Random Collision Model Represented by Random Time Change of Poisson Process

Hajime Mano

Doctor of Philosophy

Department of Statistical Science
School of Mathematical and Physical Science

1994

Contents

ABSTRACT.

Random collision models which are represented by random time changes of standard Poisson processes are studied in this paper. A strong law of large numbers for a model of a one-sided prey-predator relation between two species is discussed. Moreover, a random collision model which has a cyclic prey-predator relation is presented. In our model each component of the stochastic process is decomposed into a counting process of the number arriving over time t and a counting process of the number serviced by time t, and the number increasing over time t of the i-th component is equal to the number decreasing by time t of the $i+1$-th component with periodic boundary condition. A stochastic structure of the model is investigated. An ordinary differential equation from a weak law of large numbers and a stochastic differential equation of the Gaussian diffusion process from a central limit theorem are obtained in our model. In addition to this, by using a simulation method the behavior of a simple model where mutations are represented by random time changes containing the frequency of one species (that is the allelic state) is investigated.

1. Introduction

Problems of interspecific competitions have been studied by many authors. Lotka [22] and Volterra [35] studied interacting populations as deterministic systems. The larger populations are implicitly assumed for the deterministic systems. For smaller populations it is important to deal with the systems as the probabilistic systems. Ehrenfest's urn model was discussed by Kac [15] and Moran [27] studied an urn model for the random genetic drift introduced by Fisher [6] and Wright [37]. Itoh [11, 12] introduced a random collision model which is an urn model for competing species in finite numbers of individuals of several types interacting with each other and studied the probability of coexistence of species. The discrete systems of interacting populations are approximated by the Fokker-Planck equation. The forward Kolmogorov equation, which is the Fokker-Planck equation, is characterized by the drift coefficient for the change of the mean of the frequency and by the diffusion coefficient for the change of the variance of the frequency.

In this paper we study a system of an interacting population. An ordinary differential equation from a law of large numbers and a stochastic differential equation from a central limit theorem are derived in chapters I and II. In the course we use the method of martingales which are linked in a natural way to the concept of increasing information pattern describing the history of a stochastic process. The optional sampling theorem due to Doob [3] is a powerful tool for our analysis.

We analyze the random collision model introduced by Itoh [13, 14] which satisfies the following conditions:

(i) There are three species 1, 2 and 3 whose numbers of particles at time t are $X_1^{(M)}(t)$, $X_2^{(M)}(t)$ and $X_3^{(M)}(t)$ respectively, where $X_1^{(M)}(t)+X_2^{(M)}(t)+X_3^{(M)}(t) = M$, when M is a positive integer. We denote $X^{(M)}(t) = (X_1^{(M)}(t), X_2^{(M)}(t), X_3^{(M)}(t))$.

(ii) Each particle collides with another particle dt times on the average per time interval dt.

(iii) Each particle is in a chaotic bath of particles. Each colliding pair is equally likely chosen.

(iv) Collisions between particles of the same species do not make any change. A particle of species i and a particle of species $i+1$ collide with each other and become two particles of species i, where $i = 1, 2, 3$ and if $i = 3$ then we set $i+1 = 1$ and if $i = 1$ then we set $i-1 = 3$ from now on.

We assume the following model, represented by random time changes of three standard Poisson processes, which satisfies the above conditions:

$$\begin{cases} X_1^{(M)}(t) = X_1^{(M)}(0) + N_{12}(\frac{\lambda}{M}\int_0^t X_1^{(M)}(s)X_2^{(M)}(s)ds) \\ \qquad\qquad - N_{31}(\frac{\lambda}{M}\int_0^t X_3^{(M)}(s)X_1^{(M)}(s)ds), \\ X_2^{(M)}(t) = X_2^{(M)}(0) + N_{23}(\frac{\lambda}{M}\int_0^t X_2^{(M)}(s)X_3^{(M)}(s)ds) \\ \qquad\qquad - N_{12}(\frac{\lambda}{M}\int_0^t X_1^{(M)}(s)X_2^{(M)}(s)ds), \\ X_3^{(M)}(t) = X_3^{(M)}(0) + N_{31}(\frac{\lambda}{M}\int_0^t X_3^{(M)}(s)X_1^{(M)}(s)ds) \\ \qquad\qquad - N_{23}(\frac{\lambda}{M}\int_0^t X_2^{(M)}(s)X_3^{(M)}(s)ds), \\ X_1^{(M)}(0) + X_2^{(M)}(0) + X_3^{(M)}(0) = M, \end{cases}$$

where $X_i^{(M)}(0)$ are nonnegative initial values ($i = 1, 2, 3$) and where λ is a positive constant. The standard Poisson processes $N_{jj+1}(*)$ $(1 \le j \le 3)$ are assumed to be mutually independent, when the stochastic structure of the model is considered. We call this model paper-scissors-stone model because of the cyclic prey-predator relation, as in paper-scissors-stone game. Note that the model is an example of the system which has a cyclic prey-predator relation.

There is a large body of literature concerning random time changes. Volkonskii [34] and Helms [8] discussed the strong Markov property of general models represented by random time changes (the latter is the version of the former in multidimensional view point). For the multiparameter strong Markov model, Helms [8] showed that the model in which parameters are substituted by random time changes is strong Markov. Thus the paper-scissors-stone model is strong Markov because of the strong Markov property of standard Poisson processes. The optional sampling theorem for martingales indexed by directed sets is derived for the purpose of analyzing models represented by random time changes by Kurtz [17]. A strong law of large numbers and a central limit theorem are obtained by Kurtz [19] for general random time substituted models including the present model. A diffusion approximation indicated in Itoh [14] is obtained by Kurtz [18, 19] and Ethier and Kurtz [4].

In chapter I we study a random collision model of two species which is a special case of the paper-scissors-stone model ([31]). For the case where $X_3^{(M)}(0) = 0$, there is one-sided prey-predator relation which is represented by one random time change of one standard Poisson process. We solve the model explicitly in section 1. This enables us to prove a strong law of large numbers for this model. In section 2 the convergence of the strong law of large numbers is shown by using the explicit solution.

In chapter II the paper-scissors-stone model is studied ([32]). The cyclic prey-predator relation in the model complicates the situation. Itoh [13] studied the random collision models which involve the paper-scissors-stone model and obtained an asymptotic result for coexistence of species. Motivated by the martingale method, we analyze the paper-scissors-stone model and investigate limit theorems. Before we mention about this, we present the leading case.

A queuing model of computer networks by Kogan, Liptser and Smorodinski [16] and Liptser and Shiryayev [21] is successfully analyzed by a martingale method. Their discussion is based on the assumption that a stochastic structure has orthogonal martingales in multi-dimensional queues. In their model, queues are assumed to be of two types. Servicing of a request in queues of the first type is by the rule "the first order is served first". Each request that arrives at a queue of the second type begins to service immediately with constant intensity, that is same for all requests. There are $n+1$ queues. The 0-th queue is assumed to be the second type queue and other queues are assumed to be the first type. The martingale method was applied to the n-dimensional stochastic process of queues where the 0-th queue is removed. Each component is decomposed into a counting process of the number of requests arriving over time t and a counting process of the number of requests serviced by time t. Moreover it is assumed that there are no two jumps at the same time. The Doob-Meyer decomposition [25] of the process is such that martingales are orthogonal and bounded variations are continuous. They obtained an ordinary differential equation from a weak law of large numbers and a stochastic differential equation of the Gaussian diffusion process from a central limit theorem.

In chapter II of this paper we aim for the paper-scissors-stone model to obtain an ordinary differential equation from a weak law of large numbers and a stochastic differential equation of the Gaussian diffusion process from a central limit theorem ([32]). We solve the paper-scissors-stone model explicitly in section 1. A reference family and a stopping time are found to apply the optional sampling theorem by Doob [3], and a stochastic structure of our model is obtained in section 2 ([32]). Martingales in different components are not always orthogonal in our case. In section 3 and section 5, we refine the weak law of large numbers and the central limit theorem in [16, 21] to more generalized form in order to apply to our model. We obtain an ordinary differential equation from a weak law of large numbers in section 4 and a stochastic differential equation of the Gaussian diffusion process from a central limit theorem in section 6, starting from [32]. In section 8 for the paper-scissors-stone model the actual behavior of the weak law of large numbers and the central limit theorem is shown by using computer simulation ([23]).

In Chapter III we compare the following two simple models and discuss the simulation study ([24]).

Ohta and Kimura [29, 30] formulated a model of allelic mutation in which mutational changes are represented by stepwise movements in order to estimate electrophoretically detectable alleles in finite populations. If an amino acid is substituted in the molecule, the band observed by the electrophoresis usually varies with one unit in the positive or negative direction. Then the model explains how gametes vary at amino acid sites. The model is expressed by a Markov chain as will be discussed in section 1. Mutational changes occur by constant probability from sites of allelic states to the neighboring sites with no selections. This model is called "ladder" or "stepwise-mutation" model. Moran [27] studied the model theoretically. A modern version of the Ohta-Kimura model is discussed in Fleming and Viot [7] and Dawson and Hochberg [2].

In section 2 we discuss a continuous time model in which mutational changes are represented by random time changes of Poisson processes. These random time changes contain the frequency of one species (that is the allelic state). For simplicity we call this new model a time-change model in this paper.

As pointed out above, the time-change model has the strong Markov property. It is difficult to investigate explicitly the generator of the time-change model (Lamperti [20]). Motivated by this, we compare the time-change model with the stepwise-mutation model.

In section 3 we give a statistical method whether a new model corresponds to a known model or not. In this discussion we calculate the maximum likelihood estimator of the known model which is the stepwise-mutation model. The maximum likelihood estimation is reasonable, since the logarithmic likelihood is the estimator of the Kullback-Leibler information. By generating data for the time-change model through computer simulation, we directly fit the data to the stepwise-mutation model and compare the two models.

In section 4 we give a simulation study by using the statistical method in section 3. We make sure the correspondence between the time-change model and the stepwise-mutation model through computer simulation. We conclude that the two models well corresponds in a statistical sense.

The mathematical results are summarized as follows.

By assuming conditions given in Chapter II section 4, for any $t \in [0,\infty)$ we have a weak law of large numbers which shows a convergence of $X^{(M)}(t)/M$ to the solution $u(t) = (u_1(t), u_2(t), u_3(t))$ of the deterministic system expressed by the differential equation

$$\begin{cases} \dfrac{du_1(t)}{dt} = \lambda(u_1(t)u_2(t) - u_3(t)u_1(t)), \\ \dfrac{du_2(t)}{dt} = \lambda(u_2(t)u_3(t) - u_1(t)u_2(t)), \\ \dfrac{du_3(t)}{dt} = \lambda(u_3(t)u_1(t) - u_2(t)u_3(t)). \end{cases}$$

Put

$$Y^{(M)}(t) = \sqrt{M}(\frac{X^{(M)}(t)}{M} - u(t)).$$

By assuming conditions given in Chapter II section 6, we have a central limit theorem which shows a weak convergence of the sequence of the probability distributions of the $\mathbb{R}^3$-valued processes $Y^{(M)} = (Y^{(M)}(t))_{t\geq 0}$ to the distribution of an $\mathbb{R}^3$-valued Gaussian diffusion process $Y = (Y(t))_{t\geq 0}$ defined by the stochastic differential equation

$$dY(t) = b(t)Y(t)dt + c^{\frac{1}{2}}(t)dW(t),$$

with an $\mathbb{R}^3$-valued Wiener process $W = (W_t)_{t\geq 0}$ and with 3×3 matrix

$$b(t) = \begin{pmatrix} \lambda(u_2(t) - u_3(t)) & \lambda u_1(t) & -\lambda u_1(t) \\ -\lambda u_2(t) & \lambda(u_3(t) - u_1(t)) & \lambda u_2(t) \\ \lambda u_3(t) & -\lambda u_3(t) & \lambda(u_1(t) - u_2(t)) \end{pmatrix},$$

$$c(t) =$$

$$\begin{pmatrix} \lambda(u_1(t)u_2(t) + u_3(t)u_1(t)) & -\lambda u_1(t)u_2(t) & -\lambda u_3(t)u_1(t) \\ -\lambda u_1(t)u_2(t) & \lambda(u_2(t)u_3(t) + u_1(t)u_2(t)) & -\lambda u_2(t)u_3(t) \\ -\lambda u_3(t)u_1(t) & -\lambda u_2(t)u_3(t) & \lambda(u_3(t)u_1(t) + u_2(t)u_3(t)) \end{pmatrix}.$$

CHAPTER I

Random collision model of two species

1. Random collision model of two species and its solution

Let us consider a population of two types of individuals in which individuals randomly interact with each other. Changes occur by interactions only between particles of different types. If two individuals of different types interact, then two individuals of the dominant type result from the interaction. Hence the total number of the particles is invariant under interactions.

We set any positive integer M which denotes the total number of the particles. For each j, $j = 1, 2$, let $X_j^{(M)}(*)$ be a stochastic process which denotes the number of individual of type j. We assume that $X_1^{(M)}(*)$ is dominant and that each of the individuals is represented by the time change of a standard Poisson process $N(*)$ in a differential form as

$$(1.1) \qquad \begin{cases} dX_1^{(M)}(t) = dN(\frac{\lambda}{M}\int_0^t X_1^{(M)}(s)X_2^{(M)}(s)ds), \\ dX_2^{(M)}(t) = -\,dN(\frac{\lambda}{M}\int_0^t X_1^{(M)}(s)X_2^{(M)}(s)ds), \end{cases}$$

where λ is a positive constant. This is also written in the integral form as

$$(1.2) \qquad \begin{cases} X_1^{(M)}(t) = X_1^{(M)}(0) + N(\frac{\lambda}{M}\int_0^t X_1^{(M)}(s)X_2^{(M)}(s)ds), \\ X_2^{(M)}(t) = X_2^{(M)}(0) - N(\frac{\lambda}{M}\int_0^t X_1^{(M)}(s)X_2^{(M)}(s)ds), \\ X_1^{(M)}(0) + X_2^{(M)}(0) = M, \end{cases}$$

where $X_j^{(M)}(0)$ are nonnegative initial values of $X_j^{(M)}(*)$ $(j = 1, 2\ )$.

Now we shall prove the existence and uniqueness of the solution of equation (1.2). We denote by $\{\tau_i\}_{i\geq 0}$ the set of the jump times of the standard Poisson process $N(*)$ $(\tau_0 = 0)$.

THEOREM 1.1. *There exists a unique solution of equation (1.2) and it is represented in the form*

$$X_1^{(M)}(t) = X_1^{(M)}(0) - 1 + \sum_{i=0}^{M-X_1^{(M)}(0)} \chi_{[\sigma_i^{(M)},\infty)}(t), \tag{1.3}$$

$$X_2^{(M)}(t) = X_2^{(M)}(0) + 1 - \sum_{i=0}^{M-X_1^{(M)}(0)} \chi_{[\sigma_i^{(M)},\infty)}(t), \tag{1.4}$$

where $\sigma_k^{(M)}$ $(0 \leq k \leq M)$ *are defined by*

$$\begin{cases} \sigma_0^{(M)} =0, \\ \sigma_k^{(M)} =\infty \ \ for \ \ 1 \leq k \leq M, \ \ X_1^{(M)}(0) = 0 \ \ or \ \ M, \\ \sigma_k^{(M)} =\displaystyle\sum_{i=1}^{k} \frac{\tau_i - \tau_{i-1}}{\lambda(X_1^{(M)}(0)+i-1)(1-(X_1^{(M)}(0)+i-1)/M)} \\ \qquad\qquad for \ \ 1 \leq k \leq M - X_1^{(M)}(0), \ \ X_1^{(M)}(0) \neq 0, M, \\ \sigma_k^{(M)} =\infty \ \ \ for \ \ k \geq M - X_1^{(M)}(0) + 1 \ , \ X_1^{(M)}(0) \neq 0, M. \end{cases} \tag{1.5}$$

PROOF. Let $X_j^{(M)}(*)$ $(j = 1, 2)$ be the solution of equation (1.2). For each fixed $t \in \mathbb{R}_+ = [0, \infty)$, we define

$$T^{(M)}(t) = \frac{\lambda}{M} \int_0^t X_1^{(M)}(s) X_2^{(M)}(s) ds. \tag{1.6}$$

Every time when the function $T^{(M)}(*)$ comes to the jump time τ_k of the standard Poisson process $N(*)$, the stochastic process $X_1^{(M)}(*)$ increases in the width of one. We define $\sigma_k^{(M)}$ by

$$\begin{aligned} \sigma_0^{(M)} &= 0, \\ \sigma_k^{(M)} &= \inf\{t \geq 0; T^{(M)}(t) = \tau_k\} \quad (1 \leq k \leq M). \end{aligned} \tag{1.7}$$

When $X_1^{(M)}(0) = 0$ or M, we see that $T^{(M)}(t) = 0$, and so, $X_1^{(M)}(t) = X_1^{(M)}(0)$. It is clear that (1.3), (1.4) and (1.5) hold.

When $X_1^{(M)}(0) \neq 0, M$, if $\sigma_{k-1}^{(M)} \leq t < \sigma_k^{(M)}$ for $1 \leq k \leq M - X_1^{(M)}(0)$, and so, $\tau_{k-1} \leq T^{(M)}(t) < \tau_k$, then

$$X_1^{(M)}(t) = X_1^{(M)}(0) + N(T^{(M)}(t)) = X_1^{(M)}(0) + k - 1.$$

Hence

$$\begin{aligned}
X_1^{(M)}(t) &= X_1^{(M)}(0) - 1 + \sum_{i=0}^{k-1} 1 \\
&= X_1^{(M)}(0) - 1 + \sum_{i=0}^{M} \chi_{[\sigma_i^{(M)},\infty)}(t).
\end{aligned}$$

If $t \geq \sigma^{(M)}_{M-X_1^{(M)}(0)}$, then $T^{(M)}(t) = \tau^{(M)}_{M-X_1^{(M)}(0)}$, and so,

$$\begin{aligned}
X_1^{(M)}(t) &= X_1^{(M)}(0) + N(T^{(M)}(t)) \\
&= X_1^{(M)}(0) - 1 + \sum_{i=0}^{M-X_1^{(M)}(0)} 1 \\
&= X_1^{(M)}(0) - 1 + \sum_{i=0}^{M} \chi_{[\sigma_i^{(M)},\infty)}(t).
\end{aligned}$$

Therefore we see that (1.3) and (1.4) hold. It can be seen from (1.7) that the random times $\sigma_k^{(M)}$ satisfy a recursive relation (1.5).

Conversely, let $X_1^{(M)}(*)$ be a stochastic process defined by (1.3) and put $X_2^{(M)}(*) = M - X_1^{(M)}(*)$. It is easy to see that $X_j^{(M)}(*)$ are right continuous and have the limit from the left-hand side ($j = 1, 2$).

When the time t is involved in the interval $[\sigma_{k-1}^{(M)}, \sigma_k^{(M)})$ ($1 \leq k \leq M - X_1^{(M)}(0)$), we have the estimate:

$$\begin{aligned}
\tau_{k-1} \leq & T^{(M)}(t) \\
&= \frac{\lambda}{M} \sum_{i=1}^{k-1} \int_{\sigma_{i-1}^{(M)}}^{\sigma_i^{(M)}} X_1^{(M)}(s) X_2^{(M)}(s) ds + \frac{\lambda}{M} \int_{\sigma_{k-1}^{(M)}}^{t} X_1^{(M)}(s) X_2^{(M)}(s) ds \\
&= \sum_{i=1}^{k-1} \frac{\lambda}{M} (X_1^{(M)}(0) + (i-1))(M - X_1^{(M)}(0) - (i-1))(\sigma_i^{(M)} - \sigma_{i-1}^{(M)}) \\
&+ \frac{\lambda}{M} (X_1^{(M)}(0) + (k-1))(M - X_1^{(M)}(0) - (k-1))(t - \sigma_{k-1}^{(M)}) \\
&= \tau_{k-1} + \frac{\lambda}{M} (X_1^{(M)}(0) + (k-1))(M - X_1^{(M)}(0) - (k-1))(t - \sigma_{k-1}^{(M)}) \\
&< \tau_k.
\end{aligned}$$

Hence $N(T^{(M)}(t)) = k - 1$.

On the other hand, it follows from (1.3) that when the time t is involved in the interval $[\sigma_{k-1}^{(M)}, \sigma_k^{(M)})$, $X_1^{(M)}(t) = X_1^{(M)}(0) - 1 + k$, and so, $X_1^{(M)}(t) = X_1^{(M)}(0) + N(T^{(M)}(t))$.

When $t \geq \sigma_{M-X_1^{(M)}(0)}$, we find that

$$
\begin{aligned}
&T^{(M)}(t) \\
&= \frac{\lambda}{M} \sum_{i=1}^{M-X_1^{(M)}(0)} \int_{\sigma_{i-1}^{(M)}}^{\sigma_i^{(M)}} X_1^{(M)}(s) X_2^{(M)}(s) ds + \frac{\lambda}{M} \int_{\sigma^{(M)}_{M-X_1^{(M)}(0)}}^{t} X_1^{(M)}(s) X_2^{(M)}(s) ds \\
&= \sum_{i=1}^{M-X_1^{(M)}(0)} \frac{\lambda}{M} (X_1^{(M)}(0) + (i-1))(M - X_1^{(M)}(0) - (i-1))(\sigma_i^{(M)} - \sigma_{i-1}^{(M)}) \\
&= \tau_{M-X_1^{(M)}(0)}.
\end{aligned}
$$

Hence $X_1^{(M)}(t) = X_1^{(M)}(0) + (M - X_1^{(M)}(0)) = X_1^{(M)}(0) + N(T^{(M)}(t))$.

Consequently $X_j^{(M)}(*)$ $(j = 1, 2)$ satisfy (1.2). □

2. A strong law of large numbers

Let $u_1 = u_1(t)$ and $u_2 = u_2(t)$ $(t \in \mathbb{R}_+)$ be the solution of the deterministic system

$$
\left\{
\begin{aligned}
\frac{du_1(t)}{dt} &= \lambda u_1(t) u_2(t), \\
\frac{du_2(t)}{dt} &= -\lambda u_1(t) u_2(t).
\end{aligned}
\right. \tag{2.1}
$$

We show the weak convergence of $\frac{X_j^{(M)}(*)}{M}$ $(j = 1, 2)$ to the deterministic system in use of the martingale method in Chapter II.

As a strong law of large numbers, we shall show

THEOREM 2.1. *We assume*

$$
\left\{
\begin{aligned}
&\lim_{M \to \infty} \frac{X_1^{(M)}(0)}{M} = u_1(0) \quad a.s., \\
&0 < u_1(0) < 1 \quad \text{and} \quad u_1(0) + u_2(0) = 1.
\end{aligned}
\right.
$$

Then for any $t \in (0, \infty)$,

$$
\left\{
\begin{aligned}
&\lim_{M \to \infty} \frac{X_1^{(M)}(t)}{M} = u_1(t) \quad a.s., \\
&\lim_{M \to \infty} \frac{X_2^{(M)}(t)}{M} = u_2(t) \quad a.s.
\end{aligned}
\right.
$$

PROOF. We rewrite the solution in the integral form

$$
\frac{X_1^{(M)}(t)}{M} = \frac{X_1^{(M)}(0)}{M} + \int_0^{\frac{M - X_1^{(M)}(0)}{M}} \varphi_M(s) ds, \tag{2.2}
$$

where the function φ_M is defined by

$$(2.3)\quad \varphi_M(s) = \chi_{[\sigma_k^{(M)},\infty)}(t) \qquad for \qquad \frac{k-1}{M} \le s < \frac{k}{M}, \qquad 1 \le k \le M.$$

We fix any element $\omega \in \Omega$ and $s \in \mathbb{R}_+$ such that

$$(2.4)\qquad \lim_{M\to\infty} \frac{X_1^{(M)}(0)(\omega)}{M} = u_1(0),$$

$$(2.5)\qquad \lim_{M\to\infty} \frac{1}{M}\sum_{i=1}^{M}(\tau_i(\omega) - \tau_{i-1}(\omega)) = 1,$$

$$(2.6)\qquad \lim_{M\to\infty} \frac{1}{M}\sum_{i=1}^{M}(\tau_i(\omega) - \tau_{i-1}(\omega))^2 = 2,$$

$$(2.7)\qquad 0 \le s < 1 - u_1(0), \qquad 0 < u_1(0) < 1.$$

We note that the set of the element $\omega \in \Omega$ satisfying (2.4) - (2.6) has the probability one. In the sequal we shall abbreviate the variable ω.

[Step 1] We claim that

$$\frac{X_1^{(M)}(t)}{M} = \frac{X_1^{(M)}(0)}{M} + \int_0^{1-u_1(0)} \varphi_M(s)ds + o(1) \quad (M \to \infty).$$

By (2.2), we have

$$\frac{X_1^{(M)}(t)}{M} = \begin{cases} \dfrac{X_1^{(M)}(0)}{M} + \displaystyle\int_0^{1-u_1(0)} \varphi_M(s)ds + \int_{1-u_1(0)}^{\frac{M-X_1^{(M)}(0)}{M}} \varphi_M(s)ds \\ \qquad for\ \ 1-u_1(0) \le \dfrac{M - X_1^{(M)}(0)}{M}, \\ \dfrac{X_1^{(M)}(0)}{M} + \displaystyle\int_0^{1-u_1(0)} \varphi_M(s)ds - \int_{\frac{M-X_1^{(M)}(0)}{M}}^{1-u_1(0)} \varphi_M(s)ds \\ \qquad for\ \ \dfrac{M - X_1^{(M)}(0)}{M} \le 1 - u_1(0). \end{cases}$$

Since $0 \le \varphi_M(s) \le 1$

$$\frac{X_1^{(M)}(t)}{M} = \frac{X_1^{(M)}(0)}{M} + \int_0^{1-u_1(0)} \varphi_M(s)ds \pm O(|u_1(0) - \frac{X_1^{(M)}(0)}{M}|)$$

Hence the claim holds.

The convergence of the solution to the deterministic system is now reduced to the integrand $\varphi_M(s)$. For that purpose we shall show the convergence of $\sigma_k^{(M)}$.

We take for each $M > 0$ an integer k_M such that $\frac{k_M - 1}{M} \leq s < \frac{k_M}{M}$. It is to be noted that $\frac{k_M}{M}$ converges to s as M tends to infinity. We decompose $\sigma_{k_M}^{(M)}$ into

$$
\begin{aligned}
\sigma_{k_M}^{(M)} = \textstyle\sum_{1\leq i} & \Big[\tfrac{1}{M}(\chi_{\{i\leq k_M\}} - \chi_{\{i\leq [Ms]\}}) \frac{\tau_i - \tau_{i-1}}{\lambda \frac{(X_1^{(M)}(0)+i-1)}{M}(1 - \frac{(X_1^{(M)}(0)+i-1)}{M})} \\
& + \tfrac{1}{M}\chi_{\{i\leq [Ms]\}}\Big\{\frac{\tau_i - \tau_{i-1}}{\lambda \frac{(X_1^{(M)}(0)+i-1)}{M}(1 - \frac{(X_1^{(M)}(0)+i-1)}{M})} - \frac{\tau_i - \tau_{i-1}}{\lambda(u_1(0)+\frac{i-1}{M})(1-u_1(0)-\frac{i-1}{M})}\Big\} \\
& + \tfrac{1}{M}\chi_{\{i\leq [Ms]\}}\Big\{\frac{\tau_i - \tau_{i-1}}{\lambda(u_1(0)+\frac{i-1}{M})(1-u_1(0)-\frac{i-1}{M})} - \frac{1}{\lambda(u_1(0)+\frac{i-1}{M})(1-u_1(0)-\frac{i-1}{M})}\Big\} \\
& + \tfrac{1}{M}\chi_{\{i\leq [Ms]\}}\frac{1}{\lambda(u_1(0)+\frac{i-1}{M})(1-u_1(0)-\frac{i-1}{M})}\Big] \\
= \mathcal{S}_1 & + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4.
\end{aligned}
$$

[Step 2] We claim that $\lim_{M\to\infty} \mathcal{S}_1 = 0$.

By (2.6), there exists a positive constant C_1 such that

$$\frac{1}{M} \sum_{i=\min\{k_M,[Ms]\}+1}^{\max\{k_M,[Ms]\}} (\tau_i - \tau_{i-1})^2 < C_1.$$

Moreover, it follows from (2.4) that there is a positive integer M_0 such that for any $M \geq M_0$

$$\frac{u_1(0)}{2} < \frac{X_1^{(M)}(0)}{M} < u_1(0) + \frac{1 - u_1(0) - s}{2}. \tag{2.8}$$

Hence

$$
\begin{aligned}
& |\mathcal{S}_1| \\
& \leq \{\frac{1}{M} \sum_{i=\min\{k_M,[Ms]\}+1}^{\max\{k_M,[Ms]\}} (\tau_i - \tau_{i-1})^2\}^{\frac{1}{2}} \\
& \qquad \{\frac{1}{M} \sum_{i=\min\{k_M,[Ms]\}+1}^{\max\{k_M,[Ms]\}} \frac{1}{(\lambda \frac{X_1^{(M)}(0)+i-1}{M} \frac{M-(X_1^{(M)}(0)+i-1)}{M})^2}\}^{\frac{1}{2}} \\
& \leq \{C_1\}^{\frac{1}{2}}\{\frac{1}{M} \sum_{i=\min\{Ms,k_M\}+1}^{\max\{Ms,k_M\}} (\frac{1}{\lambda \frac{u_1(0)}{2} \frac{1-u_1(0)-s}{2}})^2\}^{\frac{1}{2}} \\
& = \{C_1\}^{\frac{1}{2}}\{\frac{|k_M - Ms|}{M}(\frac{1}{\lambda \frac{u_1(0)}{2} \frac{1-u_1(0)-s}{2}})^2\}^{\frac{1}{2}}.
\end{aligned}
$$

Therefore it follows from the convergence of $\frac{k_M}{M} \to s$ as $M \to \infty$ that $\lim_{M\to\infty} \mathcal{S}_1 = 0$.

[Step 3] We claim that $\lim_{M\to\infty} \mathcal{S}_2 = 0$.

By (2.6), there exists a positive constant C_2 such that

$$\frac{1}{M}\sum_{i=1}^{[Ms]}(\tau_i - \tau_{i-1})^2 < C_2.$$

Hence by using (2.8) of [Step 2], we have the estimate:

$$\begin{aligned}
&|\mathcal{S}_2| \\
&\leq \{\frac{1}{M}(\sum_{i=1}^{[Ms]}(\tau_i - \tau_{i-1})^2) \\
&\quad (\sum_{i=1}^{[Ms]}\frac{1}{M}(\frac{(1-\frac{X_1^{(M)}(0)}{M}-\frac{i-1}{M})-(u_1(0)+\frac{i-1}{M})}{\lambda(\frac{X_1^{(M)}(0)}{M}+\frac{i-1}{M})(1-\frac{X_1^{(M)}(0)}{M}-\frac{i-1}{M})(u_1(0)+\frac{i-1}{M})(1-u_1(0)-\frac{i-1}{M})})^2\}^{\frac{1}{2}} \\
&\qquad |\frac{X_1^{(M)}(0)}{M} - u_1(0)| \\
&\leq \{C_2\}^{\frac{1}{2}}\{s(\frac{1}{\lambda\frac{u_1(0)}{2}u_1(0)(1-u_1(0)-s)} \\
&\qquad + \frac{1}{\lambda\frac{u_1(0)}{2}\frac{1-u_1(0)-s}{2}(1-u_1(0)-s)})^2\}^{\frac{1}{2}}|\frac{X_1^{(M)}(0)}{M} - u_1(0)|.
\end{aligned}$$

Therefore we see from (2.4) that $\lim_{M\to\infty} \mathcal{S}_2 = 0$.

[Step 4] We claim that $\lim_{M\to\infty} \mathcal{S}_3 = 0$.

Now for any arbitrary real number $\epsilon > 0$, we take a natural number N such that $\frac{1}{\sqrt{N}} < \frac{\epsilon}{C}$. Here C is a positive constant, which is defined later by (2.9).

Let L be a natural number such that $[Ms]$ divided by L equals N and let r be the remainder: $[Ms] = NL + r$ and $0 \leq r < L$. We note that $M \to \infty$ iff $L \to \infty$. Put

$$\begin{aligned}
\mathcal{S}_3 &= \frac{1}{M}\sum_{i=1}^{[Ms]} a_{i,M}\xi_i, \\
Z_M &= \frac{L}{M}\sum_{k=1}^{N} a_k(\frac{1}{L}\sum_{i=(k-1)L+1}^{kL}\xi_i),
\end{aligned}$$

where

$$a_{i,M} = \frac{1}{\lambda(u_1(0)+\frac{i-1}{M})(1-u_1(0)-\frac{i-1}{M})},$$
$$a_k = \frac{1}{\lambda(u_1(0)+\frac{s(k-1)}{N})(1-u_1(0)-\frac{s(k-1)}{N})},$$
$$\xi_i = \tau_i - \tau_{i-1} - 1.$$

By Schwarz's inequality, we get

$$\begin{aligned}
&|S_3 - Z_M| \\
&\le |\frac{1}{M}\sum_{k=1}^{N}\sum_{i=(k-1)L+1}^{kL}(a_{i,M}-a_k)\xi_i| + |\frac{1}{M}\sum_{i=NL+1}^{NL+r} a_{i,M}\xi_i| \\
&\le (\frac{1}{M}\sum_{k=1}^{N}\sum_{i=(k-1)L+1}^{kL}(\xi_i)^2)^{\frac{1}{2}}(\frac{1}{M}\sum_{k=1}^{N}\sum_{i=(k-1)L+1}^{kL}|a_{i,M}-a_k|^2)^{\frac{1}{2}} \\
&\quad + (\frac{s}{N}\frac{1}{L}\sum_{i=NL+1}^{NL+r}(\xi_i)^2)^{\frac{1}{2}}(\frac{s}{N}\frac{1}{(\lambda u_1(0)(1-u_1(0)-s))^2})^{\frac{1}{2}}.
\end{aligned}$$

Since the running suffix i in the region $(k-1)L+1 \le i \le kL$ of the first term of the right-hand side means that

$$\frac{i-1}{M} - \frac{s(k-1)}{N} > -s\frac{(k-1)(r+1)}{N^2L} > -s\frac{1}{N},$$
$$\frac{i-1}{M} - \frac{s(k-1)}{N} \le s\frac{N(L-1)-r(k-1)}{N^2L} < 2s\frac{1}{N},$$

we have the estimate:

$$\begin{aligned}
&\frac{1}{M}\sum_{k=1}^{N}\sum_{i=(k-1)L+1}^{kL}|a_{i,M}-a_k|^2 = \\
&|\frac{1}{M}\sum_{k=1}^{N}\sum_{i=(k-1)L+1}^{kL}|\frac{(\frac{i-1}{M}-\frac{s(k-1)}{N})\{(u_1(0)+\frac{i-1}{M})-(1-u_1(0)-\frac{s(k-1)}{N})\}}{\lambda(u_1(0)+\frac{i-1}{M})(1-u_1(0)-\frac{i-1}{M})(u_1(0)+\frac{s(k-1)}{N})(1-u_1(0)-\frac{s(k-1)}{N})}|^2 \\
&\le \frac{(2s)^2}{MN^2}(\sum_{k=1}^{N}\sum_{i=(k-1)L+1}^{kL}(\frac{1}{\lambda(1-u_1(0)-\frac{i-1}{M})(u_1(0)+\frac{s(k-1)}{N})(1-u_1(0)-\frac{s(k-1)}{N})} \\
&\qquad + \frac{1}{\lambda(u_1(0)+\frac{i-1}{M})(1-u_1(0)-\frac{i-1}{M})(u_1(0)+\frac{s(k-1)}{N})})^2 \\
&\le \frac{4s^3}{N^2}(\frac{1}{\lambda u_1(0)(1-u_1(0)-s)^2} + \frac{1}{\lambda u_1(0)^2(1-u_1(0)-s)})^2.
\end{aligned}$$

By (2.6), there exists a positive constant C_3 such that

$$\frac{1}{NL+r}\sum_{i=1}^{NL+r}(\tau_i - \tau_{i-1} - 1)^2 < C_3.$$

We see that

$$\begin{aligned}
&\max\{\max_{1\le k\le N}\{\frac{1}{L}\sum_{i=(k-1)L+1}^{kL}(\tau_i-\tau_{i-1}-1)^2\},\frac{1}{L}\sum_{i=NL+1}^{NL+r}(\tau_i-\tau_{i-1}-1)^2\}\\
&\le \frac{NL+r}{L}\frac{1}{NL+r}\sum_{i=1}^{NL+r}(\tau_i-\tau_{i-1}-1)^2\\
&< 2NC_3.
\end{aligned}$$

Hence

$$\begin{aligned}
&|\mathcal{S}_3-Z_M|\\
&\le [\{2sC_3\}^{\frac12}\{2s^3(\frac{1}{\lambda u_1(0)(1-u_1(0)-s)^2}+\frac{1}{u_1(0)^2(1-u_1(0)-s)})^2\}^{\frac12}\\
&\qquad +\{2sC_3\}^{\frac12}\{s\frac{1}{(\lambda u_1(0)(1-u_1(0)-s))^2}\}^{\frac12}]\frac{1}{\sqrt{N}}\\
&\qquad = \frac{C}{\sqrt{N}}<\epsilon,
\end{aligned}$$

where

$$\begin{aligned}
C = s\sqrt{2C_3}[\sqrt{2}s(&\frac{1}{\lambda u_1(0)(1-u_1(0)-s)^2}+\frac{1}{\lambda u_1(0)^2(1-u_1(0)-s)})\\
&+\frac{1}{\lambda u_1(0)(1-u_1(0)-s)}].
\end{aligned}\tag{2.9}$$

This fact yields

$$\lim_{M\to\infty}|\mathcal{S}_3-Z_M|=0.$$

On the other hand, noting that $0\le\frac{L}{M}\le\frac{s}{N}$, we see from (2.5) that $\lim_{M\to\infty}Z_M=0$.

Therfore it follows that $\lim_{M\to\infty}\mathcal{S}_3=0$.

[Step 5] It is easy to see that when M tends to infinity, the fourth sum $\mathcal{S}_4$ is convergent to the non-random function $v(s)$ such that

$$\begin{aligned}
v(s) &\equiv \frac{1}{\lambda}\int_0^s\frac{1}{(u_1(0)+p)(1-u_1(0)-p)}dp\\
&= \frac{1}{\lambda}\log\frac{(u_1(0)+s)(1-u_1(0))}{u_1(0)(1-u_1(0)-s)}.
\end{aligned}$$

[Step 6] It follows from Step 1 - Step 5 that $u_1(t)\equiv\lim_{M\to\infty}\frac{X_1^{(M)}(t)}{M}$ exists and it is equal to

$$u_1(0)+\int_0^{1-u_1(0)}\chi_{[v(s),\infty)}(t)ds=\frac{u_1(0)e^{\lambda t}}{u_1(0)e^{\lambda t}+1-u_1(0)}.$$

This is a logistic distribution, which coincides with the solution of the deterministic system (2.1).

Consequently we complete the proof of Theorem 2.1. $\square$

3. Experimental study for the law of large numbers

We prove a strong law of large numbers for the model in section 2 and derive the ordinary differential equation which has a solution of the logistic function. The values of the logistic function and the values of the solution of $M = 1000$ by using the construction (1.3), (1.4) and (1.5) are shown in Figure I.1. Pseudo-random numbers generated by the mixed congruential method ($\mathbb{X}_{n+1} = \mathbb{A} \cdot \mathbb{X}_n + \mathbb{C}$ (mod $\mathbb{M}$) for $\mathbb{A} = 1229$, $\mathbb{C} = 351750$, $\mathbb{M} = 1664501$ [36]) are used in the simulation. We set $\lambda = 1$, $M = 1000$, $\frac{X_1^{(M)}(0)}{M} = \frac{X_2^{(M)}(0)}{M} = u_1(0) = u_2(0) = \frac{1}{2}$. As M is sufficiently large, we observe that the process of the random collision model is close to the deterministic system. Thus the strong law of large numbers for (1.2) is seen from the numerical experiment.

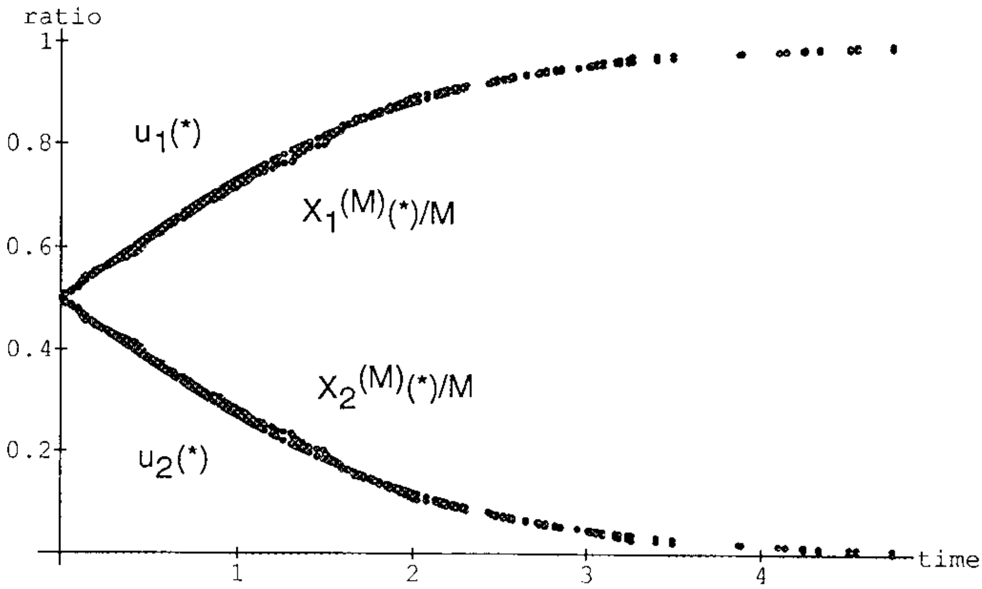

FIGURE I.1. Values of $\frac{X_1^{(M)}(*)}{M}$, $\frac{X_2^{(M)}(*)}{M}$ and values of the logistic function $u_1(*)$, $u_2(*)$ for $j = 1, 2$ and $M = 1000$ (vertical) against time (horizontal).

CHAPTER II

Paper-scissors-stone model

1. Paper-scissors-stone model and its solution

Let us investigate a model for competition between types of individuals in a population. Consider a population consisting of individuals, each of which is one of 3 types. The types may represent species, genotypes, types of consumers or other classifications. Any positive integer M is defined to be the total number of individuals. Let $X^{(M)}(*) = (X_1^{(M)}(*), X_2^{(M)}(*), X_3^{(M)}(*))$ be the stochastic process and let $X_j^{(M)}(t)$ denote the number of individuals of type j in the population at time t ($t \in [0,\infty) = \mathbb{R}_+$, $j = 1,2,3$).

For each pair of types, a dominance relation is defined such that type i dominates type $i+1$ ($i = 1,2,3$). Here we consider integers on mod 3 and if $j = 3$ then we put $j+1 = 1$ and $j = 1$ then we put $j-1 = 3$ from now on (on mod 3 we use 3 rather than 0). Random collisions between individuals are assumed to occur at the rate of $\lambda dt/M$ on the average during time interval $[t, t+dt)$ per one colliding pair, where λ is a positive constant. If two individuals of different types collide, then two individuals of the dominant type result from the collision. Note that the term "dominant" is not used here in the sense in which it is used in the genetics (see [26]). We assume that the process is written in the following form ($t \in \mathbb{R}_+$):

$$\begin{cases} dX_1^{(M)}(t) = dN_{12}(\frac{\lambda}{M}\int_0^t X_1^{(M)}(s)X_2^{(M)}(s)ds) - dN_{31}(\frac{\lambda}{M}\int_0^t X_3^{(M)}(s)X_1^{(M)}(s)ds), \\ dX_2^{(M)}(t) = dN_{23}(\frac{\lambda}{M}\int_0^t X_2^{(M)}(s)X_3^{(M)}(s)ds) - dN_{12}(\frac{\lambda}{M}\int_0^t X_1^{(M)}(s)X_2^{(M)}(s)ds), \\ dX_3^{(M)}(t) = dN_{31}(\frac{\lambda}{M}\int_0^t X_3^{(M)}(s)X_1^{(M)}(s)ds) - dN_{23}(\frac{\lambda}{M}\int_0^t X_2^{(M)}(s)X_3^{(M)}(s)ds). \end{cases}$$

This is also written in the integral form ($t \in \mathbb{R}_+$):

$$
(1.1)\qquad \begin{cases}
X_1^{(M)}(t) = X_1^{(M)}(0) + N_{12}(\dfrac{\lambda}{M}\displaystyle\int_0^t X_1^{(M)}(s)X_2^{(M)}(s)ds) \\
\qquad\qquad\qquad - N_{31}(\dfrac{\lambda}{M}\displaystyle\int_0^t X_3^{(M)}(s)X_1^{(M)}(s)ds), \\
X_2^{(M)}(t) = X_2^{(M)}(0) + N_{23}(\dfrac{\lambda}{M}\displaystyle\int_0^t X_2^{(M)}(s)X_3^{(M)}(s)ds) \\
\qquad\qquad\qquad - N_{12}(\dfrac{\lambda}{M}\displaystyle\int_0^t X_1^{(M)}(s)X_2^{(M)}(s)ds), \\
X_3^{(M)}(t) = X_3^{(M)}(0) + N_{31}(\dfrac{\lambda}{M}\displaystyle\int_0^t X_3^{(M)}(s)X_1^{(M)}(s)ds) \\
\qquad\qquad\qquad - N_{23}(\dfrac{\lambda}{M}\displaystyle\int_0^t X_2^{(M)}(s)X_3^{(M)}(s)ds), \\
X_1^{(M)}(0) + X_2^{(M)}(0) + X_3^{(M)}(0) = M,
\end{cases}
$$

where $X_j^{(M)}(0)$ are given nonnegative initial values of $X_j^{(M)}(*)$ ($j = 1, 2, 3$). We assume that there do not exist accumulation points of jump times of standard Poisson processes $N_{jj+1}(*)$ for $j = 1, 2, 3$.

We call (1.1) paper-scissors-stone model.

REMARK 1.1. *The case of the n-species is treated to have a cycle of n prey-predator relations in a similar way as the paper-scissors-stone model. The number increasing over time t of the i-th species is equal to the number decreasing by time t of the $i+1$-th species. The stopping time, the semi-martingale decomposition, the weak law of large numbers and the central limit theorem in the following sections are easily extended to the case of the n-species.*

THEOREM 1.1. *There exists a unique solution of equation* (1.1) *in* $\mathbb{R}_+$.

PROOF. We fix a sample path of $(N_{12}(*), N_{23}(*), N_{31}(*))$. We denote by $\{\tau_i^{jj+1}\}_{i\geq 0}$ the set of the jump times of three standard Poisson processes $N_{jj+1}(*)$ where we put $\tau_0^{jj+1} = 0$ ($j = 1, 2, 3$). Note that $0 = \tau_0^{jj+1} < \tau_1^{jj+1} < \tau_2^{jj+1} < \cdots < \tau_i^{jj+1} < \tau_{i+1}^{jj+1} < \cdots$ for $j = 1, 2, 3$. We define $\sigma(0) = 0$ and $K^{jj+1}(0) = 0$ for $j = 1, 2, 3$.

For an integer $I - 1$, $I \geq 1$, we define the proposition $\mathfrak{P}(I-1)$ as follows: There are nonnegative nondecreasing sequences $\{\sigma(k)\}_{0\leq k\leq I-1}$ in $\mathbb{R}_+ \cup \{\infty\}$ and $\{K^{jj+1}(k)\}_{0\leq k\leq I-1}$ in $\mathbb{N} = \{0, 1, 2, \cdots\}$ ($1 \leq j \leq 3$) such that for $X^{(M)}(t) = (X_1^{(M)}(t), X_2^{(M)}(t), X_3^{(M)}(t))$ and for $T^{(M)}(t) = (T_{12}^{(M)}(t), T_{23}^{(M)}(t), T_{31}^{(M)}(t))$ ($t \in \{u \in \mathbb{R}_+ : u \in [0, \sigma(I-1)]\}$) defined by ($1 \leq j \leq 3$)

$$
(1.2)\qquad \begin{aligned}
X_j^{(M)}(t) = X_j^{(M)}(0) &+ \sum_{i=1}^{I-1}(K^{jj+1}(i) - K^{jj+1}(i-1))\chi_{[\sigma(i),\infty)}(t) \\
&- \sum_{i=1}^{I-1}(K^{j-1j}(i) - K^{j-1j}(i-1))\chi_{[\sigma(i),\infty)}(t),
\end{aligned}
$$

$$T^{(M)}_{jj+1}(t) = \frac{\lambda}{M}\int_0^t X^{(M)}_j(s)X^{(M)}_{j+1}(s)ds,$$

with $\sum_{i=1}^{0} = 0$, the following (i) and (ii) hold:

(i) For any i satisfying $0 \le i \le I-1$, if $\sigma(i) < \infty$, then

$$P_j(i) \;:\; T^{(M)}_{jj+1}(\sigma(i)) \in [\tau^{jj+1}_{K^{jj+1}(i)}, \tau^{jj+1}_{K^{jj+1}(i)+1}) \;\; for \;\; 1 \le j \le 3.$$

For any k satisfying $1 \le k \le I-1$, if $\sigma(k-1) < \infty$, then for any t satisfying $\sigma(k-1) < t < \sigma(k)$,

$$P_j(k-1,k) \;:\; T^{(M)}_{jj+1}(t) \in [\tau^{jj+1}_{K^{jj+1}(k-1)}, \tau^{jj+1}_{K^{jj+1}(k-1)+1}) \;\; for \;\; 1 \le j \le 3.$$

(ii) For $t \in \{u \in \mathbb{R}_+ : u \in [0, \sigma(I-1)]\}$, $X^{(M)}_j(t)$ $(1 \le j \le 3)$ in (1.2) satisfies (1.1).

Assuming $\mathfrak{P}(I-1)$, by using the jump times of standard Poisson processes and by using the factors $\{X^{(M)}_j(0), \sigma(l), K^{jj+1}(l)\}_{0 \le l \le I-1, 1 \le j \le 3}$ in $\mathfrak{P}(I-1)$, we define $\sigma(I)$ and $K^{jj+1}(I)$ $(1 \le j \le 3)$ by the following (1)-(4):

(1) If $\sigma(I-1) = \infty$, we define $\sigma(I) = \infty$.
(2) If $\sigma(I-1) < \infty$, we define $\sigma(I)$ by

$$(1.3)\; \sigma(I) = \min_{1 \le j \le 3}\{\sigma(I-1) + \frac{\tau^{jj+1}_{K^{jj+1}(I-1)+1} - T^{(M)}_{jj+1}(\sigma(I-1))}{\frac{\lambda}{M}X^{(M)}_j(\sigma(I-1))X^{(M)}_{j+1}(\sigma(I-1))}\}.$$

When we have $X^{(M)}_j(\sigma(I-1)) = 0$ or $X^{(M)}_{j+1}(\sigma(I-1)) = 0$, since $\tau^{jj+1}_{K^{jj+1}(I-1)+1} - T^{(M)}_{jj+1}(\sigma(I-1)) > 0$ from $P_j(I-1)$ in $\mathfrak{P}(I-1)$, we replace the term $\sigma(I-1) + \frac{\tau^{jj+1}_{K^{jj+1}(I-1)+1} - T^{(M)}_{jj+1}(\sigma(I-1))}{\frac{\lambda}{M}X^{(M)}_j(\sigma(I-1))X^{(M)}_{j+1}(\sigma(I-1))}$ by infinity.

(3) If $\sigma(I) = \infty$, we define $K^{jj+1}(I) = K^{jj+1}(I-1)$ for $1 \le j \le 3$.
(4) If $\sigma(I) < \infty$, choose j $(= j_1$, say) for which the term $\sigma(I-1) + \frac{\tau^{jj+1}_{K^{jj+1}(I-1)+1} - T^{(M)}_{jj+1}(\sigma(I-1))}{\frac{\lambda}{M}X^{(M)}_j(\sigma(I-1))X^{(M)}_{j+1}(\sigma(I-1))}$ is the smallest in (1.3). We define $K^{j_1j_1+1}(I) = K^{j_1j_1+1}(I-1) + 1$. For $j \ne j_1$ $(1 \le j \le 3)$ we define $K^{jj+1}(I) = K^{jj+1}(I-1)$.

By mathematical induction on I, we shall prove the proposition $\mathfrak{P}(I)$.

Now we prove $\mathfrak{P}(0)$. At $t = \sigma(0)$

$$
\begin{aligned}
X_j^{(M)}(\sigma(0)) = X_j^{(M)}(0) &+ \sum_{i=1}^{0}(K^{jj+1}(i) - K^{jj+1}(i-1))\chi_{[\sigma(i),\infty)}(0) \\
&+ \sum_{i=1}^{0}(K^{j-1j}(i) - K^{j-1j}(i-1))\chi_{[\sigma(i),\infty)}(0) \\
= X_j^{(M)}(0).
\end{aligned}
$$

And we have

$$
T_{jj+1}^{(M)}(\sigma(0)) = \frac{\lambda}{M}\int_0^{\sigma(0)} X_j^{(M)}(s)X_{j+1}^{(M)}(s)ds = \frac{\lambda}{M}\int_0^{0} X_j^{(M)}(s)X_{j+1}^{(M)}(s)ds = 0.
$$

From the definition $K^{jj+1}(0) = 0$ for $j = 1, 2, 3$,

$$
\tau_{K^{jj+1}(0)}^{jj+1} = T_{jj+1}^{(M)}(\sigma(0)) = 0 < \tau_{K^{jj+1}(0)+1}^{jj+1}.
$$

For j, $1 \leq j \leq 3$, it follows that

$$
\sum_{i=1}^{0}(K^{jj+1}(i) - K^{jj+1}(i-1)) = 0 = N_{jj+1}(T_{jj+1}^{(M)}(\sigma(0))).
$$

Then at $\sigma(0)$, (1.2) satisfies (1.1). Therefore $\mathfrak{P}(0)$ hold.

We assume the proposition $\mathfrak{P}(I-1)$ for $I \geq 1$.

[Case A] We consider the case where $\sigma(l) < \infty$, $X_j^{(M)}(\sigma(l)) > 0$ for $0 \leq l \leq I-1$ and $1 \leq j \leq 3$. This case describes that the values of $X_j^{(M)}(*)$ have not reached zero in $[0, \sigma(I-1)]$.

Note that $\sigma(I) < \infty$ in [Case A]. If the term of $j = 1$ is the smallest in (1.3), for example, then we have $K^{12}(I) = K^{12}(I-1) + 1$, $K^{23}(I) = K^{23}(I-1)$ and $K^{31}(I) = K^{31}(I-1)$. If the terms of $j = 1, 2$ are the smallest in (1.3), then we have $K^{12}(I) = K^{12}(I-1)+1$, $K^{23}(I) = K^{23}(I-1)+1$ and $K^{31}(I) = K^{31}(I-1)$. If the terms of $j = 1, 2, 3$ in (1.3) take the same value, then we have $K^{12}(I) = K^{12}(I-1) + 1$, $K^{23}(I) = K^{23}(I-1) + 1$ and $K^{31}(I) = K^{31}(I-1) + 1$.

If the term of $j = 1$ is the smallest in (1.3), we have $K^{12}(I) = K^{12}(I-1) + 1$, $K^{23}(I) = K^{23}(I-1)$ and $K^{31}(I) = K^{31}(I-1)$.

This means

$$
\begin{aligned}
\sigma(I) =& \sigma(I-1) + \frac{\tau_{K^{12}(I-1)+1}^{12} - T_{12}^{(M)}(\sigma(I-1))}{\frac{\lambda}{M}X_1^{(M)}(\sigma(I-1))X_2^{(M)}(\sigma(I-1))}, \\
\sigma(I) <& \sigma(I-1) + \frac{\tau_{K^{23}(I-1)+1}^{23} - T_{23}^{(M)}(\sigma(I-1))}{\frac{\lambda}{M}X_2^{(M)}(\sigma(I-1))X_3^{(M)}(\sigma(I-1))}, \\
\sigma(I) <& \sigma(I-1) + \frac{\tau_{K^{31}(I-1)+1}^{31} - T_{31}^{(M)}(\sigma(I-1))}{\frac{\lambda}{M}X_3^{(M)}(\sigma(I-1))X_1^{(M)}(\sigma(I-1))}.
\end{aligned}
$$

From $P_1(I-1)$ all numerators are positive and all random variables $X_j^{(M)}(\sigma(I-1))$ are positive in [Case A]. Thus we have $\sigma(I) > \sigma(I-1)$ and

(1.4)

$$\tau^{12}_{K^{12}(I-1)+1} = T^{(M)}_{12}(\sigma(I-1)) + \frac{\lambda}{M} X_1^{(M)}(\sigma(I-1)) X_2^{(M)}(\sigma(I-1))(\sigma(I) - \sigma(I-1)),$$

(1.5)

$$\tau^{23}_{K^{23}(I-1)+1} > T^{(M)}_{23}(\sigma(I-1)) + \frac{\lambda}{M} X_2^{(M)}(\sigma(I-1)) X_3^{(M)}(\sigma(I-1))(\sigma(I) - \sigma(I-1)),$$

$$\tau^{31}_{K^{31}(I-1)+1} > T^{(M)}_{31}(\sigma(I-1)) + \frac{\lambda}{M} X_3^{(M)}(\sigma(I-1)) X_1^{(M)}(\sigma(I-1))(\sigma(I) - \sigma(I-1)).$$

[Step 1] We consider $P_1(I-1, I)$ and $P_1(I)$.
For $\sigma(I-1) < t < \sigma(I)$,

$$T^{(M)}_{12}(t) = T^{(M)}_{12}(\sigma(I-1)) + \frac{\lambda}{M} X_1^{(M)}(\sigma(I-1)) X_2^{(M)}(\sigma(I-1))(t - \sigma(I-1)),$$

and, from (1.4),

$$\begin{aligned} T^{(M)}_{12}(\sigma(I)) &= T^{(M)}_{12}(\sigma(I-1)) + \frac{\lambda}{M} X_1^{(M)}(\sigma(I-1)) X_2^{(M)}(\sigma(I-1))(\sigma(I) - \sigma(I-1)) \\ &= \tau^{12}_{K^{12}(I-1)+1}. \end{aligned}$$

The condition of positiveness of random variables $X_j^{(M)}(\sigma(I-1))$ $(1 \le j \le 3)$ in [Case A] leads $T^{(M)}_{12}(\sigma(I-1)) < T^{(M)}_{12}(t) < T^{(M)}_{12}(\sigma(I))$ for $\sigma(I-1) < t < \sigma(I)$. From $P_1(I-1)$ it follows that

$$\tau^{12}_{K^{12}(I-1)} \le T^{(M)}_{12}(\sigma(I-1)) < T^{(M)}_{12}(t) < T^{(M)}_{12}(\sigma(I)) = \tau^{12}_{K^{12}(I-1)+1},$$

$$\tau^{12}_{K^{12}(I-1)+1} = \tau^{12}_{K^{12}(I)} = T^{(M)}_{12}(\sigma(I)) < \tau^{12}_{K^{12}(I)+1}.$$

Therefore $P_1(I-1, I)$ and $P_1(I)$ hold. $\diamond$

[Step 2] We consider $P_2(I-1, I)$ and $P_2(I)$.
For $\sigma(I-1) < t < \sigma(I)$,

$$T^{(M)}_{23}(t) = T^{(M)}_{23}(\sigma(I-1)) + \frac{\lambda}{M} X_2^{(M)}(\sigma(I-1)) X_3^{(M)}(\sigma(I-1))(t - \sigma(I-1)),$$

and, from (1.5),

$$\begin{aligned} T^{(M)}_{23}(\sigma(I)) =& T^{(M)}_{23}(\sigma(I-1)) + \frac{\lambda}{M} X_2^{(M)}(\sigma(I-1)) X_3^{(M)}(\sigma(I-1))(\sigma(I) - \sigma(I-1)) \\ <& \tau^{23}_{K^{23}(I-1)+1}. \end{aligned}$$

From $P_2(I-1)$ it follows that

$$\tau^{23}_{K^{23}(I-1)} \le T^{(M)}_{23}(\sigma(I-1)) < T^{(M)}_{23}(t) < T^{(M)}_{23}(\sigma(I)) < \tau^{23}_{K^{23}(I-1)+1},$$

$$\tau^{23}_{K^{23}(I-1)} = \tau^{23}_{K^{23}(I)} < T^{(M)}_{23}(\sigma(I)) < \tau^{23}_{K^{23}(I-1)+1} = \tau^{23}_{K^{23}(I)+1}.$$

Therefore $P_2(I-1,I)$ and $P_2(I)$ hold. $\diamond$

We also have $P_3(I-1,I)$ and $P_3(I)$.

If the terms of $j=1,2$ are the smallest in (1.3), we have $K^{12}(I) = K^{12}(I-1)+1$, $K^{23}(I) = K^{23}(I-1)+1$ and $K^{31}(I) = K^{31}(I-1)$. In this case

$$\begin{aligned}
\sigma(I) =& \sigma(I-1) + \frac{\tau^{12}_{K^{12}(I-1)+1} - T^{(M)}_{12}(\sigma(I-1))}{\frac{\lambda}{M}X^{(M)}_1(\sigma(I-1))X^{(M)}_2(\sigma(I-1))},\\
\sigma(I) =& \sigma(I-1) + \frac{\tau^{23}_{K^{23}(I-1)+1} - T^{(M)}_{23}(\sigma(I-1))}{\frac{\lambda}{M}X^{(M)}_2(\sigma(I-1))X^{(M)}_3(\sigma(I-1))},\\
\sigma(I) <& \sigma(I-1) + \frac{\tau^{31}_{K^{31}(I-1)+1} - T^{(M)}_{31}(\sigma(I-1))}{\frac{\lambda}{M}X^{(M)}_3(\sigma(I-1))X^{(M)}_1(\sigma(I-1))}.
\end{aligned}$$

For $j=1,2$, we have $P_j(I-1,I)$ and $P_j(I)$ similarly as in [Step 1]. $P_3(I-1,I)$ and $P_3(I)$ hold in a similar way as [Step 2].

If the terms of $j=1,2,3$ in (1.3) take the same value, we have $K^{jj+1}(I) = K^{jj+1}(I-1)+1$ $(1 \le j \le 3)$. Then

$$\begin{aligned}
\sigma(I) =& \sigma(I-1) + \frac{\tau^{12}_{K^{12}(I-1)+1} - T^{(M)}_{12}(\sigma(I-1))}{\frac{\lambda}{M}X^{(M)}_1(\sigma(I-1))X^{(M)}_2(\sigma(I-1))},\\
\sigma(I) =& \sigma(I-1) + \frac{\tau^{23}_{K^{23}(I-1)+1} - T^{(M)}_{23}(\sigma(I-1))}{\frac{\lambda}{M}X^{(M)}_2(\sigma(I-1))X^{(M)}_3(\sigma(I-1))},\\
\sigma(I) =& \sigma(I-1) + \frac{\tau^{31}_{K^{31}(I-1)+1} - T^{(M)}_{31}(\sigma(I-1))}{\frac{\lambda}{M}X^{(M)}_3(\sigma(I-1))X^{(M)}_1(\sigma(I-1))}.
\end{aligned}$$

For $j=1,2,3$, $P_j(I-1,I)$ and $P_j(I)$ hold similarly as in [Step 1].

From $P_j(I-1,I)$ $(j=1,2,3)$, for any t, $\sigma(I-1) < t < \sigma(I)$, we have

$$\sum_{i=1}^{I}(K^{jj+1}(i) - K^{jj+1}(i-1))\chi_{[\sigma(i),\infty)}(t) = K^{jj+1}(I-1) = N_{jj+1}(T^{(M)}_{jj+1}(t)),$$

and from $P_j(I)$ $(j=1,2,3)$, at $\sigma(I)$, we have

$$\sum_{i=1}^{I}(K^{jj+1}(i) - K^{jj+1}(i-1))\chi_{[\sigma(i),\infty)}(\sigma(I)) = K^{jj+1}(I) = N_{jj+1}(T^{(M)}_{jj+1}(\sigma(I))).$$

Thus for any t, $\sigma(I-1) < t < \sigma(I)$, (1.2) satisfies (1.1) and at $\sigma(I)$, (1.2) satisfies (1.1). In $[0, \sigma(I-1)]$, (1.2) is assumed to satisfy (1.1).

In [Case A] we obtain the proposition $\mathfrak{P}(I)$.

[Case B] We consider the case where $\sigma(l) < \infty$, for some k satisfying $0 \leq k \leq I-1$, $X_{j-1}^{(M)}(\sigma(l)) > 0$, $X_j^{(M)}(\sigma(l')) > 0$, $X_j^{(M)}(\sigma(l'')) = 0$ and $X_{j+1}^{(M)}(\sigma(l)) > 0$ $(0 \leq l \leq I-1,\ 0 \leq l' < k,\ k \leq l'' \leq I-1$ and $1 \leq j \leq 3)$. This is the case that the value of one of $X_j^{(M)}(*)$ $(1 \leq j \leq 3)$ has come to zero and kept zero in $[\sigma(k), \sigma(I-1)]$. If the value of $X_j^{(M)}(\sigma(k))$ $(1 \leq j \leq 3,\ 0 \leq k \leq I-1)$ comes to zero, then there are no choices for j and $j+1$ in (1.3). Thus we do not consider the case where $X_j^{(M)}(*)$ does not keep the value zero in $[\sigma(k), \sigma(I-1)]$. For example we prove in the case of $j = 2$.

In this case $K^{12}(k) = \cdots = K^{12}(I-1)$ and $K^{23}(k) = \cdots = K^{23}(I-1)$ is implicitly assumed. It follows that $X_2^{(M)}(t) = 0$ for any $t \in [\sigma(k), \sigma(I-1)]$. Thus $T_{12}^{(M)}(\sigma(k)) = T_{12}^{(M)}(t)$ and $T_{23}^{(M)}(\sigma(k)) = T_{23}^{(M)}(t)$ for any $t \in [\sigma(k), \sigma(I-1)]$.

We determine $\sigma(I)$ by

$$\begin{aligned}\sigma(I) =& \min\{\sigma(I-1) + \frac{\tau_{K^{31}(I-1)+1}^{31} - T_{31}^{(M)}(\sigma(I-1))}{\frac{\lambda}{M} X_3^{(M)}(\sigma(I-1)) X_1^{(M)}(\sigma(I-1))}, \infty, \infty\} \\ =& \sigma(I-1) + \frac{\tau_{K^{31}(I-1)+1}^{31} - T_{31}^{(M)}(\sigma(I-1))}{\frac{\lambda}{M} X_3^{(M)}(\sigma(I-1)) X_1^{(M)}(\sigma(I-1))}.\end{aligned}$$

Note that $\sigma(I) < \infty$ in [Case B]. We have $K^{31}(I) = K^{31}(I-1)+1$, $K^{12}(I) = K^{12}(I-1)$ and $K^{23}(I) = K^{23}(I-1)$. Thus the implicit assumption is satisfied to the I-th step.

By $P_3(I-1)$ the numerator is positive and $\sigma(I) > \sigma(I-1)$. We have

$$\tau_{K^{31}(I-1)+1}^{31} = T_{31}^{(M)}(\sigma(I-1)) + \frac{\lambda}{M} X_3^{(M)}(\sigma(I-1)) X_1^{(M)}(\sigma(I-1))(\sigma(I) - \sigma(I-1)).$$

Similarly as in [Step 1] in [Case A], $P_3(I-1, I)$ and $P_3(I)$ hold.

[Step 3] We consider $P_1(I-1, I)$ and $P_1(I)$.

In [Case B] for $\sigma(I-1) < t < \sigma(I)$ we have

$$T_{12}^{(M)}(t) = T_{12}^{(M)}(\sigma(I-1)),$$

and

$$T_{12}^{(M)}(\sigma(I)) = T_{12}^{(M)}(\sigma(I-1)).$$

From $P_1(I-1)$ it follows that

$$\tau_{K^{12}(I-1)}^{12} \leq T_{12}^{(M)}(\sigma(I-1)) = T_{12}^{(M)}(t) < \tau_{K^{12}(I-1)+1}^{12}.$$

$$\tau_{K^{12}(I-1)}^{12} = \tau_{K^{12}(I)}^{12} \leq T_{12}^{(M)}(\sigma(I-1)) = T_{12}^{(M)}(\sigma(I)) < \tau_{K^{12}(I-1)+1}^{12} = \tau_{K^{12}(I)+1}^{12}.$$

Therefore $P_1(I-1, I)$ and $P_1(I)$ hold. $\diamond$

$P_2(I-1, I)$ and $P_2(I)$ also hold.

In $(\sigma(I-1), \sigma(I))$, (1.2) satisfies (1.1) and, at $\sigma(I)$, (1.2) staisfies (1.1). In [Case B] the proposition $\mathfrak{P}(I)$ is obtained.

[Case C] We consider the case where $\sigma(l) < \infty$, for some k satisfying $0 \le k < I-1$, $X_{j-1}^{(M)}(\sigma(l''')) > 0$, $X_{j-1}^{(M)}(\sigma(I-1)) = 0$, $X_j^{(M)}(\sigma(l')) > 0$, $X_j^{(M)}(\sigma(l'')) = 0$ and $X_{j+1}^{(M)}(\sigma(l)) > 0$ $(0 \le l \le I-1, 0 \le l' < k, k \le l'' \le I-1, 0 \le l''' < I-1$ and $1 \le j \le 3)$. This is the first case in which the values of $X_{j-1}^{(M)}(*)$ and $X_j^{(M)}(*)$ $(1 \le j \le 3)$ have come to zero at $\sigma(I-1)$, after several times of [Case B]. For example we prove in the case of $j = 2$.

By [Case B] we implicitly have $K^{12}(k) = \cdots = K^{12}(I-1)$ and $K^{23}(k) = \cdots = K^{23}(I-1)$. Thus for t, $t \in [\sigma(k), \sigma(I-1)]$, $X_2^{(M)}(t) = 0$ and $T_{ii+1}^{(M)}(\sigma(k)) = T_{ii+1}^{(M)}(t)$ $(i = 1, 2)$.

We determine $\sigma(I)$ by

$$\begin{aligned}\sigma(I) &= \min\{\infty, \infty, \infty\}\\ &= \infty.\end{aligned}$$

[Step 4] We consider $P_j(I-1, I)$ $(1 \le j \le 3)$.

For t, $\sigma(I-1) < t < \sigma(I) = \infty$, we have $(j = 1, 2, 3)$

$$T_{jj+1}^{(M)}(t) = T_{jj+1}^{(M)}(\sigma(I-1)).$$

From $P_j(I-1)$ it follows that

$$\tau_{K^{jj+1}(I-1)}^{12} \le T_{jj+1}^{(M)}(\sigma(I-1)) = T_{jj+1}^{(M)}(t) < \tau_{K^{jj+1}(I-1)+1}^{jj+1}.$$

Thus $P_j(I-1, I)$ for $1 \le j \le 3$ hold. $\Diamond$

For any t, $\sigma(I-1) < t < \sigma(I)$, (1.2) satisfies (1.1). Therefore $\mathfrak{P}(I)$ holds in [Case C].

If $\sigma(I-1) = \infty$, $\mathfrak{P}(I)$ holds.

Assuming the proposition $\mathfrak{P}(I-1)$, we have the proposition $\mathfrak{P}(I)$.

By mathematical induction (1.2) satisfies (1.1) in $\mathbb{R}_+$.

Now we shall prove that the solution constructed above is unique.

If there exist several solutions of (1.1) including the above construction, let $X^{(M)}(t) = (X_1^{(M)}(t), X_2^{(M)}(t), X_3^{(M)}(t))$ $(t \in \mathbb{R}_+)$ be any one of the solutions. Each random variable $X_j^{(M)}(t)$ has a nonnegative initial value. In the neighborhood of $t = 0$ we see that for $j = 1, 2, 3$,

$$\frac{\lambda}{M}\int_0^t X_j^{(M)}(s)X_{j+1}^{(M)}(s)ds = \frac{\lambda}{M}X_j^{(M)}(0)X_{j+1}^{(M)}(0)t \ge 0.$$

Thus the integrals are monotonically nondecreasing in the neighborhood of $t = 0$. Each random variable $X_j^{(M)}(t)$ is integer valued $(j = 1, 2, 3)$. If one of the random variables is negative valued after several jumps of the system from the nonnegative initial value, it goes through the value zero. We see that the random variables $X_j^{(M)}(*)$ $(1 \leq j \leq 3)$ are nonnegative by the following claim.

Put $X_k^{(M)}(t)$ $(1 \leq k \leq 3)$ to be a solution of the system of (1.1). We claim that when $X_{j-1}^{(M)}(t) \geq 0$, $X_{j+1}^{(M)}(t) \geq 0$ and $X_j^{(M)}(t) = 0$ for some $t \in (0, \infty)$ and for some $j \in \{1, 2, 3\}$, $X_j^{(M)}(s) = 0$ holds for any $s \geq t$.

We set u, $u > t$, to be the first jump time of both $N_{j-1j}(\frac{\lambda}{M}\int_0^* X_{j-1}^{(M)}(s)X_j^{(M)}(s)ds)$ and $N_{jj+1}(\frac{\lambda}{M}\int_0^* X_j^{(M)}(s)X_{j+1}^{(M)}(s)ds)$. Then it follows that $X_j^{(M)}(s) = 0$ for any s, $t \leq s < u$. Since $\frac{\lambda}{M}\int_0^s X_{j-1}^{(M)}(s)X_j^{(M)}(s)ds$ and $\frac{\lambda}{M}\int_0^s X_j^{(M)}(s)X_{j+1}^{(M)}(s)ds$ are continuous,

$$\frac{\lambda}{M}\int_0^u X_{j-1}^{(M)}(s)X_j^{(M)}(s)ds = \frac{\lambda}{M}\int_0^t X_{j-1}^{(M)}(s)X_j^{(M)}(s)ds,$$

$$\frac{\lambda}{M}\int_0^u X_j^{(M)}(s)X_{j+1}^{(M)}(s)ds = \frac{\lambda}{M}\int_0^t X_j^{(M)}(s)X_{j+1}^{(M)}(s)ds.$$

Therefore we have

$$N_{j-1j}(\frac{\lambda}{M}\int_0^u X_{j-1}^{(M)}(s)X_j^{(M)}(s)ds) = N_{j-1j}(\frac{\lambda}{M}\int_0^t X_{j-1}^{(M)}(s)X_j^{(M)}(s)ds),$$

$$N_{jj+1}(\frac{\lambda}{M}\int_0^u X_j^{(M)}(s)X_{j+1}^{(M)}(s)ds) = N_{jj+1}(\frac{\lambda}{M}\int_0^t X_j^{(M)}(s)X_{j+1}^{(M)}(s)ds).$$

This is in contradiction. Therefore the claim holds. $\sharp$

The random variables $X_j^{(M)}(*)$ are nonnegative, bounded and integer valued in $[0, M]$ for $1 \leq j \leq 3$. The integrals $\frac{\lambda}{M}\int_0^t X_j^{(M)}(s)X_{j+1}^{(M)}(s)ds$ $(t \in \mathbb{R}_+$ and $1 \leq j \leq 3)$ are nonnegative, monotonically nondecreasing and we have inequalities $0 \leq \frac{\lambda}{M}\int_0^t X_j^{(M)}(s)X_{j+1}^{(M)}(s)ds \leq \frac{\lambda M t}{4}$. Thus all possible classifications are covered in the following proof.

For an integer $I-1$, $I \geq 1$, we define the proposition $\mathfrak{Q}(I-1)$ as follows: For $t \in \{u \in \mathbb{R}_+ : u \in [0, \sigma(I-1)]\}$, the solution $X^{(M)}(t) = (X_1^{(M)}(t), X_2^{(M)}(t), X_3^{(M)}(t))$ constructed by (1.2) in the proposition $\mathfrak{P}(I-1)$ is the unique solution of (1.1).

The initial values are given. At $\sigma(0)$ the unique solution of (1.1) is written by (1.2). Therefore $\mathfrak{Q}(0)$ holds.

We assume $\mathfrak{Q}(I-1)$ for $I-1 \geq 0$.

We consider the case where $\sigma(I-1) < \infty$. We trace the time from $\sigma(I-1)$ and search the next jump time of the system of (1.1). If there exist several solutions of (1.1), all the solutios are nonnegative. The system changes the previous state at $s(I)$ such that, by using the factors $\sigma(I-1)$, $K^{jj+1}(I-1)$, $X_j^{(M)}(\sigma(I-1))$ and $T_{jj+1}^{(M)}(\sigma(I-1))$ $(1 \leq j \leq 3)$ which are given by $\mathfrak{P}(I-1)$ in $\mathfrak{Q}(I-1)$,

$$\begin{aligned} s(I) = \min_{1\leq j\leq 3}\{\inf\{t > \sigma(I-1) : &\frac{\lambda}{M}X_j^{(M)}(\sigma(I-1))X_{j+1}^{(M)}(\sigma(I-1))(t-\sigma(I-1)) \\ &= \tau_{K^{jj+1}(I-1)+1}^{jj+1} - T_{jj+1}^{(M)}(\sigma(I-1))\}\}. \end{aligned}$$

[Case a] We consider the case where $\sigma(l) < \infty$, $X_j^{(M)}(\sigma(l)) > 0$ for $0 \le l \le I-1$ and $1 \le j \le 3$.

We have

$$s(I) = \min_{1\le j\le 3}\{\sigma(I-1) + \frac{\tau^{jj+1}_{K^{jj+1}(I-1)+1} - T^{(M)}_{jj+1}(\sigma(I-1))}{\frac{\lambda}{M}X_j^{(M)}(\sigma(I-1))X_{j+1}^{(M)}(\sigma(I-1))}\}.$$

[Case b] We consider the case where $\sigma(l) < \infty$, for some k satisfying $0 \le k \le I-1$ $X_{j-1}^{(M)}(\sigma(l)) > 0$, $X_j^{(M)}(\sigma(l')) > 0$, $X_j^{(M)}(\sigma(l'')) = 0$ and $X_{j+1}^{(M)}(\sigma(l)) > 0$ $(0 \le l \le I-1,\ 0 \le l' < k,\ k \le l'' \le I-1$ and $1 \le j \le 3)$. For example we prove in the case of $j = 2$.

We have $\frac{\lambda}{M}X_j^{(M)}(\sigma(I-1))X_{j+1}^{(M)}(\sigma(I-1))(t-\sigma(I-1)) = 0$ for $t > \sigma(I-1)$ for $j = 1,2$. By using $P_j(I-1)$ $(j = 1,2)$

$$\{t > \sigma(I-1) : 0 = \tau^{jj+1}_{K^{jj+1}(I-1)+1} - T^{(M)}_{jj+1}(\sigma(I-1))\} = \emptyset.$$

It follows that

$$s(I) = \min\{\sigma(I-1) + \frac{\tau^{31}_{K^{31}(I-1)+1} - T^{(M)}_{31}(\sigma(I-1))}{\frac{\lambda}{M}X_3^{(M)}(\sigma(I-1))X_1^{(M)}(\sigma(I-1))}, \infty, \infty\},$$

where $\inf\emptyset = \infty$.

[Case c] We consider the case where $\sigma(l) < \infty$, for some k satisfying $0 \le k < I-1$ $X_{j-1}^{(M)}(\sigma(l''')) > 0$, $X_{j-1}^{(M)}(\sigma(I-1)) = 0$, $X_j^{(M)}(\sigma(l')) > 0$, $X_j^{(M)}(\sigma(l')) = 0$ and $X_{j+1}^{(M)}(\sigma(l)) > 0$ $(0 \le l \le I-1,\ 0 \le l' < k,\ k \le l'' \le I-1,\ 0 \le l''' < I-1$ and $1 \le j \le 3)$. For example we prove in the case of $j = 2$.

It holds that $\frac{\lambda}{M}X_j^{(M)}(\sigma(I-1))X_{j+1}^{(M)}(\sigma(I-1))(t-\sigma(I-1)) = 0$ for $j = 1,2,3$. From $P_j(I-1)$ $(1 \le j \le 3)$ it follows that

$$\{t > \sigma(I-1) : 0 = \tau^{jj+1}_{K^{jj+1}(I-1)+1} - T^{(M)}_{jj+1}(\sigma(I-1))\} = \emptyset.$$

Thus we have

$$s(I) = \min\{\infty, \infty, \infty\}.$$

The jump time $\sigma(I)$ constructed in [Case A]$\sim$[Case C] in $\mathfrak{P}(I)$ coincides with $s(I)$ of [Case a]$\sim$[Case c]. If $\sigma(I-1) = \infty$, then we do not need the solution for $t > \sigma(I-1)$ and we put $s(I) = \sigma(I) = \infty$. Thus equality $\sigma(I) = s(I)$ holds.

The jump time $\sigma(I)$ in $\mathfrak{P}(I)$ is determined uniquely by the jump times of standard Poisson processes and by the factors $\{X_j^{(M)}(0), \sigma(l), K^{jj+1}(l)\}_{0\le l\le I-1, 1\le j\le 3}$ which are given by $\mathfrak{P}(I-1)$ in $\mathfrak{Q}(I-1)$. Therefore $\sigma(I)$ is uniquely determined by $\mathfrak{Q}(I-1)$.

If $\sigma(I-1) < \infty$, the state of $\sigma(I-1)$ is kept for any t $(\sigma(I-1) < t < \sigma(I))$. If $\sigma(I) < \infty$, choose j $(= j_1$, say) whose term is the smallest in (1.3). When $t =$

$\sigma(I)$, $\frac{\lambda}{M}X_{j_1}^{(M)}(\sigma(I-1))X_{j_1+1}^{(M)}(\sigma(I-1))(t-\sigma(I-1))$ reaches at $\tau^{j_1 j_1+1}_{K^{j_1 j_1+1}(I-1)+1} - T_{j_1 j_1+1}^{(M)}(\sigma(I-1))$ and for $j_2 \neq j_1$ $(1 \leq j_2 \leq 3)$, $\frac{\lambda}{M}X_{j_2}^{(M)}(\sigma(I-1))X_{j_2+1}^{(M)}(\sigma(I-1))(t-\sigma(I-1))$ does not reach at $\tau^{j_2 j_2+1}_{K^{j_2 j_2+1}(I-1)+1} - T_{j_2 j_2+1}^{(M)}(\sigma(I-1))$. If $\sigma(I) < \infty$, we define $L^{jj+1} = K^{jj+1}(I-1)+1$ for $j = j_1$ and $L^{jj+1} = K^{jj+1}(I-1)$ for $j = j_2$. If $\sigma(I) = \infty$, we define $L^{jj+1} = K^{jj+1}(I-1)$ for $1 \leq j \leq 3$. We should give the unique solution of (1.1) for any $t \in \{u \in \mathbb{R}_+ : u \in (\sigma(I-1), \sigma(I)]\}$ by $(1 \leq j \leq 3)$

$$
\begin{aligned}
(1.6) \qquad X_j^{(M)}(\sigma(I-1)) &+ (L^{jj+1} - K^{jj+1}(I-1))\chi_{[\sigma(I),\infty)}(t) \\
&- (L^{j-1j} - K^{j-1j}(I-1))\chi_{[\sigma(I),\infty)}(t).
\end{aligned}
$$

We have $L^{jj+1} = K^{jj+1}(I)$ for $1 \leq j \leq 3$ and for any $t \in \{u \in \mathbb{R}_+ : u \in (\sigma(I-1), \sigma(I)]\}$ (1.6) coincides with $X_j^{(M)}(t)$ $(1 \leq j \leq 3)$ of (1.2) in $\mathfrak{P}(I)$. Therefore (1.2) in $\mathfrak{P}(I)$ is the unique solution of (1.1).

Assuming the proposition $\mathfrak{Q}(I-1)$, we have the proposition $\mathfrak{Q}(I)$.

By mathematical induction there exists a unique solution of (1.1) in $\mathbb{R}_+$. □

COROLLARY 1.1. *There exists a unique solution of equation* (1.1), *for* $t \in [0, t_0]$, *when* $t_0 \in \mathbb{R}_+$.

PROOF. The proof of existence and uniqueness of the solution of (1.1) is done step by step. We stop the proof when the step excess the time t_0. Thus we have the present corollary. □

For any v, $v \geq 0$, we define

$$
N_{jj+1}^v(t) = \begin{cases} N_{jj+1}(t), & 0 \leq t \leq v, \\ N_{jj+1}(v), & t > v. \end{cases}
$$

We consider the system in which $N_{12}(*)$ is replaced by $N_{12}^v(*)$ in (1.1). This system is

$$
(1.7) \qquad \begin{cases}
X_1^{(M)}(t) = X_1^{(M)}(0) + N_{12}^v(\dfrac{\lambda}{M}\displaystyle\int_0^t X_1^{(M)}(s)X_2^{(M)}(s)ds) \\
\qquad\qquad - N_{31}(\dfrac{\lambda}{M}\displaystyle\int_0^t X_3^{(M)}(s)X_1^{(M)}(s)ds), \\
X_2^{(M)}(t) = X_2^{(M)}(0) + N_{23}(\dfrac{\lambda}{M}\displaystyle\int_0^t X_2^{(M)}(s)X_3^{(M)}(s)ds) \\
\qquad\qquad - N_{12}^v(\dfrac{\lambda}{M}\displaystyle\int_0^t X_1^{(M)}(s)X_2^{(M)}(s)ds), \\
X_3^{(M)}(t) = X_3^{(M)}(0) + N_{31}(\dfrac{\lambda}{M}\displaystyle\int_0^t X_3^{(M)}(s)X_1^{(M)}(s)ds) \\
\qquad\qquad - N_{23}(\dfrac{\lambda}{M}\displaystyle\int_0^t X_2^{(M)}(s)X_3^{(M)}(s)ds), \\
X_1^{(M)}(0) + X_2^{(M)}(0) + X_3^{(M)}(0) = M.
\end{cases}
$$

We have the following theorem.

THEOREM 1.2. *There exists a unique solution of* (1.7) *in* $\mathbb{R}_+$.

PROOF. We fix a sample path of $(N_{12}^v(*), N_{23}(*), N_{31}(*))$. For the purpose of proving existence and uniqueness of the solution of (1.7) in a similar way as Theorem 1.1, we introduce the same definitions as in Theorem 1.1.

There exists an integer K_v, $K_v \geq 0$ such that $\tau_{K_v}^{12} \leq v < \tau_{K_v+1}^{12}$.

When for the fixed sample path the monotonically nondecreasing function $T_{12}^{(M)}(*)$ in Theorem 1.1 does not reach $\tau_{K_v+1}^{12}$, we prove the present theorem in just the same way as Theorem 1.1.

We consider the case in the following way. There is the smallest integer I_0, $I_0 \geq 1$, such that $K^{12}(I_0 - 1) = K_v$ and $K^{12}(I_0 - 1) + 1 = K^{12}(I_0) = K_v + 1$ in Theorem 1.1, when $\sigma(l) < \infty$, $X_1^{(M)}(\sigma(l)) > 0$ and $X_2^{(M)}(\sigma(l)) > 0$ for $0 \leq l \leq I_0 - 1$. In this situation I_0 is the smallest integer of $T_{12}^{(M)}(\sigma(I_0)) = \tau_{K_v+1}^{12}$.

Differently from Theorem 1.1, the standard Poisson process $N_{12}(*)$ in (1.1) is replaced by $N_{12}^v(*)$ in (1.7). There are no jumps of $N_{12}(*)$ after the I_0-th step and we replace $\tau_{K^{12}(I_0-1)+1}^{12}$ by infinity. By replacing (1.1) with (1.7), we define $\mathfrak{P}(I-1)$ and $\mathfrak{Q}(I-1)$.

Note that the proof from $I_0 - 1$ to I_0 is slightly different from the proof from $I - 1$ to I $(I > I_0)$ in the classification of cases of the mathematical induction.

We assume the proposition $\mathfrak{P}(I_0 - 1)$.

[Case A'1] We consider the case where $\sigma(l) < \infty$, $X_j^{(M)}(\sigma(l)) > 0$ for $0 \leq l \leq I_0 - 1$ and $1 \leq j \leq 3$. This case describes that $X_j^{(M)}(*)$ have positive values from the 0-th step to the $I_0 - 1$-th step.

From $P_1(I_0 - 1)$ in Theorem 1.1, it holds that $T_{12}^{(M)}(\sigma(I_0 - 1)) < \infty$.

We determine $\sigma(I_0)$ by

$$\sigma(I_0) = \min_{2 \leq j \leq 3} \{\sigma(I_0 - 1) + \frac{\tau_{K^{jj+1}(I_0-1)+1}^{jj+1} - T_{jj+1}^{(M)}(\sigma(I_0 - 1))}{\frac{\lambda}{M} X_j^{(M)}(\sigma(I_0 - 1)) X_{j+1}^{(M)}(\sigma(I_0 - 1))}, \infty\}.$$

Note that $\sigma(I_0) < \infty$ in [Case A'1]. If the term of $j = 2$ is the smallest in (1.3), for example, then we have $K^{12}(I_0) = K^{12}(I_0 - 1)$, $K^{23}(I_0) = K^{23}(I_0 - 1) + 1$ and $K^{31}(I_0) = K^{31}(I_0 - 1)$. If the terms of $j = 2, 3$ in (1.3) take the same value, then we have $K^{12}(I_0) = K^{12}(I_0 - 1)$, $K^{23}(I_0) = K^{23}(I_0 - 1) + 1$ and $K^{31}(I_0) = K^{31}(I_0 - 1) + 1$.

If the term of $j = 2$ is the smallest in (1.3), we have $K^{12}(I_0) = K^{12}(I_0 - 1)$, $K^{23}(I_0) = K^{23}(I_0 - 1) + 1$ and $K^{31}(I_0) = K^{31}(I_0 - 1)$. Then

$$\begin{aligned}
\sigma(I_0) &= \sigma(I_0 - 1) + \frac{\tau_{K^{23}(I_0-1)+1}^{23} - T_{23}^{(M)}(\sigma(I_0 - 1))}{\frac{\lambda}{M} X_2^{(M)}(\sigma(I_0 - 1)) X_3^{(M)}(\sigma(I_0 - 1))}, \\
\sigma(I_0) &< \sigma(I_0 - 1) + \frac{\tau_{K^{31}(I_0-1)+1}^{31} - T_{31}^{(M)}(\sigma(I_0 - 1))}{\frac{\lambda}{M} X_3^{(M)}(\sigma(I_0 - 1)) X_1^{(M)}(\sigma(I_0 - 1))}.
\end{aligned}$$

By $P_j(I_0-1)$ for $j=2,3$ the numerators are positive. Thus $\sigma(I_0)>\sigma(I_0-1)$ and

$$\begin{aligned}\tau^{23}_{K^{23}(I_0-1)+1} =&T^{(M)}_{23}(\sigma(I_0-1))+\frac{\lambda}{M}X^{(M)}_{12}(\sigma(I_0-1))X^{(M)}_3(\sigma(I_0-1))(\sigma(I_0)-\sigma(I_0-1)),\\ \tau^{31}_{K^{31}(I_0-1)+1} >&T^{(M)}_{31}(\sigma(I_0-1))+\frac{\lambda}{M}X^{(M)}_{3}(\sigma(I_0-1))X^{(M)}_1(\sigma(I_0-1))(\sigma(I_0)-\sigma(I_0-1)).\end{aligned}$$

[Step 5] We consider $P_1(I_0-1,I_0)$ and $P_1(I_0)$.
For $\sigma(I_0-1)<t<\sigma(I_0)$

$$T^{(M)}_{12}(t)=T^{(M)}_{12}(\sigma(I_0-1))+\frac{\lambda}{M}X^{(M)}_1(\sigma(I_0-1))X^{(M)}_2(\sigma(I_0-1))(t-\sigma(I_0-1)),$$

and

$$T^{(M)}_{12}(\sigma(I_0))=T^{(M)}_{12}(\sigma(I_0-1))+\frac{\lambda}{M}X^{(M)}_1(\sigma(I_0-1))X^{(M)}_2(\sigma(I_0-1))(\sigma(I_0)-\sigma(I_0-1)).$$

Since $\sigma(I_0)<\infty$, it holds that $T^{(M)}_{12}(u)<\infty$ for u, $u\in(\sigma(I_0-1),\sigma(I_0)]$. This leads

$$\tau^{12}_{K^{12}(I_0-1)}\le T^{(M)}_{12}(\sigma(I_0-1))<T^{(M)}_{12}(t)<\tau^{12}_{K^{12}(I_0-1)+1}=\infty,$$

and

$$\tau^{12}_{K^{12}(I_0-1)}=\tau^{12}_{K^{12}(I_0)}\le T^{(M)}_{12}(\sigma(I_0-1))<T^{(M)}_{12}(\sigma(I_0))<\tau^{12}_{K^{12}(I_0-1)+1}=\tau^{12}_{K^{12}(I_0)+1}=\infty.$$

Thus $P_1(I_0-1,I_0)$ and $P_1(I_0)$ hold. $\diamond$

We have $P_2(I_0-1,I_0)$ and $P_2(I_0)$ similarly as in [Step 1] in Theorem 1.1. Similarly as in [Step 2] in Theorem 1.1, $P_3(I_0-1,I_0)$ and $P_3(I_0)$ hold.

If the terms of $j=2,3$ in (1.3) take the same value, we have $K^{12}(I_0)=K^{12}(I_0-1)$, $K^{23}(I_0)=K^{23}(I_0-1)+1$ and $K^{31}(I_0)=K^{31}(I_0-1)+1$. And

$$\begin{aligned}\sigma(I_0) =&\sigma(I_0-1)+\frac{\tau^{23}_{K^{23}(I_0-1)+1}-T^{(M)}_{23}(\sigma(I_0-1))}{\frac{\lambda}{M}X^{(M)}_2(\sigma(I_0-1))X^{(M)}_3(\sigma(I_0-1))},\\ \sigma(I_0) =&\sigma(I_0-1)+\frac{\tau^{31}_{K^{31}(I_0-1)+1}-T^{(M)}_{31}(\sigma(I_0-1))}{\frac{\lambda}{M}X^{(M)}_3(\sigma(I_0-1))X^{(M)}_1(\sigma(I_0-1))}.\end{aligned}$$

$P_1(I_0-1,I_0)$ and $P_1(I_0)$ hold in a similar way as [Step 5]. Similarly as in [Step 1] in Theorem 1.1, $P_j(I_0-1,I_0)$ and $P_j(I_0)$ hold for $2\le j\le 3$.

Note that for t, $\sigma(I_0-1)<t<\sigma(I_0)$ $P_1(I_0-1,I_0)$ leads

$$\sum_{i=1}^{I_0}(K^{12}(i)-K^{12}(i-1))\chi_{[\sigma(i),\infty)}(t)=K^{12}(I_0-1)=N^{v}_{12}(T^{(M)}_{12}(t)),$$

and $P_1(I_0)$ leads

$$\begin{aligned}&\sum_{i=1}^{I_0}(K^{12}(i)-K^{12}(i-1))\chi_{[\sigma(i),\infty)}(\sigma(I_0))\\&=K^{12}(I_0)=K^{12}(I_0-1)=N_{12}^{v}(T_{12}^{(M)}(\sigma(I_0))).\end{aligned}$$

For $\sigma(I_0-1)<t<\sigma(I_0)$, (1.2) satisfies (1.7) and, at $\sigma(I_0)$, (1.2) satisfies (1.7).
Therefore the proposition $\mathfrak{P}(I_0)$ is obtained in [Case A'1].

[Case B'1] We consider the case where $\sigma(l)<\infty$, for some k satisfying $0\le k\le I_0-1$, $X_1^{(M)}(\sigma(l))>0$, $X_2^{(M)}(\sigma(l))>0$, $X_3^{(M)}(\sigma(l'))>0$ and $X_3^{(M)}(\sigma(l''))=0$ ($0\le l\le I_0-1$, $0\le l'<k$, $k\le l''\le I_0-1$ and $1\le j\le 3$). In this case the value of $X_3^{(M)}(*)$ has reached zero until the I_0-1-th step after several times of [Case B] in Theorem 1.1.

We determine $\sigma(I_0)$ by

$$\begin{aligned}\sigma(I_0)=&\min\{\infty,\infty,\infty\}\\=&\infty.\end{aligned}$$

[Step 6] We consider $P_1(I_0-1,I_0)$.

For t satisfying $\sigma(I_0-1)<t<\sigma(I_0)=\infty$, we have

$$\tau^{12}_{K^{12}(I_0-1)}\le T_{12}^{(M)}(t)<\tau^{12}_{K^{12}(I_0-1)+1}=\infty.$$

Thus $P_1(I_0-1,I_0)$ holds. $\diamondsuit$

Similarly as in [Step 4] in Theorem 1.1, $P_j(I_0-1,I_0)$ hold for $j=2,3$.

In $(\sigma(I_0-1),\infty)$, (1.2) satisfies (1.7). Therefore $\mathfrak{P}(I_0)$ holds in [Case B'1].

Assuming the proposition $\mathfrak{P}(I_0-1)$, we have the proposition $\mathfrak{P}(I_0)$.

We assume the proposition $\mathfrak{P}(I-1)$ for $I>I_0$.

[Case A'] We consider the case where $\sigma(l)<\infty$, $X_j^{(M)}(\sigma(l))>0$ for $0\le l\le I-1$ and $1\le j\le 3$. This is the case that $X_j^{(M)}(*)$ have positive values until the $I-1$-th step.

In the present system of (1.7) we implicitly assume $K^{12}(I_0-1)=\cdots=K^{12}(I-1)$. Thus $\tau^{12}_{K^{12}(I-1)+1}=\tau^{12}_{K^{12}(I_0-1)+1}=\infty$.

As $P_1(I-1)$ in $\mathfrak{P}(I-1)$ is assumed, it holds that $T_{12}^{(M)}(\sigma(I-1))<\infty$.

We determine $\sigma(I)$ by

$$\sigma(I)=\min_{2\le j\le 3}\{\sigma(I-1)+\frac{\tau^{jj+1}_{K^{jj+1}(I-1)+1}-T_{jj+1}^{(M)}(\sigma(I-1))}{\frac{\lambda}{M}X_j^{(M)}(\sigma(I-1))X_{j+1}^{(M)}(\sigma(I-1))},\infty\}.$$

Note that $\sigma(I)<\infty$ in [Case A']. If the term of $j=2$ is the smallest in (1.3), then we have $K^{12}(I)=K^{12}(I-1)$, $K^{23}(I)=K^{23}(I-1)+1$ and $K^{31}(I)=K^{31}(I-1)$. If the terms of $j=2,3$ in (1.3) take the same value, then we have

$K^{12}(I) = K^{12}(I-1)$, $K^{23}(I) = K^{23}(I-1)+1$ and $K^{31}(I) = K^{31}(I-1)+1$. The implicit assumption is satisfied to the I-th step.

If the term of $j = 2$ is the smallest in (1.3), we have $K^{12}(I) = K^{12}(I-1)$, $K^{23}(I) = K^{23}(I-1)+1$ and $K^{31}(I) = K^{31}(I-1)$. Then

$$\begin{aligned}
\sigma(I) =& \sigma(I-1) + \frac{\tau^{23}_{K^{23}(I-1)+1} - T^{(M)}_{23}(\sigma(I-1))}{\frac{\lambda}{M}X^{(M)}_2(\sigma(I-1))X^{(M)}_3(\sigma(I-1))},\\
\sigma(I) <& \sigma(I-1) + \frac{\tau^{31}_{K^{31}(I-1)+1} - T^{(M)}_{31}(\sigma(I-1))}{\frac{\lambda}{M}X^{(M)}_3(\sigma(I-1))X^{(M)}_1(\sigma(I-1))}.
\end{aligned}$$

If the term of $j = 2, 3$ in (1.3) take the same value, we have $K^{12}(I) = K^{12}(I-1)$, $K^{23}(I) = K^{23}(I-1)+1$ and $K^{31}(I) = K^{31}(I-1)+1$. And

$$\begin{aligned}
\sigma(I) =& \sigma(I-1) + \frac{\tau^{23}_{K^{23}(I-1)+1} - T^{(M)}_{23}(\sigma(I-1))}{\frac{\lambda}{M}X^{(M)}_2(\sigma(I-1))X^{(M)}_3(\sigma(I-1))},\\
\sigma(I) =& \sigma(I-1) + \frac{\tau^{31}_{K^{31}(I-1)+1} - T^{(M)}_{31}(\sigma(I-1))}{\frac{\lambda}{M}X^{(M)}_3(\sigma(I-1))X^{(M)}_1(\sigma(I-1))}.
\end{aligned}$$

In these above two cases, similarly as in [Case A'1], $P_j(I-1, I)$ and $P_j(I)$ hold for $j = 1, 2, 3$.

Note that for $\sigma(I-1) < t < \sigma(I)$ $P_1(I-1, I)$ leads

$$\begin{aligned}
&\sum_{i=1}^{I}(K^{12}(i) - K^{12}(i-1))\chi_{[\sigma(i),\infty)}(t)\\
&= K^{12}(I-1) = \cdots = K^{12}(I_0-1) = N^{v}_{12}(T^{(M)}_{12}(t)),
\end{aligned}$$

and that $P_1(I)$ leads

$$\begin{aligned}
&\sum_{i=1}^{I}(K^{12}(i) - K^{12}(i-1))\chi_{[\sigma(i),\infty)}(\sigma(I))\\
&= K^{12}(I) = K^{12}(I-1) = \cdots = K^{12}(I_0-1) = N^{v}_{12}(T^{(M)}_{12}(\sigma(I))).
\end{aligned}$$

For $\sigma(I-1) < t < \sigma(I)$, (1.2) satisfies (1.7) and, at $\sigma(I)$, (1.2) satisfies (1.7).

Therefore in [Case A'] the proposition $\mathfrak{P}(I)$ holds.

[Case B'] We consider the case where $\sigma(l) < \infty$, $X^{(M)}_1(\sigma(l)) > 0$, $X^{(M)}_2(\sigma(l)) > 0$, $X^{(M)}_3(\sigma(l')) > 0$ and $X^{(M)}_3(\sigma(I-1)) = 0$ for $0 \le l \le I-1$, $0 \le l' < I-1$ and $1 \le j \le 3$. $X^{(M)}_3(*)$ has come to the value zero at $\sigma(I-1)$, before $X^{(M)}_1(*)$ comes to the value zero.

In this case we have $K^{12}(I_0-1) = \cdots = K^{12}(I-1)$ by several times of [Case A'].

We determine $\sigma(I)$ by

$$\begin{aligned}\sigma(I) =& \min\{\infty, \infty, \infty\} \\ =& \infty.\end{aligned}$$

Similarly as in [Case B'1] we prove $P_j(I-1, I)$ for $j = 1, 2, 3$.
In $(\sigma(I-1), \infty)$, (1.2) satisfies (1.7). Therefore $\mathfrak{P}(I)$ is obtained in [Case B'].

[Case C'] We consider the case where $\sigma(l) < \infty$, for some k satisfying $I_0 - 1 < k \leq I-1$, $X_1^{(M)}(\sigma(l')) > 0$, $X_1^{(M)}(\sigma(l'')) = 0$, $X_2^{(M)}(\sigma(l)) > 0$, and $X_3^{(M)}(\sigma(l)) > 0$ ($0 \leq l \leq I-1$, $0 \leq l' < k$ and $k \leq l'' \leq I-1$ and $1 \leq j \leq 3$). In this case the value of $X_1^{(M)}(*)$ has come to zero at $\sigma(k)$ and kept it in $[\sigma(k), \sigma(I-1)]$.

In this case we implicitly assume $K^{12}(I_0-1) = \cdots = K^{12}(I-1)$ and $K^{31}(k) = \cdots = K^{31}(I-1)$. We have $X_1^{(M)}(t) = 0$ for $t \in [\sigma(k), \sigma(I-1)]$.

We determine $\sigma(I)$ by

$$\begin{aligned}\sigma(I) =& \min\{\sigma(I-1) + \frac{\tau^{23}_{K^{23}(I-1)+1} - T_{23}^{(M)}(\sigma(I-1))}{\frac{\lambda}{M} X_2^{(M)}(\sigma(I-1)) X_3^{(M)}(\sigma(I-1))}, \infty, \infty\} \\ =& \sigma(I-1) + \frac{\tau^{23}_{K^{23}(I-1)+1} - T_{23}^{(M)}(\sigma(I-1))}{\frac{\lambda}{M} X_2^{(M)}(\sigma(I-1)) X_3^{(M)}(\sigma(I-1))}.\end{aligned}$$

Note that $\sigma(I) < \infty$ in [Case C']. In this case $K^{12}(I) = K^{12}(I-1)$, $K^{23}(I) = K^{23}(I-1) + 1$ and $K^{31}(I) = K^{31}(I-1)$. Thus the implicit assumption holds to the I-th step.

In a similarly way as [Step 5], $P_1(I-1, I)$ and $P_1(I)$ hold. Similarly as in [Step 1] of Theorem 1.1, we have $P_2(I-1, I)$ and $P_2(I)$. Similarly as in [Step 3] of Theorem 1.1, $P_3(I-1, I)$ and $P_3(I)$ hold.

In $(\sigma(I-1), \sigma(I))$, (1.2) satisfies (1.7) and, at $\sigma(I)$, (1.2) satisfies (1.7). Thus we obtain the proposition $\mathfrak{P}(I)$ in [Case C'].

[Case D'] We consider the case where $\sigma(l) < \infty$, for some k satisfying $I_0 - 1 \leq k < I-1$, $X_1^{(M)}(\sigma(l')) > 0$, $X_1^{(M)}(\sigma(l'')) = 0$, $X_2^{(M)}(\sigma(l)) > 0$, $X_3^{(M)}(\sigma(l''')) > 0$ and $X_3^{(M)}(\sigma(I-1)) = 0$ ($0 \leq l \leq I-1$, $0 \leq l' < k$, $k \leq l'' \leq I-1$, $0 \leq l''' < I-1$ and $1 \leq j \leq 3$). This is the first case that the value of $X_3^{(M)}(*)$ reaches zero after several times of [Case C'].

In the present case we implicitly have $K^{12}(I_0-1) = \cdots = K^{12}(I-1)$ and $K^{23}(k) = \cdots = K^{23}(I-1)$ by several times of [Case C'].

We determine $\sigma(I)$ by

$$\begin{aligned}\sigma(I) =& \min\{\infty, \infty, \infty\} \\ =& \infty.\end{aligned}$$

Similarly as in [Step 4] in Theorem 1.1, $P_j(I-1, I)$ hold for $j = 2, 3$. We prove $P_1(I-1, I)$ similarly as in [Step 6].

In $(\sigma(I-1), \infty)$, (1.2) satisfies (1.7). Therefore $\mathfrak{P}(I)$ holds in [Case D'].

Assuming the proposition $\mathfrak{P}(I-1)$ $(I > I_0)$, we have the proposition $\mathfrak{P}(I)$. By mathematical induction (1.2) satisfies (1.7) in $\mathbb{R}_+$.

Now we shall prove that the solution constructed above is unique.

Let $X_j^{(M)}(*)$ $(1 \leq j \leq 3)$ be any one of the solutions of (1.7). In a similar way as the previous theorem, we see that the random variables $X_j^{(M)}(*)$ are nonnegative, bounded and integer valued in $[0, M]$ for $1 \leq j \leq 3$. It follows that the integrals $\frac{\lambda}{M}\int_0^t X_j^{(M)}(s)X_{j+1}^{(M)}(s)ds$ $(t \in \mathbb{R}_+$ and $1 \leq j \leq 3)$ are nonnegative, monotonically nondecreasing and inequalities $0 \leq \frac{\lambda}{M}\int_0^t X_j^{(M)}(s)X_{j+1}^{(M)}(s)ds \leq \frac{\lambda M t}{4}$ hold. Thus all possible classifications are covered in the following proof.

We assume the proposition $\mathfrak{Q}(I_0 - 1)$.

The system changes the previous state at $s(I_0)$ such that, by using the factors of $\mathfrak{P}(I_0-1)$ in $\mathfrak{Q}(I_0-1)$,

$$s(I_0) = \min_{1\leq j\leq 3}\{\inf\{t > \sigma(I_0-1) : \frac{\lambda}{M}X_j^{(M)}(\sigma(I_0-1))X_{j+1}^{(M)}(\sigma(I_0-1))(t-\sigma(I_0-1)) = \tau^{jj+1}_{K^{jj+1}(I_0-1)+1} - T_{jj+1}^{(M)}(\sigma(I_0-1))\}\}.$$

[Case a'1] We consider the case where $\sigma(l) < \infty$, $X_j^{(M)}(\sigma(l)) > 0$ for $0 \leq l \leq I_0-1$ and $1 \leq j \leq 3$.

Note that $\tau^{12}_{K^{12}(I_0-1)+1} = \infty$.

For $t > \sigma(I_0-1)$ we have

$$\frac{\lambda}{M}X_1^{(M)}(\sigma(I_0-1))X_2^{(M)}(\sigma(I_0-1))(t-\sigma(I_0-1)) < \infty.$$

As $T_{12}^{(M)}(\sigma(I_0-1))$ is bounded from $P_1(I_0-1)$ given in Theorem 1.1,

$$\inf\{t > \sigma(I_0-1) : \frac{\lambda}{M}X_1^{(M)}(\sigma(I_0-1))X_2^{(M)}(\sigma(I_0-1))(t-\sigma(I_0-1)) = \tau^{12}_{K^{12}(I_0-1)+1} - T_{12}^{(M)}(\sigma(I_0-1))\}\} = \emptyset.$$

It follows that

$$s(I) = \min_{2\leq j\leq 3}\{\sigma(I-1) + \frac{\tau^{jj+1}_{K^{jj+1}(I_0-1)+1} - T_{jj+1}^{(M)}(\sigma(I_0-1))}{\frac{\lambda}{M}X_j^{(M)}(\sigma(I_0-1))X_{j+1}^{(M)}(\sigma(I_0-1))}, \infty\}.$$

[Case b'1] We consider the case where $\sigma(l) < \infty$, for some k satisfying $0 \leq k \leq I_0 - 1$ $X_1^{(M)}(\sigma(l)) > 0$, $X_2^{(M)}(\sigma(l)) > 0$, $X_3^{(M)}(\sigma(l')) > 0$ and $X_3^{(M)}(\sigma(l'')) = 0$ $(0 \leq l \leq I_0-1$, $0 \leq l' < k$, $k \leq l'' \leq I_0 - 1$ and $1 \leq j \leq 3)$.

As $\frac{\lambda}{M}X_j^{(M)}(\sigma(I_0-1))X_{j+1}^{(M)}(\sigma(I_0-1))(t-\sigma(I_0-1)) = 0$ for $t > \sigma(I_0-1)$ $(j = 2, 3)$ and $P_j(I_0-1)$ hold, we have for $j = 2, 3$

$$\{t > \sigma(I_0-1) : 0 = \tau^{jj+1}_{K^{jj+1}(I_0-1)+1} - T_{jj+1}^{(M)}(\sigma(I_0-1))\} = \emptyset.$$

It follows that

$$s(I_0) = \min\{\infty, \infty, \infty\}.$$

The jump time $\sigma(I_0)$ in [Case A'1] and [Case B'1] coincides with $s(I_0)$ of [Case a'1] and [Case b'1]. If $\sigma(I_0 - 1) = \infty$, then we do not need the solution for $t > \sigma(I_0 - 1)$ and we put $s(I_0) = \sigma(I_0) = \infty$. Thus $\sigma(I_0) = s(I_0)$ holds.

The jump time $\sigma(I_0)$ is determined uniquely by jump times of standard Poisson processes and by the factors of $\mathfrak{P}(I_0 - 1)$ in $\mathfrak{Q}(I_0 - 1)$. Therefore $\sigma(I_0)$ is uniquely determined by $\mathfrak{Q}(I_0 - 1)$.

By using the same discussion as in Theorem 1.1, we should give the unique solution of (1.7) in $\{u \in \mathbb{R}_+ : u \in (\sigma(I_0 - 1), \sigma(I_0)]\}$ by (1.2) with $K^{jj+1}(I_0)$ $(1 \le j \le 3)$ given in $\mathfrak{P}(I_0)$. Therefore (1.2) in $\mathfrak{P}(I_0)$ is the unique solution of (1.7).

Assuming the proposition $\mathfrak{Q}(I_0 - 1)$, we have the proposition $\mathfrak{Q}(I_0)$.

We assume $\mathfrak{Q}(I - 1)$ $(I > I_0)$.

The system changes the previous state at $s(I)$ such that, by using the factors of $\mathfrak{P}(I - 1)$ in $\mathfrak{Q}(I - 1)$,

$$\begin{aligned} s(I) = \min_{1 \le j \le 3} \{\inf\{t > \sigma(I-1) : &\frac{\lambda}{M} X_j^{(M)}(\sigma(I-1)) X_{j+1}^{(M)}(\sigma(I-1))(t - \sigma(I-1)) \\ &= \tau^{jj+1}_{K^{jj+1}(I-1)+1} - T^{(M)}_{jj+1}(\sigma(I-1))\}\}. \end{aligned}$$

[Case a'] We consider the case where $\sigma(l) < \infty$, $X_j^{(M)}(\sigma(l)) > 0$ for $0 \le l \le I - 1$ and $1 \le j \le 3$.

We have $K^{12}(I_0 - 1) = \cdots = K^{12}(I - 1)$ and $\tau^{12}_{K^{12}(I-1)+1} = \cdots = \tau^{12}_{K^{12}(I_0-1)+1} = \infty$ in [Case a'].

Similarly as in [Case a'1] we have

$$s(I) = \min_{2 \le j \le 3} \{\sigma(I - 1) + \frac{\tau^{jj+1}_{K^{jj+1}(I-1)+1} - T^{(M)}_{jj+1}(\sigma(I-1))}{\frac{\lambda}{M} X_j^{(M)}(\sigma(I-1)) X_{j+1}^{(M)}(\sigma(I-1))}, \infty\}.$$

[Case b'] We consider the case where $\sigma(l) < \infty$, $X_1^{(M)}(\sigma(l)) > 0$, $X_2^{(M)}(\sigma(l)) > 0$, $X_3^{(M)}(\sigma(l')) > 0$ and $X_3^{(M)}(\sigma(I-1)) = 0$ for $0 \le l \le I - 1$, $0 \le l' < I - 1$ and $1 \le j \le 3$.

Similarly as in [Case b'1], we have

$$s(I) = \min\{\infty, \infty, \infty\}.$$

[Case c'] We consider the case where $\sigma(l) < \infty$, for some k satisfying $I_0 - 1 < k \le I - 1$, $X_1^{(M)}(\sigma(l')) > 0$, $X_1^{(M)}(\sigma(l'')) = 0$, $X_2^{(M)}(\sigma(l)) > 0$, and $X_3^{(M)}(\sigma(l)) > 0$ $(0 \le l \le I - 1$, $0 \le l' < k$ and $k \le l'' \le I - 1$ and $1 \le j \le 3)$.

We have $\frac{\lambda}{M}X_3^{(M)}(\sigma(I-1))X_1^{(M)}(\sigma(I-1))(t-\sigma(I-1))=0$ for $t>\sigma(I-1)$. From $P_3(I-1)$ in $\mathfrak{P}(I-1)$, it follows that

$$\{t>\sigma(I-1):0=\tau^{31}_{K^{31}(I-1)+1}-T^{(M)}_{31}(\sigma(I-1))\}=\emptyset.$$

We have

$$s(I)=\min\{\sigma(I-1)+\frac{\tau^{23}_{K^{23}(I-1)+1}-T^{(M)}_{23}(\sigma(I-1))}{\frac{\lambda}{M}X_2^{(M)}(\sigma(I-1))X_3^{(M)}(\sigma(I-1))},\infty,\infty\}.$$

[Case d'] We consider the case where $\sigma(l)<\infty$, for some k satisfying $I_0-1\leq k<I-1$, $X_1^{(M)}(\sigma(l'))>0$, $X_1^{(M)}(\sigma(l''))=0$, $X_2^{(M)}(\sigma(l))>0$, $X_3^{(M)}(\sigma(l'''))>0$ and $X_3^{(M)}(\sigma(I-1))=0$ ($0\leq l\leq I-1$, $0\leq l'<k$, $k\leq l''\leq I-1$, $0\leq l'''<I-1$ and $1\leq j\leq 3$).

In this case we have $K^{12}(I_0-1)=\cdots=K^{12}(I-1)$ and $K^{23}(k)=\cdots=K^{23}(I-1)$.

As we have $\frac{\lambda}{M}X_j^{(M)}(\sigma(I-1))X_{j+1}^{(M)}(\sigma(I-1))(t-\sigma(I-1))=0$ ($j=2,3$) for $t>\sigma(I-1)$ and $P_j(I-1)$ ($j=2,3$) hold, it follows that

$$\{t>\sigma(I-1):0=\tau^{jj+1}_{K^{jj+1}(I-1)+1}-T^{(M)}_{jj+1}(\sigma(I-1))\}=\emptyset.$$

Thus

$$s(I)=\min\{\infty,\infty,\infty\}.$$

The jump time $\sigma(I)$ in [Case A']~[Case D'] coincides with $s(I)$ of [Case a']~[Case d']. If $\sigma(I-1)=\infty$, then we do not need the solution for $t>\sigma(I-1)$ and we put $s(I)=\sigma(I)=\infty$. Thus $\sigma(I)=s(I)$ holds.

The jump time $\sigma(I)$ is determined uniquely by jump times of standard Poisson processes and by the factors of $\mathfrak{P}(I-1)$ in $\mathfrak{Q}(I-1)$. Therefore $\sigma(I)$ is uniquely determined by $\mathfrak{Q}(I-1)$.

In a similar way as Theorem 1.1, we should give the unique solution of (1.7) in $\{u\in\mathbb{R}_+ : u\in(\sigma(I-1),\sigma(I)]\}$ by (1.2) with $K^{jj+1}(I)$ ($1\leq j\leq 3$) given in $\mathfrak{P}(I)$. Therefore (1.2) in $\mathfrak{P}(I)$ is the unique solution of (1.7).

Assuming the proposition $\mathfrak{Q}(I-1)$ ($I>I_0$), we have the proposition $\mathfrak{Q}(I)$.

By mathematical induction there exists a unique solution of (1.7) in $\mathbb{R}_+$. □

COROLLARY 1.2. *There exists a unique solution of equation* (1.7), *when* $t\in[0,t_0]$ *for* $t_0\in\mathbb{R}_+$.

2. A stochastic structure of the model

From now on, we assume that $X_i^{(M)}(0)$, $N_{jj+1}(*)$ are mutually independent ($1\leq i\leq 3$, $1\leq j\leq 3$). We define the reference family $(\mathcal{F}_t^{jj+1})_{t\geq 0}$ ($j=1,2,3$)

by

$$\begin{aligned}\mathcal{F}_t^{jj+1} =&\sigma(X_i^{(M)}(0) : 1 \leq i \leq 3)\\ &\vee \sigma(N_{jj+1}(s) : 0 \leq s \leq t)\\ &\vee \sigma(N_{ii+1}(u) : u \geq 0, 1 \leq i \leq 3, i \neq j).\end{aligned}$$

Let random times $T_{jj+1}^{(M)}(t)$ ($t \in \mathbb{R}_+$, $j = 1,2,3$) be the same as in the previous section, that is

$$T_{jj+1}^{(M)}(t) = \frac{\lambda}{M}\int_0^t X_j^{(M)}(s)X_{j+1}^{(M)}(s)ds.$$

From equation (1.1), for $T^{(M)}(t) = (T_{12}^{(M)}(t), T_{23}^{(M)}(t), T_{31}^{(M)}(t))$ ($t \in \mathbb{R}_+$) we have the following relation:

$$\tag{2.1} \begin{cases} T_{12}^{(M)}(t) = \dfrac{\lambda}{M}\displaystyle\int_0^t (X_1^{(M)}(0) + N_{12}(T_{12}^{(M)}(s)) - N_{31}(T_{31}^{(M)}(s))) \\ \qquad\qquad (X_2^{(M)}(0) + N_{23}(T_{23}^{(M)}(s)) - N_{12}(T_{12}^{(M)}(s)))ds, \\ T_{23}^{(M)}(t) = \dfrac{\lambda}{M}\displaystyle\int_0^t (X_2^{(M)}(0) + N_{23}(T_{23}^{(M)}(s)) - N_{12}(T_{12}^{(M)}(s))) \\ \qquad\qquad (X_3^{(M)}(0) + N_{31}(T_{31}^{(M)}(s)) - N_{23}(T_{23}^{(M)}(s)))ds, \\ T_{31}^{(M)}(t) = \dfrac{\lambda}{M}\displaystyle\int_0^t (X_3^{(M)}(0) + N_{31}(T_{31}^{(M)}(s)) - N_{23}(T_{23}^{(M)}(s))) \\ \qquad\qquad (X_1^{(M)}(0) + N_{12}(T_{12}^{(M)}(s)) - N_{31}(T_{31}^{(M)}(s)))ds, \\ T^{(M)}(0) = 0. \end{cases}$$

THEOREM 2.1. *When we fix the sample path $\omega \in \Omega$, $T^{(M)}(t)(\omega)$ is uniquely determined.*

PROOF. For each $t \in \mathbb{R}_+$, we define

$$\tag{2.2} \begin{cases} X_1^{(M)}(t) =X_1^{(M)}(0) + N_{12}(T_{12}^{(M)}(t)) - N_{31}(T_{31}^{(M)}(t)), \\ X_2^{(M)}(t) =X_2^{(M)}(0) + N_{23}(T_{12}^{(M)}(t)) - N_{12}(T_{12}^{(M)}(t)), \\ X_3^{(M)}(t) =X_3^{(M)}(0) + N_{12}(T_{31}^{(M)}(t)) - N_{23}(T_{31}^{(M)}(t)). \end{cases}$$

From (2.1) and (2.2), we have

$$\tag{2.3} \begin{cases} T_{12}^{(M)}(t) =\dfrac{\lambda}{M}\displaystyle\int_0^t X_1^{(M)}(s)X_2^{(M)}(s)ds, \\ T_{23}^{(M)}(t) =\dfrac{\lambda}{M}\displaystyle\int_0^t X_2^{(M)}(s)X_3^{(M)}(s)ds, \\ T_{31}^{(M)}(t) =\dfrac{\lambda}{M}\displaystyle\int_0^t X_3^{(M)}(s)X_1^{(M)}(s)ds. \end{cases}$$

It follows that

$$\begin{cases} X_1^{(M)}(t) = X_1^{(M)}(0) + N_{12}(\frac{\lambda}{M}\int_0^t X_1^{(M)}(s)X_2^{(M)}(s)ds) \\ \qquad\qquad - N_{31}(\frac{\lambda}{M}\int_0^t X_3^{(M)}(s)X_1^{(M)}(s)ds), \\ X_2^{(M)}(t) = X_2^{(M)}(0) + N_{23}(\frac{\lambda}{M}\int_0^t X_2^{(M)}(s)X_3^{(M)}(s)ds) \\ \qquad\qquad - N_{12}(\frac{\lambda}{M}\int_0^t X_1^{(M)}(s)X_2^{(M)}(s)ds), \\ X_3^{(M)}(t) = X_3^{(M)}(0) + N_{31}(\frac{\lambda}{M}\int_0^t X_3^{(M)}(s)X_1^{(M)}(s)ds) \\ \qquad\qquad - N_{23}(\frac{\lambda}{M}\int_0^t X_2^{(M)}(s)X_3^{(M)}(s)ds). \end{cases}$$

Therefore there exists a solution of the above equation and the solution is represented by (2.2).

By the way there exists a unique solution of the above equation by Theorem1.1. If there exist two solutions $T^{(M)}(t) = (T_{12}^{(M)}(t), T_{23}^{(M)}(t), T_{31}^{(M)}(t))$ and $T^{(M)*}(t) = (T_{12}^{(M)*}(t), T_{23}^{(M)*}(t), T_{31}^{(M)*}(t))$ of equation (2.1), then by (2.3)

$$\begin{aligned} T_{12}^{(M)}(t) = T_{12}^{(M)*}(t) &= \frac{\lambda}{M}\int_0^t X_1^{(M)}(s)X_2^{(M)}(s)ds, \\ T_{23}^{(M)}(t) = T_{23}^{(M)*}(t) &= \frac{\lambda}{M}\int_0^t X_2^{(M)}(s)X_3^{(M)}(s)ds, \\ T_{31}^{(M)}(t) = T_{31}^{(M)*}(t) &= \frac{\lambda}{M}\int_0^t X_3^{(M)}(s)X_1^{(M)}(s)ds. \end{aligned}$$

Therefore $T^{(M)}(t) = T^{(M)*}(t)$.

This completes the proof. □

COROLLARY 2.1. *When we fix the sample path $\omega \in \Omega$, $T^{(M)}(t)$ is uniquely determined for $t \in [0, t_0]$ ($t_0 \in \mathbb{R}_+$).*

PROOF. Applying Corollarly 1.1 to Theorem 2.1 we have the present corollarly. □

For any v, $v \geq 0$, we define a random field $\Phi_\omega^v : \mathbb{R}_+^3 \to \mathbb{R}_+^3$ by

$$\Phi_\omega^v((x_1, x_2, x_3)) = \begin{pmatrix} \frac{\lambda}{M}(X_1^{(M)}(0) + N_{12}^v(x_1) - N_{31}(x_3))(X_2^{(M)}(0) + N_{23}(x_2) - N_{12}^v(x_1)) \\ \frac{\lambda}{M}(X_2^{(M)}(0) + N_{23}(x_2) - N_{12}^v(x_1))(X_3^{(M)}(0) + N_{31}(x_3) - N_{23}(x_2)) \\ \frac{\lambda}{M}(X_3^{(M)}(0) + N_{31}(x_3) - N_{23}(x_2))(X_1^{(M)}(0) + N_{12}^v(x_1) - N_{31}(x_3)) \end{pmatrix}'.$$

Put $S(t) = (S_1(t), S_2(t), S_3(t))$ to be the solution of

$$
(2.4) \qquad \begin{cases} S(t)(\omega) = \displaystyle\int_0^t \Phi_\omega^v(S(s)(\omega))ds, \\ S(0) = 0. \end{cases}
$$

THEOREM 2.2. *When we fix the sample path $\omega \in \Omega$, $S(t)(\omega)$ is uniquely determined.*

PROOF. Applying Theorem 1.2 to $S(t)$, the present theorem is concluded in a similar way as in Theorem 2.1. □

COROLLARY 2.2. *When we fix the sample path $\omega \in \Omega$, $S(t)$ is uniquely determined for $t \in [0, t_0]$ $(t_0 \in \mathbb{R}_+)$.*

PROOF. Corollarly 1.2 and Theorem 2.2 lead the present corollarly. □

LEMMA 2.1. *$S(t)$ is $\mathcal{F}_v^{12}$-measurable.*

PROOF. Since $\Phi_\omega^v(x)$ is represented by the generators of $\mathcal{F}_v^{12}$, $\Phi_\omega^v(x)$ is $\mathcal{F}_v^{12}$-measurable.

There exists a non-random function F^v such that

$$
S(t) = F^v(t; X^{(M)}(0), N_{12}^v(u), N_{23}(u), N_{31}(u), u \geq 0),
$$

where $X^{(M)}(0) = (X_1^{(M)}(0), X_2^{(M)}(0), X_3^{(M)}(0))$. As $S(t)$ is represented by the generators of $\mathcal{F}_v^{12}$, $S(t)$ is $\mathcal{F}_v^{12}$-measurable. □

Now, we prove the following lemma.

LEMMA 2.2. *For each j, t $(1 \leq j \leq 3,\ t \in \mathbb{R}_+)$, $T_{jj+1}^{(M)}(t)$ is a stopping time with respect to the reference family $(\mathcal{F}_t^{jj+1})_{t \geq 0}$.*

PROOF. We consider the case of $j = 1$, for example. To be proved is that, for any $v \in \mathbb{R}_+$,

$$
(T_{12}^{(M)}(t) \leq v) \equiv \{\omega; T_{12}^{(M)}(t)(\omega) \leq v\} \in \mathcal{F}_v^{12}.
$$

We claim that $(T_{12}^{(M)}(t) \leq v) = (S_1(t) \leq v)$.

For any $\omega \in (T_{12}^{(M)}(t) \leq v)$. $T_{12}^{(M)}(s)$ is a monotonically nondecreasing function for $s \geq 0$ and recall Corollary 1.1. It follows that $0 \leq T_{12}^{(M)}(u) \leq T_{12}^{(M)}(t)$ for $0 \leq u \leq t$ and that $N_{12}^v(T_{12}^{(M)}(u)) = N_{12}(T_{12}^{(M)}(u))$ for $0 \leq u \leq t$. Thus the solution of (2.1) satisfies (2.4). By uniqueness of the solution of (2.4) in $[0, t]$ (Corollary 2.2) we have $T_{12}^{(M)}(u) = S_1(u)$ for $0 \leq u \leq t$. Thus $T_{12}^{(M)}(t) = S_1(t)$.

Hence $\omega \in (S_1(t) \leq v)$. It concludes that $(T_{12}^{(M)}(t) \leq v) \subset (S_1(t) \leq v)$.

For any $\omega \in (S_1(t) \leq v)$. $S_1(s)$ is a monotonically nondecreasing function for $s \geq 0$ and recall Corollary 1.2. It follows that $0 \leq S_1(u) \leq S_1(t)$ for $0 \leq u \leq t$ and that $N_{12}(S_1(u)) = N_{12}^v(S_1(u))$ for $0 \leq u \leq t$. Thus the solution of (2.4) satisfies (2.1). By uniqueness of the solution of (2.1) in $[0,t]$ (Corollary 2.1) we have $S_1(u) = T_{12}^{(M)}(u)$ for $0 \le u \le t$. Thus $S_1(t) = T_{12}^{(M)}(t)$.

Hence $\omega \in (T_{12}^{(M)} \le v)$. We conclude $(S_1(t) \le v) \subset (T_{12}^{(M)}(t) \le v)$. ♯

Therefore the proof is completed. □

The martingale parts of $N_{jj+1}(t)$ with respect to the reference family $\sigma(N_{jj+1}(t) : 0 \le s \le t)$ for $1 \le j \le 3$ are represented by

$$\widetilde{N}_{jj+1}(t) = N_{jj+1}(t) - t.$$

Since $X_1^{(M)}(0)$, $X_2^{(M)}(0)$, $X_3^{(M)}(0)$, $N_{12}(*)$, $N_{23}(*)$ and $N_{31}(*)$ are mutually independent, $\widetilde{N}_{jj+1}(t)$ is an $\mathcal{F}_t^{jj+1}$-martingale.

Put

$$\begin{aligned}\mathcal{G}_t^{(M)} &= \sigma(X_j^{(M)}(0) : j = 1,2,3)\\ &\qquad \vee\, \sigma(N_{jj+1}(T_{jj+1}^{(M)}(s)) : 0 \le s \le t, j = 1,2,3),\end{aligned}$$

and

$$\mathcal{H}_t^{(M)} = \sigma(X_j^{(M)}(s) : 0 \le s \le t, j = 1,2,3).$$

We shall recall the general theory in Corollary to Theorem 3.2 of Chapter I of Ikeda-Watnabe [10]. We assume that $(\Omega, (\mathcal{F}_t^{jj+1})_{t\ge 0})$ is a standard measurable space for each j, $1 \le j \le 3$, and let P be a probabilty on $(\Omega, (\mathcal{F}_t^{jj+1})_{t\ge 0})$. Let $\mathcal{G}$ be a sub σ-field of $(\mathcal{F}_t^{jj+1})_{t\ge 0}$ and $P_{\mathcal{G}}(\omega, \cdot)$ be a regular conditional probability given $\mathcal{G}$. Let $\xi(\omega)$ be a mapping from Ω into a measurable space $(S, \mathcal{B})$ such that it is $\mathcal{G}/\mathcal{B}$-measurable. We assume that $\mathcal{B}$ is countably determined and $\{x\} \in \mathcal{B}$ for every $x \in S$. Then

$$P_{\mathcal{G}}(\omega, \{\omega' ; \xi(\omega') = \xi(\omega)\}) = 1 \quad a.a.\omega. \tag{2.5}$$

LEMMA 2.3. *$\mathcal{G}_t^{(M)} \subset \mathcal{F}^{jj+1}_{T_{jj+1}^{(M)}(t)}$ for t, $t \ge 0$, and j, $1 \le j \le 3$, where*

$$\mathcal{F}^{12}_{T_{12}^{(M)}(t)} = \{S \in \mathcal{F}^{12}_\infty : (T_{12}^{(M)}(t) \le u) \cap S \in \mathcal{F}^{12}_u \ \ for\ \ any\ \ u \ge 0\}.$$

PROOF. We consider that $\mathcal{G}_t^{(M)} \subset \mathcal{F}^{12}_{T_{12}^{(M)}(t)}$.

We define

$$N_{12}^{[t]}(s)(\omega) \equiv \begin{cases} N_{12}(s)(\omega), & \text{for } s \le T_{12}^{(M)}(t)(\omega),\\ 0, & \text{for } s > T_{12}^{(M)}(t)(\omega).\end{cases}$$

Since

$$N_{12}^{[t]}(u) = N_{12}(u)\chi_{(u \le T_{12}^{(M)}(t))},$$

we have $(N_{12}^{[t]}(u) \le a) \cap (T_{12}^{(M)}(t) \le v) \in \mathcal{F}_v^{12}$ for any $a \ge 0$. Hence $N_{12}^{[t]}(u)$ is $\mathcal{F}^{12}_{T_{12}^{(M)}(t)}$-measurable. We also have $(N_{23}(u) \le a) \cap (T_{12}^{(M)}(t) \le v) \in \mathcal{F}_v^{12}$

for any $a \geq 0$. Hence $N_{23}(u)$ is $\mathcal{F}^{12}_{T^{(M)}_{12}(t)}$-measurable. Also $N_{31}(u)$ is $\mathcal{F}^{12}_{T^{(M)}_{12}(t)}$-measurable.

We shall prove that $N_{12}(T^{(M)}_{12}(s))$, $N_{23}(T^{(M)}_{23}(s))$ and $N_{31}(T^{(M)}_{31}(s))$ is $\mathcal{F}^{12}_{T^{(M)}_{12}(t)}$-measurable, for $0 \leq s \leq t$.

[Step 1] Put $F = N_{23}(T^{(M)}_{23}(s))$.

We claim that

$$E[F|\mathcal{F}^{12}_{T^{(M)}_{12}(t)}](\omega) = F(\omega).$$

As the mapping in (2.5), we take an $\mathcal{F}^{12}_{T^{(M)}_{12}(t)}$-measurable mapping

$$\xi(\omega') = (N_{23}(u)(\omega') : u \geq 0).$$

It follows that

$$\begin{aligned}
&E[F|\mathcal{F}^{12}_{T^{(M)}_{12}(t)}](\omega) \\
&= \int_\Omega P_{\mathcal{F}^{12}_{T^{(M)}_{12}(t)}}(\omega, d\omega')F(\omega') \\
&= \int_{\{\omega';\xi(\omega')=\xi(\omega)\}} P_{\mathcal{F}^{12}_{T^{(M)}_{12}(t)}}(\omega, d\omega')F(\omega') \\
&= \int_{\{\omega';\xi(\omega')=\xi(\omega)\}} P_{\mathcal{F}^{12}_{T^{(M)}_{12}(t)}}(\omega, d\omega')N_{23}(T^{(M)}_{23}(s)(\omega'), \omega) \\
&= \int_\Omega P_{\mathcal{F}^{12}_{T^{(M)}_{12}(t)}}(\omega, d\omega')f(T^{(M)}_{23}(s)(\omega')),
\end{aligned}$$

where $f(u) = N_{23}(u, \omega)$.

Similarly as in (2.1), for u, $0 \leq u \leq t$, we have

$$(2.6) \quad \begin{cases}
T^{(M)}_{12}(u) = \dfrac{\lambda}{M}\displaystyle\int_0^u (X^{(M)}_1(0) + N^{[t]}_{12}(T^{(M)}_{12}(s)) - N_{31}(T^{(M)}_{31}(s))) \\
\qquad\qquad (X^{(M)}_2(0) + N_{23}(T^{(M)}_{23}(s)) - N^{[t]}_{12}(T^{(M)}_{12}(s)))ds, \\
T^{(M)}_{23}(u) = \dfrac{\lambda}{M}\displaystyle\int_0^u (X^{(M)}_2(0) + N_{23}(T^{(M)}_{23}(s)) - N^{[t]}_{12}(T^{(M)}_{12}(s))) \\
\qquad\qquad (X^{(M)}_3(0) + N_{31}(T^{(M)}_{31}(s)) - N_{23}(T^{(M)}_{23}(s)))ds, \\
T^{(M)}_{31}(u) = \dfrac{\lambda}{M}\displaystyle\int_0^u (X^{(M)}_3(0) + N_{31}(T^{(M)}_{31}(s)) - N_{23}(T^{(M)}_{23}(s))) \\
\qquad\qquad (X^{(M)}_1(0) + N^{[t]}_{12}(T^{(M)}_{12}(s)) - N_{31}(T^{(M)}_{31}(s)))ds, \\
T^{(M)}(0) = 0.
\end{cases}$$

Hence there exists a non-random function H from $\mathbb{R}_+$ to $\mathbb{N}$ such that

$$f(T^{(M)}_{23}(s)(\omega')) = H(s; X^{(M)}(0), N^{[t]}_{12}(u, \omega'), N_{23}(u, \omega'), N_{31}(u, \omega'), u \geq 0).$$

Therefore $f(T_{23}^{(M)}(s)(\omega'))$ is $\mathcal{F}^{12}_{T_{12}^{(M)}(t)}$-measurable.

$$\begin{aligned}
&\int_\Omega P_{\mathcal{F}^{12}_{T_{12}^{(M)}(t)}}(\omega, d\omega') f(T_{23}^{(M)}(s)(\omega')) \\
&= f(T_{23}^{(M)}(s)(\omega))(\omega) \\
&= N_{23}(T_{23}^{(M)}(s)(\omega), \omega) \\
&= F(\omega).
\end{aligned}$$

Hence the claim holds. It follows that $N_{23}(T_{23}^{(M)}(s))$ is $\mathcal{F}^{12}_{T_{12}^{(M)}(t)}$-measurable, for $0 \le s \le t$.

Similary, we prove that $N_{31}(T_{31}^{(M)}(s))$ is $\mathcal{F}^{12}_{T_{12}^{(M)}(t)}$-measurable, for $0 \le s \le t$.

[Step 2] Put $G = N_{12}(T_{12}^{(M)}(s))$.

We claim that

$$E[G|\mathcal{F}^{12}_{T_{12}^{(M)}(t)}](\omega) = G(\omega).$$

As the mapping in (2.5), we take $\mathcal{F}^{12}_{T_{12}^{(M)}(t)}$-measurable mappings

$$\xi_1(\omega') = (N_{12}^{[t]}(u)(\omega') : u \ge 0),$$

and

$$\xi_2(\omega') = T_{12}^{(M)}(t)(\omega'),$$

and note that $T_{12}^{(M)}(t)$, which is the solution of (2.6), is $\mathcal{F}^{12}_{T_{12}^{(M)}(t)}$-measurable. We have

$$\begin{aligned}
&E[G|\mathcal{F}^{12}_{T_{12}^{(M)}(t)}](\omega) \\
&= \int_\Omega P_{\mathcal{F}^{12}_{T_{12}^{(M)}(t)}}(\omega, d\omega') G(\omega') \\
&= \int_{\{\omega';\xi_1(\omega')=\xi_1(\omega)\}\cap\{\omega';\xi_2(\omega')=\xi_2(\omega)\}} P_{\mathcal{F}^{12}_{T_{12}^{(M)}(t)}}(\omega, d\omega') G(\omega') \\
&= \int_{\{\omega';\xi_1(\omega')=\xi_1(\omega)\}\cap\{\omega';\xi_2(\omega')=\xi_2(\omega)\}} P_{\mathcal{F}^{12}_{T_{12}^{(M)}(t)}}(\omega, d\omega') N_{12}(T_{12}^{(M)}(s)(\omega'), \omega) \\
&= \int_\Omega P_{\mathcal{F}^{12}_{T_{12}^{(M)}(t)}}(\omega, d\omega') g(T_{23}^{(M)}(s)(\omega')),
\end{aligned}$$

where $g(u) = N_{12}(u, \omega)$.

Thus $g(T_{12}^{(M)}(s)(\omega'))$ is $\mathcal{F}^{12}_{T_{12}^{(M)}(t)}$-measurable.

$$\begin{aligned}&\int_\Omega P_{\mathcal{F}^{12}_{T_{12}^{(M)}(t)}}(\omega, d\omega')g(T_{12}^{(M)}(s)(\omega'))\\&= g(T_{12}^{(M)}(s)(\omega))(\omega)\\&= N_{12}(T_{12}^{(M)}(s)(\omega), \omega)\\&= G(\omega).\end{aligned}$$

The claim holds. It follows that $N_{12}(T_{12}^{(M)}(s))$ is $\mathcal{F}^{12}_{T_{12}^{(M)}(t)}$-measurable for $0 \leq s \leq t$.

It concludes that

$$\mathcal{G}_t^{(M)} \subset \mathcal{F}^{12}_{T_{12}^{(M)}(t)}.$$

It also holds that $\mathcal{G}_t^{(M)} \subset \mathcal{F}^{23}_{T_{23}^{(M)}(t)}$ and $\mathcal{G}_t^{(M)} \subset \mathcal{F}^{31}_{T_{31}^{(M)}(t)}$. □

For $1 \leq j \leq 3$, put

$$\mathcal{M}_{jj+1}^{(M)}(*) = \widetilde{N}_{jj+1}(T_{jj+1}^{(M)}(*)).$$

THEOREM 2.3. *The process* $X^{(M)}(*) = (X_1^{(M)}(*), X_2^{(M)}(*), X_3^{(M)}(*))$ *is a* $(\mathcal{G}_t^{(M)})_{t\geq 0}$*-semi-martingale such that* ($t \in \mathbb{R}_+$)

$$\begin{cases} X_1^{(M)}(t) = X_1^{(M)}(0) + (\mathcal{M}_{12}^{(M)}(t) - \mathcal{M}_{31}^{(M)}(t)) + (T_{12}^{(M)}(t) - T_{31}^{(M)}(t)),\\ X_2^{(M)}(t) = X_2^{(M)}(0) + (\mathcal{M}_{23}^{(M)}(t) - \mathcal{M}_{12}^{(M)}(t)) + (T_{23}^{(M)}(t) - T_{12}^{(M)}(t)),\\ X_3^{(M)}(t) = X_3^{(M)}(0) + (\mathcal{M}_{31}^{(M)}(t) - \mathcal{M}_{23}^{(M)}(t)) + (T_{31}^{(M)}(t) - T_{23}^{(M)}(t)),\end{cases}$$

gives the Doob-Meyer decomposition and

(i) $\mathcal{M}_{jj+1}^{(M)}(*)$ *are square-integrable* $(\mathcal{G}_t^{(M)})_{t\geq 0}$*-martingales for* $1 \leq j \leq 3$,
(ii) $T_{jj+1}^{(M)}(*)$ *are continuous increasing* $(\mathcal{G}_t^{(M)})_{t\geq 0}$*-adapted processes for* $1 \leq j \leq 3$,
(iii) $< \mathcal{M}_{jj+1}^{(M)}(*) >_t = T_{jj+1}^{(M)}(t)$ *for* $1 \leq j \leq 3$,
(iv) $< \mathcal{M}_{jj+1}^{(M)}(*), \mathcal{M}_{kk+1}^{(M)}(*) >_t = 0$ *for* $1 \leq j, k \leq 3$, $j \neq k$.

COROLLARY 2.3. *The process* $X^{(M)}(*)$ *is an* $(\mathcal{H}_t^{(M)})_{t\geq 0}$*-semi-martingale.*

REMARK 2.1. $\mathfrak{M}_j^{(M)}(*) = \mathcal{M}_{jj+1}^{(M)}(*) - \mathcal{M}_{j-1j}^{(M)}(*)$ *are martingale parts and* $\mathfrak{A}_j^{(M)}(*) = T_{jj+1}^{(M)}(*) - T_{j-1j}^{(M)}(*)$ *are bounded variation parts* ($1 \leq j \leq 3$).

PROOF. We claim that for a counting process N_t whose martingale part is M_t and whose bounded variation part is A_t

$$< M >_t = \int_0^t (1 - \Delta A_s) dA_s.$$

By Ito's formula

$$M_t^2 = 2\int_0^t M_{s-}dM_s + \sum_{0\le s\le t}(\Delta M_s)^2.$$

As N_t is a counting process, $(\Delta N_t)^2 = \Delta N_t$ and $(\Delta M_s)^2 = (\Delta M_t + \Delta A_t) - 2(\Delta M_t + \Delta A_t)\Delta A_t + \Delta A_t^2$. Thus we have

$$M_t^2 = \int_0^t (2M_{s-} + 1 - 2\Delta A_s)dM_s + + \int_0^t (1 - \Delta A_s)dA_s.$$

Therefore the claim holds. ♯

Each counting process $N_{jj+1}(T_{jj+1}^{(M)}(*))$ has the continuous bounded variation part $(1 \le j \le 3)$. Therefore

$$< \widetilde{N}_{jj+1}(T_{jj+1}^{(M)}(*)) >_t = T_{jj+1}^{(M)}(t).$$

As there are no two jumps of the mutually independent Poisson processes $N_{jj+1}(t)$ and $N_{kk+1}(t)$ $(1 \le j,k \le 3,\ j \ne k)$ at the same time t, we have no two jumps of the processes $N_{jj+1}(T_{jj+1}^{(M)}(t))$ and $N_{jj+1}(T_{kk+1}^{(M)}(t))$ $(1 \le j,k \le 3,$ $j \ne k)$ at the same time t. Thus $N_{jj+1}(T_{jj+1}^{(M)}(*)) + N_{kk+1}(T_{kk+1}^{(M)}(*))$ is also a counting process whose bounded variation part is continuous. Hence

$$< \widetilde{N}_{jj+1}(T_{jj+1}^{(M)}(*)) + \widetilde{N}_{kk+1}(T_{kk+1}^{(M)}(*)) >_t = T_{jj+1}^{(M)}(t) + T_{kk+1}^{(M)}(t).$$

On the other hand,

$$\begin{aligned}
&< \widetilde{N}_{jj+1}(T_{jj+1}^{(M)}(*)) + \widetilde{N}_{kk+1}(T_{kk+1}^{(M)}(*)) >_t \\
&=< \widetilde{N}_{jj+1}(T_{jj+1}^{(M)}(*)) >_t + < \widetilde{N}_{kk+1}(T_{kk+1}^{(M)}(*)) >_t \\
&\quad + 2 < \widetilde{N}_{jj+1}(T_{jj+1}^{(M)}(*)), \widetilde{N}_{kk+1}(T_{kk+1}^{(M)}(*)) >_t .
\end{aligned}$$

Therefore

$$< \widetilde{N}_{jj+1}(T_{jj+1}^{(M)}(*)), \widetilde{N}_{kk+1}(T_{kk+1}^{(M)}(*)) >_t = 0.$$

□

3. A weak law of large numbers of model which has a certain stochastic structure

From now on, the norm $||x||$ of the vector $x = (x_1, x_2, \cdots, x_n)$ is to mean $\sum_{1\le i\le n}|x_i|$. We consider integers on mod n and if $j = n$ then we put $j+1 = 1$ and $j = 1$ then we put $j - 1 = n$ (on mod n we use n rather than 0).

Let $z(t) = (z_1(t), \cdots, z_n(t))$ $(t \in \mathbb{R}_+)$ be a solution of the differential equation

$$
(3.1) \qquad \begin{cases} \dfrac{dz_1(t)}{dt} = f^{12}(z_1(t), z_2(t)) - f^{n1}(z_n(t), z_1(t)), \\ \dfrac{dz_2(t)}{dt} = f^{23}(z_2(t), z_3(t)) - f^{12}(z_1(t), z_2(t)), \\ \quad \cdots\cdots\cdots \\ \dfrac{dz_i(t)}{dt} = f^{ii+1}(z_i(t), z_{i+1}(t)) - f^{i-1i}(z_{i-1}(t), z_i(t)), \\ \quad \cdots\cdots\cdots \\ \dfrac{dz_n(t)}{dt} = f^{n1}(z_n(t), z_1(t)) - f^{n-1n}(z_{n-1}(t), z_n(t)), \end{cases}
$$

with the property $\inf_{0 \le s \le t} z_i(s) > 0$ for $1 \le i \le n$ and $\sum_{i=1}^n z_i(0) = 1$. Here $f^{jj+1} = f^{jj+1}(x, y)$ are nonnegative functions on $\mathbb{R}_+^2$ with local Lipschitz conditions for each variable x, y $(1 \le j \le n)$.

By using the same method as in the queuing model by Kogan, Liptser and Smorodinski [16] and Liptser and Shiryayev [21], we show a weak law of large numbers with respect to a model which has the following stochastic structure.

For each $M > 0$, the process $Z^{(M)}(*) = (Z_1^{(M)}(*), \cdots, Z_n^{(M)}(*))$ is an $(\mathcal{H}_t^{(M)})_{t \ge 0}$-semi-martingale such that $(t \in \mathbb{R}_+)$

(i) $Z_i^{(M)}(t) = Z_i^{(M)}(0) + \mathfrak{M}_i^{(M)}(t) + \mathfrak{A}_i^{(M)}(t)$ $(1 \le i \le n)$,

(ii) $\mathfrak{M}_i^{(M)}(t) = \mathcal{M}_{ii+1}^{(M)}(t) - \mathcal{M}_{i-1i}^{(M)}(t)$ $(1 \le i \le n)$,

(iii) $\mathfrak{A}_i^{(M)}(t) = \mathcal{A}_{ii+1}^{(M)}(t) - \mathcal{A}_{i-1i}^{(M)}(t)$ $(1 \le i \le n)$,

(iv) $\mathcal{M}_{jj+1}^{(M)}(*)$ are square-integrable $(\mathcal{H}_t^{(M)})_{t \ge 0}$-martingales $(1 \le j \le n)$,

(v) $\mathcal{A}_{jj+1}^{(M)}(*)$ are continuous increasing $(\mathcal{H}_t^{(M)})_{t \ge 0}$-adapted processes $(1 \le j \le n)$,

(vi) $\mathcal{A}_{jj+1}^{(M)}(t) = \int_0^t M \chi_{\{\frac{Z_j^{(M)}(s)}{M} > 0\}} \chi_{\{\frac{Z_{j+1}^{(M)}(s)}{M} > 0\}} f^{jj+1}(\frac{Z_j^{(M)}(s)}{M}, \frac{Z_{j+1}^{(M)}(s)}{M}) ds$ $(1 \le j \le n)$,

(vii) $< \mathcal{M}_{jj+1}^{(M)}(*) >_t = \mathcal{A}_{jj+1}^{(M)}(t)$ $(1 \le j \le n)$,

(viii) $< \mathcal{M}_{jj+1}^{(M)}(*), \mathcal{M}_{kk+1}^{(M)}(*) >_t = 0$ for $j \ne k$ $(1 \le j, k \le n)$,

where $Z_i^{(M)}(0) \ge 0$ for $1 \le i \le n$, $\sum_{i=1}^n Z_i^{(M)}(0) = 1$ and $\mathcal{H}_t^{(M)} = \sigma(Z_j^{(M)}(s) : 0 \le s \le t, 1 \le j \le n)$.

We introduce random times $T_i^{(M)} = \inf\{t : \frac{Z_i^{(M)}(s)}{M} \le \frac{2}{M}\}$ $(1 \le i \le n)$ and $T_0^{(M)} = \min_{1 \le i \le n} T_i^{(M)}$.

LEMMA 3.1. *$T_i^{(M)}$ is a stopping time with respect to the reference family $(\mathcal{H}_t^{(M)})_{t \ge 0}$ for each $1 \le i \le n$. $T_0^{(M)}$ is a stopping time with respect to the reference family $(\mathcal{H}_t^{(M)})_{t \ge 0}$.*

PROOF. To be proved is, for any $s \in \mathbb{R}_+$,

$$
(3.2) \qquad (T_i^{(M)} \le s) \equiv \{\omega; T_i^{(M)}(\omega) \le s\} \in \mathcal{H}_s^{(M)}.
$$

We decompose $(T_i^{(M)} \leq s)$ into

$$(T_i^{(M)} \leq s) = \{(T_i^{(M)} \leq s) \cap (\frac{Z_i^{(M)}(0)}{M} \leq \frac{2}{M})\} \cup \{(T_i^{(M)} \leq s) \cap (\frac{Z_i^{(M)}(0)}{M} > \frac{2}{M})\}.$$

The first term is

$$(T_i^{(M)} \leq s) \cap (\frac{Z_i^{(M)}(0)}{M} \leq \frac{2}{M}) = (\frac{Z_i^{(M)}(0)}{M} \leq \frac{2}{M}) \in \mathcal{H}_0^{(M)} \subset \mathcal{H}_s^{(M)}.$$

The second term is

$$(T_i^{(M)} \leq s) \cap (\frac{Z_i^{(M)}(0)}{M} > \frac{2}{M}) = \cup_{r \leq s} (\frac{Z_i^{(M)}(r)}{M} \leq \frac{2}{M}) \cap (\frac{Z_i^{(M)}(0)}{M} > \frac{2}{M}).$$

Since $(\frac{Z_i^{(M)}(r)}{M} \leq \frac{2}{M}) \in \mathcal{H}_r^{(M)} \subset \mathcal{H}_s^{(M)}$ and $(\frac{Z_i^{(M)}(0)}{M} > \frac{2}{M}) \in \mathcal{H}_0^{(M)} \subset \mathcal{H}_s^{(M)}$, the second term $(T_i^{(M)} \leq s) \cap (\frac{Z_i^{(M)}(0)}{M} > \frac{2}{M}) \in \mathcal{H}_s^{(M)}$.

Therefore (3.2) holds.

From the general theory, $T_0^{(M)} = \min_{1 \leq i \leq n} T_i^{(M)}$ is also a stopping time. □

THEOREM 3.1. *We assume*

$$\lim_{M \to \infty} \|\frac{Z^{(M)}(0)}{M} - z(0)\| = 0 \quad in \quad probability. \tag{3.3}$$

Then for any $t \in (0, \infty)$

$$\lim_{M \to \infty} \sup_{0 \leq s \leq t} \|\frac{Z^{(M)}(s)}{M} - z(s)\| = 0 \quad in \quad probability.$$

PROOF.

$$\begin{aligned}
&\frac{Z_j^{(M)}(t)}{M} \\
&= \frac{Z_j^{(M)}(0)}{M} + \frac{1}{M}(\mathcal{M}_{jj+1}^{(M)}(t) - \mathcal{M}_{j-1j}^{(M)}(t)) \\
&\quad + \int_0^t \{\chi_{\{\frac{Z_j^{(M)}(s)}{M} > 0\}} \chi_{\{\frac{Z_{j+1}^{(M)}(s)}{M} > 0\}} f^{jj+1}(\frac{Z_j^{(M)}(s)}{M}, \frac{Z_{j+1}^{(M)}(s)}{M}) \\
&\qquad - \chi_{\{\frac{Z_{j-1}^{(M)}(s)}{M} > 0\}} \chi_{\{\frac{Z_j^{(M)}(s)}{M} > 0\}} f^{j-1j}(\frac{Z_{j-1}^{(M)}(s)}{M}, \frac{Z_j^{(M)}(s)}{M})\} ds.
\end{aligned}$$

From the previous lemma, for any $t \in \mathbb{R}_+$,

$$\begin{aligned}
&\frac{Z_j^{(M)}(t \wedge T_0^{(M)})}{M} \\
&= \frac{Z_j^{(M)}(0)}{M} + \frac{1}{M}(\mathcal{M}_{jj+1}^{(M)}(t \wedge T_0^{(M)}) - \mathcal{M}_{j-1j}^{(M)}(t \wedge T_0^{(M)})) \\
&\quad + \int_0^{t \wedge T_0^{(M)}} \{ f^{jj+1}(\frac{Z_j^{(M)}(s)}{M}, \frac{Z_{j+1}^{(M)}(s)}{M}) - f^{j-1j}(\frac{Z_{j-1}^{(M)}(s)}{M}, \frac{Z_j^{(M)}(s)}{M}) \} ds.
\end{aligned}$$

From the assumption of the local Lipschitz condition, there exists a constant C_{Lip} such that

$$\sup_{0 \le x_1 \le 1, 0 \le x_2 \le 1, 0 \le y_1 \le 1, 0 \le y_2 \le 1} \frac{|f^{jj+1}(x_1, y_1) - f^{jj+1}(x_2, y_2)|}{|x_1 - x_2|} \le C_x^{jj+1},$$

$$\sup_{0 \le x_1 \le 1, 0 \le x_2 \le 1, 0 \le y_1 \le 1, 0 \le y_2 \le 1} \frac{|f^{jj+1}(x_1, y_1) - f^{jj+1}(x_2, y_2)|}{|y_1 - y_2|} \le C_y^{jj+1},$$

$$C_{Lip} = \max_{1 \le j \le n} \{ C_x^{jj+1}, C_y^{jj+1} \}. \tag{3.4}$$

Put

$$U_t^{(M)} = || \frac{Z^{(M)}(t)}{M} - z(t) ||.$$

Thus

$$\begin{aligned}
||U_{t \wedge T_0^{(M)}}^{(M)}|| &\le ||U_0^{(M)}|| + \frac{1}{M} ||\mathfrak{M}^{(M)}(t \wedge T_0^{(M)})|| + 2C_{Lip} \int_0^t ||U_s^{(M)}|| ds \\
&\le (||U_0^{(M)}|| + \sup_{0 \le s \le t \wedge T_0^{(M)}} \frac{1}{M} ||\mathfrak{M}^{(M)}(s)||) e^{2C_{Lip} t}.
\end{aligned}$$

For any real number $\epsilon > 0$,

$$\begin{aligned}
&P(\sup_{0 \le s \le t \wedge T_0^{(M)}} ||U_s^{(M)}|| > \epsilon) \\
&\le P(\sup_{0 \le s \le t \wedge T_0^{(M)}} (||U_0^{(M)}|| + \frac{1}{M} ||\mathfrak{M}^{(M)}(s)||) > \epsilon e^{-2C_{Lip} t}).
\end{aligned}$$

Note that

$$\begin{aligned}
&P(\sup_{0 \le s \le t} ||U_s^{(M)}|| > \epsilon) \\
&\le P(T_0^{(M)} < t) + P(\sup_{0 \le s \le t \wedge T_0^{(M)}} ||U_s^{(M)}|| > \epsilon, T_0^{(M)} \ge t) \\
&\le P(T_0^{(M)} < t) + P(\sup_{0 \le s \le t \wedge T_0^{(M)}} (||U_0^{(M)}|| + \frac{1}{M} ||\mathfrak{M}^{(M)}(s)||) > \epsilon e^{-2C_{Lip} t}).
\end{aligned}$$

For any real number $\delta > 0$ we claim

$$\lim_{M\to\infty} P(\sup_{0\le s\le t\wedge T_0^{(M)}} (||U_0^{(M)}|| + \frac{1}{M}||\mathfrak{M}^{(M)}(s)||) > \delta) = 0, \tag{3.5}$$

$$\lim_{M\to\infty} P(T_0^{(M)} < t) = 0. \tag{3.6}$$

We estimate (3.5):

$$\begin{aligned}
&P(\sup_{0\le s\le t\wedge T_0^{(M)}} (||U_0^{(M)}|| + \frac{1}{M}||\mathfrak{M}^{(M)}(s)||) > \delta) \\
&\le P(||U_0^{(M)}|| > \frac{\delta}{2}) + P(\sup_{0\le s\le t\wedge T_0^{(M)}} \frac{1}{M}||\mathfrak{M}^{(M)}(s)|| > \frac{\delta}{2}) \\
&\le P(||U_0^{(M)}|| > \frac{\delta}{2}) + \sum_{1\le j\le n} P(\sup_{0\le s\le t\wedge T_0^{(M)}} \frac{1}{M}|\mathcal{M}_{jj+1}^{(M)}(s) - \mathcal{M}_{j-1j}^{(M)}(s)| > \frac{\delta}{2n}).
\end{aligned}$$

By using Chebyshev's inequality and the inequality for the martingale,

$$\begin{aligned}
&P(\sup_{0\le s\le t\wedge T_0^{(M)}} \frac{1}{M}|\mathcal{M}_{jj+1}^{(M)}(s) - \mathcal{M}_{j-1j}^{(M)}(s)| > \frac{\delta}{2n}) \\
&\le \frac{2n}{\delta} E[\sup_{0\le s\le t\wedge T_0^{(M)}} \frac{1}{M}|\mathcal{M}_{jj+1}^{(M)}(s) - \mathcal{M}_{j-1j}^{(M)}(s)|\ ] \\
&\le \frac{2n}{\delta}\frac{C_{mar}}{M^2} E[< \mathcal{M}_{jj+1}^{(M)}(*) >_{t\wedge T_0^{(M)}} + < \mathcal{M}_{j-1j}^{(M)}(*) >_{t\wedge T_0^{(M)}}] \\
&= \frac{2nC_{mar}}{\delta M} E[\int_0^{t\wedge T_0^{(M)}} f^{jj+1}(\frac{Z_j^{(M)}(v)}{M}, \frac{Z_{j+1}^{(M)}(v)}{M})dv + \int_0^{t\wedge T_0^{(M)}} f^{j-1j}(\frac{Z_{j-1}^{(M)}(v)}{M}, \frac{Z_j^{(M)}(v)}{M})dv] \\
&\le \frac{2nC_{mar}C_0}{\delta M} 2t
\end{aligned}$$

where C_{mar} is the maximum of positive constants for the martingale inqualities and

$$C_0 = \max_{1\le j\le n} \sup_{0\le x\le 1, 0\le y\le 1} f^{jj+1}(x,y). \tag{3.7}$$

By letting M tend to infinity, (3.5) holds.

Now we estimate (3.6). We define the $\{1,2,\cdot\cdot,n\}$-valued function $i_s^{(M)}$ such that $\frac{Z_{i_s^{(M)}}^{(M)}(s)}{M} = \min_{1\le l\le n}\{\frac{Z_l^{(M)}(s)}{M}\}$ for $s \in \mathbb{R}_+$. Here

$$\{T_0^{(M)} < t\} \subset \{T_0^{(M)} \le t\} \subset \{\inf_{0\le s\le t\wedge T_0^{(M)}} \frac{Z_{i_s^{(M)}}^{(M)}(s)}{M} \le \frac{2}{M}\}.$$

We estimate the third term: for any s, $s \le t \wedge T_0^{(M)}$,

$$\begin{aligned}\frac{Z^{(M)}_{i_s^{(M)}}(s)}{M} &\ge z_{i_s^{(M)}}(s) - \left|z_{i_s^{(M)}}(s) - \frac{Z^{(M)}_{i_s^{(M)}}(s)}{M}\right| \\ &\ge \inf_{0\le s\le t} z_{i_s^{(M)}}(s) - \sup_{0\le s\le t\wedge T_0^{(M)}} \|U_s^{(M)}\|.\end{aligned}$$

We put $r = \inf_{0\le s\le t}\min_{1\le i\le n} z_i(s)$ and it follows from the assumption that $r > 0$.

$$\inf_{0\le s\le t\wedge T_0^{(M)}} \frac{Z^{(M)}_{i_s^{(M)}}(s)}{M} \ge r - \sup_{0\le s\le t\wedge T_0^{(M)}} \|U_s^{(M)}\|.$$

We have the relation

$$\{T_0^{(M)} < t\} \subset \{r - \sup_{0\le s\le t\wedge T_0^{(M)}} \|U_s^{(M)}\| \le \frac{2}{M}\}.$$

Therefore

$$P(T_0^{(M)} < t) \le P(\sup_{0\le s\le t\wedge T_0^{(M)}} \|U_s^{(M)}\| \ge r - \frac{2}{M}).$$

When $M \to \infty$, (3.5) concludes (3.6).

Therefore for any $\epsilon > 0$

$$\lim_{M\to\infty} P(\sup_{0\le s\le t} \|U_s^{(M)}\| > \epsilon) = 0.$$

□

4. Application of the weak law of large numbers to paper-scissors-stone model

Let $u(t) = (u_1(t), u_2(t), u_3(t))$ $(t \in \mathbb{R}_+)$ be the solution of the deterministic system expressed by the defferential equation

$$(4.1)\qquad \begin{cases} \dfrac{du_1(t)}{dt} = \lambda(u_1(t)u_2(t) - u_3(t)u_1(t)), \\ \dfrac{du_2(t)}{dt} = \lambda(u_2(t)u_3(t) - u_1(t)u_2(t)), \\ \dfrac{du_3(t)}{dt} = \lambda(u_3(t)u_1(t) - u_2(t)u_3(t)). \end{cases}$$

Remark 4.1. *The system of* (4.1) *has two constants of motion that* $u_1(t) + u_2(t) + u_3(t) = u_1(0) + u_2(0) + u_3(0)$ *and* $u_1(t)u_2(t)u_3(t) = u_1(0)u_2(0)u_3(0)$.

Now, we shall discuss the convergence of $\frac{X^{(M)}(t)}{M}$ to $u(t)$, when M tends to infinity.

By applying the previous general theorem to our model, we have the following theorem.

THEOREM 4.1. *We assume the convergence and conditions:*

$$(4.2)\qquad \begin{cases} \lim_{M\to\infty} |\dfrac{X_1^{(M)}(0)}{M} - u_1(0)| = 0 & in\ probability, \\ \lim_{M\to\infty} |\dfrac{X_2^{(M)}(0)}{M} - u_2(0)| = 0 & in\ probability, \\ \lim_{M\to\infty} |\dfrac{X_3^{(M)}(0)}{M} - u_3(0)| = 0 & in\ probability, \\ 0 < u_1(0) < 1, \\ 0 < u_2(0) < 1, \\ 0 < u_3(0) < 1, \\ u_1(0) + u_2(0) + u_3(0) = 1. \end{cases}$$

Then for any $t \in (0, \infty)$

$$\begin{cases} \lim_{M\to\infty} \sup_{0\le s\le t} |\dfrac{X_1^{(M)}(s)}{M} - u_1(s)| = 0 & in\ probability, \\ \lim_{M\to\infty} \sup_{0\le s\le t} |\dfrac{X_2^{(M)}(s)}{M} - u_2(s)| = 0 & in\ probability, \\ \lim_{M\to\infty} \sup_{0\le s\le t} |\dfrac{X_3^{(M)}(s)}{M} - u_3(s)| = 0 & in\ probability. \end{cases}$$

PROOF. Remark (4.1) and $u_i(0) > 0$ $(i = 1, 2, 3)$ shows that $\inf_{0\le s\le t} u_i(t) > 0$ for any $t \ge 0$.

Put for $1 \le j \le 3$

$$(4.3)\qquad f^{jj+1}(x, y) = h(x, y) \equiv \lambda xy.$$

The constant C_0 of (3.7) is λ. The following estimation holds:

$$\sup_{0\le x_1\le 1, 0\le x_2\le 1, 0\le y_1\le 1, 0\le y_2\le 1} \frac{|h(x_1, y_1) - h(x_2, y_2)|}{|x_1 - x_2|} \le 4\lambda,$$

$$\sup_{0\le x_1\le 1, 0\le x_2\le 1, 0\le y_1\le 1, 0\le y_2\le 1} \frac{|h(x_1, y_1) - h(x_2, y_2)|}{|y_1 - y_2|} \le 4\lambda,$$

Thus we take the constant C_{Lip} of (3.4) to be 4λ. □

5. A central limit theorem of model which has a certain stochastic structure

Similarly as in the queuing model by Kogan, Liptser and Smorodinski [16] and Liptser and Shiryayev [21], we show the following central limit theorem with respect to the model in section 3.

Let $z(t) = (z_1(t), \cdots, z_n(t))$ $(t \in \mathbb{R}_+)$ be a solution of the differential equation (3.1), with the property $\inf_{0\le s\le t} z_i(s) > 0$ for $1 \le i \le n$ and $\sum_{i=1}^n z_i(0) = 1$. In the present case, we assume that $f^{jj+1} = f^{jj+1}(x, y)$ are nonnegative continuously differentiable functions on $\mathbb{R}_+^2$ with local Lipschitz conditions of the derivatives $f_x^{jj+1} = \frac{\partial f^{jj+1}}{\partial x}(x,y)$ for variable y and with local Lipschitz conditions of the derivatives $f_y^{jj+1} = \frac{\partial f^{jj+1}}{\partial y}(x,y)$ for variable x $(1 \leq j \leq n)$.

For each $M > 0$, the process $Z^{(M)}(*)$ has the same stochastic structure as in Theorem 3.1.

Put

$$V^{(M)}(t) = \sqrt{M}(\frac{Z^{(M)}(t)}{M} - z(t)).$$

THEOREM 5.1. *We assume* (3.3) *in Theorem* 3.1.

Let the sequence of random variables $\{V^{(M)}(0)\}_{M\geq 1}$ *converges weakly to a distribution* F.

Then the sequence of the probability distributions of the $\mathbb{R}^n$*-valued processes* $V^{(M)} = (V^{(M)}(t))_{t\geq 0}$ *converges weakly to the distribution of an* $\mathbb{R}^n$*-valued process* $V = (V(t))_{t\geq 0}$ *defined by the stochastic differential equation*

$$dV(t) = b(t)V(t)dt + c^{\frac{1}{2}}(t)dW(t),$$

with an $\mathbb{R}^n$*-valued Wiener process* $W = (W_t)_{t\geq 0}$*, with the initial condition* $V(0)$ *having the distribution* F *and with* $n \times n$ *matrix*

$$b(t) =$$
$$\begin{pmatrix} \frac{\partial f^{12}}{\partial x} - \frac{\partial f^{n1}}{\partial y} & \frac{\partial f^{12}}{\partial y} & 0 & \cdots & \cdots & \cdots & 0 & -\frac{\partial f^{n1}}{\partial x} \\ -\frac{\partial f^{12}}{\partial x} & \frac{\partial f^{23}}{\partial x} - \frac{\partial f^{12}}{\partial y} & \frac{\partial f^{23}}{\partial y} & 0 & \cdots\cdots & & & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f^{n1}}{\partial x} & 0 & \cdots & \cdots & \cdots & 0 & -\frac{\partial f^{n-1n}}{\partial x} & \frac{\partial f^{n1}}{\partial x} - \frac{\partial f^{n-1n}}{\partial y} \end{pmatrix},$$

$$c(t) =$$
$$\begin{pmatrix} f^{12} + f^{n1} & -f^{12} & 0 & \cdots & \cdots & \cdots & 0 & -f^{n1} \\ -f^{12} & f^{23} + f^{12} & -f^{23} & 0 & \cdots\cdots & & & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -f^{n1} & 0 & \cdots & \cdots & \cdots & 0 & -f^{n-1n} & f^{n1} + f^{n-1n} \end{pmatrix}.$$

The (i,j)*-th elements of* $b(t)$ *and* $c(t)$ $(1 \leq i,j \leq n)$ *are given by*

$$b_{ij}(t) =$$
$$\begin{cases} \frac{\partial f^{ii+1}}{\partial x}(z_i(t), z_{i+1}(t)) - \frac{\partial f^{i-1i}}{\partial y}(z_{i-1}(t), z_i(t)), & for\ j = i, \\ \frac{\partial f^{ii+1}}{\partial y}(z_i(t), z_{i+1}(t)), & for\quad j = i+1, \\ -\frac{\partial f^{i-1i}}{\partial x}(z_{i-1}(t), z_i(t)), & for\quad j = i-1, \\ 0, & otherwise. \end{cases}$$

$c_{ij}(t) =$

$$\begin{cases} f^{ii+1}(z_i(t), z_{i+1}(t)) + f^{i-1i}(z_{i-1}(t), z_i(t)), & \textit{for } i = j, \\ -f^{ii+1}(z_i(t), z_{i+1}(t)), & \textit{for} \quad j = i+1, \\ -f^{i-ii}(z_{i-1}(t), z_i(t)), & \textit{for} \quad j = i-1, \\ 0, & \textit{otherwise.} \end{cases}$$

PROOF. Let $V^{(M)}(*) = (V_1^{(M)}(*), \cdots, V_n^{(M)}(*))$, $B^{(M)}(*) = (B_1^{(M)}(*), \cdots, B_n^{(M)}(*))$ and $\mathfrak{m}^{(M),a}(*) = (\mathfrak{m}_1^{(M),a}(*), \cdots, \mathfrak{m}_n^{(M),a}(*))$ be defined by $(1 \le i \le n)$

$$\begin{aligned} V_i^{(M)}(t) = V_i^{(M)}(0) & \\ & + \int_0^t \sqrt{M}\{\chi_{\{\frac{Z_i^{(M)}(s)}{M} > 0\}} \chi_{\{\frac{Z_{i+1}^{(M)}(s)}{M} > 0\}} f^{ii+1}(\frac{Z_i^{(M)}(s)}{M}, \frac{Z_{i+1}^{(M)}(s)}{M}) \\ & \qquad - f^{ii+1}(z_i(s), z_{i+1}(s))\} ds \\ & - \int_0^t \sqrt{M}\{\chi_{\{\frac{Z_{i-1}^{(M)}(s)}{M} > 0\}} \chi_{\{\frac{Z_i^{(M)}(s)}{M} > 0\}} f^{i-1i}(\frac{Z_{i-1}^{(M)}(s)}{M}, \frac{Z_i^{(M)}(s)}{M}) \\ & \qquad - f^{i-1i}(z_{i-1}(s), z_i(s))\} ds \\ & + \frac{1}{\sqrt{M}}(\mathcal{M}_{ii+1}^{(M)}(t) - \mathcal{M}_{i-1i}^{(M)}(t)), \\ B_i^{(M)}(t) = & \int_0^t \sqrt{M}\{\chi_{\{\frac{Z_i^{(M)}(s)}{M} > 0\}} \chi_{\{\frac{Z_{i+1}^{(M)}(s)}{M} > 0\}} f^{ii+1}(\frac{Z_i^{(M)}(s)}{M}, \frac{Z_{i+1}^{(M)}(s)}{M}) \\ & \qquad - f^{ii+1}(z_i(s), z_{i+1}(s))\} ds \\ & - \int_0^t \sqrt{M}\{\chi_{\{\frac{Z_{i-1}^{(M)}(s)}{M} > 0\}} \chi_{\{\frac{Z_i^{(M)}(s)}{M} > 0\}} f^{i-1i}(\frac{Z_{i-1}^{(M)}(s)}{M}, \frac{Z_i^{(M)}(s)}{M}) \\ & \qquad - f^{i-1i}(z_{i-1}(s), z_i(s))\} ds, \\ \mathfrak{m}_i^{(M),a}(*) = & \chi_{\{\frac{1}{\sqrt{N}} \le a\}} \frac{1}{\sqrt{M}} \mathfrak{M}_i^{(M)}(*). \end{aligned}$$

Note that $(1 \le j, k \le n)$

$$\begin{aligned} & < \mathfrak{m}_j^{(M),a}(*), \mathfrak{m}_k^{(M),a}(*) >_t \\ = & \begin{cases} \chi_{\{\frac{1}{\sqrt{N}} \le a\}} \frac{1}{M}(\mathcal{A}_{jj+1}^{(M)}(t) + \mathcal{A}_{j-1j}^{(M)}(t)), & \text{for} \quad j = k, \\ -\chi_{\{\frac{1}{\sqrt{N}} \le a\}} \frac{1}{M} \mathcal{A}_{jj+1}^{(M)}(t), & \text{for} \quad k = j+1, \\ -\chi_{\{\frac{1}{\sqrt{N}} \le a\}} \frac{1}{M} \mathcal{A}_{j-1j}^{(M)}(t), & \text{for} \quad k = j-1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Now, we present the following general conditions which are known in [21, 16, 1, 33].

For each $T > 0$, $a \in (0, 1]$ and $j, k = 1, 2, \cdots, n$

(A) $\lim_{M \to \infty} \sup_{0 \le t \le T} ||\Delta V^{(M)}(t)|| = 0$ in probability,

(B) $\lim_{M\to\infty}\sup_{0\le t\le T}||B^{(M)}(t) - \int_0^t b(s, V^{(M)}(s))ds|| = 0$ in probability,

(C) $\lim_{M\to\infty}\sup_{0\le t\le T}| < \mathfrak{m}_j^{(M),a}(*), \mathfrak{m}_k^{(M),a}(*) >_t - \int_0^t c_{jk}(s, V^{(M)}(s))ds| =$ 0 in probability,

and so-called conditions of "linear growth" of the functions of $b(t, V(t))$ and $c(t, V(t))$ $(t \in \mathbb{R}_+)$

(I) $||b(t, V(t))|| \le L(t)(1 + \sup_{0\le s\le t}||V(s)||)$,

(II) $\sum_{j=1}^n |c_{jj}(t, V(t))| \le L(t)(1 + \sup_{0\le s\le t}||V(s)||^2)$,

(III) $\int_0^t L(s)ds < \infty$.

By assuminig the above conditions, the sequence of the $\mathbb{R}^n$-valued processes $V^{(M)} = (V^{(M)}(t))_{t\ge 0}$ converges in distribution to an $\mathbb{R}^n$-valued process $V = (V(t))_{t\ge 0}$ defined by the stochastic differential equation

$$dV(t) = b(t, V(t))dt + c^{\frac{1}{2}}(t, V(t))dW(t),$$

with an $\mathbb{R}^n$-valued Wiener process $W = (W(t))_{t\ge 0}$ consisting of independent components, as M tends to infinity ([21]).

Now we shall prove these conditions.

Condition of "linear growth" is clear because of the local Lipschitz property of the functions. We prove three conditions (A), (B) and (C) in the following steps.

[Step 1] We claim that condition (A) holds.

For any $t > 0$,

$$||\Delta V^{(M)}(t)|| = \sqrt{M}||\frac{\Delta Z^{(M)}(t)}{M}|| \le \frac{1}{\sqrt{M}}.$$

Hence condition (A) holds, since for any $\epsilon > 0$,

$$P(||\Delta V^{(M)}(t)|| > \epsilon) \le \frac{1}{\epsilon}E[||\Delta V^{(M)}(t)||] \le \frac{1}{\epsilon\sqrt{M}}.$$

[Step 2] We claim that

$$\lim_{M\to\infty} P(\int_0^t \chi_{\{\frac{Z_i^{(M)}(s)}{M}=0\}} ds > 0) = 0, \tag{5.1}$$

for any $t \in \mathbb{R}_+$.

The following estimate holds:

$$P(\int_0^t \chi_{\{\frac{Z_i^{(M)}(s)}{M}=0\}} ds > 0) \le P(\inf_{0\le s\le t}\frac{Z_i^{(M)}(s)}{M} = 0).$$

Since

$$\inf_{0\le s\le t}\frac{Z_i^{(M)}(s)}{M} \le \inf_{0\le s\le t} z_i(s) - \sup_{0\le s\le t}|\frac{Z_i^{(M)}(s)}{M} - z_i(s)|,$$

we have

$$P(\int_0^t \chi_{\{\frac{Z_i^{(M)}(s)}{M}=0\}} ds > 0) \le P(\sup_{0\le s\le t} |\frac{Z_i^{(M)}(s)}{M} - z_i(s)| \ge \inf_{0\le s\le t} z_i(s)).$$

From the weak law of large numbers (Theorem 3.1) and from the assumption of $\inf_{0\le s\le t} z_i(s) > 0$, the claim (5.1) holds.

[Step 3] We claim that $B^{(M)}(t)$ is replaced by $\overline{B^{(M)}(t)}$ in condition (B) and that $< \mathfrak{m}_j^{(M),a}(*), \mathfrak{m}_k^{(M),a}(*) >_t$ is replaced by $\overline{< \mathfrak{m}_j^{(M),a}(*), \mathfrak{m}_k^{(M),a}(*) >_t}$ in condition (C), where

$$\begin{aligned}\overline{B_i^{(M)}(t)} = & \int_0^t \sqrt{M}\{f^{ii+1}(\frac{Z_i^{(M)}(s)}{M}, \frac{Z_{i+1}^{(M)}(s)}{M}) - f^{ii+1}(z_i(s), z_{i+1}(s))\}ds \\ & - \int_0^t \sqrt{M}\{f^{i-1i}(\frac{Z_{i-1}^{(M)}(s)}{M}, \frac{Z_i^{(M)}(s)}{M}) - f^{i-1i}(z_{i-1}(s), z_i(s))\}ds,\end{aligned}$$

$$\overline{< \mathfrak{m}_j^{(M),a}(*), \mathfrak{m}_k^{(M),a}(*) >_t}$$

$$= \begin{cases} \chi_{\{\frac{1}{\sqrt{M}}\le a\}}\{\int_0^t f^{jj+1}(\frac{Z_j^{(M)}(s)}{M}, \frac{Z_{j+1}^{(M)}(s)}{M})ds \\ \quad + \int_0^t f^{j-1j}(\frac{Z_{j-1}^{(M)}(s)}{M}, \frac{Z_j^{(M)}(s)}{M})ds\}, & \text{for } j = k, \\ -\chi_{\{\frac{1}{\sqrt{M}}\le a\}} \int_0^t f^{jj+1}(\frac{Z_j^{(M)}(s)}{M}, \frac{Z_{j+1}^{(M)}(s)}{M})ds, & \text{for } k = j+1, \\ -\chi_{\{\frac{1}{\sqrt{M}}\le a\}} \int_0^t f^{j-1j}(\frac{Z_{j-1}^{(M)}(s)}{M}, \frac{Z_j^{(M)}(s)}{M})ds, & \text{for } k = j-1, \\ 0, & \text{otherwise.}\end{cases}$$

We consider the case of $\overline{B^{(M)}(t)}$. The following estimate holds:

$$\begin{aligned}&\sup_{t\le T} \|B^{(M)}(t) - \int_0^t b(s)V^{(M)}(s)ds\| \\ &\le \sup_{t\le T} \|\overline{B^{(M)}(t)} - \int_0^t b(s)V^{(M)}(s)ds\| \\ &\quad + \sum_{i=1}^n \sup_{t\le T} |\int_0^t \sqrt{M}\chi_{\{\frac{Z_i^{(M)}(s)}{M}=0 \ or \ \frac{Z_{i+1}^{(M)}(s)}{M}=0\}} f^{ii+1}(\frac{Z_i^{(M)}(s)}{M}, \frac{Z_{i+1}^{(M)}(s)}{M})ds \\ &\qquad - \int_0^t \sqrt{M}\chi_{\{\frac{Z_{i-1}^{(M)}(s)}{M}=0 \ or \ \frac{Z_i^{(M)}(s)}{M}=0\}} f^{i-1i}(\frac{Z_{i-1}^{(M)}(s)}{M}, \frac{Z_i^{(M)}(s)}{M})ds| \\ &\le \sup_{t\le T} \|\overline{B^{(M)}(t)} - \int_0^t b(s)V^{(M)}(s)ds\| \\ &\quad + C_0\sqrt{M}\sum_{i=1}^n \sup_{t\le T}\{2\int_0^t \chi_{\{\frac{Z_i^{(M)}(s)}{M}=0\}} ds + \int_0^t \chi_{\{\frac{Z_{i+1}^{(M)}(s)}{M}=0\}} ds + \int_0^t \chi_{\{\frac{Z_{i-1}^{(M)}(s)}{M}=0\}} ds\}\end{aligned}$$

$$\leq \sup_{t\leq T}||\overline{B^{(M)}(t)} - \int_0^t b(s)V^{(M)}(s)ds||$$

$$+ C_0\sqrt{M}\sum_{i=1}^{n}\{2\int_0^T \chi_{\{\frac{Z_i^{(M)}(s)}{M}=0\}} ds + \int_0^T \chi_{\{\frac{Z_{i+1}^{(M)}(s)}{M}=0\}} ds + \int_0^T \chi_{\{\frac{Z_{i-1}^{(M)}(s)}{M}=0\}} ds\},$$

where C_0 is defined by (3.7). Thus

$$\begin{aligned}
&P(\sup_{t\leq T}||B^{(M)}(t) - \int_0^t b(s)V^{(M)}(s)ds|| > \epsilon)\\
&\leq P(\sup_{t\leq T}||\overline{B^{(M)}(t)} - \int_0^t b(s)V^{(M)}(s)ds|| > \frac{\epsilon}{2})\\
&\quad + \sum_{j=1}^{n} P(C_0\sqrt{M}\int_0^T \chi_{\{\frac{Z_j^{(M)}(s)}{M}=0\}} ds > \frac{\epsilon}{8n})\\
&\leq P(\sup_{t\leq T}||\overline{B^{(M)}(t)} - \int_0^t b(s)V^{(M)}(s)ds|| > \frac{\epsilon}{2})\\
&\quad + \sum_{j=1}^{n} P(\int_0^T \chi_{\{\frac{Z_j^{(M)}(s)}{M}=0\}} ds > 0).
\end{aligned}$$

When we take the limit of $M \to \infty$, from (5.1)

$$\begin{aligned}
&\lim_{M\to\infty} P(\sup_{t\leq T}||B^{(M)}(t) - \int_0^t b(s)V^{(M)}(s)ds|| > \epsilon)\\
&\leq \lim_{M\to\infty} P(\sup_{t\leq T}||\overline{B^{(M)}(t)} - \int_0^t b(s)V^{(M)}(s)ds|| > \frac{\epsilon}{2}).
\end{aligned}$$

The estimate with respect to condition (C) is done in a similar way.
Therefore the claîm holds.

[Step 4] We claim that condition (B) holds.
From the assumption of the local Lipschitz conditions of derivatives, there is a positive constant $C_{Lip}^{(1)}$ such that

$$\sup_{0\leq x_1\leq 1, 0\leq x_2\leq 1, 0\leq y_1\leq 1, 0\leq y_2\leq 1} \frac{|f_y^{jj+1}(x_1,y_1) - f_y^{jj+1}(x_2,y_2)|}{|x_1 - x_2|} \leq C_{xy}^{jj+1},$$

$$\sup_{0\leq x_1\leq 1, 0\leq x_2\leq 1, 0\leq y_1\leq 1, 0\leq y_2\leq 1} \frac{|f_x^{jj+1}(x_1,y_1) - f_x^{jj+1}(x_2,y_2)|}{|y_1 - y_2|} \leq C_{yx}^{jj+1},$$

(5.2) $$C_{Lip}^{(1)} = \max_{1\leq j\leq n}\{C_{xy}^{jj+1}, C_{yx}^{jj+1}\}.$$

Considering [Step 3], we have the following estimate:

$$\begin{aligned}
&\sup_{t\le T}\|\overline{B^{(M)}(t)}-\int_0^t b(s)V^{(M)}(s)ds\| \\
\le &\int_0^T \|\sqrt{M}\{(f^{ii+1}(\frac{Z_i^{(M)}(s)}{M},\frac{Z_{i+1}^{(M)}(s)}{M})-f^{ii+1}(z_i(s),z_{i+1}(s))) \\
&\quad -(f^{i-1i}(\frac{Z_{i-1}^{(M)}(s)}{M},\frac{Z_i^{(M)}(s)}{M})-f^{i-1i}(z_i(s),z_{i+1}(s)))\}-\sum_{j=1}^n b_{ij}(s)V_j^{(M)}(s)\|ds \\
\le &\int_0^T \|V_i^{(M)}(s)f_x^{ii+1}(z_i(s)+\theta^{ii+1}(\frac{Z_i^{(M)}(s)}{M}-z_i(s)),z_{i+1}(s)+\theta^{ii+1}(\frac{Z_{i+1}^{(M)}(s)}{M}-z_{i+1}(s))) \\
&\quad +V_{i+1}^{(M)}(s)f_y^{ii+1}(z_i(s)+\theta^{ii+1}(\frac{Z_i^{(M)}(s)}{M}-z_i(s)),z_{i+1}(s)+\theta^{ii+1}(\frac{Z_{i+1}^{(M)}(s)}{M}-z_{i+1}(s))) \\
&\quad -V_{i-1}^{(M)}(s)f_x^{i-1i}(z_{i-1}(s)+\theta^{i-1i}(\frac{Z_{i-1}^{(M)}(s)}{M}-z_{i-1}(s)),z_i(s)+\theta^{i-1i}(\frac{Z_i^{(M)}(s)}{M}-z_i(s))) \\
&\quad -V_i^{(M)}(s)f_y^{i-1i}(z_{i-1}(s)+\theta^{i-1i}(\frac{Z_{i-1}^{(M)}(s)}{M}-z_{i-1}(s)),z_i(s)+\theta^{i-1i}(\frac{Z_i^{(M)}(s)}{M}-z_i(s))) \\
&\quad -V_i^{(M)}(s)f_x^{ii+1}(z_i(s),z_{i+1}(s))-V_{i+1}^{(M)}(s)f_y^{ii+1}(z_i(s),z_{i+1}(s)) \\
&\quad +V_{i-1}^{(M)}(s)f_x^{i-1i}(z_{i-1}(s),z_i(s))+V_i^{(M)}(s)f_y^{i-1i}(z_{i-1}(s),z_i(s))\|ds \\
\le &\sup_{t\le T}\|V^{(M)}(t)\|\sup_{t\le T}\|\frac{Z^{(M)}(t)}{M}-z(t)\|4C_{Lip}^{(1)}T,
\end{aligned}$$

where $\theta^{jj+1}\in[0,1]$ $(1\le j\le n)$ are parameters in the mean value theorem.

Hence

$$\begin{aligned}
&P(\sup_{t\le T}\|\overline{B^{(M)}(t)}-\int_0^t b(s)V^{(M)}(s)ds\|\ge\epsilon) \\
&\le P(\sup_{t\le T}\|V^{(M)}(t)\|\ge l)+P(\sup_{t\le T}\|\frac{Z^{(M)}(t)}{M}-z(t)\|\ge\frac{\epsilon}{4lC_{Lip}^{(1)}T}).
\end{aligned}$$

If

$$\lim_{l\to\infty}\overline{\lim}_{M\to\infty}P(\sup_{t\le T}\|V^{(M)}(t)\|\ge l)=0, \tag{5.3}$$

then, from the weak law of large numbers (Theorem 3.1), for any $\delta>0$ there exists an integer l such that

$$\begin{aligned}
&P(\sup_{t\le T}\|V^{(M)}(t)\|\ge l)<\delta, \\
&P(\sup_{t\le T}\|\frac{Z^{(M)}(t)}{M}-z(t)\|\ge\frac{\epsilon}{4lC_{Lip}^{(1)}T})<\delta.
\end{aligned}$$

Therefore

$$\overline{\lim}_{M\to\infty} P(\sup_{t\leq T} ||\overline{B^{(M)}(t)} - \int_0^t b(s)V^{(M)}(s)ds|| \geq \epsilon) = 0.$$

Now, we shall prove (5.3).

$$\begin{aligned}||V^{(M)}(t)|| \leq& ||V^{(M)}(0)|| + \int_0^t 2C_{Lip}||V^{(M)}(s)||ds \\ &+ C_0\sqrt{M}\sum_{i=1}^n \int_0^t \{2\chi_{\{\frac{Z_i^{(M)}(s)}{M}=0\}} + \chi_{\{\frac{Z_{i+1}^{(M)}(s)}{M}=0\}} + \chi_{\{\frac{Z_{i-1}^{(M)}(s)}{M}=0\}}\}ds \\ &+ \sup_{0\leq s\leq t}\frac{1}{\sqrt{M}}|\mathcal{M}_{ii+1}^{(M)}(s) - \mathcal{M}_{i-1i}^{(M)}(s)|,\end{aligned}$$

where C_0 and C_{Lip} are defined by (3.7) and (3.4).

By using Gromwell's inequality,

$$\begin{aligned}||V^{(M)}(t)|| \leq& \{||V^{(M)}(0)|| \\ &+ C_0\sqrt{M}\sum_{i=1}^n \int_0^t \{2\chi_{\{\frac{Z_i^{(M)}(s)}{M}=0\}} + \chi_{\{\frac{Z_{i+1}^{(M)}(s)}{M}=0\}} + \chi_{\{\frac{Z_{i-1}^{(M)}(s)}{M}=0\}}\}ds \\ &+ \sum_{i=1}^n \sup_{0\leq s\leq t}\frac{1}{\sqrt{M}}|\mathcal{M}_{ii+1}^{(M)}(s) - \mathcal{M}_{i-1i}^{(M)}(s)|\}\cdot e^{2C_{Lip}t}.\end{aligned}$$

Thus (5.3) is estimated by

$$\begin{aligned}P(\sup_{t\leq T}||V^{(M)}(t)|| \geq l) \leq& P(||V^{(M)}(0)|| \geq \frac{l}{3C_1}) \\ &+ \sum_{i=1}^n P(\int_0^T \chi_{\{\frac{Z_i^{(M)}(s)}{M}=0\}}ds > 0) \\ &+ \sum_{i=1}^n P(\sup_{0\leq s\leq T}\frac{1}{\sqrt{M}}|\mathcal{M}_{ii+1}^{(M)}(s) - \mathcal{M}_{i-1i}^{(M)}(s)| \geq \frac{l}{3nC_1}),\end{aligned}$$

where $C_1 = e^{2C_{Lip}t}$. From the assumption of the theorem, the first term is convergent to zero in probability as M tends to infinity. From (5.1), the second term is convergent to zero in probability as M tends to infinity. By using Chebyshev's inequality and the martingale inequality, the third term is estimated:

$$\begin{aligned}&P(\sup_{0\leq s\leq T}\frac{1}{\sqrt{M}}|\mathcal{M}_{ii+1}^{(M)}(s) - \mathcal{M}_{i-1i}^{(M)}(s)| \geq \frac{l}{3nC_1}) \\ &\leq \frac{3nC_1C_{mar}}{lM}E[<\mathcal{M}_{ii+1}^{(M)}(*)>_T + <\mathcal{M}_{i-1i}^{(M)}(*)>_T] \\ &\leq \frac{3nC_0C_1C_{mar}}{l}2T.\end{aligned}$$

where C_0 is defined by (3.7) and where C_{mar} is the maximum of positive constants for the martingale inequalities. Thus the third term is convergent to zero in probability, as l tends to infinity.

Therefore the claim holds.

[Step 5] We claim that condition (C) holds.

By [Step 3], we prove that $(1 \leq j, k \leq n)$

$$\lim_{M\to\infty} \sup_{t\leq T} |\overline{< \mathfrak{m}_j^{(M),a}(*), \mathfrak{m}_k^{(M),a}(*) >_t} - \int_0^t c_{jk}(s)ds| = 0 \quad in \quad probability.$$

We take the integer M as $M > \frac{1}{a^2}$.

There are no interactions between j and k for $2 \leq |j-k| \leq n-2$. Hence condition (C) holds for this case.

We consider the case of diagonal elements.

$$\begin{aligned}
&\sup_{t\leq T} |\overline{< \mathfrak{m}_i^{(M),a}(*) >_t} - \int_0^t c_{ii}(s)ds| \\
&= \sup_{t\leq T} |\int_0^t \{f^{ii+1}(\frac{Z_i^{(M)}(s)}{M}, \frac{Z_{i+1}^{(M)}(s)}{M}) - f^{ii+1}(z_i(s), z_{i+1}(s))ds \\
&\qquad + \int_0^t \{f^{i-1i}(\frac{Z_{i-1}^{(M)}(s)}{M}, \frac{Z_i^{(M)}(s)}{M}) - f^{i-1i}(z_{i-1}(s), z_i(s))ds\}ds| \\
&\leq 2C_{Lip}T \sup_{t\leq T} ||\frac{Z^{(M)}(s)}{M} - z(s)||,
\end{aligned}$$

where C_{Lip} is defined by (3.4). This term is convergent to zero in probability, from the weak law of large numbers of Theorem 3.1.

Moreover,

$$\begin{aligned}
&\sup_{t\leq T} |\overline{< \mathfrak{m}_i^{(M),a}(*), \mathfrak{m}_{i+1}^{(M),a}(*) >_t} - \int_0^t c_{ii+1}(s)ds| \\
&= \sup_{t\leq T} |\int_0^t \{-f^{ii+1}(\frac{Z_i^{(M)}(s)}{M}, \frac{Z_{i+1}^{(M)}(s)}{M}) + f^{ii+1}(z_i(s), z_{i+1}(s))\}ds| \\
&\leq C_{Lip}T \sup_{t\leq T} ||\frac{Z^{(M)}(s)}{M} - z(s)||.
\end{aligned}$$

This term is also convergent to zero in probability, from the weak law of large numbers of Theorem 3.1.

Therefore the claim holds. □

REMARK 5.1. *It is easy to see that the matrix $c(t)$ has eigenvalue zero and the eigenvector $(1, 1, \cdots, 1)$. Hence we consider the eigenvector $(*, *, \cdots, *, 0)$ which is independent of $(1, 1, \cdots, 1)$. In the restricted $(n-1) \times (n-1)$ matrix of $c(t)$ all determinants of the leading minor matrix are positive. Thus the restricted $(n-1) \times (n-1)$ matrix is positive definite. Consequently, the matrix $c(t)$ is positive semi-definite.*

6. Application of the central limit theorem to paper-scissors-stone model

We apply Theorem 5.1 to our model.
Put

$$Y^{(M)}(t) = \sqrt{M}\left(\frac{X^{(M)}(t)}{M} - u(t)\right),$$

for $t \in \mathbb{R}_+$. A sequence of the process $Y^{(M)} = (Y^{(M)}(t))_{t\geq 0}$ admits the following central limit theorem.

THEOREM 6.1. *We assume* (4.2) *in Theorem* 4.1.

Let the sequence of random variables $\{Y^{(M)}(0)\}_{M\geq 1}$ *converge weakly to a distribution* G.

Then the sequence of the probability distributions of the $\mathbb{R}^3$*-valued processes* $Y^{(M)} = (Y^{(M)}(t))_{t\geq 0}$ *converges weakly to the distribution of an* $\mathbb{R}^3$*-valued process* $Y = (Y(t))_{t\geq 0}$ *defined by the stochastic equation in the vector form*

$$dY(t) = b(t)Y(t)dt + c^{\frac{1}{2}}(t)dW(t), \tag{6.1}$$

with an $\mathbb{R}^3$*-valued Wiener process* $W = (W_t)_{t\geq 0}$, *with the initial condition* $Y(0)$ *having the distribution* G *and with* 3×3 *matrix*

$$b(t) = \begin{pmatrix} \lambda(u_2(t) - u_3(t)) & \lambda u_1(t) & -\lambda u_1(t) \\ -\lambda u_2(t) & \lambda(u_3(t) - u_1(t)) & \lambda u_2(t) \\ \lambda u_3(t) & -\lambda u_3(t) & \lambda(u_1(t) - u_2(t)) \end{pmatrix},$$

$$c(t) =$$

$$\begin{pmatrix} \lambda(u_1(t)u_2(t) + u_3(t)u_1(t)) & -\lambda u_1(t)u_2(t) & -\lambda u_3(t)u_1(t) \\ -\lambda u_1(t)u_2(t) & \lambda(u_2(t)u_3(t) + u_1(t)u_2(t)) & -\lambda u_2(t)u_3(t) \\ -\lambda u_3(t)u_1(t) & -\lambda u_2(t)u_3(t) & \lambda(u_3(t)u_1(t) + u_2(t)u_3(t)) \end{pmatrix}.$$

PROOF. Recall (4.3). We have the derivatives $f_x^{jj+1} = \frac{\partial h}{\partial x}(x,y) = \lambda y$ and $f_y^{jj+1} = \frac{\partial h}{\partial y}(x,y) = \lambda x$ $(1 \leq j \leq 3)$. The constant $C_{Lip}^{(1)}$ of (5.2) is λ. □

7. Experimental study for paper-scissors-stone model

The solution of the system of the ordinary differential equations (4.1) is analytically solved by using an elliptic function which is so-called Weierstrass type. By Theorem 4.1 the construction of the solution of the paper-scissors-stone model is convergent in probability to the solution of the system of the ordinary differential equation (4.1). Note that the solution is applicable to approximate numerical values of the elliptic function.

The actual behavior of (4.1) is computed by the fourth-order Runge-Kutta method with the step width 1/1000 (Henrici [9]). The system of (4.1) has two conserved quantities (Remark 4.1). The Runge-Kutta method is done for the vector $u(t) = (u_1(t), u_2(t), 1 - u_1(t) - u_2(t))$ $(t \in \mathbb{R}_+)$ and the conserved quantity $u_1(t) \cdot u_2(t) \cdot (1 - u_1(t) - u_2(t))$ is observed to be constant in five significant digits.

By the construction of (1.2) and (1.3) in Theorem 1.1, we perform the simulation for (1.1). Pseudo-random numbers generated by the linear congruential method which is the same method as in section 3 of chapter I, are used in the simulations. The simulation study of the system (1.1) is done in the cases of $M = 100$, $M = 1000$ and $M = 10000$. Figs II.2, II.3 and II.4 show that the deterministic process of (4.1) and the stochastic processes of $M = 100, 1000, 10000$, where we set $\lambda = 1$, $\frac{X^{(M)}(0)}{M} = u(0) = (0.4, 0.25, 0.35)$. When M tends to be larger, as we see from Figs II.2 - II.4, we observe that the process of (1.1) approaches to the deterministic system. Thus Theorem 4.1 for (1.1) is seen from the numerical experiment.

We do not know the definite solution of (6.1) in general. From Theorem 6.1 the solution of (6.1) is approximated by $Y^{(M)}(*)$ for large M. As the theorem states the convergence in distribution, several sample paths are needed in order to investigate (6.1). Figure II.5 shows, for example, 10 sample paths of $Y_1^{(M)}(*)$ with $M = 10000$, where we set $\lambda = 1$, $\frac{X^{(M)}(0)}{M} = u(0) = (0.4, 0.25, 0.35)$. The actual behavior of the density of the process $Y_1(*)$ is considered to be close to the distribution behavior in this figure, as we set M to be large. Theorem 6.1 leads the numerical observation in Figure II.5.

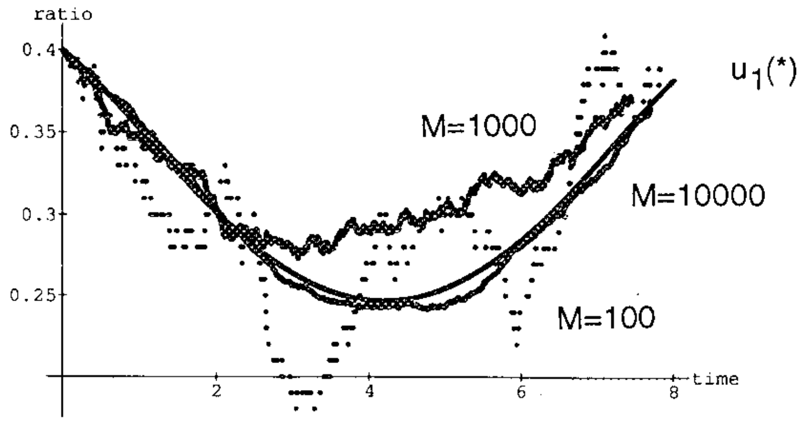

FIGURE II.2. Species 1:From inside to outside, $u_1(*)$, $\frac{X_1^{(M)}(*)}{M}$ ($M = 10000, 1000, 100$)(vertical) against time (horizontal).

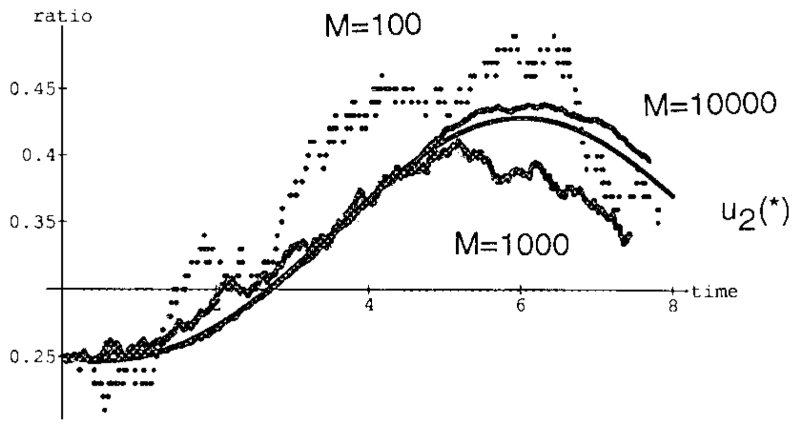

FIGURE II.3. Species 2:From inside to outside, $u_2(*)$, $\frac{X_2^{(M)}(*)}{M}$ ($M = 10000, 1000, 100$)(vertical) against time (horizontal).

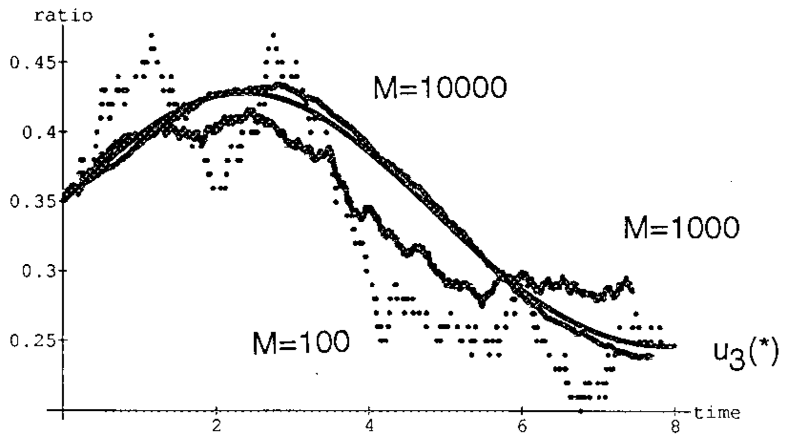

FIGURE II.4. Species 3:From inside to outside, $u_3(*)$, $\frac{X_3^{(M)}(*)}{M}$ ($M = 10000, 1000, 100$)(vertical) against time (horizontal).

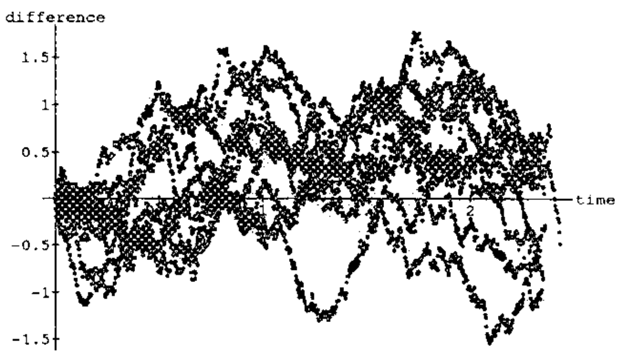

FIGURE II.5. 10 sample paths of $Y_1^{(M)}(*)$ with $M = 10000$ (vertical) against time (horizontal).

CHAPTER III

Mutation model compared with Ohta-Kimura model

1. Stepwise-mutation model

Ohta and Kimura [29, 30] formulated the following model called the stepwise-mutation model.

Let us assume that particles which are gametes exist in the integer-numbered sites of allelic states. The total number of particles is M. The set of allelic states is constructed by discrete points of n sites in a one-dimensinal lattice with periodic boundary condition. We suppose that the offspring is produced by a Markov chain. The offspring of any gamete of the site i is assumed to mutate to the site $i-1$ with probability $\frac{p}{2}$ or to the site $i+1$ with probability $\frac{p}{2}$ or remain in site i with probability $1-p$. A site is chosen with probability proportional to the frequency in the previous generation, one of the particles of the chosen site changes the allelic state with the probabilities $\frac{p}{2}$, $1-p$ and $\frac{p}{2}$ and the process is repeated $2N_e$ times. Here N_e is the effective size of population (Wright [37] and Ewens [5]). We set one step to be a process of one choice of a gamete. The next generation is considered to be obtained after $2N_e$ steps of Markov chain.

2. Time-change model

Let us consider the model which satisfies the followings:

(i) There are n integer-numbered sites of allelic states in a one-dimensional lattice with periodic boundary condition. The number of particles belonging to the i-th site at time t is denoted by $X_i(t)$. Genetically particles are interpreted as gametes. The time t is measured by the same unit as the step in the stepwise-mutation model. The total number M is to be $X_1(t) + X_2(t) + \cdots + X_n(t) = M$.

(ii) A particle of site i mutates to a particle of site $i+1$ or to a particle of site $i-1$, where $i = 1, \cdots, n$. If $i = n$ then we set $i+1 = 1$ and if $i = 1$ then we set $i-1 = n$ from now on.

(iii) Frequency of mutations to the neighboring sites per one particle is μdt during time interval $[t, t+dt)$ on the average, where μ is a positive constant.

(iv) Each particle is in a chaotic bath of particles. Each mutating particle is equally likely chosen.

We assume the following model which satisfy the above four conditions:

$$
(2.1)\quad \begin{cases}
X_1(t) = X_1(0) - N_1^r(\mu\int_0^t X_1(s)ds) - N_1^l(\mu\int_0^t X_1(s)ds) \\
\qquad\qquad + N_n^r(\mu\int_0^t X_n(s)ds) + N_2^l(\mu\int_0^t X_2(s)ds), \\
X_2(t) = X_2(0) - N_2^r(\mu\int_0^t X_2(s)ds) - N_2^l(\mu\int_0^t X_2(s)ds) \\
\qquad\qquad + N_1^r(\mu\int_0^t X_1(s)ds) + N_3^l(\mu\int_0^t X_3(s)ds), \\
\qquad\qquad \cdots\cdots\cdots \\
X_i(t) = X_i(0) - N_i^r(\mu\int_0^t X_i(s)ds) - N_i^l(\mu\int_0^t X_i(s)ds) \\
\qquad\qquad + N_{i-1}^r(\mu\int_0^t X_{i-1}(s)ds) + N_{i+1}^l(\mu\int_0^t X_{i+1}(s)ds), \\
\qquad\qquad \cdots\cdots\cdots \\
X_n(t) = X_n(0) - N_n^r(\mu\int_0^t X_n(s)ds) - N_n^l(\mu\int_0^t X_n(s)ds) \\
\qquad\qquad + N_{n-1}^r(\mu\int_0^t X_{n-1}(s)ds) + N_1^l(\mu\int_0^t X_1(s)ds), \\
X_1(0) + X_2(0) + \cdots + X_n(0) = M,
\end{cases}
$$

where $X_i(0)$ are nonnegative initial values ($i = 1, 2, \cdots, n$) and where N_j^α are standard Poisson processes ($j = 1, 2, \cdots, n$ and $\alpha = r, l$). Here we assume that there are no jumps of N_j^r and N_j^l at the same time for each j, $1 \le j \le n$ and that there are no accumulation points of jump times of standard Poisson processes.

We assume that N_j^α in (2.1) are mutually independent standard Poisson processes ($j = 1, 2, \cdots, n$ and $\alpha = r, l$). In this paper the model (2.1) is called a time-change model for simplicity.

3. Statistical method for comparing time-change model with stepwise-mutation model

We measure time by the step in the stepwise-mutation model from now on. In order to compare the continuous time model with the discrete time model, we regard that the Markov chain of the stepwise-mutation model is embedded in a Markov process of continuous time.

It is difficult to investigate the time-change model theoretically, in particular its waiting time. This is because the jump time of the system of equation (2.1) is determined by the complicatedly linked factors. Thus we study the waiting time of the time-change model by the following statistical method.

At first we have the probability function of the waiting time k to be $f(k|p) = p(1-p)^{k-1}$ for the stepwise-mutation model. The expected value of the waiting time is $\sum_{k=1}^{\infty} kf(k|p) = 1/p$. Then we expect almost constant waiting time against the number of jumps of the system. Thus we will have the nearly linear relation between the number of jumps of the system and the jump time, if we take sufficiently large steps.

When waiting time data $x = (x_1, \cdots, x_m)$ are given, the likelihood for the parameter p is calculated as $L(p|x_1, \cdots, x_m) = \prod_{i=1}^{m} f(x_i|p)$. Thus the logarithmic likelihood is

$$\begin{aligned} \log L(p|x_1, \cdots, x_m) &= \sum_{i=1}^{m} \log f(x_i|p) \\ &= m \log p + \left\{ \left(\sum_{i=1}^{m} x_i \right) - m \right\} \log(1-p), \end{aligned} \tag{3.1}$$

and the maximum likelihood estimator of p is

$$\widehat{p} = \frac{m}{\sum_{i=1}^{m} x_i}. \tag{3.2}$$

In other words the relation (3.2) corresponds to the inverse value of the mean of waiting times for a fixed sample path.

When there are sufficiently large number of data of waiting times, we have the asymptotic normality of the maximum likelihood estimator $\widehat{p}$ in (3.2). Then it holds

$$\widehat{p} \sim \mathcal{N}(p_*, \frac{1}{m} I^{-1}). \tag{3.3}$$

Here p_* is a consistent estimator and I is the Fisher information by

$$\begin{aligned} I &= E_X [\frac{\partial \log f(X|p)}{\partial p} \frac{\partial \log f(X|p')}{\partial p'}]_{p=p'=p_*} \\ &= -E_X [\frac{\partial^2 \log f(X|p)}{\partial p^2}]_{p=p_*} \\ &= \frac{1}{p_*^2 (1-p_*)}, \end{aligned}$$

where we use the relation $\sum_{k=1}^{\infty} k^2 f(k|p) = (2-p)/p^2$.

Hence the variance in (3.3) is given by

$$\frac{1}{m} I^{-1} = \frac{p_*^2 (1-p_*)}{m}. \tag{3.4}$$

Let the standard deviation of the maximum likelihood estimator be $\sigma_*(m)$, we have $\sigma_*(m) = \sqrt{p_*^2(1-p_*)/m}$. The standard deviation $\sigma_*(m)$ is a characteristic quantity of the stepwise-mutation model with its m waiting time data and is known to be the smallest from the asymptotic efficiency.

The logarithmic likelihood is a biased estimator of Kullback-Leibler information. In fact we must use the bias correction. But in this paper we use the logarithmic likelihood as the measure of the similarity between two models for simplicity with ignoring small bias. Thus the asymptotic normality (3.3) holds. We will not discuss the bias correction in this paper.

We compare the time-change model with the stepwise-mutation model as follows.

Through computer simulation of the time-change model we get data of the jump times of the system of equation (2.1) and we fit the data to the stepwise-mutation model directly. Let $p(m)$ be the right hand side of (3.2) computed from m waiting times generated through one simulation for the time-change model. Through several computer simulations we get the sample mean $\overline{p(m)}$ and the sample standard deviation $s(m)$. Then we regard $\overline{p(m)}$ as the consistent estimate p_*. By using it we have the standard deviation $\sigma_*(m)$. We compare $s(m)$ with $\sigma_*(m)$. Note that the sample standard deviation $s(m)$ is an estimate of a characteristic quantity for the time-change model. Thus we check the similarity of the two models by using these quantities.

4. Simulation study by using the statistical method

A computer simulation for the time-change model is performed by generating jump times of standard Poisson processes and by determining the jumps of the system of equation (2.1) in order from the first jump. The jump of the system occurs when one of the integrals reaches the jump time of the standard Poisson process. In Figure III.6 we show the empirical relation between the number of jumps of the system and the time at which the jump of the system occurs, where we set parameters $\mu = \frac{1}{5000}$, $M = 1000$ and $n = 10$. Here we put randomly chosen initial value and totally we have 1000 jumps of the system. This simulation is done by using pseudo-random numbers generated by the linear congruential method which is the same method as in section 3 of chapter I. Surprisingly we have the nearly linear relation of the jump of the system and the jump time in Figure III.6. The number of jumps to the site of the positive direction is 515 and the number of jumps to the site of the negative direction is 485. This near equality of the number of two kinds of jumps is expected from the symmetry of equation (2.1). In Figure III.7 we give a similar empirical relation as in Figure III.6, when we have more successive 9000 jumps of the system. The nearly linear relation is also observed in Figure III.7. By these observations we guess that the model has a kind of uniformity of the jump time of the system. This fact leads to the nearly constant mean value of the waiting time for the fixed sample path. The mean value of waiting times is 2.507 for 1000 jumps of the system and 2.500 for total 10000 jumps in this simulation. By the linear regression for the present example of 1000 jumps of the system, the slope of the jump time of the system against the number of jumps is 2.537 with the standard error 0.001, when we let the line pass the origin. This value of the slope is close to the mean value 2.507 of waiting times.

We discuss the concept of the generation in the time-change model. In genetics the concept "generation" is brought in order to explain seasonal breeding for example. When we interpret the present system of the time-change model genetically, we need to measure time t divided by $2N_e$. For example we consider the special case of $N_e = M$. Note that the range of the vertical axes of Figs. III.6 and III.7 roughly corresponds to 1.2 and 12.5 generations respectively.

Figure III.8 gives waiting times against the number of jumps of the system obtained by taking differences of the jump times in Figure III.6. We compute the logarithmic likelihood by (3.1) and plot it against the parameter $p \in [0,1]$ in Figure III.9. The inverse values of the mean of waiting times for Figs. III.6 and III.7 are $p(1000) = 0.399$ and $p(10000) = 0.400$ respectively in three significant digits.

We perform other independent 10 simulations of 10000 jumps. We use good tested physical random numbers in these and the following simulations. In Table III.1 we give $p(100)$, $p(1000)$ and $p(10000)$ for these simulations. Here we set the same values of parameters as in Figs. III.6 and III.7 with the randomly chosen initial value. From Table III.1 it is obvious that $p(m)$ is almost constant regardless of m and sample path.

In Table III.1 we see that the variance of $p(m)$ gets smaller as m tends larger. This tendency is expected also in the stepwise-mutation model from (3.4). In order to investigate it in detail, we perform independent 50 simulations with several values of μ. Here we set the same values of M and n as in Figs. III.6 and III.7 with the randomly chosen initial value. Table III.2 shows the sample mean $\overline{p(m)}$ and the sample standard deviation $s(m)$ for $m = 1000$ and 10000 through these simulations. The sample mean $\overline{p(m)}$ is observed to be almost independent of m and to be almost constant. We observe that $s(10000)$ is smaller than $s(1000)$. By this fact we guess that $s(m)$ tends to be smaller as m gets larger. If $\overline{p(10000)}$ is regarded as p_* in the stepwise-mutation model because of near equality of $\overline{p(1000)}$ and $\overline{p(10000)}$, the standard deviation of the maximum likelihood estimator is computed as $\sigma_*(m) = \sqrt{\overline{p(10000)}^2(1-\overline{p(10000)})/m}$. We put in Table III.2 its value in the 4th and 7th columns. We see that $s(m)$ is almost coincident with $\sigma_*(m)$ for $m = 1000$ and 10000 in Table III.2. Thus the efficiency of the time-change model is considered to be nearly equivalent to that of the stepwise-mutation model.

If the value of μ becomes larger than, roughly speaking, $\frac{1}{2000}$ for the parameters $M = 1000$ and $n = 10$ (found by experiments in the example of Table III.2), then $\overline{p(m)}$ is observed to exceed the value 1. As $\overline{p(m)}$ is regarded as the parameter in the stepwise-mutation model, the corespondence of the time-change model with the former breaks in this point. When the value of μ gets larger from 0 as shown in Table III.2, $s(m)$ is observed to be larger than $\sigma_*(m)$. The waiting times in the time-change model tend to distribute nearby zero. On the other hand it never occurs in the stepwise-mutation model. Thus $s(m)$ is expected to get larger in this case. On the contrary, when the value of μ is close to 0 as in Table III.2, $s(m)$ is considered to be almost coincident with $\sigma_*(m)$, as the waiting times in the time-change model tend to distribute away from the neighborhood of zero. Accordingly the correspondence between the stepwise-mutation model and the time-change model is fairly good when the parameter μ is smaller than, for example, $\frac{1}{2000}$ for $M = 1000$ and $n = 10$. This threshfold value varies for combinations of values of M and n. In relation to this, we observe the nearly linear relation between μ and $\overline{p(m)}$ as is seen in Table III.2. From this fact the threshold value $\mu = \frac{1}{2000}$ is expected in the above case.

Moreover we have done computer simulations for some combinations of values of μ, M and n ($2 \le M \le 10^6$, $2 \le n \le 10^3$ and suitable μ) with the randomly chosen initial value. In all simulations which include the simulations of Table III.2 we observe that the time-change model well corresponds to the stepwise-mutation model.

TABLE III.1. Inverse values $p(m)$ of the mean of waiting times with $M = 1000$, $n = 10$ and $\mu = \frac{1}{5000}$.

experiments	$m = 100$	$m = 1000$	$m = 10000$
1	0.420	0.412	0.407
2	0.326	0.405	0.398
3	0.414	0.425	0.403
4	0.383	0.407	0.402
5	0.456	0.420	0.397
6	0.403	0.396	0.401
7	0.352	0.391	0.399
8	0.377	0.405	0.395
9	0.475	0.406	0.400
10	0.376	0.418	0.405

TABLE III.2. Independent 50 simulations with several μ's with $M = 1000$ and $n = 10$.

μ	$\overline{p(1000)}$	$s(1000)$	$\sigma_*(1000)$	$\overline{p(10000)}$	$s(10000)$	$\sigma_*(10000)$
$\frac{1}{2500}$	0.800	0.022	0.011	0.798	0.009	0.004
$\frac{1}{5000}$	0.398	0.016	0.010	0.401	0.004	0.003
$\frac{1}{50000}$	0.0398	0.0012	0.0012	0.0400	0.0004	0.0004
$\frac{1}{500000}$	0.00399	0.00013	0.00013	0.00400	0.00004	0.00004

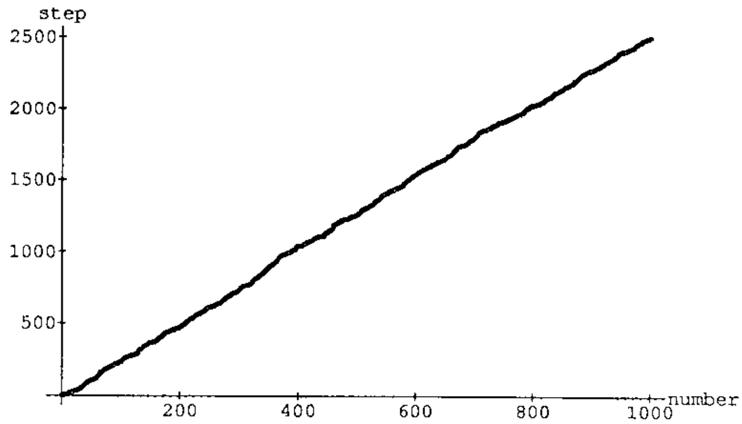

FIGURE III.6. Empirical relation between the number of jumps of the system (horizontal) and the jump time (vertical). The case of 1000 jumps.

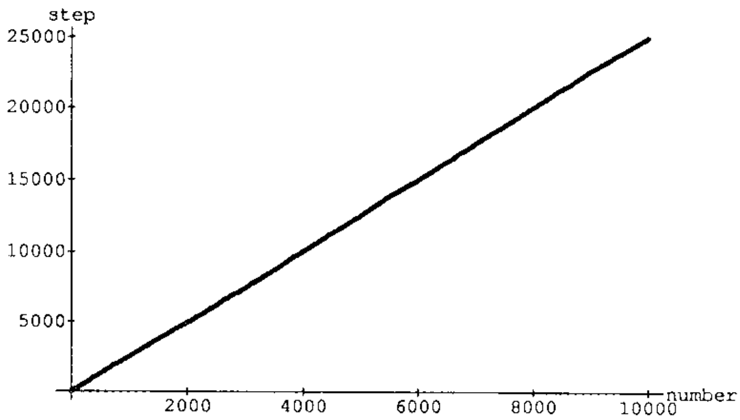

FIGURE III.7. Empirical relation between the number of jumps of the system (horizontal) and the jump time (vertical). The case of 10000 jumps.

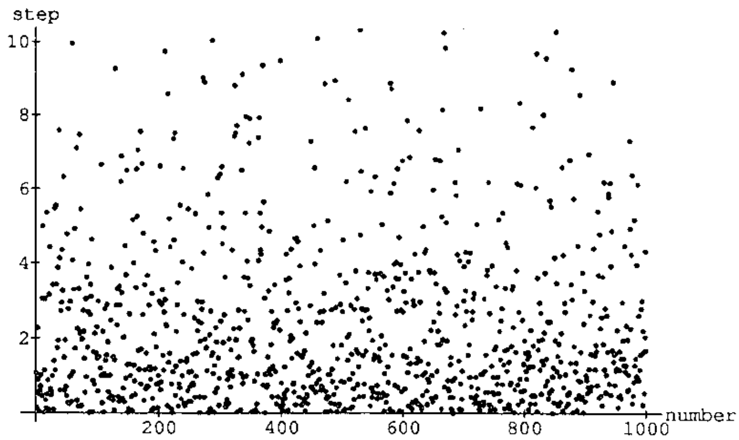

FIGURE III.8. Waiting times (vertical) against the number of jumps of the system (horizontal).

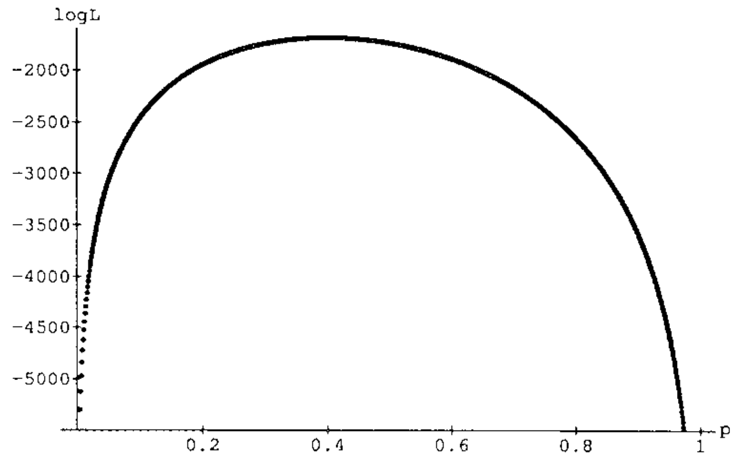

FIGURE III.9. Logarithmic likelihood (vertical) against the parameter $p \in [0, 1]$ (horizontal).

Acknowledgement

I am very grateful to Professor Y. Okabe and Professor Y. Itoh for the kind guidance in their seminar. Professor Y. Okabe taught me probability theory, in particular martingale method, and gave me many kind suggestions, comments and advice. Professor Y. Itoh suggested me the interesting subject of the present thesis.

I am encouraged by Professor H. Akaike throughout the preparation of this study. Professor S. Konishi and Professor N. Ohsumi kindly taught me basis of statistical analysis, particularly multivariate analysis. I feel gratitude to Professor T. Shiga for helpful suggestions as to the present thesis. Professor R. Shimizu kindly taught me methods of studying limit theorems in probability theory. Professor M. Tanemura kindly taught me Monte-Carlo method to make experimental study for problems. Professor N. Yoshida kindly gave me many kind suggestions.

Bibliography

[1] D. Aldous, *Stopping time and tightness*, Ann. Probab., **6** (1978), 335–340.

[2] D.A. Dawsom and K.J.Hochberg, *Wandering random measures in the Fleming–Viot model*, Ann. Probab., **10**, (1982), 554–580.

[3] J.L. Doob, *Stochastic processes*, New York, John Wiley & Sons, (1953).

[4] S. N. Ethier and T. G. Kurtz, *Markov processes*, New York, John Wiley & Sons, (1986).

[5] W.J. Ewens, *Mathematical population genetics*, New York, Springer, (1979).

[6] R.A. Fisher, *The general theory of natural selection*, New York, Dover, (1958).

[7] W.H. Fleming and M. Viot, *Some measure valued Markov processes in population genetics theory*, Indiana Univ. Math., **28**, (1979), 817–843.

[8] L.L. Helms, *Ergodic properties of several interacting Poisson particles*, Advances in Math., **12**, (1974), 32–57.

[9] P. Henrici, *Discrete variable methods in ordinary differential equations*, New York, John Wiley & Sons, (1973).

[10] N. Ikeda and S. Watanabe, *Stochastic differential equations and diffusion processes*, Amsterdam, North–Holland, (1971).

[11] Y. Itoh, *On a ruin problem with interaction*, Ann. Inst. Statist. Math., **25** (1973), 854–858.

[12] ——, *Random collision models in oriented graphs*, J. Appl. Prob., **16** (1979), 36–44.

[13] ———, *Random collision process on oriented graph*, Research Memorandum No 154, (1979),The Institute of Statistical Mathematics.

[14] ———, *Representations for Wright model in population genetics*, Research Memorandum No 201, (1981),The Institute of Statistical Mathematics.

[15] M. Kac, *Probability and related topics in physical sciences*, London, Interscience Publisher, (1959).

[16] Ya. A. Kogan, R. Sh. Liptser and A. V. Smorodinski, *Gaussian diffusion approximation of closed Markov models of computer networks*, Prob. Inf. Transmission, **22** (1986), 38–51.

[17] T. G. Kurtz, *The optional sampling theorem for martingales indexed by directed sets*, Ann. Probab., **8**, (1980), 675–681.

[18] ———, *Representations of Markov processes as multiparameter time changes*, Ann. Probab., **8**, (1980), 682–715.

[19] ———, *Approximation of population processes*, CBMS-NSF Regional Conf. Series in Appl. Math. **36**, Philadelphia, SIAM, (1981).

[20] J. Lamperti, *On random time substitutions and the Feller property*, *Markov processes and potential theory*, Joshua Chover Ed., New York, Wiley, (1967), pp.87-101.

[21] R. Sh. Liptser and A. N. Shiryayev, *Martingale theory*, Dordrecht, Kluwer Academic Publishers, (1989).

[22] A. J. Lotka, *Elements of physical biology*, Baltimore, Williams & Wilkins, (1925).

[23] H. Mano, *Limit theorem for random collision model with simulation study*, Research Memorandum No 487, (1993),The Institute of Statistical Mathematics.

[24] H. Mano and M. Tanemura, *Simualtion study for mutation model represented by random time change of Poisson process compared with Ohta-Kimura model*, Research Memorandum No 486, (1993),The Institute of Statistical Mathematics.

[25] P.A. Meyer, *Probabilities ét potentiel*, Paris, Hermann, (1966).

[26] J. W. Moon, *Topics on tournaments*, New York, Holt, Linehart & Winston, (1968).

[27] P. A. P. Moran, *Random processes in genetics*, Proc. Camb. Phil. Soc., **54** (1958), 60-71

[28] ———, *Wandering distributions and the electrophoretical profile*, Theoret. Popn. Biol., **8** (1975), 318–330.

[29] T. Ohta and M. Kimura, *A model of mutation appropriate to estimate the number of electrophoretically detectable alleles in a finite population*, Genetics, **61** (1969), 763–771.

[30] ———, *Simulation studies on electrophoretically detectable genetic variability in a finite population*, Genetics, **76** (1974), 615–624.

[31] Y. Okabe, H. Mano and Y. Itoh, *Random collision model for interacting population and its strong law of large numbers*, Research Memorandum No 484, (1993),The Institute of Statistical Mathematics.

[32] ———, *Paper-scissors-stone model for interacting population and its limit theorem*, Research Memorandum No 485, (1993),The Institute of Statistical Mathematics.

[33] R. Rebolledo, *La methode des martingale appliquee à l'etude de la convergence en loi de processus*, Bull. Soc. Math. France, Mémories, **62**, (1979), 1–125.

[34] V. A. Volkonskii, *Random substitution of time in strong Markov processes*, Theory Probab. Appl., **3**, (1958), 310–326.

[35] V. Volterra, *Leçons sur la theorie mathematique de la lutte pour la vie*, Scientifique VII, Paris, Gauthier-Villars, (1931).

[36] T. Watanabe, M. Natori and T. Okuni, *Numerical computation software by Fortran 77*, (in Japanese) Tokyo, Maruzen, (1989).

[37] S. Wright, *Evolution in Mendelian populations*, Genetics, **16**, (1931), 97–159.