Dynamics of Quantum Field Theories on Non-commutative Space-Time

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Abstract

There are some peculiar phenomena in non-commutative field theories. One is the planer limit that nonplaner diagrams arising from non-commutativity of space-time disappear in the maximal non-commmutativity. We send the parameter θ which presents the non-commutativity to ∞. The other is the UV/IR mixing that ultraviolet divergences in the commutative case change into infrared divergences due to non-commutative parameter θ. Then, we study whether nonplaner diagrams disappear, when we fix θ and take the external momentum $p \to \infty$ limit instead. We perform explicit two loop calculation to confirm the above limit by using perturbation theory. And,we calculate the effective action of non-commutative gauge theories to see infrared singular terms due to UV/IR mixing. We discuss the meaning of the terms from the point of view of instability in the system.

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Contents

§1. Introduction

Non-commutative geometry[1] has been studied for quite a long time as a simple modification of our notion of space-time at small distances possibly due to effects of quantum gravity.[2] Ref. 1) gave us an example of a Lorentz invariant discrete space-time or quantized space-time. And Ref. 2) proposed uncertainty relations for the different coordinates of space-time events, motivated by Heisenberg's principle and Einstein's theory of classical gravity. Non-commutative geometry has recently attracted much attention, since Yang-Mills theories on a non-commutative geometry have been shown to appear as a low energy limit of string theories with some background tensor field on D-brane.[3] When we think the quantum gravity, it is possible to treat more general geometry than usual Riemannian geometry. Since the non-commutative geometry actually is led from the superstring theory which describes the quantum gravity, it is important to study the field theories on the non-commutative geometry. At the classical level, introducing non-commutativity to the space-time coordinates modifies the ultraviolet dynamics of field theories, but not the infrared properties. This is not the case at the quantum level, however, due to the so-called UV/IR mixing effect.[4] This effect causes various peculiar long-distance phenomena such as spontaneous breaking of translational invariance. In the scalar field theory, this phenomenon has been predicted in Ref. 5) and confirmed by Monte Carlo simulations in Refs. 6)–8). An analogous phenomenon is also predicted in gauge theories.[9] The relation between the $\theta \to \infty$ limit of noncommutative field theories and the large N field theories are reconsidered in Ref. 10).

In this paper, first we focus rather on ultraviolet properties of non-commutative field theories. In perturbative expansion of field theories on a non-commutative geometry, planar diagrams dominate when the non-commutativity parameter θ goes to infinity.[4] This may be regarded as a manifestation of the *nonperturbative* relation between the $\theta \to \infty$ limit of non-commutative field theories and the large N matrix field theories,[10] which is based on the lattice formulation of non-commutative field theories[11] and the Eguchi-Kawai equivalence.[12),13] (Note, however, that this relation does not hold when the translational invariance is spontaneously broken.) We discuss whether the "planar dominance" occurs also in the case where θ is finite, but the external momentum goes to infinity instead. While this holds trivially at the one-loop level,[4] it is not obvious at the two-loop level in particular in the presence of UV divergences. We perform explicit two-loop calculations in the six-dimensional ϕ^3 theory, and confirm that nonplanar diagrams after renormalization do vanish in the above limit. We will consider the massive case specifically since in the massless case the equivalence of the infinite momentum limit and the $\theta \to \infty$ limit follows from dimensional arguments.

Some comments on related works are in order. In Ref. 14) correlation functions of Wilson loops in non-commutative gauge theories were studied, and it was stated that planar diagrams dominate when the external momenta become large. However, this statement was based on a regularized theory, and the issue of removing the regularization has not been discussed. In Ref. 15) Monte Carlo simulation of 2d non-commutative gauge theory was performed and the existence of a sensible continuum limit was confirmed. nonperturbative renormalizability There the expectation value of the Wilson loop agrees with the result of large N gauge theory at small area, which implies the planar dominance in the ultraviolet regime. The aim of the present work is to confirm such a statement by explicit diagrammatic calculations in a simple model taking account of possible subtleties that arise at the two-loop level. due to planar subdiagrams in nonplanar diagrams. Two-loop calculations in scalar field theories have been performed also in Refs. 16) and Refs. 17) in the case of ϕ^4 and ϕ^3 interactions, respectively, from different motivations. The issue of renormalizability to all orders in perturbation theory has been discussed in Ref. 18).

Secondly we focus on infrared properties of non-commutative field theories. There are infrared singularities due to UV/IR mixing in non-commutative field theories. We adopt $U(1)$ non-commutative gauge theory as the model in which we study the effect of infrared singularities. Because the infrared singularities of $U(1)$ non-commutative gauge theory cause instability of the theory. We calculate the effective action of non-commutative $U(1)$ gauge theory and derive infrared singularities from nonplaner diagrams. We find that the nonplaner effective potential is negative, suggesting tachyonic behavior for the low momentum modes of the gauge field. Then, we discuss a critical point of the phase transition which occurs due to instability of the theory. Regarding the instability of $U(n)$ noncommutative gauge theory, the infrared singularities are calculated in Ref. 9)

The rest of this paper is organized as follows. In section 2.1 we show how to rewrite the non-commutativity in non-commutative space-time for commutative space-time. We derive star-product from the commutator of the non-commutativity. In section 2.2, using the way of section 2.1, we derive ϕ^n non-commutative field theory. In section 2.3 we study non-commutative perturbative dynamics in ϕ^4 non-commutative field theory. We find two peculiar phenomena in non-commutative field theory. One is the planer dominance in maximal non-commutativity. The other is UV/IR mixing. In section 3.1 we set up some notations necessary for the perturbative expansion in non-commutative ϕ^3 theory. In sections 3.2 and 3.3 we investigate nonplanar two-loop diagrams of different types separately and show that the diagrams vanish in the $p^2 \to \infty$ limit. In section 3.4 we discuss the result of diagrammatic calculations. In section 4.1 we calculate the effective action to find infrared singularities in non-commuatative $U(1)$ gauge theory. In section 4.2 we find the critical point where the instability occurs in the theory. Section 5 is devoted to a summary and discussions.

§2. Non-commutative perturbation theory

2.1. *Non-commutativity and ⋆-product*

We discuss the non-commutative field theory by rewriting the events of non-commutative space for commutative space. The nature of non-commutative space is

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \tag{2·1}$$

where $\hat{x}^\mu$ is a non-commutative space coordinate and $\theta^{\mu\nu}$ is an antisymmetric tensor, which characterizes the non-commutativity of the space. Then the function of non-commutative space is defined by

$$\Phi(\hat{x}) \equiv \int \frac{d^dk}{(2\pi)^d}\,\tilde{\phi}(k)\,e^{ik\cdot\hat{x}}, \tag{2·2}$$

where $\tilde{\phi}(k)$ is the function which is obtained by fourier-transforming $\phi(x)$ in commutative space as follows.

$$\tilde{\phi}(k) = \int d^dx\,\phi(x)\,e^{-ik\cdot x}.$$

We want to resolve (2·2) in terms of $\phi(x)$ of commutative space. Multiplying the both hand of (2·2) by $e^{-il\cdot\hat{x}}$ and taking trace, we can obtain as follows.

$$\begin{aligned}
\mathrm{tr}\left[\Phi(\hat{x})\,e^{-il\cdot\hat{x}}\right] &= \int \frac{d^dk}{(2\pi)^d}\,\tilde{\phi(k)}\,e^{\frac{i}{2}k\times l}\,\mathrm{tr}\left[e^{i(k-l)\cdot\hat{x}}\right] \\
&= \int \frac{d^dk}{(2\pi)^d}\,\tilde{\phi(k)}\,e^{\frac{i}{2}k\times l}\,(2\pi)^{\frac{d}{2}}\delta^{(d)}(k-l) \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}}\,\tilde{\phi}(l),
\end{aligned}$$

where we used (2·1) for $e^{ik\cdot\hat{x}}\cdot e^{-il\cdot\hat{x}} = e^{\frac{i}{2}k\times l}\cdot e^{i(k-l)\cdot\hat{x}}$ and $k\times l \equiv k_\mu\theta_{\mu\nu}l_\nu$. Then we can obtain $\phi(x)$ as follows.

$$\phi(x) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int d^dl\,e^{il\cdot x}\,\mathrm{tr}\left[\Phi(\hat{x})\,e^{-il\cdot\hat{x}}\right] \tag{2·3}$$

Next, we want to rewrite the product $\Phi_1(\hat{x})\Phi_2(\hat{x})$ of functions on non-commutaive space for an algebra of the functions $\phi_1(x)$ and $\phi_2(x)$ on commutative space. We take the same procedure in which we derived $\phi(x)$ from $\Phi(x)$. First,

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&= \frac{1}{(2\pi)^{2d}}\int d^dk\,d^dl\,\tilde{\phi}_1(k)\tilde{\phi}_2(l)\,e^{-\frac{i}{2}k\times l}\,e^{i(k+l)\cdot\hat{x}}
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&= \int \frac{d^dk}{(2\pi)^d}\,\tilde{\phi(k)}\,e^{\frac{i}{2}k\times l}\,(2\pi)^{\frac{d}{2}}\delta^{(d)}(k-l) \\
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&= \frac{1}{(2\pi)^{2d}}\int d^dk\,d^dl\,\tilde{\phi}_1(k)\tilde{\phi}_2(l)\,e^{-\frac{i}{2}k\times l}\,e^{i(k+l)\cdot\hat{x}}
\end{aligned}$$

Secondly,

$$
\begin{aligned}
&\frac{1}{(2\pi)^{\frac{d}{2}}}\int d^dp\, e^{ip\cdot x}\,\mathrm{tr}\left[\Phi_1(\hat{x})\Phi_2(\hat{x})e^{-ip\cdot\hat{x}}\right]\\
=&\frac{1}{(2\pi)^{2d}}\int d^dk\, d^dl\, \tilde{\phi}_1(k)\tilde{\phi}_2(l)\, e^{-\frac{i}{2}k\times l}\, e^{i(k+l)\cdot x}\\
=&\phi_1(x)\exp\Big(\frac{i}{2}\overleftarrow{\partial_\mu}\theta_{\mu\nu}\overrightarrow{\partial_\nu}\Big)\phi_2(x)
\end{aligned}
$$

Here we can define a following $\star$ -product which presents the non-commutativity of the product of functions in commutative space.

$$\phi_1(x)\star\phi_2(x)\equiv\phi_1(x)\exp\Big(\frac{i}{2}\overleftarrow{\partial_\mu}\theta_{\mu\nu}\overrightarrow{\partial_\nu}\Big)\phi_2(x) \tag{2·4}$$

There is a following useful nature in the $\star$-product.

$$
\begin{aligned}
\int d^dx\phi_1(x)\star\phi_2(x) &= \int d^dx\int\frac{d^dk}{(2\pi)^d}\frac{d^dl}{(2\pi)^d}\,\tilde{\phi}_1(k)\tilde{\phi}_2(l)\, e^{ik\cdot x}\star e^{il\cdot x}\\
&=\int\frac{d^dk}{(2\pi)^d}\frac{d^dl}{(2\pi)^d}\, e^{-\frac{i}{2}k\times l}\,\tilde{\phi}_1(k)\tilde{\phi}_2(l)\,(2\pi)^d\delta^{(d)}(k+l)\\
&=\int\frac{d^dk}{(2\pi)^d}\,\tilde{\phi}_1(k)\tilde{\phi}_2(-k)\\
&=\int d^dx\,\phi_1(x)\,\phi_2(x)
\end{aligned}
\tag{2·5}
$$

Using this nature , we can obtain a following equation immediately.

$$\int d^dx\,\phi_1\star\phi_2\star\phi_3=\int d^dx\,\phi_2\star\phi_3\star\phi_1. \tag{2·6}$$

2.2. *Non-commutative ϕ^n theory*

In non-commutative space, ϕ^n theory is given by a following action .

$$\hat{S}[\Phi]=\mathrm{tr}\left[\sum_\mu\frac{1}{2}\Big((\theta^{-1})_{\mu\nu}[\,\hat{x}^\nu,\Phi(\hat{x})\,]\Big)^2+\frac{m^2}{2}\Phi(\hat{x})^2+\frac{\lambda}{n!}\Phi(\hat{x})^n\right]. \tag{2·7}$$

Using the function $\phi(x)$ in commutative space ,we can rewrite this action for the commutative space as follows. We substitute the $\Phi(\hat{x})$ of (2·2) for the above action $\hat{S}[\Phi]$. Since

$$[\,\hat{x}^\nu,\Phi(\hat{x})\,]=\int\frac{d^dk}{(2\pi)^d}\,\tilde{\phi}(k)\,[\,\hat{x}^\nu,e^{ik\cdot\hat{x}}\,]=\int\frac{d^dk}{(2\pi)^d}\,\tilde{\phi}(k)\,k_\alpha\theta^{\alpha\nu}e^{ik\cdot\hat{x}}, \tag{2·8}$$

the first term in (2·7) is as follows.

$$\mathrm{tr}\sum_\mu\frac{1}{2}\theta^{-1}_{\mu\nu}\,[\,\hat{x}^\nu,\Phi(\hat{x})\,]\,\theta^{-1}_{\mu\rho}\,[\,\hat{x}^\rho,\Phi(\hat{x})\,]$$

$$
\begin{aligned}
&= \int \frac{d^dk}{(2\pi)^d}\frac{d^dl}{(2\pi)^d}\sum_{\mu}\frac{1}{2}\,\delta^{\alpha}_{\mu}\,\delta^{\beta}_{\mu}\,k_{\alpha}l_{\beta}\,e^{-\frac{i}{2}k\times l}\mathrm{tr}\left[e^{i(k+l)\cdot\hat{x}}\right] \\
&= \frac{1}{(2\pi)^{d/2}}\int \frac{d^dk}{(2\pi)^d}\,\frac{1}{2}\,\tilde{\phi}(-k)\,k^2\,\tilde{\phi}(k) \\
&= \frac{1}{(2\pi)^{d/2}}\int d^dx\,\frac{1}{2}\left(\partial_{\mu}\phi\right)^2.
\end{aligned}
\tag{2·9}
$$

Similarly, the second term in (2·7) is

$$
\begin{aligned}
&\mathrm{tr}\,\frac{m^2}{2}\Phi(\hat{x})^2 \\
&= \frac{m^2}{2}\frac{1}{(2\pi)^{d/2}}\int \frac{d^dk}{(2\pi)^d}\,\tilde{\phi}(-k)\,\tilde{\phi}(k) \\
&= \frac{1}{(2\pi)^{d/2}}\int d^dx\,\frac{m^2}{2}\phi^2.
\end{aligned}
\tag{2·10}
$$

The third term in (2·7) is

$$
\begin{aligned}
&\mathrm{tr}\,\frac{\lambda}{n!}\,\Phi(\hat{x})^n \\
&= \frac{\lambda}{n!}\int \frac{d^dk_1}{(2\pi)^d}\cdots\frac{d^dk_n}{(2\pi)^d}\,\tilde{\phi}(k_1)\cdots\tilde{\phi}(k_n)\,e^{-\frac{i}{2}\sum_{i<j}(k_i\times k_j)}\mathrm{tr}\left[e^{i(k_1+\cdots+k_n)\cdot\hat{x}}\right] \\
&= \frac{1}{(2\pi)^{d/2}}\int d^dx\,\frac{\lambda}{n!}(\overbrace{\phi\star\cdots\star\phi}^{n}).
\end{aligned}
\tag{2·11}
$$

From (2·9), (2·10) and (2·11), we can obtain a following action in the commutative space for which the action (2·7) in the non-commutative space is rewritten.

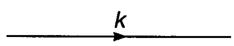
$$
S[\phi] = \int d^dx\left[\frac{1}{2}\left(\partial_{\mu}\phi\right)^2 + \frac{m^2}{2} + \frac{\lambda}{n!}\overbrace{\phi\star\cdots\star\phi}^{n}\right].
\tag{2·12}
$$

2.3. *Non-commutative perturbation theory*

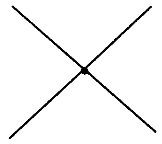
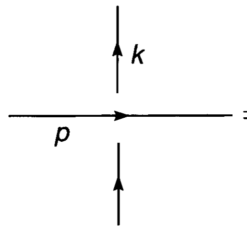
We explain the non-commutative perturbation theory by using non-commutative ϕ^4 theory in d-dimentional Euclidian space-time. We start from a followimg action.

$$
S = \int d^dx\left[\frac{1}{2}\left(\partial_{\mu}\phi\right)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi\right]
\tag{2·13}
$$

The Feynman-rules are

- 

$= \dfrac{1}{k^2+m^2},$

-  $=\lambda,$
- 

$= e^{ik\times p}.$

The diagrams for calculating the two point function up to the one loop level are shown as follows (See Fig.1). We notice that there are two types of the diagrams. One is called "planer diagram" which is the same diagram as commutative one. The other is called "nonpalaner diagram" which has some phase factors.

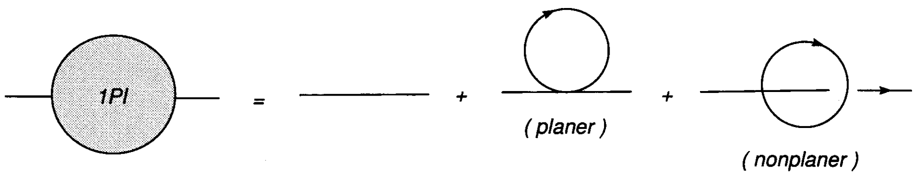
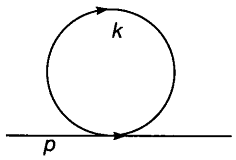


Fig. 1. The diagrams for calculating the two point function up to the one loop level in noncommutative ϕ^4 theory

First, we calculate a following planer diagram.



$$=\frac{1}{3}\cdot\lambda\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2+m^2},$$

where the factor 1/3 is a symmetric factor of this diagram,

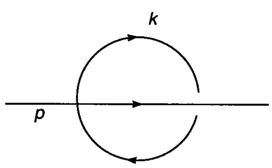
$$=\frac{\lambda}{3}\frac{1}{(4\pi)^{d/2}}\int_0^\infty d\alpha\,\frac{1}{\alpha^{d/2}}\,e^{-\alpha m^2}.$$

Taking the limit $d\to 4$, we can obtain as follows.

$$=\frac{\lambda}{3}\frac{m^2}{(4\pi)^2}\,\Gamma\big(1-d/2\big). \qquad (2\cdot14)$$

$\Gamma\big(1-d/2\big)$ of the right in this equation shows UV-divergence.

Next, we calculate a following nonplaner diagram.



$$=\frac{1}{6}\cdot\lambda\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2+m^2}\,e^{ik\times q},$$

where the factor 1/6 is a symmetric factor of this diagram,

$$=\frac{\lambda}{6}\frac{1}{(4\pi)^{d/2}}\int_0^\infty d\alpha \exp\Big[-\alpha m^2-\frac{(\theta p)^2}{4\alpha}\Big].$$

Taking the limit $d\to 4$, we can obtain as follows.

$$\begin{aligned}&=\frac{\lambda}{6}\frac{1}{(4\pi)^2}m^2\int_0^\infty dt\frac{1}{t^2}\exp\Big[-t-\frac{z^2}{4t}\Big],\\&=\frac{\lambda}{6}\frac{1}{(4\pi)^2}m^2\cdot\frac{4}{z}K_1(z),\end{aligned}\tag{2·15}$$

where $z^2=m^2(\theta p)^2$ and $K_1(z)$ is a modified Bessel function. And the right in this equation is UV finite due to $\theta\neq 0$. In (2·15), when we take the limit $\theta\to\infty$ (i.e. $z\to\infty$), (2·15) vanishes. Because the asymptotic behaviour of $K_1(z)$ is as follows.

$$K_1(z)\approx\sqrt{\frac{\pi}{2z}}e^{-z}.$$

Therefore, in the maximal non-commutativity $\theta\to\infty$, nonplaner diagrams vanish and planer diagrams dominate. This phenomenon occurs in not only one loop level, but also all orders.[4)] The phenomenon is discussed in section 3 by taking high-momentum limit $p\to\infty$ instead .

We are also interested in the behaviour of (2·15) around $z\approx 0$ (i.e. $\theta\approx 0$ or $p\approx 0$). Using a nature of $K_1(z)$ (See Appendix A), we can obtain as folows.

$$\begin{aligned}(2\cdot15)&=\frac{\lambda}{6}\frac{m^2}{(4\pi)^2}\Big[\frac{4}{z^2}+2\ln\frac{z}{2}+\mathcal{O}(1)\Big],\\&=\frac{\lambda}{6}\frac{1}{(4\pi)^2}\Big[\frac{4}{(\theta p)^2}+m^2\ln\frac{m^2(\theta p)^2}{4}+\mathcal{O}(1)\Big].\end{aligned}\tag{2·16}$$

From (2·16), we can see that the UV-divergence by taking the limit $\theta\to 0$ and IR-divergence by taking $p\to 0$ are mixing. This phenomenan is called UV/IR mixing . And the nontrivial mixture between UV and IR phenomena is the most surprising result in the non-commutative field theories. These infrared singularities are discussed in non-commuative gauge theories in section 4

§3. Ultraviolet behavior in non-commutative field theories

3.1. *Perturbative expansion in non-commutative ϕ^3 theory*

We adopt non-commutative ϕ^3 theory as the model that we study the ultraviolet behavior in non-commutative field theories. The Lagrangian density for the non-commutative ϕ^3

where the factor 1/6 is a symmetric factor of this diagram,

$$=\frac{\lambda}{6}\frac{1}{(4\pi)^{d/2}}\int_0^\infty d\alpha \exp\Big[-\alpha m^2-\frac{(\theta p)^2}{4\alpha}\Big].$$

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$$\begin{aligned}&=\frac{\lambda}{6}\frac{1}{(4\pi)^2}m^2\int_0^\infty dt\frac{1}{t^2}\exp\Big[-t-\frac{z^2}{4t}\Big],\\ &=\frac{\lambda}{6}\frac{1}{(4\pi)^2}m^2\cdot\frac{4}{z}K_1(z),\end{aligned}\tag{2·15}$$

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theory in d-dimensional Euclidean space-time can be written as

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{m_0^2}{2}\phi^2 + \frac{g_0}{3}\phi \star \phi \star \phi \ . \tag{3·1}$$

The parameters m_0 and g_0 are the bare mass and the bare coupling constant, respectively. As in the standard perturbation theory, we decompose the bare Lagrangian density into the renormalized Lagrangian density $\mathcal{L}_\mathrm{r}$ and the counter-terms $\mathcal{L}_\mathrm{c.t.}$ as $\mathcal{L} = \mathcal{L}_\mathrm{r} + \mathcal{L}_\mathrm{c.t.}$, where

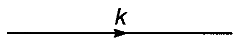
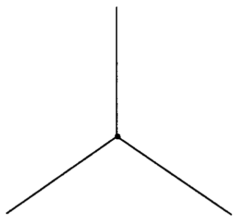
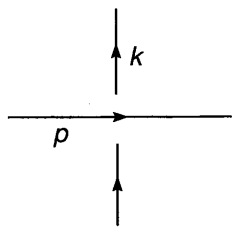
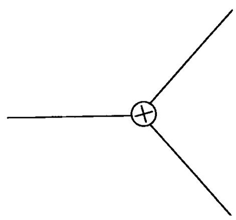
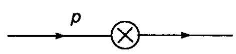
$$\mathcal{L}_\mathrm{r} = \frac{1}{2}(\partial_\mu \phi_r)^2 + \frac{m^2}{2}\phi_r^2 + \frac{g}{3}\phi_r \star \phi_r \star \phi_r \ , \tag{3·2}$$

$$\mathcal{L}_\mathrm{c.t.} = \frac{1}{2}\delta_z(\partial_\mu \phi_r)^2 + \frac{\delta_m}{2}\phi_r^2 + \frac{\delta_g}{3}\phi_r \star \phi_r \star \phi_r \ . \tag{3·3}$$

Here we have introduced the following notations.

$$\phi = Z^{\frac{1}{2}}\phi_r \ , \qquad Z = 1 + \delta_z \ , \qquad \delta_m = m_0^2 Z - m^2 \ , \qquad \delta_g = g_0 Z^{\frac{3}{2}} - g \ . \tag{3·4}$$

The Feynman rules are

-  $= \dfrac{1}{k^2 + m^2}$,
-  $= g$,
-  $= e^{ik \times p}$,
-  $= \delta_g$,
-  $= \delta_z p^2 + \delta_m$.

The diagrams that need to be evaluated in the two-loop calculation of the two-point function are listed in Fig. 2. For diagrams (e),(h),(j),(k),(l),(m),(n),(o),(p) and (q), we need to put a factor of 2 to take into account of the same contributions from analogous diagrams.

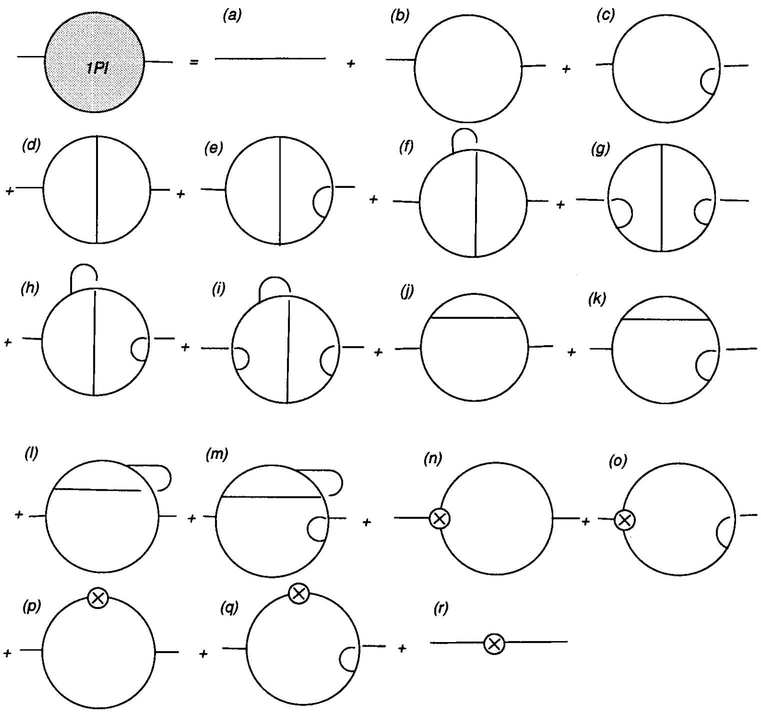


Fig. 2. The list of diagrams for calculating the two-point function up to the two-loop level.

We will focus on two types of nonplanar diagrams; i.e., the diagrams (e) and (f) in Fig. 2. The type 1 diagram is a diagram in which an external line crosses with an internal line. This type includes the ultraviolet divergence coming from the planar one-loop subdiagram. We investigate whether this diagram vanishes at infinite external momentum after appropriate renormalization. The type 2 diagram, on the other hand, is a diagram in which internal lines cross. Since the non-commutativity parameter $\theta_{\mu\nu}$ does not couple directly to the external momentum in this case, it is not obvious whether the effect of sending the external momentum to infinity is the same as that of sending $\theta_{\mu\nu}$ to infinity. We will comment on the remaining non-planar diagrams in section 3.4.

Throughout this paper we consider the massive case specifically. On the technical side, this condition simplifies the evaluation of the upper bound on the nonplanar diagrams. On the theoretical side, this is the more nontrivial case since in the massless case, the equivalence of the infinite momentum limit and the $\theta \to \infty$ limit follows from dimensional arguments.

3.2. *Type 1 diagram*

Since the type 1 nonplanar diagram includes an ultraviolet divergence, we have to renormalize it by adding a contribution from a diagram involving the one-loop counter-term for the three-point function (the diagram (o) in Fig. 2). After this procedure, we may study the behavior at infinite external momentum. We adopt the dimensional regularization and take the space-time dimensionality to be $d = 6 - \varepsilon$. Since the coupling constant g has the $(6-d)/2$ power of the mass dimension, we set $g = \mu^{\frac{\varepsilon}{2}} g_r$, where μ is the renormalization point, and g_r is the dimensionless coupling constant. The ultraviolet divergence will appear as a $1/\varepsilon$ pole in the $d \to 6$ limit.

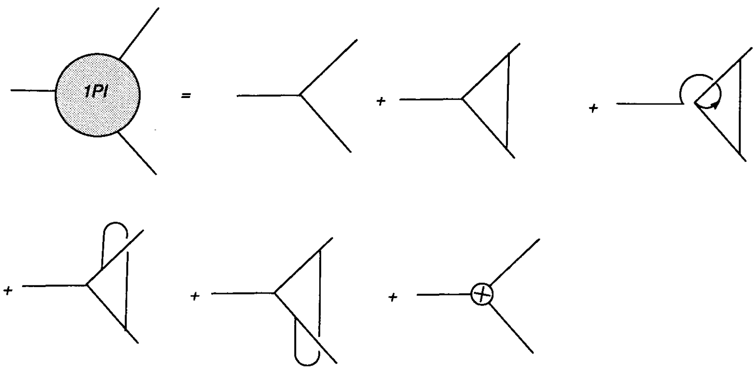


Fig. 3. The list of diagrams for calculating the three-point function up to the one-loop level.

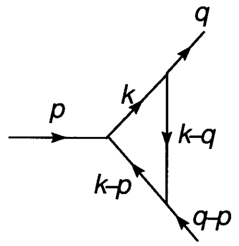


Fig. 4. The planar one-loop diagram for calculating $\Gamma(p,q)$.

The one-loop counter-term for the three-point function can be determined in such a way that the three-point function becomes finite at the one-loop level. The relevant diagrams are listed in Fig. 3. Since the nonplanar diagrams are finite due to the insertion of the momentum dependent phase factor, we only need to calculate the planar diagram depicted in Fig. 4, which can be evaluated as

$$\begin{aligned}\Gamma(p,q) &= g^3 \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2+m^2} \frac{1}{(k-q)^2+m^2} \frac{1}{(k-p)^2+m^2} \\ &= \frac{g^3}{(4\pi)^{\frac{d}{2}}} \int_0^1 d\alpha\, \alpha^{\frac{d}{2}-3}(1-\alpha)^{\frac{d}{2}-2} \int_0^1 d\beta \int_0^\infty dt \frac{1}{t^{\frac{d}{2}-2}} \exp(-t\Delta) , \end{aligned} \qquad (3\cdot5)$$

where Δ is defined by

$$\Delta = \frac{m^2}{\alpha(1-\alpha)} + q^2 - 2\beta\, p\cdot q + \frac{\beta}{\alpha}\Big(1 - (1-\alpha)\beta\Big)\, p^2 \, . \qquad (3\cdot6)$$

(The derivation of (3·5) is shown by Appendix B.) The divergent part can be extracted as

$$\Gamma(p,q) = \frac{g_r^3}{(4\pi)^3}\frac{1}{\varepsilon} + \mathrm{O}(\varepsilon^0) \,, \tag{3·7}$$

from which we determine the one-loop counter-term as

$$\delta_g = -\frac{g_r^3}{(4\pi)^3}\frac{\mu^{\frac{\varepsilon}{2}}}{\varepsilon} \,. \tag{3·8}$$

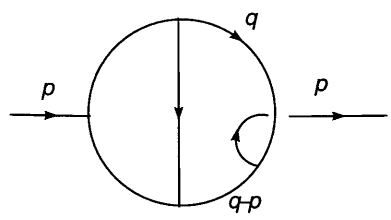


Fig. 5. The type 1 nonplanar diagram for calculating $\Pi_{\mathrm{NP1}}(p^2)$.

Using eq. (3·5) we evaluate the type 1 nonplanar diagram in Fig. 5 as

$$\begin{aligned}\Pi_{\mathrm{NP1}}(p^2) &= g\int \frac{d^dq}{(2\pi)^d}\,\Gamma(p,q)\,\frac{1}{q^2+m^2}\,\frac{1}{(q-p)^2+m^2}\,e^{iq\cdot\theta p}\\ &= \frac{g^4}{(4\pi)^d}\left[\frac{4}{(\theta p)^2}\right]^{d-5}\int_0^1 d\alpha\,\alpha^{\frac{d}{2}-3}(1-\alpha)^{\frac{d}{2}-2}\int_0^1 d\beta\int_0^1 d\zeta\,\zeta^{2-\frac{d}{2}}(1-\zeta)\\ &\quad\cdot\int_0^1 d\eta\int_0^\infty dt\,t^{d-6}\exp\Bigl(-t-\frac{\tilde{\Delta}}{4t}p^2(\theta p)^2\Bigr)\,,\end{aligned} \tag{3·9}$$

where $(\theta p)^2 \equiv -p^\mu\theta^{\mu\nu}\theta^{\nu\rho}p^\rho \geq 0$ since $\theta^{\mu\nu}$ is an antisymmetric tensor. The $\tilde{\Delta}$ is defined as

$$\tilde{\Delta} = \frac{m^2}{p^2}\left[\frac{\zeta}{\alpha(1-\alpha)}+1-\zeta\right]+\frac{1}{\alpha}\left[\zeta\,\beta(1-\beta)+\alpha(1-\zeta)\Bigl(\zeta(\beta-\eta)^2+\eta(1-\eta)\Bigr)\right] \,. \tag{3·10}$$

Extracting the divergent part from (3·9) and taking the $d\to 6$ limit for the finite part (See Appendix C), we obtain

$$\Pi_{\mathrm{NP1}}(p^2) = \frac{g_r^4}{(4\pi)^6}\left[\frac{4}{(\theta p)^2}\left\{\frac{\mathcal{A}}{\varepsilon}+\Bigl(\ln\left(\pi\mu^2(\theta p)^2\right)+1\Bigr)\mathcal{A}-\mathcal{B}-\mathcal{C}\right\}+p^2\mathcal{D}\right] \,, \tag{3·11}$$

where we have introduced

$$\mathcal{A} = \int_0^1 d\eta\int_0^\infty dt\,\exp\left[-t-\frac{\frac{m^2}{p^2}+\eta(1-\eta)}{4t}p^2(\theta p)^2\right] \,, \tag{3·12}$$

$$\mathcal{B} = \int_0^1 d\eta\int_0^\infty dt\,(\ln t)\exp\left[-t-\frac{\frac{m^2}{p^2}+\eta(1-\eta)}{4t}p^2(\theta p)^2\right], \tag{3·13}$$

$$\mathcal{C} = \int_0^1 d\alpha\int_0^1 d\beta\int_0^1 d\zeta\int_0^1 d\eta\int_0^\infty dt\,(1-\alpha)\exp\left[-t-\frac{\tilde{\Delta}}{4t}p^2(\theta p)^2\right] \,, \tag{3·14}$$

$$\mathcal{D} = \int_0^1 d\alpha\int_0^1 d\beta\int_0^1 d\zeta\int_0^1 d\eta\int_0^\infty dt\,\frac{F\ln\zeta}{\alpha t}\exp\left[-t-\frac{\tilde{\Delta}}{4t}p^2(\theta p)^2\right] \,, \tag{3·15}$$

which are functions of $\frac{m^2}{p^2}$ and $p^2(\theta p)^2$. The coefficient F in eq. (3·15) is defined by

$$F = \frac{m^2}{p^2}\Big(1-\alpha(1-\alpha)\Big) + (1-\alpha)\Big[\beta(1-\beta) + \alpha\Big((1-2\zeta)(\beta-\eta)^2 - \eta(1-\eta)\Big)\Big] \; . \tag{3·16}$$

We are going to demonstrate that the functions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ vanish in the $p^2 \to \infty$ limit. For that purpose, we first confirm the convergence of all the integrals. Then it will be suffice to show that the integrands vanish in the $p^2 \to \infty$ limit. The convergence of the integrals in $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ is evident. As for the function $\mathcal{D}$, the α- and t-integrals would have singularities at the lower ends of the integration domain if $\theta_{\mu\nu}$ were zero, but they are regularized by the term proportional to $1/(\alpha t)$ in the exponent, which appear for nonzero $\theta_{\mu\nu}$. Actually we can put an upper bound on the absolute values of these functions by integrating elementary functions, which are larger than the corresponding integrand. We may obtain upper bounds on $|\mathcal{A}|$, $|\mathcal{B}|$ and $|\mathcal{C}|$ as

$$|\mathcal{A}| < 1 \; , \quad |\mathcal{B}| < 2 \; , \quad |\mathcal{C}| < 1 \; , \tag{3·17}$$

by omitting the term proportional to $p^2(\theta p)^2$ in the exponent. Putting an upper bound on $|\mathcal{D}|$ is more involved due to the "noncommutative" regularization of the singularities mentioned above, but the calculation in Appendix D yields

$$|\mathcal{D}| < \frac{32}{p^2(\theta p)^2}\left(1 + \ln\frac{p^2}{m^2}\right)\left(4 + \ln\frac{p^2}{m^2} + \ln\frac{p^2(\theta p)^2}{4}\right) , \tag{3·18}$$

where we assumed $m^2/p^2 \ll 1$ since we are ultimately interested in the $p^2 \to \infty$ limit. This confirms the convergence. Since the integrands of the functions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ decrease exponentially at large p^2, we conclude that all the functions vanish in the $p^2 \to \infty$ limit.

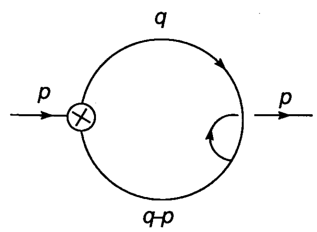


Fig. 6. The nonplanar one-loop diagram for calculating $\Pi_{\mathrm{NP1,c.t.}}(p^2)$.

Next we evaluate the diagram in Fig. 6 involving the counter-term (3·8) as

$$\begin{aligned} &\Pi_{\mathrm{NP1,c.t.}}(p^2) \\ &= \delta_g \, g \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2+m^2} \frac{1}{(k-p)^2+m^2} e^{ik\cdot\theta p} \end{aligned}$$

$$= \delta_g \frac{g}{(4\pi)^{\frac{d}{2}}} \left[\frac{4}{(\theta p)^2}\right]^{\frac{d}{2}-2} \int_0^1 d\alpha \int_0^\infty dt\, t^{\frac{d}{2}-3} \exp\left[-t - \frac{\frac{m^2}{p^2} + \alpha(1-\alpha)}{4t} p^2(\theta p)^2\right]$$
$$= \frac{4g_r^4}{(4\pi)^6} \frac{1}{(\theta p)^2} \left[-\frac{\mathcal{A}}{\varepsilon} + \left\{-\frac{1}{2} \ln\left(\pi\mu^2(\theta p)^2\right)\mathcal{A} + \frac{1}{2}\mathcal{B}\right\}\right] . \tag{3·19}$$

The first term cancels the $1/\varepsilon$ pole in (3·11), and we have taken the $d \to 6$ limit for the remaining terms.

Adding (3·11) and (3·19), we obtain a finite result

$$\Pi_{\mathrm{NP1r}}(p^2) = \frac{g_r^4}{(4\pi)^6} \left[\frac{2}{(\theta p)^2}\left\{\left(\ln\left(\pi\mu^2(\theta p)^2\right) + 2\right)\mathcal{A} - \mathcal{B} - 2\,\mathcal{C}\right\} + p^2\mathcal{D}\right] . \tag{3·20}$$

Since the functions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ vanish in the $p^2 \to \infty$ limit as we have already shown, the type 1 nonplanar diagram after renormalizing the divergence from the planar subdiagram, vanishes in the same limit. In fact, using (3·17) and (3·18), we obtain an upper bound on $|\Pi_{\mathrm{NP1r}}(p^2)|$ as

$$\left|\Pi_{\mathrm{NP1r}}(p^2)\right| < \frac{8g_r^4}{(4\pi)^6} \frac{1}{(\theta p)^2}$$
$$\cdot \left[\frac{3}{2} + \frac{1}{4}\ln(\pi\mu^2(\theta p)^2) + 4\left(1 + \ln\frac{p^2}{m^2}\right)\left(4 + \ln\frac{p^2}{m^2} + \ln\frac{p^2(\theta p)^2}{4}\right)\right] ,$$

where the right-hand side does vanish in the $p^2 \to \infty$ limit, thus confirming the above conclusion more explicitly.

3.3. *Type 2 diagram*

In this section we consider the type 2 nonplanar diagram in which internal lines cross, and study its behavior in the $p^2 \to \infty$ limit. Let us first evaluate the one-loop subdiagram in Fig. 7 as

$$\Gamma_{\mathrm{NP}}(p,q) = g^3 \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 + m^2} \frac{1}{(k-q)^2 + m^2} \frac{1}{(k-p)^2 + m^2} e^{ik\cdot\theta q}$$
$$= \frac{g^3}{(4\pi)^{\frac{d}{2}}} \int_0^1 d\alpha\, \alpha^{\frac{d}{2}-3}(1-\alpha)^{\frac{d}{2}-2} \int_0^1 d\beta \int_0^\infty dt \frac{1}{t^{\frac{d}{2}-2}}$$
$$\cdot \exp\left[-t\Delta - \frac{\alpha(1-\alpha)}{4t}(\theta q)^2 + i\,\beta\,(1-\alpha)\,p\theta\cdot q\right] \tag{3·21}$$
$$= \frac{2g_r^3}{(4\pi)^3} \int_0^1 d\alpha\,(1-\alpha) \int_0^1 d\beta\, e^{i\beta(1-\alpha)p\theta\cdot q} K_0\left(\sqrt{\alpha(1-\alpha)\Delta(\theta q)}\right) , \tag{3·22}$$

where $K_0(z)$ is the modified Bessel function, and the Δ in (3·21) is defined by

$$\Delta = \frac{m^2}{\alpha(1-\alpha)} + q^2 - 2\beta\, p\cdot q + \frac{\beta}{\alpha}\left(1 - (1-\alpha)\beta\right)p^2 . \tag{3·23}$$

When we proceed from (3·21) to (3·22), we have taken the $d \to 6$ limit. This diagram is finite since the logarithmic ultraviolet divergence, which would arise in the commutative case, is regularized by the non-commutative phase factor. We may appreciate this fact by considering the asymptotic behavior of (3·22)

$$\Gamma_{\rm NP}(p,q) \simeq -\frac{g_r^3}{2(4\pi)^3}\ln p^2(\theta q)^2 \tag{3·24}$$

for $(\theta q)^2 \approx 0$ at nonzero p^2. The logarithmic behavior reflects the ultraviolet divergence in the commutative case. This may be regarded as a result of the UV/IR mixing.

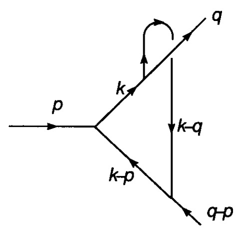


Fig. 7. The nonplanar one-loop diagram for calculating $\Gamma_{\rm NP}(p,q)$.

In what follows we assume for simplicity that all the eigenvalues of the symmetric matrix $(\theta^2)^{\mu\rho} = \theta^{\mu\nu}\theta^{\nu\rho}$ are equal and denote it as $-\theta^2$ (<0). The general case will be discussed later. In fact when some of the eigenvalues are zero, there are certain differences in the behavior at large p^2, but our final conclusion concerning the $p^2 \to \infty$ limit is the same.

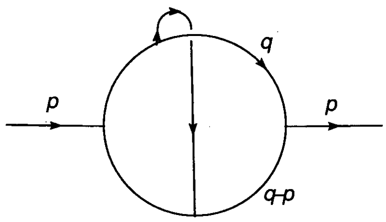


Fig. 8. The type 2 nonplanar diagram for calculating $\Pi_{\rm NP2}(p^2)$.

Using (3·21), we evaluate the diagram in Fig. 8 as

$$\begin{aligned}\Pi_{\rm NP2}(p^2) &= g\int \frac{d^dq}{(2\pi)^d}\,\Gamma_{\rm NP}(p,q)\frac{1}{q^2+m^2}\,\frac{1}{(q-p)^2+m^2}\\
&= \frac{g^4}{(4\pi)^d}(p^2)^{d-5}\int_0^1 d\alpha\,\alpha^{\frac{d}{2}-3}(1-\alpha)^{\frac{d}{2}-2}\int_0^1 d\beta\int_0^1 d\zeta\,\zeta^2(1-\zeta)\int_0^1 d\eta\\
&\cdot\int_0^\infty dt\,t^4\frac{1}{\left[t^2\zeta+\frac{\alpha(1-\alpha)}{4}\,\theta^2(p^2)^2\right]^{\frac{d}{2}}}\exp\left[-t\tilde{\Delta}-t\frac{\tilde{\Delta}_{\rm NP}}{t^2\zeta+\frac{\alpha(1-\alpha)}{4}\,\theta^2(p^2)^2}\right],\end{aligned} \tag{3·25}$$

where we have defined

$$\tilde{\Delta} = \frac{m^2}{p^2}\left[\frac{\zeta}{\alpha(1-\alpha)} + 1 - \zeta\right] + \frac{\zeta\beta(1-\beta)}{\alpha} + (1-\zeta)\Big(\zeta(\beta-\eta)^2 + \eta(1-\eta)\Big) , \quad (3\cdot 26)$$

$$\tilde{\Delta}_{\rm NP} = \frac{1-\alpha}{4}\Big[\zeta\beta^2(1-\alpha) + \alpha\big(\zeta\beta + (1-\zeta)\eta\big)^2\Big]\theta^2(p^2)^2 . \quad (3\cdot 27)$$

The ζ- and t-integrals would have singularities at the lower ends of the integration domain if θ^2 were zero, but they are regularized by the term proportional to θ^2 in the denominator. This appearance of the θ^2 term is peculiar to nonplanar diagrams in which the internal lines cross.[4)] By omitting the $t^2\zeta$ term in the denominator and omitting the terms other than the ones proportional to m^2 in the exponent, we put an upper bound on $|\Pi_{\rm NP2}(p^2)|$ as [1]

$$|\Pi_{\rm NP\,2}(p^2)| < \frac{g^4}{(4\pi)^d}\frac{4^{\frac{d}{2}}}{(\theta^2)^{\frac{d}{2}}(m^2)^5} , \quad (3\cdot 28)$$

which confirms the convergence of the multiple integral in (3·25) in the $d \to 6$ limit. Given this, the fact that the integrand in eq. (3·25) decreases as $1/(p^2)^5$ at large p^2 implies that the type 2 nonplanar diagram vanishes in the $p^2 \to \infty$ limit. As we have done in the case of type 1 diagram, we can actually put a more stringent upper bound on $|\Pi_{\rm NP2}(p^2)|$, which vanishes in the $p^2 \to \infty$ limit. This confirms our statement more explicitly. See Appendix E for the details.

Let us comment on the case in which the number of non-commutative directions is less than the space-time dimensionality d. In this case the upper bound on $|\Pi_{\rm NP2}(p^2)|$ can be evaluated as

$$|\Pi_{\rm NP2}(p^2)| < \frac{g^4}{(4\pi)^d}(p^2)^{d-5}\int_0^1 d\alpha\,\alpha^{\frac{d}{2}-3}(1-\alpha)^{\frac{d}{2}-2}\int_0^1 d\beta\int_0^1 d\zeta\,\zeta^2(1-\zeta)\int_0^1 d\eta$$

$$\cdot\int_0^\infty dt\,t^4\prod_{j=1}^{d}\frac{1}{\left[t^2\zeta + \frac{\alpha(1-\alpha)}{4}\theta_j^2(p^2)^2\right]^{\frac{1}{2}}}\exp\left[-t\tilde{\Delta}\right] , \quad (3\cdot 29)$$

where $(-\theta_j^2)$ is the j-th eigenvalue of $(\theta^2)^{\mu\nu}$ and $\tilde{\Delta}$ is the same as the one defined in (3·26). If the non-commutativity is introduced only in k directions ($2 \le k \le 6$), we obtain an upper bound from (3·29) generalizing (3·28) as

$$|\Pi_{\rm NP2}(p^2)| < \frac{g^4}{(4\pi)^d}\frac{a_k\cdot 4^{\frac{k}{2}}}{\prod_{j=1}^k(\theta_j^2)^{\frac{1}{2}}(m^2)^{k-1}} , \quad (3\cdot 30)$$

where a_k is a k-dependent constant which is irrelevant to the issue we are discussing. In the $p^2 \to \infty$ limit the integrand on the right-hand side of (3·29) decreases as $(1/p^2)^{k-1}$. Thus we conclude that the type 2 nonplanar diagram vanishes in the $p^2 \to \infty$ limit for general $\theta_{\mu\nu}$.

[1] Since the upper bound (3·28) is independent of p^2, we find that $|\Pi_{\rm NP2}(p^2)|$ is finite even in the $p^2 \to 0$ limit. This is in contrast to the type 1 diagram (after renormalization), which actually diverges in the $p^2 \to 0$ limit due to the UV/IR mixing.

3.4. *Discussions with respect to planer limit and infrared divergence*

In this section we have discussed the vanishing of nonplanar diagrams in the $p^2 \to \infty$ limit in 6d non-commutative ϕ^3 theory at the two-loop level. We have discussed two types of nonplanar diagrams separately. In the type 1 nonplanar diagram we have confirmed the statement after renormalizing the ultraviolet divergence coming from the planar subdiagram. In the type 2 nonplanar diagram the statement holds despite the fact that the non-commutative phase factor does not depend on the external momentum.

Based on the behaviors observed for these two types of diagrams, we can argue that the other types of nonplanar diagrams in Fig. 2 also vanish in the $p^2 \to \infty$ limit. The situation concerning the diagrams (k) and (l) is analogous to the type 1 and the type 2 diagrams, respectively. The diagram (k) has a planar subdiagram, which causes a UV divergence. This divergence can be cancelled by adding the contribution from the diagram (q), and the resulting finite quantity should vanish due to the crossing of an external line and an internal line. The diagram (l) is finite by itself, and it should vanish in a manner similar to the type 2 diagram due to the crossing of internal lines. The diagrams (g),(h) and (i) have more crossings of momentum lines than the type 1 diagram. Therefore one gets extra non-commutative phase factors, which make the diagrams finite by themselves. The vanishing of these diagrams then follows as in the type 1 diagram. An analogous argument applies to the diagram (m), which has more crossings of momentum lines than the diagram (k). Although we have studied a particular model for concreteness, we consider that the statement holds in general models.

We should mention that the renormalization procedure[18] in non-commutative scalar field theories encounters an obstacle due to severe infrared divergence at higher loop.[4] This problem may be overcome by resumming a class of diagrams with infrared divergence in ϕ^4 theory.[4] Indeed Monte Carlo simulations show that one can obtain a sensible continuum limit,[8] which suggests the appearance of a dynamical infrared cutoff due to *nonperturbative* effects. Introducing an infrared cutoff with such a dynamical origin in perturbation theory, we consider that the statement we have discussed up to two-loop can be generalized to all orders.

§4. Infrared behavior in non-commutative field theories

4.1. *Infrared singularities in non-commutative $U(1)$ gauge theory*

We adopt $U(1)$ non-commutative gauge theory as the model that we study the infrared behavior with respect to non-commutative field theories. We calculate the effective action of non-commutative $U(1)$ gauge theory in d-dimentional Minkowski space-time to discuss

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Based on the behaviors observed for these two types of diagrams, we can argue that the other types of nonplanar diagrams in Fig. 2 also vanish in the $p^2 \to \infty$ limit. The situation concerning the diagrams (k) and (l) is analogous to the type 1 and the type 2 diagrams, respectively. The diagram (k) has a planar subdiagram, which causes a UV divergence. This divergence can be cancelled by adding the contribution from the diagram (q), and the resulting finite quantity should vanish due to the crossing of an external line and an internal line. The diagram (l) is finite by itself, and it should vanish in a manner similar to the type 2 diagram due to the crossing of internal lines. The diagrams (g),(h) and (i) have more crossings of momentum lines than the type 1 diagram. Therefore one gets extra non-commutative phase factors, which make the diagrams finite by themselves. The vanishing of these diagrams then follows as in the type 1 diagram. An analogous argument applies to the diagram (m), which has more crossings of momentum lines than the diagram (k). Although we have studied a particular model for concreteness, we consider that the statement holds in general models.

We should mention that the renormalization procedure[18] in non-commutative scalar field theories encounters an obstacle due to severe infrared divergence at higher loop.[4] This problem may be overcome by resumming a class of diagrams with infrared divergence in ϕ^4 theory.[4] Indeed Monte Carlo simulations show that one can obtain a sensible continuum limit,[8] which suggests the appearance of a dynamical infrared cutoff due to *nonperturbative* effects. Introducing an infrared cutoff with such a dynamical origin in perturbation theory, we consider that the statement we have discussed up to two-loop can be generalized to all orders.

§4. Infrared behavior in non-commutative field theories

4.1. *Infrared singularities in non-commutative $U(1)$ gauge theory*

We adopt $U(1)$ non-commutative gauge theory as the model that we study the infrared behavior with respect to non-commutative field theories. We calculate the effective action of non-commutative $U(1)$ gauge theory in d-dimentional Minkowski space-time to discuss the meaning of infrared singularities. (Of course, we will take the limit $d \to 4$.) We use background field perturbation theory for this aim. We rescale the gauge field $A_\mu \to \frac{1}{e}A_\mu$, because the gauge coupling constant e is moved to the coefficient of the gauge field kinetic energy term. We thus start from the action as follows.

$$S = -\frac{1}{4e^2}\int d^d x\, F_{\mu\nu}^2, \tag{4·1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu \star A_\nu - A_\nu \star A_\mu].$$

Hereafter, we abbreviate $[A \star B - B \star A]$ to $[A, B]$. The action is invariant under a non-commutative gauge transformation

$$\delta A_\mu = \partial_\mu \Lambda + i[A_\mu, \Lambda]. \tag{4·2}$$

Next we split the gauge field into a classical background field $\overline{A}_\mu$ and a fluctuating quantum field a_μ as follows.

$$A_\mu = \overline{A}_\mu + a_\mu$$

The field strength decomposes as follows.

$$F_{\mu\nu} = \overline{F}_{\mu\nu} + \overline{D}_\mu a_\nu - \overline{D}_\nu a_\mu - i[a_\mu, a_\nu],$$

where

$$\overline{F}_{\mu\nu} = \partial_\mu \overline{A}_\nu - \partial_\nu \overline{A}_\mu - i[\overline{A}_\mu, \overline{A}_\nu],$$
$$\overline{D}_\mu a_\nu = \partial_\mu a_\nu - i[\overline{A}_\mu, a_\nu].$$

Thus, we can write the action that is gauge-fixed(Feynman-'t Hooft gauge) and includes Faddeev-Popov ghost as follows.[19]

$$S = \int d^d x \Big[-\frac{1}{4e^2}(\overline{F}_{\mu\nu} + \overline{D}_\mu a_\nu - \overline{D}_\nu a_\mu - i[a_\mu, a_\nu])^2 - \frac{1}{2e^2}(\overline{D}_\mu a^\mu)(\overline{D}_\nu a^\nu) + \overline{D}^\mu \overline{c} \star \big(\partial_\mu c - i[\overline{A}_\mu + a_\mu, c]\big)\Big] \tag{4·3}$$

To calculate the effective action to one-loop order, we drop terms linear in the fluctuating field a_μ and then integrate over the terms quadratic in a_μ and ghost fields. The terms quadratic in a_μ and ghost fields are

$$S' = \int d^d x \Big[-\frac{1}{2e^2}(\overline{D}_\mu a_\nu - \overline{D}_\nu a_\mu)\overline{D}^\mu a^\nu + \frac{i}{e^2}\overline{F}_{\mu\nu} \star a^\mu \star a^\nu - \frac{1}{2e^2}(\overline{D}_\mu a^\mu)(\overline{D}_\nu a^\nu) + \overline{D}^\mu \overline{c} \star \overline{D}_\mu c \Big]. \tag{4·4}$$

After integrating by parts , we can rewrite this as

$$S' = \int d^d x \Big[-\frac{1}{2e^2} a_\mu \star \big(-\overline{D}^2 g^{\mu\nu} + 2\overline{F}_{\rho\sigma} (\mathcal{J}^{\rho\sigma})^{\mu\nu} \big) \star a_\nu - \overline{c} \star \overline{D}^2 c \Big], \tag{4·5}$$

where

$$(\mathcal{J}^{\rho\sigma})^{\mu\nu} = i(g^{\rho\mu} g^{\sigma\nu} - g^{\sigma\mu} g^{\rho\nu}).$$

This detailed calculation is shown by Appendix F.

Thus, we can obtain a path integral from the Faddeev-Popov lagrangian $\mathcal{L}_{F.P.}$ and a counterterm $\mathcal{L}_{c.t.}$ as follows.

$$\begin{aligned} e^{i\Gamma(\overline{A})} &= \int Da\, Dc \exp\Big[\int d^d x \big(\mathcal{L}_{F.P.} + \mathcal{L}_{c.t.} \big) \Big] \\ &= \exp\Big[-\frac{1}{4e^2} \overline{F}^2_{\mu\nu} + \mathcal{L}_{c.t.} \Big] \cdot \big(\det \Delta_{G,1} \big)_\star^{-\frac{1}{2}} \big(\det \Delta_{G,0} \big)_\star^{+1}, \end{aligned} \tag{4·6}$$

where

$$\Delta_{G,1} = -\overline{D}^2 + 2\overline{F}_{\rho\sigma} (\mathcal{J}^{\rho\sigma}), \tag{4·7}$$

$$\Delta_{G,0} = -\overline{D}^2. \tag{4·8}$$

Solving this equation for $\Gamma(\overline{A})$, we obtain as follows.

$$\Gamma(\overline{A}) = \int d^d x \Big[-\frac{1}{4e^2} \overline{F}^2_{\mu\nu} + \mathcal{L}_{c.t.} \Big] + \frac{i}{2} \ln\big(\det \Delta_{G,1} \big) - i \ln\big(\det \Delta_{G,0} \big). \tag{4·9}$$

Now, we have to calculate $\ln\big(\det \Delta_{G,1}\big)$ and $\ln\big(\det \Delta_{G,0}\big)$. We must first calculate $-\overline{D}^2$ in (4·7) and (4·8) .

$$\begin{aligned} -\overline{D}^2 &= -\overline{D}^\mu \overline{D}_\mu = -\overline{D}^\mu \big(\partial_\mu - i[\overline{A}_\mu,\ ] \big) \\ &= -\Big[\partial^\mu \big(\partial_\mu - i[\overline{A}_\mu,\ ] \big) - i\big[\overline{A}^\mu, \partial_\mu - i[\overline{A}_\mu,\ ] \big] \Big] \\ &= -\partial^2 + \Delta^{(1)} + \Delta^{(2)}, \end{aligned}$$

where

$$\begin{aligned} \Delta^{(1)} &= i\big(\partial^\mu [\overline{A}_\mu,\ ] + [\overline{A}^\mu, \partial_\mu] \big), \\ \Delta^{(2)} &= [\overline{A}^\mu, [\overline{A}_\mu,\ ]]. \end{aligned}$$

Thus we can rewrite (4·7) and (4·8) as

$$\Delta_{G,1} = -\partial^2 + \Delta^{(1)} + \Delta^{(2)} + \Delta^{(\mathcal{J})}, \tag{4·10}$$

$$\Delta_{G,0} = -\partial^2 + \Delta^{(1)} + \Delta^{(2)}, \tag{4·11}$$

where

$$\varDelta^{(\mathcal{J})} = 2\overline{F}_{\rho\sigma}\mathcal{J}^{\rho\sigma}.$$

Now,we can calculate $\ln\bigl(\det \varDelta_{G,1}\bigr)$ or $\ln\bigl(\det \varDelta_{G,0}\bigr)$ as

$$\begin{aligned}
\ln\bigl(\det \varDelta_{G,1}\bigr) =& \ln\bigl(\det(-\partial^2 + \varDelta^{(1)} + \varDelta^{(2)} + \varDelta^{(\mathcal{J})}\bigr) \\
=& \ln\det(-\partial^2) + \ln\det\bigl[1 + (-\partial^2)^{-1}(\varDelta^{(1)} + \varDelta^{(2)} + \varDelta^{(\mathcal{J})})\bigr] \\
=& \ln\det(-\partial^2) + \mathrm{Tr}\ \ln\bigl[1 + (-\partial^2)^{-1}(\varDelta^{(1)} + \varDelta^{(2)} + \varDelta^{(\mathcal{J})})\bigr] \\
=& \ln\det(-\partial^2) + \mathrm{Tr}\ \Bigl[(-\partial^2)^{-1}(\varDelta^{(1)} + \varDelta^{(2)} + \varDelta^{(\mathcal{J})}) \\
&- \frac{1}{2}(-\partial^2)^{-1}(\varDelta^{(1)} + \varDelta^{(2)} + \varDelta^{(\mathcal{J})})(-\partial^2)^{-1}(\varDelta^{(1)} + \varDelta^{(2)} + \varDelta^{(\mathcal{J})}) \\
&+ \cdots\Bigr].
\end{aligned}$$

The first term of the right in this equation is irrelevant constant. We pick up the terms quadratic in the classical background field $\overline{A}_\mu$ from the second term. We thus obtain three terms as follows.

$$\begin{aligned}
&\mathrm{Tr}\,(-\partial^2)^{-1}\varDelta^{(2)} - \frac{1}{2}\mathrm{Tr}\,(-\partial^2)^{-1}\varDelta^{(1)}(-\partial^2)^{-1}\varDelta^{(1)} \\
&- \frac{1}{2}\mathrm{Tr}\,(-\partial^2)^{-1}\varDelta^{(\mathcal{J})}(-\partial^2)^{-1}\varDelta^{(\mathcal{J})}
\end{aligned} \tag{4·12}$$

These calculations in terms of trace correspond to the final step we must calculate to obtain the effective action. The detailed calculation is shown by Appendix G. We have to note that the trace includes a trace over spin indices. Thus there is a difference in terms of the trace between the case of the gauge field $\varDelta_{G,1}$ and the case of ghost field $\varDelta_{G,0}$. The calculation of the first term of the (4·12) in the case of gauge field is

$$\begin{aligned}
&\mathrm{Tr}\,(-\partial^2)^{-1}\varDelta^{(2)} \\
=&d\int \frac{d^dk}{(2\pi)^d}\overline{A}^\mu(k)\overline{A}_\mu(-k)\int \frac{d^dp}{(2\pi)^d}\frac{1}{p^2}(2 - e^{-ip\times k} - e^{-ik\times p}) \\
=&4\int \frac{d^4k}{(2\pi)^4}\overline{A}^\mu(k)\overline{A}_\mu(-k)\cdot 0\cdot \Gamma\Bigl(2 - \frac{d}{2}\Bigr) + \cdots \\
&+ \frac{-32i}{(4\pi)^2}\int \frac{d^4k}{(2\pi)^4}\overline{A}^\mu(k)\overline{A}_\mu(-k)\frac{1}{(k\theta)^2} + \cdots.
\end{aligned} \tag{4·13}$$

The first row of the right hand in the last line of (4·13) corresponds to planer part and the second row corresponds to nonplaner part. When we expand the second line into the third line in (4·13), we take the limit $d \to 4$, and pick up the ultraviolet singularities from the planer part and the infrared singularities around $k \approx 0$ from the nonplaner part. Similarly,the second term of (4·12) is

$$
\begin{aligned}
&-\frac{1}{2}\mathrm{Tr}\,(-\partial^2)^{-1}\Delta^{(1)}(-\partial^2)^{-1}\Delta^{(1)}\\
&=-\frac{d}{2}\int\frac{d^dk}{(2\pi)^d}\overline{A}^{\mu}(k)\overline{A}_{\mu}(-k)\int\frac{d^dp}{(2\pi)^d}\frac{(2p+k)^{\mu}(2p+k)^{\nu}}{p^2(p+k)^2}(2-e^{-ip\times k}-e^{-ik\times p})\\
&=\frac{i}{(4\pi)^2}\frac{4}{3}\int\frac{d^4k}{(2\pi)^4}\overline{A}_{\mu}(k)\overline{A}_{\nu}(-k)(g^{\mu\nu}k^2-k^{\mu}k^{\nu})\Gamma\Big(2-\frac{d}{2}\Big)+\cdots\\
&\quad+\frac{-32i}{(4\pi)^2}\int\frac{d^4k}{(2\pi)^4}\overline{A}_{\mu}(k)\overline{A}_{\nu}(-k)\left[2\frac{(k\theta)^{\mu}(k\theta)^{\nu}}{((k\theta)^2)^2}-\frac{g^{\mu\nu}}{(k\theta)^2}\right]+\cdots.
\end{aligned}
\tag{4·14}
$$

The third term of (4·12) is

$$
\begin{aligned}
&-\frac{1}{2}\mathrm{Tr}\,(-\partial^2)^{-1}\Delta^{(\mathcal{J})}(-\partial^2)^{-1}\Delta^{(\mathcal{J})}\\
&=-16\int\frac{d^dk}{(2\pi)^d}\overline{A}_{\mu}(k)\overline{A}_{\nu}(-k)(g^{\mu\nu}k^2-k^{\mu}k^{\nu})\int\frac{d^dp}{(2\pi)^d}\frac{1}{p^2(p+k)^2}\\
&=-\frac{16i}{(4\pi)^2}\int\frac{d^4k}{(2\pi)^4}\overline{A}_{\mu}(k)\overline{A}_{\nu}(-k)(g^{\mu\nu}k^2-k^{\mu}k^{\nu})\Gamma\Big(2-\frac{d}{2}\Big)+\cdots.
\end{aligned}
\tag{4·15}
$$

Adding (4·13) , (4·14) and (4·15), we can obtain $\ln\big(\det\Delta_{G,1}\big)$ as follows.

$$
\begin{aligned}
\ln\big(\det\Delta_{G,1}\big)=&\int\frac{d^4k}{(2\pi)^4}\overline{A}_{\mu}(k)\overline{A}_{\nu}(-k)(g^{\mu\nu}k^2-k^{\mu}k^{\nu})\frac{i}{(4\pi)^2}\Big(\frac{-44}{3}\Big)\Gamma\Big(2-\frac{d}{2}\Big)\\
&+\int\frac{d^4k}{(2\pi)^4}\overline{A}_{\mu}(k)\overline{A}_{\nu}(-k)\left[-\frac{4i}{\pi^2}\frac{(k\theta)^{\mu}(k\theta)^{\nu}}{((k\theta)^2)^2}\right]+\cdots.
\end{aligned}
\tag{4·16}
$$

Similarly,we can also obtain $\ln\big(\det\Delta_{G,0}\big)$ as follows.

$$
\begin{aligned}
\ln\big(\det\Delta_{G,0}\big)=&\int\frac{d^4k}{(2\pi)^4}\overline{A}_{\mu}(k)\overline{A}_{\nu}(-k)(g^{\mu\nu}k^2-k^{\mu}k^{\nu})\frac{i}{(4\pi)^2}\frac{1}{3}\Gamma\Big(2-\frac{d}{2}\Big)\\
&+\int\frac{d^4k}{(2\pi)^4}\overline{A}_{\mu}(k)\overline{A}_{\nu}(-k)\left[-\frac{i}{\pi^2}\frac{(k\theta)^{\mu}(k\theta)^{\nu}}{((k\theta)^2)^2}\right]+\cdots.
\end{aligned}
\tag{4·17}
$$

At last, substituting (4·16) and (4·17) for (4·9) and imposing a renormalization condition at the scale M, we can obtain the effective action as follows.

$$
\begin{aligned}
\Gamma(A)=&\int\frac{d^4k}{(2\pi)^4}\overline{A}_{\mu}(k)\overline{A}_{\nu}(-k)(g^{\mu\nu}k^2-k^{\mu}k^{\nu})\left[-\frac{1}{2e^2}+\frac{1}{(4\pi)^2}\frac{23}{3}\ln\frac{M^2}{k^2}\right]\\
&+\int\frac{d^4k}{(2\pi)^4}\overline{A}_{\mu}(k)\overline{A}_{\nu}(-k)\frac{1}{\pi^2}\frac{(k\theta)^{\mu}(k\theta)^{\nu}}{((k\theta)^2)^2}+\cdots.
\end{aligned}
\tag{4·18}
$$

4.2. *Critical point of instability of the theory*

Now, we can evaluate a 1-loop effective potential V.

$$V = \int \frac{d^4k}{(2\pi)^4} \overline{A}_\mu(k)\overline{A}_\nu(-k) \left[(g^{\mu\nu}k^2 - k^\mu k^\nu) \frac{1}{2\,e(k)^2} - \frac{1}{\pi^2} \frac{(k\theta)^\mu (k\theta)^\nu}{\left((k\theta)^2\right)^2} \right], \tag{4·19}$$

where

$$\frac{1}{2\,e(k)^2} = \frac{1}{2e^2} - \frac{1}{(4\pi)^2}\frac{23}{3}\ln\frac{M^2}{k^2}. \tag{4·20}$$

From the second term of the right in (4·19), we can see a tachyonic behavior for the low momentum modes of the gauge field due to infrared singularities.

We are interested in the mode which makes this potential unstable(i.e. nagative). We assume that there is the non-commutativity in the only 2 and 3 directions as follows.

$$\theta^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta \\ 0 & 0 & -\theta & 0 \end{pmatrix} \tag{4·21}$$

And so, we put $k^\mu = (0, 0, k^2, k^3)$ simply to study the effect of the non-commutativity. To tell the truth, from the result of a simulation by a lattice regularization , we have obtained the lowest mode $k^2 = k^3 = 2a/\theta$ in which the instability occurs. (a is the lattice space.) In (4·19), putting

$$M^{\mu\nu} = \left[(g^{\mu\nu}k^2 - k^\mu k^\nu) \frac{1}{2\,e(k)^2} - \frac{1}{\pi^2} \frac{(k\theta)^\mu (k\theta)^\nu}{\left((k\theta)^2\right)^2} \right], \tag{4·22}$$

and substituting (4·21), $k^2 = k^3 = 2a/\theta$ and $M^2 = -\pi^2/a^2$ for (4·22), we can obtain a following matrix.

$$M^{\mu\nu} = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & -A & 0 & 0 \\ 0 & 0 & B & -B \\ 0 & 0 & -B & B \end{pmatrix}, \tag{4·23}$$

where

$$A = -\left(\frac{4a^2}{\theta^2}\right)\frac{1}{2\,e(k)^2},$$

$$B = \left(\frac{4a^2}{\theta^2}\right)\frac{1}{2\,e(k)^2} - \frac{1}{16\,a^2\pi^2}, \tag{4·24}$$

$$\frac{1}{2\,e(k)^2} = \frac{1}{2\,e^2} - \frac{1}{(4\pi)^2}\frac{23}{3}\ln\frac{\pi^2\theta^2}{8\,a^4}. \tag{4·25}$$

To consider the direction in which the instability occurs, we take the following part of the matrix (4·23).

$$\begin{pmatrix} B & -B \\ -B & B \end{pmatrix}. \tag{4·26}$$

At this point, solving an eigenvalue equation with respect to (4·26), we can obtain eigenvalues $\lambda = 0, 2B$. Then we can obtain a critical point from the condition $B < 0$ as follows. Putting $\beta = 1/(2\, e^2)$, from (4·24) and (4·25) we can obtain as follows.

$$\beta < \frac{\theta^2}{64\pi^2 a^4} + \frac{1}{(4\pi)^2}\frac{23}{3}\ln\frac{\pi^2\theta^2}{8a^4} \qquad (4·27)$$

Using the matrix size $N = \pi\theta/a^2$ in the lattice side, above the inequality (4·27) is

$$\beta < \frac{N^2}{64\pi^4} + \frac{1}{(4\pi)^2}\frac{23}{3}\ln\frac{N^2}{8}. \qquad (4·28)$$

From the above, in the non-commutative $U(1)$ gauge theory, we have been able to derive the critical point which causes the instability of the theory due to the infrared behavior.

§5. Summary and discussions

We have investigated the ultraviolet and infrared behavior in non-commutative field theories. To study ultraviolet behavior, we adopted non-commutative ϕ^3 theory and performed explicit two loop calculation. In non-commutative field theories, there is the established theory that nonplaner diagrams which include the effect of the non-commutativity vanishes in maximal non-commutativity ($\theta \to \infty$). We investigated whether such a planer limit also occurs in the case where θ is finite but external momentum goes to ∞ (i.e. ultraviolet region). We took two types of nonplaner diagram as typical diagrams of two loop level to study above the statement. One was the diagram which we have to renormalize for the UV divergence of subdiagram. The other was the diagram in which non-commutativity parameter θ seem to not correlate the external momentum p. Consequently, we confirmed that both of the nonplaner diagrams vanished at the ultraviolet region.

Next, to study infrared behavior, we adopted non-commutative $U(1)$ gauge theory. In the theory, there were infrared singularities which cause instability of the system due to the effect of UV/IR mixing. To find the critical point which causes instability, we calculated the effective action at one loop level. We fixed the gauge and introduced ghost field. In the calculation, the effective coupling constant was derived from planer part and infrared singularities were derived from nonplaner part. We balanced the kinetic energy term which includes the effective coupling constant with the infrared singular potential term which condences at a low momentum mode to find the critical point.

We have to consider where such a result of non-commutative field theories apply. We are interested in restoring the phenomena of non-commutative field theories to D-brane physics with respect to string theories. And when we consider a physics in the region of plank length, At this point, solving an eigenvalue equation with respect to (4·26), we can obtain eigenvalues $\lambda = 0, 2B$. Then we can obtain a critical point from the condition $B < 0$ as follows. Putting $\beta = 1/(2\,e^2)$, from (4·24) and (4·25) we can obtain as follows.

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We have to consider where such a result of non-commutative field theories apply. We are interested in restoring the phenomena of non-commutative field theories to D-brane physics with respect to string theories. And when we consider a physics in the region of plank length, we may develop quantum gravity on non-commutative space-time. In the technical side, we can also change the treatment with star product into the treatment with matrix.[9] When it is difficult to calculate by using star product in non-commutative field theories, we often use a matrix model instead.

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Appendix A
—— *Leading term of $K_1(z)/z$ around $z \approx 0$* ——

We start from a following equation.

$$\int_0^\infty dt \frac{1}{t^2} \exp\left[-t - \frac{z^2}{4t}\right] = \frac{4}{z} K_1(z), \tag{A·1}$$

We can expand the $K_1(z)$ as power series in terms of z as follows.

$$K_1(z) = \frac{1}{z} + \sum_{m=0}^{\infty} \frac{1}{m!(m+1)!}\left(\frac{z}{2}\right)^{1+2m} \cdot \left[\ln\frac{z}{2} - \frac{\psi(m+1)+\psi(m+2)}{2}\right], \tag{A·2}$$

where

$$\psi(m+1) = 1 + \frac{1}{2} + \cdots + \frac{1}{m} - \gamma_E.$$

Thus, we can write down the leading term $K_1(z)/z$ around $z \approx 0$ as follows.

$$\frac{K_1(z)}{z} = \frac{1}{z^2} + \frac{1}{2}\ln\frac{z}{2} + \mathcal{O}(1) \tag{A·3}$$

Appendix B
—— *Derivation of (3·5)* ——

We start from the following equation.

$$\Gamma(p,q) = g^3 \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2+m^2} \frac{1}{(k-q)^2+m^2} \frac{1}{(k-p)^2+m^2}$$

At this point, we introduce Schwinger parameter as follows.

$$\frac{1}{k^2+m^2} = \int_0^\infty d\alpha \exp\left[-\alpha(k^2+m^2)\right].$$

we may develop quantum gravity on non-commutative space-time. In the technical side, we can also change the treatment with star product into the treatment with matrix.[9] When it is difficult to calculate by using star product in non-commutative field theories, we often use a matrix model instead.

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where

$$\psi(m+1) = 1 + \frac{1}{2} + \cdots + \frac{1}{m} - \gamma_E.$$

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—— *Derivation of (3·5)* ——

We start from the following equation.

$$\Gamma(p,q) = g^3 \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2+m^2} \frac{1}{(k-q)^2+m^2} \frac{1}{(k-p)^2+m^2}$$

At this point, we introduce Schwinger parameter as follows.

$$\frac{1}{k^2+m^2} = \int_0^\infty d\alpha \exp\big[-\alpha(k^2+m^2)\big].$$

$$\Gamma(p,q) = g^3 \int \frac{d^dk}{(2\pi)^d} \int_0^\infty d\alpha\, d\beta\, d\gamma \exp\Big[-\alpha(k^2+m^2) - \beta\big((k-q)^2+m^2\big) - \gamma\big((k-p)^2+m^2\big)\Big].$$

In the exponent, we complete the square in terms of k and integrate over k.

$$\Gamma(p,q) = \frac{g^3}{(4\pi)^{d/2}} \int_0^\infty d\alpha\, d\beta\, d\gamma \frac{1}{(\alpha+\beta+\gamma)^{d/2}} \cdot \exp\left[-(\alpha+\beta+\gamma)m^2 + \frac{2\beta\gamma p\cdot q - \beta(\alpha+\gamma)q^2 - \gamma(\alpha+\beta)p^2}{\alpha+\beta+\gamma}\right].$$

Hereafter, first we put $\alpha+\beta+\gamma = t$ and change integral variables from α, β, γ to t, β, γ. Secondly we transform in terms of scale $\beta \to t\beta, \gamma \to t\gamma$. Thirdly we transform in terms of scale $\gamma \to (1-\beta)\gamma$. Then we can obtain as follows.

$$\Gamma(p,q) = \frac{g^3}{(4\pi)^{d/2}} \int_0^1 d\beta (1-\beta) \int_0^1 d\gamma \int_0^\infty dt \frac{1}{t^{d/2-2}} \exp\big(-t\beta(1-\beta)\Delta\big),$$

where

$$\Delta = \frac{m^2}{\beta(1-\beta)} + q^2 + \frac{\gamma}{\beta}\big(1-(1-\beta)\gamma\big)p^2 - 2\gamma p\cdot q.$$

Putting $t\beta(1-\beta) = \alpha$, we can obtain as follows.

$$\Gamma(p,q) = \int_0^1 d\beta\, \beta^{\frac{d}{2}-3}(1-\beta)^{\frac{d}{2}-2} \int_0^1 d\gamma \int_0^\infty \frac{1}{\alpha^{d/2-2}} \exp(-\alpha\Delta)$$

At this point, transforming $\beta \to \alpha, \gamma \to \beta, \alpha \to t$, we can obtain (3·5).

Appendix C

—— Derivation from (3·9) to (3·11) ——

We start from (3·9) as follows.

$$\Pi_{NP1}(p^2) = \frac{g^4}{(4\pi)^d}\left[\frac{4}{(\theta p)^2}\right]^{d-5} \int_0^1 d\alpha\, \alpha^{\frac{d}{2}-3}(1-\alpha)^{\frac{d}{2}-2} \int_0^1 d\beta \int_0^1 d\zeta\, \zeta^{2-\frac{d}{2}}(1-\zeta) \cdot \int_0^1 d\eta \int_0^\infty dt\, t^{d-6} \exp\Big(-t - \frac{\tilde{\Delta}}{4t}p^2(\theta p)^2\Big),$$

where

$$\tilde{\Delta} = \frac{m^2}{p^2}\left[\frac{\zeta}{\alpha(1-\alpha)} + 1 - \zeta\right] + \frac{1}{\alpha}\Big[\zeta\,\beta(1-\beta) + \alpha(1-\zeta)\Big(\zeta(\beta-\eta)^2 + \eta(1-\eta)\Big)\Big].$$

At the lower end 0 of ζ-integral, there is the UV divergence of the three- point function at one loop in the limit $d \to 6$. Then, in the limit $d \to 6$, the divergent part and the finite part separate as follows. The divergent part is

$$\frac{g^4}{(4\pi)^d}\left[\frac{4}{(\theta p)^2}\right]^{d-5}\int_0^1 d\alpha\,\alpha^{\frac{d}{2}-3}(1-\alpha)^{\frac{d}{2}-2}\int_0^1 d\beta\int_0^1 d\zeta\,\zeta^{2-\frac{d}{2}}$$
$$\cdot\int_0^1 d\eta\int_0^\infty dt\,t^{d-6}\exp\left[-t-\frac{\frac{m^2}{p^2}+\eta(1-\eta)}{4t}p^2(\theta p)^2\right]. \qquad \text{(C·1)}$$

The finite part is

$$\frac{g_r^4}{(4\pi)^6}\frac{4}{(\theta p)^2}\int_0^1 d\alpha\,(1-\alpha)\int_0^1 d\beta\int_0^1 d\zeta\,\frac{1}{\zeta}\,(1-\zeta)\int_0^1 d\eta\int_0^\infty dt\,\exp\left[-t-\frac{\tilde{\Delta}}{4t}p^2\,(\theta p)^2\right]$$
$$-\frac{g_r^4}{(4\pi)^6}\frac{4}{(\theta p)^2}\int_0^1 d\alpha\,(1-\alpha)\int_0^1 d\beta\int_0^1 d\zeta\,\frac{1}{\zeta}\int_0^1 d\eta\int_0^\infty dt\,\exp\left[-t-\frac{\frac{m^2}{p^2}+\eta(1-\eta)}{4t}p^2(\theta p)^2\right]. \qquad \text{(C·2)}$$

Putting $d=6-\varepsilon$, (C·1) is expanded in terms of ε as follows.

$$\frac{g_r^4}{(4\pi)^6}\frac{4}{(\theta p)^2}\left(\pi\mu^2(\theta p)^2\right)^\varepsilon\int_0^1 d\alpha\,\alpha^{-\frac{\varepsilon}{2}}(1-\alpha)^{1-\frac{\varepsilon}{2}}\int_0^1 d\beta\int_0^1 d\zeta\,\zeta^{\frac{\varepsilon}{2}-1}$$
$$\cdot\int_0^1 d\eta\int_0^\infty dt\,t^{-\varepsilon}\exp\left[-t-\frac{\frac{m^2}{p^2}+\eta(1-\eta)}{4t}p^2(\theta p)^2\right],$$
$$\approx\frac{g_r^4}{(4\pi)^6}\frac{4}{(\theta p)^2}\left(\pi\mu^2(\theta p)^2\right)^\varepsilon\frac{\Gamma(1-\frac{\varepsilon}{2})\Gamma(2-\frac{\varepsilon}{2})}{\Gamma(3-\varepsilon)}\cdot\frac{2}{\varepsilon}\int_0^1 d\eta\int_0^\infty dt\,(1-\varepsilon\ln t)$$
$$\cdot\exp\left[-t-\frac{\frac{m^2}{p^2}+\eta(1-\eta)}{4t}p^2(\theta p)^2\right],$$
$$\approx\frac{g_r^4}{(4\pi)^6}\frac{4}{(\theta p)^2}\int_0^1 d\eta\int_0^\infty dt\left(\frac{1}{\varepsilon}+\ln\left(\pi\mu^2(\theta p)^2\right)+1-\ln t\right)$$
$$\cdot\exp\left[-t-\frac{\frac{m^2}{p^2}+\eta(1-\eta)}{4t}p^2(\theta p)^2\right]. \qquad \text{(C·3)}$$

And (C·2) is expanded as follows.

$$-\frac{g_r^4}{(4\pi)^6}\frac{4}{(\theta p)^2}\int_0^1 d\alpha(1-\alpha)\int_0^1 d\beta\int_0^1 d\zeta\int_0^1 d\eta\int_0^\infty dt\,\exp\left[-t-\frac{\tilde{\Delta}}{4t}p^2\,(\theta p)^2\right]$$
$$+\frac{g_r^4}{(4\pi)^6}\frac{4}{(\theta p)^2}\int_0^1 d\alpha(1-\alpha)\int_0^1 d\beta\int_0^1 d\zeta\,(\ln\zeta)'\int_0^1 d\eta\int_0^\infty dt$$
$$\cdot\left[\exp\left[-t-\frac{\tilde{\Delta}}{4t}p^2\,(\theta p)^2\right]-\exp\left[-t-\frac{\tilde{\Delta}(\zeta=0)}{4t}p^2(\theta p^2)\right]\right] \qquad \text{(C·4)}$$

The second term of (C·4) is integrated by parts in terms of ζ as follows.

$$-\frac{g_r^4}{(4\pi)^6}\frac{4}{(\theta p)^2}\int_0^1 d\alpha(1-\alpha)\int_0^1 d\beta\int_0^1 d\zeta\,\ln\zeta\int_0^1 d\eta\int_0^\infty dt\left(-\frac{p^2(\theta p)^2}{4t}\right)\frac{\partial\tilde{\Delta}}{\partial\zeta}$$

$$\cdot \exp\left[-t - \frac{\tilde{\Delta}}{4t} p^2 (\theta p)^2\right]$$

$$= \frac{g_r^4}{(4\pi)^6} p^2 \int_0^1 d\alpha \int_0^1 d\beta \int_0^1 d\zeta \int_0^1 d\eta \int_0^\infty dt \frac{F \ln \zeta}{\alpha t} \exp\left[-t - \frac{\tilde{\Delta}}{4t} p^2 (\theta p)^2\right], \qquad \text{(C·5)}$$

where

$$F = \frac{m^2}{p^2}\Big(1 - \alpha(1-\alpha)\Big) + (1-\alpha)\Big[\beta(1-\beta) + \alpha\Big((1-2\zeta)(\beta-\eta)^2 - \eta(1-\eta)\Big)\Big].$$

Consequently, adding (C·3), the first term of (C·4) and (C·5), we can obtain (3·11).

Appendix D

—— *Derivation of the upper bound (3·18)* ——

In this appendix we derive the upper bound (3·18) under the condition $\frac{m^2}{p^2} \ll 1$. Let us pay attention to $\tilde{\Delta}$ and F in eq. (3·15). Since

$$\tilde{\Delta} = \tilde{\Delta}_1 + \tilde{\Delta}_2 ,$$

$$\tilde{\Delta}_1 = \frac{m^2}{p^2}\left[\frac{\zeta}{\alpha(1-\alpha)} + 1 - \zeta\right] > \frac{m^2}{p^2}\frac{\zeta}{\alpha} ,$$

$$\tilde{\Delta}_2 = \frac{1}{\alpha}\Big[\zeta\,\beta(1-\beta) + \alpha(1-\zeta)\Big(\zeta(\beta-\eta)^2 + \eta(1-\eta)\Big)\Big] > \frac{\zeta}{\alpha}\beta(1-\beta) ,$$

we may put a lower bound on $\tilde{\Delta}$ as

$$\tilde{\Delta} > \frac{\zeta}{\alpha}\Big(\frac{m^2}{p^2} + \beta(1-\beta)\Big) . \qquad \text{(D·1)}$$

Similarly we decompose F as $F = F_1 + F_2 + F_3$, where

$$F_1 = \frac{m^2}{p^2}\Big(1 - \alpha(1-\alpha)\Big) ,$$

$$F_2 = (1-\alpha)\beta(1-\beta) ,$$

$$F_3 = \alpha(1-\alpha)\Big((1-2\zeta)(\beta-\eta)^2 - \eta(1-\eta)\Big) .$$

Since $|F_1| < 1$, $|F_2| < 1$, $|F_3| < 2$, we obtain $|F| < 4$. Thus we have

$$|\mathcal{D}| < 4\int_0^1 d\alpha \int_0^1 d\beta \int_0^1 d\zeta \int_0^1 d\eta \int_0^\infty dt \frac{|\ln \zeta|}{\alpha t} \exp\left[-t - \frac{\zeta}{4t\alpha}\Big(\frac{m^2}{p^2} + \beta(1-\beta)\Big)p^2(\theta p)^2\right] .$$

Next we change the integration variable ζ to $\lambda \equiv \frac{\zeta}{4t\alpha}\Big(\frac{m^2}{p^2} + \beta(1-\beta)\Big)p^2(\theta p)^2$, and send the upper end of the λ-integral to ∞, which yields

$$|\mathcal{D}| < \frac{16}{p^2(\theta p)^2}\int_0^1 d\beta \frac{1}{\frac{m^2}{p^2} + \beta(1-\beta)} \int_0^1 d\alpha \int_0^\infty dt \int_0^\infty d\lambda\, e^{-t-\lambda}$$

$$\cdot \left[|\ln t| + |\ln \alpha| + |\ln \lambda| + \left|\ln\Big(\frac{m^2}{p^2} + \beta(1-\beta)\Big)\right| + \left|\ln \frac{p^2(\theta p)^2}{4}\right|\right] .$$

Using the inequalities

$$\left|\ln\Big(\frac{m^2}{p^2}+\beta(1-\beta)\Big)\right| < \ln\frac{p^2}{m^2}\,,$$

$$\int_0^1 d\beta\,\frac{1}{\frac{m^2}{p^2}+\beta(1-\beta)} = \frac{1}{\sqrt{\frac{1}{4}+\frac{m^2}{p^2}}}\ln\frac{\Big(\sqrt{\frac{1}{4}+\frac{m^2}{p^2}}+\frac{1}{2}\Big)^2}{\frac{m^2}{p^2}} < 2\Big(1+\ln\frac{p^2}{m^2}\Big)\,,$$

$$\int_0^\infty dt\,|\ln t|\,e^{-t} \equiv c < \frac{3}{2}\,,$$

we arrive at the upper bound (3·18) on $|\mathcal{D}|$.

Appendix E

—— Derivation of a stringent upper bound on $|\varPi_{\rm NP2}(p^2)|$ ——

In this appendix we obtain an upper bound on $|\varPi_{\rm NP2}(p^2)|$, which is more stringent than eq. (3·28) and actually vanishes in the $p^2 \to \infty$ limit. Let us consider the integrand in the last line of (3·25). In the denominator we omit the $t^2\zeta$ term, and in the exponent we omit $\tilde{\varDelta}_{\rm NP}$ and the term $(1-\zeta)\Big(\zeta(\beta-\eta)^2+\eta(1-\eta)\Big)$ in $\tilde{\varDelta}$. Thus we obtain an upper bound

$$|\varPi_{\rm NP2}(p^2)| < \frac{g^4}{(4\pi)^d}\Big(\frac{4}{\theta^2}\Big)^{\frac{d}{2}}\frac{1}{(m^2)^5}\,G\Big(\frac{m^2}{p^2}\Big)\,, \tag{E·1}$$

where the function $G(x)$ is defined by

$$\begin{aligned} G(x) &= x^5\int_0^1 d\beta\int_0^1 d\alpha\int_0^1 d\zeta\,\frac{24\,\alpha^2(1-\alpha)^3\zeta^2(1-\zeta)}{\Big[x\big(\zeta+\alpha(1-\alpha)(1-\zeta)\big)+\zeta(1-\alpha)\beta(1-\beta)\Big]^5} \\ &= \frac{\frac{1}{2}x^2}{\big(\frac{1}{4}+x\big)^{\frac{3}{2}}}\left(\ln\frac{\sqrt{\frac{1}{4}+x}+\frac{1}{2}}{\sqrt{\frac{1}{4}+x}-\frac{1}{2}}+\frac{1}{x}\sqrt{\frac{1}{4}+x}\right)\,. \end{aligned} \tag{E·2}$$

Since $\lim_{x\to 0} G(x) = 0$, eq. (E·1) confirms explicitly that $\varPi_{\rm NP2}(p^2)$ vanishes in the $p^2 \to \infty$ limit.

Appendix F

—— Derivation from (4·4) to (4·5) ——

First we have to prove a following equation.

$$\int d^dx\,(\overline{D}_\mu a_\rho)\,(\overline{D}_\nu a_\sigma) = \int d^dx\Big[\partial_\mu(a_\rho\overline{D}_\nu a_\sigma) - a_\rho\overline{D}_\mu\overline{D}_\nu a_\sigma\Big]. \tag{F·1}$$

We expand the left in this equation as follows.

$$\int d^dx(\overline{D}_\mu a_\rho)(\overline{D}_\nu a_\sigma) = \int d^dx(\partial_\mu a_\rho - i[\overline{A}_\mu, a_\rho])\overline{D}_\nu a_\sigma$$
$$= \int d^dx\Big[\partial_\mu(a_\rho\overline{D}_\nu a_\sigma) - a_\rho\partial_\mu\overline{D}_\nu a_\sigma - i[\overline{A}_\mu, a_\rho]\,\overline{D}_\nu a_\sigma\Big] \qquad \text{(F·2)}$$

We can rewrite the third term of the right in (F·2) as follows.

$$-i\int d^dx[\overline{A}_\mu, a_\rho]\,\overline{D}_\nu a_\sigma = -\,i\int d^dx\Big[-a_\rho[\overline{A}_\mu, \overline{D}_\nu a_\sigma] + a_\rho[\overline{A}_\mu, \overline{D}_\nu a_\sigma] + [\overline{A}_\mu, a_\rho]\,\overline{D}_\nu a_\sigma\Big]$$
$$= -\,i\int d^dx\Big[-a_\rho[\overline{A}_\mu, \overline{D}_\nu a_\sigma] - a_\rho \star \overline{D}_\nu a_\sigma \star \overline{A}_\mu + \overline{A}_\mu \star a_\rho \star \overline{D}_\nu a_\sigma\Big]$$
$$= -\,i\int d^dx\Big[-a_\rho[\overline{A}_\mu, \overline{D}_\nu a_\sigma] - a_\rho \star \overline{D}_\nu a_\sigma \star \overline{A}_\mu + a_\rho \star \overline{D}_\nu a_\sigma \star \overline{A}_\mu\Big]$$
$$= +\,i\int d^dx\, a_\rho[\overline{A}_\mu, \overline{D}_\nu a_\sigma] \qquad \text{(F·3)}$$

When we removed from the second line to the third line ,we used the nature (2·6) of $\star$-product in the third term. We have been able to prove the (F·1) by using (F·2)and (F·3). Similarly, we can prove a equation in terms of the ghost field as follows.

$$\int d^dx(\overline{D}^\mu\overline{c} \star \overline{D}_\mu c) = \int d^dx\Big[\partial^\mu(\overline{c} \star \overline{D}_\mu c) - \overline{c} \star \overline{D}_\mu\overline{D}^\mu c\Big]. \qquad \text{(F·4)}$$

Using (F·1) and (F·4) , we can expand (4·4) as follows.

$$S' = \int d^dx\Big[-\frac{1}{2e^2}\, a_\mu \star \Big(-\overline{D}^2 g^{\mu\nu} - [\overline{D}^\mu, \overline{D}^\nu]\Big) \star a_\nu - \frac{i}{e^2}\, a_\mu \star \overline{F}^{\mu\nu} \star a_\nu - \overline{c} \star \overline{D}^2 c\Big]. \qquad \text{(F·5)}$$

At this point , we have to prove a following equation .

$$\int d^dx\Big(a_\mu \star [\overline{D}^\mu, \overline{D}^\nu] \star a_\nu\Big) = -2i\int d^dx\Big(a_\mu \star \overline{F}^{\mu\nu} \star a_\nu\Big). \qquad \text{(F·6)}$$

We expand the left integrand in this equation as follows.

$$a_\mu \star [\overline{D}^\mu, \overline{D}^\nu] \star a_\nu = a_\mu \star \big(\overline{D}^\mu\overline{D}^\nu - (\mu \leftrightarrow \nu)\big) \star a_\nu \qquad \text{(F·7)}$$

The first term of the right in (F·7) is

$$a_\mu \star \overline{D}^\mu\overline{D}^\nu \star a_\nu = a_\mu \star \Big[\partial^\mu(\overline{D}^\nu a_\nu) - i[\overline{A}^\mu, \overline{D}^\nu a_\nu]\Big]$$
$$= a_\mu \star \partial^\mu\partial^\nu a_\nu - i\Big(a_\mu \star \partial^\mu[\overline{A}^\nu, a_\nu] + a_\mu \star [\overline{A}^\mu, \partial^\nu a_\nu]\Big)$$
$$- a_\mu \star \Big[\overline{A}^\mu, [\overline{A}^\nu, a_\nu]\Big] \qquad \text{(F·8)}$$

Furthermore, we expand the second , the third and the fourth term to look for the terms which cancel by some terms of the $(\mu \leftrightarrow \nu)$ term in (F·7). As the result , we can pick up some following terms which remain in (F·7).

$$a_\mu \star [\overline{D}^\mu, \overline{D}^\nu] \star a_\nu = -i\Big[a_\mu \star (\partial^\mu \overline{A}^\nu - \partial^\nu \overline{A}^\mu) \star a_\nu - a_\mu \star a_\nu \star (\partial^\mu \overline{A}^\nu - \partial^\nu \overline{A}^\mu)\Big] - \Big[a_\mu \star [\overline{A}^\mu, \overline{A}^\nu] \star a_\nu + a_\mu \star a_\nu \star [\overline{A}^\nu, \overline{A}^\mu]\Big].$$

Using the nature (2·6) of $\star$-product in the second and the fourth term of the right in this equation, we can obtain as follows.

$$\begin{aligned}
&\int d^dx\Big(a_\mu \star [\overline{D}^\mu, \overline{D}^\nu] \star a_\nu\Big)\\
&= -\int d^dx\Big[i\Big(a_\mu \star (\partial^\mu \overline{A}^\nu - \partial^\nu \overline{A}^\mu) \star a_\nu - a_\nu \star (\partial^\mu \overline{A}^\nu - \partial^\nu \overline{A}^\mu) \star a_\mu\Big)\\
&\quad + \Big(a_\mu \star [\overline{A}^\mu, \overline{A}^\nu] \star a_\nu + a_\nu \star [\overline{A}^\nu, \overline{A}^\mu] \star a_\mu\Big)\Big]\\
&= -2i\int d^dx\Big[a_\mu \star \Big(\partial^\mu \overline{A}^\nu - \partial^\nu \overline{A}^\mu - i[\overline{A}^\mu, \overline{A}^\nu]\Big) \star a_\nu\Big]\\
&= -2i\int d^dx\Big(a_\mu \star \overline{F}^{\mu\nu} \star a_\nu\Big).
\end{aligned}$$

Since we have been able to prove the (F·6), substituting (F·6) for (F·5),we can obtain a following equation.

$$S' = \int d^dx\Big[-\frac{1}{2e^2} a_\mu \star (-\overline{D}^2 g^{\mu\nu} + 4i\overline{F}^{\mu\nu}) \star a_\nu - \bar{c} \star \overline{D}^2 c\Big].$$

At last, using $(\mathcal{J}^{\rho\sigma})^{\mu\nu} = i(g^{\rho\mu}g^{\sigma\nu} - g^{\sigma\mu}g^{\rho\nu})$ for $\overline{F}^{\mu\nu}$ in this equation,we can obtain (4·5) immediately.

Appendix G

——— *Derivation of (4·13), (4·14) and (4·15)* ———

First we calculate $\mathrm{Tr}\,(-\partial^2)^{-1}\Delta^{(2)}$ in detail.

$$\begin{aligned}
&\mathrm{Tr}\,(-\partial^2)^{-1}\Delta^{(2)}\\
&= \mathrm{tr}\int d^dx \int \frac{d^dk}{(2\pi)^d}\, e^{-ik\cdot x}(-\partial^2)^{-1}[\overline{A}^\mu, [\overline{A}_\mu, e^{ik\cdot x}]],
\end{aligned}$$

where the tr is taken over spin indices. In the case of gauge field, we may put tr $= d$. (In the case of ghost field , we put tr $= 1$.)

$$= d\int d^dx \int \frac{d^dk}{(2\pi)^d}\frac{d^dl_1}{(2\pi)^d}\frac{d^dl_2}{(2\pi)^d}\,\overline{A}^\mu(l_1)\,\overline{A}_\mu(l_2)\, e^{-ik\cdot x}(-\partial^2)^{-1}[e^{il_1\cdot x}, [e^{il_2\cdot x}, e^{ik\cdot x}]].$$

Using $[\,e^{il_2\cdot x}, e^{ik\cdot x}\,] = e^{i(l_2+k)\cdot x}\big(e^{-\frac{i}{2}l_2\times k} - e^{-\frac{i}{2}k\times l_2}\big)$, the above right in this equation is

$$
\begin{aligned}
=& d\int d^dx \int \frac{d^dk}{(2\pi)^d}\frac{d^dl_1}{(2\pi)^d}\frac{d^dl_2}{(2\pi)^d}\,\overline{A}^{\mu}(l_1)\,\overline{A}_{\mu}(l_2)\,\frac{1}{(l_1+l_2+k)^2}\,e^{i(l_1+l_2)\cdot x} \\
&\qquad \cdot \Big(e^{-\frac{i}{2}l_1\times(l_2+k)} - e^{-\frac{i}{2}(l_2+k)\times l_1}\Big)\big(e^{-\frac{i}{2}l_2\times k} - e^{-\frac{i}{2}k\times l_2}\big), \\
=& d\int \frac{d^dk}{(2\pi)^d}\frac{d^dl_2}{(2\pi)^d}\,\overline{A}^{\mu}(-l_2)\,\overline{A}_{\mu}(l_2)\,\frac{1}{k^2}\big(e^{-\frac{i}{2}k\times l_2} - e^{-\frac{i}{2}l_2\times k}\big)\big(e^{-\frac{i}{2}l_2\times k} - e^{-\frac{i}{2}k\times l_2}\big), \\
=& d\int \frac{d^dk}{(2\pi)^d}\,\overline{A}^{\mu}(k)\,\overline{A}_{\mu}(-k)\int\frac{d^dp}{(2\pi)^d}\frac{1}{p^2}\big(2 - e^{-ik\times p} - e^{-ip\times k}\big).
\end{aligned} \tag{G·1}
$$

Similarly, we can calculate $-\frac{1}{2}\mathrm{Tr}\,(-\partial^2)^{-1}\Delta^{(1)}(-\partial^2)^{-1}\Delta^{(1)}$ as follows.

$$
\begin{aligned}
&-\frac{1}{2}\mathrm{Tr}\,(-\partial^2)^{-1}\Delta^{(1)}(-\partial^2)^{-1}\Delta^{(1)}, \\
=&-\frac{d}{2}\int \frac{d^dk}{(2\pi)^d}\overline{A}^{\mu}(k)\overline{A}_{\mu}(-k)\int\frac{d^dp}{(2\pi)^d}\frac{(2p+k)^{\mu}(2p+k)^{\nu}}{p^2(p+k)^2}\big(2 - e^{-ip\times k} - e^{-ik\times p}\big).
\end{aligned} \tag{G·2}
$$

Next we have to explain the calculation of $-\frac{1}{2}\mathrm{Tr}\,(-\partial^2)^{-1}\Delta^{(\mathcal{J})}(-\partial^2)^{-1}\Delta^{(\mathcal{J})}$ in detail.

$$
\begin{aligned}
&-\frac{1}{2}\mathrm{Tr}\,(-\partial^2)^{-1}\Delta^{(\mathcal{J})}(-\partial^2)^{-1}\Delta^{(\mathcal{J})}, \\
=&\frac{1}{2}\mathrm{tr}\int d^dx\int\frac{d^dk}{(2\pi)^d}\,e^{-ik\cdot x}\star(-\partial^2)^{-1}2\overline{F}_{\rho\sigma}\mathcal{J}^{\rho\sigma}\star(-\partial^2)^{-1}2\overline{F}_{\mu\nu}\mathcal{J}^{\mu\nu}\star e^{ik\cdot x}, \\
=&-8\,\mathrm{tr}\,(\mathcal{J}^{\rho\sigma}\mathcal{J}^{\mu\nu})\int d^dx\int\frac{d^dk}{(2\pi)^d}\,e^{-ik\cdot x}\star(-\partial^2)^{-1}(\partial_{\rho}\overline{A}_{\sigma})\star(-\partial^2)^{-1}(\partial_{\mu}\overline{A}_{\nu})\star e^{ik\cdot x}.
\end{aligned}
$$

From this point, we take the same way that we have calculated the above $\mathrm{Tr}\,(-\partial^2)^{-1}\Delta^{(2)}$. And using $\mathrm{tr}\,(\mathcal{J}^{\rho\sigma}\mathcal{J}^{\mu\nu}) = 2(\,g^{\rho\mu}g^{\sigma\nu} - g^{\sigma\mu}g^{\rho\nu})$, we can obtain as follows.

$$
=-16\int\frac{d^dk}{(2\pi)^d}\overline{A}_{\mu}(k)\overline{A}_{\nu}(-k)(g^{\mu\nu}k^2 - k^{\mu}k^{\nu})\int\frac{d^dp}{(2\pi)^d}\frac{1}{p^2(p+k)^2}. \tag{G·3}
$$

In this case , we notice that there are no nonplaner parts.

When we derive (4·13), (4·14) and (4·15) from (G·1), (G·2) and (G·3) , we use some following equations in d-dimentional Minkowski space-time.

$$
\begin{aligned}
&\int\frac{d^dp}{(2\pi)^d}\frac{1}{p^2} = 0\cdot\Gamma\Big(2-\frac{d}{2}\Big) + \cdots, \\
&\int\frac{d^dp}{(2\pi)^d}\frac{1}{p^2(p+k)^2} = \frac{i}{(4\pi)^2}\Gamma\Big(2-\frac{d}{2}\Big) + \cdots, \\
&\int\frac{d^dp}{(2\pi)^d}\frac{p^{\mu}}{p^2(p+k)^2} = -\frac{i}{(4\pi)^2}\frac{k^{\mu}}{2}\Gamma\Big(2-\frac{d}{2}\Big) + \cdots,
\end{aligned}
$$

$$\int \frac{d^dp}{(2\pi)^d}\frac{p^\mu p^\nu}{p^2(p+k)^2} = \frac{i}{(4\pi)^2}\frac{1}{12}(-g^{\mu\nu}k^2+4k^\mu k^\nu)\Gamma\Big(2-\frac{d}{2}\Big)+\cdots,$$
$$\int \frac{d^dp}{(2\pi)^d}\frac{1}{p^2}e^{ik\times p} = \frac{i}{(4\pi)^2}\frac{4}{(k\theta)^2}+\cdots,$$
$$\int \frac{d^dp}{(2\pi)^d}\frac{1}{p^2(p+k)^2}e^{ik\times p} = -\frac{i}{(4\pi)^2}\ln k^2(k\theta)^2+\cdots,$$
$$\int \frac{d^dp}{(2\pi)^d}\frac{p^\mu}{p^2(p+k)^2}e^{ik\times p} = -\frac{(k\theta)^\mu}{(4\pi)^2}\frac{2}{(k\theta)^2}+\cdots,$$
$$\int \frac{d^dp}{(2\pi)^d}\frac{p^\mu p^\nu}{p^2(p+k)^2}e^{ik\times p} = \frac{i}{(4\pi)^2}\frac{2g^{\mu\nu}}{(k\theta)^2}-\frac{i}{(4\pi)^2}(k\theta)_\mu(k\theta)_\nu\frac{4}{\big((k\theta)^2\big)^2}+\cdots.$$

Note that in planer part, we pick up singular terms in the limit $d\to 4$, and in nonplaner part, we pick up singular terms (infrared singularities) around $k\approx 0$ in 4-dimention.

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