-diagonal part of $\tilde{A}_\mu$. y_μ is the coordinate of a mean feild D-instanton and we suppose that x_μ and y_μ are separated far enough from each other. Therefore $1/P_\mu$ is pretty small and we can expand the effective action (6.6) with $1/P_\mu$. We use the following formulas,

$$\begin{aligned} \frac{1}{P^2+2\mathcal{F}} &= \frac{1}{1+\frac{2}{P^2}\mathcal{F}}\frac{1}{P^2}, & (6.10)\\ \frac{1}{\Gamma\cdot P} &= \frac{1}{1+\frac{1}{2P^2}\Gamma\cdot\mathcal{F}}\frac{1}{P^2}\Gamma\cdot P & (6.11)\\ &= \frac{1}{2}\frac{1}{1+\frac{1}{2P^2}\Gamma\cdot\mathcal{F}}\frac{1}{P^2}\Gamma\cdot P+\frac{1}{2}\Gamma\cdot P\frac{1}{1+\frac{1}{2P^2}\Gamma\cdot\mathcal{F}}\frac{1}{P^2}, & (6.12)\end{aligned}$$

where

$$(\Gamma\cdot P)^2 = P^2+\frac{1}{2}\Gamma\cdot\mathcal{F}. \tag{6.13}$$

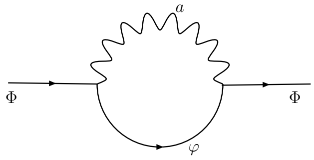
6.2 Perturbation for second order of Φ

Using Eqs. (6.10) and (6.12), the second order terms of the fermionic background Φ are given by

$$\begin{aligned} W_f &= \frac{1}{4}\mathcal{T}r\Bigg[\left(\frac{1}{1+\frac{2}{P^2}\mathcal{F}}\right)_{\mu\nu}\frac{1}{P^2}\bar{\Phi}\Gamma_\nu\frac{1}{1+\frac{1}{2P^2}\Gamma\cdot\mathcal{F}}\frac{1}{P^2}(\Gamma\cdot P)\Gamma_\mu\Phi \\ &\quad+\left(\frac{1}{1+\frac{2}{P^2}\mathcal{F}}\right)_{\mu\nu}\frac{1}{P^2}\bar{\Phi}\Gamma_\nu(\Gamma\cdot P)\frac{1}{1+\frac{1}{2P^2}\Gamma\cdot\mathcal{F}}\frac{1}{P^2}\Gamma_\mu\Phi\Bigg]. \end{aligned} \tag{6.14}$$

6.2.1 P^{-3}

The terms of this order are

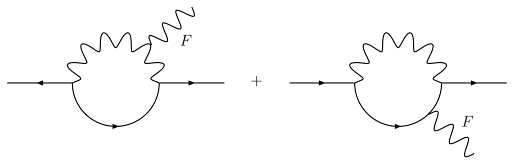


$$\begin{aligned} &= \frac{1}{4}\mathcal{T}r\left\{\frac{1}{P^2}\bar{\Phi}\Gamma_\mu\frac{1}{P^2}(\Gamma\cdot P)\Gamma_\mu\Phi+\frac{1}{P^2}\bar{\Phi}\Gamma_\mu(\Gamma\cdot P)\frac{1}{P^2}\Gamma_\mu\Phi\right\} \\ &= -2\mathcal{T}r\left(\frac{1}{P^2}\bar{\Phi}\frac{1}{P^2}\Gamma_\mu[P_\mu,\Phi]\right). \end{aligned} \tag{6.15}$$

Namely, this is propotional to the equation of motion (2.15) for the fermion.

6.2.2 P^{-5}

At this order we have the following terms,

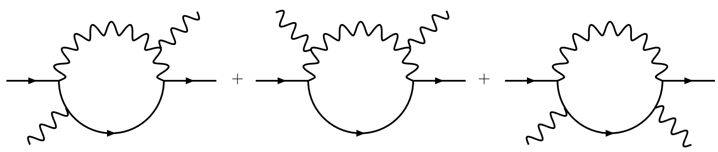


$$
\begin{aligned}
= \; & \frac{1}{4}\mathcal{T}r\left\{-\frac{2}{P^2}\mathcal{F}_{\mu\lambda}\frac{1}{P^2}\bar{\Phi}\Gamma_\lambda\frac{1}{P^2}(\Gamma\cdot P)\Gamma_\mu\Phi-\frac{2}{P^2}\mathcal{F}_{\mu\lambda}\frac{1}{P^2}\bar{\Phi}\Gamma_\lambda(\Gamma\cdot P)\frac{1}{P^2}\Gamma_\mu\Phi\right.\\
& \left.-\frac{1}{P^2}\bar{\Phi}\Gamma_\mu\frac{1}{2P^2}(\Gamma\cdot\mathcal{F})\frac{1}{P^2}(\Gamma\cdot P)\Gamma_\mu\Phi-\frac{1}{P^2}\bar{\Phi}\Gamma_\mu(\Gamma\cdot P)\frac{1}{2P^2}(\Gamma\cdot\mathcal{F})\frac{1}{P^2}\Gamma_\mu\Phi\right\} \qquad (6.16)\\
= \; & -\frac{1}{2}\mathcal{T}r\left(\frac{1}{P^2}\mathcal{F}_{\mu\lambda}\frac{1}{P^2}\Phi\frac{1}{P^2}\Gamma_{\mu\lambda}\Gamma_\nu[P_\nu,\Phi]\right)-\frac{1}{2}\mathcal{T}r\left(\frac{1}{P^2}\mathcal{F}_{\mu\lambda}\frac{1}{P^2}[\bar{\Phi},P_\nu]\Gamma_\nu\Gamma_{\mu\lambda}\frac{1}{P^2}\Phi\right)\\
& -\mathcal{T}r\left(\frac{1}{P^2}\mathcal{F}_{\mu\nu}\frac{1}{P^2}\bar{\Phi}\frac{1}{P^2}\Gamma_\mu\{P_\nu,\Phi\}\right)+\mathcal{T}r\left(\frac{1}{P^2}\mathcal{F}_{\mu\nu}\frac{1}{P^2}\{\bar{\Phi},P_\nu\}\Gamma_\mu\frac{1}{P^2}\Phi\right). \qquad (6.17)
\end{aligned}
$$

In the last equation, the first two terms are proportional to the equation of motion (2.15). It is noted that the terms in the second lines vanish if P_μ is replaced with $d_\mu(=x_\mu-y_\mu)$. Therefore these terms are actually $\mathcal{O}(d^{-6})$ in the $1/d$ expansions.

6.2.3 P^{-7}

The terms of the order P^{-7} are



$$
\begin{aligned}
= \; & \frac{1}{4}\mathcal{T}r\left[\left(\frac{2}{P^2}\mathcal{F}_{\mu\nu}\right)\frac{1}{P^2}\bar{\Phi}\Gamma_\nu\left(\frac{1}{2P^2}\Gamma\cdot\mathcal{F}\right)\frac{1}{P^2}(\Gamma\cdot P)\Gamma_\mu\Phi\right.\\
& +\left(\frac{2}{P^2}\mathcal{F}_{\mu\nu}\right)\frac{1}{P^2}\bar{\Phi}\Gamma_\nu(\Gamma\cdot P)\left(\frac{1}{2P^2}\Gamma\cdot\mathcal{F}\right)\frac{1}{P^2}\Gamma_\mu\Phi\\
& +\left(\frac{2}{P^2}\mathcal{F}_{\mu\nu}\right)\left(\frac{2}{P^2}\mathcal{F}_{\nu\rho}\right)\frac{1}{P^2}\bar{\Phi}\Gamma_\rho\frac{1}{P^2}(\Gamma\cdot P)\Gamma_\mu\Phi\\
& +\left(\frac{2}{P^2}\mathcal{F}_{\mu\nu}\right)\left(\frac{2}{P^2}\mathcal{F}_{\nu\rho}\right)\frac{1}{P^2}\bar{\Phi}\Gamma_\rho(\Gamma\cdot P)\frac{1}{P^2}\Gamma_\mu\Phi\\
& +\frac{1}{P^2}\bar{\Phi}\Gamma_\mu\left(\frac{1}{2P^2}\Gamma\cdot\mathcal{F}\right)\left(\frac{1}{2P^2}\Gamma\cdot\mathcal{F}\right)\frac{1}{P^2}(\Gamma\cdot P)\Gamma_\mu\Phi\\
& \left.+\frac{1}{P^2}\bar{\Phi}\Gamma_\mu(\Gamma\cdot P)\left(\frac{1}{2P^2}\Gamma\cdot\mathcal{F}\right)\left(\frac{1}{2P^2}\Gamma\cdot\mathcal{F}\right)\frac{1}{P^2}\Gamma_\mu\Phi\right]. \qquad (6.18)
\end{aligned}
$$

These are rewritten as

$$
\begin{aligned}
& \frac{1}{4}\mathcal{T}r\left(\frac{1}{P^2}\mathcal{F}_{\mu\nu}\frac{1}{P^2}\bar{\Phi}\frac{1}{P^2}\mathcal{F}_{\rho\sigma}\frac{1}{P^2}\Gamma_{\mu\nu\rho\sigma}\Gamma_\lambda[P_\lambda,\Phi]\right) \\
+ \;& \mathcal{T}r\left(\frac{1}{P^2}\mathcal{F}_{\mu\nu}\frac{1}{P^2}\mathcal{F}_{\nu\rho}\frac{1}{P^2}\bar{\Phi}\frac{1}{P^2}\Gamma_{\mu\rho}\Gamma_\sigma[P_\sigma,\Phi]\right)+\mathcal{T}r\left(\frac{1}{P^2}\mathcal{F}_{\mu\nu}\frac{1}{P^2}\mathcal{F}_{\nu\rho}\frac{1}{P^2}[\bar{\Phi},P_\sigma]\Gamma_\sigma\Gamma_{\mu\rho}\frac{1}{P^2}\Phi\right) \\
- \;& \mathcal{T}r\left(\frac{1}{P^2}\mathcal{F}_{\mu\nu}\frac{1}{P^2}\mathcal{F}_{\nu\mu}\frac{1}{P^2}\bar{\Phi}\frac{1}{P^2}\Gamma_\rho[P_\rho,\Phi]\right)-\mathcal{T}r\left(\frac{1}{P^2}\mathcal{F}_{\mu\nu}\frac{1}{P^2}\mathcal{F}_{\nu\mu}\frac{1}{P^2}[\bar{\Phi},P_\rho]\Gamma_\rho\frac{1}{P^2}\Phi\right) \\
- \;& \frac{1}{2}\mathcal{T}r\left(\frac{1}{P^2}\mathcal{F}_{\mu\nu}\frac{1}{P^2}\bar{\Phi}\frac{1}{P^2}\mathcal{F}_{\nu\mu}\frac{1}{P^2}\Gamma_\rho[P_\rho,\Phi]\right) \\
- \;& \frac{1}{2}\mathcal{T}r\left(\frac{1}{P^2}\mathcal{F}_{\mu\nu}\frac{1}{P^2}\mathcal{F}_{\rho\sigma}\frac{1}{P^2}P_\nu\bar{\Phi}\Gamma_{\mu\rho\sigma}\frac{1}{P^2}\Phi\right)-\frac{1}{2}\mathcal{T}r\left(\frac{1}{P^2}\mathcal{F}_{\mu\nu}\frac{1}{P^2}\mathcal{F}_{\rho\sigma}\frac{1}{P^2}P_\sigma\bar{\Phi}\Gamma_{\mu\nu\rho}\frac{1}{P^2}\Phi\right) \\
- \;& \frac{1}{2}\mathcal{T}r\left(\frac{1}{P^2}\mathcal{F}_{\mu\nu}\frac{1}{P^2}\mathcal{F}_{\rho\sigma}\frac{1}{P^2}\bar{\Phi}\Gamma_{\mu\nu\sigma}\frac{1}{P^2}\Phi P_\rho\right)-\frac{1}{2}\mathcal{T}r\left(\frac{1}{P^2}\mathcal{F}_{\mu\nu}\frac{1}{P^2}\mathcal{F}_{\rho\sigma}\frac{1}{P^2}\bar{\Phi}\Gamma_{\nu\rho\sigma}\frac{1}{P^2}\Phi P_\mu\right) \\
- \;& \frac{1}{2}\mathcal{T}r\left(\frac{1}{P^2}\mathcal{F}_{\mu\nu}\frac{1}{P^2}\bar{\Phi}\frac{1}{P^2}\mathcal{F}_{\rho\sigma}\frac{1}{P^2}P_\nu\Gamma_{\mu\rho\sigma}\Phi\right)+\frac{1}{2}\mathcal{T}r\left(\frac{1}{P^2}\mathcal{F}_{\mu\nu}\frac{1}{P^2}\bar{\Phi}P_\nu\frac{1}{P^2}\mathcal{F}_{\rho\sigma}\frac{1}{P^2}\Gamma_{\nu\rho\sigma}\Phi\right) \\
+ \;& 2\mathcal{T}r\left(\frac{1}{P^2}\mathcal{F}_{\mu\nu}\frac{1}{P^2}\mathcal{F}_{\nu\rho}\frac{1}{P^2}\bar{\Phi}\frac{1}{P^2}\Gamma_\rho P_\mu\Phi\right)+2\mathcal{T}r\left(\frac{1}{P^2}\mathcal{F}_{\mu\nu}\frac{1}{P^2}\mathcal{F}_{\nu\rho}\frac{1}{P^2}\bar{\Phi}P_\rho\frac{1}{P^2}\Gamma_\mu\Phi\right) \\
- \;& \mathcal{T}r\left(\frac{1}{P^2}\mathcal{F}_{\mu\nu}\frac{1}{P^2}\mathcal{F}_{\nu\rho}\frac{1}{P^2}\bar{\Phi}\Gamma_\mu\frac{1}{P^2}\Phi P_\rho\right)+\mathcal{T}r\left(\frac{1}{P^2}\mathcal{F}_{\mu\nu}\frac{1}{P^2}\mathcal{F}_{\nu\rho}\frac{1}{P^2}\bar{\Phi}\Gamma_\rho\frac{1}{P^2}\Phi P_\mu\right) \\
- \;& \mathcal{T}r\left(\frac{1}{P^2}\mathcal{F}_{\mu\nu}\frac{1}{P^2}\mathcal{F}_{\nu\rho}\frac{1}{P^2}P_\mu\bar{\Phi}\Gamma_\rho\frac{1}{P^2}\Phi\right)+\mathcal{T}r\left(\frac{1}{P^2}\mathcal{F}_{\mu\nu}\frac{1}{P^2}\mathcal{F}_{\nu\rho}\frac{1}{P^2}P_\rho\bar{\Phi}\Gamma_\mu\frac{1}{P^2}\Phi\right) \\
+ \;& \mathcal{T}r\left(\frac{1}{P^2}\mathcal{F}_{\mu\nu}\frac{1}{P^2}\bar{\Phi}\frac{1}{P^2}\mathcal{F}_{\nu\rho}\frac{1}{P^2}P_\mu\Gamma_\rho\Phi\right)+\mathcal{T}r\left(\frac{1}{P^2}\mathcal{F}_{\mu\nu}\frac{1}{P^2}\bar{\Phi}P_\mu\frac{1}{P^2}\mathcal{F}_{\nu\rho}\frac{1}{P^2}\Gamma_\rho\Phi\right) \qquad (6.19)
\end{aligned}
$$

The first six terms vanish each other if the fermionic background satisfies the equation of motion.

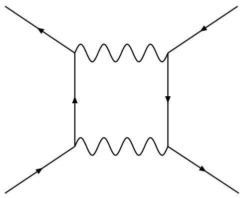
6.3 Perturbation for fourth order of Φ

Following the previous section, we calculate more higher order terms of the perturbation expansion. The fourth order terms of Φ are given by

$$
\begin{aligned}
W_f \;=\; & -\frac{1}{4}\mathcal{T}r\Bigg[\left(\frac{1}{1+\frac{2}{P^2}\mathcal{F}}\right)_{\mu\nu}\frac{1}{P^2}\bar{\Phi}\Gamma_\nu\frac{1}{1+\frac{1}{2P^2}\Gamma\cdot\mathcal{F}}\frac{1}{P^2}\left(\Gamma\cdot P\right)\Gamma_\rho\Phi \\
& \times\left(\frac{1}{1+\frac{2}{P^2}\mathcal{F}}\right)_{\rho\sigma}\frac{1}{P^2}\bar{\Phi}\Gamma_\sigma\frac{1}{1+\frac{1}{2P^2}\Gamma\cdot\mathcal{F}}\frac{1}{P^2}\left(\Gamma\cdot P\right)\Gamma_\mu\Phi\Bigg]. \qquad (6.20)
\end{aligned}
$$

6.3.1 P^{-6}

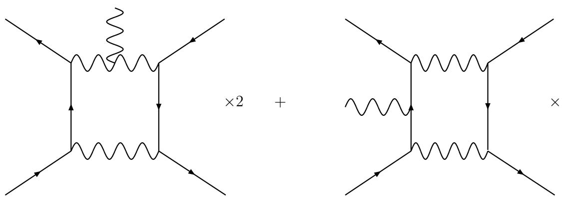
The term of this order is



$$
\begin{aligned}
&= -\frac{1}{4}\mathcal{T}r\left\{\left(\frac{1}{P^2}\bar{\Phi}\Gamma_\mu\frac{1}{P^2}(\Gamma\cdot P)\Gamma_\nu\Phi\right)\cdot\left(\frac{1}{P^2}\bar{\Phi}\Gamma_\nu\frac{1}{P^2}(\Gamma\cdot P)\Gamma_\mu\Phi\right)\right\} \\
&= -\frac{1}{4}\mathcal{T}r\left(\frac{1}{P^2}\bar{\Phi}\frac{1}{P^2}P_\lambda\Gamma_\mu\Gamma_\lambda\Gamma_\nu\Phi\frac{1}{P^2}\bar{\Phi}\frac{1}{P^2}P_\rho\Gamma_\nu\Gamma_\rho\Gamma_\mu\Phi\right) . \qquad (6.21)
\end{aligned}
$$

6.3.2 P^{-8}

This order terms are



$$
\begin{aligned}
&= \frac{1}{2}\mathcal{T}r\left[\left(\frac{2}{P^2}\mathcal{F}_{\mu\lambda}\frac{1}{P^2}\bar{\Phi}\Gamma_\lambda\frac{1}{P^2}(\Gamma\cdot P)\Gamma_\nu\Phi\right)\left(\frac{1}{P^2}\bar{\Phi}\Gamma_\nu\frac{1}{P^2}(\Gamma\cdot P)\Gamma_\mu\Phi\right)\right] \\
&\quad +\frac{1}{2}\mathcal{T}r\left[\left(\frac{1}{P^2}\bar{\Phi}\Gamma_\mu\frac{1}{2P^2}(\Gamma_{\alpha\beta}\mathcal{F}_{\alpha\beta})\frac{1}{P^2}(\Gamma\cdot P)\Gamma_\nu\Phi\right)\left(\frac{1}{P^2}\bar{\Phi}\Gamma_\nu\frac{1}{P^2}(\Gamma\cdot P)\Gamma_\mu\Phi\right)\right] . \qquad (6.22)
\end{aligned}
$$

6.4 $1/d$ expansion

Here we try to derive the result of block-block interactions by Suyama and Tsuchiya[17]. Then, we consider the terms showed in preceding sections with the expansion by d_μ. P_μ is the adjoint operator, that is, for a certain off-diagonal matrix M

$$
\begin{aligned}
P_\mu M &= [X_\mu, M] \\
&= \begin{pmatrix} x_\mu + A_\mu & 0 \\ 0 & y_\mu \end{pmatrix}\begin{pmatrix} 0 & m_{12} \\ m_{21} & 0 \end{pmatrix} - \begin{pmatrix} 0 & m_{12} \\ m_{21} & 0 \end{pmatrix}\begin{pmatrix} x_\mu + A_\mu & 0 \\ 0 & y_\mu \end{pmatrix} \\
&= \big\{(d_\mu + A_\mu)_L - (d_\mu + A_\mu)_R\big\}M , \qquad (6.23)
\end{aligned}
$$

where

$$(d_\mu + A_\mu)_L = \begin{pmatrix} d_\mu + A_\mu & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad (d_\mu + A_\mu)_R = \begin{pmatrix} 0 & 0 \\ 0 & d_\mu + A_\mu \end{pmatrix} . \tag{6.24}$$

Therefore P_μ can be effectively replaced as

$$P_\mu \longrightarrow d_\mu + A_\mu \, . \tag{6.25}$$

By $1/d$ expansion, the P^{-3} term Eq. (6.15) apparently vanishes due to the equation of motion. Moreover we can easily see that the order d^{-5} terms vanish by using the equations of motion. Thus nontrivial contributions of the $1/d$ expansion start from the order of d^{-6}.

Finally, we perform the expansion of $1/P^2$ with respect to the inverse powers of d_μ as follows;

$$\begin{aligned} \frac{1}{P^2} &= \frac{1}{d^2 + 2d\cdot A + A^2} \\ &= \frac{1}{d^2}\Bigg\{ 1 - \left(\frac{2d\cdot A}{d^2} + \frac{A^2}{d^2}\right) + \left(\frac{2d\cdot A}{d^2} + \frac{A^2}{d^2}\right)^2 \\ &\qquad - \left(\frac{2d\cdot A}{d^2} + \frac{A^2}{d^2}\right)^3 + \left(\frac{2d\cdot A}{d^2} + \frac{A^2}{d^2}\right)^4 - \cdots \Bigg\} \\ &= \frac{1}{d^2}\Bigg\{ 1 - \left[\frac{2d\cdot A}{d^2}\right] + \left[-\frac{A^2}{d^2} + 4\left(\frac{d\cdot A}{d^2}\right)^2\right] \\ &\quad + \left[2\frac{(d\cdot A)A^2 + A^2(d\cdot A)}{d^4} - 8\left(\frac{d\cdot A}{d^2}\right)^3\right] \\ &\quad + \left[\left(\frac{A^2}{d^2}\right)^2 - 4\frac{(d\cdot A)^2A^2 + (d\cdot A)A^2(d\cdot A) + A^2(d\cdot A)^2}{d^6} + 16\left(\frac{d\cdot A}{d^2}\right)^4\right] \\ &\quad + \cdots \Bigg\} . \end{aligned} \tag{6.26}$$

6.4.1 d^{-6}

There is a contribution of Φ^2 from the Eq. (6.17),

$$\frac{1}{d^6}\mathcal{T}r[A_\nu, \mathcal{F}]\bar{\Phi}\Gamma_\mu\Phi \, . \tag{6.27}$$

The following terms with Φ^4 come from Eq. (6.21),

$$\frac{1}{4}\frac{d_\rho d_\sigma}{d^2}\mathcal{T}r\left(\bar{\Phi}\Gamma_\mu\Gamma_\rho\Gamma_\nu\Phi\right)\left(\bar{\Phi}\Gamma_\nu\Gamma_\sigma\Gamma_\mu\Phi\right) \, . \tag{6.28}$$

By using the "Fierz transformations" and the "cyclic property" of the trace, this term iss simplified as

$$\frac{1}{d^6}\mathcal{T}r\left(\bar{\Phi}\Gamma_\mu\Phi\right)\left(\bar{\Phi}\Gamma_\mu\Phi\right) . \tag{6.29}$$

Therefore these two contributions cancel each other due to the bosonic equation of motion (3.14).

6.4.2 d^{-7}

Terms with Φ^2 come from the Eq. (6.17),

$$\begin{aligned}\frac{2d_\rho}{d^8}\mathcal{T}r\big(&A_\rho\mathcal{F}_{\mu\nu}\bar{\Phi}\Gamma_\nu A_\mu\Phi+\mathcal{F}_{\mu\nu}A_\rho\bar{\Phi}\Gamma_\nu A_\mu\Phi+\mathcal{F}_{\mu\nu}\bar{\Phi}A_\rho A_\mu\Gamma_\nu\Phi\\&+A_\rho\mathcal{F}_{\mu\nu}\bar{\Phi}\Gamma_\nu\Phi A_\mu+\mathcal{F}_{\mu\nu}A_\rho\bar{\Phi}\Gamma_\nu\Phi A_\mu+\mathcal{F}_{\mu\nu}\bar{\Phi}A_\rho\Gamma_\nu\Phi A_\mu\\&-A_\rho\mathcal{F}_{\mu\nu}A_\mu\bar{\Phi}\Gamma_\nu\Phi-\mathcal{F}_{\mu\nu}A_\rho A_\mu\bar{\Phi}\Gamma_\nu\Phi-\mathcal{F}_{\mu\nu}A_\mu\bar{\Phi}A_\rho\Gamma_\nu\Phi\\&-A_\rho\mathcal{F}_{\mu\nu}\bar{\Phi}A_\mu\Gamma_\nu\Phi-\mathcal{F}_{\mu\nu}A_\rho\bar{\Phi}A_\mu\Gamma_\nu\Phi-\mathcal{F}_{\mu\nu}\bar{\Phi}A_\mu A_\rho\Gamma_\nu\Phi\big) ,\end{aligned} \tag{6.30}$$

and from the Eq. (6.18),

$$\frac{2d_\rho}{d^8}\mathcal{T}r\big(\mathcal{F}_{\rho\mu}\mathcal{F}_{\mu\nu}\bar{\Phi}\Gamma_\nu\Phi+\mathcal{F}_{\nu\mu}\mathcal{F}_{\mu\rho}\bar{\Phi}\Gamma_\nu\Phi+\mathcal{F}_{\rho\mu}\bar{\Phi}\mathcal{F}_{\mu\nu}\Gamma_\nu\Phi\big) . \tag{6.31}$$

These two contributions are combined into

$$\frac{2}{d^8}\mathcal{T}r\big\{(d\cdot A)[A_\mu,\mathcal{F}_{\mu\nu}]\bar{\Phi}\Gamma_\nu\Phi+[A_\mu,\mathcal{F}_{\mu\nu}](d\cdot A)\bar{\Phi}\Gamma_\nu\Phi+[A_\mu,\mathcal{F}_{\mu\nu}]\bar{\Phi}(d\cdot A)\Gamma_\nu\Phi\big\} . \tag{6.32}$$

6.5 Condensation of the supergravity modes

We would like to discuss effective actions for the IIB matrix model under condensation of D-instantons corresponding to the massless type IIB supergravity multiplet. We here consider backgrounds produced by a mean field D-instanton. Then, the free energy is a function of the diagonal components of Eqs. (6.7) and (6.8). Namely, $W(P,\Phi)=W(A,x,\psi;y,\xi)$. We choose wave functions $f_k(y,\xi)$ of (F.1) for the mean field D-instanton, which correspond to the massless type IIB supergravity multiplet[8]. Then we obtain effective actions $S_{\rm eff}(A,x,\psi;f_k)$ under condensation of the massless modes by integrate over y,ξ;

$$\mathrm{e}^{-S_{\rm eff}(A,x,\psi;f_k)}=\int dyd\xi\quad f_k(y,\xi)\,\mathrm{e}^{-W(A_\mu,\psi,y,\xi)} . \tag{6.33}$$

6.5.1 Condensation of the antisymmetric tensor $B_{\mu\nu}$

In order to see a coupling to the antisymmetric tensor field $B_{\mu\nu}$, we choose the following wave function for the mean field D-instanton;

$$\begin{aligned} f_B(k) &\equiv \mathrm{e}^{-ik\cdot y} B_{\mu\nu}(k) k_\rho (\Gamma_{\mu\nu\rho}\Gamma_0)_{\alpha\beta} \frac{\partial}{\partial \xi_\alpha}\frac{\partial}{\partial \xi_\beta}\left(\prod_{\gamma=1}^{16}\xi_\gamma\right) \\ &= \frac{1}{3}\,\mathrm{e}^{-ik\cdot y} H_{\mu\nu\rho}(k)(\Gamma_{\mu\nu\rho}\Gamma_0)_{\alpha\beta}\frac{\partial}{\partial \xi_\alpha}\frac{\partial}{\partial \xi_\beta}\left(\prod_{\gamma=1}^{16}\xi_\gamma\right) \end{aligned} \tag{6.34}$$

Thus only second order terms of ξ in W_f are contribute the effective action under the condensation with this wave function.

Let us first see contributions from the second order terms of Φ. In these terms we can simply replace Φ with ξ and thus the terms of the order P^{-3} and P^{-5} vanish. The terms of the order P^{-7} becomes

$$\begin{aligned} \frac{1}{2}(\bar{\xi}\Gamma_{\mu\nu\rho}\xi)\mathcal{T}r\Bigg[&-\mathcal{F}_{\mu\nu}\frac{1}{P^2}\mathcal{F}_{\rho\sigma}\frac{1}{P^2}P_\nu\left(\frac{1}{P^2}\right)^2 - \mathcal{F}_{\mu\nu}\frac{1}{P^2}P_\nu\left(\frac{1}{P^2}\right)^2\mathcal{F}_{\rho\sigma}\frac{1}{P^2} \\ &+\mathcal{F}_{\mu\nu}\left(\frac{1}{P^2}\right)^2 P_\nu\frac{1}{P^2}\mathcal{F}_{\rho\sigma}\frac{1}{P^2} + \mathcal{F}_{\mu\nu}\frac{1}{P^2}\mathcal{F}_{\rho\sigma}\left(\frac{1}{P^2}\right)^2 P_\nu\frac{1}{P^2} \\ &-\mathcal{F}_{\mu\nu}\left(\frac{1}{P^2}\right)^2\mathcal{F}_{\rho\sigma}\frac{1}{P^2}P_\nu\frac{1}{P^2} + \mathcal{F}_{\mu\nu}\frac{1}{P^2}P_\nu\frac{1}{P^2}\mathcal{F}_{\rho\sigma}\left(\frac{1}{P^2}\right)^2\Bigg]\,. \end{aligned} \tag{6.35}$$

Furthermore we expand these terms with respect to the inverse powers of d_μ. For example, $1/P^2$ is expanded as in Eq. (6.26),

$$\frac{1}{P^2} = \frac{1}{d^2}\left(1 - 2\frac{d\cdot A}{d^2}\right) + \mathcal{O}\left(\frac{1}{d^4}\right)\,. \tag{6.36}$$

It is easily realized that the leading term with $1/d^7$ vanish. And the $1/d^8$ term has the following simple form,

$$-\frac{1}{2d^8}(\bar{\xi}\Gamma_{\mu\rho\sigma}\xi)\,\mathrm{tr}\,[A_\nu, F_{\mu\nu}]F_{\rho\sigma}\,. \tag{6.37}$$

After the integration of y_μ and ξ with the wave function (6.34), we obtain the effective action under condensation of the antisymmetric tensor field,

$$S_{\text{eff}}(A, x, \psi; f_B(k)) = S_{\text{IKKT}} + H_{\mu\rho\sigma}(k)\,\mathrm{e}^{ik\cdot x}\,\mathrm{tr}\,[A_\nu, F_{\mu\nu}]F_{\rho\sigma}\,. \tag{6.38}$$

Here we assumed an appropriate regularization in the infrared region of y_μ integration and renormalized the wave function $B_{\mu\nu}$. This effective action indicate that the Chern-Simons term is induced by an effect of condensation of the antisymmetric tensor.

7 Conclusion and Discussion

Firstly we have constructed a set of wave functions and vertex operators in the IIB matrix model by expanding the supersymmetric Wilson loop operator. They form a massless multiplet of the type IIB supergravity. The vertex operators satisfy conservation laws, for instance Eq. (5.24) or (5.30) by using equations of motion for A^μ and ψ. Where do the conservation laws come from? What is the origin? We have used the supersymmetric Wilson loops and the supersymmetry transformations in order to obtain the vertex operators. We did not use the explicit form of the action. Nonetheless the vertex operators satisfy the conservation laws by using the equations of motion derived from the action (2.10). This is due to the commutation relation of the supersymmetry generators (3.17). Recall that the supersymmetric Wilson loop is invariant under simultaneous supersymmetry transformations of the matrices and wave functions as Eq. (3.20) and (3.21) only by using the equations of motion. On the other hand, the supersymmetry transformations of the wave functions (D.2) contain gauge transformations. Because of it, the vertex operators satisfy conservation laws by using the equations of motion. In this sense, the conservation laws for the vertex operators follow from the supersymmetries. In string theories, conformal invariance guarantees the gauge invariance and the decoupling of unphysical modes from the S-matrix elements. It would be interesting to search for such a hidden symmetry in matrix models.

Next, we have incompletely discussed the condensation of supergravity modes with the analogy between thermodynamics and the multi-particle system of N $D(-1)$'s. The condensation of a mean field D-instanton with certain function $f_k(y, \xi)$ represents the background for N $D(-1)$'s by integrating over off-diagonal blocks of the one-loop effective action. Although we have found the emergence of the Chern-Simons term in the last of previous section, it is the gravitational terms which we want to investigate. But, technically, computations required for such a generalization become increasingly difficult. We may need some entirely new framework for developing the idea in a tractable way. What we are pursuing amounts to investigating the condensation of Goldstone bosons using the configuration space formalism. Something which can play the role of the field-theory like formalism must be a desired language, by which we can treat the matrix models with different sizes of matrices in a much more unified and dynamical manner. Only by using such a formalism, we would be able to discuss the major questions related to the present approach, such as the proof of S-duality symmetry, the background independent formulation, and so on.

As a discussion or a story about a future work, we would mention how we obtain the equation of motion for the background field of the matrix models. In string theories, conformal invariance plays an important role in deriving equation of motion for the background. In the matrix model, we expect that large N renormalization group will play such a role. Matrix models are believed to describe string theories in the large N limit. As we discussed at the end of section 2, we implicitly assume that there are background $D(-1)$'s other than the N $D(-1)$'s and a modification of the configurations of the background $D(-1)$'s leads to a modification of the background field for the N $D(-1)$'s. Hence stability of the background must be related to the stability of the background configurations under integrations of the background $D(-1)$'s. More concretely, we start from the matrix model for $(N+1)\times(N+1)$ hermitian matrices A'_μ with a graviton coupling

$$S_{\rm IKKT}[A'_\mu] + \int dk\ h^{\mu\nu}(k) V^h_{\mu\nu}[A'_\mu], \tag{7.1}$$

and integrate one $D(-1)$ (which we call a mean field $D(-1)$). Then we arrive at a matrix model action for $N\times N$ hermitian matrices A_μ with a modified graviton coupling

$$S_{\rm IKKT}[A_\mu] + \int dk\ h'^{\mu\nu}(k) V^h_{\mu\nu}[A_\mu], \tag{7.2}$$

and we can obtain a renormalization group flow for the coupling constant

$$h^{\mu\nu}(k) \to h^{\mu\nu}(k) + \delta h^{\mu\nu}(k). \tag{7.3}$$

Fixed points of this renormalization group flow will give the equations of motion for the background fields. Though the calculation itself is very difficult, we want to investigate these issues in future publications.

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Appendix

A Majorana-Weyl representation

We use the following Majorana-Weyl representation,

$$\begin{aligned}
\Gamma_0 &= i\sigma_1 \otimes 1 \otimes 1 \otimes 1 \otimes 1, \\
\Gamma_1 &= i\epsilon \otimes \epsilon \otimes \epsilon \otimes \epsilon \otimes \epsilon, \\
\Gamma_2 &= i\epsilon \otimes \epsilon \otimes 1 \otimes \sigma_1 \otimes \epsilon, \\
\Gamma_3 &= i\epsilon \otimes \epsilon \otimes 1 \otimes \sigma_3 \otimes \epsilon, \\
\Gamma_4 &= i\epsilon \otimes \epsilon \otimes \sigma_1 \otimes \epsilon \otimes 1, \\
\Gamma_5 &= i\epsilon \otimes \epsilon \otimes \sigma_3 \otimes \epsilon \otimes 1, \\
\Gamma_6 &= i\epsilon \otimes \epsilon \otimes \epsilon \otimes 1 \otimes \sigma_1, \\
\Gamma_7 &= i\epsilon \otimes \epsilon \otimes \epsilon \otimes 1 \otimes \sigma_3, \\
\Gamma_8 &= i\epsilon \otimes \sigma_1 \otimes 1 \otimes 1 \otimes 1, \\
\Gamma_9 &= i\epsilon \otimes \sigma_3 \otimes 1 \otimes 1 \otimes 1, \\
\Gamma_{11} &= \sigma_3 \otimes 1 \otimes 1 \otimes 1 \otimes 1,
\end{aligned} \tag{A.1}$$

where $\epsilon = i\sigma_2$.

B Properties of gamma matrices

- Metric

$$\eta_{\mu\nu} = diag(-1, +1, \ldots, +1) \qquad (D = 10) \tag{B.1}$$

- Clifford algebra

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\eta_{\mu\nu} \tag{B.2}$$

$$\Gamma_{11} \equiv \Gamma_0\Gamma_1 \cdots \Gamma_9 \ , \ (\Gamma_{11})^2 = 1 \tag{B.3}$$

- Hermiticity

$$(\Gamma_\mu)^\dagger = \Gamma^\mu = \Gamma_0\Gamma_\mu\Gamma_0 \tag{B.4}$$

$$(\Gamma_0)^\dagger = -\Gamma_0,\ (\Gamma_i)^\dagger = \Gamma_i \quad (i = 1, 2, \ldots, 9) \tag{B.5}$$

$$(\Gamma_{11})^\dagger = \Gamma_{11} \tag{B.6}$$

Under our representations,

$$(\Gamma_0)^{\mathrm{T}} = \Gamma_0\ ,\ (\Gamma_i)^{\mathrm{T}} = -\Gamma_i\ , \tag{B.7}$$

and

$$\Gamma_0 \Gamma_\mu \Gamma_0 = -(\Gamma_\mu)^{\mathrm{T}}. \tag{B.8}$$

- $\bar{\psi} \equiv \psi^\dagger \Gamma_0$

$$(i\bar{\psi}\psi)^* = -i\psi^\dagger (\Gamma_0)^\dagger \psi = i\bar{\psi}\psi \tag{B.9}$$

$$\left(\mathrm{tr}\ \bar{\psi}\Gamma_\mu [A^\mu, \psi]\right)^* = \mathrm{tr}\ \bar{\psi}\Gamma_\mu [A^\mu, \psi] \tag{B.10}$$

- Charge conjugation

$$\psi^c = C\bar{\psi}^{\mathrm{T}} = \psi^* \quad , \quad C = \Gamma_0 \tag{B.11}$$

- Weyl spinor

$$\psi = \Gamma_{11}\psi \tag{B.12}$$

$$\begin{aligned} \bar{\psi}_1 \Gamma_{\mu_1\mu_2\cdots\mu_n} \psi_2 &= \psi_1^\dagger \Gamma_0 \Gamma_{\mu_1\mu_2\cdots\mu_n} \Gamma_{11} \psi_2 \\ &= (-1)^{n+1} \bar{\psi}_1 \Gamma_{\mu_1\mu_2\cdots\mu_n} \psi_2 \end{aligned} \tag{B.13}$$

Therefore bilinear forms of spinors vanish unless n is odd.

- Majorana spinor

$$\psi^c = \psi \quad \longrightarrow \quad \psi = \psi^* \tag{B.14}$$

Under our representations,

$$\begin{aligned} \bar{\psi}_1 \Gamma_{\mu_1\mu_2\cdots\mu_n} \psi_2 &= -\psi_2^{\mathrm{T}} (\Gamma_{\mu_1\mu_2\cdots\mu_n})^{\mathrm{T}} (\Gamma_0)^{\mathrm{T}} \psi_1^* \\ &= -(-1)^{\frac{n(n-1)}{2}} \bar{\psi}_2 \Gamma_{\mu_1\mu_2\cdots\mu_n} \psi_1 \end{aligned} \tag{B.15}$$

When $\psi_1 = \psi_2$ is Majorana-Weyl spinor, therefore, bilinear forms of spinors vanish unless $n = 3$ or 7.

C Fierz identity

The Fierz identity is given by [19];

$$(\bar{\psi}_1 M\psi_2)(\bar{\psi}_3 N\psi_4) = -\frac{1}{32}\sum_{n=0}^{5} C_n(\bar{\psi}_1\Gamma_{A_n}\psi_4)(\bar{\psi}_3 N\Gamma_{A_n}M\psi_2), \tag{C.1}$$

$$C_0 = 2, \quad C_1 = 2, \quad C_2 = -1, \quad C_3 = -\frac{1}{3}, \quad C_4 = \frac{1}{12}, \quad C_5 = \frac{1}{120}. \tag{C.2}$$

where A_n is indexes for n-rank Gamma matrix.

We here note some useful relations related to the Fierz identity. We set

$$A = f^{\alpha\beta\gamma}(\bar{\xi}\Gamma_{\alpha\beta\gamma}\xi)\bar{\xi}, \tag{C.3}$$

$$B = f^{\alpha\beta\gamma}(\bar{\xi}\Gamma_{\nu\beta\gamma}\xi)\bar{\xi}\Gamma_{\alpha}{}^{\nu}, \tag{C.4}$$

$$C = f^{\alpha\beta\gamma}(\bar{\xi}\Gamma_{\mu\nu\alpha}\xi)\bar{\xi}\Gamma^{\mu\nu}{}_{\beta\gamma}, \tag{C.5}$$

where $f^{\alpha\beta\gamma}$ is an arbitrary antisymmetric tensor. Performing the Fierz transformation, we find

$$A = -\frac{1}{32}(2A + 6B - 3C), \tag{C.6}$$

$$B = -\frac{1}{32}(14A + 10B + 3C). \tag{C.7}$$

From these relations we obtain

$$(\bar{\xi}\Gamma_{\alpha\beta\gamma}\xi)\bar{\xi}\Gamma^{\alpha\beta} = 0. \tag{C.8}$$

The following relation holds,

$$2X + Y - Z = 0, \tag{C.9}$$

where

$$X = f^{\mu\nu}k^{\rho}(\bar{\epsilon}\not{k}\xi)(\bar{\xi}\Gamma_{\mu\nu\rho}\xi), \tag{C.10}$$

$$Y = f^{\mu\nu}k^{\rho}k_{\nu}(\bar{\epsilon}\Gamma^{a}\xi)(\bar{\xi}\Gamma_{a\mu\rho}\xi), \tag{C.11}$$

$$Z = f^{\mu\nu}k^{\rho}k^{\sigma}(\bar{\epsilon}\Gamma^{a}{}_{\nu\sigma}\xi)(\bar{\xi}\Gamma_{a\mu\rho}\xi). \tag{C.12}$$

Here $f^{\mu\nu}$ is an arbitrary antisymmetric tensor and $k^2 = 0$.

We can derive the following identity from the Fierz transformation,

$$
\begin{aligned}
b_{\mu\nu}b_{\rho\sigma} &= \frac{1}{3}(b_{\mu\nu}b_{\rho\sigma} + b_{\sigma\nu}b_{\mu\rho} - b_{\sigma\mu}b_{\nu\rho}) \\
&+ \frac{1}{6}(\eta_{\sigma\mu}b_{\nu}{}^{\alpha}b_{\alpha\rho} - \eta_{\sigma\nu}b_{\mu}{}^{\alpha}b_{\alpha\rho} + \eta_{\rho\nu}b_{\mu}{}^{\alpha}b_{\alpha\sigma} - \eta_{\rho\mu}b_{\nu}{}^{\alpha}b_{\alpha\sigma}) \\
&+ \frac{1}{6}(k_{\nu}b_{\mu}{}^{\alpha} - k_{\mu}b_{\nu}{}^{\alpha})(\bar{\lambda}\Gamma_{\rho\sigma\alpha}\lambda) \\
&+ \frac{1}{6}(k_{\sigma}b_{\rho}{}^{\alpha} - k_{\rho}b_{\sigma}{}^{\alpha})(\bar{\lambda}\Gamma_{\mu\nu\alpha}\lambda), \qquad \text{(C.13)}
\end{aligned}
$$

where $b_{\mu\nu} = k^{\rho}(\bar{\lambda}\Gamma_{\mu\nu\rho}\lambda)$.

The following relations among the gamma matrices hold,

$$
\begin{aligned}
&\Gamma^{\mu}\Gamma_{A_n}\Gamma_{\mu} = (-1)^n(10-2n)\Gamma_{A_n}, \\
&\Gamma_{\alpha\beta\gamma}\Gamma_{\mu}\Gamma^{\alpha\beta\gamma} = 288\Gamma_{\mu}, \quad \Gamma_{\alpha\beta\gamma}\Gamma_{\mu\nu\rho}\Gamma^{\alpha\beta\gamma} = -48\Gamma_{\mu\nu\rho}, \quad \Gamma_{\alpha\beta\gamma}\Gamma_{\mu\nu\rho\sigma\lambda}\Gamma^{\alpha\beta\gamma} = 0. \qquad \text{(C.14)}
\end{aligned}
$$

D Wave functions and SUSY transformations

We here summarize the wave functions for the massless multiplet and their supersymmetry transformations.

- Wave functions

$$
\begin{aligned}
\Phi(\lambda,k) &= 1, \\
\tilde{\Phi}(\lambda,k) &= \not{k}\lambda, \\
B_{\mu\nu}(\lambda,k) &= -\frac{1}{2}b_{\mu\nu}(\lambda), \\
\Psi_{\mu}(\lambda,k) &= -\frac{i}{24}(k_{\sigma}\Gamma^{\nu\sigma}\lambda)b_{\mu\nu}(\lambda), \\
h_{\mu\nu}(\lambda,k) &= \frac{1}{96}b_{\mu}{}^{\rho}b_{\rho\nu}(\lambda), \\
A_{\mu\nu\rho\sigma}(\lambda,k) &= -\frac{i}{32(4!)^2}b_{[\mu\nu}b_{\rho\sigma]}(\lambda), \\
\Psi^c_{\mu}(\lambda,k) &= -\frac{i}{4\cdot 5!}k^{\rho}\Gamma_{\rho\lambda}\lambda b^{\lambda\sigma}b_{\sigma\mu}(\lambda), \\
B^c_{\mu\nu}(\lambda,k) &= -\frac{1}{6!}b_{\mu\rho}b^{\rho\sigma}b_{\sigma\nu}(\lambda), \\
\tilde{\Phi}^c(\lambda,k) &= \frac{1}{8!}k_{\alpha}\Gamma^{\mu\nu\alpha}\lambda b_{\nu\rho}b^{\rho\sigma}b_{\sigma\mu}(\lambda), \\
\Phi^c(\lambda,k) &= \frac{1}{8\cdot 8!}b_{\mu}{}^{\nu}b_{\nu}{}^{\lambda}b_{\lambda}{}^{\sigma}b_{\sigma}{}^{\mu}(\lambda). \qquad \text{(D.1)}
\end{aligned}
$$

- SUSY transformations

$$
\begin{aligned}
\delta\Phi &= \bar{\epsilon}_2\tilde{\Phi}, \\
\delta\tilde{\Phi} &= \not{k}\epsilon_1\Phi - \frac{i}{24}\Gamma^{\mu\nu\rho}\epsilon_2 H_{\mu\nu\rho}, \\
\delta B_{\mu\nu} &= -\bar{\epsilon}_1\Gamma_{\mu\nu}\tilde{\Phi} + 2i(\bar{\epsilon}_2\Gamma_{[\mu}\Psi_{\nu]} + k_{[\mu}\Lambda_{\nu]}), \\
\delta\Psi_\mu &= \frac{1}{24\cdot 4}\left[9\Gamma^{\nu\rho}\epsilon_1 H_{\mu\nu\rho} - \Gamma_{\mu\nu\rho\sigma}\epsilon_1 H^{\nu\rho\sigma}\right] + \frac{i}{2}\Gamma^{\nu\rho}k_\rho h_{\mu\nu}\epsilon_2 \\
&\quad + \frac{i}{4\cdot 5!}\Gamma^{\rho_1\cdots\rho_5}\Gamma_\mu\epsilon_2 F_{\rho_1\cdots\rho_5} + k_\mu\xi, \\
\delta h_{\mu\nu} &= -\frac{i}{2}\bar{\epsilon}_1\Gamma_{(\mu}\Psi_{\nu)} - \frac{i}{2}\bar{\epsilon}_2\Gamma_{(\mu}\Psi^c_{\nu)} + k_{(\mu}\xi_{\nu)}, \\
\delta A_{\mu\nu\rho\sigma} &= -\frac{1}{(4!)^2}\bar{\epsilon}_1\Gamma_{[\mu\nu\rho}\Psi_{\sigma]} - \frac{1}{(4!)^2}\bar{\epsilon}_2\Gamma_{[\mu\nu\rho}\Psi^c_{\sigma]} + k_{[\mu}\xi_{\nu\rho\sigma]}, \\
\delta\Psi^c_\mu &= \frac{i}{2}\Gamma^{\nu\rho}k_\rho h_{\mu\nu}\epsilon_1 + \frac{i}{4\cdot 5!}\Gamma^{\rho_1\cdots\rho_5}\Gamma_\mu\epsilon_1 F_{\rho_1\cdots\rho_5} \\
&\quad + \frac{1}{24\cdot 4}\left[9\Gamma^{\nu\rho}\epsilon_2 H^c_{\mu\nu\rho} - \Gamma_\mu{}^{\nu\rho\sigma}\epsilon_2 H^c_{\nu\rho\sigma}\right] + k_\mu\xi^c, \\
\delta B^c_{\mu\nu} &= 2i(\bar{\epsilon}_1\Gamma_{[\mu}\Psi^c_{\nu]} + k_{[\mu}\Lambda^c_{\nu]}) - \bar{\epsilon}_2\Gamma_{\mu\nu}\tilde{\Phi}^c, \\
\delta\tilde{\Phi}^c &= -\frac{i}{24}\Gamma^{\mu\nu\rho}\epsilon_1 H^c_{\mu\nu\rho} + \not{k}\epsilon_2\Phi^c, \\
\delta\Phi^c &= \bar{\epsilon}_1\tilde{\Phi}^c,
\end{aligned}
\tag{D.2}
$$

where $\xi, \xi_\mu, \xi_{\mu\nu\rho}$ and Λ_μ are gauge parameters. This supersymmetry transformation is the same as that in [12] up to normalizations.

E Symmetrized trace

The symmetrized trace is defined in (5.10). In particular, explicit forms for two and three operators are written as

$$
\mathrm{Str}\left(e^{ik\cdot A}B\cdot C\right) = \mathrm{tr}\int_0^1 dt\; e^{ik\cdot At}Be^{ik\cdot A(1-t)}C, \tag{E.1}
$$

$$
\begin{aligned}
\mathrm{Str}\left(e^{ik\cdot A}B\cdot C\cdot D\right) &= \mathrm{tr}\int_0^1 dt_1\int_{t_1}^1 dt_2\; e^{ik\cdot At_1}Be^{ik\cdot A(t_2-t_1)}Ce^{ik\cdot A(1-t_2)}D \\
&\quad +(C\longleftrightarrow D)
\end{aligned}
\tag{E.2}
$$

where all matrices are bosonic. The definitions for fermionic matrices can be obtained by replacing the bosonic matrices on the above equations with the fermionic matrices multiplied by Grassmann odd numbers. The center-dot on the left hand side means that matrices are inserted at different places. We note useful equations related to the symmetrized trace,

$$\mathrm{Str}\left(e^{ik\cdot A}\left[ik\cdot A,\ A_\alpha\right]\cdot\psi_\beta\right) = \mathrm{tr}\left[e^{ik\cdot A},\ A_\alpha\right]\psi_\beta, \tag{E.3}$$

$$\mathrm{Str}\left(e^{ik\cdot A}\bar{\psi}\cdot\Gamma_{\mu\nu\lambda}\psi\cdot\left[ik\cdot A,\ A_\rho\right]\right) = 2\ \mathrm{Str}\left(e^{ik\cdot A}\bar{\psi}\cdot\Gamma_{\mu\nu\lambda}\left[A_\rho,\ \psi\right]\right), \tag{E.4}$$

where the following relation is used,

$$\left[e^{ik\cdot A},\ B\right] = \int_0^1 dt\ e^{ik\cdot At}\left[ik\cdot A,\ B\right]e^{ik\cdot A(1-t)}. \tag{E.5}$$

F Wave Function $f(y,\xi)$

$$\begin{aligned} f(y,\xi) &= \mathrm{e}^{-ik\cdot y}\Bigg[f^0(y^2) + if^1_{\mu\nu\lambda}(y^2)\left(\frac{\partial}{\partial\bar{\xi}}\Gamma_{\mu\nu\lambda}\frac{\partial}{\partial\xi}\right) \\ &\quad + f^2_{\mu\nu\lambda,\mu'\nu'\lambda'}(y^2)\left(\frac{\partial}{\partial\bar{\xi}}\Gamma_{\mu\nu\lambda}\frac{\partial}{\partial\xi}\right)\left(\frac{\partial}{\partial\bar{\xi}}\Gamma_{\mu'\nu'\lambda'}\frac{\partial}{\partial\xi}\right) \\ &\quad + if^3_{ABC}(y^2)\left(\frac{\partial}{\partial\bar{\xi}}\Gamma_A\frac{\partial}{\partial\xi}\right)\left(\frac{\partial}{\partial\bar{\xi}}\Gamma_B\frac{\partial}{\partial\xi}\right)\left(\frac{\partial}{\partial\bar{\xi}}\Gamma_C\frac{\partial}{\partial\xi}\right) + \cdots\Bigg]\prod_{\alpha=1}^{16}\xi_\alpha \end{aligned} \tag{F.1}$$

G "$y(d)$" integral formula

$$\int d^{10}y\ \frac{1}{y^{10}}f(y^2) = I \tag{G.1}$$

$$\int d^{10}y\ \frac{y_\mu y_\nu}{y^{12}}f(y^2) = \frac{\delta_{\mu\nu}}{10}I \tag{G.2}$$

$$\int d^{10}y\ \frac{y_\mu y_\nu y_\rho y_\sigma}{y^{14}}f(y^2) = \frac{\delta_{\mu\nu}\delta_{\rho\sigma}+\delta_{\mu\rho}\delta_{\nu\sigma}+\delta_{\mu\sigma}\delta_{\nu\rho}}{120}I \tag{G.3}$$

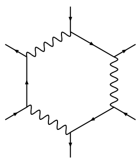
$$\begin{aligned} \int d^{10}y\ \frac{y_{\mu_1}y_{\mu_2}y_{\mu_3}y_{\mu_4}y_{\mu_5}y_{\mu_6}}{y^{16}}f(y^2) &= \frac{I}{1680}(\delta_{\mu_1\mu_2}\delta_{\mu_3\mu_4}\delta_{\mu_5\mu_6} + \delta_{\mu_1\mu_2}\delta_{\mu_3\mu_5}\delta_{\mu_4\mu_6} + \delta_{\mu_1\mu_2}\delta_{\mu_3\mu_6}\delta_{\mu_4\mu_5} \\ &\quad +\delta_{\mu_1\mu_3}\delta_{\mu_2\mu_4}\delta_{\mu_5\mu_6} + \delta_{\mu_1\mu_3}\delta_{\mu_2\mu_5}\delta_{\mu_4\mu_6} + \delta_{\mu_1\mu_3}\delta_{\mu_2\mu_6}\delta_{\mu_4\mu_5} \\ &\quad +\delta_{\mu_1\mu_4}\delta_{\mu_2\mu_3}\delta_{\mu_5\mu_6} + \delta_{\mu_1\mu_4}\delta_{\mu_2\mu_5}\delta_{\mu_3\mu_6} + \delta_{\mu_1\mu_4}\delta_{\mu_2\mu_6}\delta_{\mu_3\mu_5} \\ &\quad +\delta_{\mu_1\mu_5}\delta_{\mu_2\mu_3}\delta_{\mu_4\mu_6} + \delta_{\mu_1\mu_5}\delta_{\mu_2\mu_4}\delta_{\mu_3\mu_6} + \delta_{\mu_1\mu_5}\delta_{\mu_2\mu_6}\delta_{\mu_3\mu_4} \\ &\quad +\delta_{\mu_1\mu_6}\delta_{\mu_2\mu_3}\delta_{\mu_4\mu_5} + \delta_{\mu_1\mu_6}\delta_{\mu_2\mu_4}\delta_{\mu_3\mu_5} + \delta_{\mu_1\mu_6}\delta_{\mu_2\mu_5}\delta_{\mu_3\mu_4}) \end{aligned} \tag{G.4}$$

H "ξ" integral formula

$$
\begin{aligned}
&\int d^{16}\xi \quad (\bar{\xi}\Gamma_{\lambda_1\lambda_2\lambda_3}\xi)(\bar{\xi}\Gamma_{\tau_1\tau_2\tau_3}\xi)A_{\lambda_1\lambda_2\lambda_3\tau_1\tau_2\tau_3} \\
&\qquad\qquad \times\frac{1}{2^{14}\times 3^2}h_{\mu\nu}\frac{\partial}{\partial\xi_{\alpha_1}}\frac{\partial}{\partial\xi_{\alpha_2}}\frac{\partial}{\partial\xi_{\beta_1}}\frac{\partial}{\partial\xi_{\beta_2}}(\Gamma_{\mu\rho\sigma}\Gamma_0)_{\alpha_1\alpha_2}(\Gamma_{\nu\rho\sigma}\Gamma_0)_{\beta_1\beta_2}\prod_{i=1}^{16}\xi_i \\
= &\int d^{16}\xi \quad A_{\{\lambda\}\{\tau\}}(\Gamma_0\Gamma_{\lambda_1\lambda_2\lambda_3})_{\gamma_1\gamma_2}(\Gamma_0\Gamma_{\tau_1\tau_2\tau_3})_{\delta_1\delta_2}\xi_{\gamma_1}\xi_{\gamma_2}\xi_{\delta_1}\xi_{\delta_2} \\
&\qquad\qquad \times h_{\mu\nu}\frac{\partial}{\partial\xi_{\alpha_1}}\frac{\partial}{\partial\xi_{\alpha_2}}\frac{\partial}{\partial\xi_{\beta_1}}\frac{\partial}{\partial\xi_{\beta_2}}(\Gamma_{\mu\rho\sigma}\Gamma_0)_{\alpha_1\alpha_2}(\Gamma_{\nu\rho\sigma}\Gamma_0)_{\beta_1\beta_2}\prod_{i=1}^{16}\xi_i \\
= &\ h_{\mu\nu}A_{\mu\rho\sigma,\nu\rho\sigma} \qquad\qquad (\mathrm{H.1})
\end{aligned}
$$

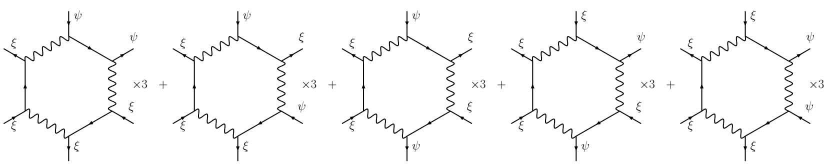
I Other expansions for $1/P$

P^{-9}



$$
= \frac{1}{6}\,\mathrm{tr}\left[\frac{1}{P^2}\bar{\Phi}\Gamma_\mu\frac{1}{P^2}(\Gamma\cdot P)\Gamma_\rho\Phi\frac{1}{P^2}\bar{\Phi}\Gamma_\rho\frac{1}{P^2}(\Gamma\cdot P)\Gamma_\nu\Phi\frac{1}{P^2}\bar{\Phi}\Gamma_\nu\frac{1}{P^2}(\Gamma\cdot P)\Gamma_\nu\Phi\right] \qquad (\mathrm{I.1})
$$

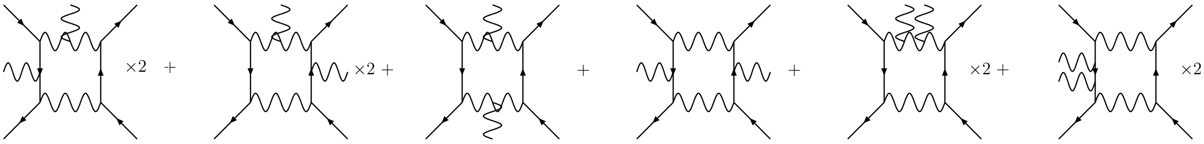
We expand $\Phi \to \psi - \xi$. Thus



$$
\begin{aligned}
= \ &\frac{1}{2}\,\mathrm{tr}\Bigg[(\bar{\xi}\Gamma_{\mu\alpha\rho}\xi)(\bar{\xi}\Gamma_{\rho\beta\nu}\xi)\left(\frac{1}{P^2}\right)^2 P_\alpha\left(\frac{1}{P^2}\right)^2 P_\beta\frac{1}{P^2}\bar{\psi}\frac{1}{P^2}P_\lambda\Gamma_\nu\Gamma_\gamma\Gamma_\mu\psi \\
&+(\bar{\xi}\Gamma_{\mu\alpha\rho}\xi)\left(\frac{1}{P^2}\right)^2 P_\alpha\left(\frac{1}{P^2}\right)^2 P_\beta(\bar{\xi}\Gamma_\rho\Gamma_\beta\Gamma_\nu\psi)\left(\frac{1}{P^2}\right)^2 P_\gamma(\bar{\xi}\Gamma_\nu\Gamma_\gamma\Gamma_\mu\psi) \\
&+(\bar{\xi}\Gamma_{\mu\alpha\rho}\xi)\left(\frac{1}{P^2}\right)^2 P_\alpha\frac{1}{P^2}(\bar{\psi}\Gamma_\rho\Gamma_\beta\Gamma_\nu\xi)\frac{1}{P^2}P_\beta\left(\frac{1}{P^2}\right)^2 P_\gamma(\bar{\xi}\Gamma_\nu\Gamma_\gamma\Gamma_\mu\psi) \\
&+(\bar{\xi}\Gamma_{\mu\alpha\rho}\xi)\left(\frac{1}{P^2}\right)^2 P_\alpha\frac{1}{P^2}(\bar{\psi}\Gamma_\rho\Gamma_\beta\Gamma_\nu\xi)\frac{1}{P^2}P_\beta\frac{1}{P^2}(\bar{\psi}\Gamma_\nu\Gamma_\gamma\Gamma_\mu\xi)\frac{1}{P^2}P_\gamma \\
&+(\bar{\xi}\Gamma_{\mu\alpha\rho}\xi)\left(\frac{1}{P^2}\right)^2 P_\alpha\left(\frac{1}{P^2}\right)^2 P_\beta(\bar{\xi}\Gamma_\rho\Gamma_\beta\Gamma_\nu\psi)\frac{1}{P^2}(\bar{\psi}\Gamma_\nu\Gamma_\gamma\Gamma_\mu\xi)\frac{1}{P^2}P_\gamma\Bigg] \qquad (\mathrm{I.2})
\end{aligned}
$$

$$
\begin{aligned}
\to\ & \frac{1}{2}(\bar\xi\Gamma_{\mu\alpha\rho}\xi)(\bar\xi\Gamma_{\rho\beta\nu}\xi)\frac{1}{d^{12}} \\
& \times \mathrm{tr}\left\{d_\beta d_\gamma A_\alpha(\bar\psi\Gamma_\nu\Gamma_\gamma\Gamma_\mu\psi) + d_\alpha d_\gamma A_\beta(\bar\psi\Gamma_\nu\Gamma_\gamma\Gamma_\mu\psi) - d_\alpha d_\beta A_\gamma(\bar\psi\Gamma_\nu\Gamma_\gamma\Gamma_\mu\psi)\right\} \\
& -4(\bar\xi\Gamma_{\mu\alpha\rho}\xi)(\bar\xi\Gamma_{\rho\beta\nu}\xi)\frac{d_\alpha d_\beta d_\gamma d_\delta}{d^{14}}\,\mathrm{tr}\,A_\delta\bar\psi\Gamma_\nu\Gamma_\gamma\Gamma_\mu\psi \\
+\ & \frac{1}{2}(\bar\xi\Gamma_{\mu\alpha\rho}\xi)\frac{1}{d^{12}} \\
& \times \mathrm{tr}\Big\{d_\beta d_\gamma A_\alpha(\bar\xi\Gamma_\rho\Gamma_\beta\Gamma_\nu\psi)(\bar\xi\Gamma_\nu\Gamma_\gamma\Gamma_\mu\psi) \\
& \qquad +d_\alpha d_\gamma A_\beta(\bar\xi\Gamma_\rho\Gamma_\beta\Gamma_\nu\psi)(\bar\xi\Gamma_\nu\Gamma_\gamma\Gamma_\mu\psi) + d_\alpha d_\beta(\bar\xi\Gamma_\rho\Gamma_\beta\Gamma_\nu\psi)A_\gamma(\bar\xi\Gamma_\nu\Gamma_\gamma\Gamma_\mu\psi)\Big\} \\
+\ & \frac{1}{2}(\bar\xi\Gamma_{\mu\alpha\rho}\xi)\frac{d_\alpha d_\beta d_\gamma d_\delta}{d^{14}}\Big\{-8\,\mathrm{tr}\,A_\delta(\bar\xi\Gamma_\rho\Gamma_\beta\Gamma_\nu\psi)(\bar\xi\Gamma_\nu\Gamma_\gamma\Gamma_\mu\psi) \\
& \qquad -4\,\mathrm{tr}\,(\bar\xi\Gamma_\rho\Gamma_\beta\Gamma_\nu\psi)A_\delta(\bar\xi\Gamma_\nu\Gamma_\gamma\Gamma_\mu\psi)\Big\} \\
+\ & \frac{1}{2}(\bar\xi\Gamma_{\mu\alpha\rho}\xi)\Big\{\frac{d_\beta d_\gamma}{d^{12}}\,\mathrm{tr}\,A_\alpha(\bar\psi\Gamma_\rho\Gamma_\beta\Gamma_\nu\xi)(\bar\xi\Gamma_\nu\Gamma_\gamma\Gamma_\mu\psi) + \frac{d_\alpha d_\gamma}{d^{12}}\,\mathrm{tr}\,(\bar\psi\Gamma_\rho\Gamma_\beta\Gamma_\nu\xi)A_\beta(\bar\xi\Gamma_\nu\Gamma_\gamma\Gamma_\mu\psi) \\
& \qquad +\frac{d_\alpha d_\beta}{d^{12}}\,\mathrm{tr}\,(\bar\psi\Gamma_\rho\Gamma_\beta\Gamma_\nu\xi)A_\gamma(\bar\xi\Gamma_\nu\Gamma_\gamma\Gamma_\mu\psi) \\
& +\frac{d_\alpha d_\beta d_\gamma d_\delta}{d^{14}}\left[-6\,\mathrm{tr}\,A_\delta(\bar\psi\Gamma_\rho\Gamma_\beta\Gamma_\nu\xi)(\bar\xi\Gamma_\nu\Gamma_\gamma\Gamma_\mu\psi) - 6\,\mathrm{tr}\,(\bar\psi\Gamma_\rho\Gamma_\beta\Gamma_\nu\xi)A_\delta(\bar\xi\Gamma_\nu\Gamma_\gamma\Gamma_\mu\psi)\right]\Big\} \\
+\ & \frac{1}{2}(\bar\xi\Gamma_{\mu\alpha\rho}\xi)\Big\{\frac{d_\beta d_\gamma}{d^{12}}\,\mathrm{tr}\,A_\alpha(\bar\psi\Gamma_\rho\Gamma_\beta\Gamma_\nu\xi)(\bar\psi\Gamma_\nu\Gamma_\gamma\Gamma_\mu\xi) + \frac{d_\alpha d_\gamma}{d^{12}}\,\mathrm{tr}\,(\bar\psi\Gamma_\rho\Gamma_\beta\Gamma_\nu\xi)A_\beta(\bar\psi\Gamma_\nu\Gamma_\gamma\Gamma_\mu\xi) \\
& \qquad +\frac{d_\alpha d_\beta}{d^{12}}\,\mathrm{tr}\,(\bar\psi\Gamma_\rho\Gamma_\beta\Gamma_\nu\xi)(\bar\psi\Gamma_\nu\Gamma_\gamma\Gamma_\mu\xi)A_\gamma \\
& -\frac{d_\alpha d_\beta d_\gamma d_\delta}{d^{14}}\left[8\,\mathrm{tr}\,A_\delta(\bar\psi\Gamma_\rho\Gamma_\beta\Gamma_\nu\xi)(\bar\psi\Gamma_\nu\Gamma_\gamma\Gamma_\mu\xi) + 4\,\mathrm{tr}\,(\bar\psi\Gamma_\rho\Gamma_\beta\Gamma_\nu\xi)A_\delta(\bar\psi\Gamma_\nu\Gamma_\gamma\Gamma_\mu\xi)\right]\Big\} \\
+\ & \frac{1}{2}(\bar\xi\Gamma_{\mu\alpha\rho}\xi)\Big\{\frac{d_\beta d_\gamma}{d^{12}}\,\mathrm{tr}\,A_\alpha(\bar\xi\Gamma_\rho\Gamma_\beta\Gamma_\nu\psi)(\bar\psi\Gamma_\nu\Gamma_\gamma\Gamma_\mu\xi) + \frac{d_\alpha d_\gamma}{d^{12}}\,\mathrm{tr}\,A_\beta(\bar\xi\Gamma_\rho\Gamma_\beta\Gamma_\nu\psi)(\bar\psi\Gamma_\nu\Gamma_\gamma\Gamma_\mu\xi) \\
& \qquad +\frac{d_\alpha d_\beta}{d^{12}}\,\mathrm{tr}\,(\bar\xi\Gamma_\rho\Gamma_\beta\Gamma_\nu\psi)(\bar\psi\Gamma_\nu\Gamma_\gamma\Gamma_\mu\xi)A_\gamma \\
& -\frac{d_\alpha d_\beta d_\gamma d_\delta}{d^{14}}\left[10\,\mathrm{tr}\,A_\delta(\bar\xi\Gamma_\rho\Gamma_\beta\Gamma_\nu\psi)(\bar\psi\Gamma_\nu\Gamma_\gamma\Gamma_\mu\xi) - 2\frac{d_\beta d_\gamma}{d^{12}}\,\mathrm{tr}\,(\bar\xi\Gamma_\rho\Gamma_\beta\Gamma_\nu\psi)A_\delta(\bar\psi\Gamma_\nu\Gamma_\gamma\Gamma_\mu\xi)\right]\Big\}
\end{aligned}
\tag{I.3}
$$

P^{-10}



$$
\begin{aligned}
= \quad & -\frac{1}{2}\operatorname{tr}\left[\frac{2}{P^2}\mathcal{F}_{\mu\lambda}\frac{1}{P^2}\bar{\Phi}\Gamma_\lambda\frac{1}{2P^2}\Gamma_{\alpha\beta}\mathcal{F}_{\alpha\beta}\frac{1}{P^2}\Gamma_\gamma P_\gamma\Gamma_\nu\Phi\frac{1}{P^2}\bar{\Phi}\Gamma_\nu\Gamma_\sigma P_\sigma\Gamma_\mu\Phi\right] \\
& -\frac{1}{2}\operatorname{tr}\left[\frac{2}{P^2}\mathcal{F}_{\mu\lambda}\frac{1}{P^2}\bar{\Phi}\Gamma_\lambda\frac{1}{P^2}\Gamma_\gamma P_\gamma\Gamma_\nu\Phi\frac{1}{P^2}\bar{\Phi}\Gamma_\nu\frac{1}{2P^2}\Gamma_{\eta\xi}\mathcal{F}_{\eta\xi}\frac{1}{P^2}\Gamma_\sigma P_\sigma\Gamma_\mu\Phi\right] \\
& -\frac{1}{4}\operatorname{tr}\left[\frac{2}{P^2}\mathcal{F}_{\mu\lambda}\frac{1}{P^2}\bar{\Phi}\Gamma_\lambda\frac{1}{P^2}\Gamma_\gamma P_\gamma\Gamma_\nu\Phi\frac{2}{P^2}\mathcal{F}_{\nu\rho}\frac{1}{P^2}\bar{\Phi}\Gamma_\rho\frac{1}{P^2}\Gamma_\sigma P_\sigma\Gamma_\mu\Phi\right] \\
& -\frac{1}{4}\operatorname{tr}\left[\frac{1}{P^2}\bar{\Phi}\Gamma_\mu\frac{1}{2P^2}\Gamma_{\alpha\beta}\mathcal{F}_{\alpha\beta}\frac{1}{P^2}\Gamma_\gamma P_\gamma\Gamma_\nu\Phi\frac{1}{P^2}\bar{\Phi}\Gamma_\nu\frac{1}{2P^2}\Gamma_{\eta\xi}\mathcal{F}_{\eta\xi}\frac{1}{P^2}\Gamma_\sigma P_\sigma\Gamma_\mu\Phi\right] \\
& -\frac{1}{2}\operatorname{tr}\left[\frac{2}{P^2}\mathcal{F}_{\mu\zeta}\frac{2}{P^2}\mathcal{F}_{\zeta\lambda}\frac{1}{P^2}\bar{\Phi}\Gamma_\lambda\frac{1}{P^2}\Gamma_\gamma P_\gamma\Gamma_\nu\Phi\frac{1}{P^2}\bar{\Phi}\Gamma_\nu\frac{1}{P^2}\Gamma_\sigma P_\sigma\Gamma_\mu\Phi\right] \\
& -\frac{1}{2}\operatorname{tr}\left[\frac{1}{P^2}\bar{\Phi}\Gamma_\mu\frac{1}{2P^2}\Gamma_{\alpha\beta}\mathcal{F}_{\alpha\beta}\frac{1}{2P^2}\Gamma_{\eta\xi}\mathcal{F}_{\eta\xi}\frac{1}{P^2}\Gamma_\gamma P_\gamma\Gamma_\nu\Phi\frac{1}{P^2}\bar{\Phi}\Gamma_\nu\frac{1}{P^2}\Gamma_\sigma P_\sigma\Gamma_\mu\Phi\right] \\
\to \quad & -\frac{1}{2}(\bar{\xi}\Gamma_\lambda\Gamma_{\alpha\beta}\Gamma_\rho\Gamma_\nu\xi)(\bar{\xi}\Gamma_{\nu\sigma\mu}\xi)\operatorname{tr}\left[\frac{1}{P^2}\mathcal{F}_{\mu\lambda}\left(\frac{1}{P^2}\right)^2\mathcal{F}_{\alpha\beta}\frac{1}{P^2}P_\rho\left(\frac{1}{P^2}\right)^2 P_\sigma\right] \\
& -\frac{1}{2}(\bar{\xi}\Gamma_{\lambda\rho\nu}\xi)(\bar{\xi}\Gamma_\nu\Gamma_{\alpha\beta}\Gamma_\sigma\Gamma_\mu\xi)\operatorname{tr}\left[\frac{1}{P^2}\mathcal{F}_{\mu\lambda}\left(\frac{1}{P^2}\right)^2 P_\rho\left(\frac{1}{P^2}\right)^2\mathcal{F}_{\alpha\beta}\frac{1}{P^2}P_\sigma\right] \\
& -(\bar{\xi}\Gamma_{\lambda\alpha\nu}\xi)(\bar{\xi}\Gamma_{\rho\beta\mu}\xi)\operatorname{tr}\left[\frac{1}{P^2}\mathcal{F}_{\mu\lambda}\left(\frac{1}{P^2}\right)^2 P_\alpha\frac{1}{P^2}\mathcal{F}_{\nu\rho}\left(\frac{1}{P^2}\right)^2 P_\beta\right] \\
& -\frac{1}{16}(\bar{\xi}\Gamma_\mu\Gamma_{\alpha\beta}\Gamma_\lambda\Gamma_\nu\xi)(\bar{\xi}\Gamma_\nu\Gamma_{\gamma\delta}\Gamma_\rho\Gamma_\mu\xi)\operatorname{tr}\left[\left(\frac{1}{P^2}\right)^2\mathcal{F}_{\alpha\beta}\frac{1}{P^2}P_\lambda\left(\frac{1}{P^2}\right)^2\mathcal{F}_{\gamma\delta}\frac{1}{P^2}P_\rho\right] \\
& -2(\bar{\xi}\Gamma_{\lambda\gamma\nu}\xi)(\bar{\xi}\Gamma_{\nu\sigma\mu}\xi)\operatorname{tr}\left[\frac{1}{P^2}\mathcal{F}_{\mu\zeta}\frac{1}{P^2}\mathcal{F}_{\zeta\lambda}\left(\frac{1}{P^2}\right)^2 P_\gamma\left(\frac{1}{P^2}\right)^2 P_\sigma\right] \\
& -\frac{1}{8}(\bar{\xi}\Gamma_\mu\Gamma_{\alpha\beta}\Gamma_{\eta\xi}\Gamma_\gamma\Gamma_\nu\xi)(\bar{\xi}\Gamma_\nu\Gamma_\sigma\Gamma_\mu\xi)\operatorname{tr}\left[\left(\frac{1}{P^2}\right)^2\mathcal{F}_{\alpha\beta}\frac{1}{P^2}\mathcal{F}_{\eta\xi}\frac{1}{P^2}P_\gamma\left(\frac{1}{P^2}\right)^2 P_\sigma\right] \\
\to \quad & \frac{7}{2d^{10}}(\bar{\xi}\Gamma_{\alpha\lambda\nu}\xi)(\bar{\xi}\Gamma_{\beta\mu\rho}\xi)\frac{d_\alpha d_\beta}{d^2}\operatorname{tr}F_{\mu\lambda}F_{\nu\rho}-\frac{3}{2d^{10}}(\bar{\xi}\Gamma_{\lambda\rho\nu}\xi)(\bar{\xi}\Gamma_{\nu\alpha\beta}\xi)\frac{d_\rho d_\mu}{d^2}\operatorname{tr}F_{\mu\lambda}F_{\alpha\beta} \\
& +\frac{1}{d^{10}}(\bar{\xi}\Gamma_{\alpha\beta\rho}\xi)(\bar{\xi}\Gamma_{\mu\nu\sigma}\xi)\frac{d_\rho d_\sigma}{d^2}\operatorname{tr}F_{\mu\nu}F_{\alpha\beta}-\frac{9}{2d^{10}}(\bar{\xi}\Gamma_{\lambda\rho\nu}\xi)(\bar{\xi}\Gamma_{\nu\beta\sigma}\xi)\frac{d_\rho d_\sigma}{d^2}\operatorname{tr}F_{\alpha\lambda}F_{\alpha\beta} \\
& -\frac{1}{8d^{10}}(\bar{\xi}\Gamma_{\mu\alpha\beta}\xi)(\bar{\xi}\Gamma_{\mu\gamma\delta}\xi)\operatorname{tr}F_{\alpha\beta}F_{\gamma\delta}+\frac{1}{2d^{10}}(\bar{\xi}\Gamma_{\gamma\delta\lambda}\xi)(\bar{\xi}\Gamma_{\mu\nu\rho}\xi)\frac{d_\lambda d_\rho}{d^2}\operatorname{tr}F_{\mu\nu}F_{\gamma\delta} \\
& -\frac{1}{d^{10}}(\bar{\xi}\Gamma_{\beta\delta\lambda}\xi)(\bar{\xi}\Gamma_{\mu\nu\rho}\xi)\frac{d_\lambda d_\rho}{d^2}\operatorname{tr}F_{\mu\beta}F_{\nu\delta}+\frac{i\epsilon_{\mu\nu\lambda\alpha\beta\gamma\delta abc}}{48d^{10}}(\bar{\xi}\Gamma_{abc}\xi)(\bar{\xi}\Gamma_{\mu\nu\rho}\xi)\frac{d_\lambda d_\rho}{d^2}\operatorname{tr}F_{\alpha\beta}F_{\gamma\delta}
\end{aligned}
\tag{I.4}
$$

Result for d^{-10} terms

Bosonic Part(from the terms of P^{-6}, P^{-8}, P^{-10})

$$
\begin{aligned}
\mathcal{W}_b \Big|_{d^{-10}} &= \frac{1}{d^{10}}(\bar{\xi}\Gamma_{\lambda\rho\nu}\xi)(\bar{\xi}\Gamma_{\nu\sigma\mu}\xi)\,\mathrm{tr}\left[\frac{4d_\rho d_\tau}{d^2}F_{\mu\lambda}F_{\sigma\tau} + \frac{1}{2}F_{\mu\lambda}F_{\rho\sigma}\right] \\
&+ \frac{7}{2d^{10}}(\bar{\xi}\Gamma_{\alpha\lambda\nu}\xi)(\bar{\xi}\Gamma_{\beta\mu\rho}\xi)\frac{d_\alpha d_\beta}{d^2}\,\mathrm{tr}\,F_{\mu\lambda}F_{\nu\rho} - \frac{3}{2d^{10}}(\bar{\xi}\Gamma_{\lambda\rho\nu}\xi)(\bar{\xi}\Gamma_{\nu\alpha\beta}\xi)\frac{d_\rho d_\mu}{d^2}\,\mathrm{tr}\,F_{\mu\lambda}F_{\alpha\beta} \\
&+ \frac{1}{d^{10}}(\bar{\xi}\Gamma_{\alpha\beta\rho}\xi)(\bar{\xi}\Gamma_{\mu\nu\sigma}\xi)\frac{d_\rho d_\sigma}{d^2}\,\mathrm{tr}\,F_{\mu\nu}F_{\alpha\beta} - \frac{9}{2d^{10}}(\bar{\xi}\Gamma_{\lambda\rho\nu}\xi)(\bar{\xi}\Gamma_{\nu\beta\sigma}\xi)\frac{d_\rho d_\sigma}{d^2}\,\mathrm{tr}\,F_{\alpha\lambda}F_{\alpha\beta} \\
&- \frac{1}{8d^{10}}(\bar{\xi}\Gamma_{\mu\alpha\beta}\xi)(\bar{\xi}\Gamma_{\mu\gamma\delta}\xi)\,\mathrm{tr}\,F_{\alpha\beta}F_{\gamma\delta} + \frac{1}{2d^{10}}(\bar{\xi}\Gamma_{\gamma\delta\lambda}\xi)(\bar{\xi}\Gamma_{\mu\nu\rho}\xi)\frac{d_\lambda d_\rho}{d^2}\,\mathrm{tr}\,F_{\mu\nu}F_{\gamma\delta} \\
&- \frac{1}{d^{10}}(\bar{\xi}\Gamma_{\beta\delta\lambda}\xi)(\bar{\xi}\Gamma_{\mu\nu\rho}\xi)\frac{d_\lambda d_\rho}{d^2}\,\mathrm{tr}\,F_{\mu\beta}F_{\nu\delta} \\
&+ \frac{i}{48d^{10}}\epsilon_{\mu\nu\lambda\alpha\beta\gamma\delta abc}(\bar{\xi}\Gamma_{abc}\xi)(\bar{\xi}\Gamma_{\mu\nu\rho}\xi)\frac{d_\lambda d_\rho}{d^2}\,\mathrm{tr}\,F_{\alpha\beta}F_{\gamma\delta}
\end{aligned} \tag{I.5}
$$

Fermionic Part(from the terms of P^{-9})

We use the Fierz identity (C.1) .

$$
\begin{aligned}
\mathcal{W}_f \Big|_{d^{-10}} &= \frac{1}{2}(\bar{\xi}\Gamma_{\mu\alpha\rho}\xi)(\bar{\xi}\Gamma_{\rho\beta\nu}\xi)\frac{1}{d^{12}} \\
&\quad \times\left\{d_\beta d_\gamma\,\mathrm{tr}\,A_\alpha\bar{\psi}\Gamma_\nu\Gamma_\gamma\Gamma_\mu\psi + d_\alpha d_\gamma\,\mathrm{tr}\,A_\beta\bar{\psi}\Gamma_\nu\Gamma_\gamma\Gamma_\mu\psi + d_\alpha d_\beta\,\mathrm{tr}\,\bar{\psi}A_\gamma\Gamma_\nu\Gamma_\gamma\Gamma_\mu\psi\right\} \\
&\quad -4(\bar{\xi}\Gamma_{\mu\alpha\rho}\xi)(\bar{\xi}\Gamma_{\rho\beta\nu}\xi)\frac{d_\alpha d_\beta d_\gamma d_\delta}{d^{14}}(\,\mathrm{tr}\,A_\delta\bar{\psi}\Gamma_\nu\Gamma_\gamma\Gamma_\mu\psi) \\
&+ \frac{1}{192}(\bar{\xi}\Gamma_{\mu\alpha\rho}\xi)(\bar{\xi}\Gamma_{abc}\xi) \\
&\quad \times\left\{\frac{1}{d^{12}}\left[-d_\beta d_\gamma\,\mathrm{tr}\,\bar{\psi}\Gamma_\mu\Gamma_\gamma\Gamma_\nu\Gamma_{abc}\Gamma_\rho\Gamma_\beta\Gamma_\nu A_\alpha\psi - d_\alpha d_\gamma\,\mathrm{tr}\,\bar{\psi}\Gamma_\mu\Gamma_\gamma\Gamma_\nu\Gamma_{abc}\Gamma_\rho\Gamma_\beta\Gamma_\nu A_\beta\psi\right.\right. \\
&\quad\quad \left. -d_\alpha d_\beta\,\mathrm{tr}\,\bar{\psi}\Gamma_\mu\Gamma_\gamma\Gamma_\nu\Gamma_{abc}\Gamma_\rho\Gamma_\beta\Gamma_\nu\psi A_\gamma\right] \\
&\quad \left. +\frac{d_\alpha d_\beta d_\gamma d_\delta}{d^{14}}\left[8\,\mathrm{tr}\,\bar{\psi}\Gamma_\mu\Gamma_\gamma\Gamma_\nu\Gamma_{abc}\Gamma_\rho\Gamma_\beta\Gamma_\nu A_\delta\psi + 4\,\mathrm{tr}\,\bar{\psi}\Gamma_\mu\Gamma_\gamma\Gamma_\nu\Gamma_{abc}\Gamma_\rho\Gamma_\beta\Gamma_\nu\psi A_\delta\right]\right\} \\
&+ \frac{1}{192}(\bar{\xi}\Gamma_{\mu\alpha\rho}\xi)(\bar{\xi}\Gamma_{abc}\xi) \\
&\quad \times\left\{\frac{d_\beta d_\gamma}{d^{12}}\,\mathrm{tr}\,\bar{\psi}\Gamma_\rho\Gamma_\beta\Gamma_\nu\Gamma_{abc}\Gamma_\nu\Gamma_\gamma\Gamma_\mu\psi A_\alpha + \frac{d_\alpha d_\gamma}{d^{12}}\,\mathrm{tr}\,\bar{\psi}\Gamma_\rho\Gamma_\beta\Gamma_\nu\Gamma_{abc}\Gamma_\nu\Gamma_\gamma\Gamma_\mu A_\beta\psi\right. \\
&\quad\quad +\frac{d_\alpha d_\beta}{d^{12}}\,\mathrm{tr}\,\bar{\psi}\Gamma_\rho\Gamma_\beta\Gamma_\nu\Gamma_{abc}\Gamma_\nu\Gamma_\gamma\Gamma_\mu A_\gamma\psi \\
&\quad\quad \left. +\frac{d_\alpha d_\beta d_\gamma d_\delta}{d^{14}}\left[-6\,\mathrm{tr}\,\bar{\psi}\Gamma_\rho\Gamma_\beta\Gamma_\nu\Gamma_{abc}\Gamma_\nu\Gamma_\gamma\Gamma_\mu\psi A_\delta - 6\,\mathrm{tr}\,\bar{\psi}\Gamma_\rho\Gamma_\beta\Gamma_\nu\Gamma_{abc}\Gamma_\nu\Gamma_\gamma\Gamma_\mu A_\delta\psi\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
+ & \frac{1}{192}(\bar{\xi}\Gamma_{\mu\alpha\rho}\xi)(\bar{\xi}\Gamma_{abc}\xi) \\
& \times\Big\{-\frac{d_\beta d_\gamma}{d^{12}}\,\mathrm{tr}\,\bar{\psi}\Gamma_\nu\Gamma_\gamma\Gamma_\mu\Gamma_{abc}\Gamma_\nu\Gamma_\beta\Gamma_\rho A_\alpha\psi-\frac{d_\alpha d_\gamma}{d^{12}}\,\mathrm{tr}\,\bar{\psi}\Gamma_\nu\Gamma_\gamma\Gamma_\mu\Gamma_{abc}\Gamma_\nu\Gamma_\beta\Gamma_\rho\psi A_\beta \\
& \quad -\frac{d_\alpha d_\beta}{d^{12}}\,\mathrm{tr}\,\bar{\psi}\Gamma_\nu\Gamma_\gamma\Gamma_\mu\Gamma_{abc}\Gamma_\nu\Gamma_\beta\Gamma_\rho A_\gamma\psi \\
& \quad +\frac{d_\alpha d_\beta d_\gamma d_\delta}{d^{14}}\big[8\,\mathrm{tr}\,\bar{\psi}\Gamma_\nu\Gamma_\gamma\Gamma_\mu\Gamma_{abc}\Gamma_\nu\Gamma_\beta\Gamma_\rho A_\delta\psi+4\,\mathrm{tr}\,\bar{\psi}\Gamma_\nu\Gamma_\gamma\Gamma_\mu\Gamma_{abc}\Gamma_\nu\Gamma_\beta\Gamma_\rho\psi A_\delta\big]\Big\} \\
+ & \frac{1}{192}(\bar{\xi}\Gamma_{\mu\alpha\rho}\xi)(\bar{\xi}\Gamma_{abc}\xi) \\
& \times\Big\{\frac{d_\beta d_\gamma}{d^{12}}\,\mathrm{tr}\,\bar{\psi}\Gamma_\nu\Gamma_\gamma\Gamma_\mu\Gamma_{abc}\Gamma_\rho\Gamma_\beta\Gamma_\nu A_\alpha\psi+\frac{d_\alpha d_\gamma}{d^{12}}\,\mathrm{tr}\,\bar{\psi}\Gamma_\nu\Gamma_\gamma\Gamma_\mu\Gamma_{abc}\Gamma_\rho\Gamma_\beta\Gamma_\nu A_\beta\psi \\
& \quad +\frac{d_\alpha d_\beta}{d^{12}}\,\mathrm{tr}\,\bar{\psi}\Gamma_\nu\Gamma_\gamma\Gamma_\mu\Gamma_{abc}\Gamma_\rho\Gamma_\beta\Gamma_\nu A_\gamma\psi \\
& \quad +\frac{d_\alpha d_\beta d_\gamma d_\delta}{d^{14}}\big[-10\,\mathrm{tr}\,\bar{\psi}\Gamma_\nu\Gamma_\gamma\Gamma_\mu\Gamma_{abc}\Gamma_\rho\Gamma_\beta\Gamma_\nu A_\delta\psi-2\,\mathrm{tr}\,\bar{\psi}\Gamma_\nu\Gamma_\gamma\Gamma_\mu\Gamma_{abc}\Gamma_\rho\Gamma_\beta\Gamma_\nu\psi A_\delta\big]\Big\}
\end{aligned} \tag{I.6}
$$

Result for d^{-12} terms

$$
\begin{aligned}
\mathcal{F}_c\Big|_{d^{-12}} = & \frac{1}{6}(\bar{\xi}\Gamma_{\mu a\nu}\xi)(\bar{\xi}\Gamma_{\nu b\lambda}\xi)(\bar{\xi}\Gamma_{\lambda c\mu}\xi)\frac{1}{d^{12}}\,\mathrm{tr}\,\Big\{-12\frac{d_a d_\alpha}{d^2}A_\alpha A_b A_c+A_a A_b A_c\Big\} \\
& -(\bar{\xi}\Gamma_{\lambda a\nu}\xi)(\bar{\xi}\Gamma_{\nu b\sigma}\xi)(\bar{\xi}\Gamma_{\sigma c\mu}\xi)\,\mathrm{tr}\,\Big\{\frac{1}{d^{16}}(-14)d_a d_b d_c d_\alpha F_{\mu\lambda}A_\alpha \\
& \qquad +\frac{1}{d^{14}}F_{\mu\lambda}(d_a d_b A_c+d_a d_c A_b+d_b d_c A_a)\Big\} \\
& -\frac{1}{4}(\bar{\xi}\Gamma_\mu\Gamma_{\alpha\beta}\Gamma_a\Gamma_\nu\xi)(\bar{\xi}\Gamma_{\nu b\lambda}\xi)(\bar{\xi}\Gamma_{\lambda c\mu}\xi)\,\mathrm{tr}\,\Big\{-14\frac{1}{d^{14}}d_a d_b d_c d_\delta F_{\alpha\beta}A_\delta \\
& \qquad +\frac{1}{d^{14}}F_{\alpha\beta}(d_a d_b A_c+d_a d_c A_b+d_b d_c A_a)\Big\}
\end{aligned} \tag{I.7}
$$

J Translation of A_μ

Choosing the following wave function with a momentum k,

$$
e^{ik\cdot d}f(\xi), \tag{J.1}
$$

the effective action becomes

$$I(A,\psi) \equiv \int ddd\xi e^{-S_{\rm eff}(d+A,\psi,\xi)} e^{ik\cdot d} f(\xi). \tag{J.2}$$

Then $I(A,\psi)e^{ik\cdot A}$ is invariant under the translation, $A_\mu \to A_\mu + c_\mu 1$,

$$\begin{aligned} I(A+c,\psi)e^{ik\cdot(A+c)} &= \int ddd\xi e^{-S_{\rm eff}(d+A+c,\psi,\xi)} e^{ik\cdot(d+A+c)} f(\xi) \\ &= \int ddd\xi e^{-S_{\rm eff}(d+A,\psi,\xi)} e^{ik\cdot(d+A)} f(\xi) \\ &= I(A,\psi)e^{ik\cdot A}. \end{aligned} \tag{J.3}$$

Similarly, $e^{ik\cdot A}I(A,\psi) = e^{ik\cdot A}\left(I(A,\psi)e^{ik\cdot A}\right)e^{-ik\cdot A}$ also has the invariance,

$$\begin{aligned} e^{ik\cdot(A+c)}I(A+c,\psi) &= e^{ik\cdot(A+c)}\left(I(A+c,\psi)e^{ik\cdot(A+c)}\right)e^{-ik\cdot(A+c)} \\ &= e^{ik\cdot A}\left(I(A,\psi)e^{ik\cdot A}\right)e^{-ik\cdot A} \\ &= e^{ik\cdot A}I(A,\psi). \end{aligned} \tag{J.4}$$

$I(A,\psi)$ can be expanded as

$$I(A,\psi)e^{ik\cdot A} = \sum_{m,n} C_{mn}(d) A^m \psi^n. \tag{J.5}$$

For fixed m and n, the leading term in $\mathcal{O}(1/d)$ on the right hand side is inavariant under the translation.

For example, taking a wave function $h_{\mu\nu}(k)e^{ik\cdot d}\xi^{12}$, it is expected that the following gravitational term will appear in $I(A,\psi)$,

$$h_{\mu\nu}(k){\rm Tr}e^{-ik\cdot A}F_{\mu\lambda}F_{\nu\lambda}. \tag{J.6}$$

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